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THE CALCULATION OF NONLINEAR SUPERSONIC CONICAL FLOWS BY THE METHOD OF INTEGRAL RELATIONS

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FOREWORD

This research program was initiated 1 January 1962 by the Control Criteria Branch of the Flight Dynamics Laboratory, Research and Technology Division (formerly the Flight Control Laboratory of the Aeronautical Systems Division). The work reported herein is a portion of the joint effort carried out by the Grumman Aircraft Engineering Corporation, Bethpage, New York, and the Aeronautical Research Associates of Princeton, Inc., Princeton, New Jersey. The remaining portions of the program are reported in FDL-TDR-64-17 and FDL-TDR-64-82. The program has been supported primarily under Contract AF33(657)-7313, which was within Project No. 8219 and Task No. 821902. Mr. Donald Hoak was the project engineer for the Control Criteria Branch. The author wishes to express his appreciation to Mr. Hoak for his encouraging aid during the course of this program. The author also would like to thank Mr. Robert O'Brien of the Digital Computing Group of the Research Department of the Grumman Aircraft Engineering Corporation for tending to the sometimes unpleasant tasks associated with the numerical computations.

ABSTRACT

The supersonic flow over cones of arbitrary shape has been investigated by using the method of integral relations. The equations are derived for one strip using a body-oriented coordinate system. Various choices for the basic equations are discussed. The solution of the boundary value problem for a number of fundamentally different cases is discussed in detail and various suggestions concerning improvements of the method and other related methods are made. Due to severe computational difficulties, however, no numerical results are presented.

This report has been reviewed and is approved.

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SYMBOLS

The principal symbols used in this report are given in the following list. These symbols plus any necessary auxiliary symbols are also defined in the text when they are first used.

a	isentropic speed of sound
h_s, h_n, h_t	scale factors in coordinate system s, n, t
K_b	reduced curvature of the body surface
M	Mach number
N	number of strips used
P	pressure
q	magnitude of crossflow velocity
r, φ, θ	spherical polar coordinate system
r, ξ, η	orthogonal conical coordinate system
R	gas constant
s, n, t	general body-oriented orthogonal coordinate system
S	specific entropy
T_i, S_i, K_i	particular terms or groupings of terms in the equations of motion
v_r, v_φ, v_θ	velocity components in the spherical coordinate system r, φ, θ
v_r, v_ξ, v_η	velocity components in the conical coordinate system r, η, ξ
v_s, v_n, v_t	velocity components in orthogonal coordinate system s, n, t
v_T, v_N	crossflow velocity components tangent and normal to the shock wave, respectively

SYMBOLS (Cont.)

V	magnitude of velocity at a point
V_{max}	maximum adiabatic velocity for a perfect gas
x_i, $\overline{x}$	vector quantities referring to $v_{\xi b}$, v_{rb}, Υ, β respectively
y_i, $\overline{y}$	vector quantity referring to perturbation quantities
α	angle of attack
β	angle between a plane normal to the shock wave and a surface ξ = constant
γ	adiabatic index
δ	angle between planes ξ = constant and φ = constant
κ	angle between free stream crossflow velocity and planes ξ = constant
ρ	density
Υ	shock layer thickness measured in surface ξ = constant
χ_ξ, χ_η	reduced conical scale factors for the coordinate system r, ξ, η
Ω_1, Ω_2, ... Ω_{10}	coefficients relating the derivatives of certain quantities to $d\Upsilon/d\xi$

Subscripts

b	conditions at the body surface
i	refers to a particular equation of motion (e.g., $i = 1$ is the continuity equation)
k	refers to quantities on the k^{th} strip boundary ($k = N$ is the shock wave)
s	conditions at the shock wave
0	stagnation conditions
∞	free stream conditions

SYMBOLS (Cont.)

Superscripts

i	denotes inverse method
~	quantities at crossflow stagnation points
*	dimensional quantities; also used with certain coefficients defined in the text
^	denotes integration from the body to the strip boundary $\eta = \Upsilon/2$
−	bar over symbol denotes vector
=	double bar over symbol denotes matrix

1. INTRODUCTION

The problem of the hypersonic flow over cones of arbitrary cross section is discussed in this report. The approximate method of solution employed is the method of integral relations due to A. A. Dorodnitsyn. The use of the term hypersonic, above, is rather loose and is not based upon the relative size of any parameters. Rather it stems from the fact that solutions, particularly surface pressures, show good agreement with exact calculations in the range $M_\infty \geq 3\text{-}5$, which is the range of interest here.

Many other authors have studied a variety of fluid mechanics problems using this technique, which is primarily applicable to problems involving two independent variables. However, by making extra approximations, problems in three independent variables can also be treated in the spirit of this report. An example of this is the work of Holt (Ref. 1) and some unpublished work of the present author. To cite a few examples, we mention: The work of Belotserkovskii (Ref. 2) and Traugott (Ref. 3) for the flow over smooth blunt bodies; that of Gold and Holt (Ref. 4) and Belotserkovskii (Ref. 5) for the flow over flat faced blunt bodies; that of Vaglio-Laurin (Ref. 6) for the flow over two dimensional asymmetric blunt bodies of quite arbitrary shape; and that of Pallone (Ref. 7) for some boundary layer problems. Additional references may be found in the article by Dorodnitsyn (Ref. 8). Of particular interest in the present context are the articles by Chushkin and Shchennikov (Ref. 9) and Kennet (Ref. 10). These papers have considered some of the problems described in the present work. However, we have taken a more general approach to the problem and although our numerical work has not been successful, we have uncovered some results not obtained in Refs. 9 and 10. The above papers are discussed at appropriate places in the text.

As mentioned, we considered the hypersonic flow over cones. Specifically, we hoped to develop a method for calculating the flow over cones which would be valid in and above the Mach number range where linear theory becomes inaccurate, but where approximate hypersonic theories are not yet applicable. Again we emphasize that this range is, to a certain extent, undefinable. For example, in Ref. 9, a calculation is presented at $M_\infty = 3.53$ for a circular cone at

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incidence which compares well with the Stone-Kopal results (Ref. 11). Also, calculations to be presented later (similar results are shown in Ref. 9) show that for circular cones at zero angle of attack the agreement with Kopal's tables (Ref. 12) is excellent for strong shock waves ($M_\infty \sin \theta_s \gtrsim 1$).

In an approximate method, such as is being employed here, one must be careful not to oversimplify the problem to the extent that important effects are ignored. This viewpoint is succinctly expressed in Ref. 13 where it is stated: "In search of a suitable approximate method one is obliged, on the one hand, to eliminate difficulties not connected with the essence of the problem; on the other hand, one endeavors to preserve difficulties one meets which are of inherent character. The specifically investigated problem is such that an approximate method claiming success must not avoid the following difficulties: 1) nonlinearity; 2) effects stemming from the mixed type of the equations; 3) explicit dependencies." The problem referred to in this case concerned blunt axisymmetric and two dimensional bodies. However, the comments apply equally well in the present case. It will be shown that each of these conditions is met by the present theory and in fact, some results which could not be discerned from the basic partial differential equations are found. The other authors, mentioned previously, have also found these three conditions to be true, the most important, of course, being the existence of a singular line, analogous to the sonic (parabolic) line of the exact theory.

In the previous paragraphs we briefly described the problem under consideration. In the following paragraphs, the work described herein will be briefly outlined. In addition, pertinent comments concerning other problems employing the method of integral relations will be made where applicable. Unfortunately, the numerical calculations performed have shown themselves to be extremely sensitive to the initial values so that no numerical results are presently available.

In the next section, we shall discuss the choice of the coordinate system, the approximations, and the basic equations. We have employed the perfect gas equations throughout.[1†] It is shown that a body-oriented coordinate system leads to singular points of the derived system of ordinary differential equations which coincide with the sonic points of the flow. More or less simultaneously, the approximations to be used and the choice of basic equations are discussed. Consideration of the results of other investigators shows that the use of linear profiles across one strip can be expected to yield satisfactory results and on this basis a set of ordinary dif-

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† Superscripts refer to notes at the end of the report.

ferential equations for certain flow properties are derived. It is also shown that different groupings of terms in the basic equations lead to restrictions on the cone geometry. It is then shown that results similar to those discussed occur in other, nonrelated, problems. The final part of the next section presents a derivation of the equations governing the flow in the two-strip approximation. In this case we note that the entropy distribution on the boundary between strips is not determined from the solution and must be found independently. From the foregoing discussions we conclude that an interesting variety of results can be obtained from the choice of coordinate system, approximations, basic equations, etc.

In the following section the properties of the derived boundary value problem are investigated. Since the equations are quite nonlinear one cannot expect to find analytic solutions, so a numerical scheme must be employed. Hence, the resulting two-point boundary value problem must be solved by an iterative procedure[2] and we must concern ourselves with the relationship between initial conditions available and end conditions to be satisfied.

For the one-strip approximation it is shown that in certain cases where the crossflow is entirely subsonic, no problems arise. However, if one must allow the locations of the crossflow stagnation point to be free, it may not be possible to obtain a solution. When the flow becomes of mixed type, it is found that the introduction of an inner shock is necessary to satisfy all the boundary conditions. Again, in this case, stagnation points may be moving and a solution may not exist satisfying <u>all</u> conditions. However, it may be possible to obtain an approximate solution by ignoring one of the conditions connected with the stagnation point. For the two-strip case similar conclusions apply, but in addition, the matrix of coefficients of the derivatives at the stagnation point becomes singular and thus it is not possible to specify independently the expected number of conditions.

The foregoing discussion applies to the case of cones with smooth profiles. However, if the cone possesses a sharp edge, as for instance a flat delta wing, then it is shown that within the one-strip approximation it is not possible to obtain a solution for configurations with the shock attached to the leading edge. It is suggested that the method of integral relations using a different method of choosing strips may apply to these cases. At high angles of attack, where the leading edge shock becomes detached, a solution is possible only if one can specify the behavior of the flow near the leading edge a priori. Unfortunately the analysis of this behavior has not as yet been carried out so that at present a proper solution of this case cannot be found.

In the section on Numerical Calculations, we discuss the numerical difficulties which have prevented the obtaining of solutions to the equations for one strip. First the flow over circular cones at zero angle of attack is calculated. This calculation has been performed primarily to provide some indication of the accuracy to be expected from the method. In this case the differential equations reduce to algebraic equations which can be solved iteratively to any degree of accuracy. The severe dependence of the solution on the initial conditions is then demonstrated by using these results in the differential equations with the angle of attack set equal to zero. Similar results are obtained for finite angles of attack, despite precautions, which are discussed, taken to eliminate some of these problems.

In the final section we present a brief description of the solution by linearization, suggested by the work of Vaglio-Laurin (Ref. 6), plus a discussion of the inverse case.

2. BASIC EQUATIONS AND APPROXIMATIONS

In the following, we shall employ an orthogonal conical coordinate system (r, ξ, η), with corresponding velocity components (v_r, v_ξ, v_η), such that the surface of the cone is the surface $\eta = 0$[3] (see Fig. 2.1). In this coordinate system the elemental arc length on the surface of the sphere r = constant is written

$$(ds)^2 = r^2\left[\chi_\xi^2(d\xi)^2 + \chi_\eta^2(d\eta)^2\right] ,$$

where χ_ξ and χ_η are surface scale factors. The possibility of using a nonorthogonal coordinate system has been discussed in Ref. 14, where the divergence form of the gas dynamics equations was developed. However, the extra complication which results from using a nonorthogonal system is not justified for most cone geometries. Other orthogonal coordinate systems could have been used, such as the spherical coordinate system employed in Ref. 9. However, depending upon the choice of basic equations, it may turn out that the singular points of the derived ordinary differential equations do not coincide with the sonic point on the body. This will be discussed later.

The method of integral relations may be summarized as follows. First, certain of the partial differential equations of gas dynamics are written in the form[4]

$$\frac{\partial}{\partial\xi}(T_i) + \frac{\partial}{\partial\eta}(S_i) = K_i \qquad (i = 1, 2, \ldots, n) , \tag{2.1}$$

where in general T_i, S_i, and K_i are functions of all the dependent and independent variables. In addition to (2.1), there may also be some integrals of the motion, such as Bernoulli's equation. These are not important here, however. We divide the region between the shock and the body into N equal[5] strips[6], and integrate (2.1) using the fact that the body is the coordinate surface $\eta = 0$ to obtain

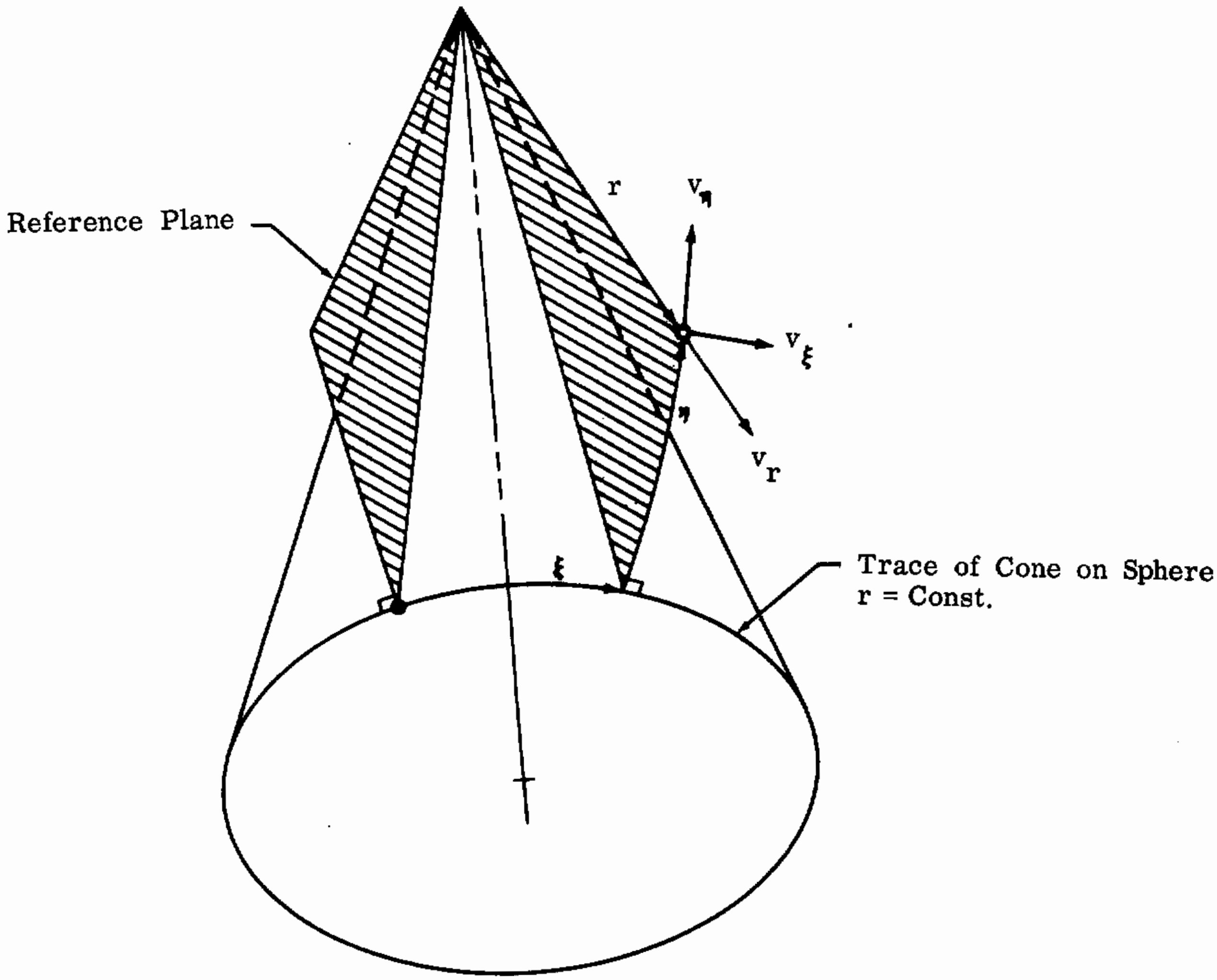

Fig. 2.1 - Body Oriented Orthogonal Conical Coordinate System

$$\frac{d}{d\xi}\int_0^{\frac{k}{N}\Upsilon(\xi)} T_i(\eta)\,d\eta - T_{ik}\frac{k}{N}\frac{d\Upsilon}{d\xi} + S_{ik} - S_{ib} = \int_0^{\frac{k}{N}\Upsilon(\xi)} K_i(\eta)\,d\eta\ , \qquad (2.2)$$

where $k = 1, 2, \ldots, N$; the subscript k denotes quantities at the k^{th} strip outer boundary and the subscript b denotes quantities at the cone surface. When $k = N$ (shock wave), we shall use the subscript s. The shock wave is defined by $\eta_s = \Upsilon(\xi)$.

Now, by assuming a variation of T_i and K_i with η, we may reduce (2.2) to a system of nN ordinary differential equations involving the unknown velocity components on the body and on the strip boundaries. The choice of this variation is arbitrary, but common sense compels us not only to satisfy as many known conditions as possible, but also to make the simplest possible choice consistent with these conditions.

The simplest scheme is to consider $N = 1$ and assume that T_i and K_i vary linearly across the shock layer.[7] In this way only flow quantities at the shock wave and at the cone surface are required, and none of their partial derivatives. Traugott (Ref. 3) has carried out this type of analysis for the smooth axisymmetric blunt body problem and obtained good results in the subsonic portion of the flow. However, when rapid variations of curvature occur, irregularities appear. Traugott tries to correct for these by taking higher order polynomials across one strip. For instance, he uses a cubic profile in n (his normal coordinate) and then employs partial derivatives of the unknown quantities at the shock and at the body to determine the extra coefficients. This complicates the analysis considerably by raising the order of the system of differential equations. To bypass this extra labor, he attempted to evaluate these partial derivatives by using the results of the linear approximation. Unfortunately, the irregularities of the linear approximation were amplified by the higher approximation in the region of rapidly changing curvature; only a slight improvement was noted upstream of this region.

Belotserkovskii (Ref. 2) has solved the axisymmetric blunt body problem by using 2 and 3 strips and approximating the equivalent of our T_i and K_i by parabolic and cubic profiles, respectively. In this case, the coefficients of the polynomials depend only upon the unknown quantities at the strip boundaries. Although the accuracy of the solution is shown to increase by increasing N, the computational labor is significantly increased and the simultaneous satisfaction of the regularity condition of the singular points becomes considerably more difficult.

Kennet (Ref. 10), in studying the flow over flat delta wings at high angle of attack, uses a parabolic profile, with no linear term, across one strip. This approximation is based on the assumption that the entropy gradient normal to the wing is zero. This would appear to be a valid assumption for this particular case; however, the possibility of a finite or infinite entropy gradient normal to the wing cannot be eliminated a priori[8] and the validity of the assumption must be established by a more exact analysis.

Chushkin and Shchennikov (Ref. 9), in solving essentially the same problem as under investigation here, employed linear variations across one strip using a spherical coordinate system and obtained good numerical results.

Thus, the results of Traugott and of Chushkin and Shchennikov using one strip lead us to assume for T_i and K_i, the form

$$T_i = T_{ib} + \frac{T_{is} - T_{ib}}{\Upsilon}\,\eta$$

$$K_i = K_{ib} + \frac{K_{is} - K_{ib}}{\Upsilon}\,\eta \,. \qquad (2.3)$$

Substituting Eq. (2.3) into Eq. (2.2) with N = 1, we obtain the system

$$\Upsilon(\xi)\left[\frac{dT_{ib}}{d\xi}+\frac{dT_{is}}{d\xi}\right] + [T_{ib} - T_{is}]\,\frac{d\Upsilon}{d\xi} = \Upsilon(\xi)\,[K_{ib}+K_{is}] - 2[S_{is} - S_{ib}]\ . \quad (2.4)$$

Up to this point, we have not specified T_i, S_i, and K_i. This depends not only on the partial differential equations used but also on how they are written. For example, in Refs. 9, 10, 14, 16a, and 16b, the equations employed were the continuity equation (i = 1), the ξ-momentum equation (i = 2) and the η-momentum equation (i = 3). The system is closed by the use of Bernoulli's equation and the assumption of constant entropy on the body.

In Ref. 16a, the ξ and η momentum equations were written such that the terms

$$\frac{\partial}{\partial\xi}(P^* + \rho^* v_\xi^{*2})$$

and

$$\frac{\partial}{\partial\eta}(P^* + \rho^* v_\eta^{*2})$$

appeared. Here P^*, ρ^*, v_ξ^*, and v_η^* are the dimensional pressure, density and velocity components. Writing the equations in this way leads to terms in K_{2s} and K_{3s} which involve $\partial\chi_\xi/\partial\xi$. If we wish to have continuous coefficients in the system of ordinary differential equations, we must require $\partial\chi_\xi/\partial\xi$ to be continuous. Since χ_ξ is related to the body curvature, this requires that the body have a continuous first derivative of curvature. However, if the system of partial differential equations is written with terms

$$\frac{\partial}{\partial \xi}\left[(P^* + \rho^* v_\xi^{*2}) \chi_\eta \right]$$

and

$$\frac{\partial}{\partial \eta}\left[(P^* + \rho^* v_\eta^{*2}) \chi_\xi \right] ,$$

appearing, as in Refs. 16b and 14, this difficulty is eliminated and the requirement of continuous coefficients requires only the weaker condition that the body curvature be continuous.[9] Kennet (Ref. 10) also noticed that the divergence form of the equations can be written in more than one way, but since he employed a spherical coordinate system, the restriction on body curvature does not explicitly appear. Using the requirement of continuous curvature, we find (Ref. 14)

$$T_1 = \rho^* \chi_\eta v_\xi^*$$

$$S_1 = \rho^* \chi_\xi v_\eta^*$$

$$K_1 = - 2\rho^* \chi_\eta \chi_\xi v_r^*$$

$$T_2 = (P^* + \rho^* v_\xi^{*2}) \chi_\eta$$

$$S_2 = \rho^* \chi_\xi v_\xi^* v_\eta^*$$

$$K_2 = (P^* + \rho^* v_\eta^{*2}) \frac{\partial \chi_\eta}{\partial \xi} - 3\rho^* v_r^* v_\xi^* \chi_\eta \chi_\xi - \rho^* v_\xi^* v_\eta^* \frac{\partial \chi_\xi}{\partial \eta}$$

$$T_3 = \rho^* \chi_\eta v_\xi^* v_\eta^*$$

$$S_3 = (P^* + \rho^* v_\eta^{*2}) \chi_\xi$$

$$K_3 = (P^* + \rho^* v_\xi^{*2}) \frac{\partial \chi_\xi}{\partial \eta} - 3\rho^* v_r^* v_\eta^* \chi_\xi \chi_\eta - \rho^* v_\xi^* v_\eta^* \frac{\partial \chi_\eta}{\partial \xi} \quad .$$

In addition to these relations, we have Bernoulli's equation

$$\frac{\gamma}{\gamma-1}\frac{P^*}{\rho^*} + \tfrac{1}{2}(v_r^{*2} + v_\xi^{*2} + v_\eta^{*2}) = \frac{\gamma}{\gamma-1}\frac{P_{0\infty}^*}{\rho_{0\infty}^*} ,$$

and the entropy equation

$$\frac{P^*}{\rho^{*\gamma}} = \frac{P_0^*}{\rho_0^{*\gamma}} ,$$

where the subscript 0 denotes local stagnation conditions, and γ is the adiabatic index.

Let us introduce the nondimensional variables P, ρ, v_r, v_ξ, and v_η such that $P^* = PP_{0\infty}^*$, $\rho\rho_{0\infty}^*$, $v_r^* = v_r V_{max}$, $v_\xi^* = v_\xi V_{max}$, and $v_\eta^* = v_\eta V_{max}$ where $P_{0\infty}^*$ is the free stream stagnation pressure, $\rho_{0\infty}^*$ the free stream stagnation density, and V_{max} the maximum adiabatic velocity. In terms of the free stream physical variables

$$V_{max}^2 = \frac{2\gamma}{\gamma-1}\frac{P_{0\infty}^*}{\rho_{0\infty}^*} .$$

Then the quantities appearing in Eq. (2.4) become (employing $v_{\eta b} = 0$)

$$\begin{aligned} T_{1b} &= \rho_b v_{\xi b}\chi_{\eta b} & T_{1s} &= \rho_s v_{\xi s}\chi_{\eta s} \\ S_{1b} &= 0 & S_{1s} &= \rho_s v_{\eta s}\chi_{\xi s} \\ K_{1b} &= -\,2\rho_b\chi_{\eta b}\chi_{\xi b}v_{rb} & K_{1s} &= -2\rho_s\chi_{\eta s}\chi_{\xi s}v_{rs} \end{aligned} \qquad (2.5)$$

$$T_{2b} = \left(\frac{\gamma-1}{2\gamma} P_b + \rho_b v_{\xi b}^2\right)\chi_{\eta b} \qquad T_{2s} = \left(\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\xi s}^2\right)\chi_{\eta s}$$

$$S_{2b} = 0 \qquad S_{2s} = \rho_s \chi_{\xi s} v_{\xi s} v_{\eta s}$$

$$K_{2b} = \frac{\gamma-1}{2\gamma} P_b \left.\frac{\partial \chi_\eta}{\partial \xi}\right|_b - 3\rho_b v_{rb} v_{\xi b} \chi_{\eta v} \chi_{\xi b} \qquad K_{2s} = \left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\eta s}^2\right]\left.\frac{\partial \chi_\eta}{\partial \xi}\right|_s - \rho_s v_{\xi s} v_{\eta s} \left.\frac{\partial \chi_\xi}{\partial \eta}\right|_s - 3\rho_s v_{rs} v_{\xi s} \chi_{\xi s} \chi_{\eta s}$$

$$T_{3b} = 0 \qquad T_{3s} = \rho_s \chi_{\eta s} v_{\xi s} v_{\eta s} \tag{2.5 (Cont.)}$$

$$S_{3b} = \frac{\gamma-1}{2\gamma} P_b \chi_{\xi b} \qquad S_{3s} = \left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\eta s}^2\right]\chi_{\xi s}$$

$$K_{3b} = \left[\frac{\gamma-1}{2\gamma} P_b + \rho_b v_{\xi b}^2\right]\left.\frac{\partial \chi_\xi}{\partial \eta}\right|_b \qquad K_{3s} = \left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\xi s}^2\right]\left.\frac{\partial \chi_\xi}{\partial \eta}\right|_s - \rho_s v_{\xi s} v_{\eta s} \left.\frac{\partial \chi_\eta}{\partial \xi}\right|_s - 3\rho_s v_{rs} v_{\eta s} \chi_{\xi s} \chi_{\eta s} .$$

Bernoulli's equation takes the form

$$\frac{P}{\rho} + V^2 = 1 , \tag{2.6}$$

where $V^2 = v_r^2 + v_\xi^2 + v_\eta^2$, and the entropy equation becomes

$$\frac{P}{\rho^\gamma} = \frac{P_0}{\rho_0^\gamma} . \tag{2.7}$$

In order to reduce Eqs. (2.4) to the desired system of ordinary differential equations, we must derive the shock boundary conditions for our coordinate system. In specifying the shock wave, we employ two angular type variables: $\eta_s = \Upsilon(\xi)$ which specifies the shock surface, and β which is the angle between a plane normal to the shock wave and a surface ξ = constant passing through the point in question. Referring to Fig. 2.2, we may deduce the relations

$$v_{N\infty} = -\, q_\infty \cos(\kappa - \beta)$$
$$v_{T\infty} = \quad q_\infty \sin(\kappa - \beta) \tag{2.8}$$

where $v_{N\infty}$ and $v_{T\infty}$ are the crossflow[10] components of velocity normal and tangential to the shock, respectively, on the upstream side; q_∞ is the magnitude of the free stream crossflow velocity given by

$$q_\infty = \sqrt{V_\infty^2 - v_{r\infty}^2} \ , \tag{2.9}$$

where V_∞ is the free stream velocity, and $v_{r\infty}$ the radial component of velocity at the shock on the upstream side. The angle κ is given by

$$\kappa = \sin^{-1} \frac{v_{\xi\infty}}{q_\infty} \ , \tag{2.10}$$

where $v_{\xi\infty}$ is the upstream component of v_ξ at the shock.

The downstream conditions at the shock (subscript s) are obtained from Prandtl's relation

$$v_{N\infty} v_{Ns} = \frac{\gamma-1}{\gamma+1} \left\{ 1 - v_{T\infty}^2 - v_{r\infty}^2 \right\} , \tag{2.11}$$

and the condition of constant tangential velocity, which give the relations

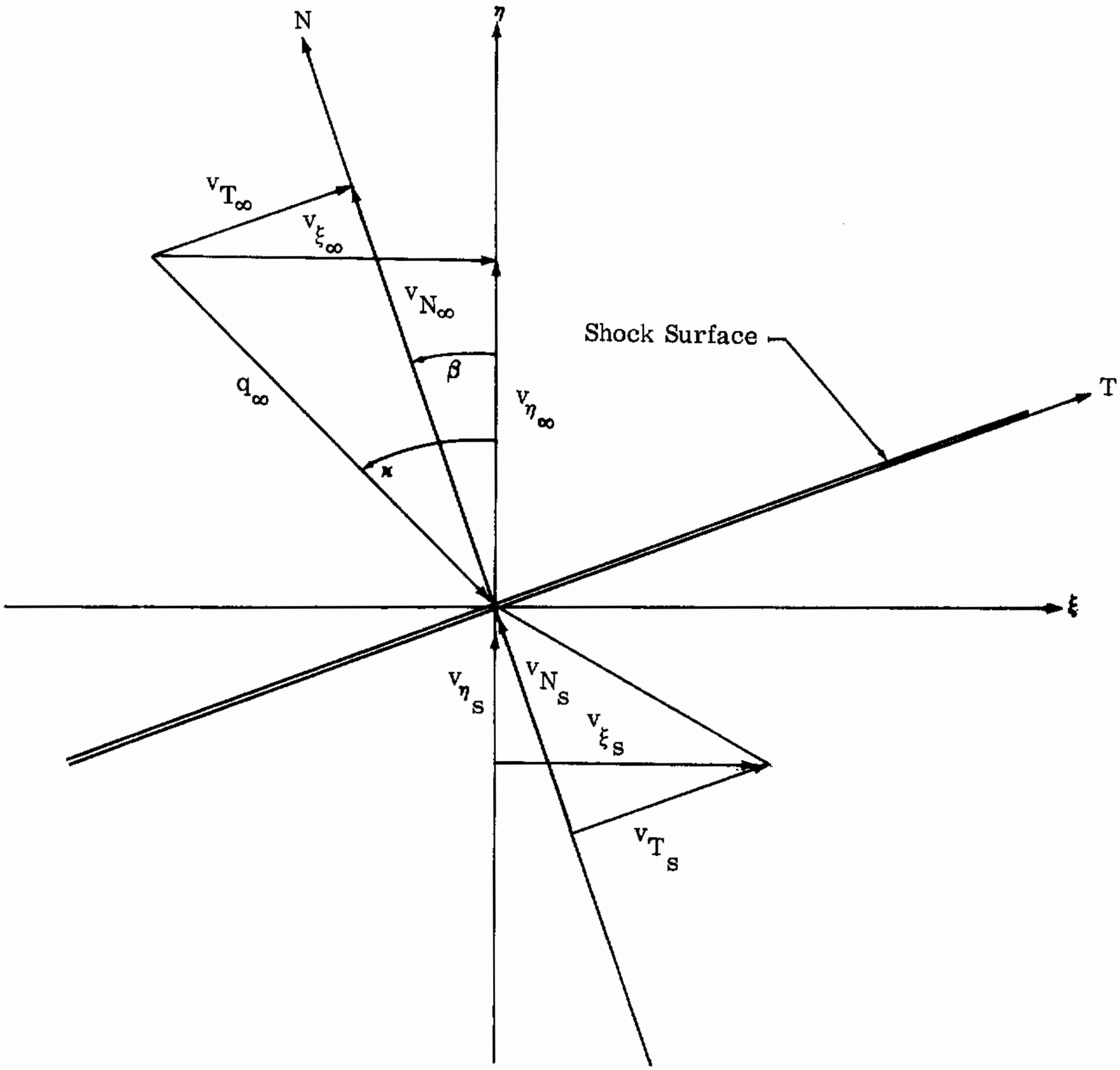

Fig. 2.2 - The Geometry of Velocity Components at Shock Wave

$$v_{rs} = v_{r\infty} \tag{2.12}$$

and

$$v_{Ts} = v_{T\infty} = q_\infty \sin(\kappa - \beta) \; . \tag{2.13}$$

Using Fig. 2.3, we may deduce

$$\begin{aligned} v_{\xi s} &= v_{Ts} \cos\beta - v_{Ns} \sin\beta \\ v_{\eta s} &= v_{Ns} \cos\beta + v_{Ts} \sin\beta \; . \end{aligned} \tag{2.14}$$

Furthermore, it is possible to relate β and $d\Upsilon/d\xi$, an expression we shall need further on to complete our system of equations. Figure 2.3 shows that

$$\frac{d\Upsilon}{d\xi} = \frac{\chi_{\xi s}}{\chi_{\eta s}} \tan\beta = R_5 \; . \tag{2.15}$$

In terms of the nondimensional free stream velocity, the total pressure and total density behind the shock may be written

$$P_{0s} = \rho_{0s} = \left[\frac{\gamma+1}{\gamma-1} \frac{V_{N\infty}^2}{V_{N\infty}^2 + 1 - V_\infty^2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{(\gamma^2-1)(1 - V_\infty^2)}{4\gamma V_{N\infty}^2 - (\gamma-1)^2(1 - V_\infty^2)}\right]^{\frac{1}{\gamma-1}} \; . \tag{2.16}$$

Finally, combining Eqs. (2.6) and (2.7), the density and pressure may be written

$$\begin{aligned} \rho &= \rho_0 (1 - V^2)^{1/(\gamma-1)} \\ P &= P_0 (1 - V^2)^{\gamma/(\gamma-1)} \; . \end{aligned} \tag{2.17}$$

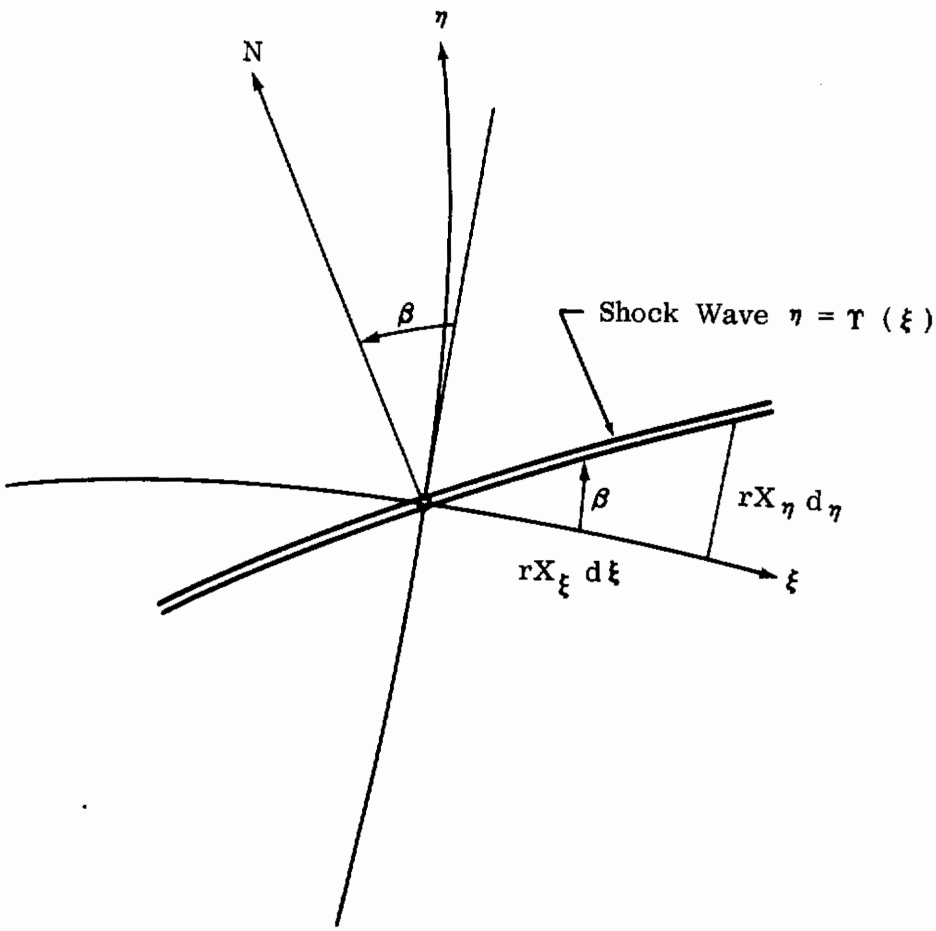

Fig. 2.3 - Differential Geometry of the Shock Wave

Before obtaining the set of ordinary differential equations, we must express the variation of certain quantities along the shock wave. We do this by the relations

$$\begin{aligned} q_\infty &= q_\infty(\Upsilon, \xi) \\ \kappa &= \kappa(\Upsilon, \xi) \\ v_{rs} &= v_{rs}(\Upsilon, \xi) = v_{r\infty}(\Upsilon, \xi) \\ \chi_{\eta s} &= \chi_{\eta s}(\Upsilon, \xi) \quad , \end{aligned} \tag{2.18}$$

which leads to the expressions

$$\begin{aligned} \frac{dq_\infty}{d\xi} &= \frac{\partial q_\infty}{\partial \xi} + \frac{\partial q_\infty}{\partial \Upsilon} \frac{d\Upsilon}{d\xi} = \Omega_1 + \Omega_2 \frac{d\Upsilon}{d\xi} \\ \frac{d\kappa}{d\xi} &= \frac{\partial \kappa}{\partial \xi} + \frac{\partial \kappa}{\partial \Upsilon} \frac{d\Upsilon}{d\xi} = \Omega_3 + \Omega_4 \frac{d\Upsilon}{d\xi} \\ \frac{dv_{rs}}{d\xi} &= \frac{\partial v_{r\infty}}{\partial \xi} + \frac{\partial v_{r\infty}}{\partial \Upsilon} \frac{d\Upsilon}{d\xi} = \Omega_5 + \Omega_6 \frac{d\Upsilon}{d\xi} \\ \frac{d\chi_{\eta s}}{d\xi} &= \frac{\partial \chi_{\eta s}}{\partial \xi} + \frac{\partial \chi_{\eta s}}{\partial \Upsilon} \frac{d\Upsilon}{d\xi} = \Omega_7 + \Omega_8 \frac{d\Upsilon}{d\xi} \ . \end{aligned} \tag{2.19}$$

Substituting Eqs. (2.5) through (2.19) into the set of Eqs. (2.4) leads to the result

$$J_1^*\left(1 - \frac{v_{\xi b}^2}{a_b^2}\right)\frac{dv_{\xi b}}{d\xi} + J_2 \frac{dv_{rb}}{d\xi} + J_3 \frac{d\Upsilon}{d\xi} + J_4 \frac{d\beta}{d\xi} = J_5 \tag{2.20a}$$

$$M_1^*\left(1 - \frac{v_{\xi b}^2}{a_b^2}\right)\frac{dv_{\xi b}}{d\xi} + M_2 \frac{dv_{rb}}{d\xi} + M_3 \frac{d\Upsilon}{d\xi} + M_4 \frac{d\beta}{d\xi} = M_5 \tag{2.20b}$$

$$Q_3 \frac{d\Upsilon}{d\xi} + Q_4 \frac{d\beta}{d\xi} = Q_5 \ , \tag{2.20c}$$

where

$$a^2 = \frac{\gamma-1}{2}\left(1 - v_r^2 - v_\xi^2 - v_\eta^2\right),$$

which, with (2.15), completes the system of ordinary differential equations. The coefficients in Eqs. (2.20) are listed in APPENDIX I.

It is evident from the determinant of the coefficients of Eqs. (2.20) and (2.15) that the singular points of the system of equations occur at crossflow sonic points on the body. This circumstance is true only because we have written the equations in a body-oriented system. We point out here that this result is not true in the work of Chushkin and Shchennikov (Ref. 9) except for the case of the circular cone, even though in the form in which they write their equations it appears to be so. What they have done is as follows.[11] First, they derived a set of equations analogous to Eqs. (2.20) and (2.15). In their case the coefficients Q_1 and Q_2 appear because T_3 does not vanish on the cone. Most important, however, is the fact that the coefficients of the derivative of the crossflow velocity do not all contain the same factor $1 - v_{\xi b}^2/a_b^2$ as in Eqs. (2.20). Only the continuity-equation yields this term, the other equations giving coefficients which depend on the body slope. Hence, the determinant of their system does not vanish at sonic crossflow velocity and in fact the singularity condition becomes so algebraically complicated that the a priori location of the singular point becomes impossible. Chushkin and Shchennikov have reduced the solution of their equations in the following way. First, they eliminate β by Eq. (2.15). Next, they solve the system (2.20) for $d^2\Upsilon/d\xi^2$. Then two of the equations are used to eliminate $dv_{\xi b}/d\xi$ and obtain $dv_{rb}/d\xi$ in terms of $d^2\Upsilon/d\xi^2$. Finally the continuity equation is used to solve for $dv_{\xi b}/d\xi$ in terms of $dv_{rb}/d\xi$ and $d^2\Upsilon/d\xi^2$ and since, as mentioned above, the continuity equation yields the term $1 - v_{\xi b}^2/a_b^2$, this gives the false impression that the singularity is at the sonic point. However, their stated purpose is to solve the equations for entirely subsonic crossflow so that this reduction of the equations presented no computational difficulty and probably does not affect the obtained pressure distributions.

One finds a similar result when applying the method of integral relations to nozzle flows. Here, the basic equations are the continuity equation and the axial momentum equation in Cartesian or cylindrical coordinates. Bernoulli's equation and the constancy of entropy are used to close the system. Then, using profiles of the form $A + By^2$ across one strip, one finds the result that the continuity equation yields a singularity at $M = 1$, but that the momentum equation has a singularity which depends on the nozzle shape. It is suggested here that this circumstance can be corrected by the use of intrinsic coordinates.[12]

Now, returning to Eqs. (2.20) and (2.15) we solve for the individual derivatives:

$$\frac{d\Upsilon}{d\xi} = R_5 \tag{2.21a}$$

$$\frac{d\beta}{d\xi} = \frac{Q_5 - R_5 Q_3}{Q_4} = U_5 \tag{2.21b}$$

$$\frac{dv_{rb}}{d\xi} = \frac{J_1^*(M_5 - M_4 U_5 - M_3 R_5) - M_1^*(J_5 - J_4 U_5 - J_3 R_5)}{J_1^* M_2 - J_2 M_1^*} \tag{2.21c}$$

$$\frac{dv_{\xi b}}{d\xi} = \frac{W_1'}{1 - v_{\xi b}^2/a_b^2} \tag{2.21d}$$

$$W_1' = \frac{M_2(J_5 - J_4 U_5 - J_3 R_5) - J_2(M_5 - M_4 U_5 - M_3 R_5)}{J_1^* M_2 - J_2 M_1^*} . \tag{2.21e}$$

This system of equations, which results from our choice of basic partial differential equations, can in principle be integrated for specific boundary conditions to give the solution to a variety of conical flow problems. However, it is easy to show that Eq. (2.20c) apparently contradicts the assumption that the entropy on the body is constant. To see this, we write Crocco's equations in our general orthogonal coordinate system. There results

$$v_\eta\left(\frac{\partial v_r}{\chi_\eta \partial\eta} - v_\eta\right) - v_\xi\left(v_\xi - \frac{\partial v_r}{\chi_\xi \partial\xi}\right) = 0 \tag{2.22a}$$

$$\frac{v_\xi}{\chi_\xi}\left[\frac{\partial}{\partial\xi}(\chi_\eta v_\eta) - \frac{\partial}{\partial\eta}(\chi_\xi v_\xi)\right] - v_r\chi_\eta\left[\frac{\partial v_r}{\chi_\eta \partial\eta} - v_\eta\right] = -\frac{a^2}{\gamma R}\frac{\partial S}{\partial\eta} \tag{2.22b}$$

$$v_r\chi_\xi\left[v_\xi - \frac{\partial v_r}{\chi_\xi \partial\xi}\right] - \frac{v_\eta}{\chi_\eta}\left[\frac{\partial}{\partial\xi}(\chi_\eta v_\eta) - \frac{\partial}{\partial\eta}(\chi_\xi v_\xi)\right] = -\frac{a^2}{\gamma R}\frac{\partial S}{\partial\xi}, \tag{2.22c}$$

where S is the entropy and R the gas constant. If we evaluate Eq. (2.22c) on the cone surface where $v_\eta = 0$ and $\partial S/\partial\xi = 0$, we obtain the result

$$v_{\xi b} = \frac{dv_{rb}}{\chi_{\xi b} d\xi}, \tag{2.23}$$

as consistent with constant entropy on the body. It is unlikely that the right-hand side of Eq. (2.21c) will equal $\chi_{\xi b} v_{\xi b}$ at all points, so that we propose here to replace the ξ-momentum equation with the condition of irrotationality on the cone surface, Eq. (2.23).[13] Referring back to Eq. (2.20b), this is equivalent to replacing M_1^*, M_3, and M_4 by zero, M_2 by one, and M_5 by $\chi_{\xi b} v_{\xi b}$. Eqs. (2.21) then take the form

$$\frac{d\Upsilon}{d\xi} = R_5 \tag{2.24a}$$

$$\frac{d\beta}{d\xi} = U_5 \tag{2.24b}$$

$$\frac{dv_{rb}}{d\xi} = \chi_{\xi b} v_{\xi b} \tag{2.24c}$$

$$\frac{dv_{\xi b}}{d\xi} = \frac{W_1}{1 - v_{\xi b}^2 / a_b^2} \tag{2.24d}$$

$$W_1 = \frac{J_5 - J_4 U_5 - J_3 R_5 - J_2 \chi_{\xi b} v_{\xi b}}{J_1^*} \tag{2.24e}$$

which is one form of the equations we shall use.

It is interesting to see why this discrepancy arises. As mentioned, the first step in utilizing the method of integral relations is to write the basic equations in divergence form. The previous papers on conical flow have used continuity and two momentum equations as basic, and Bernoulli's equations and the entropy equation as extra conditions. All of these papers have overlooked the fact that for steady flow the entropy equation

$$\vec{V} \cdot \nabla S = 0$$

when combined with the continuity equation, and

$$\nabla \cdot (\rho \vec{V}) = 0$$

can be written in divergence form as

$$\nabla \cdot (\rho \vec{V} S) = 0 \ . \tag{2.25}$$

Hence, for a strictly consistent approximation to the flow, Eq. (2.25) should be used along with the other 3 equations. The system is still closed by the use of Bernoulli's equation which provides a true constant of the flow.

An analysis of the basic equations in this latter system (not presented herein) leads to the result that the entropy on the body is not constant, but is determined by the solution. This circumstance is similar to that which occurs in calculations of flows over cones by other methods, for example in Ref. 19. There, the non-constancy of entropy on the cone is due to ignoring the entropy layer near the cone where normal derivatives of entropy, which have been neglected, are large. In the present case we can attribute the variation of the entropy on the surface to a similar cause; the linear, one-strip approximation to Eq. (2.25) which is an averaging process cannot possibly be valid locally, i.e., within the entropy layer. However, it may be possible to include, to the same degree of approximation employed thus far, a description of the entropy layer or "inner" solution which matches the solution discussed above ("outer" solution). However, previous work on entropy layers (e.g., Ref. 20) shows that the extra labor involved does not significantly alter the pressure distribution on the cone.

Thus we see that within the basic framework of the method of integral relations there is a wide range of results to be obtained. The choice of the coordinate system, the basic equations used and the approximate profiles chosen all lead to systems of ordinary differential equations with different properties. From among these possibilities, one must choose those which are as consistent with known facts as possible. In the numerical calculations we have employed both the system (2.24) and the system (2.21) for the linear approximation across one strip. Due to the numerical instabilities encountered, however, it was not possible to assess the relative merits of the two approximations.

The conclusions drawn from the above discussion apply as well to the other flows. For instance, Vaglio-Laurin (Ref. 21) has derived a set of first order partial differential equations for the flow over general three dimensional bodies using the one-strip approximation of the method of integral relations. In that work, a result similar to Eq. (2.21c) is obtained, that is, a result which is apparently not consistent with the constancy of entropy on the body. We propose that this equation can be replace with the corresponding equation of irrotationality. Using a body-oriented orthogonal coordinate system (s, t, n) with corresponding velocity components (Fig. 2.4) Crocco's equations take the form

$$
\begin{aligned}
-\frac{a^2}{\gamma R}\frac{\partial S}{h_s \partial s} &= \frac{v_n}{h_s h_n}\left[\frac{\partial}{\partial s}(v_n h_n) - \frac{\partial}{\partial n}(v_s h_s)\right] - \frac{v_t}{h_s h_t}\left[\frac{\partial}{\partial t}(v_s h_s) - \frac{\partial}{\partial s}(v_t h_t)\right] \\
-\frac{a^2}{\gamma R}\frac{\partial S}{h_n \partial n} &= \frac{v_t}{h_n h_t}\left[\frac{\partial}{\partial n}(v_t h_t) - \frac{\partial}{\partial t}(v_n h_n)\right] - \frac{v_s}{h_s h_n}\left[\frac{\partial}{\partial s}(v_n h_n) - \frac{\partial}{\partial n}(v_s h_s)\right] \qquad (2.26) \\
-\frac{a^2}{\gamma R}\frac{\partial S}{h_t \partial t} &= \frac{v_s}{h_s h_t}\left[\frac{\partial}{\partial t}(v_s h_s) - \frac{\partial}{\partial s}(v_t h_t)\right] - \frac{v_n}{h_n h_t}\left[\frac{\partial}{\partial n}(v_t h_t) - \frac{\partial}{\partial t}(v_n h_n)\right]
\end{aligned}
$$

where (h_s, h_n, h_t) are the scale factors of the coordinate system. If we evaluate the first and third of these equations on the body surface where $v_n = 0$ and S = constant, we find

$$\frac{\partial}{\partial t}(h_s v_s) - \frac{\partial}{\partial s}(h_t v_t) = 0 \qquad (2.27)$$

as the desired condition of irrotationality.

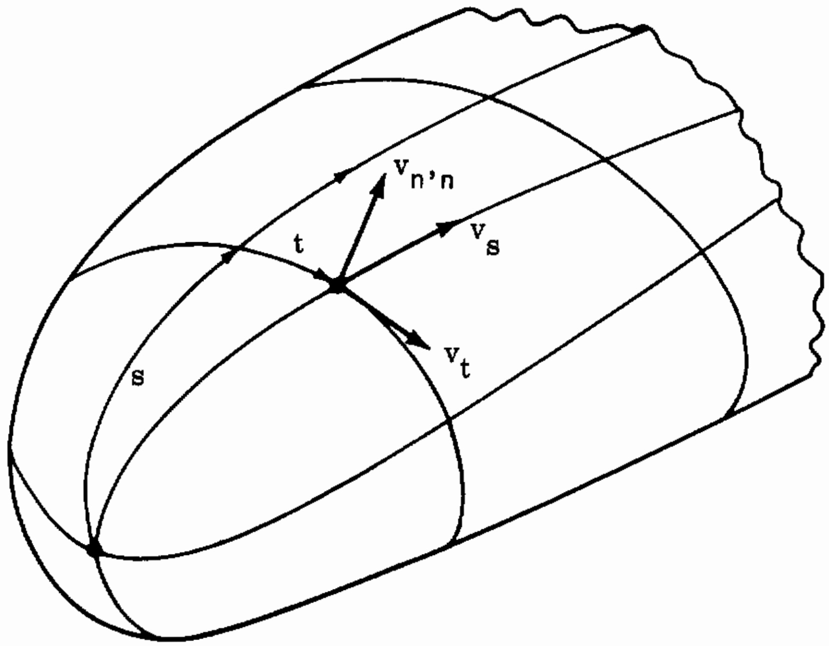

Fig. 2.4 - General Body Oriented Orthogonal Coordinate System

We shall now outline the derivation of the ordinary differential equations corresponding to the two-strip approximation. In this approximation we are faced with the difficulty of determining the entropy distribution on the boundary between the strips. This difficulty will be glossed over because an approximation to the entropy can be made simultaneously with the solution, or some iterative procedure can be used. Here we are interested in the form of the equations for purposes of discussion. The exact entropy distribution does not alter this form.

For this approximation we assume T_i and K_i vary as

$$\begin{aligned} T_i &= a_i + a_2\eta + a_3\eta^2 \\ K_i &= b_1 + b_2\eta + b_3\eta^2 , \end{aligned} \tag{2.28}$$

where the a's and b's are determined by setting

$$\begin{aligned} &T_i(0) = T_{ib} \qquad T_i(\tfrac{1}{2}\Upsilon) = T_{i2} \qquad T_i(\Upsilon) = T_{is} \\ &K_i(0) = K_{ib} \qquad K_i(\tfrac{1}{2}\Upsilon) = K_{i2} \qquad K_i(\Upsilon) = K_{is} . \end{aligned}$$

Carrying out the calculation we find that Eqs. (2.28) become

$$\begin{aligned} T_i &= T_{ib} + (4T_{i2} - 3T_{ib} - T_{is})\,\frac{\eta}{\Upsilon} + 2(T_{is} + T_{ib} - 2T_{i2})\left(\frac{\eta}{\Upsilon}\right)^2 \\ K_i &= K_{ib} + (4K_{i2} - 3K_{ib} - K_{is})\,\frac{\eta}{\Upsilon} + 2(K_{is} + K_{ib} - 2K_{i2})\left(\frac{\eta}{\Upsilon}\right)^2 \end{aligned} \tag{2.29}$$

Substituting Eqs. (2.28) into Eq. (2.2) for $k = 1, 2,$ we obtain the system of equations

$$J_6^*(1 - v_{\xi b}^2/a_b^2)\frac{dv_{\xi b}}{d\xi} + J_7^*(1 - v_{\xi 2}^2/a_2^2)\frac{dv_{\xi 2}}{d\xi} + J_8\frac{dv_{rb}}{d\xi} + J_9\frac{dv_{r2}}{d\xi} + J_{10}\frac{dv_{\eta 2}}{d\xi} + J_{11}\frac{d\Upsilon}{d\xi} + J_{12}\frac{d\beta}{d\xi} = J_{13} \tag{2.30a}$$

$$\hat{J}_6^*(1 - v_{\xi b}^2/a_b^2)\frac{dv_{\xi b}}{d\xi} + \hat{J}_7^*(1 - v_{\xi 2}^2/a_2^2)\frac{dv_{\xi 2}}{d\xi} + \hat{J}_8\frac{dv_{rb}}{d\xi} + \hat{J}_9\frac{dv_{r2}}{d\xi} + \hat{J}_{10}\frac{dv_{\eta 2}}{d\xi} + \hat{J}_{11}\frac{d\Upsilon}{d\xi} + \hat{J}_{12}\frac{d\beta}{d\xi} = \hat{J}_{13} \tag{2.30b}$$

$$M_6^*(1 - v_{\xi b}^2/a_b^2)\frac{dv_{\xi b}}{d\xi} + M_7^*(1 - v_{\xi 2}^2/a_2^2)\frac{dv_{\xi 2}}{d\xi} + M_8\frac{dv_{rb}}{d\xi} + M_9\frac{dv_{r2}}{d\xi} + M_{10}\frac{dv_{\eta 2}}{d\xi} + M_{11}\frac{d\Upsilon}{d\xi} + M_{12}\frac{d\beta}{d\xi} = M_{13} \tag{2.30c}$$

$$\hat{M}_6^*(1 - v_{\xi b}^2/a_b^2)\frac{dv_{\xi b}}{d\xi} + \hat{M}_7^*(1 - v_{\xi 2}^2/a_2^2)\frac{dv_{\xi 2}}{d\xi} + \hat{M}_8\frac{dv_{rb}}{d\xi} + \hat{M}_9\frac{dv_{r2}}{d\xi} + \hat{M}_{10}\frac{dv_{\eta 2}}{d\xi} + \hat{M}_{11}\frac{d\Upsilon}{d\xi} + \hat{M}_{12}\frac{d\beta}{d\xi} = \hat{M}_{13} \tag{2.30d}$$

$$Q_7^*(1 - v_{\xi 2}^2/a_2^2)\frac{dv_{\xi 2}}{d\xi} + Q_9\frac{dv_{r2}}{d\xi} + Q_{10}\frac{dv_{\eta 2}}{d\xi} + Q_{11}\frac{d\Upsilon}{d\xi} + Q_{12}\frac{d\beta}{d\xi} = Q_{13} \tag{2.30e}$$

$$\hat{Q}_7^*(1 - v_{\xi 2}^2/a_2^2)\frac{dv_{\xi 2}}{d\xi} + \hat{Q}_9\frac{dv_{r2}}{d\xi} + \hat{Q}_{10}\frac{dv_{\eta 2}}{d\xi} + \hat{Q}_{11}\frac{d\Upsilon}{d\xi} + \hat{Q}_{12}\frac{d\beta}{d\xi} = \hat{Q}_{13}\ , \tag{2.30f}$$

where the continuity equation, ξ-momentum equation and η-momentum equation have been used. Equation (2.15) closes the system. The coefficients of Eqs. (2.30) are listed in APPENDIX II.

Note that the condition of irrotationality has not been included among Eqs. (2.30). In the two-strip approximation it is not clear where this condition belongs. Apparently, either Eq. (2.30c) or Eq. (2.30d) could be replaced by Eq. (2.23) but there is no reason why it should be one or the other. An alternative would be to solve Eqs. (2.30) for the individual derivatives and then replace $dv_{rb}/d\xi$ by $\chi_{\xi b} v_{\xi b}$ leaving the other equations intact. This could have been done also with Eqs. (2.20) to obtain a system of equations different from Eqs. (2.21); however, this is a less satisfactory procedure than that which has been employed, and in any case using Eq. (2.25) should become more useful as the number of strips increases. However, Eqs. (2.30) are sufficient for purposes of discussion.

It is easy to see from Eqs. (2.30) that the singular points of the system occur at the sonic point on the body and at $v_{\xi 2} = a_2$ on the strip boundary. Thus, we see that with two strips (or any arbitrary number), the singular points of the system of ordinary differential equations do not correspond precisely to the sonic line of the flow. This, of course, is due to the fact that the strip boundary of the flow $\eta = \frac{1}{2}\Upsilon$ is not a stream surface. However, for small $v_{\eta 2}$, the singular point $v_{\xi 2} = a_2$ will lie close to the sonic point. Naturally, for more than two strips, the discrepancy between the singular points and the sonic points will increase as the shock is approached. A similar result occurs for the blunt body problem and is discussed in Ref. 22.

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3. THE BOUNDARY VALUE PROBLEM

The boundary values one must specify vary according to the geometry of the body. Initially, we shall consider smooth cones in the one-strip approximation. Other cases, including the two-strip approximation, will be discussed later. In all cases we shall assume that the cone is at zero angle of yaw[14] so that a plane of symmetry exists. No new features are introduced by non-zero yaw, however, so that the extension to this case follows easily.

First let us investigate the elliptic cone at zero angle of attack.[15] In this case, for the correct solution of the two-point boundary value problem, Eqs. (2.24) must satisfy the symmetry conditions

$$v_{\xi b}(\pi/2) = \beta(\pi/2) = v_{\xi b}(\pi) = \beta(\pi) = 0 \ ,$$

where the argument in the above relations is the meridian angle φ_b of Fig. 3.1. Since Eqs. (2.24) constitute a fourth order system, the above conditions are sufficient to determine the solution completely. Because of the highly complex nature of the system (2.24), an analytical treatment is not conceivable. Hence let us inquire into the procedure for a numerical solution.

The solution should start at the stagnation point (i.e., the saddle point) since the entropy on the cone is constant and must be determined by its value at crossflow stagnation points. In the case of elliptic cones at zero angle of attack [α is measured as negative in the plane $\varphi_b = 0$ and positive in the plane $\varphi_b = \pi$ (see Fig. 3.1)] the stagnation streamlines lie in the half-planes of symmetry (major axis) $\varphi_b = \pi/2,\ 3\pi/2$ as shown by AB in Fig. 3.1. (For a circular cone at angle of attack the stagnation streamline is on the windward

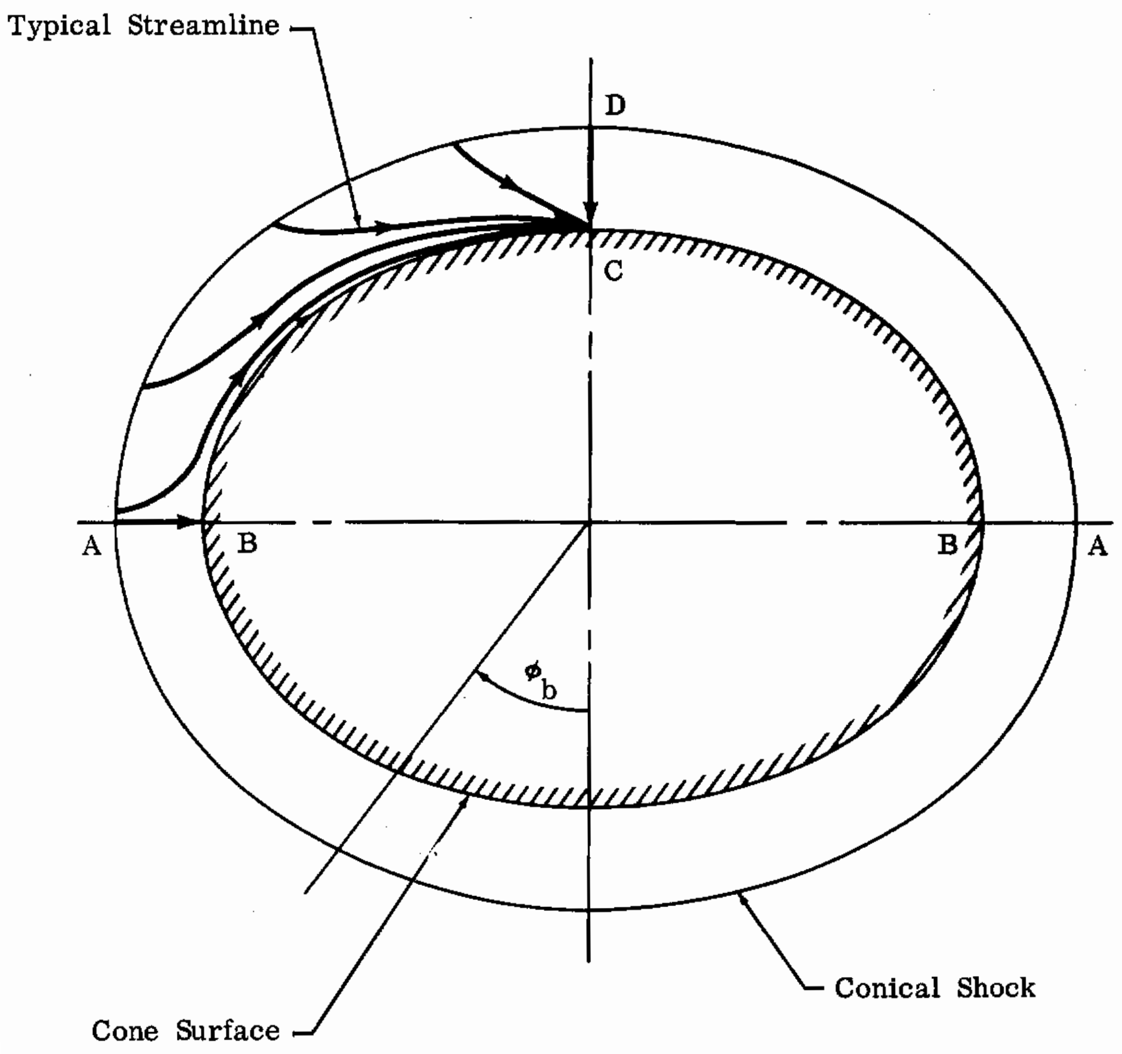

Fig. 3.1 - Flow Pattern Around an Elliptic Cone

side.[16]) It is easily shown that in either plane of symmetry, $v_{\xi b} = \beta = 0$ implies $dv_{rb}/d\xi = d\Upsilon/d\xi = 0$, and that if any two of the quantities v_{rb}, Υ, $dv_{\xi b}/d\xi$, or $d\beta/d\xi$ are chosen, the other two may be found (a similar result does not hold true in the two-strip approximation). Thus, for instance, one may choose $\tilde{v}_{rb}$ and $\tilde{\Upsilon}$ (the $\sim$ will henceforth denote quantities at the coordinate of crossflow stagnation points) as initial values and integrate Eq. (2.24) from $\pi/2$ to π to obtain $v_{\xi b}$ and β at DC of Fig. 3.1. If these are not zero, as is likely, an iteration procedure can be used to obtain the correct initial values and hence the entire solution. This procedure is satisfactory for the case when the crossflow is all subsonic. However, the situation changes when the singularity in Eq. (2.24a) must be passed. If, as in the case of the elliptic cone, the conical body is smooth and analytic, then we should expect the velocity distribution to be analytic through the singular point $v_{\xi b} = a_b$, and hence $dv_{\xi b}/d\xi$ will be continuous. At such points, then, we must satisfy the regularity conditions $W_1 = 0$. However, the boundary condtions require the flow to be elliptic in the neighborhood of both planes of symmetry. Hence, if one parabolic point exists on the body, then at least two parabolic points must exist on the body except in the exclusive case of a single sonic point (see Fig. 3.1). Thus, we find that both initial values $\tilde{v}_{rb}$ and $\tilde{\Upsilon}$ are required to satisfy regularity conditions at the two sonic points and no conditions remain to satisfy the downstream symmetry conditions, where by downstream we mean generally the boundary of the flow in either direction away from the stagnation point. This eventuality suggests the introduction of an inner shock wave (Fig. 3.2) to bring about the transition from supersonic to subsonic crossflow. As a consequence, the regularity condition and one downstream symmetry condition can be satisfied by varying $\tilde{v}_{rb}$ and $\tilde{\Upsilon}$ while the location of the inner shock is free to satisfy the remaining condition.

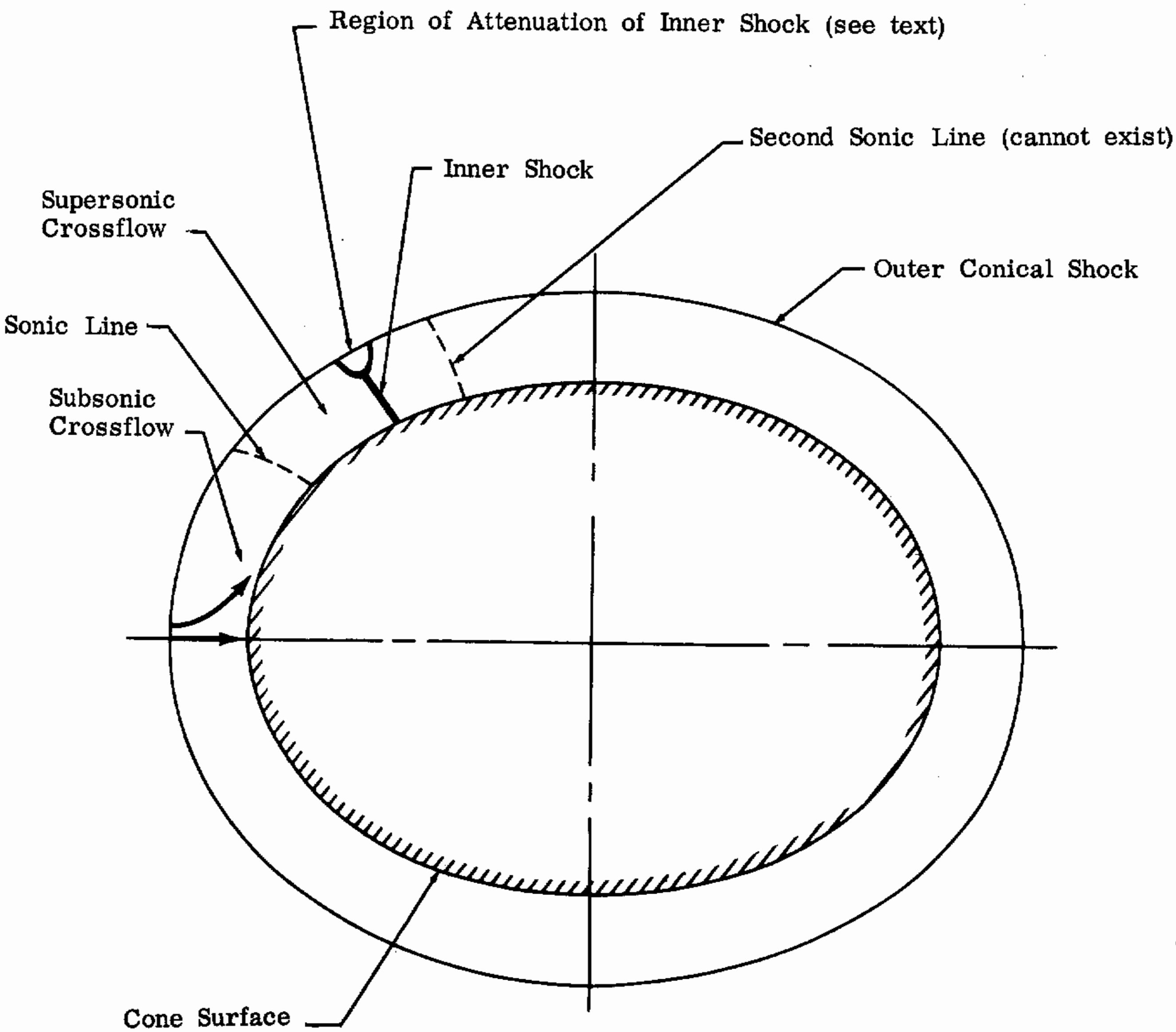

Fig. 3.2 - Flow Pattern Around Elliptic Cone for Mixed Crossflow

By introducing the inner shock wave we are undoubtedly straining the validity of the assumptions of linear variations across the shock layer. In Ref. 14 various configurations were conjectured upon for the flow structure in the hyperbolic region. In all of these, it is possible that the inner shock will penetrate planes ξ = constant which certainly violates the linear assumption. However, it was also shown in Ref. 14 that insofar as the one-strip approximation is concerned, all of these configurations are equivalent to the case where the shock does lie in a plane ξ = constant. Hence, we shall assume that the introduction of the inner shock does not violate the linear profile approximation. As we shall see this is no longer possible in the two-strip case.

Because of the boundary condition at the cone, the inner shock must be normal to the cone surface. Hence, the shock will produce a jump in $v_{\xi b}$ and correspondingly a jump in $dv_{\xi b}/d\xi$ (v_{rb} remains constant). By referring to Eq. (2.24), we find that if we impose Υ = constant and β = constant[17] at the value of ξ corresponding to the inner shock position, then the effect of the inner shock on the main conical shock is to produce a jump in curvature (i.e., $d\beta/d\xi$) of the latter. Thus we see that the inner shock produces no unusual effects in the flow[18] and theoretically may be easily incorporated into the numerical solution.

The foregoing discussion applies to the elliptic cone at zero angle of attack and the circular cone at angle of attack. However, when the elliptic cone is placed at an angle of attack α, a new problem arises; the stagnation point moves from the plane of symmetry to a position which must be determined by the solution itself.[19] On the cone surface, the stagnation point is determined by the condition $v_{\xi b} = 0$. To see what this implies, let us refer back to the linear approximation for T_i [Eq. (2.3)].

The expressions for T_1 and T_3 are (with $v_{\eta b} = 0$)

$$\rho v_\xi = \rho_b v_{\xi b} + \frac{\rho_s v_{\xi s} - \rho_b v_{\xi b}}{\Upsilon}\,\eta$$

and (3.1)

$$\rho v_\xi v_\eta = \rho_s v_{\xi s} v_{\eta s} \frac{\eta}{\Upsilon} .$$

At the stagnation point $\xi = \tilde{\xi}$, $\tilde{v}_{\xi b} = 0$; hence

$$\tilde{\rho}\tilde{v}_\xi = \tilde{\rho}_s \tilde{v}_{\xi s} \frac{\eta}{\tilde{\Upsilon}}$$

(3.2)

$$\tilde{\rho}\tilde{v}_\xi \tilde{v}_\eta = \tilde{\rho}_s \tilde{v}_{\xi s} \tilde{v}_{\eta s} \frac{\eta}{\tilde{\Upsilon}} .$$

Combining Eqs. (3.2) leads to the result $\tilde{v}_\eta = \tilde{v}_{\eta s}$ = constant which is clearly impossible. Only if we allow $\tilde{v}_\xi$ to be identically zero can $\tilde{v}_{\eta b} = 0$ (in fact the distribution of $\tilde{v}_\eta$ becomes undetermined, but easily can be shown to be linear). Hence, we conclude that the stagnation streamline lies in a surface $\xi = \tilde{\xi}$ = constant. As a consequence of the requirement $\tilde{v}_{\xi s} = 0$, we find from Eq. (2.14)

$$\tilde{v}_{Ts} \cos \tilde{\beta} = \tilde{v}_{Ns} \sin \tilde{\beta} . \tag{3.3}$$

Using Eqs. (2.8) and (2.18), Eq. (3.3) reduces to a relationship of the form

$$\tilde{\beta} = \tilde{\beta}(\tilde{\Upsilon}, \tilde{\xi}) . \tag{3.4}$$

Thus we see that the stagnation streamline does not necessarily pass through the shock normally and also that within our approximations the stagnation streamline need not possess the maximum entropy in the flow.

This latter point has been the subject of recent controversy. Vaglio-Laurin (Ref. 6) originally made the assumption of maximum entropy on the stagnation streamline in his paper concerning the solution of the two dimensional asymmetric blunt body problem by the method of integral relations. Swigart, (Ref. 23) contested this point of view and presented calculations showing that the line of maximum entropy does not wet the body. However, the difference between the body entropy and maximum entropy proved to be small. Vaglio-Laurin (Ref. 24) noted this to be true in a later work by a derivation similar to that above, but suggested that since the difference is small, the maximum entropy should be assumed for entropy on the body for the sake of convenience. Our results, although for conical flows, tend to confirm these views but it is this writer's opinion that the point is still open to question since each of the methods used has been an approximate one.[20]

Now let us ascertain the effect on the numerical solution of not knowing a priori the location of the stagnation point. First, one assumes a location $\tilde{\xi}$ for the stagnation point. Since $\tilde{v}_{\xi b} = 0$ and $\tilde{\beta} = \tilde{\beta}(\tilde{\Upsilon}, \tilde{\xi})$, we have $\tilde{\Upsilon}$ and $\tilde{v}_{rb}$ at our disposal to satisfy the downstream symmetry conditions in one direction away from $\tilde{\xi}$ (see Fig. 3.3), say at CD. Note that during this integration we may encounter the sonic point and inner shock wave mentioned previously. Once the conditions at CD are satisfied, we integrate in the opposite direction to determine the conditions at AB. In general, the symmetry conditions will not be satisfied here. Unfortunately, we now have only one parameter at our disposal to satisfy two conditions at AB, that is, the location $\tilde{\xi}$ of the stagnation point. Hence, we must adjust $\tilde{\xi}$, $\tilde{v}_{rb}$, and $\tilde{\Upsilon}$ in such a way that the boundary conditions at AB and CD are satisfied, if this is possible. Thus, it cannot be said with any assurance whether a solution can be found for this configuration.

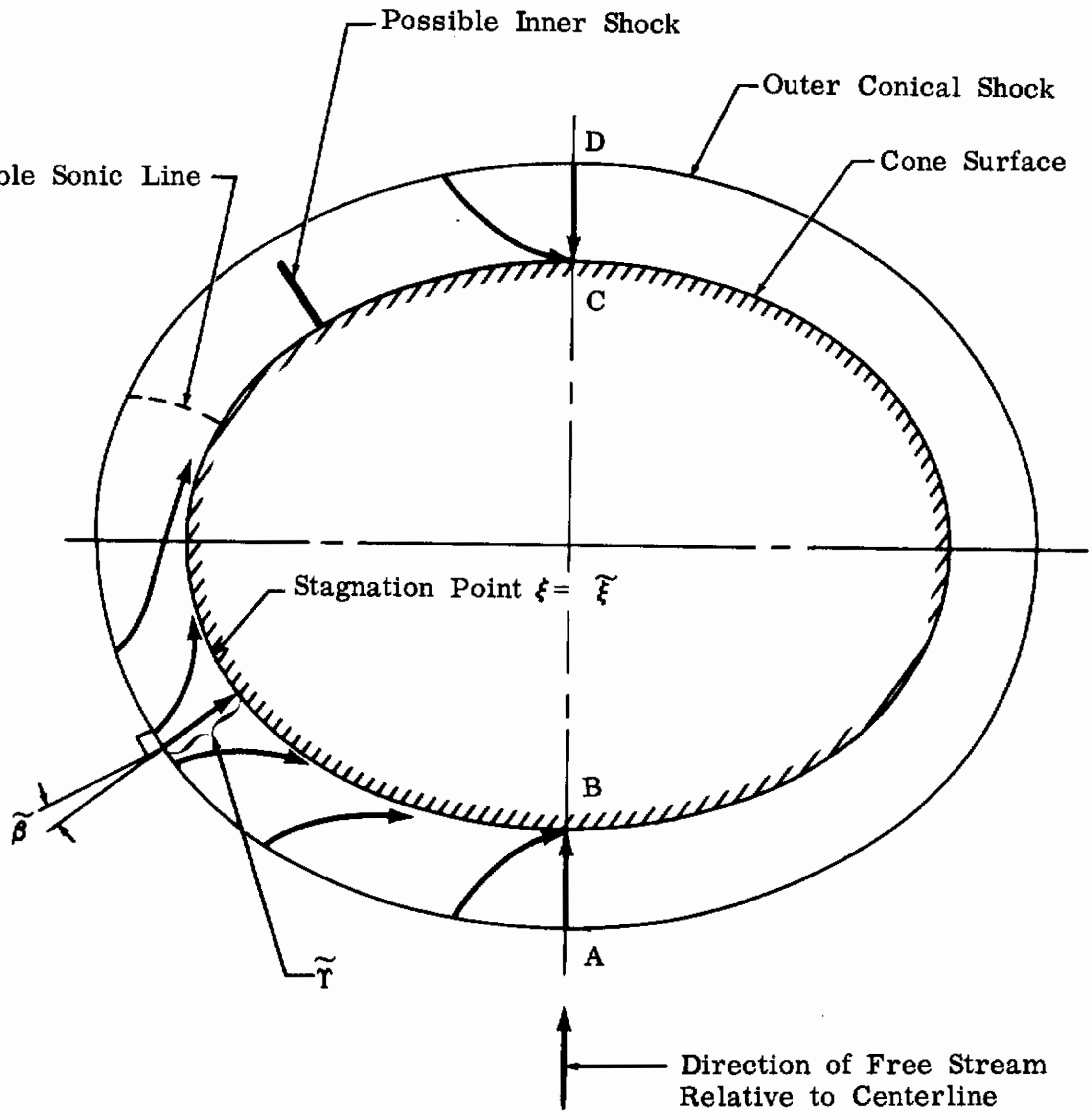

Fig. 3.3 - Flow Pattern Around an Elliptic Cone at Angle of Attack

Alternatively, the solution can be carried out by starting at AB (Fig. 3.3) and integrating up to CD using an assumed value for the entropy on the body. By iterating on the value of the body entropy (obtained in each step from the location of the stanation point $\tilde{\xi}$), a solution satisfying the symmetry conditions can be obtained. However, it is doubtful whether Eq. (3.3) will be satisfied at $\xi = \tilde{\xi}$, but this may prove to be the only way a solution for this configuration can be obtained.

Let us now consider the flow over an elliptic cone at zero angle of attack by the two-strip approximation. At the plane of symmetry in the one-strip approximation it was found that the two symmetry conditions $\tilde{v}_{\xi_b} = \tilde{\beta} = 0$, and arbitrary values of $\tilde{v}_{rb}$ and $\tilde{\Upsilon}$ were sufficient to compute the four derivatives in Eq. (2.24) (two of which are zero). In the two-strip approximation, we have a different result.

It can be shown from the coefficients listed in APPENDIX II that at the stagnation point (which is in a plane of symmetry here), the system of Eq. (2.30) reduces to:

$$\tilde{J}_6^* \frac{dv_{\xi b}}{d\xi} + \tilde{J}_7^* \frac{dv_{\xi 2}}{d\xi} + \tilde{J}_{12} \frac{d\beta}{d\xi} = \tilde{J}_{13} \tag{3.5a}$$

$$\tilde{\hat{J}}_6^* \frac{dv_{\xi b}}{d\xi} + \tilde{\hat{J}}_7^* \frac{dv_{\xi 2}}{d\xi} + \tilde{\hat{J}}_{12} \frac{d\beta}{d\xi} = \tilde{\hat{J}}_{13} \tag{3.5b}$$

$$\tilde{M}_8 \frac{dv_{rb}}{d\xi} + \tilde{M}_9 \frac{dv_{r2}}{d\xi} + \tilde{M}_{10} \frac{dv_{\eta 2}}{d\xi} = 0 \tag{3.5c}$$

$$\tilde{\hat{M}}_8 \frac{dv_{rb}}{d\xi} + \tilde{\hat{M}}_9 \frac{dv_{r2}}{d\xi} + \tilde{\hat{M}}_{10} \frac{dv_{\eta 2}}{d\xi} = 0 \tag{3.5d}$$

$$\tilde{Q}_7 \frac{dv_{\xi 2}}{d\xi} + \tilde{Q}_{12} \frac{d\beta}{d\xi} = \tilde{Q}_{13} \tag{3.5e}$$

$$\tilde{\hat{Q}}_7 \frac{dv_{\xi 2}}{d\xi} + \tilde{\hat{Q}}_{12} \frac{d\beta}{d\xi} = \tilde{\hat{Q}}_{13} , \tag{3.5f}$$

where $\tilde{\beta} = d\tilde{\Upsilon}/d\xi = \tilde{v}_{\xi b} = \tilde{v}_{\xi 2}$ has been used. All the coefficients in (3.5) depend upon $\tilde{v}_{rb}$, $\tilde{v}_{r2}$, $\tilde{v}_{\eta 2}$, and $\tilde{T}$. Now, it is easy to show that the determinant of the coefficients of Eq. (3.5) is zero so that the system (3.5) is singular. In fact, Eqs. (3.5c) and (3.5d) form a set of two equations in three unknowns and Eqs. (3.5a), (3.5b), (3.5c), and (3.5f) form a set of four equations in three unknowns. The solution of the former system is seen to be

$$\frac{dv_{rb}}{d\xi} = \frac{dv_{r2}}{d\xi} = \frac{dv_{\eta 2}}{d\xi} = 0 \quad . \tag{3.6}$$

The latter system, however, has a unique solution only if two of the equations are dependent. Equations (3.5e) and (3.5f) give the solutions

$$\frac{dv_{\xi 2}}{d\xi} = \frac{\tilde{Q}_{13}\tilde{\hat{Q}}_{12} - \tilde{\hat{Q}}_{13}\tilde{Q}_{12}}{\tilde{Q}_7\,\tilde{\hat{Q}}_{12} - \tilde{\hat{Q}}_7\,\tilde{Q}_{12}} = \tilde{N}_1 \tag{3.7a}$$

$$\frac{d\beta}{d\xi} = \frac{\tilde{Q}_{13}\tilde{\hat{Q}}_7 - \tilde{\hat{Q}}_{13}\tilde{Q}_7}{\tilde{Q}_7\,\tilde{\hat{Q}}_{12} - \tilde{\hat{Q}}_7\,\tilde{Q}_{12}} = \tilde{N}_2 \ , \tag{3.7b}$$

and from (3.5a) and (3.5b) we then get

$$\tilde{J}_6^* \frac{dv_{\xi b}}{d\xi} = \tilde{J}_{13} - \tilde{J}_{12}\tilde{N}_2 - \tilde{J}_7\tilde{N}_1 \tag{3.8a}$$

$$\tilde{\hat{J}}_6^* \frac{dv_{\xi b}}{d\xi} = \tilde{\hat{J}}_{13} - \tilde{\hat{J}}_{12}\tilde{N}_2 - \tilde{\hat{J}}_7\tilde{N}_1 \ , \tag{3.8b}$$

which is only possible if

$$\tilde{\hat{J}}_6^*(\tilde{J}_{13} - \tilde{J}_{12}\tilde{N}_2 - \tilde{J}_7\tilde{N}_1) = \tilde{J}_6^*(\tilde{\hat{J}}_{13} - \tilde{\hat{J}}_{12}\tilde{N}_2 - \tilde{\hat{J}}_7\tilde{N}_1) \ . \tag{3.9}$$

Thus, since the coefficients in Eq. (3.9) are functions of $\tilde{v}_{rb}$, $\tilde{v}_{r2}$, $\tilde{v}_{\eta 2}$, and $\tilde{\Upsilon}$, we obtain a relationship, e.g.

$$\tilde{v}_{\eta 2} = \tilde{v}_{\eta 2}(\tilde{v}_{rb}, \tilde{v}_{r2}, \tilde{\Upsilon}) , \tag{3.10}$$

and we see that although the number of equations has been increased from four to seven by the addition of another strip, we are at liberty to prescribe only six initial conditions in the numerical solution, e.g., $\tilde{\Upsilon}$, $\tilde{v}_{rb}$, $\tilde{v}_{r2}$ and $\tilde{v}_{\xi b} = \tilde{v}_{\xi 2} = \tilde{\beta} = 0$. In general, the matrix of coefficients of Eq. (2.30) will not be singular at succeeding points and $v_{\eta 2}$ can be found as part of the numerical integration.

At the downstream plane of symmetry, we must specify $v_{\xi b} = v_{\xi 2} = \beta = 0$. These conditions can be satisfied by varying $\tilde{v}_{r2}$, $\tilde{v}_{rb}$, and $\tilde{\Upsilon}$ when the crossflow is all subsonic. However, when the crossflow is mixed, we must again consider the appearance of a shock wave. If the flow first becomes supersonic on the body then the procedure outlined previously will work since we have one new initial condition and one extra boundary condition to satisfy. If the flow becomes supersonic on the strip boundary, the regularity condition again absorbs the extra condition so that a shock must be introduced to satisfy the downstream boundary condition.

On the strip boundary, however, the flow can no longer be taken normal to the shock wave so that an extra parameter, namely, the slope of the inner shock, is introduced. There is no criteria for determining this quantity so that it appears that one mustcalculate it from an assumed shape of the inner shock, for instance, a parabola.

We emphasize here that the quadratic profile assumption for the two-strip approximation almost certainly will be violated in the vicinity of the inner shock. However, the method may still provide useful results for the pressures on the body. This can only be ascertained by comparison with known results.

Now, let us shift our attention to wing-like conical bodies (sharp edges), for instance a flat plate delta wing. At small angles of attack, the significant features of the flow on the compression side of the wing are as shown in Fig. 3.4. According to Ref. 25, the sonic line AB is determined uniquely by the uniform flow in the hyperbolic region ABC, so that we can restrict our attention to the elliptic region ABDE. We shall also employ an orthogonal coordinate system such that the plane of symmetry DE and the sonic line AB are coordinate surfaces[21] (see Fig. 3.4). We assume that the calculation is to be started at the sonic line AB, with conditions on AB determined from the hyperbolic region, and to proceed to the plane of symmetry DE where the conditions $\beta = v_{\xi b} = 0$ are to be satisfied. Hence, we must have two quantities free at the sonic line. Physically, we expect these two quantities to be $dv_{\xi b}/d\xi$ and $d\beta/d\xi$.[22] However, referring to Eq. (2.21), we see that $d\beta/d\xi = U_5$ is determined by the known values of v_{rb}, $v_{\xi b}$, Υ, β on AB and consequently is determined leading to, in general, $dv_{\xi b}/d\xi$ becoming infinite. Thus, there is no freedom of the initial conditions and we must conclude that this is unsolvable by the one-strip approximation. It is easy to see that similar reasoning and results apply to the many-strip case.

Generally it is true that any attempt to start calculations by the one-strip approximation at a known sonic line will not succeed because of the lack of freedom in choosing initial conditions. In fact, it is impossible to solve for the flow over any wing-like configuration (see Fig. 3.5) for the same reason. In these cases a

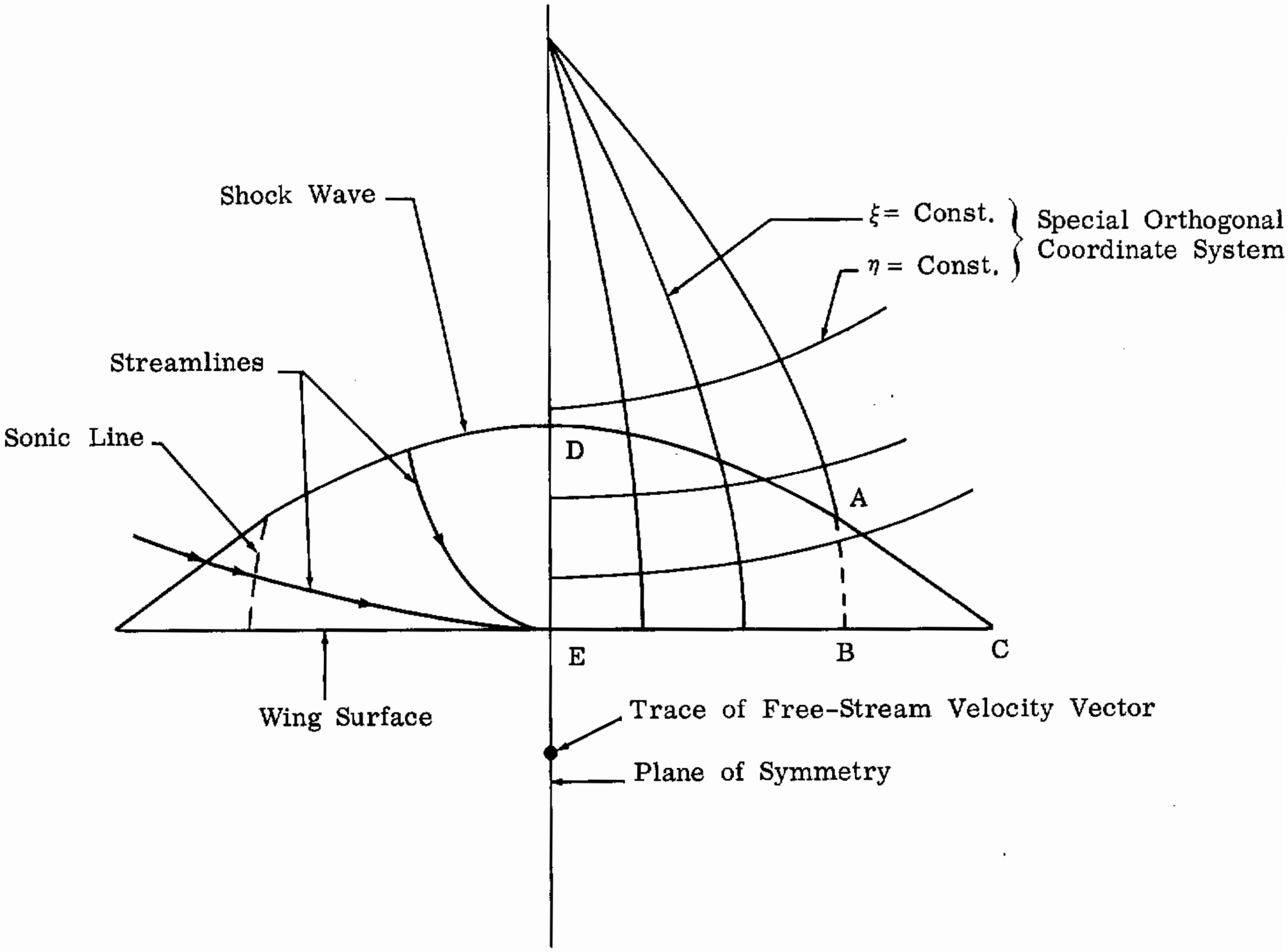

Fig. 3.4 - Flow Pattern on Compression Side of Flat Plate Delta Wing

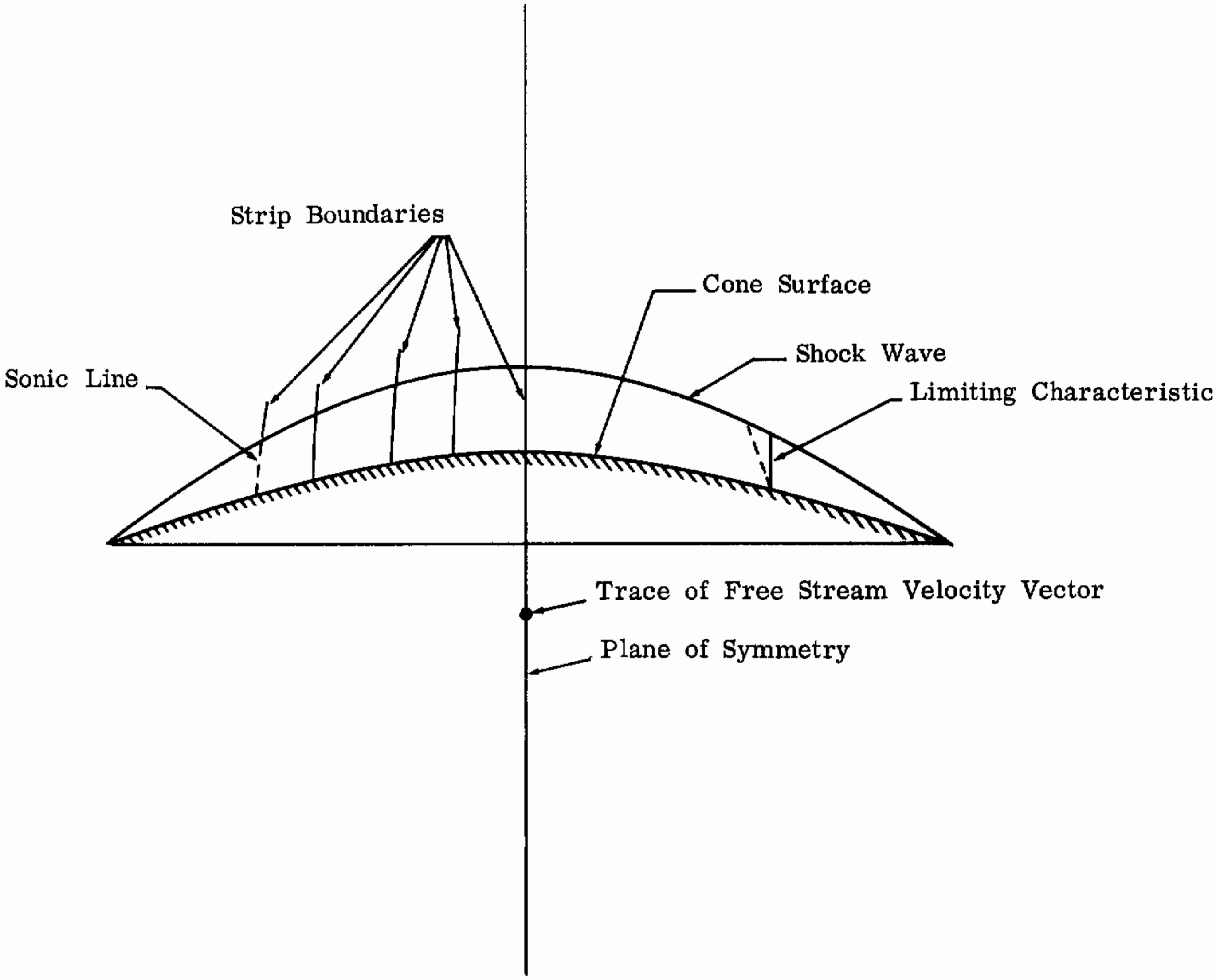

Fig. 3.5 - Flow Over Winglike Conical Surface with Attached Leading Edge Shock Wave

solution may possibly be obtained, however, by taking the strips normal to the body (Fig. 3.5) and using a coordinate system which contains the plane of symmetry and either the sonic line or limiting characteristic (Fig. 3.5) as coordinate surfaces.[23] This analysis has not been performed and is a subject for further investigation.

Now if the angle of attack is high enough, the shock wave will detach from the leading edge and the flow pattern will appear as shown in Fig. 3.6 (illustrated for the flat plate wing). In this case we start the integration at the plane of symmetry AB (which is now a stagnation point) with unknown values of (say) $\tilde{v}_{rb}$ and $\tilde{\Upsilon}$. The integration is continued up to the leading edge where we must have a singularity corresponding to the sonic line CD. Therefore, one of the initial conditions must be varied until the sonic point lies at the leading edge of the wing. The remaining initial value is then free to satisfy another condition. Vaglio-Laurin (Ref. 27) has shown that the flow in the vicinity of the corner of a flat-nosed body exhibits a boundary layer behavior and that on the body the velocity varies like $(s_c - s)^{2/5}$, where s is the coordinate along the body and s_c the coordinate of the corner. Hence the velocity derivative becomes infinite like $(s_c - s)^{-3/5}$.[24] Extrapolating this result to our case, we see that the second initial value must be varied until the flow in the vicinity of the corner behaves according to a certain law. Unfortunately the corresponding analysis for conical flow has not been carried out so that it is impossible at present to prescribe the second boundary condition precisely.[25]

Kennet (Ref. 10), as mentioned, has analyzed the same problem but obtains a system of equations for which it is only necessary to specify one initial value, say $\tilde{\Upsilon}$, to obtain a solution.[26] This result is somewhat unexpected but a possible explanation is as follows.

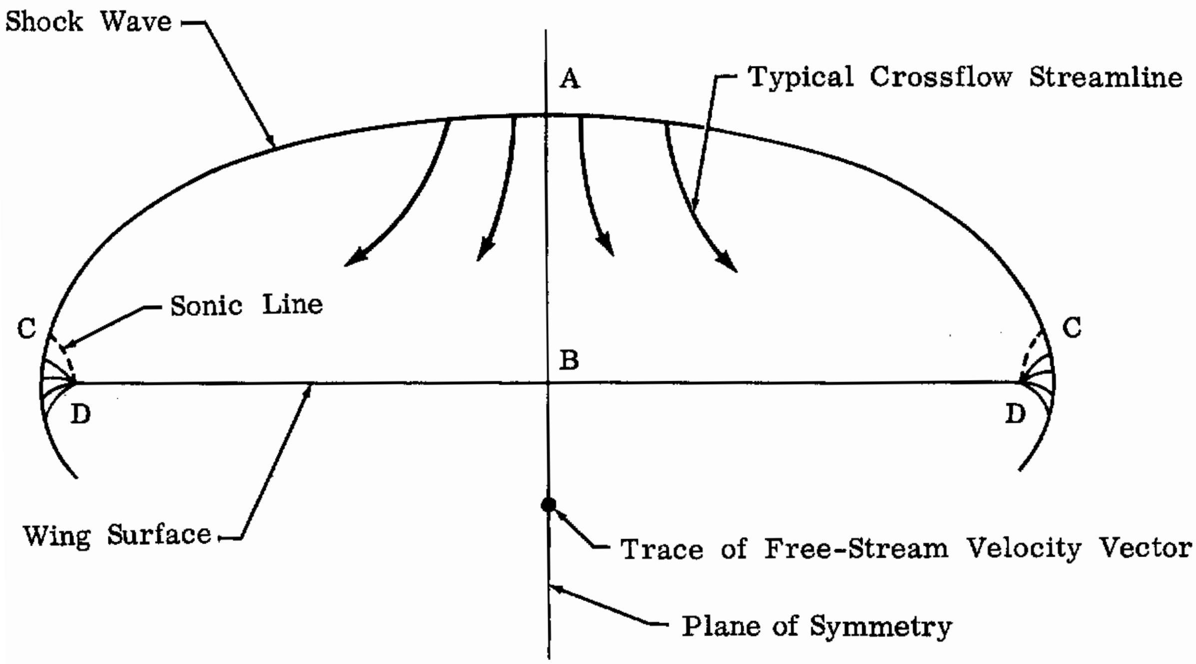

Fig. 3.6 - Flat Plate Delta Wing at Large Angle of Attack

In Kennet's basic equations, the term corresponding to our $dv_{rb}/d\xi$ is missing. However, by investigating the derivation of the equations it appears that this term can be removed only by the use of the irrotationality condition, Eq. (2.23). Thus, if this equation is used <u>in addition to</u> the result from the ξ-momentum equation, the effect is to reduce the system from fourth to third order and hence only one initial condition is required. Kennet, however, has obtained good results, which are hard to argue with.

This completes the discussion of the boundary value problem. In it we have shown that not all conical flow problems can be handled by the approximations employed (e.g., flat plate delta wing at low angle of attack) and that others are questionable (e.g., moving stagnation points). It has also been shown that an unexpected phenomenon (inner shock)[27] occurs for other geometries. In the following section the numerical work performed to date will be described.

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4. NUMERICAL CALCULATIONS

Before proceeding with the discussion of the numerical calculations performed, let us now make definite our coordinate system. Many orthogonal body-oriented coordinate systems are possible; however, we shall employ a boundary layer type, that is, the ξ = constant surfaces will be taken as planes normal to the cone surface. Intersection of the ξ = constant and r = constant surfaces then form great circles on the spherical surface. The coordinate η then is analogous to the polar angle of a spherical coordinate system, to which it reduces for circular cones.

With this definition we have $\chi_\eta \equiv 1$ and (Ref. 29)

$$\chi_\xi = \cos\eta - K_b(\xi)\sin\eta ,$$

where $K_b(\xi)$ is the nondimensional principle curvature of the conical body.[28] In what follows, we shall refer the body-oriented system to a spherical coordinate system (see Fig. 4.1). $K_b(\xi) = K_b(\varphi_b)$ can then be expressed as (Ref. 29)

$$K_b(\varphi_b) = -\left\{(1 + \sin^2\delta_b)\cos\delta_b \cot\theta_b - \frac{\cos^3\delta_b}{\sin^2\theta_b}\frac{d^2\theta_b}{d\varphi_b^2}\right\} , \qquad (4.1)$$

where the body is given by

$$\theta_b = \theta_b(\varphi_b) , \qquad (4.2)$$

and

$$\tan\delta_b = \frac{1}{\sin\theta_b}\frac{d\theta_b}{d\varphi_b} . \qquad (4.3)$$

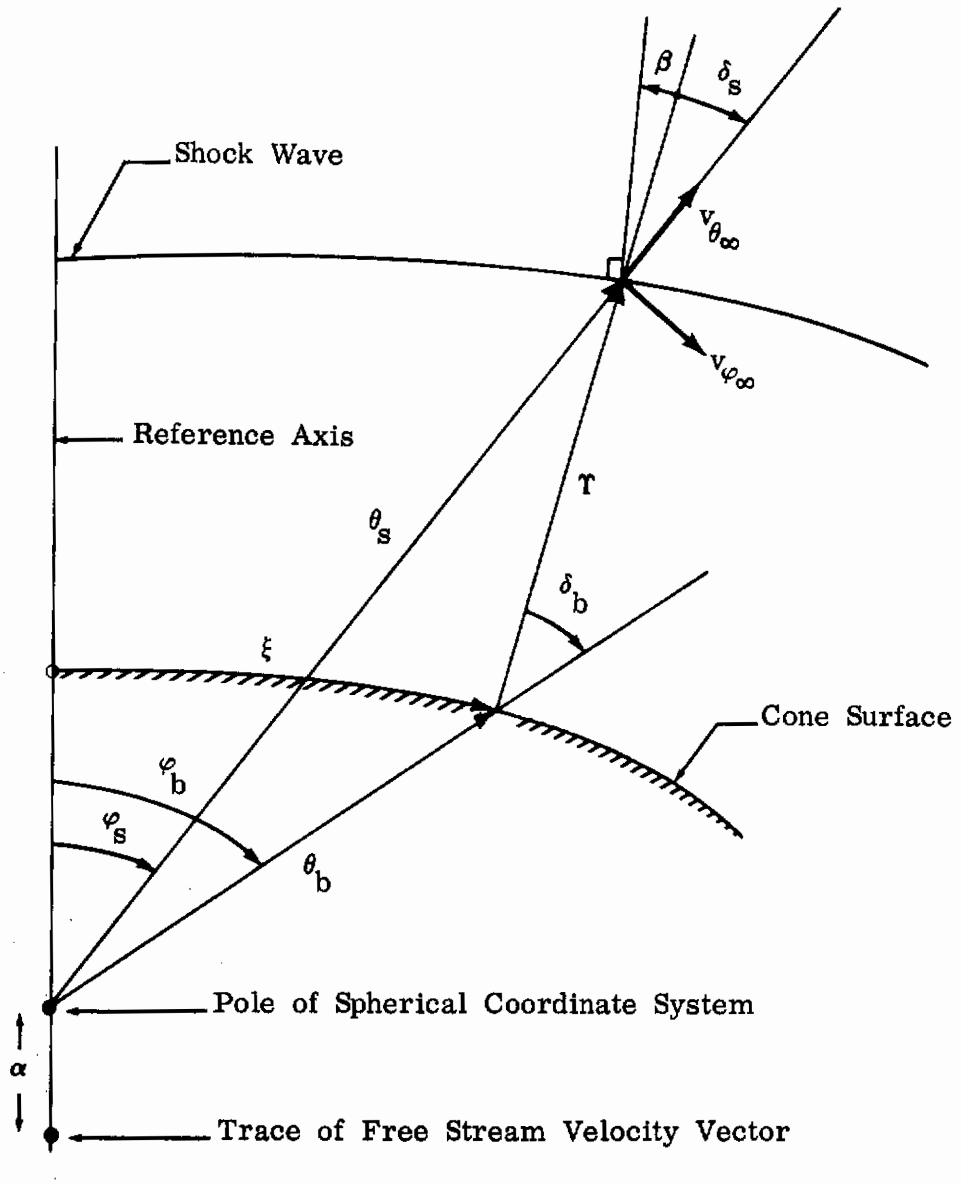

Fig. 4.1 - Relation Between Body Oriented and Spherical Coordinate Systems

The relationships between the two coordinate systems are expressed by

$$\xi = \int_0^{\varphi_b} \frac{\sin\,\theta_b}{\cos\,\delta_b}\, d\varphi_b \qquad (4.4)$$

$$\cos\,\theta_s = \cos\,\theta_b\,\cos\,\Upsilon - \cos\,\delta_b\,\sin\,\theta_b\,\sin\,\Upsilon \qquad (4.5)$$

$$\sin(\varphi_b - \varphi_s) = \sin\,\delta_b\,\frac{\sin\,\Upsilon}{\sin\,\theta_s} \qquad (4.6)$$

$$\sin\,\delta_s = \sin\,\delta_b\,\frac{\sin\,\theta_b}{\sin\,\theta_s}\quad . \qquad (4.7)$$

Other expressions which are necessary to calculate Ω_1 through Ω_8 are found in APPENDIX III.

To determine what accuracy could be expected from the method of approximation, we first investigated the case of a circular cone at zero angle of attack. For this configuration the coordinate system reduces to a spherical polar system. To simplify the algebra we have actually used the latter system for this case. As mentioned, there are two ways to write the basic equations, and a secondary purpose of the present analysis was to ascertain the relative accuracy of the two. All calculations were performed on an IBM 7094 computer.

In a spherical polar coordinate system, the basic equations can be written:

<u>Method 1</u>

$$\frac{d}{d\theta}(\rho v_\theta\,\sin\,\theta) = -\,2\rho v_r\,\sin\,\theta \qquad (4.8a)$$

$$\frac{d}{d\theta}\left(\frac{\gamma-1}{2\gamma} P + \rho v_\theta^2\right) = - \rho v_\theta (3v_r + v_\theta \cot \theta) \tag{4.8b}$$

$$\frac{P}{\rho} + V^2 = 1 \tag{4.8c}$$

$$\frac{P}{\rho^\gamma} = \frac{P_0}{\rho_0^\gamma} = \text{constant} . \tag{4.8d}$$

Method 2

Equations (4.8a), (4.8c), and (4.8d) remain unchanged and Eq. (4.8b) becomes

$$\frac{d}{d\theta}\left[\left(\frac{\gamma-1}{2\gamma} P + \rho v_\theta^2\right)\sin \theta\right] = \frac{\gamma-1}{2\gamma} P \cos \theta - 3\rho v_r v_\theta \sin \theta . \tag{4.9}$$

Performing the integration between θ_b and θ_s, we obtain the relations

$$\bar{\rho}_s v_{\theta s} \sin \theta_s + [\bar{\rho}_s v_{rs} \sin \theta_s + \bar{\rho}_b v_{rb} \sin \theta_b](\theta_s - \theta_b) = 0 , \tag{4.10}$$

and

$$v_{rb} = \sqrt{1 - \left(\frac{2\gamma}{\gamma-1} A\right)^{\frac{\gamma-1}{2\gamma}}} , \tag{4.11}$$

where

$$A = B/C \tag{4.12}$$

$$\begin{aligned} B = &\left(\frac{\gamma-1}{2\gamma} \bar{P}_s + \bar{\rho}_s v_{\theta s}^2\right)\sin \theta_s + \left[\frac{3}{2} \bar{\rho}_s v_{rs} v_{\theta s} \sin \theta_s\right. \\ &\left. - \frac{1}{2}(\nu-1)^2 \frac{\gamma-1}{2\gamma} \bar{P}_s \cos \theta_s + \frac{1}{2}(\nu-2)^2 \bar{\rho}_s v_{\theta s}^2 \cos \theta_s\right](\theta_s - \theta_b) \end{aligned} \tag{4.13}$$

$$C = \left[(\nu-2)^2 \sin \theta_s + (\nu-1)^2(\sin \theta_b + \frac{1}{2} \cos \theta_b[\theta_s - \theta_b])\right] . \tag{4.14}$$

Here $\nu = 1$ denotes Method 1 and $\nu = 2$ denotes Method 2; $\bar{\rho}_s = \rho_s/\rho_0$, $\bar{P}_s = P_s/P_0$.

The solution is obtained by choosing θ_s, calculating v_{rb} from Eq. (4.11) and substituting this result into Eq. (4.10) until it becomes equal to zero. The results of some sample calculations are presented in Table 1 along with the corresponding exact results from Kopal (Ref. 12). It can be seen that both methods give results which become increasingly more accurate as M_∞ and θ_b increase, with Method 2 being consistently more accurate. By comparing the difference between the results of Method 1 and Method 2 and the exact results of Kopal with the quantity $M_\infty \sin \theta_c$, we see that for $M_\infty \sin \theta_c \gtrsim 1$ we obtain good accuracy. This corresponds to the "strong shock" condition of Hayes and Probstein (Ref. 22) so that we may anticipate reasonable results in the general case when the "strong shock" assumption is satisfied locally. It is gratifying to observe that Method 2 gives the more accurate results since in the general case the equations written with the weaker condition on body curvature should then also give more accurate results.

The first case we chose to test the applicability of the method to nonsymmetric flows was the circular cone at angle of attack. However, before proceeding it was decided to first test the numerical sensitivity of the equations by trying to reproduce the zero angle of attack results using Eqs. (2.21) and Eqs. (2.24). To carry out the calculations we first transformed the equations, in a manner similar to the regularization technique of Temple (Ref. 30), in order to eliminate the violent oscillations which occur if a singular point is inadvertently passed. Using Eqs. (2.24) as an example [the same transformation holds for Eqs. (2.21)], we write, using φ_b' as a new independent variable

TABLE 4.1

COMPARISON BETWEEN METHOD OF INTEGRAL RELATIONS AND EXACT CALCULATION FOR AXISYMMETRIC CONICAL FLOW

M_∞	θ_b	θ_s, degrees			v_{rb}			C_{P_b}			$M_\infty \sin \theta_b$
		Method 1	Method 2	Kopal	Method 1	Method 2	Kopal	Method 1	Method 2	Kopal	
3.3041	10°	20.421	21.113	20.184	.79189	.80599	.80000	.11229	.06455	.08452	.574
5.4223	10°	15.072	15.242	15.013	.89795	.90050	.90000	.08158	.07071	.07338	.942
15.146	10°	11.580	11.590	11.568	.96987	.96998	.97000	.06574	.06469	.06488	2.630
3.2188	25°	33.814	33.946	33.651	.69528	.69928	.70000	.43639	.41334	.41366	1.360
5.1233	25°	30.119	30.157	30.066	.79859	.79967	.80000	.39865	.39015	.39042	2.165
8.0962	25°	28.617	28.633	28.589	.84937	.84981	.85000	.38509	.38064	.38087	3.422
3.3691	40°	50.587	50.641	50.327	.54422	.54725	.55000	.94830	.92917	.92348	2.166
5.8336	40°	46.733	46.756	46.650	.64795	.64905	.65000	.89709	.88788	.88664	3.750
13.101	40°	45.284	45.297	45.235	.69881	.69942	.70000	.87979	.87376	.87322	8.421

$$\frac{dv_{\xi b}}{d\varphi_b'} = W_1 \frac{\sin\theta_b}{\cos\delta_b} \frac{Q_4}{Q_4(0)} \tag{4.15a}$$

$$\frac{dv_{rb}}{d\varphi_b'} = v_{\xi b} \frac{\sin\theta_b}{\cos\delta_b} \frac{Q_4}{Q_4(0)} \left(1 - \frac{v_{\xi b}^2}{a_b^2}\right) \tag{4.15b}$$

$$\frac{d\Upsilon}{d\varphi_b'} = Rs \frac{\sin\theta_b}{\cos\delta_b} \frac{Q_4}{Q_4(0)} \left(1 - \frac{v_{\xi b}}{a_b^2}\right) \tag{4.15c}$$

$$\frac{d\beta}{d\varphi_b'} = \frac{(Q_s - Q_3R_5)}{Q_4(0)} \frac{\sin\theta_b}{\cos\delta b} \left(1 - \frac{v_{\xi b}^2}{a_b^2}\right) \tag{4.15d}$$

$$\frac{d\varphi_b}{d\varphi_b'} = \frac{Q_4}{Q_4(0)} \left(1 - \frac{v_{\xi b}^2}{a_b^2}\right) \quad . \tag{4.15e}$$

All we have done here is to remove all the possible sources of zeros from the denominators of Eqs. (2.24)[29] and grouped them into the new independent variable φ_b'. The singular point $v_{\xi b} = ab$ or (possibly) $Q_4 = 0$ is now determined by $d\varphi_b/d\varphi_b' = 0$. If either of these cases occurred, the program was made to stop and indicate which condition prevailed.

The case chosen to test was: $M_\infty = 5.5457$, θ_b = constant = 20°, $\gamma = 1.405$. At zero angle of attack the solution of Eqs. (4.10) through (4.14), when carried out to the limit of machine capacity (with single precision arithmetic), provided the following solution:

$$\theta_s = 24.520304918°$$

$$v_{rb} = .849812187 \quad .$$

Previous experience showed that the solution is more sensitive to $v_{rb}(0)$ than to $\Upsilon(0)\cdot(\theta_s(0) - \theta_b)$; hence the following values were used as initial conditions for both Eqs. (2.21) and (2.24) (transformed):

$$\Upsilon(0) = 4.520304918°$$

$$v_{rb} = .84981209 \qquad \text{Case 1}$$

$$= .84981219 \qquad \text{Case 2}$$

$$= .84981229 \qquad \text{Case 3}$$

The rather distressing results are shown in Figs. 4.2 and 4.3.

As can be seen from Figs. 4.2 and 4.3, the solutions proceeded smoothly near zero[30] up until a certain point at which the solution rapidly blows up to sonic velocity or reaches $\varphi_b = 180°$ with incorrect values. We can conjecture from this that two factors are at work to cause the disintegration of the solution. First, not enough significant figures have been retained in the calculation. This is apparent from a close inspection of the results near $\varphi_b = 0$. For Eqs. (2.24) with Case 2, we find a slope $dv_{\xi b}/d\varphi_b$ of about 1.4×10^{-5}. Changing $v_{rb}(0)$ by $+10^{-7}$ changes this derivative to about -5.5×10^{-2}, a factor of about -4000. Secondly, as the calculation proceeds a further loss of significant figures can cause the sonic singularity to become influential, resulting in a complete breakdown of the solution.

In conclusion, it seems apparent that the solution of either Eqs. (2.21) or (2.24) requires the use of a double precision method of solving simultaneous differential equations. Such a system is not available presently at Grumman, hence the calculations cannot be completed.[31] In any case, even if such a scheme were available, it is doubtful whether the method "as is" would be of practical value since

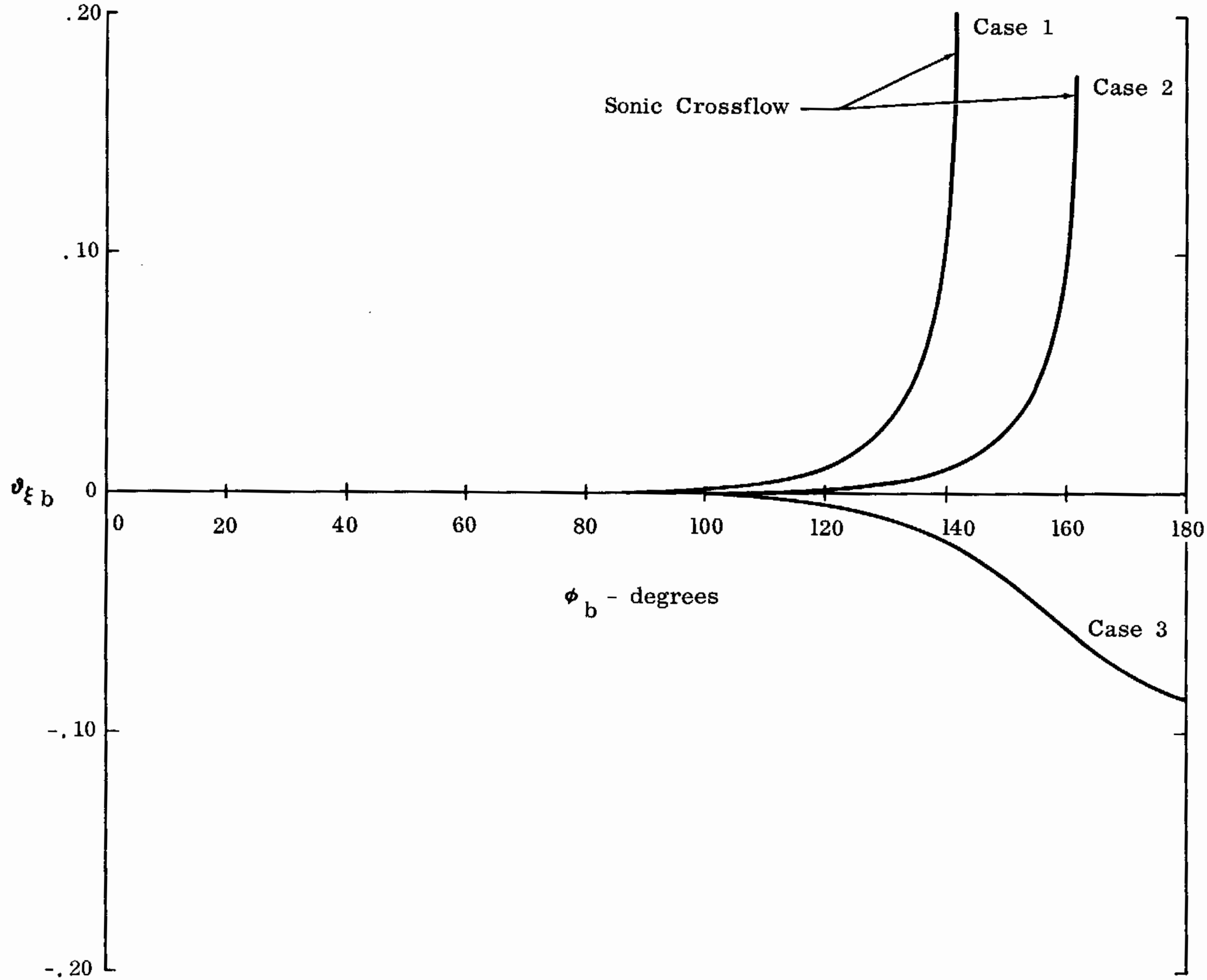

Fig. 4.2a - Distribution of $v_{\xi b}$ for a 20° Cone at $\alpha = 0°$, M = 5.5457 [Using Eqs. (2.24)]

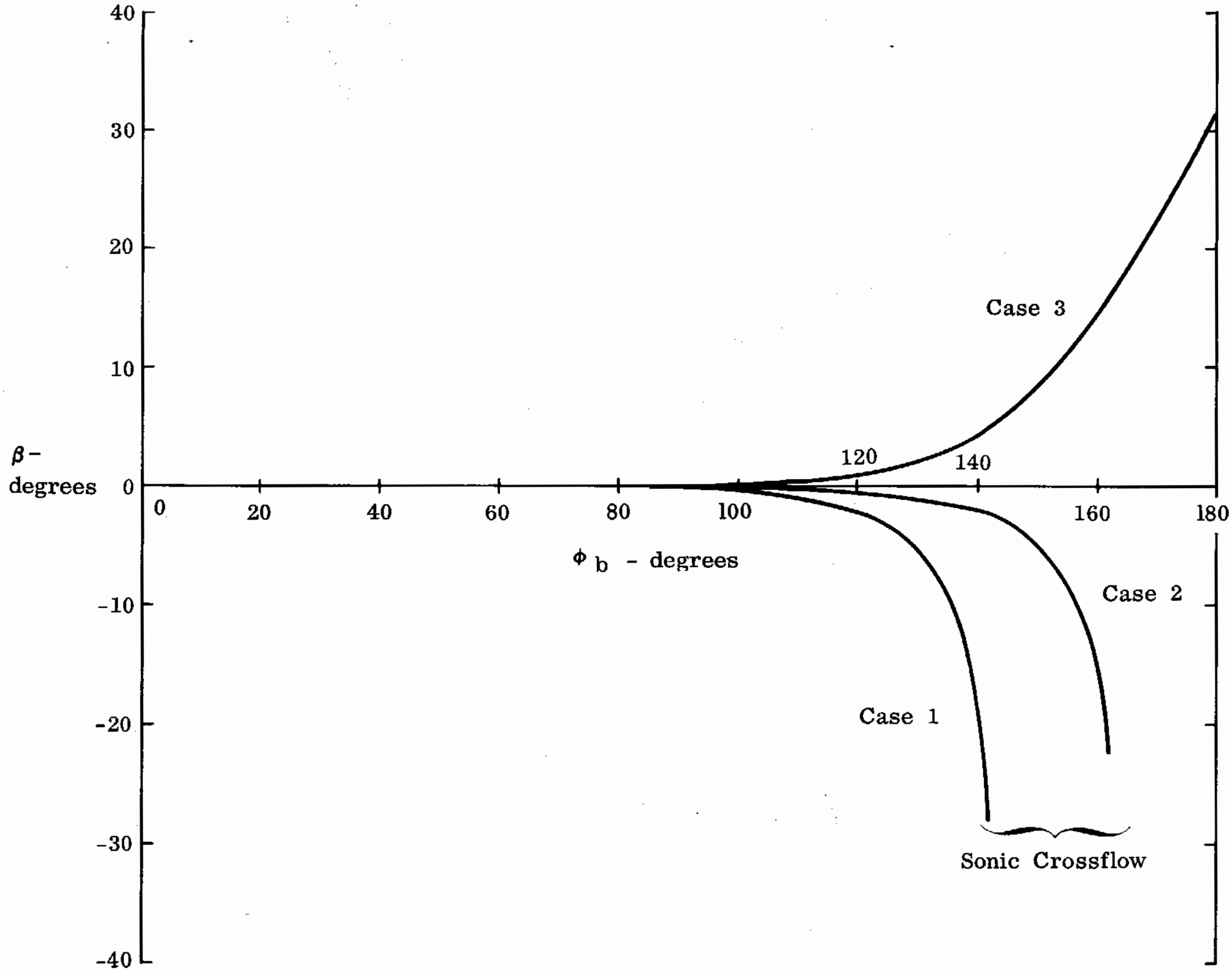

Fig. 4.2b - Distribution of β for a 20° Cone at $\alpha = 0°$, $M = 5.5457$ [Using Eqs. (2.24)]

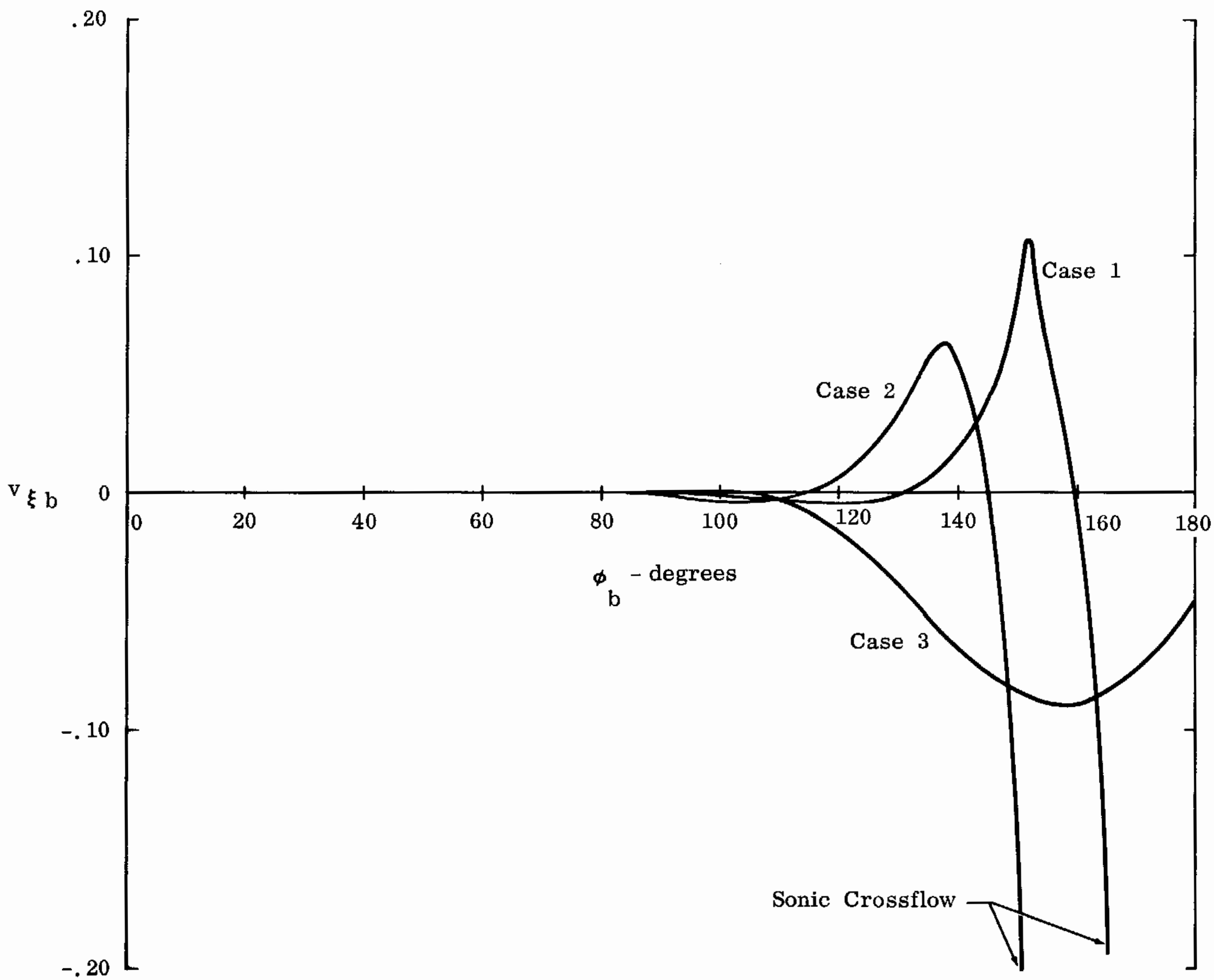

Fig. 4.3a - Distribution of $v_{\xi b}$ for a 20° Cone at $\alpha = 0°$, M = 5.5457 [Using Eqs. (2.21)]

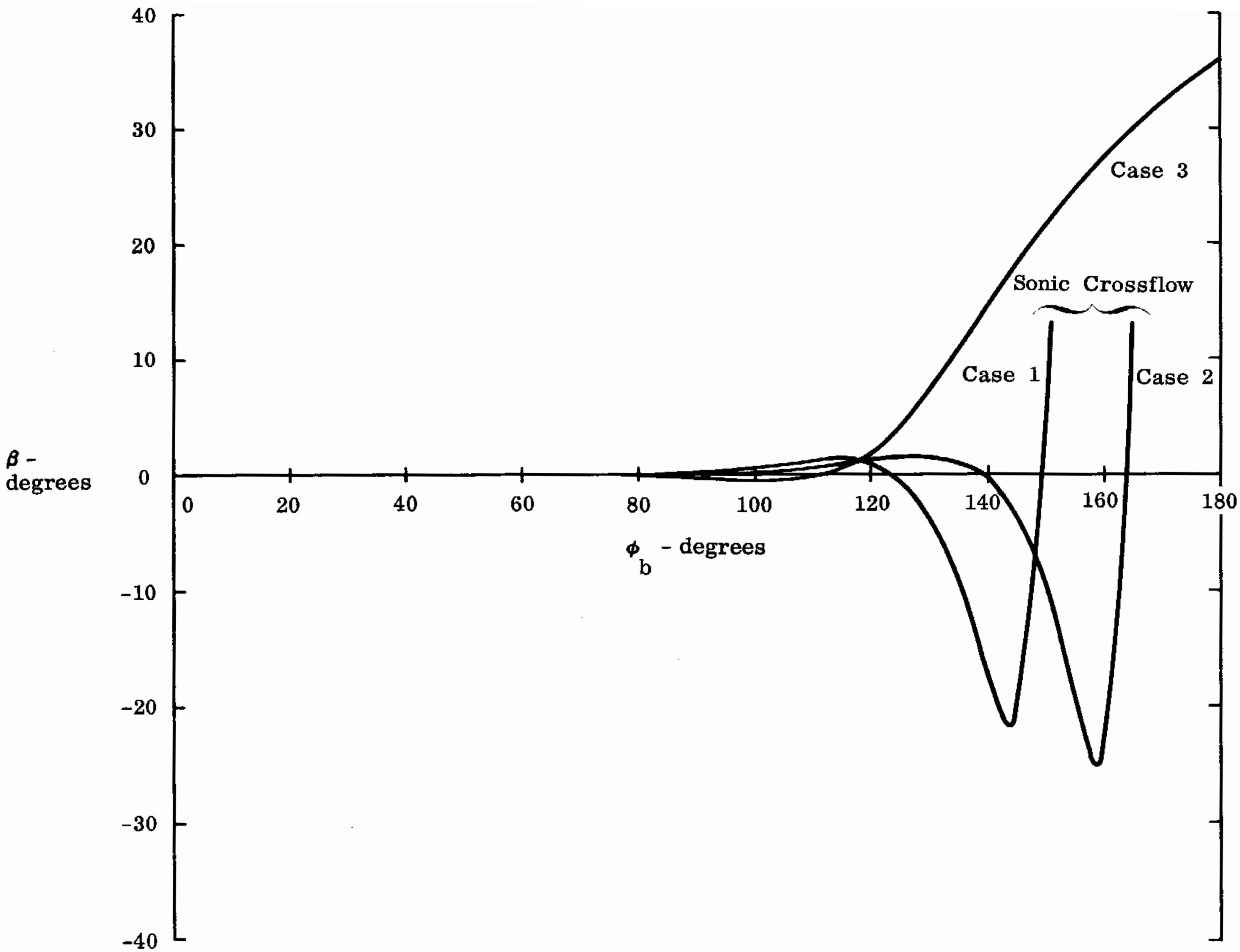

Fig. 4.3b - Distribution of β on a 20° Cone at $\alpha = 0°$,
M = 5.5457 [Using Eqs. (2.21)]

it would require the choice of two initial conditions by the user, whose values must certainly be correct to 9 or 10 significant figures. If practically useful solutions can be obtained by the method of integral relations,[32] it appears necessary to choose the strips differently, as mentioned on page 43, or to introduce additional approximations as Mikheev (Ref. 31) has done.

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5. SOME OTHER APPROACHES TO THE PROBLEM

In this section we shall describe some other approaches to the posed problem which have been investigated during the course of the contract work. These include the solution of the inverse (shock given) problem and the linearization of the equations.

The solution of the inverse problem of conical flow, that is, given a shock wave to find the body which produces it, has been attacked previously by several authors (Refs. 33, 34, and 35), using a step-by-step numerical solution of the governing partial differential equations. It was felt that a similar solution, using the method of integral relations, might be useful, at least for comparison, since these are the most exact solutions known at present.

The governing equations for the inverse method can be obtained by manipulating Eqs. (2.24) so that derivatives of the quantities specifying the cone surface, θ_b and δ_b appear. In addition, it was found to be convenient to retain the angle β as a dependent variable. The angle φ_b has been used as the independent variable. Carrying out the manipulations, we find the governing equations

$$
\begin{aligned}
{}^{i}J_1^{*}\left(1 - \frac{v_{\xi b}^2}{a_b^2}\right)\frac{dv_{\xi b}}{d\varphi_b} + {}^{i}J_2\frac{dv_{rb}}{d\varphi_b} + {}^{i}J_4\frac{d\beta}{d\varphi_b} + {}^{i}J_{14}\frac{d\delta_b}{d\varphi_b} &= {}^{i}J_5 \\
\frac{dv_{rb}}{d\varphi_b} &= {}^{i}M_5 \\
{}^{i}Q_4\frac{d\beta}{d\varphi_b} + {}^{i}Q_{14}\frac{d\delta_b}{d\varphi_b} &= {}^{i}Q_5 \qquad (5.1) \\
{}^{i}R_4\frac{d\beta}{d\varphi_b} + {}^{i}R_{14}\frac{d\delta_b}{d\varphi_b} &= {}^{i}R_5 \\
\frac{d\theta_b}{d\varphi_b} &= {}^{i}N_5
\end{aligned}
$$

where the coefficients are listed in APPENDIX IV. Note that the irrotationality condition on the body has been used here.

The properties of the system (5.1) are much the same as those for Eqs. (2.24). For instance it can be shown that $v_{\xi b} = \delta_b = 0$ at a plane of symmetry implies $\beta = 0$ so that the free initial conditions are θ_b and v_{rb}. Also, the singular points lie at crossflow sonic points. However, if one desires to pass through the singularity of Eq. (5.1), one again runs out of initial conditions to satisfy the downstream boundary conditions. Thus, some nonuniformity is required downstream of the singular point. By inspecting Eq. (5.1) and the coefficients listed in APPENDIX IV, we see that it is again necessary to have an inner shock wave somewhere downstream of the sonic point. In addition, we must also have a jump in the curvature $(d\delta_b/d\varphi_b)$ of the body at this point.

Thus we see that in mixed flow, according to our approximate equations, an analytic body produces a nonanalytic outer shock and an analytic outer shock produces a nonanalytic body. Due to the afore-mentioned computational difficulties, this case has not been investigated further.

The second approach we will discuss is the solution of Eqs. (2.24) by linearization. This has not been investigated in any great detail until now, but since it may form the basis for future work, we will outline the method. The following discussion is an expansion of the work reported on in Ref. 36. First let us rewrite Eqs. (2.24) in the form

$$\frac{dx_i}{d\xi} = f_i(x_i;\ \xi,\ P_{0b}) \ , \tag{5.2}$$

where $x_1 = v_{\xi b}$, $x_2 = v_{rb}$, $x_3 = T$, and $x_4 = \beta$. The f_i are self-explanatory. Note that the explicit dependence on P_{0b} has been indicated. Now, assume that some solution $x_i^{(0)}$ is known, which presumably satisfies the boundary conditions, and which is assumed to be near the correct solution. Then, we can represent the correct solution as

$$x_i = x_i^{(0)} + y_i \qquad |y_i|/|x_i^{(0)}| \ll 1 \ . \tag{5.3}$$

Then, substituting Eq. (5.3) into Eq. (5.2), we obtain

$$\frac{dy_i}{d\xi} = \mathfrak{z}_i^{(0)}(\xi) + \sum_{j=1}^{4} c_{ij}^{(0)}(\xi)\, y_j + \wp_i(\xi) \ , \tag{5.4}$$

where

$$\mathfrak{z}_i^{(0)}(\xi) = f_i^{(0)}(\xi) - \frac{dx_i^{(0)}}{d\xi} \ , \tag{5.5}$$

and

$$c_{ij}^{(0)}(\xi) = \frac{\partial f}{\partial x_i}\bigg|^{(0)} \ . \tag{5.6}$$

The $\wp_i(\xi)$ term arises from the fact that the total pressure on the cone surface is unknown and this forms one point of departure from the two dimensional solutions of Vaglio-Laurin (Ref. 6) where, as mentioned, one could assume the entropy on the body is the maximum flow entropy. In this case, the entropy is determined at cross-flow stagnation points and hence, from Eq. (3.4), it must be related to $\tilde{y}_3$. It can be shown that the $\wp_i(\xi)$ term then has the form

$$\wp_i(\xi) = \wp_i^{(0)}(\xi)\varkappa^{(0)}(\tilde{\xi})\tilde{y}_3 \ . \tag{5.7}$$

The coefficients of the above equations will not be listed here. Let it suffice to say that they occupy a quite substantial amount of space.

At present no detailed discussion of Eq. (5.4) for the mixed flow case will be undertaken. It is obvious from Ref. 6, however, that in the vicinity of the sonic point one must perform a PLK type of expansion and that solutions in several different regions must be matched. In addition, some iterative procedure will be necessary to account for the $\rho_i(\xi)$ term.

In the all subsonic case, however, it is possible to obtain a solution to Eqs. (5.4) in terms of a Green's matrix. Briefly, the solution to the vector boundary value problem

$$\frac{d\bar{y}}{d\xi} = \bar{\bar{c}}(\xi)\bar{y} + H(\bar{\xi}) \quad , \tag{5.8}$$

subject to the boundary conditions

$$\bar{\bar{M}}\,\bar{y}(a) + \bar{\bar{N}}\,\bar{y}(b) = 0 \quad , \tag{5.9}$$

where (a, b) is the interval of interest and $\bar{\bar{M}}$ and $\bar{\bar{N}}$ are constant matrices, can be written (Ref. 37)

$$\bar{y}(\xi) = \int_a^b \bar{\bar{G}}(\xi,\ \psi)\bar{H}(\psi)\,d\psi \ . \tag{5.10}$$

Here, $\bar{H}(\psi) = \bar{\mathfrak{z}}^{(0)}(\psi) + \bar{\rho}^{(n)}(\psi)$ where $\bar{\rho}^{(n)}(\psi)$ corresponds to the n^{th} approximation to the total pressure on the body. The matrices $\bar{\bar{M}}$ and $\bar{\bar{N}}$ can be written

$$\bar{\bar{M}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\tilde{k} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \bar{\bar{N}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.11}$$

where $\tilde{k} = 0$ if the stagnation point is fixed in a plane of symmetry, and $\tilde{k} \neq 0$ if the stagnation point is not in a plane of symmetry. $\bar{\bar{G}}(\xi, \psi)$ is the Green's matrix given by

$$\bar{\bar{G}}(\xi, \psi) = \begin{cases} \bar{\bar{\Phi}}(\xi)\bar{\bar{\Phi}}^{-1}(\psi) + \bar{\bar{\Phi}}(\xi)\bar{\bar{J}}(\psi) & \psi < \xi \\ \bar{\bar{\Phi}}(\xi)\bar{\bar{J}}(\psi) & \psi > \xi , \end{cases} \tag{5.12}$$

where

$$\bar{\bar{J}}(\psi) = - [\bar{\bar{M}}\,\bar{\bar{\Phi}}(a) + \bar{\bar{N}}\,\bar{\bar{\Phi}}(b)]^{-1}\bar{\bar{N}}\,\bar{\bar{\Phi}}(b)\bar{\bar{\Phi}}^{-1}(\psi) , \tag{5.13}$$

and $\bar{\bar{\Phi}}(\xi)$ is the solution of

$$\frac{d\bar{\bar{\Phi}}}{d\xi} = \bar{\bar{c}}(\xi)\bar{\bar{\Phi}} \quad \text{with} \quad \bar{\bar{\Phi}}(a) = \bar{\bar{I}} . \tag{5.14}$$

where $\bar{\bar{I}}$ is the unit matrix.

Now, in the case of the circular cone at angle of attack, $a = 0$, $b = \pi$, the stagnation point is fixed and the necessary iteration procedure is straightforward. But, for the elliptic cone at incidence, we encounter the same difficulties mentioned in Section 3, that is, we must apply Eq. (5.10) in the region from $\tilde{\xi}$ to π and again from $\tilde{\xi}$ to 0, and match the solutions at $\xi = \tilde{\xi}$. However, we again have the problem of matching two quantities with one free parameter $(\tilde{\xi})$ so that the method apparently will not work in this case.

As stated previously, neither of the methods discussed above has been carried to the computational stage. They may be of interest for future work, however, particularly the linearized solution for the circular cone over a wide range of angle of attack.

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6. FOOTNOTES

1. The extension of the method to a real gas in equilibrium, while perfectly feasible, only tends to complicate the analysis and except for very strong shocks should not greatly influence the results obtained.

2. It is possible to linearize but this brings up new problems. These are briefly discussed in Section 5.

3. In addition, the coordinate system should take into account any known conditions of symmetry. Aside from this, it can be arbitrarily chosen.

4. This is possible for quite arbitrary coordinate systems (see Ref. 14).

5. This is convenient but not necessary.

6. By strips we mean, of course, the regions into which the shock layer is divided by conical surfaces lying between the shock and the body.

7. This is equivalent to using the trapezoidal role for integration.

8. In fact, Melnik (Ref. 15) has shown that $\partial s/\partial n = \infty$ (s = entropy, n = normal coordinate) for circular cones at small angle of attack. However, this solution comes from an analysis of the entropy layer and thus, while Kennet's assumption may be rigorously incorrect, the fact that the entropy layer is so thin probably does not affect the over-all result.

9. This requirement would not appear to be important. However, Koppenfels (Ref. 17) has shown for two dimensional, incompressible flow that strong local accelerations occur near jumps in curvature. In order to pass such jumps in the present case, the local behavior of the velocity must be known, but this problem has not yet been solved.

10. We shall refer to velocity components tangent to spheres r = constant as crossflow components of velocity. When referring to streamlines or sonic lines we mean the intersection of stream surfaces and sonic surfaces with spheres r = constant. Similarly, reference to points means the intersection of conical rays with spheres r = constant.

11. Chushkin and Shchennikov have used a spherical coordinate system. However, the discussion which follows is carried out by using the present system of equations as an example.

12. This difficulty can also be overcome by using the condition of irrotationality which is valid everywhere in this case instead of the momentum equation. Holt (Ref. 18) has done this in his calculations for nozzles.

13. Note that this assumption is not valid everywhere in the flow field in contrast to Holt's (Ref. 18) nozzle calculations wherein the entire flow field is irrotational.

14. Note that we have distinguished between angle of attack and angle of yaw here.

15. In the present context, the circular cone at angle of attack has the same features as the elliptic cone at zero angle of attack.

16. We shall consider only cases where there is one crossflow stagnation point in the region of interest. Extension of the following discussion to more than one stagnation point is possible but considerably increases the complexity of the analysis.

17. This allows $dv_{\xi_b}/d\xi$ downstream of the shock to be calculated.

18. This is true in our approximation. Actually, the outer shock is affected by the inner shock through some attenuation process which is beyond the scope of this discussion.

19. As the angle of attack increases, the stagnation point will move from the major axis of the cone until it reaches the minor axis where it becomes stationary again and the preceding discussion holds. This is for the case when the angle of attack vector lies in the plane of the minor axis. If the angle of attack vector lies in the plane of the major axis, the stagnation point will always lie at the major axis on the windward side.

20. We note here that the assumption of maximum flow entropy on the body is of no use in conical flows since we do not know this quantity anyhow.

21. It may not be possible to construct such a system beforehand, but it can be determined along with the solution.

22. $dv_{\xi b}/d\xi$ can be shown (Ref. 26, APPENDIX III) to be finite at point B on the subsonic side. $d\beta/d\xi$ which is essentially the curvature of the shock, must jump since the segment AC is straight.

23. Depending on the geometry, one or the other may be the correct boundary to take for the elliptic region (Ref. 25). Generally the calculation should include both the elliptic region and transonic region, if any.

24. Note that Belotserkovskii (Ref. 5) has employed the regularity condition for this case and obtained good comparison with experimental results. This agreement may be due to the fact that the regularity conditions gives a velocity derivative which while finite, is very large.

25. Note in Fig. 3.6 that the flow on the rear side of the wing is unknown and is immaterial for our purposes.

26. In this case one only need specify that the singular point is at the leading edge.

27. This is not entirely unexpected. Many authors have conjectured the existence of this shock on the lee side of a flat delta wing and Tracy (Ref. 28) has obtained a similar shock experimentally, although the strong viscous effects on his experiment mask the true origin of the shock.

28. The actual curvature = $rK_b(\xi)$.

29. The $Q_4(0)$ term is included solely to make $\left. d\varphi_b/d\varphi_b' \right|_{Q_b'=0} = 1$.

30. The oscillations seen at the higher values of φ_b in Fig. 4.3 were also present for smaller values of φ_b but with negligible amplitude.

31. Attempts to obtain a solution at $\alpha = 5°$ were made, but the severe dependence of the initial derivatives on the initial conditions made the proper choice of the latter impossible. We also mention here that the reverse scheme was also tried, that is, to pick the derivatives and calculate $v'_{rb}(0)$ and $\Upsilon(0)$. However, this required the iterative solution of transcendental equations which, due to the necessity of picking an error criterion, also introduced errors causing the subsequent disintegration of the solution. Various accuracy criteria in the solution of the system of simultaneous equations were also employed but produced no improvement in the results.

32. Chushkin and Shchennikov (Ref. 9) and Holt and Lee (Ref. 32) have obtained solutions by the method of integral relations. However, they used different formulations and did not comment in detail upon the numerical difficulties encountered.

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APPENDIX I

COEFFICIENTS OF THE ORDINARY DIFFERENTIAL EQUATIONS (2.20) FOR THE ONE-STRIP APPROXIMATION

$$J_1^* = \Upsilon \rho_b \chi_{\eta b}$$

$$J_2 = - \frac{\rho_b v_{rb} v_{\xi b} \chi_{\eta b} \Upsilon}{a_b^2}$$

$$J_3 = \Upsilon [F_2 \chi_{\eta s} v_{\xi s} + \Omega_8 \rho_s v_{\xi s} + D_2 \rho_s \chi_{\eta s}] + (T_{1L} - T_{1s}) \qquad \text{(I-1)}$$

$$J_4 = \Upsilon [F_3 v_{\xi s} + D_3 \rho_s]$$

$$J_5 = \Upsilon \left[K_{1b} + K_{1s} - \frac{1}{\chi_{\eta b}} \frac{\partial \chi_\eta}{\partial \xi} \bigg|_b - (F_1 \chi_{\eta s} v_{\xi s} + \Omega_7 \rho_s v_{\xi s} + D_1 \chi_{\eta s} \rho_s) \right] - 2S_{1s}$$

$$M_1^* = \Upsilon \rho_b v_{\xi b} \chi_{\eta b}$$

$$M_2 = - \Upsilon \rho_b v_{rb} \chi_{\eta b} \left[1 + \frac{v_{\xi b}^2}{a_b^2} \right]$$

$$M_3 = \Upsilon \left[F_2 v_{\xi s}^2 + \frac{\gamma - 1}{2\gamma} G_2 + 2\rho_s v_{\xi s} D_2 \right] \chi_{\eta s} + T_{2b} - T_{2s} + \Upsilon \Omega_8 \left[\frac{\gamma - 1}{2\gamma} P_s + \rho_s v_{\xi s}^2 \right] \qquad \text{(I-2)}$$

$$M_4 = \Upsilon \left[F_3 v_{\xi s}^2 + \frac{\gamma - 1}{2\gamma} G_3 + 2\rho_s v_{\xi s} D_3 \right] \chi_{\eta s}$$

$$M_5 = \Upsilon \left[\left(- F_1 v_{\xi s}^2 - \frac{\gamma - 1}{2\gamma} G_1 - 2\rho_s v_{\xi s} D_1 \right) \chi_{\eta s} + K_{2b} + K_{2s} \right] - 2S_{2s}$$

$$- \Upsilon \left\{ \left[\frac{\gamma - 1}{2\gamma} P_s + \rho_s v_{\xi s}^2 \right] \Omega_7 + \left[\frac{\gamma - 1}{2\gamma} P_b + \rho_b v_{\xi b}^2 \right] \frac{d\chi_{\eta b}}{d\xi} \right\}$$

$$Q_3 = \Upsilon\left[F_2\chi_{\eta s}v_{\xi s}v_{\eta s} + \Omega_8\rho_s v_{\xi s}v_{\eta s} + D_2\rho_s\chi_{\eta s}v_{\eta s} + E_2\rho_s\chi_{\eta s}v_{\xi s}\right] - T_{3s}$$

$$Q_4 = \Upsilon\left[F_3\chi_{\eta s}v_{\xi s}v_{\eta s} + D_3\rho_s v_{\eta s}\chi_{\eta s} + E_3\rho_s\chi_{\eta s}v_{\xi s}\right] \tag{I-3}$$

$$Q_5 = \Upsilon\left\{K_{3s} + K_{3b} - \left[F_1\chi_{\eta s}v_{\xi s}v_{\eta s} + \Omega_7\rho_s v_{\xi s}v_{\eta s} + D_1\rho_s v_{\eta s}\chi_{\eta s} + E_1\rho_s\chi_{\eta s}v_{\xi s}\right]\right\} + 2(S_{3b} - S_{3s})$$

where

$$A_1 = X\left[\frac{\Omega_1}{q_\infty} - \Omega_3\tan(\kappa - \beta)\right]$$

$$A_2 = X\left[\frac{\Omega_2}{q_\infty} - \Omega_4\tan(\kappa - \beta)\right] \tag{I-4}$$

$$A_3 = X\tan(\kappa - \beta)$$

and

$$X = \frac{2\gamma}{\gamma-1}P_{0s}\left[1 - \frac{v_{N\infty}^2}{v_{N\infty}^2 + 1 - V_\infty^2} - \frac{4v_{N\infty}^2}{4\gamma v_{N\infty}^2 - (\gamma-1)^2(1 - V_\infty^2)}\right]$$

$$B_1 = q_\infty\Omega_3\cos(\kappa - \beta) + \Omega_1\sin(\kappa - \beta)$$

$$B_2 = q_\infty\Omega_4\cos(\kappa - \beta) + \Omega_2\sin(\kappa - \beta) \tag{I-5}$$

$$B_3 = -q_\infty\cos(\kappa - \beta)$$

$$C_1 = \frac{1}{v_{N\infty}}\left\{- 2\,\frac{\gamma-1}{\gamma+1}\left[\Omega_5 v_{r\infty} + B_1 v_{T\infty}\right] + v_{Ns}\left[\Omega_1 \cos(\kappa - \beta) - \Omega_3 v_{T\infty}\right]\right\}$$

$$C_2 = \frac{1}{v_{N\infty}}\left\{- 2\,\frac{\gamma-1}{\gamma+1}\left[\Omega_6 v_{r\infty} + B_2 v_{T\infty}\right] + v_{Ns}\left[\Omega_2 \cos(\kappa - \beta) - \Omega_4 v_{T\infty}\right]\right\} \qquad \text{(I-6)}$$

$$C_3 = - \frac{v_{T\infty}}{v_{N\infty}}\left\{2\,\frac{\gamma-1}{\gamma+1} B_3 - v_{Ns}\right\}$$

$$D_1 = B_1 \cos\beta - C_1 \sin\beta$$

$$D_2 = B_2 \cos\beta - C_2 \sin\beta \qquad \text{(I-7)}$$

$$D_3 = (B_3 - v_{Ns})\cos\beta - (C_3 + v_{Ts})\sin\beta$$

$$E_1 = B_1 \sin\beta + C_1 \cos\beta$$

$$E_2 = B_2 \sin\beta + C_2 \cos\beta \qquad \text{(I-8)}$$

$$E_3 = (B_3 - v_{Ns})\sin\beta + (C_3 + v_{Ts})\cos\beta$$

$$F_1 = \rho_s\left[\frac{A_1}{\rho_{0s}} - \frac{1}{a_s^2}(\Omega_5 v_{rs} + D_1 v_{\xi s} + E_1 v_{\eta s})\right]$$

$$F_2 = \rho_s\left[\frac{A_2}{\rho_{0s}} - \frac{1}{a_s^2}(\Omega_b v_{rs} + D_2 v_{\xi s} + E_2 v_{\eta s})\right] \qquad \text{(I-9)}$$

$$F_3 = \rho_s\left[\frac{A_3}{\rho_{0s}} - \frac{1}{a_s^2}(D_3 v_{\xi s} + E_3 v_{\eta s})\right]$$

$$G_1 = P_s\left[\frac{A_1}{P_{0s}} - \frac{\gamma}{a_s^2}(\Omega_5 v_{rs} + D_1 v_{\xi s} + E_1 v_{\eta s})\right]$$

$$G_2 = P_s\left[\frac{A_2}{P_{0s}} - \frac{\gamma}{a_s^2}(\Omega_6 v_{rs} + D_2 v_{\xi s} + E_2 v_{\eta s})\right]$$

$$G_3 = P_s\left[\frac{A_3}{P_{0s}} - \frac{\gamma}{a_s^2}(D_3 v_{\xi s} + E_3 v_{\eta s})\right] \quad .$$

Above, we have written, for example

$$\frac{dP_{0s}}{d\xi} = A_1 + A_2 \frac{d\Upsilon}{d\xi} + A_3 \frac{d\beta}{d\xi} \quad .$$

Similarly, derivatives corresponding to B, C, D, E F, G are those of v_{Ts}, v_{Ns}, $v_{\xi s}$, $v_{\eta s}$, ρ_s, and P_s.

APPENDIX II

COEFFICIENTS OF THE ORDINARY DIFFERENTIAL EQUATIONS (2.30) FOR THE TWO-STRIP APPROXIMATION

NOTE

[The total pressure on the strip boundary $\eta = \frac{1}{2}\Upsilon(\xi)$ is denoted by Θ_2 in the following. The value of Θ_2, as noted in the text, is not given by the system (2.30) but must be determined independently.]

$$J_6^* = \frac{1}{6} J_1^*$$

$$J_7^* = \frac{2}{3} \rho_2 \chi_{\eta 2} \Upsilon$$

$$J_8 = \frac{1}{6} J_2$$

$$J_9 = -\frac{2}{3} \frac{\rho_2 v_{r2} v_{\xi 2} \chi_{\eta 2} \Upsilon}{a_2^2}$$

$$J_{10} = -\frac{2}{3} \frac{\rho_2 v_{\eta 2} v_{\xi 2} \chi_{\eta 2} \Upsilon}{a_2^2} \qquad \text{(II-1)}$$

$$J_{11} = \frac{1}{6}\left\{T_{1b} + 4T_{12} - 5T_{1s}\right\} + \frac{1}{3} \rho_2 v_{\xi 2} \Omega_{10} \Upsilon + \frac{1}{6} \Upsilon\left\{F_2 \chi_{\eta s} v_{\xi s} + \Omega_8 \rho_s v_{\xi s} + D_2 \rho_s \chi_{\eta s}\right\}$$

$$J_{12} = \frac{1}{6} \Upsilon\left\{F_3 \chi_{\eta s} v_{\xi s} + D_3 \rho_s \chi_{\eta s}\right\}$$

$$J_{13} = S_{1b} - S_{1s} + \frac{1}{6} \Upsilon\left\{K_{1b} + 4K_{12} + K_{1s}\right\} - \frac{1}{6} \rho_b \Upsilon \left.\frac{\partial \chi_\eta}{\partial \xi}\right|_b - \frac{2}{3} \rho_2 \Upsilon \Omega_9 v_{\xi 2} - \frac{2}{3} \Upsilon T_{12} \frac{1}{\Theta_2} \frac{d\Theta_2}{d\xi} - \frac{1}{6} \Upsilon\left\{F_1 v_{\xi s} \chi_{\eta s} + \Omega_7 \rho_s v_{\xi s} + D_1 \rho_s \chi_{\eta s}\right\}$$

$$\hat{J}_6^* = \frac{5}{24} J_1^*$$

$$\hat{J}_7^* = \frac{1}{3} \rho_2 \Upsilon \chi_{\eta b}$$

$$\hat{J}_8 = \frac{5}{24} J_2$$

$$\hat{J}_9 = - \frac{1}{3} \frac{v_{r2} v_{\xi 2} \rho_2 \Upsilon \chi_{\eta 2}}{a_b^2}$$

$$\hat{J}_{10} = - \frac{1}{3} \frac{v_{\eta 2} v_{\xi 2} \rho_2 \Upsilon \chi_{\eta 2}}{a_b^2}$$

$$\hat{J}_{11} = \frac{1}{24}\left\{5T_{1b} - 4T_{12} - T_{1s}\right\} + \frac{1}{6} \Upsilon \rho_2 v_{\xi 2} \Omega_{10} - \frac{1}{24} \Upsilon \left\{\chi_{\eta s} v_{\xi s} F_2 + \rho_s v_{\xi s} \Omega_8 + \rho_s \chi_{\eta s} D_2\right\} \qquad \text{(II-2)}$$

$$\hat{J}_{12} = - \frac{1}{24} \Upsilon \left\{F_3 \chi_{\eta s} v_{\xi s} + D_3 \rho_s \chi_{\eta s}\right\}$$

$$\hat{J}_{13} = S_{1b} - S_{12} + \frac{1}{24} \Upsilon \left\{5K_{1b} + 8K_{12} - K_{1s}\right\} - \frac{5}{24} \rho_b \Upsilon v_{\xi b} \left.\frac{\partial \chi_\eta}{\partial \xi}\right|_b - \frac{1}{3} \rho_2 \Upsilon v_{\xi 2} \Omega_9 - \frac{1}{3} \Upsilon T_{12} \frac{1}{\Theta_2} \frac{d\Theta_2}{d\xi} + \frac{1}{24} \Upsilon \left\{F_1 \chi_{\eta s} v_{\xi s} + D_1 \rho_s \chi_{\eta s} + \Omega_7 \rho_s v_{\xi s}\right\}$$

$$M_6^* = \frac{1}{6} v_{\xi b} \chi_{\eta b} \rho_b \Upsilon$$

$$M_7^* = \frac{2}{3} \rho_2 v_{\xi 2} \chi_{\eta 2} \Upsilon$$

$$M_8 = -\frac{1}{6}\left\{1 + \frac{v_{\xi 2}^2}{a_b^2}\right\} v_{rb} \chi_{\eta b}$$

$$M_9 = -\frac{2}{3} \rho_2 \Upsilon v_{r2} \chi_{\eta 2}\left\{1 + \frac{v_{\xi 2}^2}{a_2^2}\right\}$$

$$M_{10} = -\frac{2}{3} \rho_2 \Upsilon v_{\eta 2} \chi_{\eta 2}\left\{1 + \frac{v_{\xi 2}^2}{a_2^2}\right\}$$

(II-3)

$$M_{11} = \frac{1}{6} \Upsilon \chi_{\eta s}\left\{F_2 v_{\xi s}^2 + \frac{\gamma-1}{2\gamma} G_2 + 2D_2 \rho_s v_{\xi s}\right\} + \frac{1}{6}\left\{T_{2b} + 4T_{22} - 5T_{2b}\right\}$$

$$+ \frac{1}{6} \Upsilon\left\{\Omega_{18}\left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\xi s}^2\right] + 2\Omega_{10}\left[\frac{\gamma-1}{2\gamma} P_2 + \rho_2 v_{\xi 2}^2\right]\right\}$$

$$M_{12} = \frac{1}{6} \Upsilon \chi_{\eta s}\left\{F_3 v_{\xi s}^2 + \frac{\gamma-1}{2\gamma} G_3 + 2D_3 \rho_s v_{\xi s}\right\}$$

$$M_{13} = S_{2b} - S_{2s} + \frac{1}{6} \Upsilon\left\{K_{2b} + 4K_{22} + K_{26} - \chi_{\eta s}\left[F_1 v_{\xi s}^2 + \frac{\gamma-1}{2\gamma} G_1 + 2D_1 \rho_s v_{\xi s}\right]\right\}$$

$$- \frac{2}{3} \rho_2 \Upsilon\left\{v_{\xi 2}^2 + \frac{1}{\gamma} a_2^2\right\} \frac{1}{\Theta_2} \frac{a\Theta_2}{d\xi} - \frac{1}{6} \Upsilon\left\{\left[\frac{\gamma-1}{2\gamma} P_b + \rho_b v_{\xi b}^2\right] \frac{d\chi_{\eta b}}{d\xi}\right.$$

$$\left. + 4\left[\frac{\gamma-1}{2\gamma} P_2 + \rho_2 v_{\xi 2}^2\right]\Omega_9 + \left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\xi s}^2\right]\Omega_{17}\right\}$$

$$\hat{M}_6^* = \frac{5}{24}\rho_b \Upsilon v_{\xi b}\chi_{\eta b}$$

$$\hat{M}_7^* = \frac{1}{3}\rho_2 \Upsilon v_{\xi 2}\chi_{\eta 2}$$

$$\hat{M}_8 = -\frac{5}{24} v_{rb}\rho_b \Upsilon\left\{1 + \frac{v_{\xi b}^2}{a_b^2}\right\}\chi_{\eta b}$$

$$\hat{M}_9 = -\frac{1}{3}\rho_2 \Upsilon v_{r2}\left\{1 + \frac{v_{\xi 2}^2}{a_2^2}\right\}\chi_{\eta 2}$$

$$\hat{M}_{10} = -\frac{1}{3}\rho_2 \Upsilon v_{\eta 2}\left\{1 + \frac{v_{\xi 2}^2}{a_2^2}\right\}\chi_{\eta 2}$$

$$\hat{M}_{11} = \frac{1}{24}\left\{5T_{2b} - 4T_{22} - T_{2s}\right\} - \frac{1}{24}\Upsilon\chi_{\eta s}\left\{F_2 v_{\xi s}^2 + \frac{\gamma-1}{2\gamma}G_2 + 2D_2\rho_s v_{\xi s}\right\}$$
$$- \frac{1}{24}\Upsilon\left\{\Omega_8\left[\frac{\gamma-1}{2\gamma}P_s + \rho_s v_{\xi s}^2\right] + 2\Omega_{10}\left[\frac{\gamma-1}{2\gamma}P_2 + \rho_2 v_{\xi 2}^2\right]\right\} \qquad \text{(II-4)}$$

$$\hat{M}_{12} = -\frac{1}{24}\Upsilon\chi_{\eta s}\left\{F_3 v_{\xi s}^2 + \frac{\gamma-1}{2\gamma}G_3 + 2D_3\rho_s v_{\xi s}\right\}$$

$$\hat{M}_{13} = S_{2b} - S_{22} + \frac{1}{24}\Upsilon\left\{5K_{2b} + 8K_{22} - K_{2s} + \chi_{\eta s}\left[F_1 v_{\xi s}^2 + \frac{\gamma-1}{2\gamma}G_1\right.\right.$$
$$\left.\left. + 2D_1\rho_s v_{\xi s}\right]\right\} - \frac{1}{3}\rho_2\Upsilon\left\{v_{\xi 2}^2 + \frac{1}{\gamma}a_2^2\right\}\frac{1}{\Theta_2}\frac{d\Theta_2}{d\xi} - \frac{1}{24}\Upsilon\left\{5\left[\frac{\gamma-1}{2\gamma}P_b + \rho_b v_{\xi b}^2\right]\right.$$
$$\left. - 4\left[\frac{\gamma-1}{2\gamma}P_2 + \rho_2 v_{\xi 2}^2\right]\Omega_9 - \left[\frac{\gamma-1}{2\gamma}P_s + \rho_s v_{\xi s}^2\right]\Omega_7\right\}$$

$$Q_7^* = \frac{2}{3}\rho_2 \Upsilon \chi_{\eta 2} v_{\sigma 2}$$

$$Q_9 = -\frac{2\Upsilon\rho_2 v_{r2} v_{\xi 2} v_{\eta 2}\chi_{\eta 2}}{3a_2^2}$$

$$Q_{10} = \frac{2}{3}\Upsilon\rho_2 v_{\xi 2}\chi_{\eta 2}$$

$$Q_{11} = \frac{1}{6}\left\{4T_{32} - 5T_{3s}\right\} + \frac{1}{3}\Upsilon\frac{T_{32}}{\chi_{\eta 2}}\Omega_{10} + \frac{1}{6}\Upsilon\left\{F_2\chi_{\eta s}v_{\xi s}v_{\eta s} + \Omega_8\rho_s v_{\eta s}v_{\xi s} + D_2\rho_s v_{\eta s}\chi_{\eta s} + E_2\rho_s\chi_{\eta s}v_{\xi s}\right\} \quad \text{(II-5)}$$

$$Q_{12} = \frac{1}{6}\Upsilon\left\{F_3\chi_{\eta s}v_{\eta s}v_{\xi s} + D_3\rho_s\chi_{\eta s}v_{\eta s} + E_3\rho_s\chi_{\eta s}v_{\xi s}\right\}$$

$$Q_{13} = S_{3b} - S_{3s} + \frac{1}{6}\Upsilon\left\{K_{3b} + 4K_{32} + K_{3s}\right\} - \frac{2}{3}\Upsilon T_{32}\left\{\frac{1}{\Theta_2}\frac{d\Theta_2}{d\xi} + \frac{\Omega_9}{\chi_{\eta 2}}\right\} - \frac{1}{6}\Upsilon\left\{F_1\chi_{\eta s}v_{\eta s}v_{\xi s} + \Omega_7\rho_s v_{\eta s}v_{\xi s} + D_1\rho_s\chi_{\eta s}v_{\eta s} + E_1\rho_s v_{\xi s}\chi_{\eta s}\right\}$$

$$\hat{Q}_7^* = \frac{1}{3}\rho_2\Upsilon\chi_{\eta 2}v_{\eta 2}$$

$$\hat{Q}_9 = -\frac{\Upsilon v_{r2}\rho_2 v_{\xi 2}v_{\eta 2}\chi_{\eta 2}}{a_2^2} \quad \text{(II-6)}$$

$$\hat{Q}_{10} = \frac{1}{3}\rho_2\Upsilon\chi_{\eta 2}v_{\xi 2}$$

$$\hat{Q}_{11} = -\frac{1}{24}\left\{4T_{32} + T_{3s}\right\} + \frac{1}{6}\,\Upsilon T_{32}\,\frac{\Omega_{10}}{\chi_{\eta 2}} - \frac{1}{24}\,\Upsilon\left\{F_2\chi_{\eta 2}v_{\eta s}v_{\xi s}\right.$$

$$\left. + \Omega_8\rho_s v_{\eta s}v_{\xi s} + D_2\rho_s\chi_{\eta s}v_{\eta s} + E_2\rho_s\chi_{\eta s}v_{\xi s}\right\}$$

(II-6)
(Cont.)

$$\hat{Q}_{12} = -\frac{1}{24}\,\Upsilon\left\{F_3\chi_{\eta s}v_{\eta s}v_{\xi s} + D_3\rho_s\chi_{\eta s}v_{\eta s} + E_3\rho_s\chi_{\eta s}v_{\xi s}\right\}$$

$$\hat{Q}_{13} = S_{3b} - S_{32} + \frac{1}{24}\,\Upsilon\left\{8K_{32} - K_{3s}\right\} - \frac{1}{3}\,\Upsilon T_{32}\left\{\frac{1}{\Theta_2}\,\frac{d\Theta_2}{d\xi} + \frac{\Omega_9}{\chi_{\eta s}}\right\}$$

$$+ \frac{1}{24}\,\Upsilon\left\{F_1\chi_{\eta s}v_{\eta s}v_{\xi s} + \Omega_7\rho_s v_{\eta s}v_{\xi s} + D_1\rho_s\chi_{\eta s}v_{\eta s} + E_1\rho_s\chi_{\eta s}v_{\xi s}\right\}$$

where

$$T_{12} = \rho_2 v_{\xi 2}\chi_{\eta 2}$$

$$S_{12} = \rho_2 v_{\eta 2}\chi_{\eta 2} \qquad \text{(II-7)}$$

$$K_{12} = -\,2\chi_{\eta 2}\chi_{\xi 2}\rho_2 v_{r2}$$

$$T_{22} = \left\{\frac{\gamma-1}{2\gamma}\,P_2 + \rho_2 v_{\xi 2}^2\right\}\chi_{\eta 2}$$

$$S_{22} = \rho_2 v_{\xi 2}v_{\eta 2}\chi_{\xi 2} \qquad \text{(II-8)}$$

$$K_{22} = \left\{\frac{\gamma-1}{2\gamma}\,P_2 + \rho_2 v_{\eta 2}^2\right\}\frac{\partial\chi_\eta}{\partial\xi}\bigg|_2 - \rho_2 v_{\xi 2}v_{\eta 2}\,\frac{\partial\chi_\xi}{\partial\eta}\bigg|_2 - 3\rho_2 v_{r2}\chi_{\eta 2}\chi_{\xi 2}$$

$$T_{32} = \rho_2 v_{\eta 2} v_{\xi 2} \chi_{\eta 2}$$

$$S_{32} = \left(\frac{\gamma-1}{2\gamma} P_2 + \rho_2 v_{\eta 2}^2\right)\chi_{\xi 2} \qquad \text{(II-9)}$$

$$K_{32} = \left(\frac{\gamma-1}{2\gamma} P_2 + \rho_2 v_{\xi 2}^2\right) \frac{\partial \chi_\xi}{\partial \eta}\bigg|_2 - \rho_2 v_{\xi 2} v_{\eta 2} \frac{\partial \chi_\eta}{\partial \xi}\bigg|_2 - 3\rho_2 v_{r2} v_{\xi 2} \chi_{\eta 2}$$

where Ω_9 and Ω_{10} are defined by:

$$\frac{d\chi_{\eta 2}}{d\xi} = \Omega_9 + \frac{1}{2}\Omega_{10}\frac{d\Upsilon}{d\xi} = \frac{\partial \chi_\eta}{\partial \xi}\bigg|_\Upsilon + \frac{1}{2}\frac{\partial \chi_\eta}{\partial(\Upsilon/2)}\frac{d\Upsilon}{d\xi} \quad . \qquad \text{(II-10)}$$

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APPENDIX III

DETERMINATION OF Ω_1 THROUGH Ω_8

The radial component of velocity at the shock wave, v_{rs}, may be expressed

$$v_{rs} = v_{r\infty} = V_\infty \left\{ \cos\theta_s \cos\alpha - \sin\theta_s \sin\alpha \cos\varphi_s \right\} \qquad \text{(III-1)}$$

Hence, differentiating, we find (using Eqs. (2.19))

$$\Omega_5 = v_{\theta\infty} \frac{\partial\theta_s}{\partial\xi} + v_{\varphi\infty} \sin\theta_s \frac{\partial\varphi_s}{\partial\xi} \qquad \text{(III-2)}$$

$$\Omega_6 = v_{\theta\infty} \frac{\partial\theta_s}{\partial T} + v_{\varphi\infty} \sin\theta_s \frac{\partial\varphi_s}{\partial T} \quad . \qquad \text{(III-3)}$$

Then, we find from Eqs. (2.19), (2.9) and (2.10)

$$\Omega_1 = -\frac{v_{r\infty}}{q_\infty} \Omega_5 \qquad \text{(III-4)}$$

$$\Omega_2 = -\frac{v_{r\infty}}{q_\infty} \Omega_6 \qquad \text{(III-5)}$$

$$\Omega_3 = \frac{1}{q_\infty \cos\kappa} \frac{\partial v_{\xi\infty}}{\partial\xi} - \frac{\Omega_1}{q_\infty} \tan\kappa \qquad \text{(III-6)}$$

$$\Omega_4 = \frac{1}{q_\infty \cos\kappa} \frac{\partial v_{\xi\infty}}{\partial T} - \frac{\Omega_2}{q_\infty} \tan\kappa \qquad \text{(III-7)}$$

where

$$\frac{\partial\theta_s}{\partial\xi} = \frac{1}{\sin\theta_s}\left\{(\sin\theta_b \cos\Upsilon + \cos\delta_b \cos\theta_b \sin\Upsilon)\sin\delta_b - \sin\delta_b \cos\delta_b \sin\Upsilon \frac{d\delta_b}{d\varphi_b}\right\} \tag{III-8}$$

$$\frac{\partial\varphi_s}{\partial\xi} = \frac{\cos\delta_b}{\sin\theta_b} - \frac{\sin\Upsilon}{\sin\theta_s \cos(\varphi_b - \varphi_s)}\left\{\frac{\cos^2\delta_b}{\sin\theta_b}\frac{d\delta_b}{d\varphi_b} - \sin\delta_b \cos\theta_s \frac{\partial\theta_s}{\partial\xi}\right\} \tag{III-9}$$

$$\frac{\partial\theta_s}{\partial\Upsilon} = \frac{1}{\sin\theta_s}\left\{\cos\theta_b \sin\Upsilon + \cos\delta_b \sin\theta_b \cos\Upsilon\right\} \tag{III-10}$$

$$\frac{\partial\varphi_s}{\partial\Upsilon} = -\frac{\sin\delta_b}{\cos(\varphi_b - \varphi_s)\sin\theta_s}\left\{\cos\Upsilon - \sin\Upsilon \cot\theta_s \frac{\partial\theta_s}{\partial\Upsilon}\right\} . \tag{III-11}$$

Also, from Fig. 4.1

$$v_{\xi\infty} = v_{\varphi\infty} \cos\delta_s + v_{\theta\infty} \sin\delta_s \tag{III-12}$$

where

$$v_{\theta\infty} = - V_\infty\left\{\cos\alpha \sin\theta_s + \sin\alpha \cos\theta_s \cos\varphi_s\right\} \tag{III-13}$$

$$v_{\varphi\infty} = V_\infty \sin\alpha \sin\varphi_s \tag{III-14}$$

so that

$$\frac{\partial v_{\xi\infty}}{\partial \xi} = (v_{\theta\infty} \cos \delta_s - v_{\varphi\infty} \sin \delta_s)\frac{\partial \delta_s}{\partial \xi} + \sin \delta_s \frac{\partial v_{\theta\infty}}{\partial \xi} + \cos \delta_s \frac{\partial v_{\varphi\infty}}{\partial \xi} \qquad \text{(III-15)}$$

$$\frac{\partial v_{\xi\infty}}{\partial \Upsilon} = (v_{\theta\infty} \cos \delta_s - v_{\varphi\infty} \sin \delta_s)\frac{\partial \delta_s}{\partial \Upsilon} + \sin \delta_s \frac{\partial v_{\theta\infty}}{\partial \Upsilon} + \cos \delta_s \frac{\partial v_{\varphi\infty}}{\partial \Upsilon} \qquad \text{(III-16)}$$

where

$$\frac{\partial \delta_s}{\partial \xi} = \tan \delta_s \left\{ \cot \theta_b \sin \delta_b - \cot \theta_s \frac{\partial \theta_s}{\partial \xi} \right\} + \frac{\cos^2 \delta_b}{\cos \delta_s \sin \theta_b} \frac{d\delta_b}{d\varphi_b} \qquad \text{(III-17)}$$

$$\frac{\partial \delta_s}{\partial \Upsilon} = - \tan \delta_s \cos \theta_s \frac{\partial \theta_s}{\partial \Upsilon} \qquad \text{(III-18)}$$

$$\frac{\partial v_{\theta\infty}}{\partial \xi} = - v_{r\infty} \frac{\partial \theta_s}{\partial \xi} + v_{\varphi\infty} \cos \theta_s \frac{\partial \varphi_s}{\partial \xi} \qquad \text{(III-19)}$$

$$\frac{\partial v_{\varphi\infty}}{\partial \xi} = V_\infty \sin \alpha \cos \varphi_s \frac{\partial \varphi_s}{\partial \xi} \qquad \text{(III-20)}$$

$$\frac{\partial v_{\theta\infty}}{\partial \Upsilon} = - v_{r\infty} \frac{\partial \theta_s}{\partial \Upsilon} + v_{\varphi\infty} \cos \theta_s \frac{\partial \varphi_s}{\partial \Upsilon} \qquad \text{(III-21)}$$

$$\frac{\partial v_{\varphi\infty}}{\partial \Upsilon} = V_\infty \sin \alpha \cos \varphi_s \frac{\partial \varphi_s}{\partial \Upsilon} \quad . \qquad \text{(III-22)}$$

Also, since $\chi_\eta \equiv 1$, we have

$$\Omega_7 = \Omega_8 \equiv 0 \qquad \text{(III-23)}$$

Finally,

$$\left.\frac{\partial \chi_\eta}{\partial \xi}\right|_s = \left.\frac{\partial \chi_\eta}{\partial \xi}\right|_b \equiv 0 \qquad \text{(III-23)}$$

and

$$\left.\frac{\partial \chi_\xi}{\partial \eta}\right|_b = - K_b(\xi) \qquad \text{(III-24)}$$

$$\left.\frac{\partial \chi_\xi}{\partial \eta}\right|_s = \cos \Upsilon - K_b(\xi) \sin \Upsilon \quad . \qquad \text{(III-25)}$$

APPENDIX IV

COEFFICIENTS OF THE EQUATIONS OF THE INVERSE METHOD

The shock wave is defined by $\theta_s = \theta_s(\varphi_s)$

$$ {}^{i}J_1^* = \Upsilon\rho_b $$

$$ {}^{i}J_2 = -\frac{\rho_b v_{rb} v_{\xi b}}{a_b^2}\Upsilon $$

$$ {}^{i}J_4 = J_4 + J_3 \frac{r_4}{T_1}\frac{p_3}{p_1}\frac{d\theta_s}{d\varphi_s} $$

$$ {}^{i}J_{14} = \frac{J_3}{r_1}\left\{r_3 + r_4 \frac{p_5}{p_1}\frac{d\theta_s}{d\varphi_s}\right\} - m_2 \frac{\sin\theta_b}{\cos\delta_b} \qquad \text{(IV-1)} $$

$$ {}^{i}J_5 = m_1 \frac{\sin\theta_b}{\cos\delta_b} - J_3 \frac{r_4}{r_1}\frac{p_2}{p_1}\frac{d\theta_s}{d\varphi_s} - {}^{i}J_{15}\sin\theta_b \tan\delta_b $$

$$ {}^{i}J_{15} = \frac{J_3}{r_1}\left\{r_2 + r_4 \frac{p_4}{p_1}\frac{d\theta_s}{d\varphi_s}\right\} $$

$$ {}^{i}M_5 = v_{\xi b}\frac{\sin\theta_b}{\cos\delta_b} \qquad \text{(IV-2)} $$

$$ {}^{i}Q_4 = Q_4 + \frac{r_4}{r_1}\frac{p_3}{p_1} Q_3 \frac{d\theta_s}{d\varphi_s} $$

$$ {}^{i}Q_{14} = \frac{Q_3}{r_1}\left\{r_3 + r_4 \frac{p_5}{p_1}\frac{d\theta_s}{d\varphi_s}\right\} - m_6 \frac{\sin\theta_b}{\cos\delta_b} \qquad \text{(IV-3)} $$

$$ {}^{i}Q_5 = m_5 \frac{\sin\theta_b}{\cos\delta b} - \frac{r_4}{r_1}\frac{p_2}{p_1} Q_3 \frac{d\theta_s}{d\varphi_s} - {}^{i}Q_{15} \sin\theta_b \tan\delta_b $$

$$ {}^{i}Q_{15} = \frac{Q_3}{r_1}\left\{r_2 + r_4 \frac{p_4}{p_1}\frac{d\theta_s}{d\varphi_s}\right\} \qquad \text{(IV-3) (Cont.)} $$

$$ {}^{i}R_4 = \frac{\cos\delta_b}{\sin\theta_b} P_3 \frac{r_4}{r_1}\frac{d\theta_s}{d\varphi_s} $$

$$ {}^{i}R_{14} = \frac{\cos\delta_b}{\sin\theta_b}\left\{r_3 + r_4 \frac{p_5}{p_1}\frac{d\theta_s}{d\varphi_s}\right\} + r_1 k_3 \tan\beta \sin\Upsilon \qquad \text{(IV-4)} $$

$$ {}^{i}R_5 = r_1\left\{\cos\Upsilon - k_1 \sin\Upsilon\right\}\tan\beta - \frac{\cos\delta_b}{\sin\theta_b} r_4 \frac{p_2}{p_1}\frac{d\theta_s}{d\varphi_s} $$

$$ - \left\{\frac{\cos\delta_b}{\sin\theta_b}\left[r_2 + r_4 \frac{p_4}{p_1}\frac{d\theta_s}{d\varphi_s}\right] + r_1 \tan\beta\, k_2 \sin\Upsilon\right\}\sin\theta_b \tan\delta_b $$

where

$$ m_1 = -\Upsilon\left\{2\rho_s v_{rs}[\cos\Upsilon - (k_1 + k_2 \sin\theta_b \tan\delta_b)\sin\Upsilon] + 2\rho_b v_{rb} \right. $$

$$ \left. + [x_{61} v_{\xi s} + x_{41}\rho_s]\right\} - 2\rho_s v_{\eta s}[\cos\Upsilon - (k_1 + k_2 \sin\theta_b \tan\delta_b)\sin\Upsilon] $$

$$ m_2 = \Upsilon\left\{2\rho_s v_{rs} k_3 \sin\Upsilon - x_{62} v_{\xi s} - x_{42}\rho_s\right\} + 2\rho_s v_{\eta s} k_3 \sin\Upsilon \qquad \text{(IV-5)} $$

$$ m_s = -\Upsilon\left\{\left[\frac{\gamma-1}{2\gamma} P_b + \rho_b v_{\xi b}^2\right][k_1 + k_2 \sin\theta_b \tan\delta_b] \right. $$

$$ \left. + \left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\xi s}^2\right][\sin\Upsilon + (k_1 + k_2 \sin\theta_b \tan\delta_b)\cos\Upsilon] \right. $$

$$+ 3\rho_s v_{rs} v_{\eta s}[\cos \Upsilon - (k_1 + k_2 \sin \theta_b \tan \delta_b)\sin \Upsilon]$$

$$+ x_{61} v_{\xi s} v_{\eta s} + x_{41} \rho_s v_{\eta s} + x_{51} \rho_s v_{\xi s}\Big\} + 2\Big\{\frac{\gamma-1}{2\gamma} P_b$$

$$- \left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\xi s}^2\right][\cos \Upsilon - (k_1 + k_2 \sin \theta_b \tan \delta_b)\sin \Upsilon]\Big\}$$

(IV-5) (Cont.)

$$m_6 = - \Upsilon\left\{\left[\frac{\gamma-1}{2\gamma} P_b + \rho_b v_{\xi b}^2\right]k_3 + \left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\xi s}^2\right]k_3 \cos \Upsilon\right.$$

$$- 3\rho_s v_{rs} v_{\eta s} k_3 \sin \Upsilon + x_{62} v_{\xi s} v_{\eta s} + x_{42} \rho_s v_{\eta s}$$

$$\left. + x_{52} \rho_s v_{\xi s}\right\} + 2\left[\frac{\gamma-1}{2\gamma} P_s + \rho_s v_{\xi s}^2\right]k_3 \sin \Upsilon \quad .$$

The r_j come from the relation

$$r_1 \frac{d\Upsilon}{d\varphi_b} = r_2 \frac{d\theta_b}{d\varphi_b} + r_3 \frac{d\delta_b}{d\varphi_b} + r_4 \frac{d\theta_s}{d\varphi_b}$$

which is obtained from Eq. (4.5). Thus

$$r_1 = \cos \theta_b \sin \Upsilon + \cos \delta_b \sin \theta_b \cos \Upsilon$$

$$r_2 = - \left\{\sin \theta_b \cos \Upsilon + \cos \delta_b \cos \theta_b \sin \Upsilon\right\} \qquad \text{(IV-6)}$$

$$r_3 = \sin \delta_b \sin \theta_b \sin \Upsilon$$

$$r_4 = \sin \theta_s \quad .$$

The k_j come from the expression for K_b, Eq. (4.1) which can be expressed

$$K_b = k_1 + k_2 \frac{d\theta_b}{d\varphi_b} + k_3 \frac{d\delta_b}{d\varphi_b}$$

where

$$k_1 = -(1 + \sin^2\delta_b)\cos\delta_b \cot\theta_b$$

$$k_2 = \frac{\cos^2\delta_b \sin\delta_b \cos\theta_b}{\sin^2\theta_b} \qquad \text{(IV-7)}$$

$$k_3 = \frac{\cos\delta_b}{\sin\theta_b}$$

The p_j come from the relation

$$p_1 \frac{d\varphi_s}{d\varphi_b} = p_2 + p_3 \frac{d\beta}{d\varphi_b} + p_4 \frac{d\theta_b}{d\varphi_b} + p_5 \frac{d\delta_b}{d\varphi_b}$$

which results from Eq. (4.6). Here,

$$p_1 = \cos\delta_s \frac{d\sigma_s}{d\varphi_s} + \frac{r_4}{r_1} \cot\Upsilon \sin\delta_s \frac{d\theta_s}{d\varphi_s} + \frac{\sin\varphi_b}{\sin\Upsilon} \cos(\varphi_b - \varphi_s)$$

$$p_2 = \frac{\sin\theta_b}{\sin\Upsilon} \cos(\varphi_b - \varphi_s)$$

$$p_3 = \cos\delta_s \qquad \text{(IV-8)}$$

$$p_4 = -\sin\delta_3 \left\{\frac{r_2}{r_1} \cot\Upsilon - \cot\theta_b\right\}$$

$$p_5 = -\frac{r_3}{r_1} \cot\Upsilon \sin\delta_s$$

where

$$\tan\sigma_s = \frac{1}{\sin\theta_s} \frac{d\theta_s}{d\varphi_s} \quad . \qquad \text{(IV-9)}$$

In addition, from the results of APPENDIX III we find

$$\Omega_i = t_{i1} + t_{i2} \frac{d\delta_b}{d\varphi_b}$$

where

$$t_{11} = - \frac{vr_\infty}{q_\infty} t_{51}$$

$$t_{12} = - \frac{vr_\infty}{q_\infty} t_{52}$$

$$t_{21} = - \frac{vr_\infty}{q_\infty} t_{61}$$

$$t_{22} = 0$$

$$t_{31} = \frac{s_{11,\ 1}}{q_\infty \cos \kappa} - \frac{\tan \kappa}{q_\infty} t_{11}$$

$$t_{32} = \frac{s_{11,\ 2}}{q_\infty \cos \kappa} - \frac{\tan \kappa}{q_\infty} t_{12} \qquad \text{(IV-10)}$$

$$t_{41} = \frac{s_{12,\ 1}}{q_\infty \cos \kappa} - \frac{\tan \kappa}{q_\infty} t_{21}$$

$$t_{42} = 0$$

$$t_{51} = s_{31} v_{\theta\infty} + s_{41} v_{\varphi\infty} \sin \theta_s$$

$$t_{52} = s_{32} v_{\theta\infty} + s_{42} v_{\varphi\infty} \sin \theta_s$$

$$t_{61} = s_{11} v_{\theta\infty} + s_{21} v_{\varphi\infty} \sin \theta_s$$

$$t_{62} = 0 \quad .$$

The $s_{j,1}$, $s_{j,2}$ come from the expansions of $\partial\theta_s/\partial\Upsilon$, etc. For instance, $\partial\theta_s/\partial\Upsilon = s_{11}$, $\partial\theta_s/\partial\xi = s_{31} + s_{32}(d\delta_b/d\varphi_b)$.

$$s_{11} = \partial\theta_s/\partial\Upsilon$$

$$s_{21} = \partial\varphi_s/\partial\Upsilon$$

$$s_{31} = \frac{\sin\delta_b}{\sin\theta_s}\left\{\sin\theta_b\cos\Upsilon + \cos\delta_b\cos\theta_b\sin\Upsilon\right\}$$

$$s_{32} = -\frac{\sin\delta_b\cos\delta_b\sin\Upsilon}{\sin\theta_s}$$

$$s_{41} = \frac{\cos\delta_b}{\sin\theta_b} + \frac{\sin\Upsilon}{\sin\theta_s\cos(\varphi_s - \varphi_b)}\sin\delta_b\cot\theta_s s_{31}$$

$$s_{42} = \frac{\sin\Upsilon}{\sin\theta_s\cos(\varphi_s - \varphi_b)}\left\{\sin\delta_b\cot\theta_s s_{32} - \frac{\cos^2\delta_b}{\sin\theta_b}\right\} \qquad \text{(IV-11)}$$

$$s_{51} = \tan\delta_s\left\{\cot\theta_b\sin\delta_b - s_{31}\cot\theta_s\right\}$$

$$s_{52} = -\tan\delta_s\cot\theta_s s_{32} + \frac{\cos^2\delta_b}{\cos\delta_s\sin\theta_s}$$

$$s_{61} = \frac{\partial\delta_s}{\partial\Upsilon}$$

$$s_{71} = V_\infty\sin\alpha\cos\varphi_s s_{41}$$

$$s_{72} = V_\infty\sin\alpha\cos\varphi_s s_{42}$$

$$s_{81} = \frac{\partial v_{\varphi\infty}}{\partial\Upsilon}$$

$$s_{91} = -v_{r\infty}s_{31} + v_{\varphi\infty}\cos\theta_s s_{41}$$

$$s_{92} = -v_{r\infty}s_{32} + v_{\varphi\infty}\cos\theta_s s_{42}$$

$$s_{10,\,1} = \frac{\partial v_{\theta\infty}}{\partial \Upsilon}$$

(IV-11)
(Cont.)

$$s_{11,\,1} = (v_{\theta\infty}\cos\delta_s - v_{\varphi\infty}\sin\delta_s)s_{51} + \sin\delta_s s_{91} + \cos\delta_s s_{71}$$

$$s_{11,\,2} = (v_{\theta\infty}\cos\delta_s - v_{\varphi\infty}\sin\delta_s)s_{52} + \sin\delta_s s_{92} + \cos\delta_s s_{72}$$

$$s_{12,\,1} = \frac{\partial v_{\xi\infty}}{\partial \Upsilon}\ .$$

The x_j come from expressions of the form

$$A_1,\ B_1,\ \ldots = x_{j1} + x_{j2}\frac{d\delta_b}{d\varphi_b}$$

A_2, A_3, B_2 ... are independent of the body geometry because $t_{j2} = 0$ for j even. Hence

$$x_{11} = X\left\{\frac{t_{11}}{q_\infty} - t_{31}\tan(\kappa - \beta)\right\}$$

$$x_{12} = X\left\{\frac{t_{12}}{q_\infty} - t_{32}\tan(\kappa - \beta)\right\}$$

(IV-12)

$$x_{21} = q_\infty t_{31}\cos(\kappa - \beta) + t_{11}\sin(\kappa - \beta)$$

$$x_{22} = q_\infty t_{32}\cos(\kappa - \beta) + t_{12}\sin(\kappa - \beta)$$

$$x_{31} = \frac{1}{v_{N\infty}}\left\{- 2\,\frac{\gamma-1}{\gamma+1}[t_{51}v_{r\infty} + x_{21}v_{T\infty}] + v_{Ns}[t_{11}\cos(\kappa - \beta) - t_{31}v_{T\infty}]\right\}$$

$$x_{32} = \frac{1}{v_{N\infty}}\left\{- 2\,\frac{\gamma-1}{\gamma+1}[t_{52}v_{r\infty} + x_{22}v_{T\infty}] + v_{Ns}[t_{12}\cos(\kappa - \beta) - t_{32}v_{T\infty}]\right\}$$

$$x_{41} = x_{21}\cos\beta - x_{31}\sin\beta$$

$$x_{42} = x_{22}\cos\beta - x_{32}\sin\beta$$

$$x_{51} = x_{21}\sin\beta + x_{31}\cos\beta$$

$$x_{52} = x_{22}\sin\beta + x_{32}\cos\beta$$

(IV-12)
(Cont.)

$$x_{61} = \rho_s\left\{\frac{x_{11}}{\rho_{0s}} - \frac{1}{a_s^2}[t_{51}v_{rs} + x_{41}v_{\xi s} + x_{51}v_{\eta s}]\right\}$$

$$x_{62} = \rho_s\left\{\frac{x_{12}}{\rho_{0s}} - \frac{1}{a_s^2}[t_{52}v_{rs} + x_{42}v_{\xi s} + x_{52}v_{\eta s}]\right\}$$

$$x_{71} = P_s\left\{\frac{x_{11}}{P_{0s}} - \frac{\gamma}{a_s^2}[t_{51}v_{rs} + x_{41}v_{\xi s} + x_{51}v_{\eta s}]\right\}$$

$$x_{72} = P_s\left\{\frac{x_{12}}{P_{0s}} - \frac{\gamma}{a_s^2}[t_{52}v_{rs} + x_{42}v_{\xi s} + x_{52}v_{\eta s}]\right\} .$$