A METHOD FOR THE DETERMINATION OF THE MATRIX OF IMPULSE RESPONSE FUNCTIONS WITH SPECIAL REFERENCE TO APPLICATIONS IN RANDOM VIBRATION PROBLEMS

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The impulse response function, $h_{jk}(t)$, of a linear structure gives the response at point j due to a unit impulse excitation at point k applied at time $t = 0$. When the damping in a structure is such that the normal modes of vibration do not exist, the matrix of impulse response functions is usually difficult to determine. In the present paper the determination of this matrix is discussed under rather general conditions. A method due essentially to K. A. Foss is reviewed first, then an alternative scheme more convenient in some cases is presented. An example is given to illustrate the alternative procedure. The role of the matrix of impulse functions in the random vibration analysis is also noted.

INTRODUCTION

In the analysis of forced vibrations of a linear structure idealized as a multiple degrees of freedom system, it is convenient to use the matrix of impulse response functions. A typical element of this matrix, $h_{jk}(t)$, represents the response at point j due to a unit impulse excitation at point k applied at time $t = 0$. Assuming that the structure is at rest prior to the commencement of excitations at $t = 0$, the response of the structure under arbitrary excitations $f_k(t)$ is given by

$$x_j(t) = \sum_{k=1}^{n} \int_0^t h_{jk}(t-\tau)\, f_k(\tau)\, d\tau \qquad j = 1, 2, \dots, n \tag{1}$$

Or, in matrix notation,

$$\mathbf{x}(t) = \int_0^t \mathbf{h}(t-\tau)\, \mathbf{f}(\tau)\, d\tau \tag{1a}$$

To be physically realizable, the matrix $\mathbf{h}(t)$ must be null for $t \leq 0$.

To compute the matrix of impulse response functions, consider the equation of motion

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{f} \tag{2}$$

where $\mathbf{m}$, $\mathbf{c}$, and $\mathbf{k}$ are the inertia, damping, and stiffness matrices, respectively. The simplest case is one in which the normal modes of vibration exist. The conditions for the existence of the normal modes have been investigated by Caughey (Reference 1). The normal modes exist whenever $\mathbf{m}^{-1}\mathbf{c}$ is expressible as a fairly general type of function of $\mathbf{m}^{-1}\mathbf{k}$. Then both $\mathbf{m}^{-1}\mathbf{c}$ and $\mathbf{m}^{-1}\mathbf{k}$ can be diagonalized by the same collinear transformation in view of the well-known Cayley-Hamilton theorem. Then the matrix of impulse response functions can be obtained rather easily by the usual decomposition into normal modes.

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When normal modes of a system do not exist a procedure essentially due to Foss (Reference 2) can be applied. Let

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}, \quad \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{2n} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{3}$$

Then Equation 2 may be replaced by a first order matrix equation for $\mathbf{y} = \left\{ y_1(t), y_2(t), \ldots, y_{2n}(t) \right\}$,

$$\mathbf{r}\,\dot{\mathbf{y}} + \mathbf{s}\,\mathbf{y} = \left[\frac{0}{\mathbf{f}}\right] \tag{4}$$

where

$$\mathbf{r} = \left[\begin{array}{c|c} 0 & \mathbf{m} \\ \hline \mathbf{m} & \mathbf{c} \end{array}\right], \quad \mathbf{s} = \left[\begin{array}{c|c} -\mathbf{m} & 0 \\ \hline 0 & \mathbf{k} \end{array}\right] \tag{5}$$

and the partitioning of the matrices is indicated by the dotted lines.

Assuming that $\mathbf{m}^{-1}$ exists, Equation 4 is readily transformed into

$$\dot{\mathbf{y}} + \mathbf{g}\,\mathbf{y} = \left[\frac{\mathbf{m}^{-1}\,\mathbf{f}}{0}\right] \tag{6}$$

where

$$\mathbf{g} = \left[\begin{array}{c|c} \mathbf{m}^{-1}\mathbf{c} & \mathbf{m}^{-1}\mathbf{k} \\ \hline -\mathbf{I} & 0 \end{array}\right] \tag{7}$$

Note that $\mathbf{g}$ is generally unsymmetrical. The solution to Equation 6 may be written as

$$\mathbf{y} = \exp(-\mathbf{g}\,t)\,\mathbf{y}_0 + \int_0^t \exp\left(-\mathbf{g}\,(t-\tau)\right)\left[\frac{\mathbf{m}^{-1}}{0}\right]\mathbf{f}\,(\tau)\,d\tau \tag{8}$$

The (2nxn) matrix

$$\exp(-\mathbf{g}\,t)\left[\frac{\mathbf{m}^{-1}}{0}\right]$$

gives the impulse response functions for $\mathbf{y}$, the last n components of which are the impulse response functions for $x_1, x_2, \ldots, x_n$. By retaining only the impulse response functions for x, we have

$$\mathbf{h}(t) = \begin{cases} \left[0 \,\vdots\, \mathbf{I}\right] \exp(-\mathbf{g}\,t) \left[\frac{\mathbf{m}^{-1}}{0}\right], & t > 0 \\ \left[0\right] & t \le 0 \end{cases} \tag{9}$$

To evaluate the matrix exp ($-\mathbf{g}$ t), we note that if $\mathbf{T}$ is a 2nx2n matrix whose columns are the modal columns of $\mathbf{g}$, then

$$\exp(-\mathbf{g}\, t) = \mathbf{T} \exp(-\mathbf{b}\, t)\ \mathbf{T}^{-1}$$

where $\mathbf{b} = \mathbf{T}^{-1}\, \mathbf{g}\, \mathbf{T}$ is a diagonal matrix of the latent roots of $\mathbf{g}$. The matrix exp ($-\mathbf{b}$ t) is also a diagonal matrix having the elements exp ($-b_{jj}$ t). The main task in the application of Equation 9 is to find the latent roots and the corresponding modal columns of $\mathbf{g}$ Since $\mathbf{g}$ is unsymmetrical, the latent roots are either real or complex conjugate pairs. An iterative procedure for determining the complex latent roots is outlined by Frazer, Duncan and Collar (Reference 3), but this procedure is quite tedious.

In this paper there is presented an alternative method to determine the matrix of impulse response functions which is also without the restriction of the existence of the normal modes. The alternative method requires the knowledge of the modal columns and eigenvalues of two (2nx2n) symmetric matrices instead of an unsymmetrical one; therefore, it may be more convenient in some cases. Furthermore, in the development of this alternative method, a proof is obtained that the matrix of impulse response functions is symmetrical, thus demonstrating the reciprocal theorem in dynamics. Finally, an example is offered to illustrate each step of the new method. Applications to random vibration problems are also noted.

THEORY

If Equation 2 is derived by use of the Lagrange equations, then it is known that the matrices $\mathbf{m}$ and $\mathbf{k}$ are both symmetrical. We will assume that matrix $\mathbf{c}$ is also symmetrical, as is usually the case. Then the matrices $\mathbf{r}$ and $\mathbf{s}$ on the left-hand side of Equation 4 are symmetrical Assume that there exists a matrix of transformation $\mathbf{U} = \mathbf{G}\,\mathbf{D}$, where the columns of $\mathbf{G}^T$ are the modal columns of $\mathbf{r}$, each of which is normalized to have a unit modulus, and $\mathbf{D}$ is a diagonal matrix constructed from the latent roots λ_j of $\mathbf{r}$ as follows:

$$\mathbf{D} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & \frac{1}{\sqrt{\lambda_{2n}}} \end{bmatrix} \tag{10}$$

Since $\mathbf{r}$ is symmetrical and $\mathbf{G}$ is normalized, $\mathbf{G}^T = \mathbf{G}^{-1}$.
Therefore,

$$\begin{aligned} \mathbf{U}^T\,\mathbf{r}\,\mathbf{U} &= \mathbf{D}^T\ \mathbf{G}^T\ \mathbf{r}\,\mathbf{G}\,\mathbf{D} \\ &= \mathbf{D}\ \mathbf{G}^{-1}\ \mathbf{r}\ \mathbf{G}\ \mathbf{D} = \mathbf{I} \end{aligned} \tag{11}$$

If $\mathbf{y} = \mathbf{U}\,\mathbf{v}$ is substituted into Equation 4 and the result is premultiplied by $\mathbf{U}^T$, we obtain

$$\dot{\mathbf{v}} + \mathbf{U}^T\ \mathbf{s}\ \mathbf{U}\ \mathbf{v} = \mathbf{U}^T \left[\frac{\mathbf{0}}{\mathbf{f}}\right] = \mathbf{U}^T\ \left[\frac{\mathbf{0}}{\mathbf{I}}\right] \mathbf{f} \tag{12}$$

Solving,

$$\mathbf{v} = \exp(-\mathbf{U}^T \mathbf{s} \mathbf{U} t)\ \mathbf{v}(0) + \int_0^t \exp\left(-\mathbf{U}^T \mathbf{s} \mathbf{U}(t-\tau)\right) \mathbf{U}^T \begin{bmatrix} \mathbf{O} \\ \mathbf{I} \end{bmatrix} \mathbf{f}(\tau)\, d\tau \tag{13}$$

The required matrix of the impulse response functions, $\mathbf{h}(t)$, is thus given by

$$\mathbf{h}(t) = \begin{cases} \begin{bmatrix} \mathbf{O} \mid \mathbf{I} & \mathbf{U} \end{bmatrix} \exp\left(-\mathbf{U}^T \mathbf{s} \mathbf{U} t\right) \mathbf{U}^T \begin{bmatrix} \mathbf{O} \\ \mathbf{I} \end{bmatrix}, & t > 0 \\ \mathbf{O}, & t \leq 0 \end{cases} \tag{14}$$

An application of Equation 14 requires a knowledge of the latent roots and the modal columns of the matrices $\mathbf{r}$ and $\mathbf{U}^T \mathbf{s} \mathbf{U}$, but both of these matrices are symmetrical. The latent roots of $\mathbf{r}$ are real, and the iterative process for finding them and the associated modal columns is simpler than the one used for the complex latent roots. However, we cannot always make the same claim for the matrix $\mathbf{U}^T \mathbf{s} \mathbf{U}$ since it may contain imaginary elements.

Equation 14 also demonstrates the important fact that the matrix of impulse response functions $\mathbf{h}(t)$ is always symmetrical.

AN EXAMPLE

As an example consider a massless uniform cantilevered beam supporting a heavy rigid block as shown in Figure 1. The structural model is a two degrees of freedom system since the configuration of the system is specified by the deflection and the rotation at the right end of the beam, $x_1(t)$ and $x_2(t)$, which will be chosen as the generalized displacements. It can be shown by use of the elementary techniques of strength of materials that the strain energy is given by

$$V = \frac{6EI}{\ell^3} x_1^2 + \frac{6EI}{\ell^2} x_1 x_2 + \frac{2EI}{\ell} x_2^2 \tag{15}$$

The kinetic energy can be expressed most simply in terms of the translational and rotational velocities, $\dot{x}_C$ and $\dot{\alpha}$, of the centroid C of the heavy block, i.e.

$$T = \frac{1}{2} m \dot{x}_C^2 + \frac{1}{2} I_C \dot{\alpha}^2 \tag{16}$$

where m and I_C are the mass and the mass polar moment of inertia about C of the block. Now $\alpha = x_2$, and, within the scope of a small deflection theory, $x_C = x_1 + a\, x_2$. Then, in terms of the generalized velocities,

$$T = \frac{1}{2} m \dot{x}_1^2 + m a \dot{x}_1 \dot{x}_2 + \frac{1}{2}(I_C + m a^2)\dot{x}_2^2 \tag{17}$$

With virtual displacements Δx_1 and Δx_2, the virtual work done by the excitation and the damping force is

$$\Delta W = (\Delta x_1 + a\Delta x_2)P + \Delta x_2 Q - \eta(\dot{x}_1 + a\dot{x}_2)(\Delta x_1 + a\Delta x_2)$$

Therefore

$$\frac{\partial W}{\partial x_1} = P - \eta \dot{x}_1 - a\eta \dot{x}_2$$
$$\frac{\partial W}{\partial x_2} = aP + Q - a\eta \dot{x}_1 - a^2 \eta \dot{x}_2 \tag{18}$$

Application of the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) + \frac{\partial V}{\partial x_j} = \frac{\partial W}{\partial x_j} \qquad j = 1, 2 \tag{19}$$

leads to

$$\begin{bmatrix} m & ma \\ ma & (I_C + ma^2) \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \eta & a\eta \\ a\eta & a^2\eta \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} \\ \frac{6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} P \\ (aP + Q) \end{bmatrix} \tag{20}$$

To determine the matrix of the impulse response functions $h(t)$ for this structure by the present method, we would have to work with numerical data. However, for illustration purposes we will consider a special case of the present example so that we may continue with a symbolic representation. Let $a = 0$ in Equation 20, then the system is coupled only through the spring matrix. When this equation is changed to the form of Equation 4, we find

$$r = \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & I_c \\ m & 0 & \eta & 0 \\ 0 & I_c & 0 & 0 \end{bmatrix}, \quad s = \begin{bmatrix} -m & 0 & 0 & 0 \\ 0 & -I_c & 0 & 0 \\ 0 & 0 & \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} \\ 0 & 0 & \frac{6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix} \tag{21}$$

The latent roots of r are found to be

$$\gamma_{1,2} = \pm I_c , \quad \gamma_{3,4} = \frac{\eta}{2} \pm \sqrt{\frac{\eta^2}{4} + m^2} \tag{22}$$

and the normalized modal columns are

$$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \quad \begin{bmatrix} \frac{m}{\sqrt{m^2+\gamma_3^2}} \\ 0 \\ \frac{\gamma_3}{\sqrt{m^2+\gamma_3^2}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{m}{\sqrt{m^2+\gamma_4^2}} \\ 0 \\ \frac{\gamma_4}{\sqrt{m^2+\gamma_4^2}} \\ 0 \end{bmatrix} \tag{23}$$

The required matrix of transformation **U** for the application of Equation 14 is therefore

$$\mathbf{U} = \begin{bmatrix} 0 & \frac{1}{i\sqrt{2I_c}} & 0 & \frac{1}{\sqrt{2\gamma_4}} \\ 0 & \frac{1}{i\sqrt{2I_c}} & 0 & -\frac{1}{\sqrt{2\gamma_4}} \\ \frac{m}{\sqrt{I_c}\sqrt{m^2+\gamma_3^2}} & 0 & \frac{\sqrt{\gamma_3}}{\sqrt{m^2+\gamma_3^2}} & 0 \\ \frac{m}{\sqrt{I_c}\sqrt{m^2+\gamma_4^2}} & 0 & \frac{\gamma_4}{\sqrt{\gamma_3}\sqrt{m^2+\gamma_4^2}} & 0 \end{bmatrix} \tag{24}$$

Let $\mathbf{E} = \mathbf{U}^T \mathbf{s}\, \mathbf{U}$; the elements of **E** are readily computed to be

$$e_{11} = \frac{4EIm^2}{\ell I_c} \quad \frac{3}{\ell^2(m^2+\gamma_3^2)} + \frac{1}{m^2+\gamma_4^2} + \frac{3}{\ell\sqrt{(m^2+\gamma_3^2)(m^2+\gamma_4^2)}}$$

$$e_{22} = \frac{1}{2}\left(\frac{m}{I_c} + 1\right)$$

$$e_{33} = \frac{4EI}{\ell\gamma_3}\left[\frac{3\gamma_3^2}{\ell^2(m^2+\gamma_3^2)} + \frac{\gamma_4^2}{m^2+\gamma_4^2} + \frac{3\gamma_3\gamma_4}{\ell\sqrt{(m^2+\gamma_3^2)(m^2+\gamma_4^2)}}\right]$$

$$e_{44} = -\frac{1}{2\gamma_4}(m + I_c)$$

$$e_{13} = e_{31} = \frac{2\,EIm}{\ell\sqrt{I_c\gamma_3}}\left[\frac{6\gamma_3}{\ell^2(m^2+\gamma_3^2)} + \frac{2\gamma_4}{m^2+\gamma_4^2} + \frac{3}{\ell\sqrt{(m^2+\gamma_3^2)(m^2+\gamma_4^2)}}\right](\gamma_3+\gamma_4) \tag{25}$$

$$e_{24} = e_{42} = \frac{i}{\sqrt{\gamma_4}}\left(\frac{m}{\sqrt{I_c}} - \sqrt{I_c}\right)$$

The other elements in **E** are zero. To determine the latent roots and the modal columns of **E** which are required for the evaluation of exp(- **E** t), we note that for the present case

$$|\lambda\mathbf{I} - \mathbf{E}| = \left[(\lambda - e_{11})(\lambda - e_{33}) - e_{13}^2\right]\left[(\lambda - e_{22})(\lambda - e_{44}) - e_{24}^2\right] \tag{26}$$

Therefore,

$$\lambda_{1,2} = \frac{e_{11}+e_{33}}{2} \pm \sqrt{\left(\frac{e_{11}+e_{33}}{2}\right)^2 - (e_{11}e_{33} - e_{13}^2)}$$

$$\lambda_{3,4} = \frac{e_{22}+e_{44}}{2} \pm \sqrt{\left(\frac{e_{22}+e_{44}}{2}\right)^2 - (e_{22}e_{44} - e_{24}^2)} \tag{27}$$

and the modal columns for **E** are

$$
\begin{bmatrix} e_{13} \\ 0 \\ \lambda_1 - e_{11} \\ 0 \end{bmatrix}, \begin{bmatrix} e_{13} \\ 0 \\ \lambda_2 - e_{11} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ e_{24} \\ 0 \\ \lambda_3 - e_{22} \end{bmatrix} \begin{bmatrix} 0 \\ e_{24} \\ 0 \\ \lambda_4 - e_{22} \end{bmatrix} \tag{28}
$$

Using these columns as the columns of a matrix $\mathbf{L}$, then

$$
\exp(-\mathbf{E}t) = \exp(-\mathbf{U}^T \mathbf{s} \mathbf{U} t)
$$

$$
= \mathbf{L} \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{-\lambda_2 t} & 0 & 0 \\ 0 & 0 & e^{-\lambda_3 t} & 0 \\ 0 & 0 & 0 & e^{-\lambda_4 t} \end{bmatrix} \mathbf{L}^{-1} \tag{29}
$$

We now have all the constituents necessary to construct the matrix $\mathbf{h}(t)$ according to Equation 14.

CONCLUDING REMARKS: RANDOM VIBRATIONS OF A LINEAR SYSTEM

The representation of a linear structure by a matrix of impulse response functions is also convenient in the analysis of the random vibrations (Reference 4) of the structure. In such an analysis, we usually assume that the excitations are not predictable, but are described by probability laws. The emphasis is often placed on the determination of the statistical averages of the responses in terms of those of the excitations. In particular, we seek the statistical averages of the product $X_j(t_1)\, X_k(t_2)$ where the responses $\mathbf{X}$ are represented by a capital letter to emphasize their probabilistic nature. Called a correlation function and denoted by $E\,(X_j(t_1)\, X_k(t_2))$, this statistical average can be computed from

$$
E(X_j(t_1)\, X_k(t_2)) = \sum_{p=1}^{n} \sum_{q=1}^{n} \int_0^{t_1}\int_0^{t_2} h_{jp}(t_1 - \tau_1)\, h_{kq}(t_2 - \tau_2)\; E(F_p(\tau_1)\, F_q(\tau_2)\, d\tau_1\, d\tau_2 \tag{30}
$$

where, in the integrand, $E(F_p(\tau_1)\; F_q(\tau_2))$ is the correlation function of the excitations at location p and at location q. We, again, assume that the structure is at rest at t = 0. Equation 30 can also be conveniently expressed in matrix form,

$$
E\left[\mathbf{X}(t_1)\;(\mathbf{X}(t_2))^T\right] = \int_0^{t_1}\int_0^{t_2} \mathbf{h}(t_1 - \tau_1)\; E\,(\mathbf{F}(\tau_1)(\mathbf{F}(\tau_2))^T)(\mathbf{h}(t_2 - \tau_2))^T d\tau_1\, d\tau_2 \tag{30a}
$$

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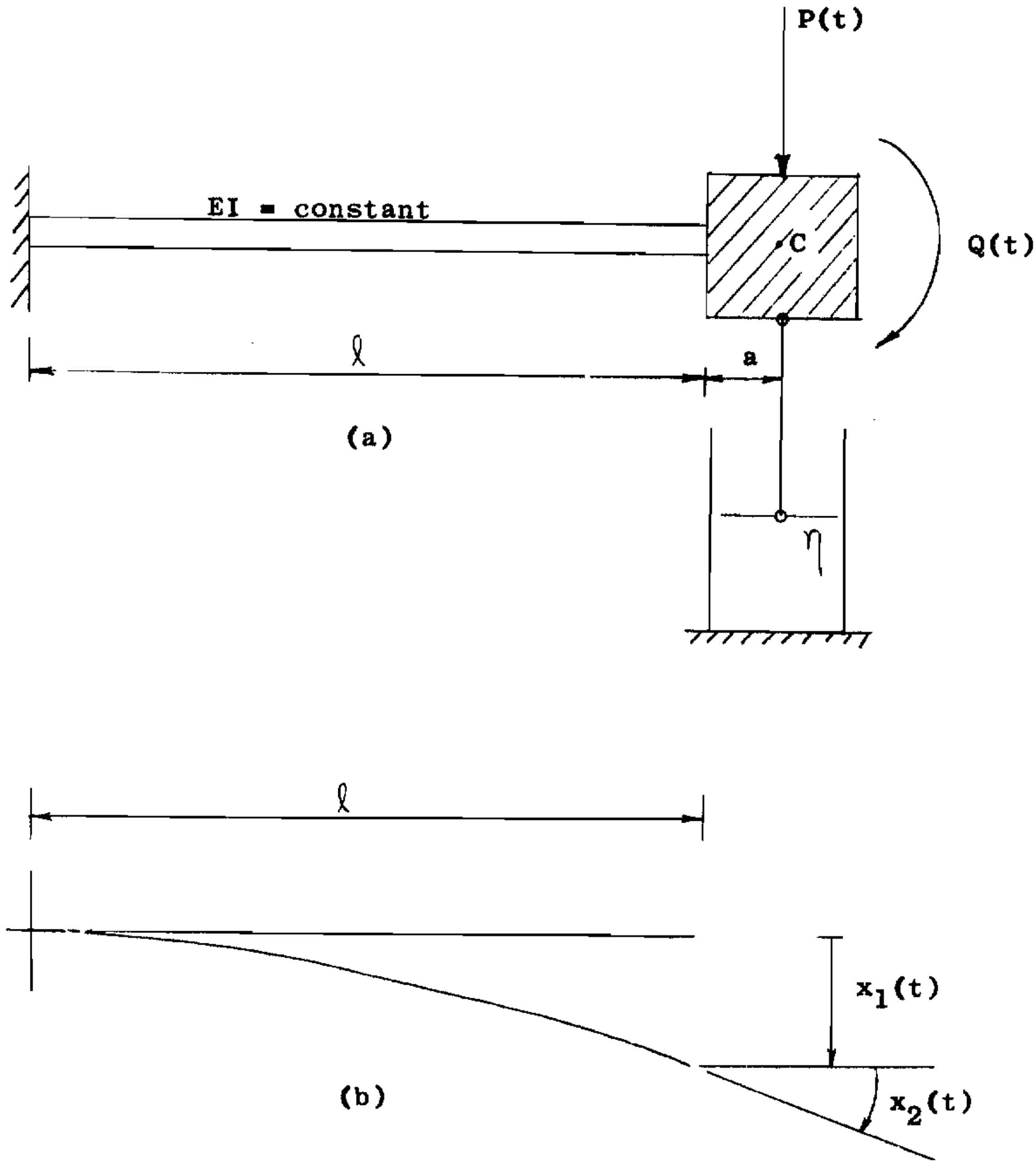

Figure 1. A Structure with Two Degrees of Freedom
(a) Physical Arrangement, (b) Selected Generalized Coordinates