WADC TECHNICAL REPORT 54-250
PART 6
ASTIA DOCUMENT No. AD 110625

DYNAMIC SYSTEMS STUDIES

OPERATION AND MAINTENANCE PROCEDURES FOR ANALOG COMPUTERS

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UNIVERSITY OF CHICAGO

SEPTEMBER 1956

AERONAUTICAL RESEARCH LABORATORY
PROJECT 7060

ADVISORY BOARD ON SIMULATION
CONTRACT No. AF 33(038)-15068, SUPPLEMENTS 2 AND 11

WRIGHT AIR DEVELOPMENT CENTER
AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

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FOREWORD

The Advisory Board on Simulation has concluded a three-year research program in air weapon system dynamics sponsored by Wright Air Development Center, with P. W. Nosker/WCRR as project engineer. This volume is one of the following 16 comprising the final report, WADC TR 54-250, entitled Dynamic System Studies:

Part No.	Subtitle	Editing Agency
1	Conclusion and Recommendations	University of Chicago
2	The Design of a Facility	" " "
3	The Mission of a Facility (Confidential)	" " "
4	Technical Staff Requirements	" " "
5	Analog Computation	Naval Ord. Lab., Corona
6	Operation & Maintenance Procedures for Analog Computers	University of Chicago
7	Digital Computers	" " "
8	Recorders	" " "
9	Flight Tables (Confidential)	" " "
10	Performance Requirements for Flight Tables	Mass. Inst. of Tech.
11	Load Simulators (Confidential)	Cook Research Lab.
12	Guidance Simulation (Secret)	Naval Ord. Lab., Corona
13	Error Studies	University of Chicago
14	Error Analysis for Differential Analyzers (written by F. J. Murray, Columbia U., and K. S. Miller, N.Y.U.)	" " "
15	Air Vehicle Characteristics (Secret)	" " "
16	Aerodynamic Studies (written by M. Z. Krzywoblocki, U. of Ill.)	" " "

The history of the project and a complete bibliography may be found in Part 1. All reports may be obtained through the project engineer.

This report represents the culmination of an assignment to determine the proper mission, equipmentation, operating procedures, and personnel for an engineering facility in the field of air weapon systems dynamics. The subdivisions of the report correspond to these four basic objectives and the subsidiary work in their support, and reflect the role of simulation as a dominant technique. The functions of each part and the relations among them are indicated in the technical summary, Part 2.

The following organizations have participated directly in the program:

Organization	Contract No.	Time of Performance
University of Chicago	AF33(038)-15068 Supplements 2 and 11	1 Feb. '51-31 Aug. '54
J. B. Rea Company	AF33(038)-15068 Subcontract 2	1 Feb. '51-31 Oct. '52
Cook Research Laboratories	AF33(038)-15068 Subcontracts 3 and 9	1 Feb. '51-31 May '54
RCA Laboratories	AF33(038)-15068 Subcontract 4	1 Feb. '51-1 Mar. '53
Armour Res. Foundation of Ill. Inst. of Technology	AF33(038)-15068 Subcontract 5	1 Feb. '52-30 Nov. '52
Northwestern University, Aerial Meas. Lab.	AF33(038)-15068 Subcontract 8	17 July '52-22 Aug. '52
Mass. Inst. of Technology, Flight Control Lab.	AF33(038)-15068 Purchase Order A2086	20 Apr. '54-31 Aug. '54
Mass. Inst. of Technology, Dynamic Analysis & Control Laboratory	AF33(038)-15068 Purchase Order A23883	22 July '53-30 Nov. '53
Mass. Inst. of Technology, D.A.C.L.	AF33(616)-2263 Task Statement 2	1 Dec. '53-30 Sept. '54
Nat. Bur. of Standards Corona, which became	AF33(038)-51-4345-E	25 Feb. '51-Sept. '53
Naval Ordnance Lab., Corona	MIPR(33-616)54-154	20 Nov. '53-31 Dec. '55

This is a record of formal participation only; the program was aided immeasurably by the splendid cooperation of all governmental, industrial and educational organizations (particularly the simulation laboratories) contacted. Although it is impractical to mention them all here, the extent of their assistance is evident throughout the reports and is hereby gratefully acknowledged. Details of these affiliations, including statements of work, may be found throughout the 21 Bimonthly Progress Reports issued by the University of Chicago during the course of the work. (All formal participation in the program is recorded above; missing supplement and subcontract numbers do not pertain to this project.)

The University of Chicago was assigned prime responsibility for integration of the program. This has been effected by a full time staff at the University, and by speriodic meetings of the following advisory committee, selected by the Air Force:

Dean Walter Bartky, Chairman	University of Chicago	1 Feb. '51-31 Aug. '54
Prof. C. S. Draper	Mass. Inst. of Tech.	1 Feb. '51-28 Feb. '53
Mr. Donald McDonald	Cook Research Labs.	1 Feb. '51-31 Aug. '54
Prof. F. J. Murray	Columbia University	1 Apr. '52-31 Aug. '54
Dr. J. B. Rea	J. B. Rea Company	1 Feb. '51-28 Feb. '53
Prof. R. C. Seamans, Jr.	Mass. Inst. of Tech.	1 Sept. '53-31 Aug. '54
Mr. R. J. Shank	Hughes Aircraft Co.	1 July '51-31 Aug. '54
Dr. H. K. Skramstad	NBS-NOLC	1 Feb. '51-31 Aug. '54
Mr. A. W. Vance	RCA Laboratories	1 Feb. '51-31 Aug. '54
ex officio:		
Mr. P. W. Nosker, Project Eng.	WADC	1 Feb. '51-31 Aug. '54
Dr. B. E. Howard, Secretary	University of Chicago	1 Feb. '51-31 Aug. '54

The meetings have been recorded in the Bimonthly Progress Reports previously mentioned. Except for Dr. Skramstad, who has participated through direct arrangement between NBS-NOLC and WADC, members of the advisory committee who are not connected directly with the University have participated in the program through consulting agreements with the University of Chicago. In addition, similar consulting agreements with the University have provided for the participation of:

Dr. R. R. Bennett	Hughes Aircraft Co.	1 Jan. '52-31 Jan. '54
Mr. J. P. Corbett	Libertyville, Ill. (formerly with the University)	11 May '54-31 Aug. '54
Mr. G. L. Landsman	Motorola, Inc.	1 May '54-31 Aug. '54
Dr. Thornton Page	Johns Hopkins Univ. (formerly with the University, and Secretary to the Board until 1 Aug. '51)	7 Aug. '51-1 Mar. '53
Prof. M. Z. Krzywoblocki	University of Illinois	15 Jan. '52-31 Aug. '54
Prof. K. S. Miller	New York University	2 Nov. '53-31 Aug. '54
Dr. J. Winson	Riverside, N. Y. (formerly consultant to Project Cyclone)	1 Mar. '53-30 June '54

Many others have contributed significantly to the progress of the work. Among those from other organizations in regular attendance at most of the meetings of the committee have been Mr. Charles F. West, Air Force Missile Test Center; Prof. L. L. Rauch, University of Michigan, representing Arnold Engineering Development Center; Col. A. I. Lingard, WADC; and Dr. F. W. Bubb, WADC.

Coordination of the program and administration of the prime contract at the University of Chicago has been under the charge of Dr. Walter Bartky, Dean of the Division of Physical Science and Director of the Institute for Air Weapons Research; Dr. B. E. Howard, Assistant to the Director; and Messrs. William R. Allen and William J. Riordan, Group Leaders. The work at the cooperating institutions has been directed by the appropriate member of the advisory committee and his assistants: Dr. H. K. Skramstad and Mr. Gerald L. Landsman at the National Bureau of Standards-Naval Ordnance Laboratory, Corona; Messrs. Donald McDonald and Jay Warshawsky at Cook Research Laboratories; Messrs. A. W. Vance, J. Lehman, and Dr. E. C. Hutter at RCA Laboratories; Dr. J. B. Rea at J. B. Rea Company; Prof. R. C. Seamans at the Flight Control Laboratory and Dr. W. W. Seifert and Mr. H. E. Blanton at the Dynamic Analysis and Control Laboratory, Mass. Inst. of Technology. W. H. Disney, S. Hori, and G. F. Warnke at Armour Research Foundation and J. C. MacAnulty and George Goelz at Northwestern University, Aerial Measurements Laboratory have directed the contributory studies at their respective organizations. More explicit credit is found in appropriate places throughout the reports; biographical sketches are in Part 1. Space does not allow full credit that is due to all the workers on the combined project, but special mention is certainly due the project engineer for his conception of the project and for his cooperation during its execution.

For Part 6 we are indebted to Messrs. J. C. MacAnulty and S. J. Horowitz of the Aerial Measurements Laboratory and D. Abramis and W. Morrow of Convair, Pomona, Calif., for making available to us, for analysis, the maintenance and operating records of their computer facilities. Mrs. Dorothea Minden of the University of Chicago has carried the major responsibility for reducing this data to a form usable in our study. Messrs. R. Farrell and W. Feurzig have given much helpful advice in planning the punch card code and machine reduction for the data.

Among the many who have been exceedingly helpful to us in various stages of the program of preparing this part of the report, we are particularly indebted to the following:

R. E. Gaskell, Boeing Airplane Co.
R. R. Bennett, Ramo Woolridge Corp.
G. Eisenhardt, Douglas Aircraft Co.
S. Sherman, Bell Laboratories
W. W. Seifert, Mass. Inst. Tech.
E. Buell, S. J. Horowitz, H. Nelson, Northwestern University
D. Abramis, W. Morrow, Convair, Pomona, Calif.
J. P. Corbett, R. Farrell, W. J. Riordan, University of Chicago

Organizations cooperating in our survey were:

Aerial Measurements Lab., Northwestern University
Applied Physics Lab., John Hopkins University
Askania Regulator Corp.
Bell Telephone Labs.
Boeing Airplane Co.
Consolidated Vultee Aircraft Corp., Pomona and San Diego divisions
Douglas Aircraft Co.
Dynamics Analysis and Control Lab., Mass. Inst. Tech.
Glenn L. Martin Co.
Goodyear Aircraft Co.
Hughes Aircraft Co.
Naval Air Development Center
Naval Ordnance Lab., Corona, Calif.
Project Cyclone, Reeves Instrument Co.
Radio Corporation of America, Laboratories Division
Rand Corp.

The authors of Appendix 3 wish in particular to acknowledge the benefit of Mr. W. E. Nickelson's considerable experience through his contributions as consultant and critic in the work described in that appendix. Thanks are also due the M.I.T. Transonic Aircraft Control Project, which provided a particularly good example for the purposes of the appendix.

Tables 2, 3, 4, and 5 originally appeared in Rand Research Memorandum RM-525, Rand REAC Manual, and we are grateful to the Rand Corporation for permission to reprint the material in this volume.

ABSTRACT

Getting useful results from analog computation calls for careful study of (1) the limits of the problem representation, (2) purpose to which computational results will be put, (3) suitability of the computer for the problem, and (4) methods for preventing and discovering malfunctions on the part of both operating staff and computer. There is no substitute for careful scientific procedures in obtaining such useful results.

Close cooperation between a specialized computer staff, that knows how best to use the computer, and the design staff, that knows the physical problem and its mathematical representation, will aid in assuring useful results from analog computation.

It may be necessary to do some experimentation in order to determine the most desirable maintenance procedures for a given computer facility.

PUBLICATION REVIEW

The publication of this report does not constitute approval by the Air Force of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

FOR THE COMMANDER:

Nathan L. Krisberg
NATHAN L. KRISBERG, Colonel, USAF
Chief, Aeronautical Research Laboratory
Directorate of Research

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1. PRELIMINARY DISCUSSION

When one starts a discussion of the use of analog computers, it is more often than not considered necessary to preface the remarks with a lengthy discourse on the philosophy and methods of instrumentation of analog computation. The present discussion shall not break with this precedent, since it will be helpful in pinpointing the area in this field with which this part of the report is concerned. The very name "analog computer" is suggestive. Analog, derived through French and Latin from the Greek "analogos," meaning up to proportion, $\alpha\nu\alpha$ (according to) + $\lambda o\gamma os$ (ratio), has the additional meanings of "similar (parallel) word or thing" and in biology "a form or function of an organ resembling that of another essentially different organ." We find in studying different concepts of analog computation that each of the definitions plays a role.

The "direct analog" computer is largely tailor-made for each physical system. The construction is such that a direct functional correspondence exists between the components of the "direct analog" and those of the physical system, the quantities occurring in both the "direct analog" and the physical system. This correspondence must be such that for every subsystem of the physical system and for the corresponding subsystem of the "direct analog" the relationship

$$\frac{\text{input of direct analog}}{\text{input of physical system}} = \frac{\text{output of direct analog}}{\text{output of physical system}} = K$$

where K is some fixed number for this physical system and the choice of "direct analog." (Naturally any physical system can be its own "direct analog.") The simplest examples of such direct analogs are the well-known spring-condenser, mass-inductor, and dashpot-resistor analogies found in most electrical engineering, mechanical engineering, and elementary physics texts. This "direct analog" technique has been exploited by, among others, Professor G. McCann of the California Institute of Technology primarily for aircraft structure and engine studies, Professor V. Paschkis of Columbia University for the study of heat transfer problems, the communications industry for studying communications networks, and the utility companies for studying power distribution systems.

Manuscript released by the author in August 1954 for publication as a WADC Technical Report.

"Mathematical analog" computation depends on being able to represent the physical system by a mathematical model and then constructing a computer that performs the mathematical operations required by the model, such as addition, multiplication, integration, differentiation, function generation, convolution, etc. The best known of this type of analog computer is the "differential analyzer," of which the GEDA, EA, REAC, etc. are the familiar electronic types. The Bush type of mechanical differential analyser and the DACL electromechanical differential analyzer are two other representatives of this class. Another type of "mathematical analog" computer which is relatively familiar is a correlation computer which, given a distribution function P(t) on a region A and a fixed value η, computes the integral

$$r(\eta) = \int_A f(t - \eta)\, g(t) dP(t).$$

The two types of analog computers appear to be distinguished here to a large extent by whether we use the mathematical model or not. It should be noted that in actual fact the setup of a "direct analog" computer is often made by assuring that the subsystems of the computer satisfy the same mathematical model as the corresponding physical system. Furthermore, the fact that many subsystems are difficult to represent mathematically has forced the use of actual components in connection with "mathematical analog" computers or has caused these components to be represented by "direct analog" components obtained by empirical testing.

The type of computer with which this part of the report is concerned is basically a differential analyzer, but may include for any given problem certain direct analog components. The computer we are considering has as its independent variable the time of machine operation and will perform the operations of sign inversion, summation, multiplication by a constant, integration with respect to the independent variable, multiplication of two functions of dependent variables, generation of functions of time, generation of functions of dependent variables, and certain transformations with respect to the independent variables which are rational functions of the integral and differential transformations.* We shall not be concerned for the most part with the opera-

*If we use the LaPlace s operator notation, $g = s(f) = \frac{d}{dt} f$ and $h = \frac{1}{s}(f) = \int f dt$, these transformations are of the form $g^* = \frac{p(s)}{q(s)}(f)$ where p and q are polynomials in s and are relatively prime.

tional circuitry of these operators, but for convenience a list of these operations, methods of performing the operations, and the symbols we shall use to represent the operations in a computer diagram is given in Table 1. As one of the operations of the differential analyzer we have included differentiation. This operation when performed electronically is in general avoided, since differentiators have a tendency to amplify noise pulses, and their inherently good high frequency response may lead to unwanted high frequency oscillation in the computing machine. In Table 1 we have listed a number of purposes for which "operational amplifiers" and passive networks can be used. To clarify this and to give more concrete examples of such networks we have included in Table 2 a list of 18 operational amplifiers with transfer functions, and in Table 3 a list of computer passive networks and their transfer functions in the LaPlace notation.

The name indicates that differential analyzers are suited to studying systems of ordinary differential equations. But, a quick glance through the literature shows that they have been used with varying degrees of success to solve algebraic systems of equations, find the roots of polynomials, evaluate definite integrals, map out regions represented by systems of differential inequalities, approximate the solution of partial differential equations, analyze frequency response spectra, to name only some of the problems attacked with such analog computers. However, it is fair to stress that the electronic differential analyzer has had its greatest success in solving systems of differential equations.

Our interest in the use of differential analyzers is for the most part restricted to problems connected with unitary aerial weapons systems. In Chapter 2 we will give a few very simplified examples, and the Bibliography refers to many more such problems.

In subsequent chapters this part will treat topics connected with setting up problems, checking problem setups and/or solutions, analyzing computer solutions, examining the type of staff to operate the computer, and deciding on a maintenance procedure. In addition, appendices are included, covering special computer techniques of current interest, presenting a mathematical model which may be of some use in setting up spare parts inventories, and discussing fast time computers.

Table 1: OPERATIONS PERFORMED BY DIFFERENTIAL ANALYZER

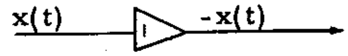
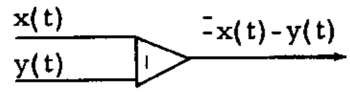
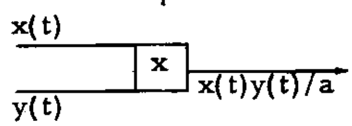
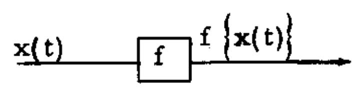
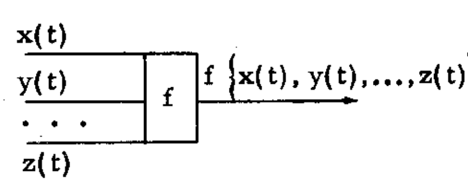
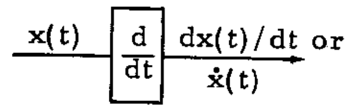
Operation	Methods	Symbolism for Machine Diagram
1. Change of sign	Operational amplifier	
2. Summation	Operational amplifier	
3. Integration	Operational amplifier	
4. Multiplication by a constant	(a) Change ratio of feed-back to input resistor or operational amplifier	
	(b) Attenuator (potentiometer)	
5. Multiplication of two dependent variables*	(a) Servo-driven potentiometer	
	(b) Square law devices	
	(c) Time division multipliers	
	(d) AM-FM modulation schemes	
6. Function generation for functions of a single variable	(a) Specially wound potentiometers**	
	(b) Special potentiometer boards	
	(c) Mechanical curve followers used to follow graphs formed with conducting ink or wire	
	(d) Photoformers	
	(e) Diodes for forming polygonal approximations	
	(f) Generation of the function by solving a differential equation	
	(g) Power series (polynomial) approximations	
7. Function generation for functions of several variables	(a) Series methods (polynomial) approximations involving sums of products	
	(b) Pot settings or a mesh with interpolation formulas	

Table 1 (continued)

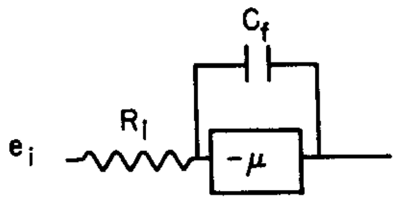
	(c) Constant voltage on conducting contour lines with interpolation between contours	
8. Differentiation	(a) Operational amplifier	
	(b) Tachometer on friction servo	
	(c) Passive networks	
9. Rational functions of integration and differentiation operations	(a) Operational amplifiers	
	(b) Passive networks	

* For more complete listing see item 29, Bibliography, and Part 5 of this report.

** See Part 5.

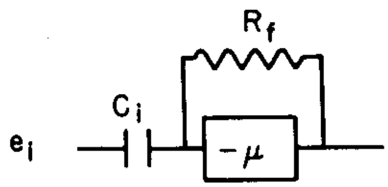
Table 2: SEVERAL OPERATIONAL AMPLIFIERS

1.



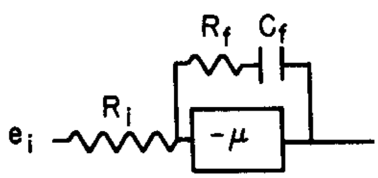
$$e_o = -\frac{1}{R_i C_f s} e_i$$

2.



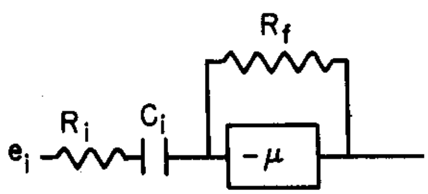
$$e_o = -R_f C_i s e_i$$

3.



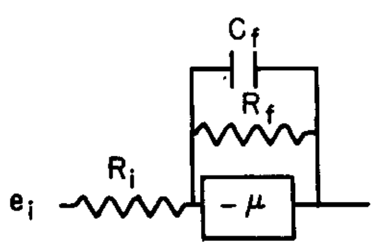
$$e_o = -\frac{(1 + R_f C_f s)}{R_i C_f s} e_i$$

4.



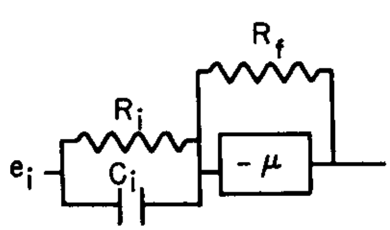
$$e_o = -\frac{R_f C_i s}{1 + R_i C_i s} e_i$$

5.



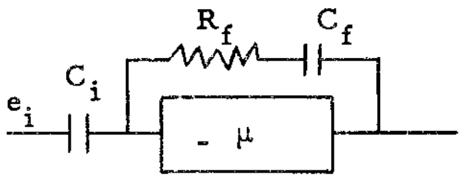
$$e_o = -\frac{R_f}{R_i(1 + R_f C_f s)} e_i$$

6.

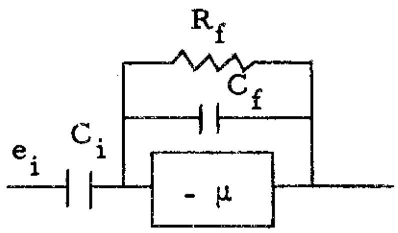


$$e_o = -\frac{R_f}{R_i}(1 + R_i C_i s) e_i$$

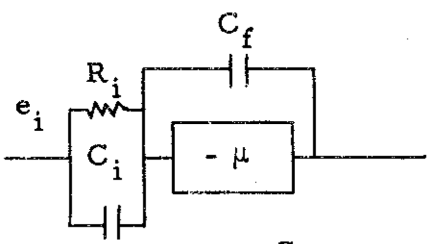
Table 2 (continued)

7. 

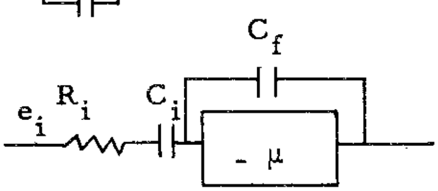
$$e_o = -\frac{C_i}{C_f}(1 + R_fC_fs)\ e_i$$

8. 

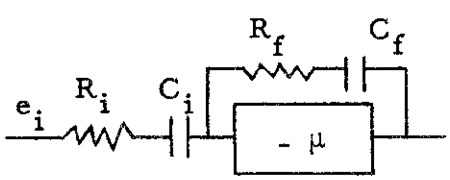
$$e_o = -\frac{R_fC_is}{1 + R_fC_fs}\ e_i$$

9. 

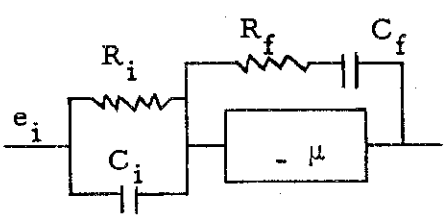
$$e_o = -\frac{1 + R_iC_is}{R_iC_fs}\ e_i$$

10. 

$$e_o = -\frac{C_i}{C_f(1+R_iC_is)}\ e_i$$

11. 

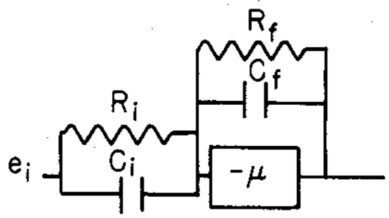
$$e_o = -\frac{C_i(1 + R_fC_fs)}{C_f(1 + R_iC_is)}\ e_i$$

12. 

$$e_o = -\frac{(1 + R_iC_is)(1 + R_fC_fs)}{R_iC_fs}\ e_i$$

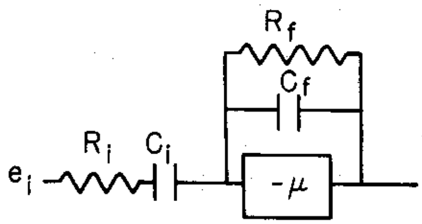
Table 2 (continued)

13.



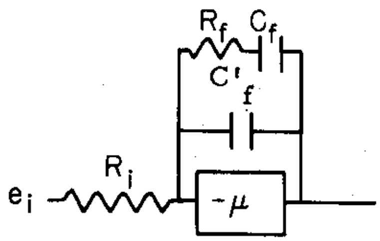
$$e_o = -\frac{R_f}{R_i}\frac{(1 + R_i C_i s)}{(1 + R_f C_f s)}\, e_i$$

14.



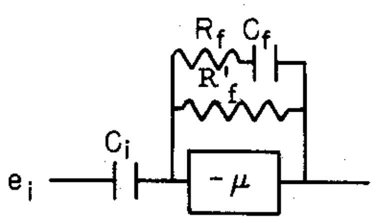
$$e_o = -\frac{R_f C_i s}{(1 + R_f C_f s)(1 + R_i C_i s)}\, e_i$$

15.



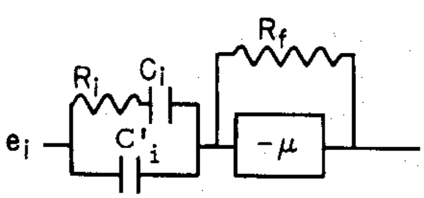
$$e_o = -\frac{1 + R_f C_f s}{R_i (C_f + C'_f)\, s \left(1 + \frac{R_f C_f C'_f}{C_f C'_f} s\right)}\, e_i$$

16.



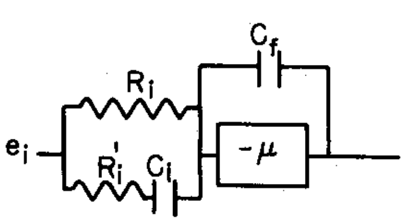
$$e_o = -\frac{R'_f C_i s\,(1 + R_f C_f s)}{1 + (R_f + R'_f)\, C_f s}\, e_i$$

17.



$$e_o = -\frac{R_f (C_i + C'_i)\, s \left(1 + \frac{R_i C_i C'_i s}{C_i + C'_i}\right)}{1 + R_i C_i s}\, e_i$$

18.

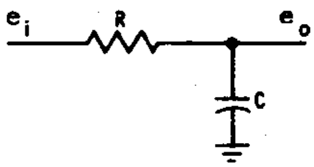


$$e_o = -\frac{\left[1 + (R_i + R'_i)\, C_i s\right]}{R'_i C_f s\,(1 + R_i C_i s)}\, e_i$$

Table 3

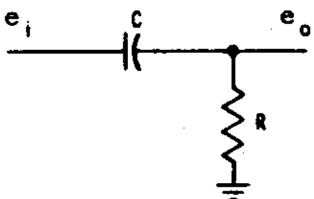
REAC COMPUTER
SUGGESTED PASSIVE NETWORKS

1.



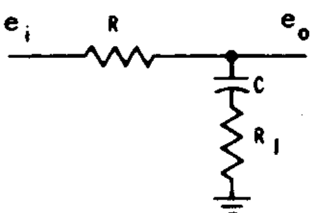
$$\frac{e_o}{e_i} = \frac{1}{1 + RCs} = \frac{1}{1 + as}$$

2.



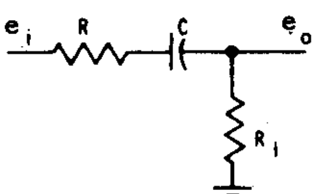
$$\frac{e_o}{e_i} = \frac{RCs}{1 + RCs} = \frac{as}{1 + as}$$

3.



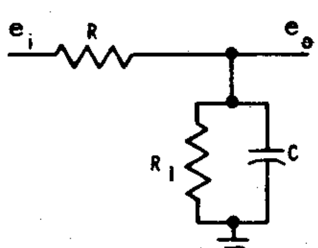
$$\frac{e_o}{e_i} = \frac{1 + R_1Cs}{1 + (R + R_1)Cs} = \frac{1 + as}{1 + bs}$$

4.



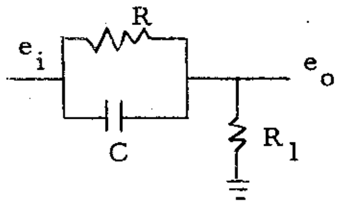
$$\frac{e_o}{e_i} = \frac{R_1Cs}{1 + (R + R_1)Cs} = \frac{as}{1 + bs}$$

5.

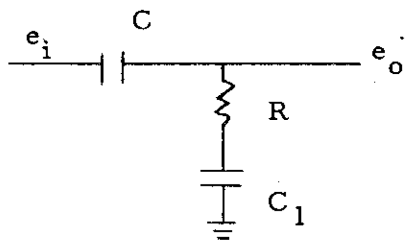


$$\frac{e_o}{e_i} = \frac{R_1}{(R + R_1) + RR_1Cs} = \frac{a}{b + cs}$$

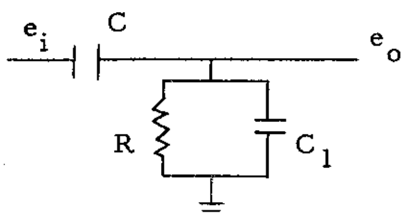
Table 3 (continued)

6. 

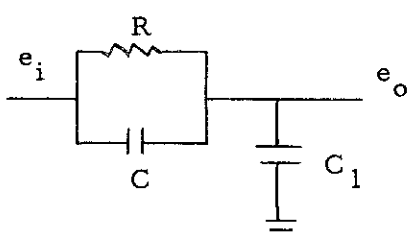
$$\frac{e_o}{e_i} = \frac{R_1 + RR_1Cs}{(R+R_1) + RR_1Cs} = \frac{a + bs}{c + bs}$$

7. 

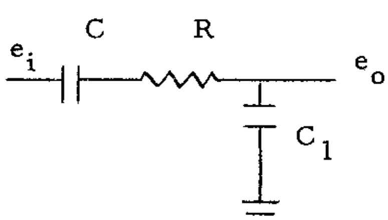
$$\frac{e_o}{e_i} = \frac{C + RCC_1s}{(C+C_1) + RCC_1s} = \frac{a + bs}{c + bs}$$

8. 

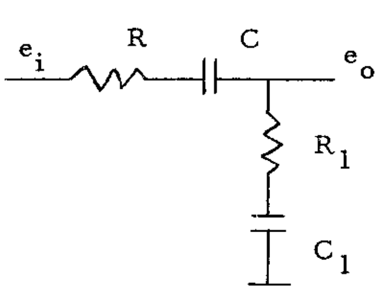
$$\frac{e_o}{e_i} = \frac{RCs}{1 + R(C+C_1)s} = \frac{as}{1 + bs}$$

9. 

$$\frac{e_o}{e_i} = \frac{1 + RCs}{1 + R(C+C_1)s} = \frac{1 + as}{1 + bs}$$

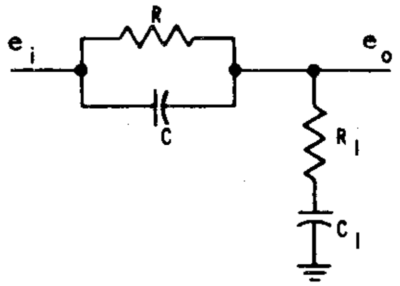
10. 

$$\frac{e_o}{e_i} = \frac{C}{(C+C_1) + RCC_1s} = \frac{a}{b + cs}$$

11. 

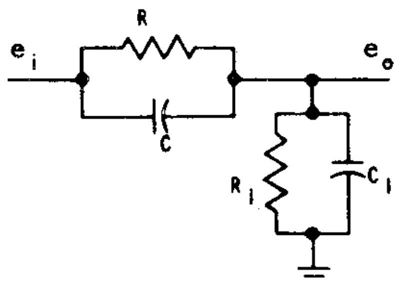
$$\frac{e_o}{e_i} = \frac{C + R_1CC_1s}{(C+C_1) + (R+R_1)\,CC_1s} = \frac{a + bs}{c + ds}$$

Table 3 (continued)

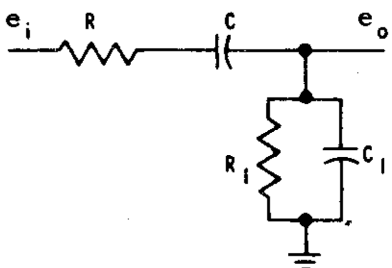
12. 

$$\frac{e_o}{e_i} = \frac{1 + (CR + C_1R_1)s + RR_1CC_1s^2}{1 + (CR + C_1R_1 + C_1R)s + RR_1CC_1s^2}$$

$$= \frac{1 + as + bs^2}{1 + cs + bs^2}$$

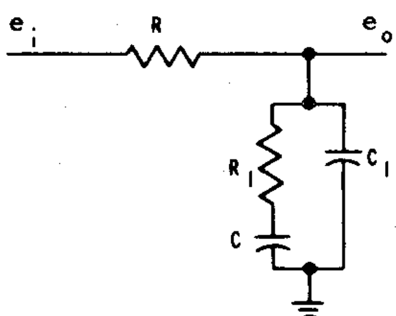
13. 

$$\frac{e_o}{e_i} = \frac{R_1 + RR_1Cs}{(R + R_1) + RR_1(C + C_1)s} = \frac{a + bs}{c + ds}$$

14. 

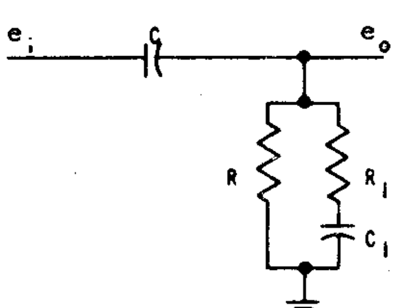
$$\frac{e_o}{e_i} = \frac{R_1Cs}{1 + (RC + R_1C_1 + R_1C)s + RR_1CC_1s^2}$$

$$= \frac{1 + as}{1 + bs + cs^2}$$

15. 

$$\frac{e_o}{e_i} = \frac{1 + R_1Cs}{1 + (RC + R_1C + RC_1)s + RR_1CC_1s^2}$$

$$= \frac{1 + as}{1 + bs + cs^2}$$

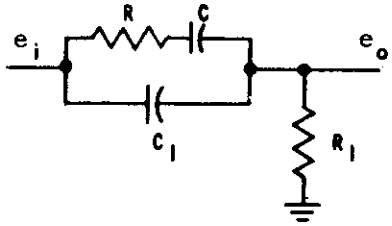
16. 

$$\frac{e_o}{e_i} = \frac{RCs + RR_1CC_1s^2}{1 + (RC + R_1C_1 + RC_1)s + RR_1CC_1s^2}$$

$$= \frac{as + bs^2}{1 + cs + bs^2}$$

Table 3 (continued)

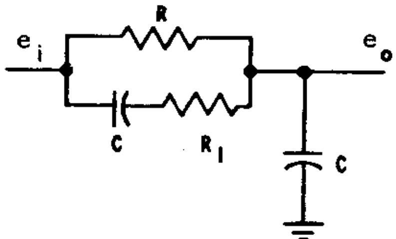
17.



$$\frac{e_o}{e_i} = \frac{R_1(C + C_1)s + RR_1CC_1s^2}{1 + (RC + R_1C_1 + R_1C)s + RR_1CC_1s^2}$$

$$= \frac{as + bs^2}{1 + cs + bs^2}$$

18.



$$\frac{e_o}{e_i} = \frac{1 + as}{1 + bs + cs^2}$$

2. PROBLEM SETUP PROCEDURES

Since the differential analyzer is a machine to perform the necessary operations for solving systems of differential equations, it is probably wise to examine two examples of how such setups can be instrumented. The first example is the instrumentation of the first order linear homogeneous differential equation and is given to illustrate the "closed loop" principle of solution.

$$\dot{x}(t) + ax(t) = 0 \tag{1}$$

or rewriting in the form we prefer for setting up analogs

$$\dot{x}(t) = -ax(t) \tag{1a}$$

We know the solution of this equation is

$$x(t) = \exp(-at) + c \tag{2}$$

where c is an arbitrary constant. We must specify one initial condition to obtain a unique answer, so let us specify $x(0) = 1$, which gives us $c = 0$. Now we see how this would look on a differential analyzer.

We note that this solution requires one integration, one multiplication by a constant, generation of a constant voltage, a summation of terms, and the setting of an initial condition. Our diagram, Figure 1, will not show the setting of the initial condition.*

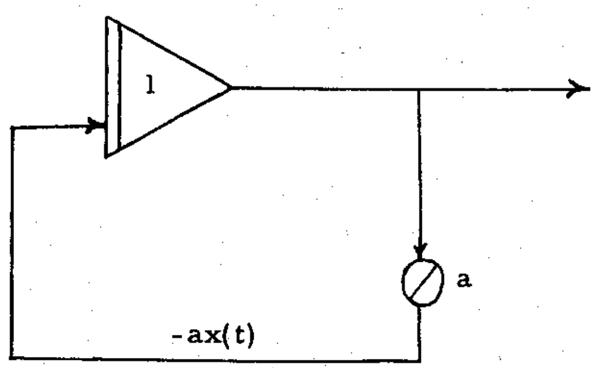


Figure 1: Closed Loop Principle

* These can be set in by releasing the proper voltage into the integration computation at the start of the problem. For details see Reference 5.

In Figure 1 if we start with the integrator and assume its input is $\dot{x}(t)$, then the output will be $-x(t)$. The problem is how to get $x(t)$. Equation (1a) gives us the method to pull ourselves up by our bootstraps. We reason as follows: if we multiply $-x(t)$, the output of the integrator, by a constant a (set on a potentiometer*), add a constant voltage $-b$, and feed them into the integrator, we will be putting in $\dot{x}(t)$.

The second problem is the nonlinear second order differential equation

$$\ddot{x}(t) + a\left[\dot{x}(t)\right]^2 + bx(t) = 0 \qquad (3)$$

or rewriting

$$-\ddot{x}(t) = a\left[\dot{x}(t)\right]^2 + bx(t). \qquad (3a)$$

This example is given to illustrate that many (in this case two) different instrumentations for solving the same problem are possible and, at least in theory, equally attractive. The first instrumentation, Figure 2, involves the use of multipliers. The alternative, Figure 3, involves a function generator to generate the square.

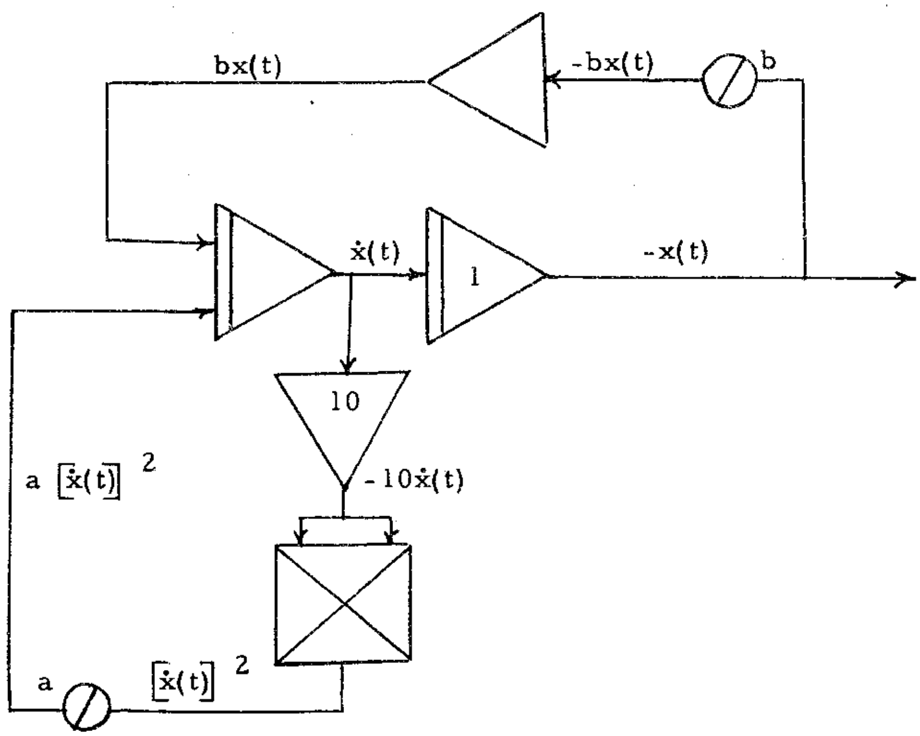


Figure 2: Instrumentation with Multipliers

* This would restrict $|a| \leqslant 1$. For values of $|a| \geqslant 1$ we can use the fact that $a = (|a^*|/10^n)$ where $|a^*| \leqslant 1$ and use two sign changing amplifiers with total gain 10^n in series with the potentiometer.

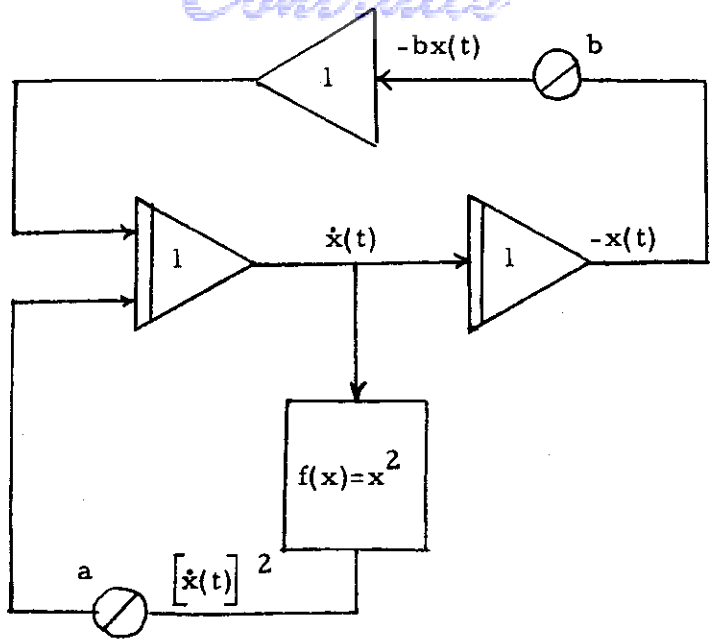


Figure 3: Instrumentation with Function Generator

In the instrumentations of these simple examples we have used certain facts about computing components. In Figure 2 we have implicitly stated that the computing system has a range of $\pm$ 100 volts by having the multiplier give the product divided by 100, and we have followed the rule of having an odd number of amplifiers in the closed computing loops. However, we have ignored, among other things, the problem of scaling the functions (variables) that appear in the equations so that the corresponding computer functions stay within the computer voltage range. We shall now describe two methods for scaling all variables simultaneously in order to assure that they fall in the computer voltage range.

The first is a method of scaling in which the zero of all the functions coincides with the electrical zero of the computer. We assume in all that follows that the voltage extremes of the computer are $\pm V$. Let $f_i(t)$, $i = 1, \ldots, n$, be the functions of time that appear in the system under consideration. Then for the range of the independent variable, $0 \leq t \leq t_o$, we consider the number K defined by equation (4)*

$$K = \max \left[|f_i(t)| \; ; \; 0 \leq t \leq t_o, \;\; i = 1, \ldots, n \right] \tag{4}$$

Define the new functions $\phi_i(t)$ to be

$$\phi_i(t) = (v/K)\, f_i(t), \;\; i = 1, \ldots, n \tag{5}$$

This method of scaling is not optimal in some ways, but it always assures that all functions stay within the voltage range and are the best of their type in that at least one function actually attains either V or -V.

One difficulty with this first method is that it does not take full advantage of the range 2V possible for the transformed variable. A second method does take advantage of this whole range; and the zeros of the functions, while not necessarily coinciding with the electrical zero, all translate to a common value. Define numbers K+ and K- on the range of the independent variable and for the functions $f_i(t)$

$$\begin{aligned} K+ &= \max \left[f_{i+}(t); \;\; 0 \leq t \leq t_o, \;\; i = 1, \ldots, n \right] \\ K- &= \min \left[f_{i-}(t); \;\; 0 \leq t \leq t_o, \;\; i = 1, \ldots, n \right] \end{aligned} \tag{6}$$

where $f_{i+}(t)$ and $f_{i-}(t)$ are defined by

$$\begin{aligned} f_{i+}(t) &= f_i(t), \;\; f_i(t) \geq 0 \\ &= 0, \;\; f_i(t) \leq 0 \\ f_{i-}(t) &= f_i(t), \;\; f_i(t) \leq 0 \\ &= 0, \;\; f_i(t) \geq 0 \end{aligned} \tag{7}$$

We then define the function $\Psi(t)$ by

$$\Psi(t) = \frac{2V}{K+ \; - \; K-} \left(f_i(t) - \frac{K+ \; + \; K-}{2} \right) \tag{8}$$

This method assures that all functions remain within the range $\pm$ V and that both V and -V are attained during the interval.

Both of these methods have the following obvious disadvantages:

(1) They require extensive knowledge about the range of the functions.

(2) If one of the functions has a very high maximum or low minimum compared to the maximum or minimum of the other functions, then these functions will not be able to take full advantage of the voltage range of the computer.

* If this max is plus infinity this method of scaling is, of course, impossible.

The first objection is an inherent problem which must be lived with. To do any scaling we must know something of the range of the functions in our systems, and the more optimal the scaling the more must be known. However, the fact that so little is generally known about the functions gives rise to the usual practice of conservatism, overestimating the maxima and minima.

The second objection is somewhat more basic. We shall here discuss some ways of getting around this problem if (a) the function with a very high maximum is the derivative or the integral of another function appearing in the problem and (b) the problem involves a system of equations.

For case (a) we note that a linear transformation of the independent variable t

$$\eta = kt \tag{9}$$

gives us the result that

$$\frac{d^n f(\eta)}{d\eta^n} = \frac{1}{k^n} \frac{d^n f(t)}{dt^n} \tag{10}$$

This tells us that by picking the proper time scale we can reduce or increase the maximum of a derivative as much as we desire. We should note that if the nth iterated integral with respect to time of f(t) occurs in the problem, we will have with the transformation of (9)

$$\overset{n}{\int} \dots \int f(\eta) d\eta = k^n \overset{n}{\int} \dots \int f(t) dt. \tag{11}$$

For case (b) we transform each equation separately using the transformation (5). This means that we will be forced to do some additional multiplications of functions by constants in order to use results from one equation in the set to evaluate another equation in the set. This generally means more equipment.

In addition to the problem of adjusting the range of the functions to the range of the machine, there is the problem of adjusting the range of frequencies in the problem to the range of frequencies in the machine. The most common way of doing this is by using the transformation (9). Suppose Ω_{max} is the maximum frequency in the problem and ω_{max} is the maximum frequency acceptable to the machine. Then the transformation of the independent variable t

$$\eta = \frac{\Omega_{max}}{\omega_{max}} t \tag{12}$$

will give transformed equations all of whose frequencies are less than or equal to ω_{max}.

This transformation has the following disadvantages:

(1) Unless zero is the minimum of both frequency bands the transformation does not necessarily carry one band of frequencies onto the other.

(2) Even if the bands are carried one onto the other by this change, the machine time necessary to observe one cycle of the low frequencies may be greater than is practically feasible.

It is worth noting that there are transformations which will transform from one finite band width exactly to any other finite band width, but these involve multiplying the functions of the problem by a complex valued function of the maxima and minima of the band width. It is, however, possible to transform the function so that it uses part of the machine frequency band width. We assume that the problem has a frequency band width of $\Omega_{min} \leqslant \omega \leqslant \Omega_{max}$ and the machine has a frequency band width $\omega_{min} \leqslant \omega \leqslant \omega_{max}$. First transform the independent variable t as in (12) by setting

$$\eta = \frac{2\ \Omega_{max}}{\omega_{max} - \omega_{min}} \tag{13}$$

Then define the new function $\phi(\eta)$ as

$$\phi(\eta) = \sin\left(\frac{\omega_{max} + \omega_{min}}{2}\eta\right) f(\eta) \tag{14}$$

This function will have its band width contained in the band width of the machine. In practice this means first changing time scale and then using a modulator on the transformed variables.

Frequently problems involve functions $z(t) = Z(x_1(t) \ldots x_n(t))$ where Z may be either difficult or impossible to compute, while the equivalent implicit formulation $F(x_1(t) \ldots x_n(t), z(t)) = 0$ offers no such problem. Some common examples are $z(t) = \frac{x(t)}{y(t)}$ and $z(t)y(t) - x(t) = 0$; $z(t) = \sqrt{x(t)^2 + y(t)^2}$ and $z(t)^2 - x(t)^2 - y(t)^2 = 0$; $z(t) = \tan^{-1}\frac{x(t)}{y(t)}$ and $y(t) \sin z(t) - x(t) \cos z(t) = 0$.

To do such problems it is customary to use a high gain amplifier, with gain μ, as in Figure 4.

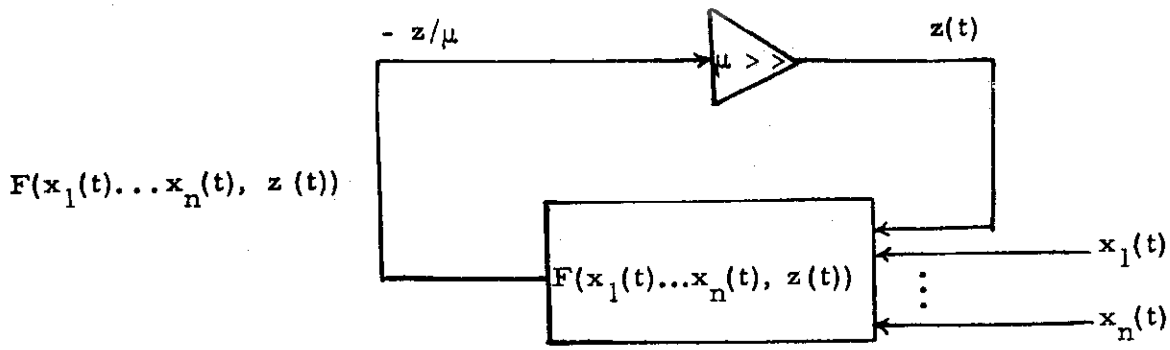


Figure 4: Instrumentation with High Gain Amplifier

If we call the output of the amplifier $z(t)$, then $z(t)$ and $x_1(t)$ to $x_n(t)$ form F. If we feed F into the high gain amplifier, we have then set $F = -z(t)/\mu$, which if μ is of, say, the order 10^7 usually leads to negligible error. A sufficient condition for this circuit to be stable with +F being fed back is

$$\frac{\partial F(x_1 \ldots x_n(t),\ z(t))}{\partial z(t)} > 0 \tag{15}$$

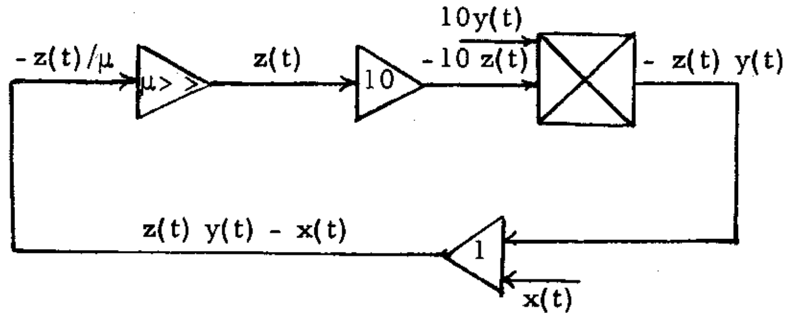
and with -F fed back is

$$\frac{\partial F(x \ldots x(t),\ z(t))}{\partial z(t)} < 0 \tag{15a}$$

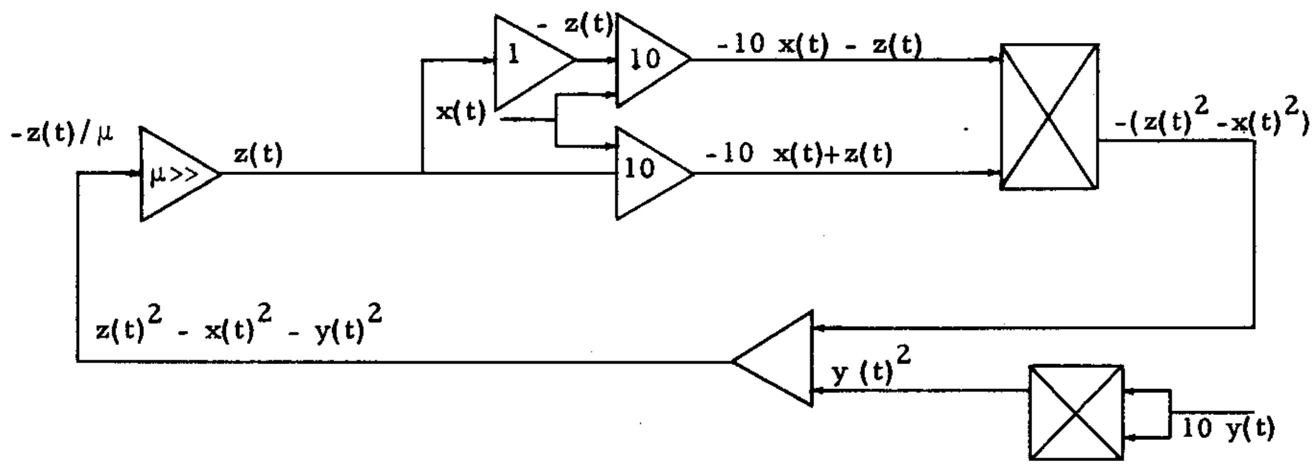
If it should happen that $\partial F/\partial z$ changes sign, then it is necessary to minimize the function $F(x_1(t) \ldots x_n(t)z)^2$, which can be done by finding the root of

$$F(x_1(t) \ldots x_n(t)\ )\ \frac{\partial F(x_1(t) \ldots x_n(t)\ )}{\partial z(t)} = 0 \tag{16}$$

$$z(t) = \frac{x(t)}{y(t)} \text{ and } z(t)\,y(t) - x(t) = 0$$



$$z(t) = \sqrt{x(t)^2 + y(t)^2} \text{ and } z(t)^2 - x(t)^2 - y(t)^2 = 0$$



$$z(t) = \tan^{-1}\frac{x(t)}{y(t)} \text{ and } y(t)\sin z(t) - x(t)\cos z(t) = 0$$

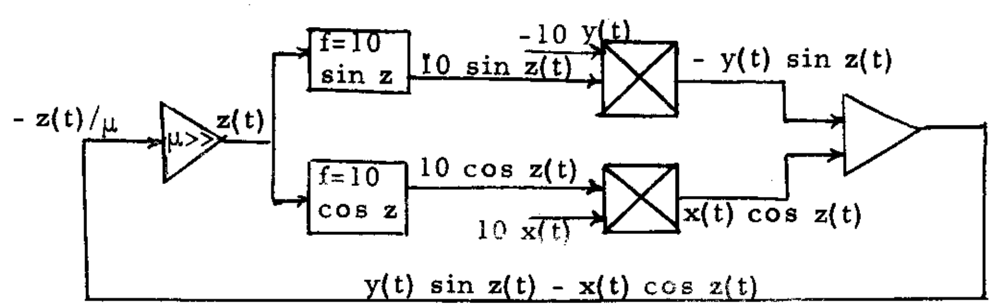


Figure 5: EXAMPLES OF THE USE OF HIGH GAIN AMPLIFIERS FOR COMPUTING WITH IMPLICIT FUNCTIONS.

In Figure 5 circuit diagrams are given for all of the foregoing examples.

In a large number of problems it is necessary to use function generators to obtain functions of the functions that appear as "machine variables." In practice it is often necessary to approximate these functions in some manner. One possible method of approximation is to select a number of points of the independent variable of the function being generated, obtain the functional values at these points, and then interpolate between these points. For example, if we were to use biased diode function generators the interpolation between points would in general be linear polygonal approximations, whereas if we were to use photoformers or curve followers the interpolation would generally be a smooth curve. Another possibility for polygonal approximation is to find the functional values at the points, compute the derivatives at the points, draw the tangents to the curve at these points, and consider this polygonal approximation. Quite often it is convenient to fit the function with an approximating polynomial over the range of its independent variable, since then all that is needed for function generation are multipliers, summing and sign changing amplifiers, and potentiometers. Some such polynomial approximations may be obtained by using

(1) the truncation of the Taylor series expansion of an analytic function;

(2) least squares approximation for passing a polynomial of degree n through m points where $m + 1 > n$;

(3) Lagrange formula for passing a polynomial of degree n through n + 1 points where the spacing on the axis of independent variable is arbitrary;

(4) finite difference methods for passing a polynomial of degree n through n + 1 points with equal spacing on the axis of the independent variable;

(5) fitting a polynomial to a set of points so the maximum deviation is minimized.

We have listed in Table 4 several series approximations and their ranges for a given error. In Table 5 is listed the Tchevycheff approximations to x^n by a polynomial of degree $n - 1$.

Table 4: SOME USEFUL APPROXIMATIONS

Approximation	Error	Limit
1) $(1 \pm x)^m = 1 \pm mx$	ε	$\|x^2\| < \frac{2\varepsilon}{m(m-1)}$
2) $e^{\pm x} = 1 \pm x + \frac{x^2}{2}$	.01	$\|x\| < .39$
	.001	$\|x\| < .18$
3) $a^{\pm x} = 1 \pm (x \ \ln a) + \frac{(x \ln a)^2}{2}$	ε	$\|x^3\| < \frac{6\varepsilon}{(\ln a)^3}$
4) $\sin x = x$	.01	$\|x\| < .31 = 18^\circ$
	.001	$\|x\| < .15 = 8.5^\circ$
5) $\sin x = x - \frac{x^3}{6}$	.01	$\|x\| < 1 = 57.3^\circ$
	.001	$\|x\| < .65 = 37.3^\circ$
6) $\cos x = 1$	.01	$\|x\| < .1 = 6^\circ$
	.001	$\|x\| < .03 = 2^\circ$
7) $\cos x = 1 - \frac{x^2}{2}$	.01	$\|x\| < .70 = 40^\circ$
	.001	$\|x\| < .39 = 22.5^\circ$
8) $\tan x = x$	.01	$\|x\| < .24 = 14^\circ$
	.001	$\|x\| < .11 = 6.3^\circ$
9) $\tan x = x + \frac{x^3}{3}$	.01	$\|x\| < .59 = 34^\circ$
	.001	$\|x\| < .37 = 21.3^\circ$
10) $\sin^{-1} x = x$	.01	$\|x\| < .31 = 18^\circ$
	.001	$\|x\| < .15 = 8.5^\circ$
11) $\sin^{-1} x = x + \frac{x^3}{6}$	.01	$\|x\| < .63 = 36^\circ$
	.001	$\|x\| < .41 = 23.5^\circ$
12) $\tan^{-1} x = x$	.01	$\|x\| < .24 = 14^\circ$
	.001	$\|x\| < .11 = 6.3^\circ$
13) $\tan^{-1} x = x - \frac{x^3}{3}$	.01	$\|x\| < .68 = 39^\circ$
	.001	$\|x\| < .42 = 24^\circ$

Table 5: TCHEVYCHEFF POLYNOMIAL APPROXIMATIONS

For $0 < x < 1, \quad -1 < T_i(x) < +1$

$$x = \frac{1}{2} + \frac{1}{2} T_1(x)$$

$$x^2 = -\frac{1}{8} + x + \frac{T_2(x)}{8}$$

$$x^3 = \frac{1}{32} - \frac{9x}{16} + \frac{3x^2}{2} + \frac{T_3(x)}{32}$$

$$x^4 = -\frac{1}{128} + \frac{32x}{128} - \frac{160x^2}{128} + 2x^3 + \frac{T_4(x)}{128}$$

$$x^5 = \frac{1}{512} - \frac{50x}{512} + \frac{400x^2}{512} - \frac{1120x^3}{512} + \frac{1280x^4}{512} + \frac{T_5(x)}{512}$$

$$x^6 = -\frac{1}{2048} + \frac{72x}{2048} - \frac{840x^2}{2048} + \frac{3584x^3}{2048} - \frac{6912x^4}{2048} + \frac{6144x^5}{2048} + \frac{T_6(x)}{2048}$$

$$x^7 = \frac{1}{8192} - \frac{98x}{8192} + \frac{1568x^2}{8192} - \frac{9480x^3}{8192} + \frac{26880x^4}{8192} - \frac{39424x^5}{8192} + \frac{28672x^6}{8192} + \frac{T_7(x)}{8192}$$

$-1 < x < +1, \quad -1 < \overline{T}_i(x) < +1$

$$x = 0 + \overline{T}_1(x)$$

$$x^2 = \frac{1}{2} + \frac{1}{2}\overline{T}_2(x)$$

$$x^3 = \frac{3x}{4} + \frac{1}{4}\overline{T}_3(x)$$

$$x^4 = x^2 - \frac{1}{8} + \frac{1}{8}\overline{T}_4(x)$$

$$x^5 = \frac{5}{4}x^3 - \frac{5}{16}x + \frac{1}{16}\overline{T}_5(x)$$

$$x^6 = \frac{3}{2}x^4 - \frac{9}{16}x^2 + \frac{1}{32} + \frac{1}{32}\overline{T}_6(x)$$

$$x^7 = \frac{112}{64}x^5 - \frac{56}{64}x^3 + \frac{7}{64}x + \frac{1}{64}\overline{T}_7(x)$$

Still another method is to set up on the analog a differential equation that has the required function as its solution.

The problem of doing statistical studies on analog computers has received considerable attention recently. For linear systems this is discussed in Appendix 2 of this part. For nonlinear systems this is discussed by R. R. Bennett, E. Lakatos and S. Sherman, and W. R. Allen* among others. One of the main problems is the question of "random noise." We believe that the most important feature in doing such problems is having a noise source that is thoroughly known and that is not itself subject to radical changes. This implies tests of randomness that are as rigorous as those applied by statisticians to their noise generators, the tables of random numbers. Most of the tests of randomness involving filters and electronic correlation devices fall down in that the characteristics of these pieces of test apparatus are only approximately known. We feel that either the output of a noise generator must be constantly monitored and tested or it must be put in permanent form by recording tested random noise on magnetic tape or on other storage devices** suitable for feeding into computing machinery. If it is put on tape or recorded in any way, the problem occurs of assuring a random start, that is, we wish to make sure we are not biased in the selection of the noise sample. A discussion of this can be found in any standard text on statistical sampling.

Since we are feeding the noise into a machine that is subject to malfunction, it is perhaps a good idea to record the noise fed in on a given run, to use if later the question arises as to whether the results are truly due to our noise sample or to a machine malfunction.

An excellent discussion of techniques used in connection with random digits was presented by J. von Neumann at a symposium on Monte Carlo techniques in June 1949.***

There are many special purposes for which differential analyzers can be used. The bibliography lists references to some of the papers on these topics.

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* Bibliography, items 9, 11, 19.

** Lakatos and Sherman, item 11.

***A summary appears in National Bureau of Standards Applied Math Series 12, "Monte Carlo Methods," issued 11 June 1949.

In discussing setup procedures it would be inappropriate not to mention that prior to running the problem it is essential to plan a simulation program with specific goals in mind. The use and analysis of the results planned should to a large extent determine the admissibility of the approximations used, the types of components permissible in the computer analog, plan of the runs taken, and the type of recording equipment used.

The first essential step in planning runs is to examine critically the mathematical representation and its relation to the physical system being studied. A careful analysis of what is possible in the physical system combined with some analytic work on the representation will very often give good indications as to the range over which the representation is adequate. It should be kept in mind that the differential analyzer is itself a physical system with its own representation. All that can be done is to plan the computer setup so that the representations of the physical system and of the differential analyzer coincide as nearly as possible over some range of interest. Such planning requires extensive knowledge of the computer, the physical system, and the mathematics of the representations.

When a representation and a goal for the simulation program are determined, a careful plan should be made of how the runs will be made. For example, if the goal is to minimize the miss distance of a missile with respect to several parameters, there are several ways of varying parameters in order to find this minimum.* A plan should be selected which is most suitable to the equipment, problem, and goal. Before the run is started, this plan should be worked out in detail. If the plan is sequential so that what has happened on previous runs determines what is to be done, a decision procedure must be selected in advance and maintained.**

When the problem is set on the computer checks should be made that the setup on the computer is the one planned, that the computer components are doing what they should, and that the computer can in fact handle this representation. A record of the computer setup for every run should be kept. The runs should be repeated sufficiently often to provide a check on the operation of the computer components. Some methods of checking are discussed in Chapter 3.

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* A discussion of various such plans is given in item 18 of the Bibliography.

**A discussion of a plan for using a simulator in connection with hardward testing is given in item 19.

In determining the method for analysis of the results, it should be ascertained that the recording equipment is sufficiently precise to make possible the desired analysis. If more than one recording channel is used, it should be provided that the results on all channels can be compared. If functions of time are recorded, and the paper speed and paper centering mechanism are no more reliable than on most current recorders, a timing pulse should be printed on each channel so that it is possible to compare all channels for some fixed time t. If the analysis is determined in advance, it may be possible to instrument the analysis automatically to save engineering man hours.

In surveying the use of analog computers for large scale flight simulation problems, the following unhappy conclusions concerning some major causes of difficulties have been reached:

(1) Problems are put on computers without sufficient study of the mathematical representation used.

(2) Insufficient thought is given to the intended uses of the results of the computation and what methods of analysis of these results will be adequate for their intended use.

(3) Inadequate records are kept of computer setup, changes in setup, and conditions under which a given run is made.

(4) Too little thought is given to planning the sequence of computation runs, resulting in unsuitable data and inefficient use of the machine.

(5) There is a tendency, partially because of the difficulty in resetting the problem on the computer, to continue runs beyond a time when the engineers are reasonably certain they are obtaining useful results.

The following possible remedies, realizing full well that the cure can be worse than the disease, are suggested:

(a) More careful analysis of the problem will help to eradicate difficulties (1) and (2).

(b) Better bookkeeping systems or automatic printing devices will tend to eliminate (3). Automatic printout or "setup verifier" devices are to be preferred, since the Steig principle is now generally conceded: people are no damn good.

(c) Careful, detailed planning of the computer "experiment" is the only way known to correct (4).

(d) Automatic and "off the computer" setup devices will alleviate (5).

3. CHECKING PROCEDURES

Checking procedures, other than those for machine malfunctioning, which is discussed in Chapter 5, can be listed in three main categories:

(1) procedures to see if the computer is actually set up to solve the problem as programmed,

(2) procedures to check a given computing circuit for suitability in a given problem,

(3) procedures for checking a machine solution to see what relation it has to reality.

We shall discuss in this chapter a few of the more common techniques for checking and in some cases try to evaluate the usefulness of the technique.

3.1 Checking Machine Programming and Setup

The simplest method for checking accuracy of setup is to have one person program and plug in the problem, have another person independently read off the connections, and from this reconstruct the program and the original equations. If an automatic printout device or "problem verifier" is available, the printed sheet of connections can be used by several people to reproduce the program and original equations independently. This procedure should be followed at frequent intervals during the running of the problem, particularly every time a major change is made in the setup.

A more sophisticated method, called "dynamic substitution check,"* has been advocated for performing this check and for isolating that portion of the circuit which is not correctly set up. The problem is run so that all parameters and available derivatives are recorded. The recorded traces are then read at small intervals. The values thus obtained are then substituted into the original differential equations. If no errors are present in the computation, recording, or reading, this should give a system of identities. In addition, the traces are checked to see if, for example, $x(t)$ is actually the integral of $\dot{x}(t)$. In practice the two sides of the original "equations" will differ by some quantity, and it must be determined if this quantity is negligible.

*Bibliography, item 46

Still another problem in this method is the need to be certain that some one function does not dominate and mask the effects of the other functions. An even more serious difficulty in analog computation is that some of the important derivatives may not be readily available in a manner useful for this procedure.

To illustrate, let us consider the second order homogeneous differential equation

$$\ddot{x}(t) = -\dot{x}(t) - x(t), \qquad x(0) = 0 \qquad \dot{x}(0) = 1 \tag{17}$$

Figure 6 gives the circuit diagram for this equation.

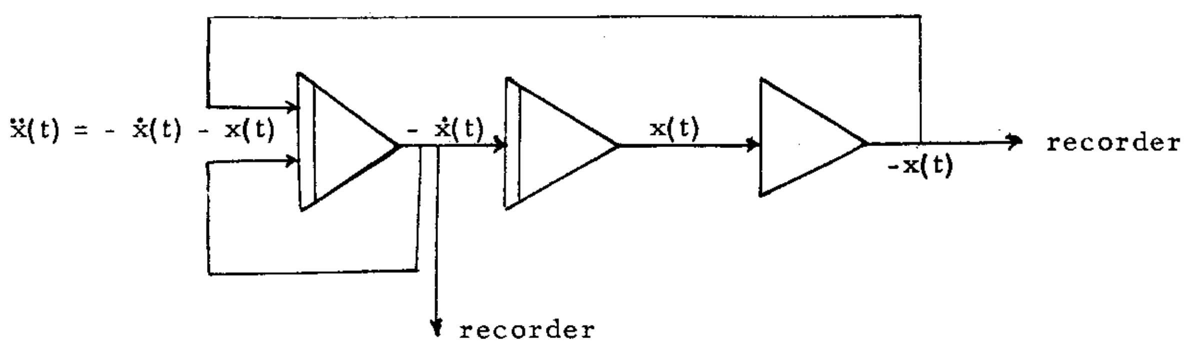


Figure 6: Circuit Diagram for $\ddot{x}(t) = -\dot{x}(t) - x(t)$

Figure 6: Circuit Diagram for $\ddot{x}(t) = \dot{x}(t) - x(t)$

Note that $\dot{x}(t)$ and $x(t)$ are both available to record from the machine. However, $\ddot{x}(t)$, which is necessary for this technique, is available from the machine only as the sum $- (\dot{x}(t) + x(t)$ if we introduce an additional adder. Such is not a very useful method, since, except for errors introduced by an adder that is not in the original circuit, this is exactly the quantity with which we wish to compare $\ddot{x}(t)$. We could get $\ddot{x}(t)$ by differentiating $\dot{x}(t)$ either manually from the traces or with a differentiator, but of these two possibilities the first is exceedingly tedious as well as inaccurate and the second is simply inaccurate.

To show the difficulty of determining whether a quantity is negligible, consider the equation (17) and the following equation, which is the same multiplied by a constant.

$$10\ddot{x} = -10x(t) \tag{18}$$

If on substituting the variables $\tilde{\ddot{x}}$, $\tilde{\dot{x}}$, and $\tilde{x}$ obtained from the machine in (17) we obtained

$$\tilde{\ddot{x}} + \tilde{\dot{x}} + \tilde{x} = \varepsilon \tag{19}$$

then we would obtain

$$10\tilde{\ddot{x}} + 10\tilde{\dot{x}} + 10\tilde{x} = 10\,\varepsilon \tag{19a}$$

If we determined that the machine was in error whenever the difference between the two sides of the equation exceeds .05, ε = .006 would cause us to accept the setup for (17) and reject the setup for (18). This is something that can give us constant difficulty in analog computation in view of our practice of "scaling."

3.2 Checking a Given Computing Circuit for Problem Suitability

This type of checking is to see if a given circuit is stable for the frequencies encountered, to see if an implicit function circuit gives adequate answers, to see if a method of function generation is adequate for problem needs, etc.

A technique in quite general use for testing the stability of a circuit in the frequency band of the problem is to measure or estimate the range of frequencies that this circuit will be forced to accept in the problem and then feed in a sine wave with a sine wave generator, with frequencies generally increasing somewhat beyond the estimated bandwidth.

To test function generators, feed in first a linear function covering the entire voltage range of the expected input and record both input and output. A timing pulse should be placed in both channels to make the comparison of input and output easy. This gives what is essentially a static check of the accuracy of the function generator. Next feed in a sine wave with the maximum frequency expected in the problem. Again record both input and output of the function generator with timing pulses, and compare them. This gives a dynamic check of the ability of the function generator to follow the inputs generated by the problem.

3.3 Checking a Machine Solution

Some general procedures which have been used or proposed for checking the machine solutions are as follows:

(1) Compare the solution with a carefully prepared digital check solution for some fixed values of the parameters.

(2) Obtain an analytic solution of the equations for some fixed values of the parameters.

(3) Obtain a solution of a simplified set of equations.

(4) Check by rerunning the problem on the analog computer with a different time scale.

(5) Check the solution to see if it coincides with known physical results for certain special cases.

The use of digital check is discussed at some length in Part 7. There Mr. Farrell concludes:

(a) The satisfactory comparison of a digital with an analog solution really allows one to infer only that for parameter values "near" to those used is the analog valid.

(b) The use of digital checks at several points of interest in the parameter space combined with the use of the perturbation equations to study the solutions in the neighborhood appears to be a useful technique.

The use of the analytic solution suffers from the obvious fault that it is often difficult or impossible to obtain. Furthermore, the conclusions concerning the use of digital solutions also apply to the use of the analytic solution.

Obtaining a solution of a simplified set of equations is not a good method unless the unamplified equations were really unnecessary.* To illustrate this consider the following example:

*This can be inferred from Mr. Farrell's remarks on checking the validity on checking the validity of simplified sets for equations.

$$\ddot{x}(t) + 2\dot{x}(t) + 2x(t) = 0 \qquad |\dot{x}(t)| < a \tag{20}$$

$$\ddot{x}(t) + 2x(t) + 2a = 0 \qquad \dot{x}(t) \geq a$$

$$\ddot{x}(t) + 2x(t) + 2a = 0 \qquad \dot{x}(t) \leq -a$$

initial conditions: $x(0) = 1,\ \dot{x}(0) = 0$

One check would be to examine the solution of (21):

$$\ddot{x} + 2\dot{x} + 2x = 0 \tag{21}$$

initial conditions: $x(0) = 1,\ \dot{x}(0) = -1$

The solutions of this equation and its first derivatives are respectively

$$x(t) = \exp(-t) \cos t$$

$$\dot{x}(t) = -\exp(-t)(\cos t + \sin t) \tag{22}$$

Note that in this equation $\dot{x}(t) \leq a$ if $t > \log \sqrt{2}/a$. We might thus expect perfect correspondence beyond that value of t. However, suppose this correspondence is indeed shown by our computer; all we have been told is that we have in fact corresponded, but in a region in which we are not interested. Actually the situation is worse than described, for we do not necessarily have such correspondence. Suppose $\dot{x}(t)$ exceeds a for the first time at t_o. We then are solving the equation

$$\ddot{x}_1(t) + 2x_1(t) + 2a = 0$$

$$\text{with conditions} \quad x_1(t_o) = \exp(-t_o)(\cos t_o) \tag{23}$$

$$\dot{x}_1(t_o) = a$$

which has as its solution

$$x_1(t) = \Big[\exp(t_o)(\cos t_o \cos \sqrt{2}\, t_o) + a \sin \sqrt{2}\, t_o - (a/\sqrt{2})(\cos \sqrt{2}\, t_o)\Big] \sin \sqrt{2}\, t + \Big[(a/\sqrt{2}) \cos \sqrt{2}\, t_o + \exp(t_o) \cos t_o \sin \sqrt{2}\, t_o + a \cos t_o \sin \sqrt{2}\, t_o\Big] \cos \sqrt{2}\, t + a$$

---

*This can be inferred from Mr. Farrell's remarks on checking the validity of simplified sets of equations.

When $\dot{x}_1(t)$ becomes less than a at time t_1 we then will be solving

$$\ddot{x}_2(t) + 2\dot{x}_2(t) + 2x_2(t) = 0 \qquad (24)$$

with conditions: $x_2(t_1) = x_1(t_1)$

$\dot{x}_2(t) = a$

which does not necessarily have the solution

$\exp(-t) \cos t.$

Hence we see our check of the simplified equation gave us no information or, even worse, misleading information in that we might have rejected the machine solution when in fact it was correct.

The procedure of checking the problem by rerunning it with different time scales on the analog computer checks only the bandwidth of the computer, its repeatability, and the programmer's ability to change time scale.

Checking the solution against results of experimentation is an excellent check in that if there are sufficient numbers of these experimental checks spread over the parameter space and they verify the computer results, we can have some confidence that the computer is at least a good interpolation device.

4. WHO SHOULD OPERATE THE COMPUTER?

Our survey indicates that in actual practice the choice of who operates and programs the analog computer varies considerably. At one extreme, the people who generate the problem also program, plug in, and run the computer. At the other extreme, the people who originate the problem give the formulation to the computer specialists and then go away until the computer group is able to give them many miles of recorder tape and several acres of plotting table graphs. Most practice falls somewhere in between; that is, the people who generate the problem cooperate in programming it for the computer to be sure that their representation is not violated and stand by during the running of the problem for consultation on the difficulties that arise.

As is noted in Part 4, we favor the middle way, particularly for large scale simulation problems. A specialized computer staff should handle the actual programming, plugging in of circuits, and running of the problem, but they should have available to them the advice, assistance, and restraining influence of the people who generated the problem.

Some of the reasons for favoring the existence of a special computer staff are:

(1) Efficiently programming and operating an analog computer is a special skill that is distinct from the ability to design equipment and formulate problems.

(2) It is generally more economical of computer time if the actual programming, plugging in, and running are performed by a specialized staff.

(3) Computer specialists are likely to have more respect for the physical limitations of the computer and will tend to program a problem in a manner that will minimize maintenance problems.

(4) Having a special group whose only responsibility is computer programming and operation enhances the possibility of keeping careful records of problem programming and setup which, current practice to the contrary notwithstanding, is quite essential to careful analysis of results.

It is only fair to note that in the installations surveyed in which the engineers who formulate the problems also operate the computer, generally some engineers are present who in fact qualify as computer specialists, and are available, at least on an informal basis, to advise on programming and running the computer.

In addition, these groups have found that the main advantage of letting the design engineers do their own machine operation is that, having produced the results themselves, they trust them further and are generally happier with the procedure. One reason for their happiness is that these analog computers have knobs to turn, plugs to plug in, switches to switch, etc. It might be desirable in the cases where a special computer group operates the computer either to eliminate such attractions by programming from, say, punched cards or punched tape or, alternatively, having a special board, not connected with the computer, containing knobs, meters, etc. for the design engineers to use while the problem is being run.

5. COMPUTER MAINTENANCE

The three conflicting schools of thought on computer maintenance are:

(1) Repair the computer only when there is a breakdown.

(2) Repair each breakdown and at more or less regular intervals overhaul the entire computer.

(3) Repair each breakdown and every day replace a certain number of components with spares that have been checked for utility.

The first of these schools feels that delicate electronic equipment is much better off with the barest minimum of handling. For example, it is felt that if a tube has endured 100 hours of operation it is likely to last for quite a long time more, providing it is not subject to mechanical shocks, sudden cooling, etc. This school of thought holds that the mechanical shock of simply removing the component introduces failure where none existed.

The second school reasons that after a long time of running, the computer will need a considerable number of adjustments, that the probability of tube failure has increased, and that choppers are near the end of their normal life; therefore, the computer should be closed down and given a thorough overhaul.

The third school feels that it is possible to prevent failure by maintaining a supply of components which can be rotated in use in the machine. For example, suppose that every day we replace three amplifiers in the computer by three newly tested amplifiers, and proceed daily in this manner systematically through the machine. The reasoning here is much the same as in the second school, but this group believes it can forestall failure more effectively by systematic replacement while still keeping the machine in daily operation.

Which of these techniques is best for a given facility depends on the type of equipment, personnel, and operation existing at the facility. A computer installation such as the Aerial Measurements Laboratory at Northwestern University, which generally commits all of its equipment to a single problem for a long period of time, could well adopt the second method. On the other hand, Convair in Pomona, Calif., which uses its equipment on a two-shift basis, with problems changing every shift, may have as many as six problems on the machine at one time and cannot afford the time for overhaul; it will adopt either the first or third method.

One disconcerting thing is that separate computing groups with similar types of equipment, personnel, and operation give radically different reports on the effectiveness of the various methods. A computer group can determine which method of maintenance is most advantageous by running a simple experiment, necessitating only some extra record keeping and a supervisor who can restrain engineers and/or technicians from "repairing" equipment that should not be repaired. After dividing the computer into three structurally similar parts, the maintenance of each part is practiced respectively according to the first, second, or third kind of maintenance procedure. In that part of the computer on which the third type is practiced, rotation of components should cause them to remain in as long as they would if the rotation were being practiced on the whole computer, i.e., there should be a lag in the rotation of the one section as if meanwhile the other two sections were similarly having their components rotated. Since in rotating parts, the length of cycle is probably connected with the failure rate, one may wish to divide the computer into more than three sections in order to test different cycle lengths. It is possible to compute the cycle length that will theoretically minimize down time due to failure of a given component providing the probability distribution of failure of the component as a function of time is known.* In running such an experiment it is desirable to coordinate with a statistician in order that the experiment is designed to wring the maximum amount of information from the data.

Whether or not a failing part should be repaired while the machine is down or should be replaced immediately with a spare and repaired at the staff's leisure is another question which a maintenance procedure must face. Our studies (see Appendix 2, Part 4) indicate that the most efficient use of the maintenance staff is obtained when a spare part is placed in service and the faulty part repaired in spare time. This, however, may not be the most efficient way to operate from the standpoint of expected cost; to determine this it would be necessary to use some model similar to the model we have used for inventory problems (see Appendix 1).

Not only does an adequate set of records help to judge machine efficiency; it

* If this has an exponential failure distribution, then life tests procedures and estimates of the distribution parameters based on these are given by B. Epstein, Bibliography, item 30.

can also, if used properly, aid in determining (1) whether the maintenance staff is of proper size (see Part 4), (2) what inventory of parts should be maintained (see Appendix 1), (3), which components need modification or greater reliability, (4) what the life distribution of components is, (5) what the relation of problem size, e.g., number of amplifiers, is to the failure rate, (6) what the relation of one type of component failure is to another type of component failure, (7) what the effectiveness of component modification is, and (8) what machine operating procedures are undesirable.

Items (1) and (2) are discussed elsewhere as noted. Graphs, such as Figure 7, of failures of components of a given type during a given time interval plotted as percent of total numbers of this type of component in the computer and as percent of total number of machine failures against time are good indicators for (3) and (8). In determining (7) these same charts analyzed before and after the modification give a good indication of its effectiveness. Linear regression and correlation technique combined with statistical hypothesis testing applied to the data will give (5) and possibly (6). It may be necessary to obtain (6) by means of serial correlation techniques. Component life distributions can be obtained by keeping files on each component, such as an amplifier, and then fitting (if possible) a failure distribution.

Table 6 is a sample code for transferring failure records to IBM punched cards in a manner suitable for computing the relations we desire. Table 7 is a list of information that is useful to have on maintenance records.

Actual maintenance routines fall into two categories:

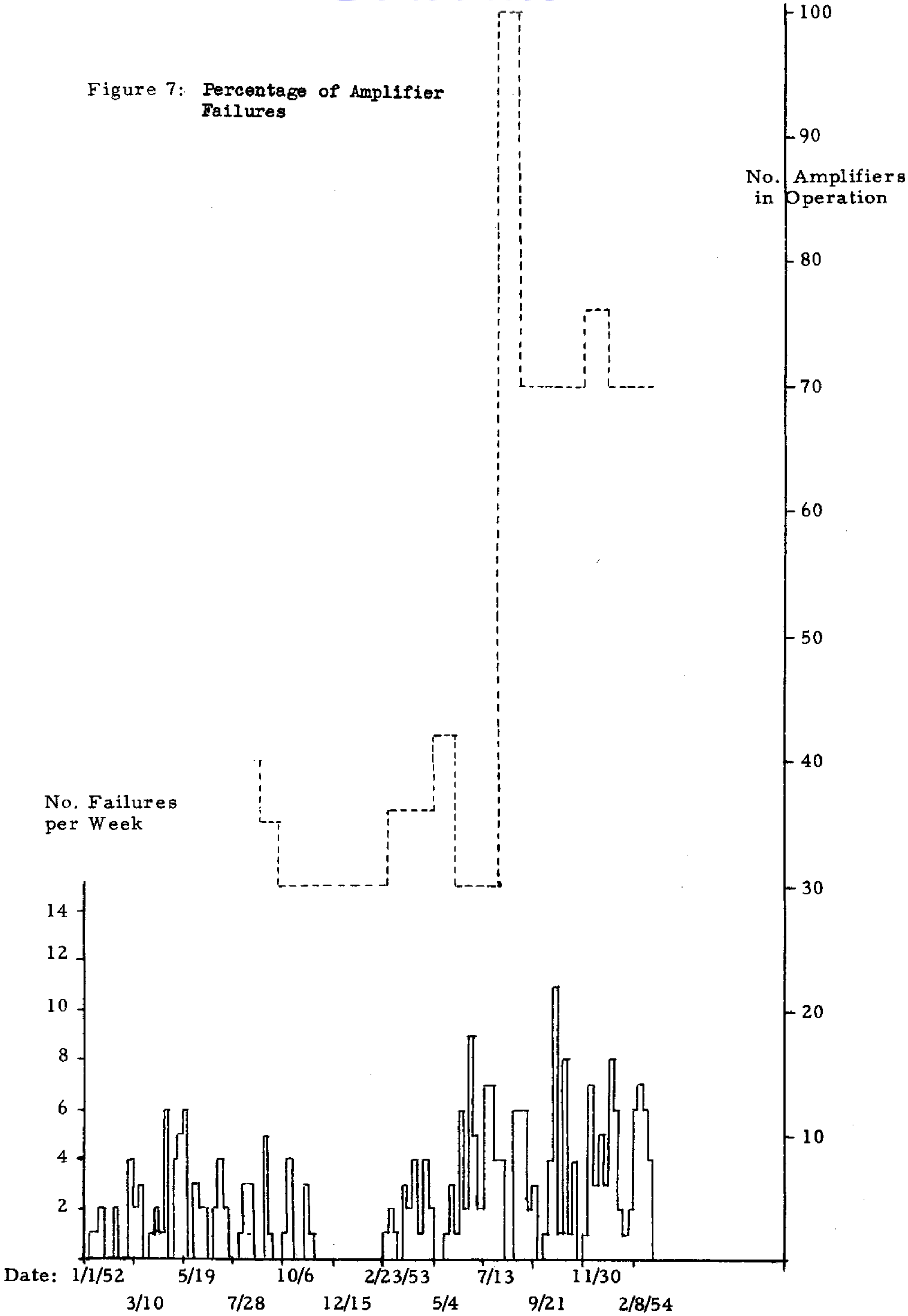
(1) routine checks to see if the machine is operating properly,

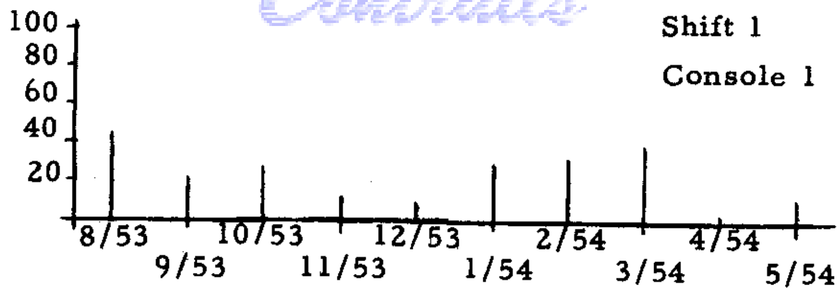
(2) trouble shooting a malfunction.

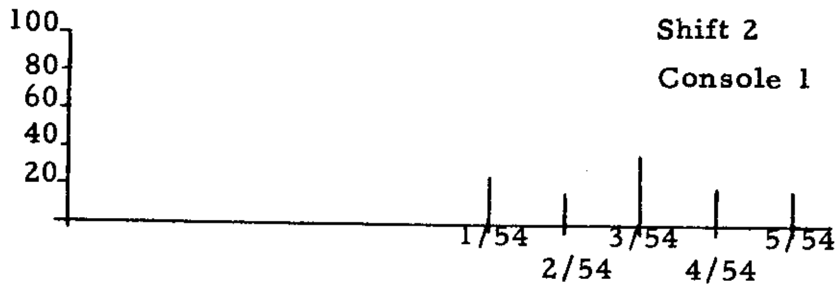
It is desirable that a routine to check all components be devised so that the machine can be checked at the beginning and end of every computer shift. This routine should take no more than a half hour per check for a machine comparable in size to one containing 900 amplifier equivalents* and 24 function generators. In addition, a more detailed check should be made regularly, say once a week, to be sure that all components are in reasonable operating condition.

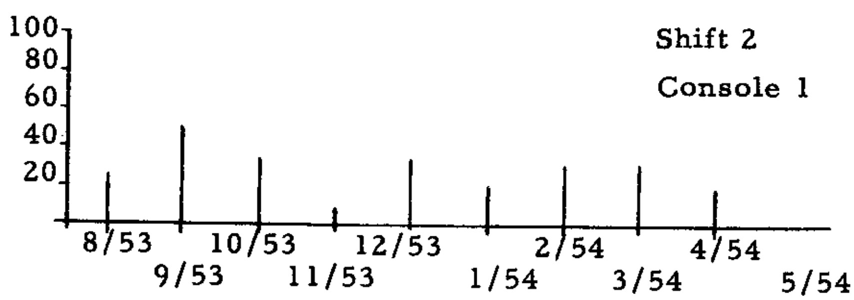
* We shall call one multiplier the equivalent of three amplifiers.

Figure 7: **Percentage of Amplifier Failures**









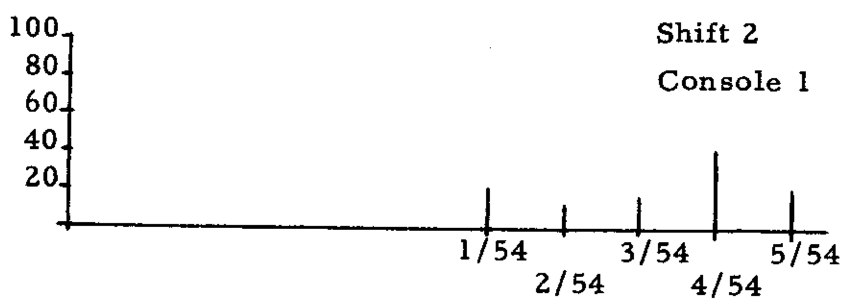


Figure 7a: % AMPLIFIER FAILURES TO TOTAL FAILURES

Table 6: IBM CARD LAYOUT

Columns	Information
1-3	Failure reference number
4-5	Month of failure
6-7	Day of failure
8-9	Year of failure
10	Shift
11	Console
12-14	Location in machine
15-16	Computing block
17-20	Type of repair
21-22	Number of parts replaced
23-24	Comments on kind of failure when no immediate repair was indicated
25-26, 32-33, 39-40, 46-47, 53-54	Number of tubes replaced
27-31, 34-38, 41-45, 48-52, 55-59	Manufacturer's tube number
78-80	Card number

Table 6a: COLUMN DETAILS IN IBM CARD LAYOUT

Columns 15-16

No.	Computing Block	No.	Computing Block
11	Amplifier	47	Tape Recorder
12	Summer	48	Stopper
13	Integrator		
14	Inverter	51	DVM
15	Servo Amplifier.	52	VT VM
16	Relay Amplifier	53	Balance Indicator
17	Balance Amplifier	54	Galvo Box
18	Chopper Amplifier	55	Servo VM
		56	Circuit Breaker
21	EM		
22	Servo Multiplier	60	Power Supply
23	Multiplier		
		71	Attenuator
31	EFG	72	Condenser
32	Servo	73	Diode
33	Resolver	74	Potentiometer
34	Limiter	75	Relay
35	Geda	76	Resistor
36	Noise Generator	77	Switch
		78	Transformer
40	Recorder	79	Ovld Alarm
41	Sanborn Recorder		
42	Mid-Century	81	Pot Setter
43	R101	82	Patch Board
44	R104	83	Switch
45	Plotting Board	84	Keyboard
46	Variplotter	85	Control

Columns 17-20

1000	Repair	1012	DVM
1001	Potentiometer	1013	Stopper
1002	Amplifier	1014	EFG
1003	Power Supply	1015	Relay
1004	Dial	1016	Multiplier
1005	Servo	1017	Spring
1006	Recorder	1018	VTVM
1007	EM		
1008	Pen	2000	Adjust
1009	Pin	2001	Potentiometer
1010	TP 10	2002	Amplifier
1011	Switch		

Table 6a continued

2003	Power Supply
2004	Dial
2005	Stop
2006	Current
2007	Gain
2008	Pot Diaph.
2009	Contacts
2010	Overload
2011	Relay
2012	EFG
2013	Unloader
2014	C4018
2015	Trimmer
2016	EM
2100	<u>Align</u>
2101	Geda
2102	EM
2103	Scope
2104	Dial
2105	Stop
2200	<u>Clean</u>
2201	Contacts
2202	Pins
2203	Wipers
2204	Amplifier
2300	<u>Balance</u>
2301	EM
2302	PS
2303	Gates
2304	Geda Channel
2305	Potentiometers
2400	<u>Calibrate</u>
2401	Diode
2402	DVM
2403	Pot Set
2500	<u>Check</u>
2501	Tubes
2502	Amplifiers
2503	Potentiometers
3000	<u>Rotate</u>
3001	Servo
3002	Switch
3003	Potentiometer
3004	Servo Multiplier
4000	<u>Install</u>
4001	Potentiometer
4002	Power Supply
4003	Ovld Alarm
4004	CRT
4005	Servo Multiplier
4006	Amplifier
4007	Servo
4008	Recorder
5000	<u>Remove</u>
5001	Resistor
5002	Resolver
5003	Overload
5004	Servo
5005	Amplifier
5006	Recorder
6000	<u>Wire</u>
6001	Solder
6002	Remove short
6003	Servo
6004	Bridge Input
7000	<u>Replace</u>
7001	Tubes
7002	Battery
7003	Chopper
7004	Pen
7005	Potentiometer
7006	Pin
7007	Light
7008	Spring
7009	Dial
7010	Screw
7011	Capacitor
7012	Pin
7013	Resistor
7014	Transformer
7015	Fuse
7016	Fuse holder
7017	Components
7018	Switch
7019	Regulator
7020	Power Supply
7021	Servo
7022	Amplifier
7023	Relay
7024	EM

Table 6a continued

7025	Plotting Board	7029	Multiplier
7026	Motor	7030	Diode
7027	Resolver	7031	Inverter
7028	Input	7032	Servo M

Columns 23-24

10	NG	32	Ring around center of tube
11	Bad Contacts	33	Glowing
12	Bad Timing	34	Arcing
13	Blown Fuse	35	Bad Tube
14	Bad Output		
15	Short	41	Noise
16	Grounded	42	Drift
17	Open lead	43	Unbalanced
18	Calibration	44	Overload
19	Bad adjustment	45	Intermittent
		46	Jitter
21	Loose assembly	47	Oscillation
22	Slipping	48	Broken stop
23	Open	49	Bad Battery
24	Won't set		
31	Gassy tube	51	Pen clogged
		52	Broken pin

When the machine operator reports a malfunction, current practice appears to rely largely on the ingenuity of the maintenance men in locating the malfunction; it is considered something of an art to locate certain types of malfunction. Art and ingenuity are to be encouraged, but it appears desirable to have these formalized into a search routine, if possible. It also appears desirable that after the alleged malfunction is corrected, the portion of the computer in which the malfunction occurred should be subjected to at least the cursory check that was used at the beginning of the shift in order to be sure that no other obvious difficulty exists. A good automatic programming device facilitates such search routines.

The major sources of failures in computers appear to be (1) faulty tubes in the amplifiers and power supplies, (2) failure of mechanical choppers in the A.C. stabilizing amplifier, (3) changed values of input resistors, (4) changed values of capacitors in the feedback circuits of integrating amplifiers, (5) burned out cathode ray tubes in the photoformers, and (6) faulty jack cords. It is reported that chopper life has been increased by using thyrite protection on contacts and changing the sampling circuit slightly. Trimmers placed on resistors and capacitors make it possible to adjust the values to those desired, and placing the input resistors in an environment of stabilized temperature and humidity cuts down on the number of such failures. Modification of the photoformer circuit has increased useful C-R tube life. Regular testing of jack cords will eliminate problem (6). Tube failures if too frequent may make desirable the use of aged tubes to eliminate those that tend to quick burnout. The experience with ruggedized tubes in computing circuits has been that they introduce more troubles than they correct. However, if tube failure is such a major problem that it is necessary to assess the comparative merits of various types of tubes, it is possible to test these experimentally with the computer by much the same technique used for testing maintenance procedures.

In closing this chapter is is desirable to take account of the possibility of predicting failures.* Some tests have been devised which purport to do this. However, we do not know of any experimental verification that these tests are in fact good predictors and feel that before much faith is placed in such

*See Bibliography, item 41.

methods they should receive careful scrutiny; they should be tested experimentally as to their efficacy in preventing machine failure in much the same way vaccines are tested as to their efficacy in preventing disease.

Table 7: TROUBLE REPORT SHEET

Date_ _ _ _ _ _ _ _ _ _ _ _ Shift_ _ _ _ _ Console_ _ _ _ _ _ Problem No. _ _ _ _

Computing Block_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ Location in Console_ _ _ _ _ _ _ _ _ _

Type of Repair (or Failure, if no repair is made):

Bench Time (including time spend on parts removed and immediately replaced):

Number of Parts Replaced:

Note: If more than one IBM card is necessary for a given failure, all but one should have an identifying punch in a specified column so that these can be discarded when counting the number of failures.

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Appendix 1*

AN APPLICATION OF WAITING LINE THEORY TO A SIMPLE SPARE PARTS INVENTORY PROBLEM

by

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1. Introduction

This appendix concerns the inventory problem in which no repair is possible and the length of the ordering period is fixed. The problem is to find the size of the spare parts order that will minimize the expected cost for the lot and per part, respectively, if the number of failures during the period follows a fixed distribution. A partial table of expected costs and expected costs per spare part when the distribution of failures is Poisson is included.

2. The "No Repair" Model

In this model, we consider that the length of the ordering period is some fixed time T, n is the number of spare parts purchased, β is the cost per spare part, m is the number of failures during the period, P_m is the probability of m failures during the period, γ is the penalty paid for each failure for which a spare part is not available, α is the penalty for each spare part ordered but not used during the period, $C_{n,m}$ is the cost of servicing m failures during the period when n spare parts were ordered.

We have

$$C_n, m = \beta n + \gamma (m - n)_+ + \alpha (n - m)_+ \tag{1.1}$$

where $(m - n)_+ = m - n, \quad m > n$
$\qquad\qquad\quad = 0 \quad , \quad m < n.$

The expected cost,

$$E(C_{n,m}) = \beta n + \sum_{m=n+1}^{\infty} P_m \, \gamma (m-n)_+ + \sum_{m=o}^{n} P_m \, \alpha (n-m) \tag{1.2}$$

*From Advisory Board on Simulation Technical Note 35, 1 March 1954 will have a minimum at $n = s$ (smallest $n \geq 0$) when the difference

$$E(C_{n,m}) - E(C_{n+1,m}) = -\beta + \gamma - \gamma \sum_{m=o}^{n} P_m - \alpha \sum_{m=o}^{n} P_m \qquad (1.3)$$

as a function of n is nonnegative for all $n < s$ and nonpositive for all $n > s$. Using (1.3) we note this s is obtained when n is such that

$$\sum_{m=o}^{n} P_m = \frac{\gamma - \beta}{\gamma + \alpha} \; . \qquad (1.4)$$

Note that if $s = 0$ we have $P_o = \frac{\gamma - \beta}{\gamma - \alpha}$ which says, since P_o is nonnegative, we are certain no spare parts should be stocked if the penalty for one unit of down time is less than the cost for one spare part ($\beta > \gamma$). (This is something everyone knows anyway; it is consoling that our model also gives this result.)

Another criterion is to make the expected cost per spare part a minimum, that is to say, minimize

$E(\frac{C_{n,m}}{n})$, which requires that we determine the value of s such that

$$E(\frac{C_{n,m}}{n}) - E(\frac{C_{n+1,m}}{n+1}) = \frac{1}{n+1} E(C_{n,m} - C_{n+1,m}) + \frac{1}{n(n+1)} E(C_{n,m}) \qquad (1.5)$$

$$= \frac{1}{n(n+1)} (\gamma \sum_{m=n+1}^{\infty} m P_m - \alpha \sum_{m=o}^{n} m P_m)$$

which satisfies the conditions of being nonpositive for all $n > s$ and nonnegative for all $n < s$.

This occurs for n such that

$$\sum_{m=o}^{n} mP_m = \frac{\gamma}{\gamma + \alpha} E(m) \; . \qquad (1.6)$$

Note that if there were no penalty, $\alpha = 0$, one should order as many parts as possible (assuming of course there is no N such that for all $m > N$, $P_m = 0$; if there are such N order a number of spare parts equal inf. N). This again agrees with our intuition.

In case the distribution of failures is Poisson with parameter λ we have in place of (1.4)

$$\sum_{m=o}^{n} \frac{\lambda^m}{m!} = \exp(\lambda)(\gamma - \beta)/(\gamma + \alpha) \qquad (1.4a)$$

and in place of (1.6)

$$\sum_{m=o}^{n-1} \frac{\lambda^m}{m!} = \exp(\lambda)\,\gamma/(\gamma+\alpha) \qquad (1.6a)$$

In the Poisson case we note that we have, $\gamma, \alpha, \beta > 0$ from (1.4a) and (1.6a)

$$\sum_{m=o}^{n} \frac{\lambda^m}{m!} - \sum_{m=o}^{n'-1} \frac{\lambda^m}{m!} = \frac{e}{\gamma+1}(-\beta) \leq 0 \qquad (1.7)$$

which implies

$n' - 1 \geq n$ or $n' > n$

We have not in general discovered under what conditions this result is true for arbitrary distributions, but so far know only the following condition:

$n <$ (the greatest integer less than $\frac{E(m)}{1-\frac{\beta}{\gamma}}$) implies $n' > n$, since from (1.4) and (1.6) we have

$$\sum_{m=o}^{n'} m\,P_m = \sum_{m=o}^{n} \left(\frac{E(n)}{1-\frac{\beta}{\gamma}}\right) P_m \qquad (1.8)$$

Glossary for Tables 8 and 9

Symbols

m = number of failures

n = number of spare parts

P_m = probability of m failures

α = penalty for not using a part

β = cost per part

γ = cost of down time per part not available

Table 8 lists the expected value of the cost of m failures for n spare parts ($E(C_n) = \sum_{m=o}^{\infty} P_m C_{m,n}$), when the probability of m failures is given by a Poisson distribution

$$\left(\sum_{m=o}^{\infty} P_m = \sum_{m=o}^{\infty} \exp(-\lambda) \frac{\lambda^m}{m!} \right)$$

Table 9 is a refinement of Table 8 when the following values are assigned: $\beta = 1$

$\gamma = \alpha,\ \gamma = 5\alpha,\ \gamma = 10\alpha,\ \gamma = 20\alpha.$

Also listed is the minimum expected value of the cost per one part.

Table 8: EXPECTED VALUE OF FAILURES

$\lambda = 5$

n	$E(C_n)$
1	$\beta + 4.0067\,\gamma + .0067\,\alpha$
2	$2\beta + 3.0472\,\gamma + .0472\,\alpha$
3	$3\beta + 2.1718\,\gamma + .1718\,\alpha$
4	$4\beta + 1.4368\,\gamma + .4368\,\alpha$
5	$5\beta + .8773\,\gamma + .8773\,\alpha$
6	$6\beta + .4933\,\gamma + 1.4933\,\alpha$
7	$7\beta + .2555\,\gamma + 2.2555\,\alpha$
8	$8\beta + .1221\,\gamma + 3.1221\,\alpha$
9	$9\beta + .0540\,\gamma + 4.0540\,\alpha$
10	$10\beta + .0222\,\gamma + 5.0222\,\alpha$
11	$11\beta + .0085\,\gamma + 6.0085\,\alpha$
12	$12\beta + .0030\,\gamma + 7.0030\,\alpha$
13	$13\beta + .0010\,\gamma + 8.0010\,\alpha$
14	$14\beta + .0003\,\gamma + 9.0003\,\alpha$
15	$15\beta + .0001\,\gamma + 10.0001\,\alpha$

Table 8 (Cont'd)

$\lambda = 10$

n	$E(C_n)$
1	$\beta + 9.0000\gamma$
2	$2\beta + 8.0005\gamma + .0005\alpha$
3	$3\beta + 7.0033\gamma + .0033\alpha$
4	$4\beta + 6.0136\gamma + .0136\alpha$
5	$5\beta + 5.0429\gamma + .0429\alpha$
6	$6\beta + 4.1100\gamma + .1100\alpha$
7	$7\beta + 3.2401\gamma + .2401\alpha$
8	$8\beta + 2.4603\gamma + .4603\alpha$
9	$9\beta + 1.7932\gamma + .7932\alpha$
10	$10\beta + 1.2511\gamma + 1.2511\alpha$
11	$11\beta + .8341\gamma + 1.8341\alpha$
12	$12\beta + .5309\gamma + 2.5309\alpha$
13	$13\beta + .3225\gamma + 3.3225\alpha$
14	$14\beta + .1869\gamma + 4.1869\alpha$
15	$15\beta + .1035\gamma + 5.1035\alpha$
16	$16\beta + .0547\gamma + 6.0547\alpha$
17	$17\beta + .0277\gamma + 7.0277\alpha$
18	$18\beta + .0134\gamma + 8.0134\alpha$
19	$19\beta + .0062\gamma + 9.0062\alpha$
20	$20\beta + .0028\gamma + 10.0028\alpha$

Table 8 (Cont'd)

λ = 15

n	$E(C_n)$				
1	β	+	14.0000 γ		
2	2β	+	13.0000 γ		
3	3β	+	12.0000 γ		
4	4β	+	11.0002 γ	+	.0002 α
5	5β	+	10.0011 γ	+	.0011 α
6	6β	+	9.0039 γ	+	.0039 α
7	7β	+	8.0115 γ	+	.0115 α
8	8β	+	7.0295 γ	+	.0295 α
9	9β	+	6.0670 γ	+	.0670 α
10	10β	+	5.1368 γ	+	.1368 α
11	11β	+	4.2553 γ	+	.2553 α
12	12β	+	3.4401 γ	+	.4401 α
13	13β	+	2.7077 γ	+	.7077 α
14	14β	+	2.0709 γ	+	1.0709 α
15	15β	+	1.5365 γ	+	1.5365 α
16	16β	+	1.1046 γ	+	2.1046 α
17	17β	+	.7687 γ	+	2.7687 α
18	18β	+	.5176 γ	+	3.5176 α
19	19β	+	.3371 γ	+	4.3371 α
20	20β	+	.2123 γ	+	5.2123 α
21	21β	+	.1293 γ	+	6.1293 α
22	22β	+	.0762 γ	+	7.0762 α
23	23β	+	.0435 γ	+	8.0435 α
24	24β	+	.0240 γ	+	9.0240 α
25	25β	+	.0128 γ	+	10.0128 α
26	26β	+	.0066 γ	+	11.0066 α
27	27β	+	.0033 γ	+	12.0033 α
28	28β	+	.0016 γ	+	13.0016 α
29	29β	+	.0008 γ	+	14.0008 α
30	30β	+	.0004 γ	+	15.0004 α
31	31β	+	.0002 γ	+	16.0002 α
32	32β	+	.0001 γ	+	17.0001 α

Table 8 (Cont'd)

$\lambda = 20$

n	$E(C_n)$				
2	2β	+	18.0000 γ		
3	3β	+	17.0000 γ		
4	4β	+	16.0000 γ		
5	5β	+	15.0000 γ		
6	6β	+	14.0001 γ	+	.0001 α
7	7β	+	13.0003 γ	+	.0003 α
8	8β	+	12.0011 γ	+	.0011 α
9	9β	+	11.0032 γ	+	.0032 α
10	10β	+	10.0082 γ	+	.0082 α
11	11β	+	9.0190 γ	+	.0190 α
12	12β	+	8.0404 γ	+	.0404 α
13	13β	+	7.0794 γ	+	.0794 α
14	14β	+	6.1455 γ	+	.1455 α
15	15β	+	5.2504 γ	+	.2504 α
16	16β	+	4.4069 γ	+	.4069 α
17	17β	+	3.6280 γ	+	.6280 α
18	18β	+	2.9250 γ	+	.9250 α
19	19β	+	2.3064 γ	+	1.3064 α
20	20β	+	1.7767 γ	+	1.7767 α
21	21β	+	1.3358 γ	+	2.3358 α
22	22β	+	.9795 γ	+	2.9795 α
23	23β	+	.7001 γ	+	3.7001 α
24	24β	+	.4876 γ	+	4.4876 α
25	25β	+	.3308 γ	+	5.3308 α
26	26β	+	.2186 γ	+	6.2186 α
27	27β	+	.1407 γ	+	7.1407 α
28	28β	+	.0883 γ	+	8.0883 α
29	29β	+	.0539 γ	+	9.0539 α
30	30β	+	.0321 γ	+	10.0321 α
31	31β	+	.0186 γ	+	11.0186 α
32	32β	+	.0105 γ	+	12.0105 α
33	33β	+	.0058 γ	+	13.0058 α
34	34β	+	.0031 γ	+	14.0031 α
35	35β	+	.0016 γ	+	15.0016 α
36	36β	+	.0008 γ	+	16.0008 α
37	37β	+	.0004 γ	+	17.0004 α
38	38β	+	.0002 γ	+	18.0002 α

Table 9: REFINEMENT OF VALUES

λ = 5

n	γ = α: $E(C_n)$	$\frac{E(C_n)}{n}$ (min)	γ = 5α: $E(C_n)$	$\frac{E(C_n)}{n}$ (min)
1	1 + 4.0134 α		1 + 20.0402 α	
2	2 + 3.0942 α		2 + 15.2826 α	
3	3 + 2.3436 α		3 + 11.0308 α	
4	4 + 1.8736 α		4 + 7.6208 α	
5	5 + 1.7546 α	1 + .3509 α	5 + 5.2638 α	
6	6 + 1.9866 α	1 + .3311 α	6 + 3.9598 α	
7	7 + 2.5110 α	1 + .3587 α	7 + 3.5330 α	1 + .5047 α
8	8 + 3.2442 α		8 + 3.7326 α	1 + .4666 α
9	9 + 4.1080 α		9 + 4.3240 α	1 + .4804 α
10	10 + 5.0444 α		10 + 5.1332 α	
11	11 + 6.0170 α		11 + 6.0510 α	
12	12 + 7.0060 α		12 + 7.0180 α	
13	13 + 8.0020 α		13 + 8.0060 α	
14	14 + 9.0006 α		14 + 9.0018 α	
15	15 + 10.0002 α		15 + 10.0006 α	

λ = 10

n	γ = α: $E(C_n)$	$\frac{E(C_n)}{n}$ (min)	γ = 5α: $E(C_n)$	$\frac{E(C_n)}{n}$ (min)
1	1 + 9.0000 α		1 + 45.0000 α	
2	2 + 8.0010 α		2 + 40.0030 α	
3	3 + 7.0066 α		3 + 35.0198 α	
4	4 + 6.0272 α		4 + 30.0816 α	
5	5 + 5.0858 α		5 + 25.2574 α	
6	6 + 4.2200 α		6 + 20.6600 α	
7	7 + 3.4802 α		7 + 16.4406 α	
8	8 + 2.9206 α		8 + 12.7618 α	
9	9 + 2.5864 α		9 + 9.7592 α	
10	10 + 2.5022 α	1 + .2502 α	10 + 7.5066 α	
11	11 + 2.6682 α	1 + .2426 α	11 + 6.0046 α	
12	12 + 3.0618 α	1 + .2553 α	12 + 5.1854 α	
13	13 + 3.6450 α		13 + 4.9350 α	1 + .3796 α
14	14 + 4.3738 α		14 + 5.1214 α	1 + .3658 α
15	15 + 5.2070 α		15 + 5.6210 α	1 + .3747 α
16	16 + 6.1094 α		16 + 6.3282 α	
17	17 + 7.0554 α		17 + 7.1662 α	
18	18 + 8.0272 α		18 + 8.0804 α	
19	19 + 9.0124 α		19 + 9.0372 α	
20	20 + 10.0056 α		20 + 10.0168 α	

Table 9 (Cont'd)

$\lambda = 5$

$\gamma = 10\alpha$ n	$E(C_n)$	$\frac{E(C_n)}{n}$ (min)	$\gamma = 20\alpha$	$E(C_n)$	$\frac{E(C_n)}{n}$ (min)
1	1 + 40.0737 α			1 + 80.1407 α	
2	2 + 30.5182 α			2 + 60.9892 α	
3	3 + 21.8898 α			3 + 43.6078 α	
4	4 + 14.8048 α			4 + 29.1728 α	
5	5 + 9.6503 α			5 + 18.4233 α	
6	6 + 6.4923 α			6 + 11.3593 α	
7	7 + 4.8105 α			7 + 7.3655 α	
8	8 + 4.3431 α	1 + .5429 α		8 + 5.5641 α	
9	9 + 4.5940 α	1 + .5104 α		9 + 5.1340 α	1 + .5704 α
10	10 + 5.2442 α	1 + .5244 α		10 + 5.4662 α	1 + .5466 α
11	11 + 6.0935 α			11 + 6.1785 α	1 + .5617 α
12	12 + 7.0330 α			12 + 7.0630 α	
13	13 + 8.0110 α			13 + 8.0210 α	
14	14 + 9.0033 α			14 + 9.0063 α	
15	15 + 10.0011 α			15 + 10.0021 α	

$\lambda = 10$

$\gamma = 10\alpha$ n			$\gamma = 20\alpha$		
1	1 + 90.0000 α			1 + 180.0000 α	
2	2 + 80.0055 α			2 + 160.0105 α	
3	3 + 70.0363 α			3 + 140.0693 α	
4	4 + 60.1496 α			4 + 120.2856 α	
5	5 + 50.4719 α			5 + 100.9009 α	
6	6 + 41.2100 α			6 + 82.3100 α	
7	7 + 32.6411 α			7 + 65.4210 α	
8	8 + 25.0633 α			8 + 49.6663 α	
9	9 + 18.7252 α			9 + 36.6572 α	
10	10 + 13.7621 α			10 + 26.2731 α	
11	11 + 10.1751 α			11 + 18.5161 α	
12	12 + 7.8399 α			12 + 13.1489 α	
13	13 + 6.5475 α			13 + 9.7725 α	
14	14 + 6.0559 α	1 + .4326 α		14 + 7.9249 α	
15	15 + 6.1385 α	1 + .4092 α		15 + 7.1735 α	
16	16 + 6.6017 α	1 + .4126 α		16 + 7.1487 α	1 + .4468 α
17	17 + 7.3047 α			17 + 7.5817 α	1 + .4460 α
18	18 + 8.1474 α			18 + 8.2814 α	1 + .4601 α
19	19 + 9.0682 α			19 + 9.1302 α	
20	20 + 10.0308 α			20 + 10.0588 α	

Table 9 (Cont'd)

$\lambda = 15$

$\gamma = \alpha$ n	$E(C_n)$	$E(C_n)$ n (min)	$\gamma = 5\alpha$ $E(C_n)$	$E(C_n)$ n (min)
1	1 + 14.0000 α		1 + 70.0000 α	
2	2 + 13.0000 α		2 + 65.0000 α	
3	3 + 12.0000 α		3 + 60.0000 α	
4	4 + 11.0004 α		4 + 55.0012 α	
5	5 + 10.0022 α		5 + 50.0066 α	
6	6 + 9.0078 α		6 + 45.0234 α	
7	7 + 8.0230 α		7 + 40.0069 α	
8	8 + 7.0590 α		8 + 35.1770 α	
9	9 + 6.1340 α		9 + 30.4020 α	
10	10 + 5.2736 α		10 + 25.8208 α	
11	11 + 4.5106 α		11 + 21.5318 α	
12	12 + 3.8802 α		12 + 17.6406 α	
13	13 + 3.4154 α		13 + 14.2462 α	
14	14 + 3.1418 α		14 + 11.4254 α	
15	15 + 3.0730 α	1 + .2049 α	15 + 9.2190 α	
16	16 + 3.2092 α	1 + .2006 α	16 + 7.6276 α	
17	17 + 3.5374 α	1 + .2081 α	17 + 6.6122 α	
18	18 + 4.0352 α		18 + 6.1056 α	
19	19 + 4.6742 α		19 + 6.0226 α	1 + .3170 α
20	20 + 5.4246 α		20 + 6.2738 α	1 + .3137 α
21	21 + 6.2586 α		21 + 6.7758 α	1 + .3227 α
22	22 + 7.1524 α		22 + 7.4572 α	
23	23 + 8.0869 α		23 + 8.2605 α	
24	24 + 9.0480 α		24 + 9.1440 α	
25	25 + 10.0256 α		25 + 10.0768 α	
26	26 + 11.0132 α		26 + 11.0396 α	
27	27 + 12.0066 α		27 + 12.0198 α	
28	28 + 13.0032 α		28 + 13.0096 α	
29	29 + 14.0016 α		29 + 14.0048 α	
30	30 + 15.0008 α		30 + 15.0024 α	

Table 9 (Cont'd)

$\lambda = 15$

$\gamma = 10\alpha$ n	$E(C_n)$	$\frac{E(C_n)}{n}$ (min)	$\gamma = 20\alpha$	$E(C_n)$	$\frac{E(C_n)}{n}$ (min)
1	1 + 140.0000 α			1 + 280.0000 α	
2	2 + 130.0000 α			2 + 260.0000 α	
3	3 + 120.0000 α			3 + 240.0000 α	
4	4 + 110.0022 α			4 + 220.0042 α	
5	5 + 100.0121 α			5 + 200.0231 α	
6	6 + 90.0429 α			6 + 180.0819 α	
7	7 + 80.1265 α			7 + 160.2415 α	
8	8 + 70.3245 α			8 + 140.6195 α	
9	9 + 60.7370 α			9 + 121.4070 α	
10	10 + 51.5048 α			10 + 102.8728 α	
11	11 + 42.8983 α			11 + 85.3613 α	
12	12 + 34.8411 α			12 + 69.2421 α	
13	13 + 27.7847 α			13 + 54.8617 α	
14	14 + 21.7799 α			14 + 42.4889 α	
15	15 + 16.9015 α			15 + 32.2665 α	
16	16 + 13.1506 α			16 + 24.1966 α	
17	17 + 10.4557 α			17 + 18.1427 α	
18	18 + 8.6936 α			18 + 13.8696 α	
19	19 + 7.7081 α			19 + 11.0791 α	
20	20 + 7.3353 α	1 + .3668 α		20 + 9.4583 α	
21	21 + 7.4223 α	1 + .3534 α		21 + 8.7153 α	
22	22 + 7.8382 α	1 + .3563 α		22 + 8.6002 α	1 + .3909 α
23	23 + 8.4775 α			23 + 8.9115 α	1 + .3875 α
24	24 + 9.2640 α			24 + 9.5040 α	1 + .3960 α
25	25 + 10.1408 α			25 + 10.2688 α	
26	26 + 11.0726 α			26 + 11.1386 α	
27	27 + 12.0363 α			27 + 12.0693 α	
28	28 + 13.0176 α			28 + 13.0336 α	
29	29 + 14.0088 α			29 + 14.0168 α	
30	30 + 15.0044 α			30 + 15.0084 α	

Table 9 (Cont'd)

$\lambda = 20$

n	$\gamma = \alpha$ $E(C_n)$	$\frac{E(C_n)}{n}$ (min)	$\gamma = 5\alpha$ $E(C_n)$	$\frac{E(C_n)}{n}$ (min)
1				
2	2 + 18.0000 α		2 + 90.0000 α	
3	3 + 17.0000 α		3 + 85.0000 α	
4	4 + 16.0000 α		4 + 80.0000 α	
5	5 + 15.0000 α		5 + 75.0000 α	
6	6 + 14.0002 α		6 + 70.0006 α	
7	7 + 13.0003 α		7 + 65.0018 α	
8	8 + 12.0022 α		8 + 60.0066 α	
9	9 + 11.0064 α		9 + 55.0192 α	
10	10 + 10.0164 α		10 + 50.0492 α	
11	11 + 9.0380 α		11 + 45.1140 α	
12	12 + 8.0808 α		12 + 40.2424 α	
13	13 + 7.1588 α		13 + 35.4764 α	
14	14 + 6.2910 α		14 + 30.8730 α	
15	15 + 5.5008 α		15 + 26.5024 α	
16	16 + 4.8138 α		16 + 22.4414 α	
17	17 + 4.2560 α		17 + 18.7680 α	
18	18 + 3.8410 α		18 + 15.5500 α	
19	19 + 3.6128 α		19 + 12.8384 α	
20	20 + 3.5534 α	1 + .1777 α	20 + 10.6602 α	
21	21 + 3.6716 α	1 + .1748 α	21 + 9.0148 α	
22	22 + 3.9590 α	1 + .1800 α	22 + 7.8770 α	
23	23 + 4.4002 α		23 + 7.2006 α	
24	24 + 4.9752 α		24 + 6.9256 α	1 + .2886 α
25	25 + 5.6616 α		25 + 6.9848 α	1 + .2794 α
26	26 + 6.4372 α		26 + 7.3116 α	1 + .2812 α
27	27 + 7.2814 α		27 + 7.8442 α	
28	28 + 8.1765 α		28 + 8.5293 α	
29	29 + 9.1078 α		29 + 9.3234 α	
30	30 + 10.0642 α		30 + 10.1926 α	
31	31 + 11.0372 α		31 + 11.1116 α	
32	32 + 12.0210 α		32 + 12.0630 α	
33	33 + 13.0116 α		33 + 13.0348 α	
34	34 + 14.0062 α		34 + 14.0186 α	
35	35 + 15.0032 α		35 + 15.0096 α	
36	36 + 16.0016 α		36 + 16.0048 α	
37	37 + 17.0008 α		37 + 17.0024 α	
38	38 + 18.0004 α		38 + 18.0012 α	

Table 9 (Cont'd)

$\lambda = 20$

$\gamma = 10\alpha$ n	$E(C_n)$	$\frac{E(C_n)}{n}$ (min)	$\gamma = 20\alpha$ $E(C_n)$	$\frac{E(C_n)}{n}$ (min)
2	2 + 180.0000 α		2 + 360.0000 α	
3	3 + 170.0000 α		3 + 340.0000 α	
4	4 + 160.0000 α		4 + 320.0000 α	
5	5 + 150.0000 α		5 + 300.0000 α	
6	6 + 140.0011 α		6 + 280.0021 α	
7	7 + 130.0033 α		7 + 260.0063 α	
8	8 + 120.0121 α		8 + 240.0231 α	
9	9 + 110.0352 α		9 + 220.0672 α	
10	10 + 100.0902 α		10 + 200.1722 α	
11	11 + 90.2090 α		11 + 180.3990 α	
12	12 + 80.4444 α		12 + 160.8484 α	
13	13 + 70.8734 α		13 + 141.6674 α	
14	14 + 61.6005 α		14 + 123.0555 α	
15	15 + 52.7554 α		15 + 105.2584 α	
16	16 + 44.4759 α		16 + 88.5449 α	
17	17 + 36.9080 α		17 + 73.1880 α	
18	18 + 30.1750 α		18 + 59.4250 α	
19	19 + 24.3704 α		19 + 47.4344 α	
20	20 + 19.5437 α		20 + 37.3107 α	
21	21 + 15.6938 α		21 + 29.0518 α	
22	22 + 12.7745 α		22 + 22.5695 α	
23	23 + 10.7011 α		23 + 17.7021 α	
24	24 + 9.3636 α		24 + 14.2396 α	
25	25 + 8.6388 α		25 + 11.9468 α	
26	26 + 8.4046 α	1 + .3233 α	26 + 10.5906 α	
27	27 + 8.5477 α	1 + .3166 α	27 + 9.9547 α	
28	28 + 8.9703 α	1 + .3204 α	28 + 9.8523 α	1 + .3519 α
29	29 + 9.5929 α		29 + 10.1319 α	1 + .3494 α
30	30 + 10.3531 α		30 + 10.6741 α	1 + .3558 α
31	31 + 11.2046 α		31 + 11.3906 α	
32	32 + 12.1155 α		32 + 12.2205 α	
33	33 + 13.0638 α		33 + 13.1218 α	
34	34 + 14.0341 α		34 + 14.0651 α	
35	35 + 15.0176 α		35 + 15.0336 α	
36	36 + 16.0088 α		36 + 16.0168 α	
37	37 + 17.0044 α		37 + 17.0084 α	
38	38 + 18.0022 α		38 + 18.0042 α	

Appendix 2

THE ADJOINT METHOD IN ANALOG COMPUTATION

by
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1. Introduction

In analog computation, it is frequently desirable to have available in explicit form the weighting function for a linear differential system. This function is defined to be the impulsive response of the system, and as such is a function of two variables. Thus it is easy to evaluate this function as a function of time for an impulse applied at a certain fixed time. However, it is quite often necessary to have the function available in the form of a response at a fixed time as a function of the time of impulsion. In order to evaluate the function in this way, it is convenient to use the adjoint system. The purpose of this appendix is to describe this use of the adjoint system.

We begin with a mathematical development of the theory of linear systems, in which the weighting function is defined. A mathematical exposition of the impulse response, or Dirac delta function, method is given. A serious effort has been made to describe the process by which the delta function is made mathematically rigorous. Further details may be found in references cited in Section 6 of this appendix.

The adjoint system of a linear differential system and the relation between the weighting functions of the two systems are described. This is followed by a treatment of the simulation of the adjoint system, and by a description of the procedures used in calculating the weighting function.

We use the letters t, T, and s, both with and without subscripts and primes, to denote the independent variable or variables ("time") under consideration, or particular values of these variables. The letter n is used to denote the dimension of the system, which remains constant throughout this appendix. The letters j, k are used to denote integer indices. All other lower case Latin letters denote n-dimensional (column) vectors. Upper case Latin letters denote n by n matrices. The notation $y(t)$ or $A(t)$ indicates that the components (entries) of y or of A are functions of t. The notation $\dot{y}(t)$ or $\dot{A}(t)$ denotes the derivative with respect to t, that is, the vector or matrix whose components are the derivatives of the components of $y(t)$ or $A(t)$. The notation $\lim_{k \to \infty} y_k = y$ or $\lim_{k \to \infty} A_k = A$ means that the components converge in the ordinary sense. Lower case Greek letters indicate real valued functions of the independent variable.

The notation A^* is used to indicate the transpose of the matrix A. Thus, if the components of A are α_{jk}, the components of A^* are β_{jk}, where $\beta_{jk} = \alpha_{kj}$. The letter I denotes the identity matrix.

2. The Weighting Function

Consider the linear differential system

$$\dot{x}(t) = A(t)\, x(t) + f(t) \tag{2.1}$$

subject to the initial condition $x(0) = 0$. If it is supposed that the function $f(t)$ does not vanish identically, the solution may be obtained by the method of vari- of parameters, due to Lagrange, from a fundamental system of solutions of the homogeneous system. Let $M(t)$ denote a solution of the homogeneous (matricial) system

$$\dot{M}(t) = A(t)\, M(t). \tag{2.2}$$

where $M(t)$ is subject to any non-singular initial condition. Then $M(t)$ is non-singular for all values of t. Let $u(t)$ be the solution of the system

$$M(t)\, \dot{u}(t) = f(t), \tag{2.3}$$

subject to the initial condition $u(0) = 0$. An easy computation shows that $x(t) = M(t)\, u(t)$ is the desired solution of (2.1). That is, for any t, the solution of (2.1) is given by

$$x(t) = \int_0^t M(t)\, M(s)^{-1} f(s)\, ds \tag{2.4}$$

Let a function $K(t, s)$, of two variables, be defined by the relations

$$K(t, s) = 0, \quad s > t, \tag{2.5}$$

$$K(t, s) = M(t)\, M(s)^{-1}, \quad s \leq t.$$

Then the formula (2.4) may be written as

$$s(t) = \int_0^\infty K(t, s)\, f(s)\, ds \tag{2.6}$$

(provided we are interested only in the case $t \geq 0$).

The function K(t, s) is called the weighting function, or Green's function, of the system (2.1). It is also called the impulsive response of the system (2.1). To see the reason for this last name, we must investigate the Dirac delta function.

3. Delta Functions

Let ξ_k be a sequence of Lebesgue-integrable functions on the real line having the following properties:

$$\begin{aligned} &(a) \quad \int_{-\infty}^{\infty} \xi_k(t)\, dt = 1 \;; \\ &(b) \quad \lim_{n \to \infty} \xi_k(t) = 0, \text{ if } t \neq 0 \;; \\ &(c) \quad \xi_k(t) = \xi_k(-t), \text{ for all } t \;; \\ &(d) \quad \xi_k(t) \geq 0 \text{ for all } t . \end{aligned} \tag{2.7}$$

For example, we may define ξ_k by $\xi_k(t) = \frac{k}{\sqrt{\pi}} \exp(-kt^2)$. Such a sequence is said to define a Dirac delta function, or an approximate identity in the convolution algebra of the real line. We shall always assume that the functions ξ_k are continuous (as in the example).

By standard methods in the theory of the Lebesgue integral, it can be shown that for any integrable function ϕ on the real line, the functions ψ_k defined by

$$\psi_k(t) = \int_{-\infty}^{\infty} \phi(s)\, \xi_k(t-s)\, ds$$

converge in mean to the function $\phi(t)$. If we require the function ϕ to be continuous on the whole real line, then the functions ψ_k will also be continuous.

The convergence of the functions ψ_k in the mean to the function ϕ holds for all ϕ simultaneously. However, convergence in mean does not imply pointwise convergence. If one is concerned only with one specified function ϕ, it is possible to select a subsequence of the functions ψ_k, and hence a subsequence of the functions ξ_k, for which pointwise convergence almost everywhere is also valid. If ϕ is continuous, then this convergence pointwise will be valid everywhere.

Suppose then that we have a specified continuous, integrable (in the sense of Lebesgue) function ϕ, and a sequence of functions ξ_k, continuous and satisfying the conditions (2.7), and satisfying the further condition that

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} \phi(s)\, \xi_k(t-s)\, ds = \phi(t)$$

for all t. Let T_o, T_1 be two fixed real numbers, and let μ denote the characteristic function of the interval $T_o \leq t \leq T_1$. That is, $\mu(t) = 1$ if $T_o \leq t \leq T_1$, and $\mu(t) = 0$ otherwise. Then we have

$$\int_{-\infty}^{\infty} \mu(s)\, \phi(s)\, \xi_k(t-s)\, ds = \int_{T_o}^{T_1} \phi(s)\, \xi_k(t-s)\, ds.$$

Thus it is easily seen that

$$\mu(t)\, \phi(t) = \lim_{k \to \infty} \int_{-\infty}^{\infty} \mu(s)\, \phi(s)\, \xi_k(t-s)\, ds =$$

$$\lim_{k \to \infty} \int_{T_o}^{T_1} \phi(s)\, \xi_k(t-s)\, ds.$$

Thus we have the result

$$\lim_{k \to \infty} \int_{T_o}^{T_1} \phi(s)\, \xi_k(t-s)\, ds = \phi(t), \text{ if } T_o \leq t \leq T_1; \tag{2.8}$$

$$\lim_{k \to \infty} \int_{T_o}^{T_1} \phi(s)\, \xi_k(t-s)\, ds = 0 \text{ otherwise.}$$

Similar results hold for the case $T_o = -\infty$ or $T_1 = \infty$.

The above remarks will also hold for the so-called "n-dimensional delta function." By this we mean a sequence of matrix functions $X_k(t) = \xi_k(t)\, I$, where ξ_k is a sequence defining an ordinary delta function.

Now suppose K(t, s) is the weighting function of the differential system (2.1), and suppose $X_k(t)$ defines an n-dimensional delta function. Suppose we choose any fixed value T of time. Then we have

$$K(t, T) = \lim_{k \to \infty} \int_{-\infty}^{\infty} K(t, s)\, X_k(T-s)\, ds \tag{2.9}$$

Suppose we have chosen $T \geq 0$. Then we can write this in the form

$$K(t, T) = \lim_{k \to \infty} \int_{-\infty}^{0} K(t, s)\, X_k(T-s)\, ds + \lim_{k \to \infty} \int_{0}^{\infty} K(t, s)\, X_k(T-s)\, ds$$

$$= \lim_{k \to \infty} \int_{0}^{\infty} K(t, s)\, X_k(T-s)\, ds$$

Thus we have, for $T \geq 0$,

$$K(t, T) = \lim_{k \to \infty} \int_{0}^{\infty} K(t, s)\, X_k(T-s)\, ds. \qquad (2.10)$$

Let us denote by $K_{k, T}(t)$ the function defined by

$$K_{k, T}(t) = \int_{0}^{\infty} K(t, s)\, X_k(T-s)\, ds \qquad (2.11)$$

A comparison of (2.11) with (2.6) shows that the function $K_{k, T}$ satisfies the differential system

$$\dot{K}_{k, T}(t) = A(t)\, K_{k, T}(t) + X_k(T-t) \qquad (2.12)$$

Then formula (2.10) expresses the relation

$$K(t, T) = \lim_{k \to \infty} K_{k, T}(t) \qquad (2.13)$$

This, then, gives us a theoretical method of evaluating K(t. T) in terms of solutions of the system (2.12), which is nothing more than a matrix version of the system (2.1). We shall discuss the problem of actually computing $K(t, T)$ by analog methods later.

4. Adjoint Systems

As indicated in section 1, it is often desirable to have the function $K(t, s)$ as a function of s for some fixed values of t. That is, given T, we wish to know $K(T, s)$. In this section, we shall indicate how the "adjoint system" of (2.1) enables us to do this.

Let us begin by introducing a pair of variables t', s', which are related to the variables s, t by the relations

$$t' = T - t, \quad s' = T - s, \tag{2.14}$$

where T is any (fixed) real number. Let us define a function L(t', s') by the relation

$$L(t', s') = K(s, t)^* , \tag{2.15}$$

where K is the weighting function of the system (2. 1). Now suppose $t' < s'$; then $t > s$, and hence $K(s,t) = 0$, and thus $L(t', s') = 0$, For $t' \geq s'$, we have $t \leq s$, and hence $L(t', s') = K(s,t)^* = \left[M(s)\ M(t)^{-1} \right]^* = M(t)^{-1*}\ M(s)^*$.

We now define a function z(t') by the formula

$$z(t') = \int_0^\infty L(t', s')\ g(s')\ ds', \tag{2.16}$$

where g(s') is any Lebesque integrable and continuous function on the real line. We may write this in the form

$$z(t') = \int_0^{t'} L(t', s')\ g(s')\ ds', \tag{2.17}$$

since $L(t', s') = 0$ for $t' < s'$. Let $\bar{z}(t')$ denote $\frac{d}{dt'} z(t')$. Then differentiation of (2. 17) yields

$$\begin{aligned} \bar{z}(t') &= \int_0^{t'} \frac{\partial}{\partial t'} L(t', s')\ g(s')\ ds' + L(t'. t')\ g(t') \\ &= \int_0^{t'} \frac{\partial}{\partial t'} L(t', s')\ g(s')\ ds' + g(t') \end{aligned} \tag{2.18}$$

since $L(t', t') = 1$. We now evaluate $\frac{\partial}{\partial t'} L(t', s')$ for $s' \leq t'$.

In this case, $s \geq t$, and $L(t', s') = M(t)^{-1*}\ M(s)^*$. Hence

$$\frac{\partial}{\partial t'} L(t', s') = \frac{\partial}{\partial t} \left[M(t)^{-1*}\ M(s)^* \right] \frac{dt}{dt'} = - \left[\frac{d}{dt} \left\{ M(t)^{-1*} \right\} \right] M(s)^* ,$$

since

$\frac{dt}{dt'} = -1$, by (2. 14). Now

$$\frac{d}{dt} \left\{ M(t)^{-1*} \right\} = \left[\frac{d}{dt} \left\{ M(t)^{-1} \right\} \right]^* .$$

It thus remains for us to evaluate $\frac{d}{dt}\left\{M(t)^{-1}\right\}$.

Since $M(t)\,M(t)^{-1} = 1$, then

$$\dot{M}(t)\,M(t)^{-1} + M(t)\,\frac{d}{dt}\left\{M(t)^{-1}\right\} = 0,$$

and hence

$$\frac{d}{dt}\left\{M(t)^{-1}\right\} = -\,M(t)^{-1}\,\dot{M}(t)\,M(t)^{-1}.$$

But $\dot{M}(t) = A(t)\,M(t)$, so that

$$M(t)^{-1}\,\dot{M}(t)\,M(t)^{-1} = M(t)^{-1}\,A(t).$$

Hence

$$\frac{d}{dt}\left\{M(t)^{-1}\right. = -\,M(t)^{-1}\,A(t).$$

Thus we may conclude that

$$\frac{\partial}{\partial t'}\,L(t', s') = A(t)^*\,M(t)^{-1*}\,M(s)^* = A(t)^*\,L(t', s'). \tag{2.19}$$

Then (2.18) becomes

$$\bar{z}(t') = \int_0^{t'} A(t)^*\,L(t', s')\,g(s')\,ds' + g(t'). \tag{2.20}$$

If we let $B(t') = A(t)^*$, we can rewrite (2.20) as

$$\bar{z}(t') = B(t')\,z(t') + g(t'). \tag{2.21}$$

It is clear that the weighting function of (2.21) is $L(t', s')$.

This system (2.21) may be referred to as the adjoint system of the system (2.1). It is more common to refer to a slightly different form of this as the adjoint system. Suppose we define a function $y(t)$ by the relation

$$y(t) = z(t'). \tag{2.22}$$

Then we have $\dot{y}(t) = \bar{z}(t')\frac{dt'}{dt} = -\,\bar{z}(t')$. Then equation (2.21) is transformed into

$$\dot{y}(t) = -\,A(t)^*\,y(t) + h(t). \tag{2.23}$$

The system (2. 23) is also referred to as the adjoint of the system (2. 1).

Thus if we need to evaluate K(t, s) as a function of s for a given, fixed value of t, we need only evaluate L(s', t') for the corresponding t' by the method of section 3. In particular, if we wish to evaluate K(T, s), we may evaluate L(s', o); then K(T, s) = L(s', o)*.

5. Analog Computation of the Weighting Function

Suppose the system (2. 1) has been simulated on an analog computer. Then it is possible to give a physical interpretation to the evaluation of K(t, T) by formula (2. 10). According to (2. 12), each column of the matrix $K_{k, T}(t)$ is a solution of (2. 1), subject to zero initial condition, where f(t) is replaced by the corresponding column of $X_k(T-t)$. Then (2. 13) shows that K(t, T) is the limit of the $K_{k, T}(t)$ as $k \longrightarrow \infty$. Since the j-th column of the sequence $X_k(T-t)$ consists of zero except in the j-th entry, where a delta function about T is defined, we may regard the j-th column of K(t, T) as the solution of the system (2. 1), where the forcing function f(t) is an impulse at time T in the j-th variable only. This leads us to the following procedure for evaluating K(t, T).

(1) Simulate the given linear system.

(2) Set all initial conditions at zero.

(3) Start the computer, and at time T apply an impulse in the first variable.

(4) Iterate the process, applying at T the impulse to each variable successively, and record.

(5) The n solutions so obtained form the columns of the matrix K(t, T).

Suppose that we wish to evaluate K(T, t). We may do this by evaluating the weighting function of the adjoint system. The procedure for doing this is as follows:

(1) Simulate the system (2. 21).

(2) Set all initial conditions at zero.

(3) Start the computer; at time 0 apply the impulse to the first variable, and record.

(4) Iterate the process, applying at time 0 the impulse to each successive variable, and record.

(5) The n-solutions so obtained form the rows of K(T, t), provided it is assumed that the computer has been running backward in time, starting at time T.

The principal problem encountered in this procedure is to simulate the adjoint system (2.21) when the computer is set to simulate the system (2.1). This is, however, easily done. The matrix B(t') of (2.21) and the matrix A(t) of (2.1) are related by the identity $B(t') = A(t)^*$. Hence the simulation of (2.21) can be derived from the simulation of (2.1) in the following two steps:

(1) The function generators for the components of A(t) are disconnected, and are reconnected adjointly. That is, where the generator for the component $\alpha_{jk}(t)$ has been multiplied by $\xi_k(t)$, it is now multiplied by $\xi_j(t)$.

(2) The function generators for the components of A(t) are set to start at time T, and run in reverse fashion to that in which they run in (2.1).

6. Remarks

Any standard text on differential equations discusses the solutions of linear systems and the method of variation of parameters. The adjoint system was developed by Bliss, and is discussed in:

"The Use of Adjoint Systems in the Problem of Differential Corrections for Trajectories" by G. A. Bliss, J. U. S. Artillery, li (1919), 296-311.

The Dirac delta function is standard methematical equipment of applied science today. It is discussed in

Theory of Servo-Mechanisms, by H. M. James, et al (N. Y., 1947), pp. 30 et seq.

A more rigorous and complete description of the meaning of the delta function will be found (in complete generality) in

Introduction to Modern Integration Theory, by I. E. Segal, mimeographed lecture notes, U. of Chicago, 1950.

A discussion of the uses of the weighting function for analog computation will be found in

Simulation Council Newsletter, May 1954.

"The Adjoint Method Applied to Guided Missile System Design," by R. R. Bennett, R. R. Favreau, I. Pfeffer, Typhoon Symposium III on Simulation and Computing Techniques, Part 2, U. S. Naval Air Development Center, Johnsville, Pa., 1953, pp. 74-111 (Confidential).

"An Application of Analogue Computers to Problems of Statistical Analysis," by J. H. Laning, Jr., and R. H. Battin, Project Cyclone Symposium II on Simulation and Computing Techniques, Part 2, N. Y., 1952, pp. 79-87.

Other references will be found in this latter paper. These papers indicate reasons for wishing to know K(T, t) rather than K(t, T), when T is fixed.

Appendix 3*

THE HIGH SPEED ANALOG COMPUTER IN SYSTEM ENGINEERING

by

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1. The Unique Role of High Speed Analog Computation

From its beginning in 1950, the M. I. T. Flight Control Laboratory has assumed responsibility for the design of a number of automatic systems for the guidance and control of air and water craft. At least three aspects of this work have been materially assisted by machine computation; (1) preliminary design investigations, (2) system evaluation by analog and simulator tests, and (3) system evaluation by actual performance tests.

The authors have found that each of three types of machine computation, high-speed analog, real-time analog, and digital, enjoys certain exclusive features which make its role unique in this field. Analog computation refers here to the study of an engineering system's behavior by studying instead the behavior of the approximate analog (or model) of that system (i. e., the approximate dynamical equations of the system and of the model are analogous). Digital computation, on the other hand, refers to numerical methods of approximating solutions to a set of dynamical equations. As the names imply, the principal difference between high-speed and real-time analog computers is in the speed of computation. The first device is capable of the order of ten different solutions per second or faster, while the latter seldom permits more than two or three per minute.

The exclusive feature of the high-speed computer, which makes its role unique, is the rapidity with which a whole series of different solutions may be scanned by the operator at will and either discarded or recorded. This feature requires certain equipment not needed in real-time operation for

*Adapted for the purposes of this volume by E. R. Spangler from M. I. T. Flight Control Laboratory Report FCL-7231-R5, 11 June 1954.

rapidly resetting the computer after each solution and inserting new driving functions and initial conditions as desired for the next solution. In contrast with real-time analog or digital computation, high-speed analog computation, as presently practiced at this laboratory, suffers from both poorer accuracy and poorer precision and entails greater difficulty with problems which are either nonlinear or have time-varying coefficients. In spite of these drawbacks, however, the high-speed advantages have made it unique in many applications where thousands of solutions are to be considered in short periods of time. For example, the role of the high-speed computer is unique for preliminary design investigations of automatic systems, such as for aircraft guidance and control. The primary goal of these exploratory studies is usually to establish for the entire system or for component systems functional relationships together with parameter values and tolerances which will permit not only practical engineering configurations, but also acceptable performance over extreme variations in conditions such as altitude, speed, loading, and atmospheric turbulence. In addition, only moderate accuracy is required in these investigations since they are preliminary, and an acceptable system is rarely affected appreciably by minor variations and uncertainties. Finally, for the purpose of preliminary design, simplifying assumptions can be made in a great many cases which permit approximating the system's behavior with equations having not more than one or two nonlinearities or time dependent coefficients. For similar reasons, this high-speed feature has also been used successfully at M.I.T. for demonstrating performance of a number of dynamical systems, a particular advantage in teaching. With the aid of a projection-type oscilloscope, which produces an enlarged image of the analog solutions on a motion-picture screen, such demonstrations have been conducted for groups as large as 80 people.

In contrast, the real-time analog computer performs best where only moderate numbers of solutions are required with good accuracy, and the digital computer where relatively few solutions are needed with very high accuracy. Consequently, real-time analog computers serve unique functions in such tasks as analog and simulator evalation studies of proposed or existing systems. Relatively good accuracy is important because the validity of early assumptions are to be verified (e.g., the effects of cross-coupling terms, nonlinearities, etc., may need to be checked), and only a moderate number of solutions is required because the suitability of the design is to be established as firmly as possible. Finally, digital computers serve control system engineering in the unique capacity of providing accurate solutions for use in checking either type of analog computer.

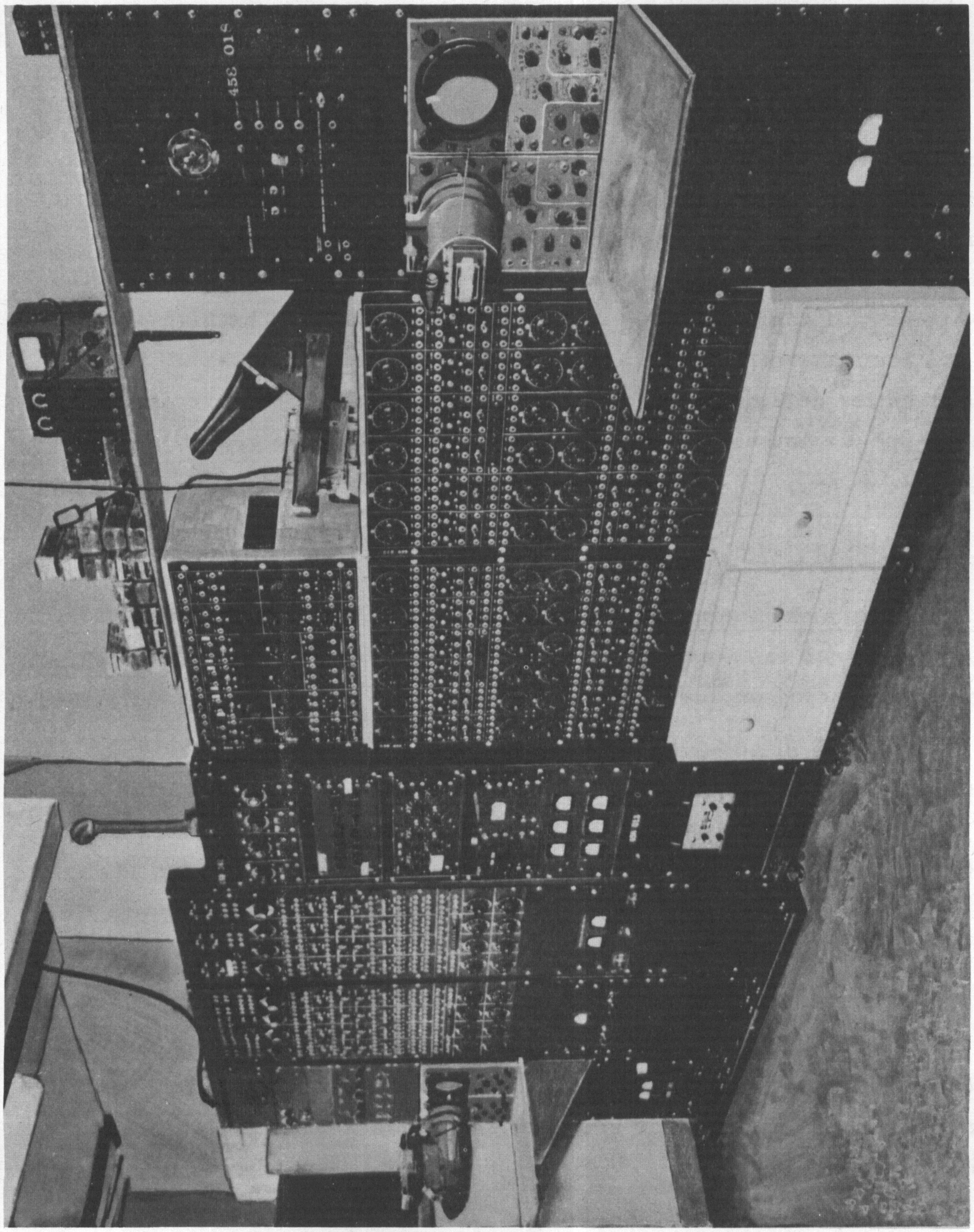


FIG. 8. VIEW OF M. I. T. FLIGHT CONTROL LABORATORY HIGH-SPEED ANALOG COMPUTER.

2. The M. I. T. Flight Control Laboratory High-Speed Analog Computer Facility

2.1 Physical Characteristics

The Flight Control Laboratory high-speed computer (see Fig. 8) is basically composed of 20 integrators, 20 amplifiers, 80 potentiometers, two master generators, and several power supplies. Also available are recording devices and several nonlinear elements. For details, see Section 6. Compared with real-time equipment of equal functionality the high-speed equipment requires much less floor space. The high-speed over-all accuracy of 5 to 15%, however, is markedly less than the order of 1% or better obtainable with real-time equipment. The precision of the high-speed computer is about the same as that of the real-time equipment.

2.2 Problem Types

2.2.1 Kinetic and Kinematic. Kinematics deals with the motion of bodies without reference to force or mass, while kinetics deals with the action of forces in producing or changing the motions.

The degree of accuracy usually required to solve kinematic problems necessitates the use of an extremely accurate real-time analog computer such as, for instance, the M. I. T. Flight Simulator.* It should be borne in mind that the high-speed computer was not developed for the purpose of solving kinematic problems since there are no resolvers available for resolving one coordinate system into another, and the accuracy of the computer is not considered to be sufficient for most problems of this type.

Purely kinetic problems, however, are easily handled by the high-speed computer, provided, of course, that the nature of the problem does not make it unsatisfactory for some other reason (for example, too many nonlinearities).

An example of the use of the high-speed computer in conjunction with the Flight Simulator is given in Section 7.

*The M. I. T. Flight Simulator was developed for the Bureau of Ordnance, United States Navy, by the Dynamic Analysis and Control Laboratory, M. I. T. This facility is currently operated by DACL for the Research and Development Command, United States Air Force.

2.2.2 Linear and Nonlinear Systems. The Flight Control Laboratory high-speed computer is capable of solving linear systems up to the 20th order, or with both master generators operating, it can solve simultaneously any two linear systems the sum of whose orders is equal to or less than 20. This laboratory has never had the need to solve a system of greater than 20th order on the high-speed computer because the systems of this complexity have been unsatisfactory for high-speed solution for other reasons.

A number of nonlinear elements are available for use with the high-speed equipment. These include six electronic multipliers as well as simulators of arbitrary functions, dead zones, hysteresis, and limiters. Any one of these components is easily handled by the high-speed computer operator. However, if a problem calls for more than about three nonlinearities, it is best studied on a good real-time computer, because more than two or three nonlinearities in a problem call for very careful watch of voltage levels throughout the system and frequent return to the check solution. Depending on the nature of the nonlinearities and the experience of the operator, a point will be reached where it is uneconomical to attempt to obtain information from the high-speed computer if a real-time machine is also available.

If the coefficients of the terms in the equations are not constant, then time-varying coefficients may be used. However, such problems are best handled by real-time or digital equipment for reasons similar to those given in the previous paragraph.

2.2.3 Preliminary and Exploratory Problems. Variational Problems. In the early stages of an extensive control problem, the high-speed analog can be used for a rapid survey of possible control systems in order to arrive at an early decision concerning the desirable feedback quantities.

When these have been established, the computer is useful for determining the relative importance of each of the physical parameters appearing in the problem, and this information may facilitate the study through considerable simplification of the equations.

The actual design of an aircraft control system is best carried out on a high-speed computer principally because of its ability to scan a large number of solutions rapidly. All anticipated flight conditions must be explored before a design is proposed. If the computer reveals that a certain combination of loading, altitude, and velocity leads to an unacceptable system, then it is a simple matter to find out what is necessary to improve the system and re-explore the range of flight conditions with a new set of control system parameters. Even though the high-speed computer is admittedly inaccurate, and may at times introduce as much as a 15% error, a good control system should nevertheless perform well even with large changes in all the aerodynamic parameters. In other words, a control system cannot be considered satisfactory if its performance is so marginal that it depends on the extra accuracy of simulation achieved by real-time computing equipment.

However, the final verification of any proposed design should usually be carried out by the more accurate real-time analog computation, by simulation with a flight table, or by actual flight tests where the number of solutions or tests will usually be small. The latter three investigate, for example, the non-linear and cross-coupling effects omitted from the high-speed analog studies and also test the validity of other assumptions which had to be made in the early investigations.

2.2.4 Input Functions. Each of the master generators associated with the high-speed equipment is capable of producing both step and pulse input functions. The step function, however, is the one most widely used for most systems work. The pulse is approximately square-shaped, and an adjustment on the master generator is available for changing the pulse width.

In addition, ramp and limited ramp input functions can be obtained by integrating step and pulse functions respectively.

The high-speed computer is not easily adapted to handling sinusoidal signals. To make steady-state frequency measurements it is necessary to adjust the phase of the incoming signal (obtained, for example, from a high frequency oscillator) so that there is no initial transient when the integrator clamps are opened, and also to adjust the repetition rate of the computer to be some sub-multiple of the driving frequency so that the solution always starts at the same section of the sine wave. At present, no such phase shifter is available; hence all solutions to sinusoidal inputs show starting transients. Because of the limited display time, this transient may take up most of this time so that there may not be enough of the steady-state solution visible to make a measurement. In addition, because of the limited range available for changing the repetition rate of the computer, it is not always possible to start every solution at the same portion of the sine wave, and even when it is possible, it is a tedious adjustment since both the master generator's repetition rate and the sweep speed of the oscilloscope must be simultaneously synchronized with the oscillator.

In general, sinusoidal steady-state measurements are convenient means for checking analog accuracy and for studying the behavior of dynamic systems. If sufficiently drift-free components were available, they could be unclamped for long enough periods to make such sinusoidal measurements possible.

To generate arbitrary inputs to the high-speed computing equipment, there are two units available: a function fitter and a function generator (see Section 6).

It has been found that high-speed simulation with random noise generators is possible if certain precautions are taken. Introducing noise having too high a frequency content has caused trouble from high frequency coupling between computer components whose individual shielding is not sufficient at very high frequencies. This difficulty can be overcome by choosing the computer time scale such that the troublesome, high frequency noise may be filtered out without affecting the results of the investigation.

2.2.5 Stable and Unstable Solutions. A solution is called stable if the dependent variable converges to a finite limit as the independent variable becomes infinitely large; otherwise the solution is unstable.

A good high-speed computer should be capable of presenting unstable solutions since the nature of the instability is an important piece of information. Not all high-speed computers have this ability; however, the basic circuitry of the F.C.L. computer was designed to have this advantage.

2.2.6 Flight Test Comparisons. Nearly every flight test supervised by this laboratory has first been checked out on the high-speed computer. This procedure has helped to avert possible unsuccessful flights. Unexpected results of a flight can also be partially explained by means of the computer.

The high-speed computer solutions have also been used comparatively with flight data to obtain further quantitative information on the correctness of aerodynamic coefficients based on theory and wind-tunnel tests.

2.3 Operators

When a problem is on the computer, the system engineer responsible for the problem operates the computer with whatever assistance is needed from the computer engineer. Operators who use the equipment frequently need little or no assistance, but those with limited experience sometimes require almost full-time assistance.

Several qualifications are desirable for machine operators. First, and most important, is a feeling for the physical situation represented by the analog. Secondly, he should have at least a basic knowledge of computer theory and be thoroughly familiar with the limitations of the machine he is using. Third, he should know how to apply all the usual methods for checking correct machine operation and solution accuracy. Finally, he should be blessed with patience and perseverance. The second and third of these in particular may be acquired during operator training.

The high-speed computer is an excellent device for use in training inexperienced operators primarily because they cannot cause the machine severe damage. Furthermore, problems can be set up and changed rapidly and the trainee can see the results of his manipulations immediately.

2.4 Programming, Machine Operation, and Data Handling

2.4.1 Preparation of Equations. The methods of preparing equations for the high-speed computation are essentially the same as for other types of analog computers. However, there are no resolvers available with the high-speed equipment; hence only aircraft problems which can be formulated and solved in aircraft coordinates are considered suitable for this machine.

Usually a computer circuit is prepared such that the differential equations of the circuit are term by term analogous to the equations to be solved. There is then a circuit voltage analogous to each variable in the equations. It is rarely the practice, on the other hand, to represent merely the system's transfer function because only one of the system's physical output variables would be obtained by this process. Furthermore, the coefficients of the transfer function are usually not simple functions of the system's parameters.

Whenever there is a clear advantage in normalizing the equations or in nondimensionalizing them, these will be done. Generalized design studies of control equipment for airplanes and other craft can probably best be accomplished with normalized or nondimensional equations, since these processes usually result in considerable saving in the total number of solutions required. However, when perhaps only a very few specific answers are sought regarding problems of a particular system, the dimensional and unnormalized forms of the equations are usually the most convenient to use.

Tabular values are prepared when these will facilitate the analog studies. An aircraft's aerodynamic parameters can be listed as a function of its speed and altitude. Resistor values for the relay-driven attenuators used in varying coefficients with time can be tabulated as functions of time.

2.4.2 Computer Circuit Diagrams. Computer circuit diagrams are always prepared in detail. This procedure facilitates checking the machine setup at any time and duplicating the setup at a future date. A typical example is given in Fig. 16. This example, together with some general precautions that should be observed in constructing computer diagrams, are discussed in Section 7.

2.4.3 Preparation of Checks. Gain-phase plots, root-locus plots, and numerical solutions (by Laplace transform techniques in linear cases, and the Runge-Kutta method in nonlinear cases) may be prepared in advance. The details of their preparation are discussed in the standard modern reference texts on servomechanisms and numerical analysis.

2.4.4 Preparation of Special Equipment. Special equipment is designed and constructed as necessary. Passive networks, clamping relays, and special attenuators are among the most common requests.

2.4.5 Computer Hook-Up Procedure. As shown in Fig. 8, all outputs and inputs are through phone jacks, thus precluding the use of patch boards. The use of phone jacks and plugs is necessary as an adjunct to the extensive shielding required in such wide band-pass equipment. In setting up a problem, therefore, patch cords are used for direct interconnection of components. Outputs of the machine may be viewed on a cathode ray oscilloscope.

Caution must be exercised in the use of patch cords to prevent introduction of difficulties. Excessive strain on the cords must be avoided to prevent breakage that might not be obvious; and as the center element of a phone jack has a tendency, if loose, to twist and short with the outside element, each jack should be checked for tightness before use.

2.4.6 Use of Checks. A problem should be plugged into the computer in the following order with the prescribed checks being made:

(a) Plug in first and second order lags, and simple lead-lag circuits without interconnection between these isolated circuits. Then applying a step input to each in turn, observe the output to determine if proper gains, time constants, and damping factors have been used. This procedure is also a check on proper operation of the machine.

(b) Complete plugging problem into the machine; then make a detailed wiring recheck against the computer diagram, retracing all inputs and outputs for all computer components used. The recheck is an important aid for preventing wiring errors.

(c) Check open loop and closed loop gains against calculated data.

(d) Check gain and frequency for two or more oscillatory conditions against gain-phase plots.

(e) Compare digitally computed check solution with the corresponding computer solution. If all computer components are operating properly, the computer should match the check solution closely. An example of such a comparison is given in Fig. 14.

Checks (d) and/or (e) should be applied to the machine before further work is done each day and should be repeated before leaving the machine to insure that the solutions obtained intermediately are correct. These checks should also be applied at any time that doubt regarding correctness of solution exists.

When errors do exist, physical interpretation of the results obtained as compared to the expected results, plus a logical application of this comparison will very often pin-point the source of the error.

2.4.7 Data Recording. While investigating a problem, the operator may observe several hundred solutions on the oscilloscope screen. Only those few which are pertinent to conclusions or are intended for use in future work need be recorded. A permanent record may be obtained either by tracing out the desired solutions on graph paper with the aid of an Oscillo-tracer or by photographing the solutions on the oscilloscope screen.

Date sheets should be used when photographing or tracing solutions from the oscilloscope screen. These sheets should indicate for each film or trace: (1) The computer setup used, (2) the system parameters, potentiometer designations, and settings used, and (3) the designation and magnitude of the input and output quantities. Typical data sheets are shown in Tables 10 and 11, and discussed in Section 7.

2.4.8 Problem Exploration Difficulties. In problem exploration, the voltage limits of amplifiers and integrators and the rate of discharge of integrator capacitors during the clamped period must be considered. Input levels must be kept low enough so that saturation does not occur at any point during a solution, and, to prevent base line rotation, the integrators should be balanced for the input levels used. Also, checks of the outputs with no input should be made occasionally to see if unbalance due to drift is producing an effect in the output.

2.4.9 Data Processing. In exploratory problems where simplified setups may be used, the information received from the high-speed analog computer is often used qualitatively to reach a decision about more detailed studies on a real-time computer. For most purposes, however, the use is quantitative (within the limitations of the accuracy of the computer and the values of the parameters used on the computer). Whatever the use, there is generally very little processing of the solutions needed.

Tracings may be labelled carefully, placed in manila folders, and filed in cabinets. Photos should each be identified by a code number. This code number is then referred to in the data sheets for complete information. The film strips may then be inserted in celluloid jackets (see Fig. 11 and Section 6) and appropriately labelled, and then filed in cabinets.

With the aid of data sheets and the formula for the calculation of scale factors, a technical artist may easily and readily reproduce the traces and films to any size with the proper scales. Proportional dividers are employed; and with photographs, a viewer is used.

2.5 Maintenance

Maintenance of the high-speed computer installation is essentially a matter of correction of failures as they occur, plus periodic balancing, calibration, and overhaul of the equipment. Balancing is scheduled weekly, while calibration and overhaul are annual. Following a failure, however, both balancing and calibration are done if necessary. The laboratory keeps a detailed record of all work done on the computer.

Maintenance problems and work are usually handled by the computer technician; although sometimes it is necessary to call in the computer engineer. The engineer should be well trained and experienced in the fields of servomechanisms, electronics, and analog computation. It is also helpful if he grasps easily the physical meaning of computer solutions. The technician should be thoroughly experienced in the fields of electronic circuit trouble-shooting, and repair.

3. Comparison of High-Speed and Real-Time Analog Computers

3.1 Cost

Comparison is made on the basis of computer operation for problems which are adaptable to both high-speed and real-time electronic analog computers.

Basic equipment for real-time computers is about twice as expensive as for high-speed computers. Accessory equipment, such as nonlinear components, are much more costly for the former.

The operational costs of the two types of computers are about the same, with the real-time computer using a little more electricity because of its power needs. Data handling costs are also about the same for both.

Since computer operational difficulties and parts replacement are more frequent for a real-time computer than for a high-speed computer, maintenance costs for the former are greater than those of the latter.

3.2 Speed of Solutions

The high-speed portion of the Flight Control Laboratory analog computer operates on a compressed time scale of 1600 to 1, so that individual solutions from a high-speed machine are 1600 times as fast as those on a real-time computer. Of course, either computer can be made to work faster or slower than this merely by changing the time scale of the integrators.

While individual solution times are in the ration of 1600 to 1, complete problem times are about in the ratio of one day of high-speed analog computer operation to one week of real-time operation. There are several reasons for this:

(1) To draw a computer hookup diagram takes about the same amount of time with either computer.

(2) While it is somewhat faster to set up the high-speed computer and change from one setup to another, these operations cannot be performed 1600 times as fast.

(3) Not every solution from the high-speed computer is actually used since it takes time for the operator to grasp the significance of the solutions. Thus, while the operator is thinking about a result, the computer is producing still more solutions. For a real-time computer, the operator's reaction time is of the same order of magnitude as the solution time.

(4) Visual inspection of high-speed solutions on the cathode ray oscilloscopes permits recording of only those solutions needed for permanent record, whereas all real-time solutions must be recorded.

3.3 Demonstrations

The high-speed computer, equipped with a projection type oscilloscope and screen, is ideal for demonstrations to large groups. To attempt to use a real-time computer for the same purpose would introduce the inconvenience of passing recorder outputs from one person to another.

While it is possible to use a real-time machine for demonstrations to small groups, more information can be presented in a shorter period of time with the high-speed analog computer.

3.4 Operator Training

Operator training for high-speed computers is considerably less than and faster than that for REAC equipment. Preliminary training for REAC can be done using high-speed equipment and is recommended because relatively less damage can be done to the latter.

3.5 Accuracy and Precision

Accuracy and precision of high-speed computers are one or more orders less than that of real-time computers.

3.6 Patch-Boards

Patch boards are impractical with high-speed equipment, while both practical and available for real-time computers.

3.7 Problem Types

A facility which is equipped with both high-speed and real-time computers can use each for the type of problem it can do best. For example, it is not economical to do exploratory problems on real-time equipment, nor is it advisable to attempt solutions of problems with a great many nonlinearities on a high-speed machine.

3.8 Checkout Difficulties

All analog equipment is subject to approximately the same checkout difficulties.

3.9 Open Loop Gain, Band Width, and Output Impedance

	HIGH SPEED	REAC*
Open loop gain:		
amplifiers	200	30,000,000
integrators	2,000	30,000,000
Bandwidth (half-power point):	250,000 cps	10,000 cps
Output impedance:	Approximately 2 ohms	Exact value not calculated but it is a couple of orders less than that of high-speed.

*Note: (Based on item 13 of the Bibliography)

4. Recommendations

On the basis of their personal experience with high-speed analog equipment, the authors make the following general recommendations regarding its selection and use in facilities engaged in control system engineering.

4.1 Need

In small installations where it is felt that the number of problems is small enough so that only one type of analog computer is necessary, the real-time equipment should probably be obtained in preference to the high-speed computer. However, if the demand for analog computer time is fairly heavy, then the acquiring of a high-speed computer is recommended. The time-saving element due to the high-speed computer's ability to scan a large number of solutions in a relatively short time (and in addition, possibly, its advantages in operator training and demonstration) would then justify its relatively low cost.

The size of the installation and the number and types of equipment obtained should be determined by such considerations as

(1) the highest order system with which the laboratory usually deals,

(2) the number of different problems which the laboratory may explore simultaneously.

(3) the type of problems and systems of equations that are customarily encountered in the laboratory's work.

For a typical installation, the high-speed analog facility might include three units, each capable of handling up to a tenth order system when used separately, and able to handle up to a 30th order system when used together. Suitable accessory equipment, such as some nonlinear components and/or time-varying coefficient equipment, would be included.

4.2 Selection of Computer Components

Selection of high-speed analog computer components should be governed by the following considerations, listed approximately in order of importance:

1. Ability to handle unstable solutions.
2. Freedom from drift.
3. Simplicity of balancing and calibration.
4. Ease of access for trouble-shooting.
5. Accuracy and precision.
6. Availability of accessory equipment.
7. Cost.

4.3 Personnel Requirements

The system engineer involved should oversee the problem when it is on the computer. His need for assistance from the computer engineer will vary inversely with his experience in operating the equipment.

For maintenance, one engineer and one technician should be adequate for up to a 50th order installation. Requirements for the computer engineer are thorough training and experience in the fields of servomechanism, electronics, and analog computation. Similarly, the computer technician should be experienced in the fields of electronic circuit trouble-shooting and repair.

4.4 Use as a Supplement to Other Computers

A high-speed computer is quite useful for exploring broad ranges of solutions and for indicating what problems are worthwhile for more careful study on the real-time computers.

Also, a problem engineer can learn a great deal about his system if he first studies it on a high-speed machine. In this way, he will use his time more effectively when he is ready to use the real-time equipment.

The high-speed computer is also quite useful for aiding in the setup of larger real-time machines which may be acting as aerodynamic simulators with actual control system components being tested on a flight table. Such setups are usually quite complex and any additional checks on the validity of the simulation are helpful. The high-speed computer can provide such checks and actually aid in the isolation of faulty hookup or equipment.

4.5 Machine Operation

It is recommended that detailed computer circuit diagrams be drawn for every problem and that a systematic procedure such as given in Section 2.4.6 be followed for putting the problem on the computer and checking it out.

4.6 Handling and Use of Data

Data may be recorded by means of tracing or photography. In either case, each solution should be identified so that complete information as to input and output quantities and the exact computer setup for the solution may be obtained from data sheets. It has been found desirable to file the film strips in celluloid jackets rather than in rolls. These solutions may be reproduced to any desired size by a technical artist, and may be used quantitatively within the limits of computer accuracy.

4.7 Maintenance

Complete records of machine operation, balancing, trouble-shooting, and repair should be kept. At first, balancing should be done frequently to acquaint the staff with any machine peculiarities. Then a regular maintenance schedule should be set up involving balancing, overhaul, power supply adjustment, etc. In the replacement of tubes, particular care should be exercised to prevent use of tubes that have not been properly aged. (Fifty hours' aging is recommended.) In a large installation, it may prove worthwhile to perform a statistical analysis of the maintenance data in order to determine the optimum maintenance procedure.

5.1 Basic Theory: Instrumentation Techniques. Loading Errors. Time Scales.

Reduced to fundamentals, the theoretical basis for electronic analog computation is the same for real-time and high-speed computers, voltages being used as the analogs of the variables in both cases. The principal differences between the two types of electronic computers lie in the methods of instrumenting the analogs, but even here great similarities exist.

Elementary instrumentation techniques can be illustrated by operation on a simple second order differential equation such as

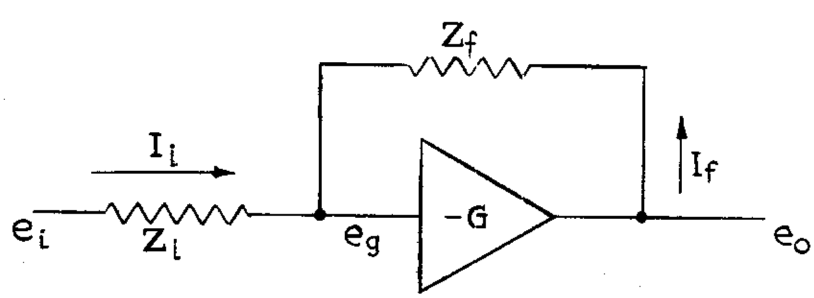
$$a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = f(t)$$

where x represents displacement and f(t) an arbitrary driving function. Rewritten in the form

$$a\ddot{x} = -b\dot{x} - cx + f(t)$$

it is readily seen that the velocity and displacement terms can be obtained by successive integrations of the acceleration term, which in turn is obtained by summation, with proper signs, of velocity, displacement, and some function of time.

Therefore, the operations to be instrumented for solution of the above linear equation are summation, integration, and inversion. Using feedback theory, we can assume an amplifier of gain G that draws no input grid current and yields a 180° phase shift. If we use an impedance Z_i between input and grid, and a feedback impedance Z_f from output to input grid, the currents through Z_i and Z_f are identical; and, as many be seen from the figure below



$$\frac{e_i - e_g}{Z_i} + \frac{e_o - e_g}{Z_f} = 0$$

where e_i, e_g, and e_o are input, grid, and output voltages respectively. Now,

$$e_o = -G\, e_g$$

so that we obtain:

$$\frac{e_o}{e_i} = \frac{-Z_f}{Z_i + \frac{1}{G}\,(Z_i + Z_f)}$$

Here the inversion is already instrumented, and, if the gain G is very large, the amplitude ratio is closely determined by the ratio of Z_i to Z_f. Thus, if both are resistances, a gain and inversion are accomplished.

$$\frac{e_o}{e_i} = -\frac{R_f}{R_i}$$

and, if Z_f is capacitive, i.e., $Z_f = 1/p\,C_f$,

$$\frac{e_o}{e_i} = -\frac{1}{p\,C_f\,R_i}$$

and integration, gain, and inversion result. Here p denotes the time-derivative operator d/dt. Similarly, differentiation requires resistive feedback and a capacitive input, but differentiators are not used if such use can be avoided because high pass characteristics give increased noise problems. If sufficiently high gains are not available for the accuracy desired, the feedback impedances can be made variable in order to permit calibration.

Summation necessitates the use of a separate Z_i for each input. In this case the transfer ratio becomes

$$\frac{e_o}{Z_{i_2}Z_{i_3}e_{i_1} + Z_{i_1}Z_{i_3}e_{i_2} + Z_{i_1}Z_{i_2}e_{i_3}}$$

$$= \frac{-Z_f}{Z_{i_1}Z_{i_2}Z_{i_3} + \frac{1}{G}\left(Z_{i_2}Z_{i_3}Z_f + Z_{i_1}Z_{i_3}Z_f + Z_{i_1}Z_{i_2}Z_f + Z_{i_1}Z_{i_2}Z_{i_3}\right)}$$

for these inputs. Here again a large G can make the error term negligible and the equation can be expressed as

$$e_o = -\frac{Z_f}{Z_{i_1}}e_{i_1} - \frac{Z_f}{Z_{i_2}}e_{i_2} - \frac{Z_f}{Z_{i_3}}e_{i_3}$$

If the coefficients of our equations are less than one, a simple resistive voltage divider (a potentiometer can be used for convenience) is adequate for simulation, but if a coefficient is greater than one, amplifier gains in conjunction with voltage dividers are used for simulation.

The voltage dividers or potentiometers are subject to error due to loading imposed by the following impedances. For maximum accuracy correction of loading errors is necessary. If the calculated potentiometer setting P_1, the total potentiometer resistance R_T, and the loading resistance R_L are known, than a correction δ_1 can be calculated, using the equation

$$\delta_1 = \frac{\left[1 - 2P_1 - \frac{R_L}{R_T P_1}\right] + \sqrt{1 - \frac{2R_L}{R_T P_1} + \frac{4R_L}{R_T} + \frac{R_L^2}{R_T^2 P_1^2}}}{2}$$

so that the corrected potentiometer setting P_c becomes

$$P_c = P_1 + \delta_1$$

When the reverse procedure is desired, and P_c, R_L, and R_T are known, a correction δ_2 can be obtained from

$$\delta_2 = \frac{R_T P_c^2 (1 - P_c)}{R_L + R_T P_c (1 - P_c)}$$

so that

$$P_1 = P_c - \delta_2$$

In integration and differentiation the gain of the units is not only dependent on the input and feedback impedances, but also on the time scale employed. In a real-time computer the time-scale K is a factor denoting the relation of real, or machine, time T_m to solution time T_s. The relationship may be stated as

$$T_m = KT_s$$

Restated as

$$T_s = \frac{1}{K} T_m$$

it is more obvious that for time scales greater than unity, problems are solved faster than for real time while for time scales less than unity, some fraction of a second of solution time is represented by each second of real time. As an example, take the integrator previously discussed. If we have a unity gain and unity time scale,

$$\frac{e_o}{e_i} = -\frac{1}{p}$$

but if K = 10 then

$$\frac{e_o}{e_i} = -\frac{10}{p}$$

and each second of T_m represents 10 seconds of T_s. Thus, for simplification, time scale and gain A_I could be included in the integration expression as

$$\frac{e_o}{e_i} = -\frac{KA_I}{p}$$

Solution of simultaneous, linear differential equations involves the same procedures as outlined above.

Nonlinear problems are also soluble but involve the use of specialized equipment.

Further details may be found in the references.

5.2 High-Speed Versus Real-Time Computers

The principal difference between high-speed and real-time computers is that the high-speed integrators and differentiators operate on a very fast time scale relative to the real-time units, and to permit observation of the high-speed solutions on a cathode ray oscilloscope, high-speed computers are made repetitive.

6. High-Speed Analog Computer Equipment

6.1 Definitions

Certain terms used in the text are defined here for convenience and to prevent confusion:

Drift — A change in the output voltage level, as a function of time, of a computer unit, assuming zero input. Drift may be due to component aging, temperature changes, or changes in the levels of the power supply voltages.

Unbalance — A deviation above or below zero voltage in the output of a computer unit, assuming zero input. Unbalance is usually due to drift.

Accuracy - Percentage error of full output from a computer unit, calculated for a particular instant of solution time.

Precision - Percentage error of full output of a computer unit, comparing repetitions of a particular instant of solution time.

The reader may wish to consult Figs. 8, 9, and 10, when reading the description of components given in the ensuing section.*

*The equipment shown in Fig. 9 was modified from a prototype design developed by the Instrumentation Laboratory, M.I.T.

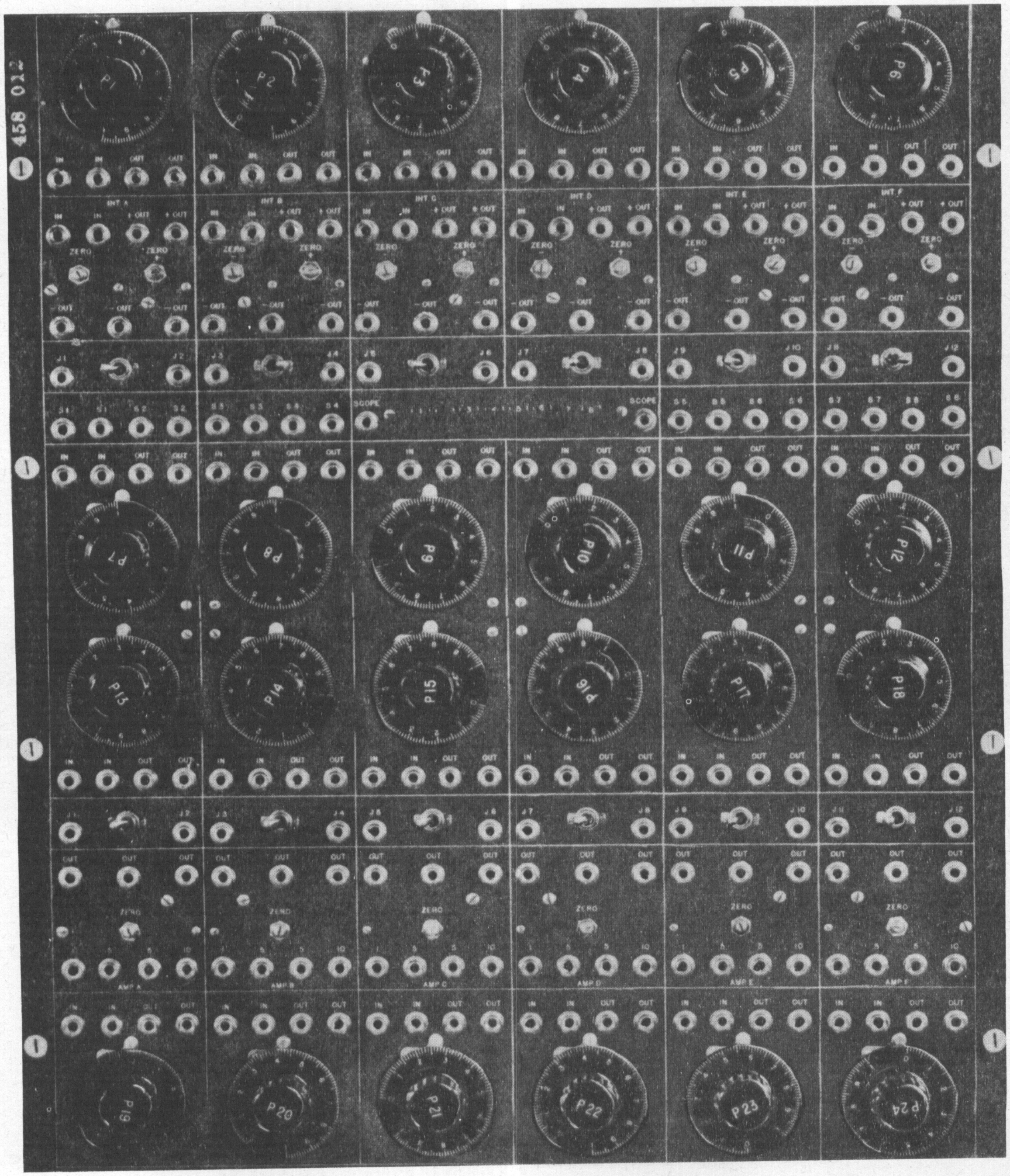


FIG. 9. CLOSE-UP VIEW OF LINEAR EQUIPMENT UNIT OF M. I. T. COMPUTER.

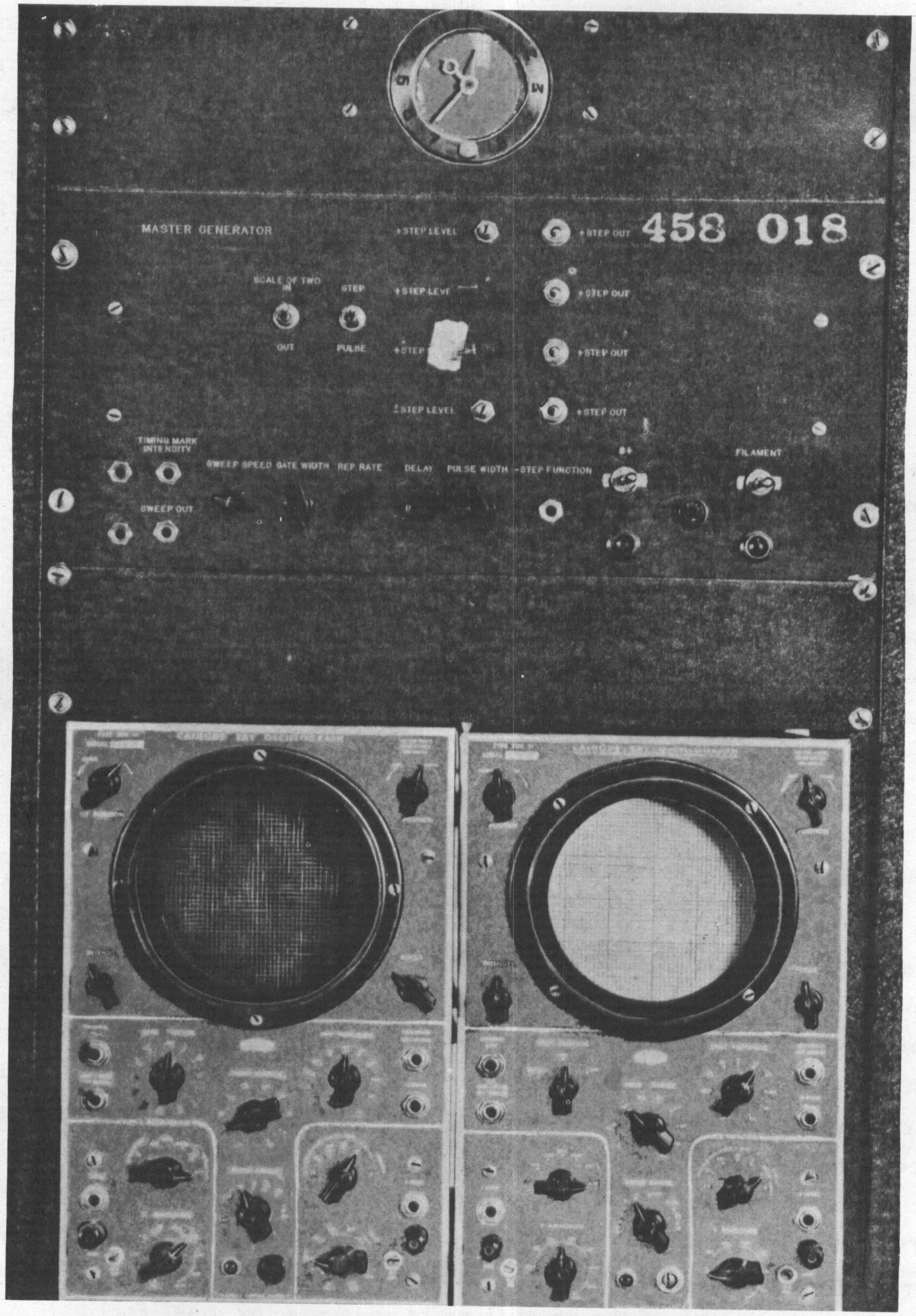


FIG. 10. CLOSE UP VIEW OF MASTER GENERATOR AND OSCILLOSCOPES OF M.I.T. COMPUTER.

6.2 Summing Amplifiers

Twenty summing and inverting amplifiers* each providing four inputs (1, 5, 5, 10) are included in the regular high-speed equipment. Twelve of the units have three outputs, the other eight have four outputs. Each has an output zeroing potentiometer located on the front panel.

The relatively small number of inputs to these amplifiers is partially due to limited panel space. As A.C. pickup is a serious problem in wide-band systems all input and output leads and patch cords must be shielded. Phone plugs are used as patch cord terminations and phone jacks for inputs and outputs. Thus, the physical size of the terminations limits the number used in a given space to considerably less than would be practical if pin jacks could be used as in REAC equipment. Physical size also militates against the use of pre-patch boards and consoles.

The impedance into each amplifier input is 100,000 ohms, and the output impedance is about 2 ohms. Open loop gain is approximately 200 and is obtained with a two-tube circuit: a pentode amplifier followed by a power pentode cathode follower. Any error caused by the low gain may be compensated for by adjustment of a potentiometer in the feedback channel. Phase shift of the output with respect to the input is 180 degrees throughout the flat portion of the pass band. The pass band is flat from zero to 30 kilocycles per second, with the half-power point occurring at 250 kilocycles per second. The output voltage range of the amplifiers is 100 volts, from plus 50 to minus 50 volts.

Maintenance and troubleshooting of the amplifiers is simplified considerably by the use of tubes that have been aged 50 hours before insertion into the machine. After aging, the tube characteristics change little with time. Therefore, recalibration is only occasionally necessary and drift is minimized. Calibration is necessary each time a new tube is inserted in the machine, but tube replacement is a relatively rare occurrence and is usually necessary only for a burned out filament. Drift results in output unbalance, but this is easily corrected using the zeroing potentiometer and a sensitive zero-center voltmeter in an output jack. Slight unbalance has little or no effect because capacitive coupling is used at the integrator inputs; therefore a weekly rebalancing of the amplifiers is adequate for most applications.

* Constructed by Laboratory for Electronics, Leon Street, Roxbury, Massachusetts, to M.I.T. specifications.

Accuracy of the amplifiers to within 1 percent can be maintained with reasonable ease of calibration, and precision of the order of 0.1 percent can be obtained after the computer temperature has stabilized.

6.3 Integrators

Twenty integrators are included in the equipment, each having two unity inputs (unity time scale). Twelve of the units have three negative (180° phase shift) and two positive (0° phase shift) outputs. The other eight units also have two positive but only two negative outputs and provide, in addition, an initial condition input and a special input that permits use as a dual input inverter when not needed as an integrator. The latter type is considered better because of the increased utility and convenience.

Input and output impedances, output voltage range, and frequency range are of the same order as those of the amplifiers except that very low frequencies are attenuated, with a half-power point at about 10^{-1} cycles per second, due to the use of capacitive input coupling. The open loop gain is high, about 25,000, and minimizes the need for a corrective element in the feedback loop. The circuit consists of a Miller input stage followed by a pentode amplifier and power pentode cathode follower. An inverter is included on the same chassis to provide the positive outputs.

Between solutions, the integrators are clamped by shunting the feedback capacitors with resistive elements so as to bleed the charge and yield amplifier action. The capacitive input coupling prevents D.C. feed-through during the clamped period. The shunting is accomplished by use of relays controlled by pulses from a Master Generator.

Maintenance and troubleshooting are approximately the same as for the summing amplifiers. Separate zeroing controls for integrator and inverter are provided on the front panel, and the procedure for zeroing the inverters is the same as that described for the summing amplifiers. Integrator balancing involves application of a step to the input with the integrator driven on a Scale of Two (see section 6.8), and observance of the ouput during the force-free solution; balance is achieved when the integrator output remains at zero during the force-free period. Integrator balance is a function of input voltage level. The principal cause of difficulty in these circuits is failure of the clamping relays. This failure is usually due to sticking contacts, and replacement is the proper corrective action.

The accuracy and precision of the integrators is of the same order as that of the amplifiers.

6.4 Scale Factor and Coefficient Potentiometers

The high-speed equipment includes 80 single-turn, 10,000 ohm, wire-wound potentiometers having a linearity of ± 1 percent of total resistance. The bandwidth of these units is comparable to that of the electronic equipment.*

Each unit has two jacks paralleled at the input and at the output, and the low side is grounded. The twin input jacks serve to provide an extra amplifier or integrator output when a potentiometer loads such a unit, while the twin outputs provide for two separate loads on the potentiometer.

At present, ten-turn potentiometers are not used in the high-speed equipment because of their limited band-width. With adequate band-width, however, there would be a distinct advantage to their use due to the improvement in resolution over the single-turn units.

The accuracy and precision of the potentiometers in use is about ± 1 percent. This figure suffers, however, at less than full scale settings, from errors introduced by loading. In most practical cases this error is not large and application of a correction is not feasible because of potentiometer resolution, or not expedient because of errors introduced in the problem by simplifying assumptions. Large errors should always be corrected, and, for maximum accuracy, all errors must be corrected if possible.

Maintenance and troubleshooting of potentiometers involves periodic calibration and linearity checks, and occasionally a potentiometer must be replaced because of a broken winding.

6.5 Patch Cords, Multiple Jacks, and Plug-In Attenuators

About 200 patch cords in five different lengths are available for use with the high-speed equipment. They are made up with heavy coaxial cable terminated by standard phone plugs. These phone plugs sometimes give trouble because when the center conductor of the plug loosens, the solder lug inside the unit tends to twist and short-circuit the cord. Consequently, cords must be checked frequently to prevent this.

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*Technology Instrument Corporation, Waltham, Massachusetts, Type RV 3-8 precision potentiometers.

To facilitate multiple connections when needed, two small boxes holding four paralleled jacks have been constructed, and, for attaining less than standard amplifier or integrator gains, several plug-in resistive attenuators with gains of 1/5 or 1/10 are available. Occasionally, a need for extra summations can be met by use of resistive summers having a gain of 1/10. With some care these resistance networks can be made with an accuracy and precision of better than 1 percent.

6.6 Interconnect and Switch Jack Units

Four switch jack units, each having 8 twin input jacks and 2 output jacks are provided, permitting push button selection to the output of any of the eight inputs. The principal use of these units is rapid selection of variables for the output equipment.

Also available are 24 pairs of jacks (see Fig. 9) with one jack of each pair above and one below the middle rows of potentiometers. Each pair of jacks is paralleled, with the cable connections behind the panel. The principal purpose of these jacks is to minimize the number of patch cords crossing the panel in front of the middle rows of potentiometers, thus maintaining potentiometer accessibility.

6.7 Accessory Equipment

The installation has several accessories, including four unity gain lag networks having variable time constants, two differentiators, having two unity gain (unity time scale) inputs each, four hysteresis networks, three limiter networks, three dead zone networks, a fifth order function generator, * a function fitter utilizing linear segments, seven electronic multipliers, a ten-channel time-varying coefficient or function unit that utilizes automatic sequential switching to control an arbitrary resistive divider network.

Most of the accessory electronic equipment is subject to severe drift, but, when unbalance due to drift is present, its effects can be minimized or eliminated altogether by clamping. Clamping is accomplished by inserting a capacitor between the nonlinear unit and the following input. The capacitor is grounded between solutions by a

*George A. Philbrick Researches, Inc., 230 Congress Street, Boston 10, Massachusetts. Units: K3-H (2 each), K3-B (1), K3-Z (1), K4-FG (1), K4-FF (1), K4-MU (1).

relay controlled by the master generator. F.C.L. real-time equipment is so constructed that, if necessary, it may be converted to high-speed operation. When used in this manner an extra 16 integrators and 32 amplifiers (all chopper-stabilized) are made available.

6.8 Master Generator

Having two master generators serves several purposes. It provides variable amplitude, variable width pulse and step functions for use as problem driving function, and a pulse for driving the relays to unclamp the integrators. A variable time delay is provided so that the input functions may lag the relay drive, and the relay drive pulse can be varied by a gate in order to control the solution period. Solution repetition rate is variable, and a variable speed sweep circuit is provided for cathode ray oscillographs to make them independent of their own sweep speed variations.

Timing marks are available for insertion into cathode ray tube (CRT) displays by means of Z-axis intensity modulation, the modulating pulses being generated by a second order resonant circuit driven at line frequency. Forty marks per solution are inserted, spaced at four second intervals in computer time for a unity time scale.

Two types of CRT display are available. The first, called Scale of One, is produced by applying an input to the problem each time the integrators are unclamped. This yields a trace of the solution without a baseline. The second, called Scale of Two, alternates between applying an input function and zero input for consecutive integrator operations. This yields consecutive traces of solution and base line which appear as a unified picture on a long persistence screen.

The circuits used in these units are well known, having been individually discussed in many texts on electronic circuits. The only difference is that here all circuits are synchronized. The arrangement of the relay drive permits use of as many integrators as desired in one problem, or in two problems if two master generators are available, or in two problems if only one master generator is available and the time scales are identical.

Precision of the driving functions is of the order of $\pm$ 0.1 percent while accuracy is a function of measurement. Accuracy and precision of the timing marks are unknown, but are as good as the accuracy and precision of the frequency of the line voltage supplied the laboratory and are considered more than ample for the laboratory's purposes.

Maintenance and troubleshooting are minor. Replacement of burned-out tubes when necessary, occasional recalibration, and a quarterly check appear to be adequate.

6.9 Power Supplies and Distribution

Each unit of the installation requires a 6.3 volt A.C. filament supply, + 300 volts at 0.5 amp. D.C.,* -300 volts at 0.5 amp D.C.,** and + 100 volts at 0.2 amp D.C.* The number of supply units that this laboratory has on hand are three, two, and two, respectively, for the latter three types. Regulation of all supplies must be of a high order to minimize drift in the computing elements, and the ouput impedance must be very low. To maintain the low output impedance much heavier wire than normal must be used for distribution.

Circuit resonance in the D.C. power supplies must occur at some frequency far below the computer repetition rate if power supply and computer stability is to be maintained. When the resonant frequency of a power supply falls near the computer repetition rate it is possible for computer oscillations at the resonant frequency to drive the supply hard enough to cause severe damage.

6.10 Recording Equipment

Direct visual observation of computer solutions is obtained on the screens of cathode ray oscillographs CRO***. Two sets of oscilloscopes are available, each set consisting of one CRO with a long persistence screen (P-7 phosphor) for visual observation, and one with a high actinic, short persistence screen (P-11 phosphor) for photography (using a blue filter). Each unit is provided with an illuminated, rectangular-coordinate grid.

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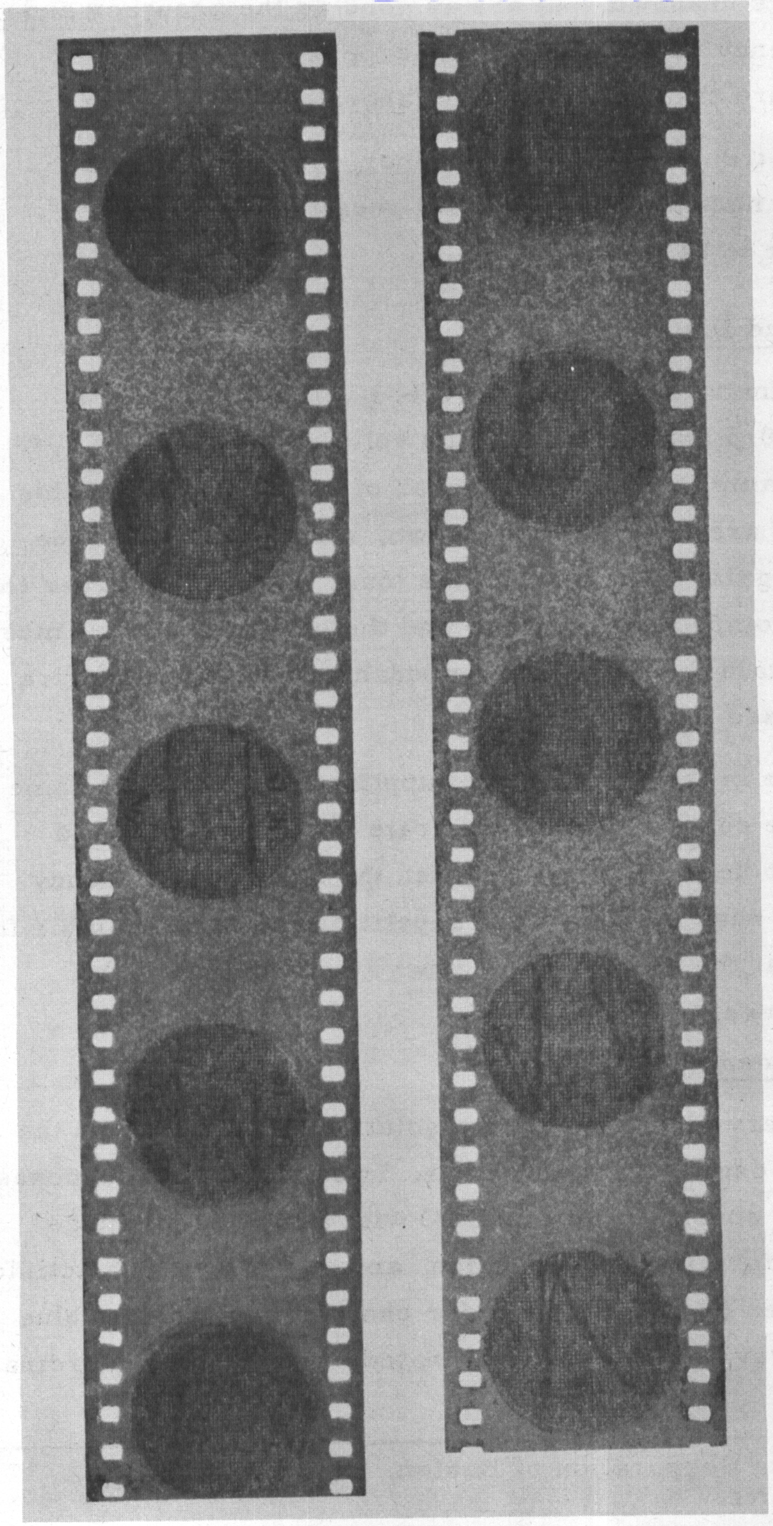
*Associated Engineering Corporation of Boston, 65 Kent Street, Brookline, Massachusetts.

**Polarad Electronics Corporation, 100 Metropolitan Avenue, Brooklyn 11, New York.

***Allen B. Dumont Laboratories, Inc., Instrument Division, 1500 Main Avenue, Clifton, New Jersey. Units: 304-H(4), 297(1), 296(1), 321(1), 314-A(1).



Missile Pitch Systems: Flight Test Conditions.
December, 1952.

FIG. 11. CELLULOID JACKET AS USED FOR STORAGE OF ANALOG SOLUTIONS ON 35mm FILM.

For permanent records an oscillo-tracer (a semi-transparent, mounted mirror)** is available to simplify manual tracing of display, but most records are made photographically. The installation has three cameras for this purpose; one is a finished-print camera* utilizing the Polaroid-Land process, for use when only a few pictures are needed; second is a 35 mm oscillograph-record camera* for use when numerous records are needed, and last is a continuous motion record camera* for use in recording high-speed transients and effectively lengthening the observed trace. Also available (on loan) is a dual purpose, single frame or continuous motion camera* which can be used to supplement other equipment when more than one problem is running.

Films are usually placed in celluloid jackets*** and filed for storage (see Fig. 11). Reading of films is facilitated by use of projection type viewers*** available in the laboratory.

Accuracy and precision of CRO displays depend to a large extent on the oscilloscope used. Assuming a CRO of high quality in peak condition, precision of the order of $\pm$ 0.1 percent can be obtained and accuracy of the order of $\pm$ 1 percent is feasible, but $\pm$ 5 percent is more likely.

Finally, an RMS Meter**** for use in noise measurements is available, the unit being accurate to with $\pm$ 2 percent.

7. Examples

C.1 Supplement to Flight Simulator

In a typical case, the M.I.T. Flight Simulator was used for the purpose of evaluating the operational characteristics of the pitch-system aircraft hardware. The kinematics of craft-trajectories and the aerodynamics of the aircraft were simulated. Comparison of the transient response of pitch, yaw, and roll control systems as obtained with the

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*Allen B. Dumont Laboratories, Inc., Instrument Division, 1500 Main Avenue, Clifton, New Jersey. Units: 304-H(4), 297(1), 296(1), 321(1), 314-A(1).

**R. A. Waters, Inc., 4 Gordon Street, Waltham, Massachusetts.

***Film-N-File, Inc., 330 West 42 Street, New York, New York.

****Reed Researches, Inc., Washington, D. C.

high-speed analog computer was made with the response given by the Flight Table simulation. An example of such a comparison for the pitch system is shown in Fig. 12.

7.2 Comparison with Actual Flight Test Data

A three-axis autopilot for a craft was designed on the basis feedback theory based on sinusoidal steady-state performance computations and on high-speed analog computer solutions, both of these for a single axis at a time. After the craft was tested, transients from the flight records were compared with corresponding solutions from the high-speed analog computer. (See Fig. 13.) Since these were in good agreement, it was concluded that under appropriate circumstances it was possible to use this method as a basis for design and that the corresponding performance could be predicted in many instances.

7.3 Comparison of Digital and High-Speed Analog Solutions.

A study of a pitch attitude autopilot with control surface rate-limiting was undertaken. Because of this nonlinearity, it was decided to obtain an analytic solution for the check. In nonlinear cases, the increased complexity of analog operation and the lack of intuitive interpretation require a greater number of check solutions. Such a comparison is shown in Fig. 14, where the digital solution was obtained using the Runge-Kutta method.

7.4 Complex System

A pitch accelerometer autopilot was proposed for a pilotless aircraft It was desired to evaluate this system, including the effects of body bending. A program of investigation of this ninth order system at various altitudes and Mach numbers was undertaken on the high-speed analog computer. Considering the numerous conditions and the complexity of the problem, the investigation was successfully completed in a relatively short time.

7.5 Equations - Functional and Computer Diagrams

The functional diagram of a typical proposed control system for a pilotless aircraft is given in Fig. 15. In order to investigate this system on

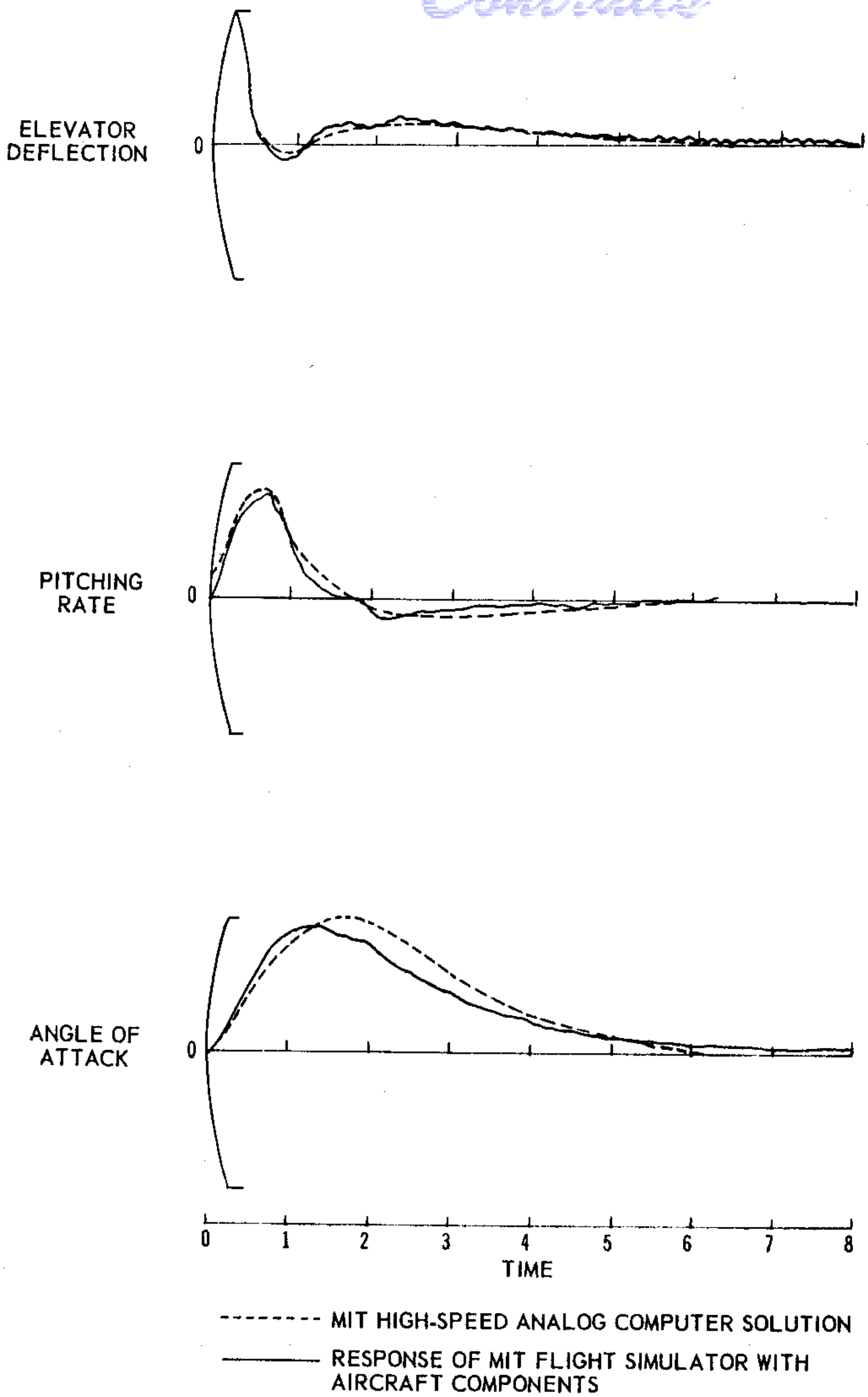

FIG. 12. HIGH-SPEED COMPUTER CHECK ON VALIDITY OF FLIGHT TABLE SIMULATOR TESTS OF ACTUAL AIRCRAFT COMPONENTS.

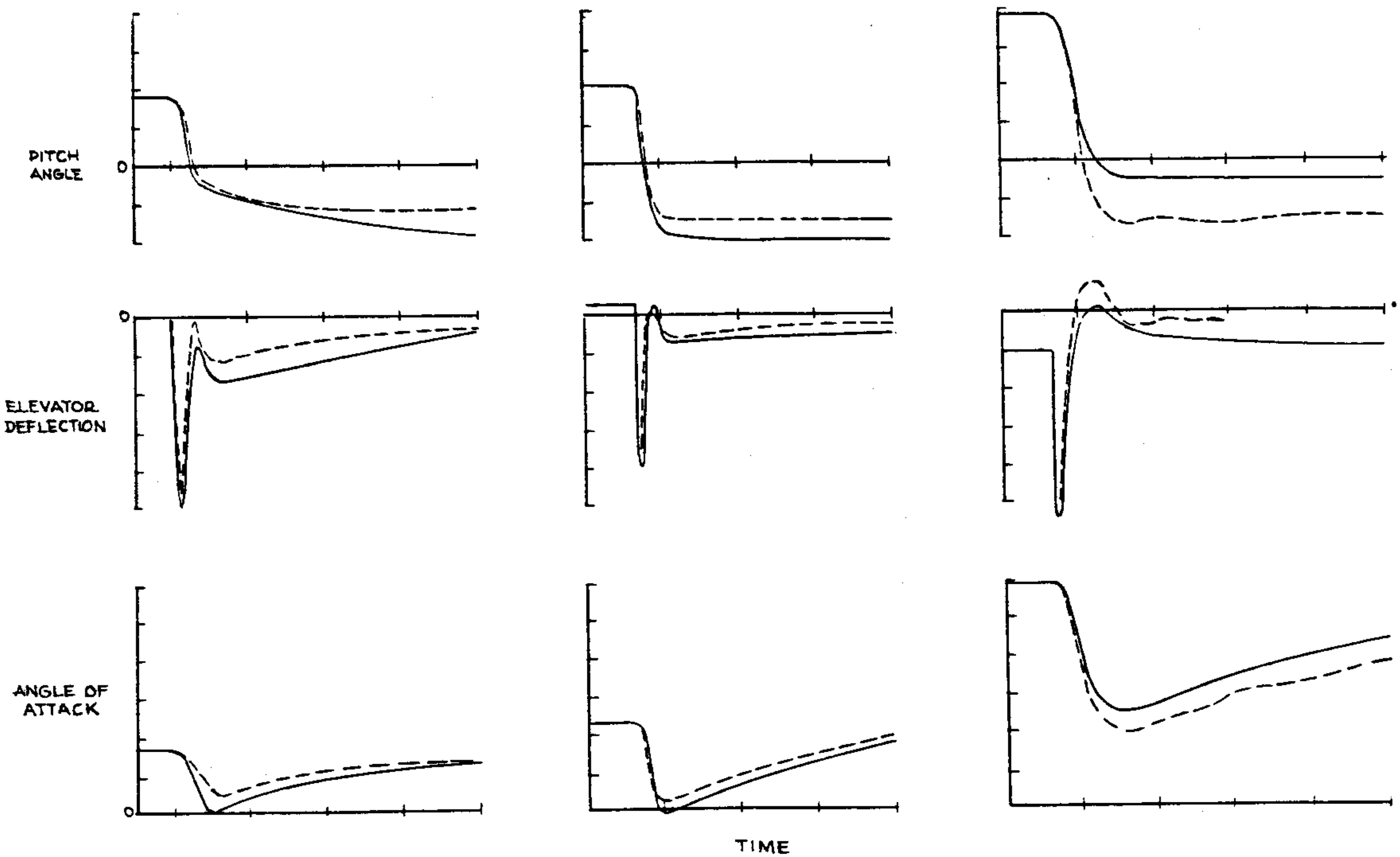

FIG. 13. COMPARISON BETWEEN HIGH-SPEED ANALOG SOLUTIONS USED TO PREDICT PERFORMANCE OF AIRCRAFT PITCH ORIENTATION CONTROL SYSTEM WITH ACTUAL PERFORMANCE. SOLID LINE: HIGH-SPEED ANALOG COMPUTED DATA. BROKEN LINE: DATA FROM TELEMETERED FLIGHT RECORDS.

the high-speed analog computer, we might take the equations of the system to be the following:

For the hydraulic servo:

$$\frac{\delta}{e_c - e_p} = S_{HS} \frac{1}{\frac{p^2}{\omega^2_{HS}} + \frac{2\xi_{HS}}{\omega_{HS}} p+1}$$

For the aerodynamics of the aircraft itself:

$$p_\alpha - A_\alpha - W_y = - B_\delta$$

$$C_\alpha + pW_y + DW_y = E_\delta$$

For the integrating gyro:

$$\frac{e_{IG}}{W_y} = S_{IG} \frac{1}{p(\tau_{IG} p + 1)}$$

For the signal modifier:

$$\frac{e_P}{e_{IG}} = S_P \frac{\alpha_d \tau_d p + 1}{\tau_d p + 1}$$

We may then proceed to draw the computer circuit diagram. The general techniques for constructing the diagrams for a high-speed analog computer are the same as those for a real-time computer. A few precautions should be mentioned, however. A lead going from a potentiometer to a potentiometer should be avoided if at all possible, since it introduces a gross error and necessitates a correction for loading error. Similarly, cascading of more than two amplifiers should be avoided, since the internal noise of the amplifiers is additive and may be amplified, so that with cascading of several amplifiers it is possible that the noise generated may be of greater amplitude than the solution. While attenuators may be used, their use should be minimized for greater accuracy.

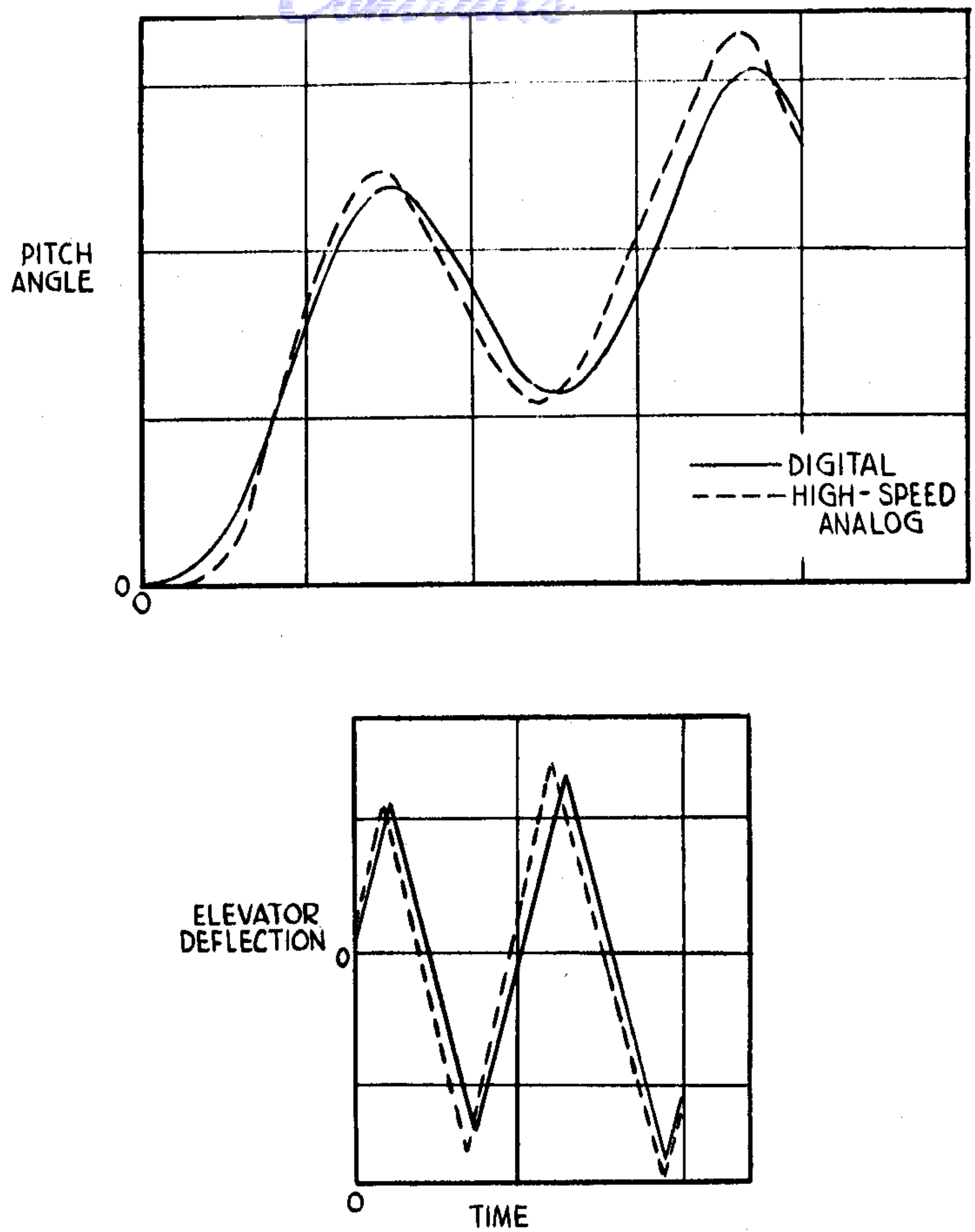

FIG. 14. COMPARISON BETWEEN DIGITAL SOLUTION AND HIGH-SPEED ANALOG SOLUTION SHOWING UNSTABLE RESPONSE OF A PITCH ORIENTATION CONTROL SYSTEM CAUSED BY DEFLECTION-RATE LIMITS OF THE ELEVATOR SERVO.

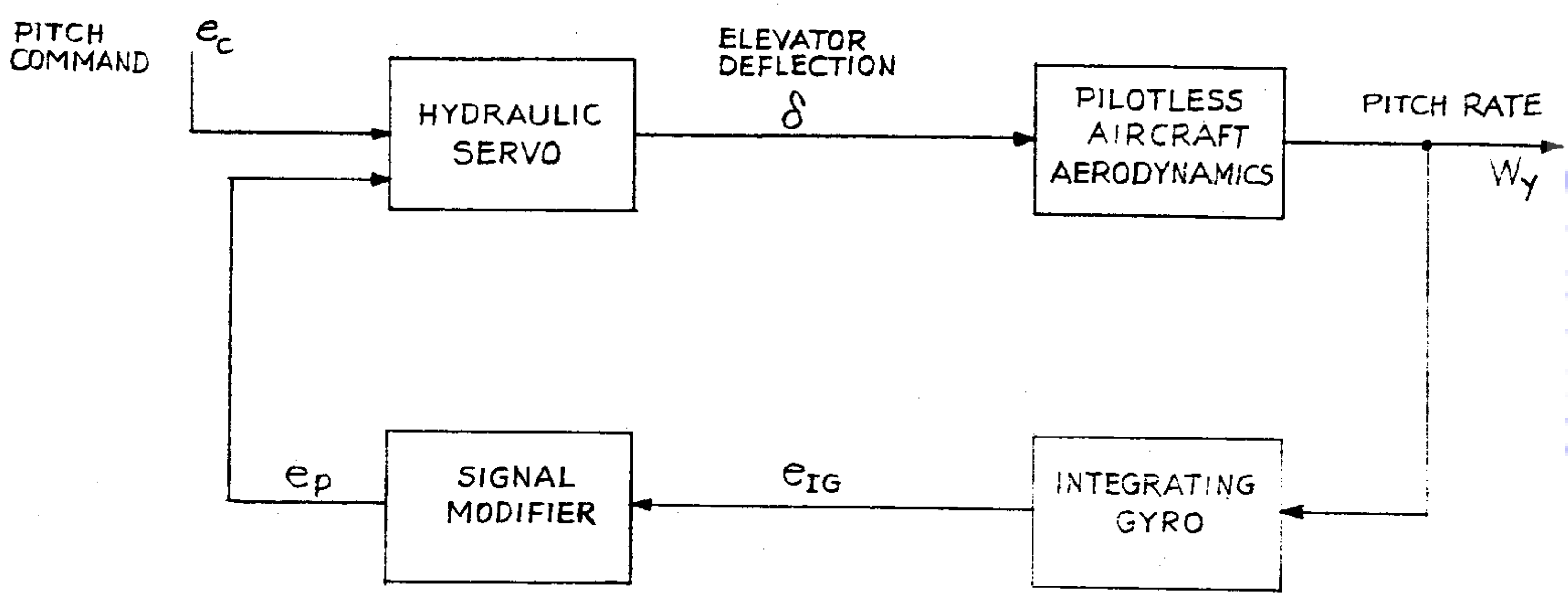

FIG. 15. FUNCTIONAL DIAGRAM OF AIRCRAFT PITCH ORIENTATION CONTROL SYSTEM.

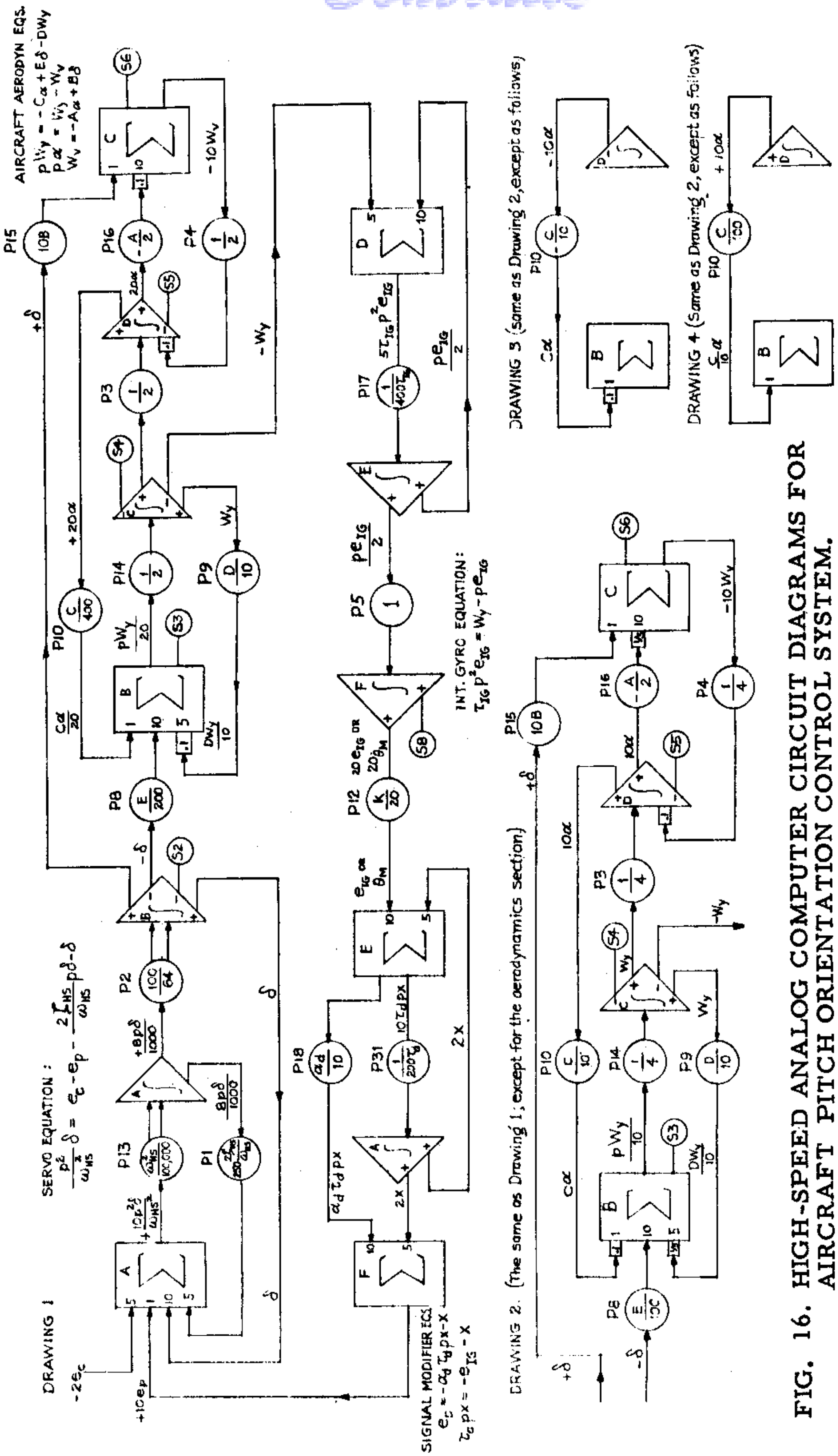

FIG. 16. HIGH-SPEED ANALOG COMPUTER CIRCUIT DIAGRAMS FOR AIRCRAFT PITCH ORIENTATION CONTROL SYSTEM.

NOTES (1) ALL INTEGRATORS EQUIVALENT TO $\frac{40}{p}$

(2) CIRCLES DESIGNATED BY S-- INDICATE THOSE VARIABLES WHICH ARE PRECONNECTED FOR OSCILLOSCOPE OBSERVATION.

The basic computer diagram corresponding to the functional diagram of Fig. 15 and the equations given above is shown in Drawing 1 of Fig. 16. The quantitites that each potentiometer represents may be placed directly on the computer diagram in the appropriate circle if so desired, or may be included in the data sheets (see column 6 of Table 10). The gains through the integrators if they are other than unity may also be indicated on the computer diagram. We observe that it is rather simple to check the gains across each section and the equations around each loop. We also observe that the physical limitations of the equipment must be considered, e.g., the number of inputs and outputs of each component. Since potentiometers may only take on values between zero and plus one, and since it is advisable not to have potentiometer settings less than 0.1 (in order to avoid excessive introduction of errors), large variations in parameters may necessitate modifying the computer diagram at various stages of the investigation. This may be done as shown in drawings 2, 3, and 4 of Fig. 15. This method of drawing the computer circuits so that section changes may be included without necessitating the redrawing of the entire diagram has been found quite practical. Note that if we should decide to simplify the setup so that the integrating gyro were represented by a simple integration (instead of an integration with a lag), this could easily be incorporated by a simple diagram indicating the integrator, the input W_y and the output e_{IG}, (with the proper signs and magnitudes, of course).

7.6 Data Sheets

The use of data sheets in conjunction with data recording has been found very helpful. The uniform procedure established calls for two such sheets. One is the "Potentiometer Setting Sheet" (see Table 10), and the other is the "Film Description Sheet" (see Table 11). On both sheets, we give the title of the study, name of the operators performing the study, and the date. When the film description sheet is used in conjunction with the computer diagram and the potentiometer setting sheet, we are able to duplicate at any time the exact setup that was employed to obtain a picture.

POTENTIOMETER SETTING SHEET

STUDY Pitch Control System
OPERATORS
DATE

POTENTIOMETER NUMBER	VARIABLE	VARIABLE VALUES			POTENTIOMETER REPRESENTATION	POTENTIOMETER SETTINGS		
		SCHEDULE A	SCHEDULE B	SCHEDULE C		SCHEDULE A	SCHEDULE B	SCHEDULE C
1	ζ_{HS}/ω_{HS}	$.7/120\pi$			$500\zeta_{HS}/\omega_{HS}$	.928		
2								
3								
4								
5								
8								
9								
10								
12								
14								
15								
16								
17								
18								
31								

SCHEDULE A represents the setting for h = 10,000 ft., M = 2.0
SCHEDULE B
SCHEDULE C

DIAGRAM 1 goes with Schedule A
DIAGRAM 2
DIAGRAM 3
DIAGRAM 4

CHECK SOLUTION : Unstable when $P_{12} = .216$,
f_{osc} = 25 cycles/second

TABLE 10. DATA SHEETS FOR INVESTIGATION OF AIRCRAFT PITCH ORIENTATION CONTROL SYSTEM POTENTIOMETER SETTING SHEET.

FILM DESCRIPTION SHEET

STUDY Pitch Control System
OPERATORS
DATE

INPUT			FILM NUMBER	OUTPUT		DRAWING NUMBER	POTENTIOMETER SETTING SCHEDULE	X_{amp}
QUANTITY	SCOPE INCHES	y ATTENUATION		QUANTITY	y ATTENUATION			
$-2e_c$	−4.0	10/26	VE1	$+20\,\theta_M$	100/26	1	A	48
$-2e_c$	−1.0	100/51	VE2	$-20\,\alpha$	100/51	1	A	24

SCHEDULE A represents the setting for h = 10,000 ft., M = 2.0
SCHEDULE B
SCHEDULE C

TABLE 11. DATA SHEETS FOR INVESTIGATION OF AIRCRAFT PITCH ORIENTATION CONTROL SYSTEM. FILM DESCRIPTION SHEET

On the film description sheet, the columns marked y attenuation refer to the vertical attenuation employed on the oscilloscope when the picture was taken (coarse/fine). The fine adjustment should not be altered until all photographs for a given input have been made. The dial reading for the fine adjustment does not enter into the computation of scale factors, but is nevertheless recorded as a check on its remaining fixed during the shooting of a series of photographs. The coarse adjustment gives a convenient picture height for photographing a given variable.

For linear cases only, the calculation of scale factors may be computed by means of the following formula:

$$\frac{q_{(out)}}{\text{scope inch}} = \frac{A_0\,(out)\; K(in)\; q(in)}{A_0(in)\; K(out)\; N}$$

where

$A_0(out)$	=	Oscilloscope attenuator coarse setting while photographing output quantity q(out)
$A_0(in)$	=	Oscilloscope attenuator coarse setting while photographing input quantity q(in)
K(in)q(in)	=	Input quantity designated on computer diagram
K(out)q(out)	=	Output quantity designated on computer diagram
N	=	Number of scope inches displayed on oscilloscope screen to represent a given value of K(in) q(in).

If we apply this formula to Film Number VE1 (see Table 11), then $A_0(out)$ - 100, $A_0(in) = 10$, $K(in) = -2$, $q(in) = e_c$, $K(out) = 20$, $q(out) = \theta_M$, and $N = -4.0$ inches.

The column in the film description sheet labeled X_{amp} gives the horizontal scale attenuation used.

We may observe that on the potentiometer setting sheet, we correlate the Potentiometer Setting Schedules with their corresponding circuit diagrams; and on the film description sheet, we correlate each film with its appropriate schedule and diagram. We also note that on both these data sheets, each schedule has been identified as to the general conditions or circumstances that it represents.

Finally, it is often convenient to refer to a check solution on the potentiometer setting sheet.