Correlation Techniques To Determine Model Form In Robust Nonlinear System Realization/Identification

Greselda I. Stry[1] and D. Joseph Mook[2]

Department of Mechanical and Aerospace Engineering
State University of New York at Buffalo
Buffalo, New York 14260
716–636–3058

ABSTRACT

The fundamental challenge in identification of nonlinear dynamic systems is determining the appropriate form of the model. A robust technique is presented in this paper which essentially eliminates this problem for many applications.

The technique is based on the Minimum Model Error (MME) optimal estimation approach. A detailed literature review is included in which fundamental differences between the current approach and previous work is described. The most significant feature of the current work is the ability to identify nonlinear dynamic systems without prior assumptions regarding the form of the nonlinearities, in contrast to existing nonlinear identification approaches which usually require detailed assumptions of the nonlinearities. Model form is determined via statistical correlation of the MME optimal state estimates with the MME optimal model error estimates. The example illustrations indicate that the method is robust with respect to prior ignorance of the model, and with respect to measurement noise, measurement frequency, and measurement record length.

---

1 Graduate Resarch Assistant; NASA Graduate Researcher

2 Assistant Professor

INTRODUCTION

The widespread existence of nonlinear behavior in many dynamic systems is well-documented, e.g, Thompson and Stewart [1]; Nayfeh and Mook [2]. In particular, virtually every problem associated with orbit estimation, flight trajectory estimation, spacecraft dynamics, etc., is known to exhibit nonlinear behavior. Many excellent methods for analyzing nonlinear system models have been developed. However, a key practical link is often overlooked, namely: How does one obtain an accurate mathematical model for the dynamics of a particular complicated nonlinear system? General methods for actually obtaining accurate models for real physical systems are not nearly as widespread or well developed as are the techniques available for analyzing models.

Accurate dynamic models are necessary for many tasks, including basic physical understanding, analysis, performance prediction, evaluation, life cycle estimation, control system design, etc. For example, most filter design assumes white process noise, yet many nonlinear effects are inherently non-zero mean; e.g., quadratic nonlinearities are always positive. In order to obtain a model with truly zero mean process noise for filter design purposes, all of the quadratic terms (and many other nonlinearities) must be well modeled. However, the complexity of many real systems greatly diminishes the possibility of accurately constructing a dynamic model purely from analysis using the laws of physics.

Identification is the process of developing an accurate mathematical model for a system, given a set of output measurements and knowledge of the input. Many well developed and efficient identification algorithms already exist for linear systems (e.g., [3]-[7]). These often may be employed to model nonlinear systems when the system nonlinearities are small, and/or the system operates in a small linear regime. However, linearization does not work well (if at all) in every application, and even when it does provide a reasonable approximation, the approximation is normally limited to a small region about the operating point of linearization. Consequently, there is a real need for nonlinear identification algorithms. If nonlinearities are a predominant part of a system's behavior, using a linear model to describe such a system leads to inconsistencies ranging from inaccurate numerical results to misrepresentation of the system's qualitative behavior. Many important characteristics of nonlinear behavior, such as multiple steady-states, limit cycles, hysteresis, softening or hardening systems, chaos, etc., have no linear equivalent. Since nonlinearities are seldomly easily characterized, identification techniques may prove beneficial in developing accurate mathematical representations of nonlinear systems.

Numerous methods for the identification of nonlinear systems have been developed in the past two decades. Many of these techniques are reviewed in Natke, Juang and Gawronski [8], Billings [9], and Bekey [10]. Most methods fall into one of the following categories:

- ☐ describing the nonlinear system using a linear model
- ☐ the direct equation approach
- ☐ representing the nonlinear system in a series expansion, and obtaining the respective coefficients either by using a regression estimation technique, by minimizing a cost functional,

by using correlation techniques, or by some other approach

- obtaining a graphical representation of the nonlinear term(s), then finding an analytical model for the nonlinearity

With such diversity of nonlinear identification techniques, the choice of a particular algorithm may be based on criteria such as: the degree to which prior assumptions of the model form affect the user's effort in applying the algorithms; the number of iterations required; the sensitivity to the presence of measurement noise in the data; the number of state measurements needed; whether or not knowledge of the initial conditions is required; the kind of forcing input(s) required or permitted (step, white gaussian noise, sinusoidal, etc.); the ability to handle hysteritic or discontinuous nonlinearities; the degree of a priori knowledge of system properties required; and the computational requirements. Most algorithms differ widely in at least some of these comparisons; the choice of a particular technique depends on the needs of the particular application.

Among the methods which linearize the nonlinear system are those presented by Jedner and Unbehauen [11] and Ibanez [12]. Jedner and Unbehauen represent a nonlinear system, which may often operate in small regions around a number of operating points, by an equivalent number of linear submodels. It is assumed that the system operates at only a few points. Although the model may work well for controller design, the points at which the system is operating must be known and the linear models apply only within the operating regions. Ibanez takes a slightly different approach by assuming the system response to be periodic at the forcing frequency. An approximate transfer function is constructed. The transfer function is dependent on the amplitude as well as on the exciting frequency and is valid only within the region of exciting frequencies.

The direct equation approach is used by Yasuda, Kawamura and Watanabe [13], [14]. The input and output measurements of a dynamic process are expressed in a Fourier Series using, for example, an FFT algorithm. The system nonlinearity is represented as a sum of polynomials with unknown coefficients. Applying the principle of harmonic balance, the polynomial coefficients as well as the other system parameters are obtained accurately. Knowledge of the nonlinearity is needed to construct the polynomial. Truncation in the Fourier Series expansion of the input or output may lead to error.

The regression estimation approach is used by Billings and Voon [15] and Greblick and Pawlak [16]. Billings and Voon use the NARMAX model (Nonlinear Auto Regressive Moving Average model with eXogenous inputs) to represent the nonlinear system. A stepwise regression method determines the significant terms in the NARMAX model. Then a prediction-error algorithm provides optimal estimates of the final model parameters. Greblick and Pawlak represent the linear dynamic submodel by an ARMA model and the nonlinearities by a Borel function. A non-parametric kernel regression estimation is employed to obtain the final analytical model.

Kortman and Unbehauen [17] and Distefano and Rath [18] use the minimization of an error cost function as a means of obtaining the coefficients of the functions used to represent the

nonlinearities. The method presented by Kortman and Unbehauen uses only system input and output information to estimate the polynomial representing the nonlinearities and the parameters of the linear components. It is robust in the presence of noise, although iteration is necessary. Distefano and Rath present two techniques, a non-iterative direct identification and an iterative direct identification. In the first technique, measurement of all variables is required and the model parameters are obtained through the minimization of an error function. In the second technique, iteration is used to minimize a cost function yielding the system parameters in addition to the state trajectories. In Distefano and Rath, the nonlinear model form is also taken to be known.

In other techniques, as in statistical linearization, a nonlinear relation is replaced by a linear equivalent gain. Broersen [19] extends the technique of statistical linearization by representing the nonlinearity as a linear combination of a number of arbitrary functions. Correlation techniques are then used to determine the coefficients of these functions. The number and type of functions selected depends on the desired accuracy as well as some knowledge of the system nonlinearity. Reasonable accuracy is obtained in the presence of noise and no iterations are necessary. Although some of the basic properties of the true nonlinear output are preserved, it is limited to only random excitation, and knowledge of all states and forcing terms is required.

In the method of multiple scales (Hanagud, Mayyappa and Craig [20]), a perturbation solution to the nonlinear equation of motion is obtained. An objective function is built employing an integral least squares approach. The minimization of the functional yields the unknown parameters. Data on only one field variable is necessary, and the method is effective in the presence of high noise. The method of multiple scales, however, is restricted to systems with small damping and slight nonlinearities and, as in most other methods, the form of the nonlinearity is assumed a priori. The method typically requires some algebraic manipulations which may be quite involved, and these manipulations are only valid for a particular assumed nonlinear form. If the assumed nonlinear form is changed, the algebra must be repeated.

Several techniques describe the nonlinear system using the Volterra or Wiener kernels. The Volterra series consists of the summation of impulse responses of increasing dimensionality. The Wiener series is also a set of orthogonal functions in which the input is white gaussian noise. Marmarelis and Udwadia [21], for example, estimate the first and higher order kernels appearing in the Volterra series using correlation techniques. Chen, Ishii and Suzumura [22] use cross-correlation functions in addition to the Volterra and Wiener series to describe nonlinear models and to show the relation between the system inner structure and the series. Although weakly nonlinear systems can be described by the first few kernels, for strongly nonlinear systems these series give accurate numerical results only at the expense of an excessive number of coefficients. This renders the analytical model impractical for control applications.

Other popular series used in nonlinear identification are orthogonal polynomials such as Legendre (Wang and Chan [23]), Chebyshev, and Jacobi (Horn and Chou [24]). Horn and Chou expand the variables of the system into a shifted Jacobi series, reducing the nonlinear state equation into a linear algebraic matrix equation. The unknown parameters of the nonlinear

system are then estimated using least squares. Even though the algorithm works well in the presence of noise, the nonlinear form must be known a priori.

Methods for the identification of nonlinear systems have also been developed based on the extended Kalman filter. The extended Kalman filter is the linear Kalman filter applied to nonlinear systems by linearizing the nonlinear model into a Taylor series expansion about the estimated state vector. Yun and Shinozuka [25] apply the extended Kalman filter for the parameter estimation of a quadratic term. The state vector is augmented by including the unknown parameters in addition to the state variables. Through a series of iterations, the response, as well as the unknown parameters, are estimated by the Kalman filter. Among its disavantages are high sensitivity to initial conditions, in particular if the initial conditions are barely known. The nonlinear form must be chosen a priori in order to estimate the corresponding parameter(s).

Hammond, Lo and Seager-Smith [26] use an optimal control technique based on optimal control methods employed for linear system deconvolution. The form of the linear model is assumed to be known as well as the input and the output. A cost functional consisting of the weighted sum of the square of the error (between the actual and estimated output) yields an optimal estimated input. The estimated input and the actual input are used to obtain the nonlinearity as a function of the state variables. Although no previous assumption is made of the nonlinearities, there is no provision to deal with noise.

All of the techniques outlined above have proven useful in certain applications. However, all of them are subject to one or more of the following shortcomings:

1. The form of the nonlinearity (quadratic, cubic, exponential, etc.) must be assumed a priori. This is a very serious drawback, because the identification algorithm can only attempt to find the best model in the assumed form. If the form is assumed incorrectly, the resulting model may be so poor as to be useless, or it may appear to fit the data well enough that the user erroneously concludes that the correct model has been obtained. Also, for many techniques of this type, the effort required to test a given form is considerable, which greatly diminishes the effectiveness since multiple form tests are less likely to be conducted.
2. Techniques which attempt to avoid the problem of a priori model form assumption through the use of series expansions generally eliminate any possibility of understanding the underlying physics. Thus, although a good fit of the data might be achieved using a sufficient number of terms in the series, physical insight is lost. Moreover, large systems and/or particularly complicated behavior may require that a very large number of terms be used to obtain a given level of accuracy.
3. The presence of noise in the measurement data is not rigorously treated, yet noise is generally unavoidable.
4. Initial conditions must be known in order to implement the algorithm.
5. The algorithm can only be implemented if the data is obtained using very specific system excitations.

The algorithm of the current paper compares favorably with existing algorithms in most of

the categories listed above. It is robust with respect to measurement noise; does not require knowledge of the initial conditions; is independent of the forcing (but, like all methods, assumes that it is known); is not computationally prohibitive; and, most importantly, it requires minimal a priori assumptions regarding the form of the model or the system properties. In fact, using the correlation technique outlined in the next section, the algorithm essentially eliminates the need to ever assume the nonlinear model form.

The identification algorithm is based on a combination of Minimum Model Error (MME) state estimation, correlation techniques, and least squares. MME was first described by Mook and Junkins [27]. The MME combines the available measurements and an assumed model of the system to produce optimal estimates of the states and the model error. The assumed model represents an initial attempt to model the system using direct analysis, but may be extremely poor. Given the noisy output measurements of the system, MME estimates the state histories as well as the error in the assumed model. In previous work, the correct form and corresponding parameters of the nonlinear model were then estimated in a trial-and-error fashion, by assuming a nonlinear (in the states) form of the error terms, and then determining the best least-squares fit between the state estimates and the model error estimates. Thus, although the MME portion of the algorithm did not require the model form to be assumed, the subsequent least-squares fit between the state estimates and the model error estimates did. In Mook [28] it was shown that this approach could accurately identify terms in a Duffing oscillator, in the presence of noise and sparse measurements. The method worked well even when only a crude model of the dynamic system was assumed, and the error model used for the least-squares fit contained numerous terms in addition to the correct one(s). Later, in Mook and Stry [29], a simple harmonic oscillator with quadratic feedback was simulated on an analog computer. The algorithm was shown to accurately identify the nonlinear model from analog measurements.

In this paper, the identification of the model from the MME-produced state and model error estimates is improved by using correlation techniques to select the form of the correction terms. The correction terms, when added to the initially assumed model, yield the true model of the system. The correction terms may consist of a combination of linear and nonlinear functions. An extensive library of linear and nonlinear functions has been assembled. The correlation technique is used to select the true forms from the library. Even when the true form of the nonlinearity was not present in the library, the correlation technique picks the closest form(s), typically, the first term(s) in the Taylor Series expansion. Once the forms have been selected by the correlation algorithm, least-squares is used to determine the model parameters.

IDENTIFICATION ALGORITHM

In this section, the identification algorithm is explained. First, the MME technique is briefly reviewed, and then the correlation technique used to automate the model form determination is explained in detail.

The MME may be summarized as follows (a more detailed explanation may be found

in Mook and Junkins [27]). Suppose there is a nonlinear system whose exact analytical representation is unknown, but for which output measurements are available. Using whatever means are available (analysis, finite elements, etc.), a system model is constructed. As shown in [27]-[29], the MME works well even if this system model is poor. The MME combines the assumed model with the measurements to produce optimal estimates of (i) the state trajectories, and (ii) the error in the model. In the present work, these state and model error estimates are used for system identification.

Consider a forced nonlinear dynamic system which may be modeled in state-space form by the equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{F}(t) + \underline{f}(\underline{x}(t), \dot{\underline{x}}(t)) \tag{1}$$

where $\underline{x}(t)$ is the $n \times 1$ state vector consisting of the system states, A is the $n \times n$ state matrix, $\underline{F}(t)$ is an $n \times 1$ vector of known external excitation, and $\underline{f}(\underline{x}(t), \dot{\underline{x}}(t))$ is an $n \times 1$ vector which includes all of the system nonlinearities. State-observable discrete time domain measurements are available for this system in the form

$$\tilde{\underline{y}}(t_k) = \underline{g}_k(\underline{x}(t_k), t_k) + \underline{v}_k, \qquad t_0 \le t_k \le t_f \tag{2}$$

where $\tilde{\underline{y}}(t_k)$ is an $m \times 1$ measurement vector at time t_k, $\underline{g}_k$ is the accurate model of the measurement process, and $\underline{v}_k$ represents measurement noise. $\underline{v}_k$ is assumed to be a zero-mean, gaussian distributed process of known covariance R_k. The measurement vector $\tilde{\underline{y}}(t_k)$ may contain one or more of the system states. To implement MME, assume that a model, which is generally not the true system model because of the difficulties inherent in obtaining the true system model, is constructed in state-vector form as

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{F}(t) \tag{3}$$

Here, we show a linear model because in practice, linearization is the most common approach to modeling nonlinear systems. MME uses the assumed linear model in Eq. (3) and the noisy measurements in Eq. (2) to find optimal estimates of the states and of the model error.

The model error, which includes the unknown nonlinear terms of the system, is represented by the addition of a term to the assumed linear model as

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{F}(t) + \underline{d}(t) \tag{4}$$

where $\underline{d}(t)$ is the $n \times 1$ model error to be estimated along with the states.

A cost functional, J, that consists of the weighted integral square of the model error term plus the weighted sum square of the measurement-minus-estimated measurement residuals, is formed:

$$J = \sum_{k=1}^{M} \left\{ [\tilde{\underline{y}}(t_k) - \underline{g}_k(\hat{\underline{x}}(t_k), t_k)]^T R_k^{-1} [\tilde{\underline{y}}(t_k) - \underline{g}_k(\hat{\underline{x}}(t_k), t_k)] \right\}$$

$$+\int_{t_o}^{t_f} \underline{d}(\tau)^T W\, \underline{d}(\tau) d\tau \tag{5}$$

where M is the number of measurement times, $\underline{\hat{x}}(t_k)$ is the estimated state vector and W is a weight matrix to be determined.

J is minimized with respect to the model error term, $\underline{d}(t)$. The necessary conditions for the minimization lead to the following two point boundary value problem (TPBVP), (see Geering [30]),

$$\underline{\dot{x}}(t) = A\underline{x}(t) + \underline{F}(t) + \underline{d}(t) \tag{5a}$$

$$\underline{\dot{\lambda}}(t) = -A^T \underline{\lambda}(t) \tag{5b}$$

$$\underline{d}(t) = -\frac{1}{2} W \underline{\lambda}(t) \tag{5c}$$

$$\underline{\lambda}(t_k^+) = \underline{\lambda}(t_k^-) + 2H_k R_k^{-1}[\underline{\tilde{y}}(t_k) - \underline{g}_k(\underline{\hat{x}}(t_k), t_k)] \tag{5d}$$

$$H_k = \frac{\delta \underline{g}}{\delta \underline{x}} \Big|_{\underline{\hat{x}}(t_k), t_k}$$

$$\underline{x}(t_o) = \underline{x}_o \quad or \quad \underline{\lambda}(t_o) = 0 \tag{5e}$$

$$\underline{x}(t_f) = \underline{x}_f \quad or \quad \underline{\lambda}(t_f) = 0 \tag{5f}$$

where $\underline{\lambda}(t)$ is a vector of costates (Lagrange multipliers). Estimates of the states and of the model error are produced by the solution of this two-point boundary value problem. The estimates depend on the particular value of W. The solution is repeated until a value of W is obtained which produces state estimates which satisfy the "covariance constraint", explained next.

According to the covariance constraint, the measurement-minus-estimated measurement residual covariance matrix must match the measurement-minus-truth error covariance matrix. This may be written as

$$[\underline{\tilde{y}}(t_k) - \underline{g}_k(\underline{\hat{x}}(t_k), t_k)]^T [\underline{\tilde{y}}(t_k) - \underline{g}_k(\underline{\hat{x}}(t_k), t_k)] \approx R_k \tag{6}$$

During the minimization, the weight W is varied until the state estimates satisfy the covariance constraint, i.e., the left hand side of Eq. (6) is approximately equal to the right hand side. The model error is, therefore, the minimum adjustment to the model required for the estimated states to predict the measurements with approximately the same covariance as the measurement error.

The TPBVP represented by Eqs. (5a) to (5f) contains jumps in the costates and, consequently, in the model error. As evident from Eq. (5d), the size of the jump is directly proportional to the measurement residual at each measurement time. The noisier the measurements, the larger the jump size. A multiple shooting algorithm, developed by Mook and Lew [31], converts this jump-discontinuous TPBVP into a set of linear algebraic equations which may be solved using any linear equation solver. Multiple shooting also facilitates the analysis of a large number of measurements, by processing the solution at the end of every set of jumps.

Correlation is a measure of the relationship that exists between two variables. The more highly correlated two variables are, the more closely will the change in one variable correspond to

a change in the other variable. The cross-correlation coefficient between two discrete variables, say x and y, is defined as (see Newland [32] or Witte [33])

$$C(x,y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sigma_x \sigma_y n} \tag{7}$$

where n is the number of data points and the overbar denotes the mean of those n points. σ_x is the standard deviation of the variable x and is defined as

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})}{n}}$$

$C(x,y)$ is a measure of the linear relationship between variables x and y. The value of $C(x,y)$ lies in the range $-1 < C(x,y) < 1$. If, for instance, changes in the value of x correspond to perfectly predictable (linearly) changes in the value of y, where the changes in both variables are of the same sign, then the value of $C(x,y)$ is 1. If the changes are of opposite sign but still perfectly predictable, then the value of $C(x,y)$ is -1. If changes in the values of x and y tend to correspond in sign but are not perfectly predictable, then $0 < C(x,y) < 1$. If changes in the values of x and y tend to be of opposite sign but are not perfectly predictable, then $-1 < C(x,y) < 0$. If there is no linear relationship between the values of x and y, then $C(x,y) = 0$. For example, suppose x and y are multiples of each other, $x = K * y$, where K is an arbitrary constant of proportionality. Then

$$C(x,y) = \frac{\sum_{i=1}^{n} K(x_i - \bar{x})^2}{\sum_{j=1}^{n} K(x_j - \bar{x})^2} = 1.0 \tag{8}$$

The true functional form of the model error can be found by calculating the correlation of the MME model error estimates with functions of the MME state estimates. If the functional form of the actual system is used, and if the estimates from MME are perfect, then $C(x,y) = 1.0$. Thus, an algorithm may be constructed which performs nonlinear system identification by (i) utilizing the MME to process the available measurements and the initial model in order to produce state estimates and model error estimates, and (ii) testing the correlation between the state estimates and the model error estimates using a "sufficient number" of functional forms so that the actual form is included among those tested. The MME does not require that the correct form of the model be known a priori. The correlation tests may be performed using an existing library of nonlinear functional forms, without input from the user. Thus, if the library is complete (in the sense that it contains the actual model form), the identification of the nonlinear model is accomplished, yet at no point in the algorithm is the user required to assume the correct model form.

The success of the algorithm is determined by the ability of the MME to produce accurate state and model error estimates, and by the completeness of the library of nonlinear functions to be used in the correlation test. We now address these issues in order.

The MME has been shown to consistently produce state and model error estimates of high accuracy in the presence of high measurement noise, low measurement frequency, and poor

initial model [27–29]. Generally, however, some noise is still present in both the state estimate and the model error term, although these noise levels are considerably less than the noise in the original data. Let the model error term be given by $x_{correction} = x + \xi$ where ξ is the noise. The cross-correlation between the error term and the test function y becomes

$$C(x,y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) + \sum_{j=1}^{n}\xi(y_j - \bar{y})}{n\sigma_y\sqrt{\sigma_x^2 + \frac{1}{n}\sum_{k=1}^{n}(2\xi(x - \bar{x}) + \xi^2)}} \approx 1.0 \tag{9}$$

As long as the noise is negligible all terms containing ξ are small and affect the result only slightly. Thus, the correlation calculated for the actual function is close to, but not exactly equal to, 1, while the correlation calculated for incorrect terms remains close to 0. If the level of noise is excessive, say, of comparable magnitude to one or more of the actual nonlinear model terms, then the ability of the correlation test to distinguish this term from similar terms may be greatly reduced or eliminated. However, subsequent least-squares fit of the terms has, in every case tested, correctly selected the actual nonlinear function from among those which the correlation test could not distinguish. An example of this is shown in the next section.

The issue of completeness of the library is now addressed. The error term may be composed of more than one function from the library, or the actual function may be missing from the library. Consider first the case where the actual error is a combination of library terms, say, two terms. The error term may be written $x_{correction} = x_1 + x_2$ and the cross-correlation has the form

$$C(x,y) = \frac{\sum_{i=1}^{n}(x_{1i} - \bar{x}_1)(y_i - \bar{y}) + \sum_{j=1}^{n}(x_{2j} - \bar{x}_2)(y_j - \bar{y})}{n\sigma_y\sqrt{{\sigma_{x_1}}^2 + {\sigma_{x_2}}^2 + \frac{1}{n}\sum_{k=1}^{n}2(x_{1k} + \bar{x}_1)(x_{2k} + \bar{x}_2)}} \tag{10}$$

The cross-correlation is highest for the term which constitutes the largest part of the error. Thus, it is desirable to execute the algorithm iteratively. The library term which constitutes the largest portion of the actual model error is identified first and then added to the MME model. The entire process (including MME) is then repeated, so that new state and model error estimates are obtained (note that the change in state estimates should be minimal, while the change in model error estimates should be a large reduction in magnitude). The largest term remaining in the model error is identified in each pass, then added to the initial MME model.

An alternative to iterative application of the algorithm is to test the correlation of combinations of the library functions. An algorithm can be constructed which tests every possible combination of the functions explicitly contained in the library. This approach has not been attempted in the examples which follow.

If the actual model error is not present in the library, then test cases show that the highest correlation values are calculated for the terms in the series expansion of the actual function. Thus, for example, if the actual model error was of the form $sin(x)$, but $sin(x)$ was not present in the library, the correlation coefficients are highest for the terms x, x^3, x^5, etc. However, the test described by Eq. 7 is very fast, so the library may contain a very large number of terms.

The final step in the identification procedure is to use a least-squares algorithm to fit the model error to the functional forms (i.e., perform parameter identification once the true nonlinear form has been determined). The error term is expanded into a combination of the functional forms such as

$$d(t) = \alpha f_1(\underline{x}(t)) + \beta f_2(\underline{x}(t)) + \gamma f_3(\underline{x}(t)) + \dots \tag{11}$$

where α, β, γ, ... are unknown coefficients to be determined by least squares, and f_1, f_2, f_3, ... are functions which are selected as a result of the correlation test (often, however, only one function is used at a time). Other parameters may be present inside the functions (such as, for example, coefficients of exponents). Eq. (11) may be sampled repeatedly (using the MME estimates) to obtain

$$\begin{aligned}
d(t_1) &= \alpha f_1(\underline{x}(t_1)) + \beta f_2(\underline{x}(t_1)) + \gamma f_3(\underline{x}(t_1)) + \dots \\
d(t_2) &= \alpha f_1(\underline{x}(t_2)) + \beta f_2(\underline{x}(t_2)) + \gamma f_3(\underline{x}(t_2)) + \dots \\
\vdots &= \qquad\qquad \vdots \\
d(t_l) &= \alpha f_1(\underline{x}(t_l)) + \beta f_2(\underline{x}(t_l)) + \gamma f_3(\underline{x}(t_l)) + \dots
\end{aligned}$$

or, in matrix form,

$$\underline{D}_{l\times 1} = M_{l\times p}\underline{P}_{p\times 1} \tag{12}$$

where $\underline{P} = [\alpha \quad \beta \quad \gamma \quad \dots]^T$ is the vector of coefficients for the terms in $d(t)$. Since estimates of $d(t)$ are available continuously throughout the time domain, the parameter l may be chosen quite large to improve the least squares fit. Generally, because of the potential jump discontinuities in the model error estimates at the measurement times, it is desirable to pick the least squares sampling times in Eq. (12) at points other than the measurement times. The least squares estimate is found by minimizing the following cost functional with respect to P:

$$\Phi = [\underline{D} - M\underline{P}]^T[\underline{D} - M\underline{P}] \tag{13}$$

The solution is given by

$$\underline{P} = (M^T M)^{-1} M^T \underline{D} \tag{14}$$

If the functions include parameters to be estimated, the equivalent nonlinear least-squares problem is constructed.

The multiple shooting algorithm presented by Mook and Lew [31] was used to obtain the MME solutions used in the tests presented in this paper. It was assumed in the examples that MME obtained the dynamic error term without knowledge of the boundary conditions on $\underline{x}$, so some distortion of the correction term at the initial and final times was expected due to the constraints of Eqs. (5e-5f), i.e., by assuming no state knowledge is available at t_0 or t_f, we constrain $\lambda(t_0) = 0$ and $\lambda(t_f) = 0$. Therefore, in all test cases, the initial and final ten percent of the correction term data was ignored in the least squares fit.

EXAMPLES

For illustrative purposes, the true system was chosen as a simple harmonic oscillator with various forms of nonlinear feedback. The true system can be modeled as

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(x,v) \end{pmatrix} \tag{15}$$

where x is position, v is velocity and the dot indicates differentiation with respect to time. For simplicity, the system was unforced. The term $f(x,v)$ represents the nonlinear terms to be identified by the MME-based identification algorithm. Measurements were generated from the true system, Eq. (15), with different kinds of nonlinear functions $f(x,v)$. The ability of the identification algorithm to identify the model with no prior knowledge of $f(x,v)$ is tested. Table 1 shows the functions used in each simulation. Note that the unknown error term may be a combination of linear and nonlinear functions. Table 1 also shows the initial conditions and the amount of noise used to generate measurements for each test. The noise levels represent the percentage of the peak system response (actual percentages are higher for the majority of the measurements since the response is only at peak amplitude for brief periods).

Table 1
SUMMARY OF TEST CASES

TEST #	TRUE ERROR: f(x,v)	x(0)	v(0)	NOISE
1	3.0*x*x	0.175	0	0
2	-0.1*x*x*v	0.175	0	0
3	-0.5*cos(x)*cos(v)	0.175	0	0
4	-1.0*v*sin(x)	0.175	0	0
5	-1.0*x*x - 0.25*v	0.350	0	0
6	-1.0*x*x*x - 0.1*tan(v)	0.873	0	0
7	-1.0/cos(x) - 1.0*sin(v)	1.750	0	0
8	3.0*x*x	0.175	0	10%
9	-1.0*x*x - 0.25*v	0.350	0	10%
10	-1.0*x*x*x - 0.1*tan(v)	0.873	0	10%

The assumed model used for the MME analysis consisted of the undamped linear oscillator part of the system,

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \tag{16}$$

For each test, 200 measurements of position were obtained from the digital simulation of Eq. (15) at a sampling rate of 10 Hz. The functional form of the dynamic error, $f(x, v)$, was determined solely from the least-squares fit of the functions identified during the correlation tests on the MME state and model error estimates obtained using only the model in Eq. (16).

A library of functions was built consisting of approximately 300 of the most commonly found nonlinear and linear forms. For a particular test, after the model error term was found from MME, it was correlated with each one of the functions in the library. The correlation test of the entire library of functions did not take more than a few seconds to execute, since the calculations are simple. The functional form of the unknown nonlinear term was chosen as the one for which the absolute value of the cross-correlation coefficient was closest to 1. Table 2 shows the results for all 10 tests, including the true dynamic error, the highest cross-correlation coefficient obtained, the corresponding functional form, and the respective coefficient computed from the least squares fit. The star (*) indicates tests performed from noisy measurements.

Table 2.
IDENTIFICATION RESULTS FOR EACH TEST CASE

TEST#	TRUE ERROR(S)	C(d(t),f)	SELECTED	L.S.
1	3.0*x*x	0.999	x*x	2.99
2	-0.1*x*x*v	0.999	x*x*v	-0.10
3	-0.5*cos(x)*cos(v)	0.999	cos(x)*cos(v)	-0.49
4	-1.0*v*sin(x)	0.999	v*sin(x)	-1.00
5	-1.0*x*x	0.999	x*x	-0.99
	-0.25*v	0.746	v	-0.24
6	-1.0*x*x*x	0.936	x*x*x	-1.00
	-0.1*tan(v)	0.999	tan(v)	-0.10
7	-1.0/cos(x)	0.927	1/cos(x)	-0.99
	-1.0*sin(v)	0.999	sin(v)	-1.00
8*	3.0*x*x	0.797	x*x	3.12
9*	-1.0*x*x	0.937	x*x	-0.90
	-0.25*v	0.772	v	-0.22
10*	-1.0*x*x*x	0.838	x*x*x	-0.98
	-0.1*tan(v)	0.583	tan(v)	-0.10

For tests 1, 2, 3, and 4, the exact form of the nonlinearity was contained in the library and the measurements did not contain noise. The calculated value of $C(d(t), f)$ was 1 for the true forms. In test 8, the library contained the exact form of the nonlinearity but the measurements contained significant noise. The correlation for the correct term was much higher than for any other term, but was approximately 0.8 instead of 1 due to the noise. In the cases where the

error term consisted of two functions but the measurements were noise-free (tests 5, 6 and 7), $C(d(t), f)$ was close to one for both functions after applying the algorithm iteratively as described in the previous section.

When noise and more than one function was present in the dynamic error term (tests 9 and 10), the maximum value of the cross-correlation coefficients dropped significantly and in some cases did not immediately identify the actual form over other similar forms. As an example, Table 3 shows the top five cross-correlation values for the identification of the $tan(v)$ term in test case 10. Note that the functions with the highest cross-correlation values are all similar in form to $tan(v)$, and the corresponding correlation coefficients are of similar magnitude. Since $C(d(t), f)$ did not clearly identify $tan(v)$ as the missing term, the five functions yielding the highest $C(d(t), f)$ values were individually least-squares fit to the model error term. In all cases (i.e., repeating this test for a number of different random noise samples), the function with the smallest least squares error cost was the correct function ($tan(v)$). Thus, the least-squares fit of the parameters to the functional forms also serves as a second test if the correlation test is inconclusive due to high noise levels.

Table 3.
HIGHEST CROSS-CORRELATION COEFFICIENTS OBTAINED FOR THE TAN(V) TERM OF TEST CASE 10

FUNCTION	C(d(t),f)	L.S.	L.S. cost
tan(v)	0.583	-0.104	0.588
v	0.584	-0.119	0.623
v*cos(x)*cos(v)	0.584	-0.150	0.659
v*cos(x)	0.586	-0.126	0.607
sin(v)*cos(x)	0.586	-0.133	0.621

The number of data points used in the MME algorithm was irrelevant as long as there were enough points to reasonably span the qualitative aspects of the system (e.g., sinusoidal terms cannot be identified if the data only spans a small fraction of the period).

If the exact functional form of the dynamic error term was not in the function library, the correlation procedure would pick the first term in the Taylor Series expansion of the exact form. For example in a test case where the dynamic error term corresponded to $x*sin(v)$ and $x*sin(v)$ was deleted from the library, the function with the largest $C(d(t), f)$ was $x * v$. Similarly, in several examples which are not shown the magnitude of the states, x and v, were small. Thus, the trigonometric functions of position and velocity were approximately equal to the first term in their Taylor Series expansions, i.e., $cos(x) \approx 1.0$, $sin(x) \approx x$, $cos(v) \approx 1.0$ and $sin(v) \approx v$. In these cases, assumptions of linearity are clearly valid, and are not of interest in the present work.

SUMMARY AND CONCLUSIONS

In this paper, an algorithm based on the MME estimation technique, coupled with correlation tests and least squares, has been developed for identification of nonlinear systems. The results of the examples indicate that the correlation technique applied to the MME-produced state and model error estimates enables the form of the model to be accurately determined, thus eliminating the requirement that the form be assumed a priori. Once the form is determined, the least-squares fit provides excellent parameter identification. In cases of high noise, where the correlation test may not be able to distinguish the actual form from similar forms, the least-squares fit also proved to be a reliable second test for determining the actual form.

At no point in the algorithm is the user required to assume the form of the model, representing a tremendous advantage over existing techniques, including the previous MME-based work. The MME does not require an accurate model in order to produce accurate state and model error estimates, and the correlation tests are automatically performed on a large existing library of functions. Additional functions and more sophisticated methods of combining existing functions can be added to the correlation testing portion of the algorithm (the authors are currently pursuing this), virtually eliminating the likelihood that the actual model error terms are not tested.

REFERENCES

1. Thompson, J.M.T., and Stewart, H.B., **Nonlinear Dynamics and Chaos**, Wiley, New York, 1986.
2. Nayfeh, A. H., and Mook, D. T., **Nonlinear Oscillations**, Wiley, New York, 1979.
3. Ibrahim, S.R., and Mikulcik, E.C., "A Method for the Direct Identification of Vibration Parameters from the Free Response," *Shock and Vibration Bulletin*, No. 47, Pt. 4, pp. 183-198, Sept. 1977.
4. Rajaram, S., and Junkins, J.L.,"Identification of Vibrating Flexible Structures", *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 4, pp. 463-470, July-Aug. 1985.
5. Hendricks, S.L., et. al., "Identification of Mass, Damping, and Stiffness Matrices for Large Linear Vibratory Systems", *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 2, pp. 244-245, March-April 1984.
6. Chen, J.C., et.al., "Direct Structural Parameter Identification by Modal Test Results", *24th Structures, Structural Dynamics, and Materials Conference*, Pt. 2, 1983.
7. Juang, J.-N., and Pappa, R.S., "An Eigensystem Realization Algorithm (ERA) for Modal Parameter Identification and Model Reduction", *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 5, pp. 620-627, Sept.-Oct. 1985.
8. Natke, H.G., Juang, J.-N., and Gawronski, W., "Identification of Nonlinear Mechanical Systems: A Brief Review," NASA Langley Research Center, Hampton, VA, U.S.A. .
9. Billings, S.A., "Identification of Nonlinear Systems-a Survey," *IEE Proc.*, Vol. 127, Pt. D, No. 6, 1980, pp.272–285.
10. Bekey, G.A., "System Identification-an Introduction and a Survey," *Simulation*, October 1970, pp. 151–166.

11. Jedner, U., and Unbehauen, H., "Identification of a Class of Nonlinear Systems by Parameter Estimation of a Linear Multi-Model," *IMACS, Modeling and Simulation for Control of Lumped and Distributed Parameter Systems*, June 3-6, 1986, pp. 11–15.
12. Ibanez, P., "Identification of Dynamic Parameters of Linear and Non-Linear Structural Models from Experimental Data," *Nuclear Engineering and Design*, Vol. 25, 1973, pp. 32–41.
13. Yasuda, K., Kawamura, S., and Watanabe, K., "Identification of Nonlinear Multi-Degree-of-Freedom Systems (Identification Under Noisy Measurements)," *JSME International Journal* , Vol. 3, No. 1, 1988, pp. 502–509.
14. Yasuda, K., Kawamura, S., and Watanabe, K., "Identification of Nonlinear Multi-Degree-of-Freedom Systems (Presentation of an Identification technique)," *JSME International Journal* , Vol. 3, No. 1, 1988, pp. 8–14.
15. Billings, S.A., and Voon, W.S.F., "A Prediction-Error and Stepwise Regression Estimation Algorithm for Nonlinear Systems," *Int. J. Control*, Vol. 44, No. 3, 1986, pp. 803–822.
16. Greblick, W., and Pawlak, M., "Hammerstein System Identification by Non-Parametric Regression Estimation," *Int. J. Control*, Vol. 45, No. 1, 1987, pp. 343–354.
17. Kortmann, M., and Unbehauen, H., "Application of a Recursive Prediction Error Method to the Identification of Nonlinear Systems Using the Weiner Model," *IMACS, Modeling and Simulation for Control of Lumped and Distributed Parameter Systems*, June 3-6, 1986, pp. 3–9.
18. Distefano, N., and Rath, A., "System Identification in Nonlinear Seismic Dynamics," *Computer Methods in Applied Mechanics and Engineering*, Vol. 5, 1975, pp. 353–372.
19. Broersen, P.M.T., "Estimation of Parameters of Non-Linear Dynamical Systems," *Int. J. Non-Linear Mechanics*, Vol. 9, 1974, pp. 355–361.
20. Hanagud, S.V., Meyyappa, M., and Craig, J.I., "Method of Multiple Scales and Identification of Nonlinear Dynamical Systems," *AAIA J.*, Vol. 23, No. 5, 1985, pp. 802–807.
21. Marmarelis, P.Z., and Udwadia, F.E., "The Identification of Building Structural Systems-PartII: The Nonlinear Case," *Bulletin of the Seismological Society of America*, Vol. 66, 1979, pp. 153–171.
22. Chen, H., Ishii, N., and Suzumura, N., "Structural Classification of Non-Linear Sytems by Input and Output," *Int. J. Systems Sci.*, Vol. 17, No. 5, 1986, pp. 741–774.
23. Wang, M.L., and Chang, R.Y., "Model Reduction and Control System Design by Shifted Legendre Polynomial Functions," *A.S.M.E. J. Dynam. Sys. Meas. Control*, Vol. 105, 1983, pp. 52–55.
24. Horn, I-R., and Chou, J-H., "Analysis and Identification of Non-Linear Systems via Shifted Jacobi Series," *Int. J. Control*, Vol. 45, No. 1, 1987, pp. 279–290.
25. Yun, C-B., and Shinozuka, M., "Identification of Nonlinear Structural Dynamic Systems," *J. Struct. Mech.*, Vol. 8, No. 2, 1980, pp. 187–203.
26. Hammond, J.K., Lo, H.R., and Seager-Smith, E.J., "Identification of Nonlinearities in Vibrating Systems Using Optimal Control Techniques," *IMAC*, 1987, pp. 1467–1473.
27. Mook, D.J., and Junkins, "Minimum Model Error Estimation for Poorly Modeled Dynamic Systems", *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 3, 1988, pp. 256–261.

28. Mook, D.J., "Estimation and Identification of Nonlinear Dynamic Systems," *AIAA Journal*, Vol. 27, No. 7, 1989, pp. 968–974.
29. Mook, D.J., and Stry, G.I., "An Analog Experimental Study of Nonlinear System Identification," to appear, *Nonlinear Dynamics*.
30. Geering, H.P., "Continuous TIme Optimal Control Theory for Cost Functionals Including Discrete State Penalty Terms," *IEEE Trans. A. C.*, Vol. AC-21, 1976, pp. 86–869.
31. Mook, D.J., and Lew, J-H., "Solution of Linear Two-Point Boundary Value Problems with Jump Discontinuities," to appear in the *IEEE Trans. A.C.* .
32. Newland, D.E., *"(An Introduction to) Random Vibrations and Spectral Analysis,"* 3rd. ed., Longman, Great Britain, 1980, pp. 29.
33. Witte, R.S., *"Statistics,"* Holt, Rinehart and Winston, New York, USA, 1980, pp. 75.