STATISTICAL PROPERTIES OF THE PRESSURE FIELD IN A TURBULENT BOUNDARY LAYER*

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A brief description of measurements of the fluctuating wall pressure in a turbulent boundary layer is given. The measurements consist of the root-mean-square pressure, the spectrum of the pressure and the streamwise space-time correlation of the pressure. The space-time correlation measurements show that the pressure field is carried downstream at a convection speed of approximately $0.83U_{\infty}$. On the other hand, the non-dimensional spectral measurements of the pressure appear to be attenuated at high frequencies. Since the transducer is of a finite size, it is concluded that the attenuation is the result of local pressure cancellation on the sensitive area of the transducer. The pressure spectra can be corrected for the effect of the finite area of the transducer if one assumes that the entire pressure field is convected at the same speed, $0.83U_{\infty}$. The correction is not successful. It is therefore concluded that only the large-scale pressure fluctuations are convected at $0.83U_{\infty}$, and the smaller-scale pressure fluctuations must be convected at speeds considerably less than $0.83U_{\infty}$. Support for this idea is obtained from Favre's measurements of the space-time correlation for velocity fluctuations. However, additional experiments are necessary to determine the details of the relationship between the scale, both transverse and longitudinal, and the convection speed of the pressure fluctuations.

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I. INTRODUCTION

This paper presents a brief review of our knowledge of the pressure fluctuations in a turbulent boundary layer. These pressure fluctuations are of interest in problems of acoustics and possibly structural fatigue because they cause motion of the surface on which the boundary layer is developed. The resulting sound radiation and structural motion can be studied statistically once the statistical properties of the random surface pressure produced by the boundary layer are known. The present review is concerned with the turbulent flow and resulting pressure field in the boundary layer developed on a rigid surface. It is supposed that, for small surface displacement, one can study the surface motion and resulting sound radiation by assuming that the turbulent pressure field is approximately the same on a moving surface as it is on a rigid surface. The problem is thus simplified; however, one cannot discuss problems of "flutter" or other large amplitude motion without considering the interaction between the surface motion and the flow field which causes the motion.

The pressure field within a turbulent boundary layer developed on a rigid surface is of considerable interest in itself since there exists no adequate theory of the turbulent boundary layer. Consider the equations of motion for an incompressible fluid.

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad (1)$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_i}{\partial x_j{}^2} \qquad (2)$$

If the divergence of the momentum equation is taken and one uses Eq. (1), the resulting equation,

$$\frac{\partial^2 p}{\partial x_i{}^2} = -\rho \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} \qquad (3)$$

describes the fluctuating pressure field in terms of the fluctuating velocities. The fluctuating pressure may then be written

$$p = \frac{\rho}{4\pi}\iiint \frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}(\xi_1, \xi_2, \xi_3, t)\,\frac{d\xi_1 d\xi_2 d\xi_3}{r} \qquad (4)$$

where

$$r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2 \quad (5)$$

The other term within the integral represents quadrapoles of various orientations and structure (depending on the value of i and j). This term is the one Lighthill[1] has used to describe the sound radiation from a turbulent jet. Here, sound generation and radiation directly from turbulence will not be discussed; hence, the fluid has been assumed incompressible. Indeed, measurements[2] show that pressure fluctuations caused by sound radiation from the turbulent boundary layer on a rigid surface are small compared with the incompressible surface pressure fluctuations, see Eq. (4).

Using Eq. (4) one can obtain a relation for the pressure correlation, in space and time, in terms of the quadrapole correlation, in space and time. However, the nature of the quadrapole correlation is not yet known. There are nine possible quadrapoles for an incompressible medium; since, with zero divergence

$$\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} = \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \quad (6)$$

thus, the task of determining how the pressure is produced in a given turbulent flow is not simple.

II. REVIEW OF EXISTING EXPERIMENTAL INFORMATION

The approach to the problem of pressure fluctuations in turbulent boundary layers has been the experimental measurement of the statistical properties of the surface pressure. The published measurements[2,3,4] have been made in air at subsonic speeds. The subsonic measurements are being extended to supersonic speeds by Kistler and Chen at the Jet Propulsion Laboratory and will be briefly mentioned later.

The statistical property of the surface pressure that is of interest for our purposes is the surface pressure correlation in space and time,

$$P(\xi_1,\xi_2,\tau) = \lim_{\substack{L \to \infty \\ T \to \infty}} \frac{1}{8L^2T} \int_{-T}^{T}\int_{-L}^{L}\int_{-L}^{L} p(x_1,x_2,t)\,p(x_1 + \xi_1, x_2 + \xi_2, t + \tau)\,dx_1\,dx_2\,dt \quad (7)$$

The surface pressure correlation, $P(\xi_1,\xi_2,\tau)$, has not yet been completely investigated. Information about the mean-square pressure,[2,3,4,5] $P(0,0,0)$, the streamwise space-time correlation,[5] $P(\xi_1,0,\tau)$, and the instantaneous transverse correlation,[4] $P(0,\xi_2,0)$, has been obtained.

The spectrum of $P(0,0,\tau)$[3,4,5] (e.g., the "power" spectrum) and the spectrum of the streamwise space-time correlation,[4] $P(\xi_1,0,\tau)$, have also been measured. The measurements are discussed below.

A. Mean-Square Pressure, $P(0,0,0)$

Early measurements by the author[2] have shown that the ratio of root-mean-square fluctuating surface pressure to dynamic pressure is a constant independent of Mach number and Reynolds number for $0.2 < M_\infty < 0.8$ and $1.5 \times 10^6 < Re_x < 20 \times 10^6$. In addition, it was found[2] that almost all the wall surface pressure fluctuations at a given point on the surface are caused by incompressible pressure fluctuations in the turbulent boundary layer near the given point. In fact, very little sound is radiated at subsonic speeds since only 6 per cent of the root-mean-square surface pressure was caused by background noise, vibration, and sound radiation from the turbulence in the boundary layer at $M_\infty = 0.45$. Later measurements[5] with an improved transducer[6] calibrated in a shock tube showed that the ratio of root-mean-square pressure fluctuations to dynamic pressure was 6.55×10^{-3}, see Fig. 1 and Sec. III. The transducer used earlier[2] had resonances and was not accurately calibrated.

Other measurements of the root-mean-square surface pressure are those of Mull and Algranti,[3] Harrison,[4] and unpublished measurements by the NACA that are referred to by Callaghan.[7] Mull and Algranti[3] found $\sqrt{\overline{p^2}}/q_\infty \approx 0.0013$ in flight tests. Harrison[4] found $\sqrt{\overline{p^2}}/q_\infty \approx 9.5 \times 10^{-3}$. Callaghan refers to unpublished measurements by the NACA in flight and in an acoustic channel. Callaghan[7] states that he believes that $\sqrt{\overline{p^2}}/q_\infty \approx 4.5 \times 10^{-3}$.

Additional measurements of $\sqrt{\overline{p^2}}/q_\infty$ have been made by Corcos* on the inner surface of a pipe containing a fully developed turbulent flow at subsonic speeds. Corcos found that $\sqrt{\overline{p^2}}/q_\infty$ was a constant greater than 6×10^{-3} but less than 9×10^{-3}.

Measurements by Kistler and Chen* at supersonic speeds up to $M_\infty = 5$ on the surface of a flat plate are in progress. They find that $\sqrt{\overline{p^2}}/q_\infty$ is of the same order as that measured at subsonic speeds. They will have exact values when they complete a shock tube calibration of their transducers.

B. Streamwise Space-Time Correlation, $P(\xi_1,0,\tau)$

The author's subsonic measurements[8,5] of the space-time correlation in the stream direction were made using two separate transducers and a time delay correlator. The results of these measurements, see Figs 2, 3 and 4, show that a substantial portion of the fluctuating pressure field

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*Private communication.

is convected in the direction of the free stream at a convection speed $U_c \doteq 0.83U_\infty$.

At subsonic speeds, Harrison[4] has measured the Fourier transform with respect to time of $P(\xi_1,0,\tau)$, (e.g., the cross-spectral density of $P(\xi_1,0,\tau)$). He also finds that the pressure field is convected at a speed $U_c \doteq 0.8U_\infty$. Recently, Kistler and Chen* have made supersonic measurements of $P(\xi_1,0,\tau)$ in a turbulent boundary layer. They also find a convected pressure field with approximately the same speed as that measured at subsonic speeds, e.g., $U_c \doteq 0.8U_\infty$.

C. Instantaneous Transverse Correlation, $P(0, \xi_2, 0)$

Harrison[4] has also reported measurements of the instantaneous, $\tau = 0$, surface pressure correlation between two transducers at the same streamwise station but separated by a variable distance, ξ_2, normal to the stream, see Fig. 7. He finds that the correlation $P(0, \xi_2, 0)$ had decreased to a value, $P(0, \xi_2, 0) \doteq 0.5$, when the separation distance, ξ_2, was equal to approximately one boundary layer displacement thickness. Further separation of the transducers resulted in a much slower decay of the transverse correlation. At $\xi_2 = 10\delta$,* the largest separation distance he has investigated, the value of the correlation $P(0, \xi_2, 0)$ is still approximately 0.2. Note that $8\delta^*$ is approximately the thickness of the turbulent boundary layer. The transverse correlation is discussed and compared with the longitudinal correlation in Sec. IV.

D. The Spectrum of $P(0,0,\tau)$, (The "Power" Spectrum)

The author's measurements,[5] at subsonic speeds, of the spectrum, $\tilde{p}(\omega)$, of $P(0,0,\tau)$ are shown in Fig. 5 in non-dimensional form, for various values of the ratio of transducer diameter to boundary layer displacement thickness, d/δ^*. It is apparent that the effect of finite transducer size is to cause high frequency attenuation of the fluctuating pressure measured by the transducer. If there were no attenuation (a transducer with $d/\delta^* \approx 0$) all the data of Fig. 6 when expressed in the dimensionless form of Fig. 6 would lie on a single curve. Approximate corrections of the spectra of Fig. 5 for the effect of finite transducer size have been made, see Sec. III.

The spectrum at subsonic speeds of $P(0,0,\tau)$ has also been measured by Harrison.[4] He finds that the spectrum has a shape similar to that for $d/\delta^* = 1.1$ of Fig. 5, but his spectra contain somewhat more energy at all frequencies. His spectrum and that of Fig. 5 for $d/\delta^* = 1.1$ can be brought into approximate agreement by multiplying the ordinate of Fig. 5 by a factor of approximately 3. This is in agreement with the ratio of the value of $\overline{P^2}$ measured by Harrison[4] to the value of $\overline{P^2}$ measured by the author,[5] namely,

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*Private communication.

$$\frac{\overline{p}^{2(4)}}{\overline{p}^{2(5)}} = \left(\frac{9.5}{5.5}\right)^2 \approx 3 \qquad (8)$$

The spectrum of $P(0,0,\tau)$ at subsonic speeds on the inner wall of a fully developed turbulent pipe flow has been investigated by Corcos.* He found that when his data were expressed in the same non-dimensional form as that of Fig. 5 the spectra also had approximately the same shape as the boundary layer spectra.

At supersonic speeds on a flat plate for Mach numbers as high as M = 5, Kistler and Chen* report that the spectra again have the same shape as before, see Fig. 5, and that all their spectra may be expressed in the same non-dimensional variables as those of Fig. 3.

III. APPROXIMATE CORRECTION OF THE SPECTRA AND THE MEAN SQUARE PRESSURE FOR ATTENUATION CAUSED BY A TRANSDUCER OF FINITE SIZE

Uberoi and Kovaszuay[9] have given the method by which spectral measurements may be corrected for the abberations introduced by a measuring instrument with linear response characteristics. In the present case corrections of the measured spectra are necessary in order to account for the loss in response caused by cancellation of signals from the small scale pressure fluctuations on the finite sensitive area of the pressure transducer. These corrections are analagous to those made for hot wire anemometers of finite length.

In order to correct the measured pressure spectra it is assumed that the entire pressure field is "frozen" and is swept over the face of the stationary pressure transducer at the convection speed, $U_c \approx 0.83U_\infty$. Furthermore, the two dimensional pressure field on the wall is assumed isotropic when viewed by an observer moving at the convection speed. In the coordinate system moving at the convection speed one has a two dimensional spatial correlation and spectrum of the pressure fluctuations which are related by a two dimensional Fourier transform in the following way,

$$\left.\begin{aligned} P(\xi) &= \int_0^{2\pi}\int_0^{\infty} E(k)\exp\left[-i\,\vec{k}\cdot\vec{\xi}\right]\xi d\xi d\theta \\ E(k) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} P(\xi)\exp\left[i\,\vec{k}\cdot\vec{\xi}\right]kdkd\theta \end{aligned}\right\} \qquad (9)$$

*Private communication.

$$\xi^2 = \xi_1^2 + \xi_2^2 \quad \text{and} \quad k^2 = k_1^2 + k_2^2 \tag{9}$$

where circular symmetry (isotropy) has been introduced.

The spectrum measured by the stationary pressure transducer is a one-dimensional spectrum with frequency as a variable. In order to convert the stationary spectra into the spectra that would be measured by an observer moving at the convection speed one interprets the circular frequency as the product of wave number and convection speed

$$\omega = kU_c \; ; \quad U_c = 0.83U_\infty \tag{10}$$

One also needs a relation between the corrected two-dimensional spectrum and the measured one-dimensional spectrum. This relation has been obtained by Uberoi,[10]

$$E(k) = - \left[2\pi \, S(k) \right]^{-1} \int_k^\infty \frac{d\widetilde{p}(k_1)}{dk_1} \left(k_1^2 - k^2\right)^{-\frac{1}{2}} dk_1 \tag{11}$$

where S(k) is the correction factor which accounts for the finite size of the transducer face. For a circular transducer S(k) is

$$S(k) = 4 \left[\frac{J_1(kr_o)}{kr_o} \right]^2 \tag{12}$$

The corrections to the measured spectra were made by fitting smooth curves to the measured one-dimensional spectra and then computing the two-dimensional spectra using Eq. (11). The corrected two-dimensional spectra are shown in Fig. 6 in non-dimensional form. The variables used in Fig. 6 are: the non-dimensional spectrum

$$\mathcal{E}(k\delta^*) = \frac{E(k)}{q_\infty^2 \, \delta^{*2}} \tag{13}$$

and the non-dimensional wave number

$$k\delta^* \tag{14}$$

One can also correct the measured value of mean square pressure for the attenuation caused by a finite size transducer by simply integrating over the corrected spectra of Fig. 6. This integral for the corrected mean-square-pressure is obtained from the first of Eq. (9) with $\xi = 0$. The integral was computed for $d/\delta^* \doteq 1.0$. It was found that the corrected root-mean-square pressure is 1.19 times larger than the uncorrected value. This gives $\sqrt{\overline{P^2}}/q_\infty = 6.55 \times 10^{-3}$ for the corrected root-mean-square pressure and is somewhat greater than is obtained by extrapolating the straight line of Fig. 1 to the origin, $d/\delta^* = 0$.

IV. DISCUSSION OF THE PRESSURE MEASUREMENTS AND CORRECTED SPECTRA

There are a few observations that can be made about the surface pressure measurements that have been obtained. It must be recognized that the detailed structure of the flow in the turbulent boundary layer is still not clear. We will discuss here only the pressure measurements. However, the discussion could obviously be greatly enlarged if we were to bring in the great mass of data already accumulated from hot wire measurements of the fluctuating velocities, see for instance Ref. 11 for a review and extensive bibliography.

First, let us consider the instantaneous transverse pressure correlation measured by Harrison,[4] see Fig. 7. It is of interest to compare his transverse correlation with the instantaneous longitudinal correlation. The instantaneous longitudinal correlation (e.g., in space) can be obtained, approximately, from the spectrum of the pressure. We need to assume that the pressure pattern is convected past the transducer at the convection speed $U_c \doteq 0.83 U_\infty$. Then the Fourier transform of the pressure spectra (the auto correlation function) gives us the instantaneous longitudinal correlation if we substitute for the time delay τ the value

$$\tau = \frac{\xi_1}{U_c} \qquad (15)$$

The auto correlation of the pressure was computed by fitting a smooth curve to the spectra for $d/\delta^* \doteq 1.1$ after substituting Eq. (15) for τ, the relation

$$\frac{P(\xi_1,0,0)}{P(0,0,0)} = e^{-0.36\left(\frac{U_\infty}{U_c}\right)^2\left(\frac{1}{\delta^*}\right)^2} \qquad (16)$$

was obtained. Equation (16) is plotted along with Harrison's transverse correlation in Fig. 7. The longitudinal correlation dies out much more rapidly than the transverse correlation. The difference between transverse and longitudinal correlation is really striking. It would appear from a comparison of Harrison's data[4] and the computed longitudinal correlation, Eq. (16), that the pressure fluctuations have a marked anisotropic structure with large transverse extent compared to the longitudinal extent.

The transverse and longitudinal pressure correlations may be compared with the work of Favre, Gaviglio, and Dumas[12,13] who measured longitudinal and transverse velocity correlations and found that the transverse correlation of the fluctuating velocity in the ξ_1, free stream, direction decreased to a value of two-tenths very rapidly (in a distance less than $\xi_2/\delta^* \doteq 1.6$) while their longitudinal correlation of the same velocity had decreased to a value of two-tenths in a distance $\xi_1/\delta^* \doteq 8\delta^*$ or less depending on the distance of the hot wires from the surface. This is completely different from the results for the pressure, see Fig. 7. These observations point out the need for further experiments to increase our understanding of the structure of the turbulence in the boundary layer.

Additional information which increases our understanding of the turbulent structure of the boundary layer can be obtained from the lack of success of the corrections that were applied to the pressure spectra, see Sec. III and Fig. 6. If the corrections had been successful all the spectra of Fig. 6 would lie on one curve. However, the spectra at high frequencies appear attenuated. The amount of attenuation increases as d/δ^* increases.

It is the author's belief that the attenuation of the corrected spectra at high frequencies, i.e., large k, is caused by the assumption of too large a convection velocity at high frequencies. If a lower convection velocity were used, then at a given frequency, ω, the wave number from Eq. (10), $k = \omega/U_c$, would be larger and the correction factor $\left[S(k)\right]^{-1}$ would be larger.

The postulate is made that while low frequency, small k, pressure fluctuations are convected at $U_c = 0.83U_\infty$; the higher frequency, large k, pressure fluctuations must be convected at lower speeds.

In support of this idea, the work of Favre, Gaviglio and Dumas[12,13] may be examined. They find that velocity fluctuations in the ξ_1, stream, direction are convected at various speeds depending on the distance normal to the surface. In fact, at a given distance normal to the surface the maximum of the space-time correlation between the velocity fluctuations occurs at a ratio of ξ_1/τ which is almost exactly equal to the mean velocity at that distance from the surface. In Fig. 8 are shown the values of ξ_1/τ for which the space-time correlation of the velocity was a maximum. It can be seen that the apparent convection velocity for the ξ_1 velocity fluctuations is the same as the mean velocity profile that was measured in the boundary layer.

Thus, the smaller scale velocity fluctuations near the wall are convected at speeds which decrease as one approaches the wall. The author believes that the pressure field produced by the small scale velocity fluctuations near the wall is convected at speeds considerably less than $U_c = 0.83U_\infty$.

Additional measurements are being performed to discover whether or not this conjecture is correct.

SYMBOLS

d	diameter of transducer face, ft
E	spatial spectrum of pressure, see Eq. (9)
$\mathcal{E}$	nondimensional spatial spectrum of pressure, see Eq. (13)
k	wave number, $(ft)^{-1}$
M_{∞}	free stream Mach number
p	static pressure, lb/ft^2
$\tilde{p}(\omega)$	one-dimensional spectrum of the pressure, with respect to time
P	pressure correlation, see Eq. (7)
q_{∞}	free stream dynamic pressure, lb/ft^2
t	time, sec
u_i	velocity in i^{th} direction, ft/sec
U_c	convection velocity, ft/sec
U_{∞}	free stream velocity, ft/sec
x_i	spatial coordinate in i^{th} direction with x_1 in stream direction, x_3 normal to surface, (ft)
δ	boundary layer thickness, ft
δ^*	boundary layer displacement thickness, ft
ρ	density, $slugs/ft^3$
ν	kinematic viscosity, ft^2/sec
ξ_i	spatial separation of two points in i^{th} direction, ft
τ	temporal separation of two events, sec
$\overline{(\ )}$	time average

REFERENCES

1. Lighthill, M. J., "On Sound Generated Aerodynamically, I General Theory," Proc. Roy. Soc., Ser. A, Vol. 211, No. 1107, March, 1952.

2. Willmarth, W. W., "Wall Pressure Fluctuations in a Turbulent Boundary Layer," J. Acous. Soc. Am., Vol. 20, 1956.

3. Mull, H. R., and J. S. Algranti, Preliminary Flight Survey of Aerodynamic Noise on an Airplane Wing, NACA, RM E55k07, 1956.

4. Harrison, M., "Pressure Fluctuations on the Wall Adjacent to a Turbulent Boundary Layer," David Taylor Model Basin, Report 1260, December, 1958.

5. Willmarth, W. W., Space-Time Correlations and Spectra of Wall Pressure in a Turbulent Boundary Layer, NASA Memo 3-17-59W, March, 1959.

6. Willmarth, W. W., "Small Barium Titanate Transducer for Aerodynamic or Acoustic Pressure Measurements," Rev. Sci. Inst., Vol. 29, No. 3, March, 1958.

7. Callaghan, E. E., An Estimate of the Fluctuating Surface Pressures Encountered in the Reentry of a Ballistic Missile, NACA, TN4315, July, 1956.

8. Willmarth, W. W., "Space-Time Correlations of the Fluctuating Pressure in a Turbulent Boundary Layer," J. Aero. Sci., Vol. 25, No. 5, May, 1958.

9. Uberoi, M. S., and L. S. G. Kovasznay, "On Mapping and Measurement of Random Fields," Quar. Appl. Math, Vol. 10, No. 4, 1952.

10. Uberoi, M. S., "On the Solar Granules," The Astrophysical Journal, Vol. 122, No. 3, November, 1955.

11. Liepmann, H. W., "Aspects of the Turbulence Problem," Z. angew. Math Phys., Vol. 3, pp. 321-342 and 407-426, 1952.

12. Favre, A. J., J. J. Gaviglio, and R. Dumas, "Space-Time Double Correlations and Spectra in a Turbulent Boundary Layer," J. Fl. Mech., Vol. 2, Part 4, June, 1957.

13. Favre, A. J., J. J. Gaviglio, and R. Dumas, "Further Space-Time Correlations of Velocity in a Turbulent Boundary Layer," J. Fl. Mech., Vol. 3, Part 4, January, 1958.

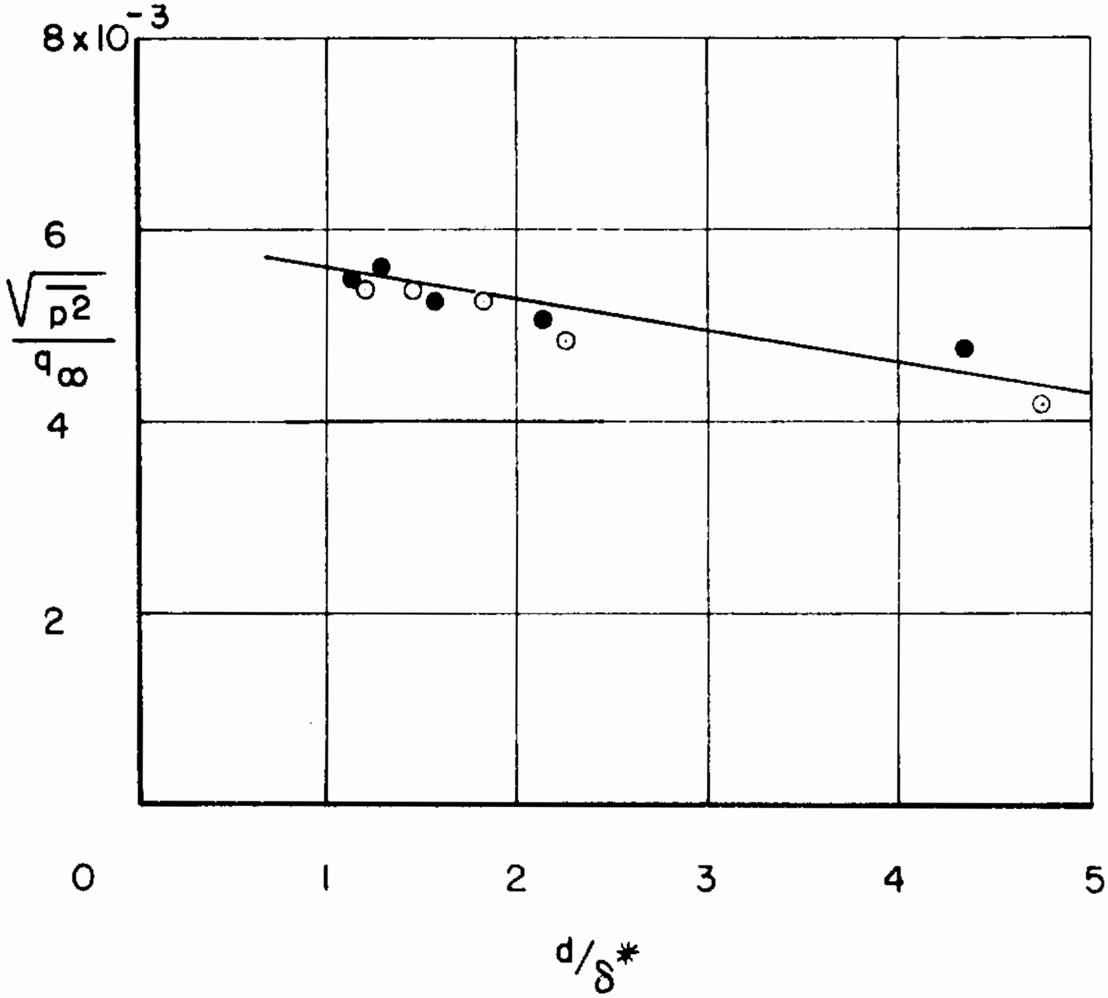

Figure 1 - Root-Mean-Square Wall Pressure

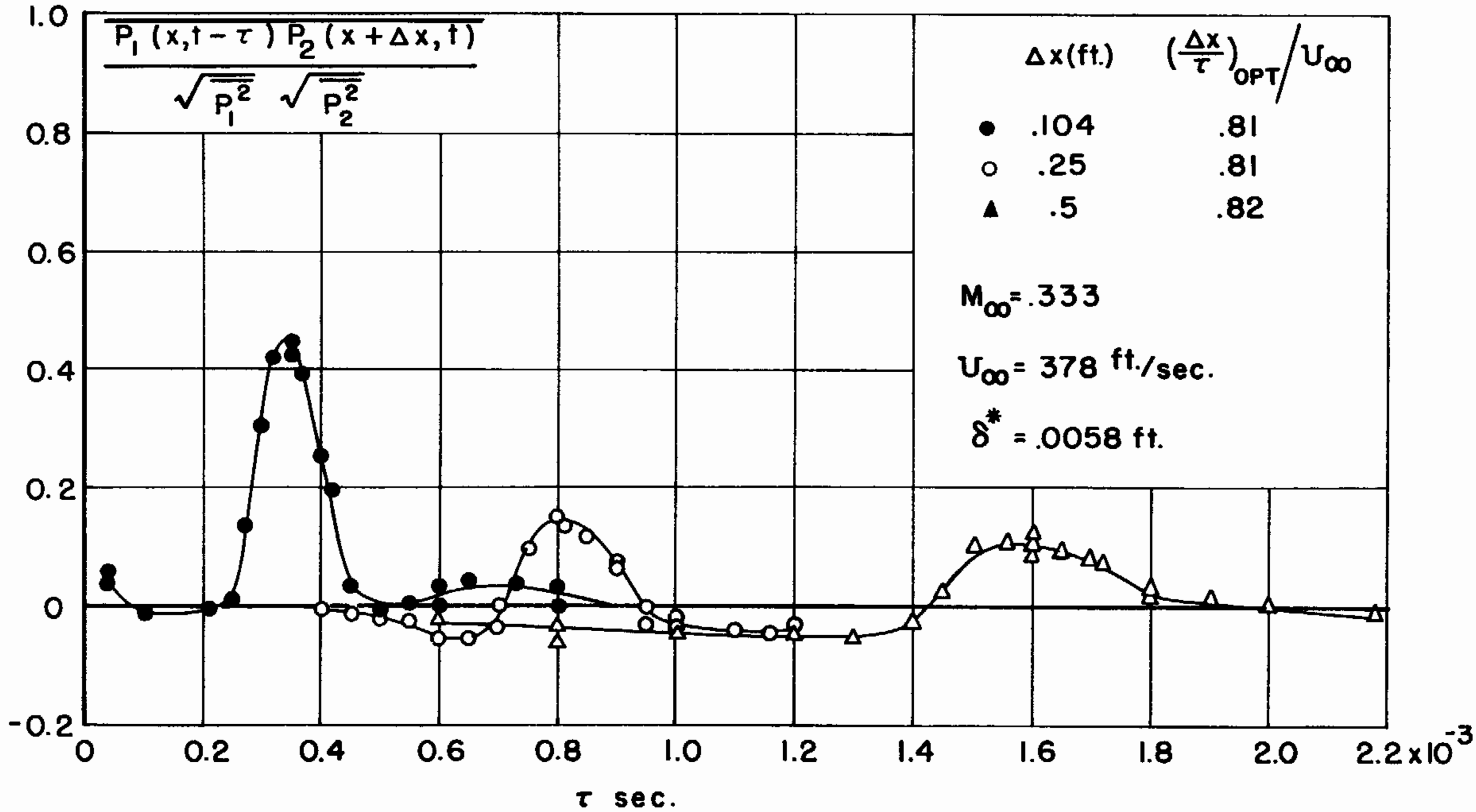

Figure 2 - Streamwise Space-Time Correlation of Pressure

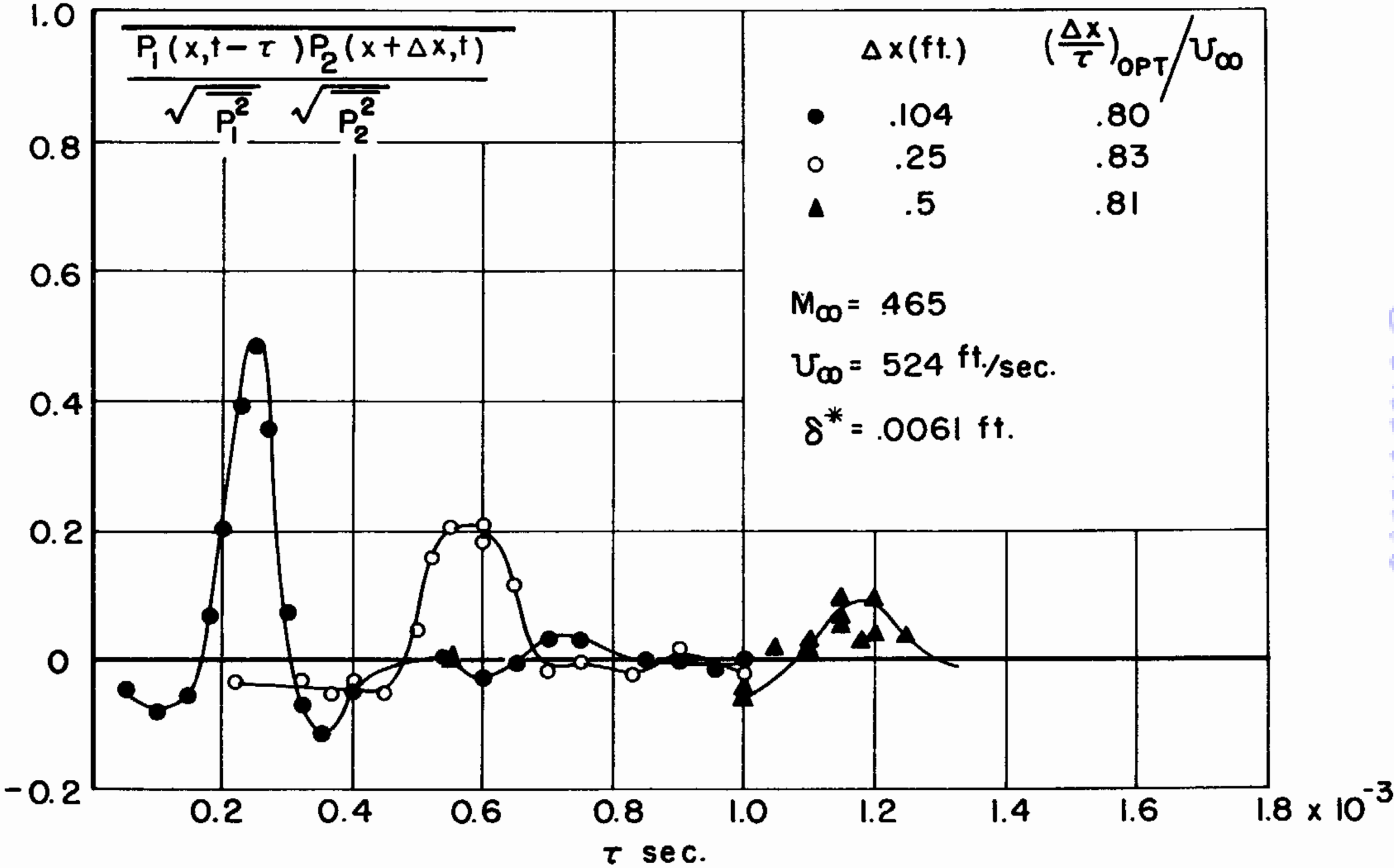

Figure 3 - Streamwise Space-Time Correlation of Pressure

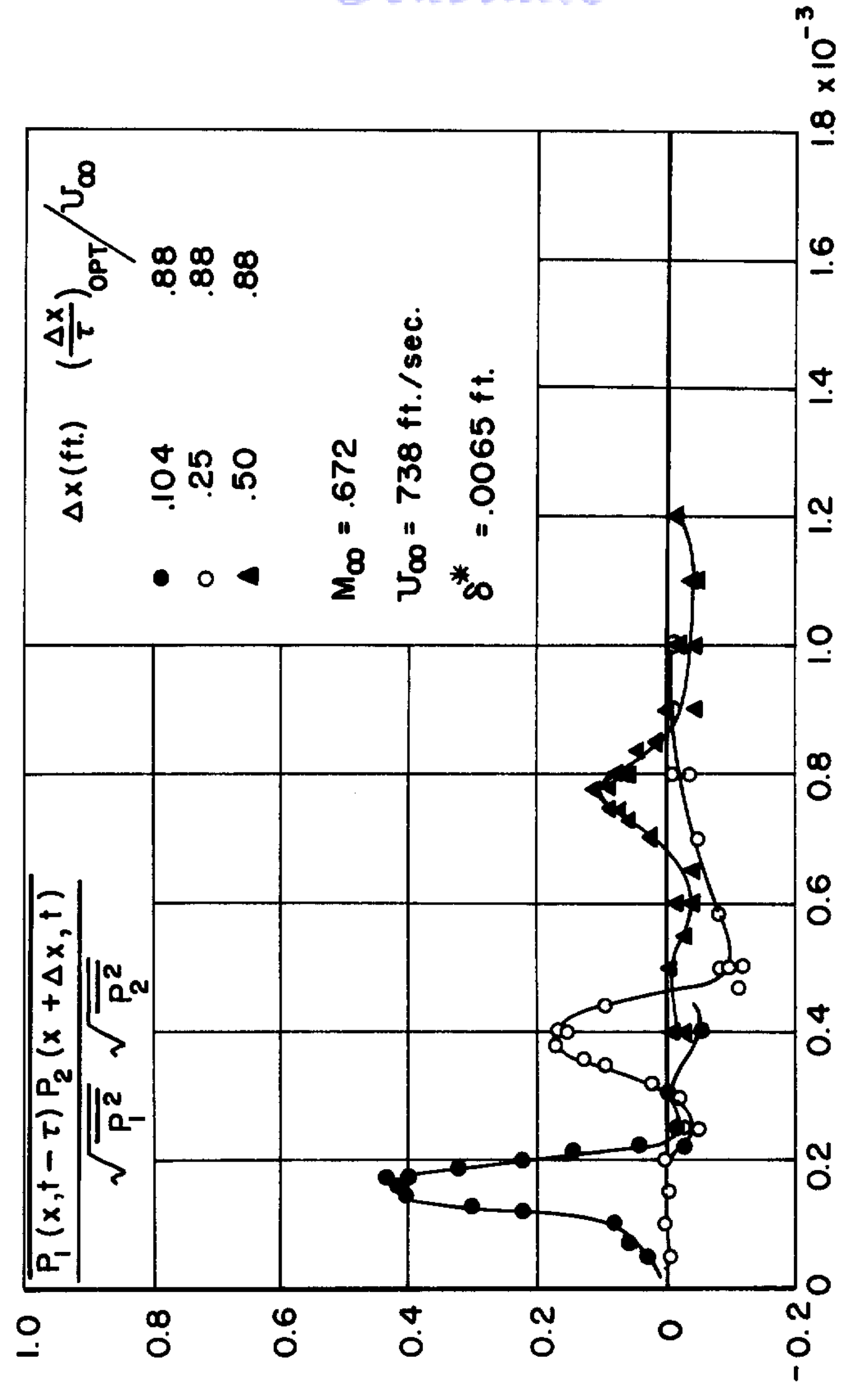

Figure 4 - Streamwise Space-Time Correlation of Pressure

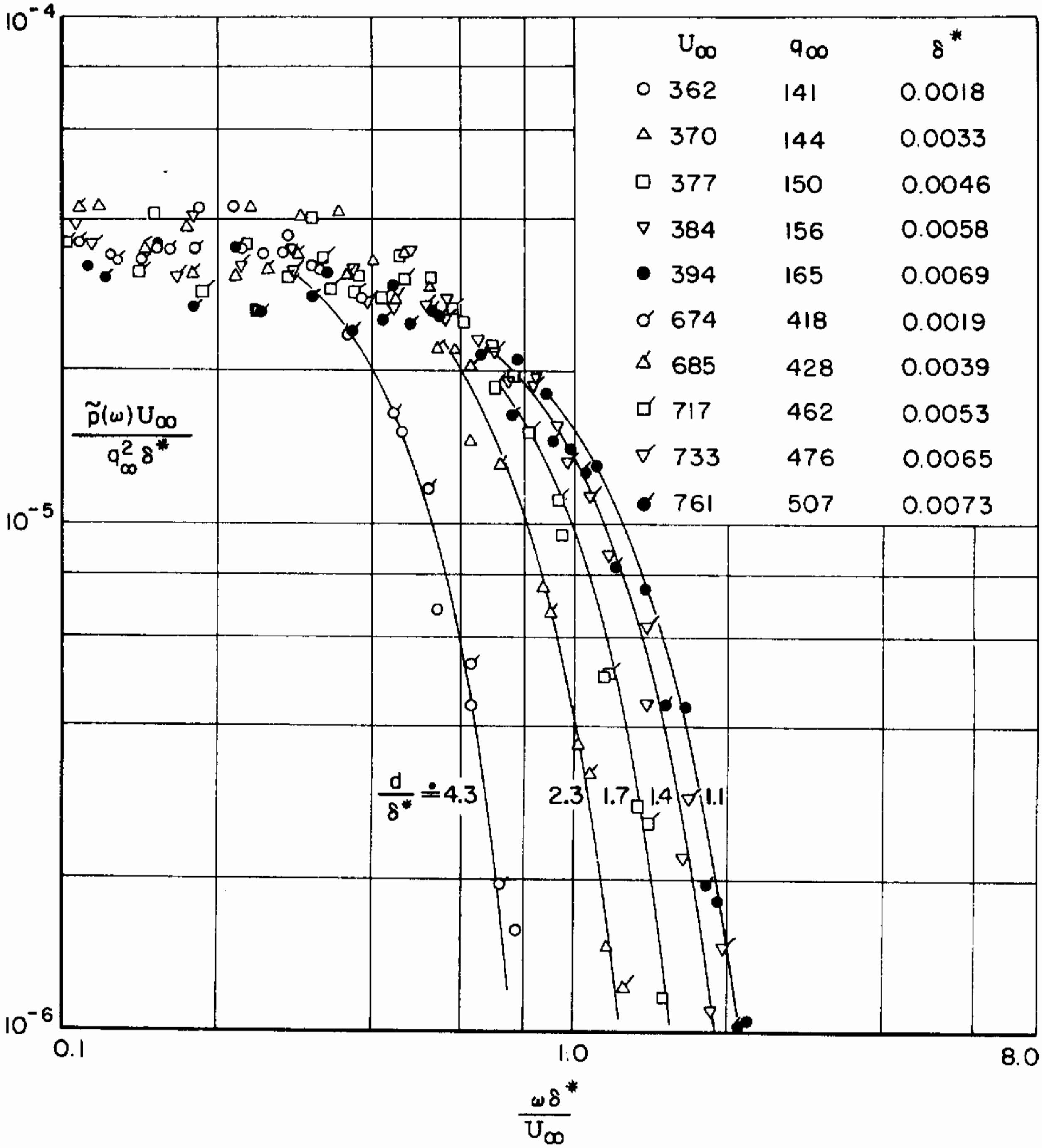

Figure 5 - One-Dimensional, Uncorrected Spectrum of the Wall Pressure

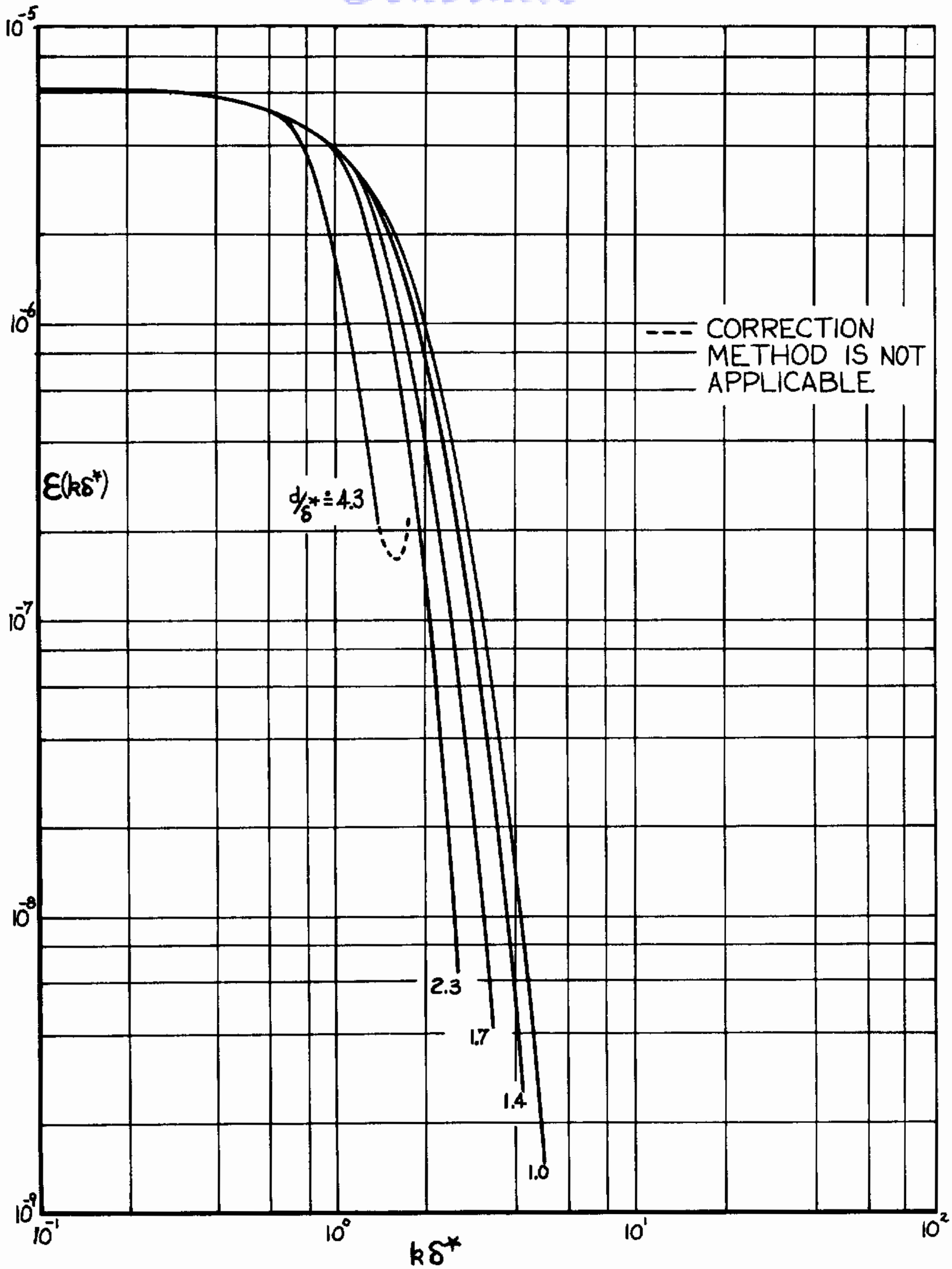

FIGURE 6

Figure 6 - Corrected Spatial Spectrum of the Wall Pressure

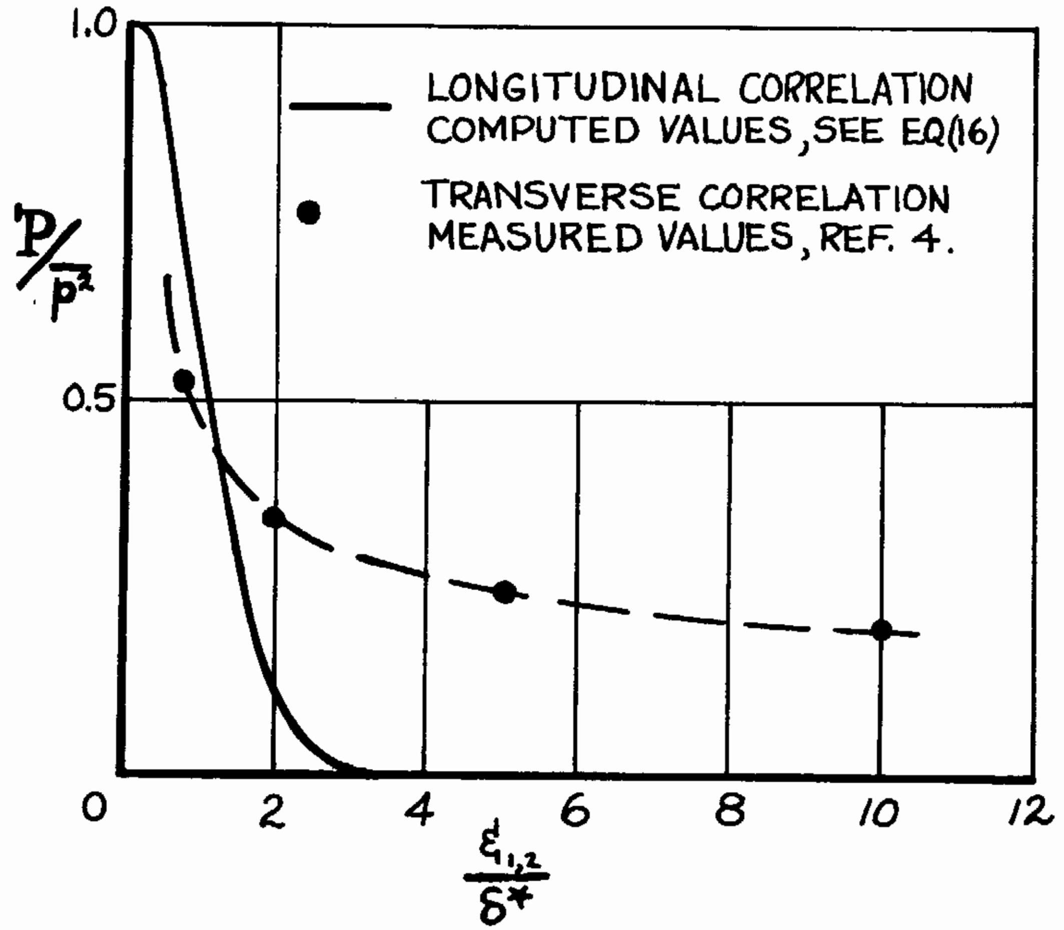

Figure 7 - Instantaneous Transverse and Longitudinal Pressure Correlations

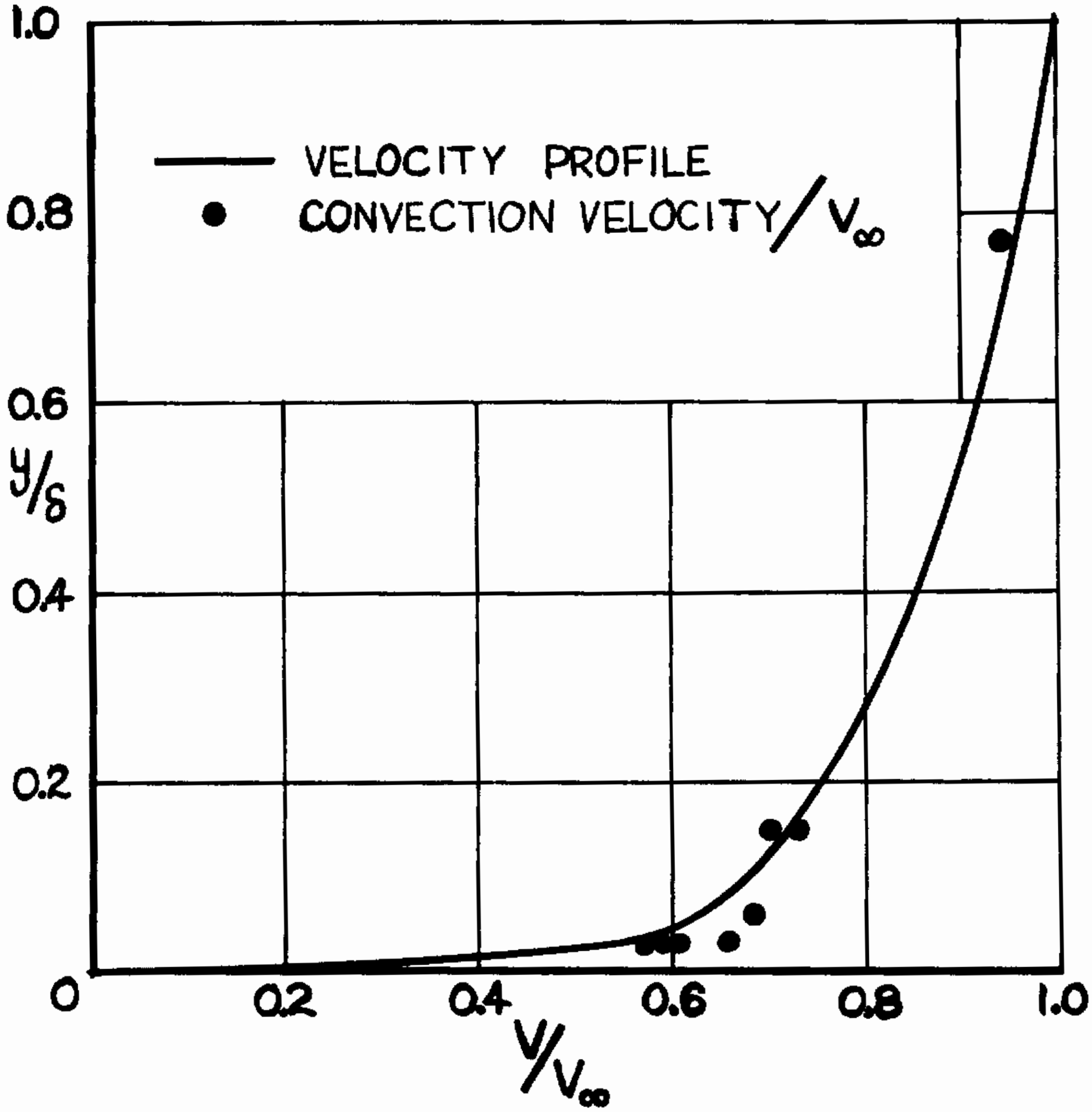

Figure 8 - Apparent Convection Velocities for Streamwise Velocity Fluctuations from Refs. 12 and 13