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RESEARCH ON NONLINEAR ACOUSTICS

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FOREWORD

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This technical report has been reviewed and is approved.

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ABSTRACT

The possibility that nonlinear acoustic flows may be represented by spherical progressive waves (in the sense of Courant and Friedrichs) was examined and found to be unlikely. An iterative finite-difference method for the calculation of continuous periodic spherical flows was developed together with a FORTRAN code, SPHERE, that implements the method. Sample calculations have shown that the code is effective but slow, and several ways for reducing the computation time are suggested. Convergence difficulties in one of the iterative loops were overcome by the use of a semi-iterative underrelaxation scheme. When applied to linear systems, such semi-iterative schemes were found to be equivalent to a class of summability methods that may be regarded as generalizations of Euler summation.

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SECTION I

INTRODUCTION AND SUMMARY

G. M. Muller and C. M. Ablow

The purpose of the present research was to investigate possible new methods for the calculation of nonlinear acoustic fields, with some emphasis on the eventual applicability of these methods to the calculation of axisymmetric flows, especially those generated by a high-intensity siren. However, the difficulties encountered in the course of the work proved sufficiently great to limit the actual investigations to flows having spherical symmetry.

Under the assumptions of classical acoustic theory, acoustic pressure fields are governed by a system of linear partial differential equations. For many purposes, the dissipative effects of heat conduction and viscosity may be neglected, and the governing equations reduce simply to the linear wave equation. The calculation of the acoustic pressure field imposed on an ambient atmosphere by a periodic source is one of the important areas of classical acoustics; because of the linearity of the underlying equations, a great variety of solution techniques is available, corresponding to the diversity of possible boundary conditions. (An extensive compendium both of techniques and of solved problems may be found in [8], especially Chapters 7 and 11.)

For acoustic pressure fielas generated by sources of sufficiently high intensity, the approximations made in deriving the linear wave equation are no longer valid,[†] and an adequate theory of the acoustic field has to be based on the nonlinear equations of gas dynamics. The mathematical complications arising from the nonlinearity are so severe that the only problems that have been investigated in any depth are those associated with propagation of plane waves. A good survey of this field, with some attention to dissipative effects, is an article by Beyer [7, Chapter 10]; more detailed discussions and references to very recent work may be found in a series of papers by Blackstock [1-4].

Much less is known about spherically symmetric waves. Heaps [5] and Blackstock [1] have shown that under assumptions valid at large distances from the center of symmetry, plane-wave solutions may be made to represent spherical waves under a simple change of variables. Naugolnykh [7, p. 247] has derived a perturbation solution valid near the surface of a pulsating sphere; this solution contains, to first order, the effects of both nonlinearity and dissipation. A significant but little-known contribution to the literature of nonlinear spherical waves in a nondissipative medium was made by Laird, et al. [6] who described two different perturbation methods (valid, respectively, close to and at some distance from the surface of a pulsating sphere) and used these to calculate several representative acoustic flow fields.

Two basic approaches were used in this investigation, and these are reflected in the organization of this report: Sections II, III, and IV deal with the general subject of oscillatory self-similar flows and

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† Significant nonlinear effects may appear with acoustic pressure amplitudes of no more than a few percent of the ambient pressure.

related theoretical questions; Sections V and VI are concerned with the calculation of continuous, periodic, spherically symmetric flows by a novel finite-difference method, with emphasis on the underlying numerical analysis.

In Section II it is shown that no spherical progressive wave provides a continuous oscillatory flow. Discontinuous periodic flows produced by a succession of spherically symmetric puffs of gas at the origin, a highly idealized model of siren behavior, are considered in Section III. The puffs are separated by shock waves. Conditions at the origin (that each puff contains a finite mass, momentum, and energy) and at infinity (that the shocks weaken to sound waves) have been used to eliminate a parameter and to locate the phase plane trajectory of the solution. The remaining parameter is determined by the strength of each puff or by the time between puffs, so that for a progressive wave flow the strength and timing of puffs are related. A modified phase plane has been defined in which the shock condition for arbitrary points along the trajectory can be expressed geometrically. In this plane it is clear that for a spherical progressive flow to be bounded by periodic shocks is highly unlikely; however, this negative conjecture has not been rigorously established.

Section IV makes a start toward the exploration of general methods for finding types of flow similarity other than that given by dimensional analysis. One such method could be based on Noether's theorem which gives a partial integration of equations derived from a variational principle. The section contains an account of the theorem as a basis for possible further work.

Section V describes a method for the numerical calculation of continuous periodic flows with spherical symmetry. It is shown that under near-acoustic conditions such flows are, strictly speaking, impossible even within a subregion of the flow field if the entire flow field extends to infinity. Accordingly, a modified problem is defined for flows confined to a region $1 < r < R$, with the boundary condition at infinity replaced by one at $r = R$. Various forms of this boundary condition are discussed; of these, the <u>acoustic</u> boundary condition--that the flow at R have the character of an outgoing wave in the sense of classical acoustics--is chosen for use in the numerical computations. Specification of the problem is completed by prescribing at $r = 1$ either the velocity or the mass flow as a periodic function of time, $Q(\tau)$, and imposing on the flow variables the requirement that they have the same periodicity in time as $Q(\tau)$. The boundary and periodicity conditions together define what is essentially a boundary value problem, whereas the usual problem posed for the hyperbolic partial differential equations of unsteady compressible flow is either of the initial value or the mixed initial-boundary value type. Not surprisingly, therefore, the method of solution involves forward-stepping (in approximately characteristic directions) within the framework of an outer iterative loop designed to satisfy the boundary conditions. Convergence difficulties are removed by use of an appropriate underrelaxation scheme.

A FORTRAN code, SPHERE, that implements the numerical method has been developed and is described in sufficient detail in Section V to permit its use by interested persons. Preliminary calculations are described and suggestions both for further calculations and for modifications of the code are made. Finally, attention is drawn to the possibility of the existence of subharmonic solutions and the consequent possible lack of uniqueness in asymptotic behavior (time $\to \infty$) of the corresponding mixed initial-boundary value problem.

In Section V the convergence difficulties are shown to be essentially associated with the linear part of the problem, i.e., with the location of the eigenvalues of a certain iteration matrix $\mathcal{L}$. Section VI contains a general discussion of methods for removing such difficulties and establishes the close relationship between these methods and various concepts of classical summability theory.

SECTION II

CONTINUOUS OSCILLATORY FLOWS

C. C. Ablow and L. D. McCulley

Solutions to the equations of gas dynamics are found under some restrictive condition which serves to define the physical situation and to direct attention toward appropriate solution techniques. The condition that the flow is a spherical progressive wave is a restriction to flows spreading from a single point in space-time and provides a simple system of first-order ordinary differential equations for solution. The spherical progressive blast wave of Taylor [1] and the shock reflection solution of Guderley [2] are important contributions to our understanding of the flow of gases.

Flows of use in acoustics are oscillatory. This section reports on a search for continuous oscillatory spherical progressive waves, and contains a rigorous proof that no such waves can exist.

A. Differential Equations

For spherically symmetric wave motion in a polytropic medium with adiabatic constant γ, the radial velocity u, the pressure p, and the density ρ as functions of the radius r and the time t satisfy

$$u_t + uu_r + p_r/\rho = 0$$

$$\rho_t + u\rho_r + \rho(u_r + 2u/r) = 0$$

$$(p\rho^{-\gamma})_t + u(p\rho^{-\gamma})_r = 0 \qquad (A.1)$$

where subscripts indicate partial differentiation.

The class of solutions may be restricted to those of the "progressing wave" type [1, p. 416] in the following way. If c is sound speed in the medium, define new variables U, C, Ω, P, and η by

$$u = r(\lambda t)^{-1}U(\eta)$$

$$c = r(\lambda t)^{-1}C(\eta)$$

$$\rho = r^{\kappa}\Omega(\eta)$$

$$p = r^{\kappa+2}(\lambda t)^{-2}P(\eta)$$

$$\eta = r^{-\lambda}t \qquad (A.2)$$

where κ and λ are constants, with $\lambda > 0$. Then $\gamma P/\Omega = C^2$, and from (A.1) a system of ordinary differential equations with variables U, P, and R can be derived:

$$U_\eta = A/\eta D$$

$$R_\eta = 2BR/\eta D$$

$$P_\eta = EP/\eta D \qquad (A.3)$$

where

$$R = C^2$$

$$D = (U - 1)^2 - R$$

$$A = \lambda^{-1}[U(U - 1)(U - \lambda) - 3R(U - \sigma_1)]$$

$$B = \lambda^{-1}[(U - 1)^2(U - \lambda) + (\gamma - 1)U(U - 1)(U - \sigma_2) - R(U - \sigma_3)]/(U - 1)$$

$$\sigma_1 = (2\lambda - 2 - \kappa)/3\gamma$$

$$\sigma_2 = (3 - \lambda)/2$$

$$\sigma_3 = 1 + [2\lambda - 2 + (\lambda - 1)\kappa]/2\gamma$$

$$E = (U - 1)[\lambda^{-1}(U - \lambda)(2 + 2\gamma + \kappa) + (3\gamma + \kappa)] - \gamma\lambda^{-1}(U - \lambda) - (2 + \kappa)\lambda^{-1}R \ .$$

B. Oscillatory Solutions

Acoustically significant solutions are taken to be progressing waves that contain oscillations and that die away with distance from the origin. Points with zero sound speed are permitted only at the origin or at infinity in physical space.

The method of solution is to examine the solutions to

$$\frac{dR}{dU} = \frac{2BR}{A} \tag{B.1}$$

in the (U,R) phase plane. An acceptable solution curve starts ($\eta = 0$) at a point P_1 on $U = 0$ so as to represent a flow dying away to $u = 0$ at $r = \infty$. Since the differential equation determining η, the first equation of (A.3), may be written

$$\frac{d(\ln \eta)}{dU} = \frac{D}{A} , \tag{B.2}$$

one sees that starting point P_1 is a singularity for $\ln \eta$ and so requires $A = 0$. This, with $U = 0$, generally puts P_1 at the origin $U = R = 0$. (In case $\sigma_1 = 0$, P_1 can be elsewhere on $U = 0$. The special case $\sigma_1 = 0$ is discussed in subsection F.)

An acceptable solution curve continues from initial point P_1 toward terminal point P_2, at which $\eta = \infty$. At P_2, $A = 0$ since $\eta = \infty$. For an oscillatory solution, the phase plane solution curve will spiral around in approaching P_2. Thus P_2 is a singular point for the phase-plane differential equation (B.1) of the type called a focus [3].

Let $\bar{A}$ and $\bar{B}$ be the polynomials $\bar{A} = \lambda A$, $\bar{B} = \lambda(U - 1)B$. Then equation (B.1) becomes

$$\frac{dR}{dU} = \frac{2\bar{B}R}{\bar{A}(U - 1)} \tag{B.3}$$

Singular points in the phase plane occur where $\bar{B}R = \bar{A}(U - 1) = 0$. Such points on $R = 0$ cannot be used for terminal points, P_2, since the spiraling solution curve would cross into negative R where sound speed function C is imaginary. Singular points on $U = 1$ also require $R = 0$, except in the special case $\sigma_1 = \sigma_3 = 1$ discussed in subsection G. Thus points (U_0, R_0), which can serve as end points for oscillatory solution curves, are found from $\bar{A} = \bar{B} = 0$. These equations can be written in the form

$$\begin{aligned} R_0 &= U_0(U_0 - 1)(U_0 - \lambda)/3(U_0 - \sigma_1) \\ &= (U_0 - 1)[(U_0 - 1)(U_0 - \lambda) + (\gamma - 1)U_0(U_0 - \sigma_2)]/(U_0 - \sigma_3) \; . \end{aligned} \tag{B.4}$$

Eliminating R_0 gives a cubic equation which can be factored:

$$[U_0 - 2\lambda/(3\gamma - 1)][U_0^2 + U_0\{\lambda - 3(1 + \sigma_1)\}/2 + 3\sigma_1/2] = 0 \; . \tag{B.5}$$

It is easily verified that the two roots of the quadratic factor give points lying on $D = 0$. Such points cannot serve as end points of oscillatory phase plane curves because in spiraling toward one of them the solution curve would cross $D = 0$ repeatedly. By equation (B.2), η would not steadily increase along the curve.

In summary, an acceptable phase plane curve representing an oscillatory solution starts at P_1, the origin, and ends at P_2: $U_0 = 2\lambda/(3\gamma - 1)$ with R_0 given by (B.4). The curve joining P_1 and P_2 should not touch $R = 0$ and so, since solution curves cross $U = 1$ only at $R = 0$, it is necessary that $U_0 < 1$.

C. The Focus Condition

The condition necessary for singular point P_2 to be a focus will now be derived. Let $a = \bar{B}_R$, $b = \bar{B}_U$, $c = \bar{A}_R$, and $d = \bar{A}_U$ where the subscripts indicate partial differentiation and the partial derivatives are evaluated at (U_0, R_0). Then

$$a = -(U_0 - \sigma_3)$$

$$b = 2(U_0 - \lambda)(U_0 - 1) + (U_0 - 1)^2 + (\gamma - 1)[(U_0 - 1)(U_0 - \sigma_2) + U_0(U_0 - 1) + U_0(U_0 - \sigma_2)] - R_0$$

$$c = -3(U_0 - \sigma_1)$$

$$d = (U_0 - 1)(U_0 - \lambda) + U_0(U_0 - 1) + U_0(U_0 - \lambda) - 3R_0 \, , \tag{C.1}$$

and if $(ad - bc) \neq 0$, one may write equation (B.3) near (U_0, R_0):

$$\frac{dR}{dU} = \frac{2R_0[a(R - R_0) + b(U - U_0)]}{(U_0 - 1)[c(R - R_0) + d(U - U_0)]} \, . \tag{C.2}$$

Following Lefschetz[5], (U_0, R_0) is a focus if

$$[2R_0 a - (U_0 - 1)d]^2 + 8R_0(U_0 - 1)bc < 0. \tag{C.3}$$

Proper choice of the two rational expressions for R_0, given in (B.4), allows writing $R_0 a$ and $R_0 c$ as polynomial expressions in U_0. Substitution and rearrangement give

$$9R_0^2 + q(\lambda, \gamma)R_0 + r(\lambda, \gamma) < 0 \tag{C.4}$$

where

$$q = m - 6s$$

$$r = s^2 - m(b + R_0)$$

$$m = 8U_0(U_0 - \lambda)$$

$$s = 3(U_0 - 1)(U_0 - \lambda) + 2(\gamma - 1)U_0(U_0 - \sigma_2) + U_0(U_0 - 1) + U_0(U_0 - \lambda) \, .$$

Since κ is arbitrary, R_0 may be assigned any value. Thus condition (C.4) can be satisfied by proper choice of κ if

$$q^2 - 36r > 0. \tag{C.5}$$

Set $\zeta = \gamma - 1$. Then, using the fact that $U_0 = 2\lambda/(3\gamma - 1) < \lambda$, one finds that

$$8(\zeta + 1)\lambda^2 - (3\zeta + 2)(3\zeta + 10)\lambda + 3(3\zeta + 2)^2 < 0 \tag{C.6}$$

is equivalent to equation (C.5). Thus equation (C.5) holds in a region of the ζ-λ plane bounded by the curves

$$\lambda = (3\zeta + 2)(3\zeta + 10 \pm \sqrt{9\zeta^2 - 36\zeta + 4})/16(\zeta + 1).$$

Since $\lambda > 0$ and $\gamma > 1$, we may restrict the region under consideration to the first quadrant of the ζ-λ plane. These curves are sketched in Fig. 1.

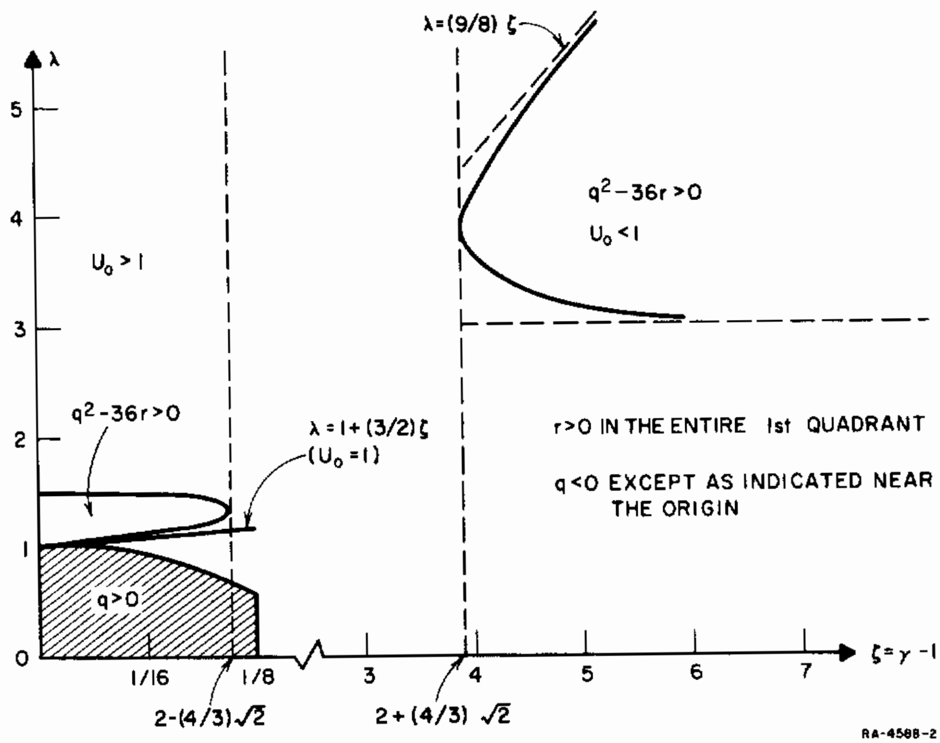

FIG. 1 PARAMETER PLANE — OSCILLATORY FLOW

One may record

$$q = -6U_0[U_0(6\mu^2 - 2\mu + 1) + 3\mu - 1]$$

$$r = U_0^2\,[U_0^2\,(36\mu^4 + 72\mu^3 + 96\mu^2 + 12\mu + 1)$$
$$- U_0(36\mu^3 + 168\mu^2 + 30\mu + 2)$$
$$+ (81\mu^2 + 18\mu + 1)]$$

$$\mu = \zeta/2 = (\gamma - 1)/2$$

$$U_0 = 2\lambda/(3\gamma - 1) \ . \qquad (C.7)$$

The discriminant of the bracketed quadratic in the expression for r is

$$-864\mu^3(12\mu^3 + 16\mu^2 + 7\mu + 1)$$

and is negative for positive μ. Therefore, r is positive for positive λ and ζ. From the factored form above, q is readily seen to be negative in the first quadrant of the (λ,ζ) plane, except for a small region near the origin. Since this region does not intersect that in which equation (C.5) holds, as shown in Fig. 1, q is negative where the polynomial in equation (C.4) has real roots. Those real roots are then positive values for R_0 and so give real, positive sound speeds C_0.

The existence of foci with $U_0 < 1$ is assured for values of parameters ζ and λ falling in the proper region of Fig. 1. From the fact that the region lies between the straight lines $\lambda = 3$ and $\lambda = 9\zeta/8$, one finds that the foci lie between $U_0 = 0$ and $U_0 = 3/4$.

D. Existence of Oscillatory Solutions

In just what region of the (U_0, R_0) plane foci can lie remains to be determined. For the finding of significant solutions, it is necessary to know whether this region is on the same side of D = 0 as the origin

$R = U = 0$. For, if so, many solution curves may be expected for a given parameter pair (λ,γ) running from the origin, P_1, to a termination at the focus P_2. If the region in which equation (C.4) is satisfied lies across $D = 0$ from the origin, then only solution curves crossing $D = 0$ at singular points can provide a proper solution.

One can see that the region does lie across $D = 0$ from the origin, as follows: First, the line $D = 0$ is not in the region where equation (C.4) is satisfied because on $D = 0$, $R_0 = (1 - U_0)^2$, and the polynomial in equation (C.4) becomes

$$[2U_0^2(3\mu^2 + 3\mu + 1) - U_0(3\mu + 5) + 3]^2.$$

Second, for the particular values $\lambda = 5$ and $\gamma = 7$, one may compute the range on which R_0 lies to be $(65 - 4\sqrt{30})/12 < R_0 < (65 + 4\sqrt{30})/12$ or $3.60 < R_0 < 7.24$. This range is well above $(1 - U_0)^2 = 1.4$. It follows that the region lies above $D = 0$ for all λ and γ since it is a connected set, the continuous image of a connected set in the parameter plane.

In summary, an oscillatory solution exists as a curve in the (U,R) phase plane connecting the origin P_1 with the focus P_2, provided there is a singular point P_3 on $D = 0$ in $U < 1$ of a type that permits the solution curve to cross $D = 0$ in going from P_1 to P_2.

E. Phase Plane Geometry

The curve $\bar{A} = 0$, along which solution curves in the (U,R) phase plane have vertical tangents, passes through the origin, P_1, with positive slope if σ_1 is negative, or with negative slope if σ_1 is positive.

If σ_1 is positive, $A = 0$ has a branch on $0 < U < 1$ in $R > 0$ only for $\sigma_1 < 1$. This branch drops monotonically from $R = \infty$ at $U = \sigma_1$ to

$R = 0$ at $U = 1$. A solution curve starting at the origin P_1 and passing through P_3, the intersection of $\bar{A} = 0$ and $D = 0$, cannot continue on toward the singular point P_2. This is because P_2 is above and to the left of P_3, i.e., $R_2 > R_3$ and $U_2 < U_3$ so that the solution curve, which has positive slope from P_1 through P_3, would have to turn back through a point of infinite slope, a point on $\bar{A} = 0$. But the solution curve beyond P_3 is above and to the right of $\bar{A} = 0$ and, with its positive slope, tends away from $\bar{A} = 0$. Hence no solution curve can run from P_1 through P_3 to P_2 as needed.

For $\sigma_1 \leq 0$, the following algebraic argument shows that no oscillatory solution exists. By (B.4), with U_0 from (C.7), the coordinates of the focus singularity P_2 are related by $R_0 = \mu U_0^2(1 - U_0)/(U_0 - \sigma_1)$. Since $0 < U_0 < 1$, $\sigma_1 \leq 0$ implies $R_0 \leq \mu U_0(1 - U_0)$. From focus condition (C.4) the least possible value for R_0 is $(-q - \sqrt{q^2 - 36r})/18$. Therefore, for an oscillatory solution, this becomes

$$-\sqrt{q^2 - 36r} \leq 18\mu U_0(1 - U_0) + q. \qquad \text{(E.1)}$$

Using (C.7) one may simplify the right side of (E.1) to $-6U_0[U_0(6\mu^2 + \mu + 1) - 1]$. If this quantity were positive or zero, (E.1) would be satisfied. However, with $\lambda \geq 3$, one finds $U_0 \geq 3/(3\mu + 1)$, and $U_0(6\mu^2 + \mu + 1) - (18\mu^2 + 2)/(3\mu + 1) > 0$, so that the right side of (E.1) is negative. With both sides of (E.1) negative, the inequality requires

$$[18\mu U_0(1 - U_0) + q]^2 \leq q^2 - 36r.$$

or

$$36U_0^2[U_0^2(36\mu^4 + 108\mu^3 + 93\mu^2 + 18\mu + 1) - U_0(72\mu^3 + 156\mu^2 + 42\mu + 2) + (72\mu^2 + 24\mu + 1)] \leq 0.$$

The discriminant of the quadratic factor is $-576\mu^3(9\mu^3+21\mu^2+12\mu+2)$ so that the quadratic is positive for positive μ, and the inequality fails to be satisfied.

F. Special Case $\sigma_1 = 0$

This case yields no oscillatory solution since none of the singular points provides an acceptable focus. The singular points are

P_1: $(0,0)$

P_2: $(2\lambda/[3\gamma - 1],\ \{\lambda[3\gamma - 1] - 2\lambda^2\}/[3\gamma - 1]^2)$

P_3: $(0,1)$

P_4: $(1,0)$

P_5: $(\lambda,0)$

P_6: $([3 - \lambda]/2,\ [\lambda - 1]^2/4)$.

Points P_1, P_4, and P_5 cannot provide a useful focus since, in spiraling about one of these points, the integral curve would dip into regions where $R = C^2 < 0$. The argument in the last paragraph of subsection E above applies here to show that P_2 is not a focus. Points P_3 and P_6 lie on $D = 0$ so that integral curves spiraling around them would unacceptably cross $D = 0$ at nonsingular points.

This special case provides no oscillatory solution.

G. Special Case $\sigma_1 = \sigma_3 = 1$

In this case, $\lambda = (3\gamma - 1)/2$, and singular point P_2 lies on $U = 1$. One readily checks that $U_0 = 1$, $R_0 = 3(\gamma - 1)^2/4$, and that (C.2) becomes

$$\frac{dR}{dU} = \frac{2R_0[4(R - R_0) - b'(U - U_0)]}{d'(U - 1)}$$

where $b' = (3\gamma - 5)(\gamma - 1)$ and $d' = 9\gamma^2 - 12\gamma + 11$. Focus condition (C.3) becomes $(8R_0 - d')^2 < 0$ so that P_2 is no longer a focus. The other singular points lie, as before, on $R = 0$ or $D = 0$ and so can provide no acceptable foci.

No oscillatory solution is present in this case.

SECTION III

DISCONTINUOUS OSCILLATORY FLOWS

C. M. Ablow and Y.D.S. Rajapakse

Periodic flows could conceivably be produced by a succession of shocks propagating from an origin. The construction of such flows is attempted in this section under the assumption that the region between shocks is occupied by a spherical progressive wave.

Necessary conditions for a solution show that the source emits a finite burst of energy in each period but no mass or momentum. The quantity of energy and the period are related to one another, and the sound speed of the gas is zero at the source and at infinity. Just a few flows are found that satisfy these conditions.

A major condition, that the bounding discontinuities be shock waves at intermediate points as they are at the source point and at infinity, has not been applied. Since even without this condition the solution is all but unique, the requirement of spherical progressive wave character for the intershock flow is apparently too stringent, so that only an approximation to a possible periodic flow has been found.

A. The Source

For an oscillatory flow resembling that emanating from a siren, one may consider a succession of puffs or spurts of gas at an origin, each of which creates a spherical progressive wave, the waves being separated from one another by shock discontinuities.

According to the progressive wave assumption, the continuous flow between the shocks is represented by a single curve K in the (U,C) phase plane, the curve being traced out by parameter η, $\eta = tr^{-\lambda}$. Thus curve K corresponds to a whole region of the (r,t) physical plane and also to a whole region of the (u,c) hodograph plane.

The shock curve in the physical plane provides two curves, A and B, in the hodograph plane, with A the curve representing points just ahead of the shock, and B the points just behind it. To be in a single continuous flow region, the A points are considered to be ahead of the shock which ends the region, while the B points are behind the shock which begins it.

The content of each puff may be taken to be a certain mass of gas, outward momentum, or quantity of energy. If M, Q, and K are the amounts of mass, outward momentum, and energy passing through a spherical surface of radius r between shocks, then

$$M = 4\pi r^2 \int_{t_B}^{t_A} \rho u \; dt \tag{A.1}$$

$$Q = 4\pi r^2 \int_{t_B}^{t_A} \rho u^2 \; dt \tag{A.2}$$

$$K = 4\pi r^2 \int_{t_B}^{t_A} [\rho(e + u^2/2) + p] u \; dt \tag{A.3}$$

where t_A and t_B are the times of passage of the shocks past the sphere, and ρ, u, and p are the density, speed, and pressure in the gas. For a perfect gas, the internal energy, e, is given by $[p/(\gamma - 1)\rho]$, γ being the ratio of specific heats. In terms of the variables describing spherical progressive waves, defined in Section II, we have

$$M = (4\pi/\lambda)\ r^{3+\kappa} \int_{\eta_B}^{\eta_A} U(\eta)\ \Omega(\eta)\ d\eta/\eta \tag{A.4}$$

$$Q = (4\pi/\lambda^2)\ r^{\kappa-\lambda+4} \int_{\eta_B}^{\eta_A} \Omega U^2\ d\eta/\eta^2 \tag{A.5}$$

$$K = (2\pi/\lambda^3)\ r^{\kappa-2\lambda+5} \int_{\eta_B}^{\eta_A} \left\{\Omega U^3 + [2\gamma/(\gamma-1)]PU\right\}\ d\eta/\eta^3. \tag{A.6}$$

The general form of these integrals reads

$$I = k_I r^j \int_{\eta_B}^{\eta_A} f(\eta)\ d\eta \tag{A.7}$$

$$j = \kappa + 3 - n(\lambda - 1), \qquad n = \begin{cases} 0 \text{ for } I = M \\ 1 \text{ for } I = Q \\ 2 \text{ for } I = K \end{cases}$$

and k_I is the proper constant.

The basic assumption of the present model of siren action is that at least one of the three particular forms of the integral I is bounded and positive in the limit as r tends to zero, while all three forms are bounded.

Near $r = 0$, $t_B = 0$ and $t_A = T$ where T is the period between puffs, a given constant. Hence

$$\eta_A = T/r^\lambda \tag{A.8}$$

so that η_A becomes infinite as r tends to zero. Assuming that near $r = 0$ the B shock curve can be approximated by

$$r = k_B t^b, \qquad 0 < b \leq 1, \quad k_B > 0, \tag{A.9}$$

one finds

$$\eta_B = \left[k_B\ r^{\lambda b-1}\right]^{(-1/b)}. \tag{A.10}$$

Thus η_B becomes infinite or remains bounded as r tends to zero, depending on whether $\lambda b > 1$ or $\lambda b \leq 1$.

If $\lambda b > 1$, carrying through the integration in equation (A.7) shows that near $r = 0$ integral I tends to a sum of two terms proportional to

$$r^{j-\lambda\mu} \quad \text{and} \quad r^{j-(\lambda b-1)(\mu/b)}, \tag{A.11}$$

respectively, where integrand $f(\eta)$ has been supposed to tend smoothly to a constant multiple of $\eta^{\mu-1}$ as η becomes infinite. To obtain a properly bounded I, it is necessary that

$$j - \lambda\mu \leq \begin{cases} 0 & \text{if } \mu > 0 \\ -\mu/b & \text{if } \mu < 0, \end{cases} \qquad \lambda b > 1, \tag{A.12}$$

with equality in one of the three cases I = M, Q, or K. If $\mu = 0$, logarithmic terms arise so that no bounded value for I can be obtained.

If $\lambda b = 1$, η_B remains bounded and positive as r tends to zero. Assuming that $f(\eta)$ is integrable to a finite value over any closed finite interval excluding $\eta = 0$, one finds that integral I becomes a sum of terms proportional to

$$r^{j} \quad \text{and} \quad r^{j-\lambda\mu} \tag{A.13}$$

so that for a finite I,

$$j \geq \begin{cases} \lambda\mu & \text{if } \mu > 0, \\ 0 & \text{if } \mu < 0, \end{cases} \qquad \lambda b = 1. \tag{A.14}$$

As before, equality is required in one of the three cases. Also, $\mu \neq 0$ in order for I to be finite.

If $\lambda b < 1$, integral I becomes a sum of terms proportional to

$$r^{j}, \quad r^{j-\lambda\mu}, \quad r^{j-(\lambda b-1)(\nu/b)} \tag{A.15}$$

where integrand $f(\eta)$ has been supposed to tend to a constant multiple of $\eta^{\nu-1}$ as η tends to zero. For a bounded I,

$$j \geq \max\left[0,\ \lambda\mu,\ (\lambda b-1)(\nu/b)\right], \qquad \lambda b < 1, \tag{A.16}$$

with equality required for one of the cases I = M, Q, or K. Neither μ nor ν can be zero for a bounded I.

The assumption that $f(\eta)$ becomes proportional to $\eta^{\mu-1}$ as η becomes infinite implies that

$$\mu = 1 + \lim_{\eta \to \infty} \eta[\ln f(\eta)]. \tag{A.17}$$

The same formula applies to give ν if the limit is taken as η tends to zero. From the explicit forms for $f(\eta)$ and the equations governing spherical progressive waves, presented in Section II, one finds

$$\left.\begin{matrix}\mu\\ \nu\end{matrix}\right\} = 1 + \begin{cases} -1 + [E + (A/U) - 2B]/D & \text{if } I = M \\ -2 + [E + 2(A/U) - 2B]/D & \text{if } I = Q \\ -3 + \max\left\{[E + A/U]/D,\ [E + 3(A/U) - 2B]/D\right\} & \text{if } I = K, \end{cases} \tag{A.18}$$

where the functions are evaluated at $\eta = \infty$ to obtain μ or at $\eta = 0$ to obtain ν.

B. Infinite Time

For large values of t, the flow and sound speeds are assumed to be finite or zero and the shock weak. The B-shock curve will then approach

$$r = k_\beta t^\beta, \qquad 0 \le \beta \le 1, \quad k_\beta > 0. \tag{B.1}$$

From this,

$$\eta_B = k_\beta^{-\lambda} t^{1-\lambda\beta}$$

$$\eta_A = k_\beta^{-\lambda}(t + T)^{1-\lambda\beta}. \tag{B.2}$$

For $\lambda\beta \neq 1$,

$$\eta_A \doteq \eta_B[1 + (1 - \lambda\beta)(T/t)] \tag{B.3}$$

so that for large times,

$$I = k_I k_\beta^j T(1 - \lambda\beta) t^{\beta j-1}\, \eta_B\, f(\eta_B). \tag{B.4}$$

Since mass, momentum, and energy are conserved, quantity I is necessarily bounded as t becomes infinite. In terms of exponents μ and ν, defined in equations (A.18) and (A.19), one finds

$$\beta j - 1 \leq \begin{cases} (\lambda\beta - 1)\nu & \text{if } \lambda\beta > 1 \\ (\lambda\beta - 1)\mu & \text{if } \lambda\beta < 1. \end{cases} \tag{B.5}$$

If $\lambda\beta = 1$, similar considerations show that for bounded 1 as t becomes infinite,

$$\beta j - 1 \leq 0, \qquad \lambda\beta = 1. \tag{B.6}$$

C. <u>The Shock Hodograph</u>

The conservation relations connecting quantities on the two sides of a shock may be reduced [1, Sect. 67] to

$$\begin{aligned}(\gamma - 1)(u_0 - w)^2 + 2c_0^2 &= (\gamma - 1)(u_1 - w)^2 + 2c_1^2 \\ &= (\gamma + 1)(u_0 - w)(u_1 - w)\end{aligned} \tag{C.1}$$

where w is the shock speed. These equations may be rewritten as:

$$\begin{aligned} S(u_0, c_0, u_1, c_1) &= 0 \\ V(u_0, c_0, u_1, c_1) &= w \end{aligned} \tag{C.2}$$

where

$$S = \gamma(\gamma - 1)^2(u_0 - u_1)^4 + 2(\gamma - 1)^2(u_0 - u_1)^2(c_0^2 + c_1^2) - 4(c_0^2 - c_1^2)^2$$

$$V = [(\gamma - 1)(u_0^2 - u_1^2) + 2(c_0^2 - c_1^2)]/2(\gamma - 1)(u_0 - u_1).$$

Limiting values for a weak, forward-moving shock are

$$\begin{aligned} u_0 &= u_1, & \frac{dc_1}{du_1} &= \gamma_1 \\ c_0 &= c_1, & \gamma_1 &= (\gamma - 1)/2 \\ w &= u_0 + c_0 \end{aligned} \tag{C.3}$$

where the derivative is taken holding u_0 and c_0 fixed. At the strong shock limit, defined by $c_0/c_1 = 0$, one finds

$$u_0/c_1 = 0, \qquad c_1/u_1 = \gamma_2,$$
$$w/u_1 = (\gamma + 1)/2 \qquad \gamma_2 = \sqrt{\gamma(\gamma - 1)/2}. \tag{C.4}$$

In the hodograph plane, for each point (u_0, c_0) on one side of the shock, a whole locus of possible points (u_1, c_1) is found. For a forward-moving shock, the locus is a curve with positive slope. It passes through its "center" (u_0, c_0) with slope γ_1, has infinite slope at its point of intersection with the u-axis ($c_1 = 0$, $u_1 = u_0 - c_0/\gamma_2$), and has asymptotic slope γ_2 as u_1 and c_1 become infinite. The slope of the straight line joining center point (u_0, c_0) to any point (u_1, c_1) lies between γ_1 and γ_2.

If center point (u_0, c_0) is on curve B behind the shock, then the corresponding point (u_1, c_1) on curve A ahead of the shock is on the shock jump locus below the center, i.e., with $u_1 < u_0$ and $c_1 < c_0$. It follows that the slope of the line joining corresponding points on the A and B curves in the hodograph plane lies between γ_1 and γ_2.

D. The Phase Plane

Trajectories in this plane are solutions of

$$\frac{dC}{dU} = \frac{\bar{B}C}{(U-1)A} . \tag{D.1}$$

Along a trajectory, η is found from

$$\frac{d(\ln \eta)}{dU} = \frac{D}{A} \tag{D.2}$$

where

$$A = \lambda^{-1}[U(U-1)(U-\lambda) - 3C^2(U-\sigma_1)]$$

$$\bar{B} = (U-1)B = \lambda^{-1}[(U-1)^2(U-\lambda) + (\gamma-1)U(U-1)(U-\sigma_2) - C^2(U-\sigma_3)]$$

$$D = (U-1)^2 - C^2$$

$$\sigma_1 = (2\lambda - 2 - \kappa)/3\gamma$$

$$\sigma_2 = (3 - \lambda)/2$$

$$\sigma_3 = 1 + [2\lambda - 2 + (\gamma - 1)\kappa]/2\gamma.$$

Singular points of differential equation (D.1) where $\bar{B}C = (U-1)A = 0$ are found to be

$$S_0:\ (0,0),\quad S_1:\ (1,0),\quad S_2:\ (\lambda,0),\quad S_i:\ (U_i, C_i),\qquad i = 3,4,5$$

where

$$U_3 = 2\lambda/(3\gamma - 1)$$

$$C_3^2 = U_3(U_3 - 1)(U_3 - \lambda)/3(U_3 - \sigma_1) \tag{D.3}$$

and

$$2(U_i - 1)^2 - (3\sigma_1 - 1 - 2\lambda)(U_i - 1) + 2\lambda - 1 = 0, \qquad i = 4,5$$

$$C_i = |U_i - 1|, \qquad i = 4,5. \tag{D.4}$$

Note that S_3 exists only if $C_3^2 \geq 0$.

In the neighborhood of S_0, equation (D.1) simplifies to

$$\frac{dC}{dU} = \frac{C}{U}$$

with solution

$$C = kU \tag{D.5}$$

where k is an arbitrary constant of integration as are k', k'', etc., below. From (D.2),

$$U = k'\eta. \tag{D.6}$$

Hence S_0 is a node at which $\eta = 0$.

Near S_1 equation (D.1) reduces to

$$\frac{dC}{dU'} = \frac{C[(\gamma - 1)(1 - \sigma_2)U' - (1 - \sigma_3)C^2]}{U'[(1 - \lambda)U' - 3(1 - \sigma_1)C^2]} \tag{D.7}$$

where $U' = U - 1$. The substitution of $C^2 = VU'$ gives

$$\frac{U'}{V}\frac{dV}{dU'} = \frac{a_1 V + a_2}{a_3 V + a_4}$$

where

$$a_1 = \kappa + 3, \qquad a_3 = (2\lambda - 3\gamma - \kappa - 2)/\gamma$$

$$a_2 = \gamma(\lambda - 1), \qquad a_4 = 1 - \lambda$$

with solution

$$|U'|^{\gamma}\ |V| = k\ |a_1V + a_2|^{a_5}, \qquad a_5 = 1 + \gamma a_3/a_1. \tag{D.8}$$

This is seen to be a nonelementary singular point with saddle-point character for small $|V|$ and possibly nodal character for large. One finds the value η takes on at this point by integrating

$$\frac{d(\ln \eta)}{dU'} = \frac{\lambda(U' - V)}{a_3V + a_4}\ . \tag{D.9}$$

The integration can be carried through when a particular trajectory is considered so that U' and V become related.

Except for the special solutions, $U' = 0$ and $V = 0$, by equation (D.8) any trajectory which reaches $U' = 0$ does so along $V = -a_2/a_1$. Assuming $U_3 \neq 1$ so that $a_1/a_4 \neq a_2/a_3$, equation (D.9) will be regular at $U' = 0$ so that $\ln \eta$ will be finite there.

In the vicinity of S_2, equation (D.1) becomes

$$\frac{dC}{dU} = \frac{3(\gamma - 1)C}{2(U - \lambda)}$$

with solution

$$C = k|U - \lambda|^{3(\gamma-1)/2}. \tag{D.10}$$

Equation (D.2) simplifies to

$$\frac{d(\ln \eta)}{dU} = \frac{\lambda - 1}{U - \lambda}$$

so that

$$\eta = k'\ |U - \lambda|^{\lambda-1}. \tag{D.11}$$

Hence S_2 is a node at which η becomes infinite if $\lambda < 1$, or zero if $\lambda > 1$. The case $\lambda = 1$ involves a confluence of singularities S_1 and S_2. Such special cases will be treated later as required.

Behavior of the trajectories in the neighborhood of any of singular points S_3, S_4, and S_5 is difficult to assess in general. When more definite values are available for the parameters, the nature of these points can be determined. Since S_4 and S_5 lie on $D = 0$, $\ln \eta$ remains finite at these points.

It is also of value to determine how trajectories may leave the finite part of the phase plane to end at infinity. Since the slope dC/dU is given in equation (D.1) by a rational fraction in U and C, the slope is smooth and monotonic at great distances. Trajectories approach infinity in a definite direction. We may therefore obtain all trajectories that tend to infinity by examining equation (D.1) with large U or C in the three cases with the value of the ratio C/U zero, finite, or infinite.

If C/U is zero, C is to be neglected with respect to U, and equation (D.1) reduces for large U to

$$\frac{dC}{dU} = \frac{\gamma C}{U}$$

with solution

$$C = kU^{\gamma}. \tag{D.12}$$

This solution satisfies the conditions under which it was derived, i.e., U large and C/U small, only if $k = 0$. Thus only the trivial solution $C = 0$ is obtained in this case.

If $C/U = m$ with m finite, equation (D.1) becomes

$$m = (\gamma - m^2)m/(1 - 3m^2)$$

or

$$m(2m^2 + \gamma - 1) = 0.$$

Since $m \neq 0$ and $\gamma - 1 > 0$, there are no proper solutions for m. No trajectories approach infinity with finite slope.

If U is neglected with respect to C in equation (D.1) and C is large, one obtains

$$\frac{dC}{dU} = \frac{C(U - \sigma_3)}{3(U - 1)(U - \sigma_1)}$$

with solution

$$C^3 = k(U-\sigma_1)^{(1-\sigma)}(U - 1)^{\sigma}, \qquad \sigma = (\sigma_3 - 1)/(\sigma_1 - 1). \qquad (D.13)$$

If $\sigma > 1$ there are solutions approaching infinity along $U = \sigma_1$; if $\sigma < 0$ there are such solutions along $U = 1$; if $0 \le \sigma \le 1$ there are no solutions tending to infinity in C with finite U. The special solutions $U = \sigma_1$ and $U = 1$, corresponding to infinite or zero values of k, are always present.

If U is neglected with respect to C in equation (D.2), one obtains

$$\frac{d(\ln \eta)}{dU} = \frac{\lambda}{3(U - \sigma_1)}$$

with solution

$$\eta = k'(U - \sigma_1)^{\lambda/3}. \qquad (D.14)$$

Since λ is positive, η approaches a finite limit on any of these trajectories. If the trajectories approach $U = \sigma_1$, η tends to zero.

E. Exponents μ and ν

These exponents are found at the extreme values of η, $\eta = 0$ and $\eta = \infty$. At S_0, η becomes zero and equation (A.19) gives

$$\nu = 1. \qquad (E.1)$$

By equation (D.13)

$$\sigma = 3(2\lambda - 2 - K + \gamma K)/2(2\lambda - 2 - K - 3\gamma). \qquad (E.2)$$

If $\sigma > 1$, equation (D.14) shows that η can become zero for trajectories approaching ($U = \sigma_1$, $C = \infty$), and one finds

$$\nu = \begin{cases} (K + 2\sigma)/\lambda & \text{if } I = M \\ (K + 2\sigma - \lambda)/\lambda & \text{if } I = Q \\ (K + 2 - 2\lambda)/\lambda & \text{if } I = K. \end{cases} \qquad (E.3)$$

At S_2 and S_3, η can take either extreme value. If η becomes zero, one can evaluate ν; if η becomes ∞, one can evaluate μ. Thus

$$(E/D) - n = \begin{cases} \mu & \text{if } \eta = \infty \\ \nu & \text{if } \eta = 0 \end{cases} \tag{E.4}$$

where

$$E/D = \begin{cases} (3\gamma + \kappa)/(\lambda - 1) & \text{at } S_2 \\ 2(\kappa + 3)/(3\gamma - 1)(U_3 - 1) & \text{at } S_3 \end{cases} \tag{E.5}$$

and integer n has been defined in equation (A.7):

$$n = \begin{cases} 0 & \text{if } I = M \\ 1 & \text{if } I = Q \\ 2 & \text{if } I = K. \end{cases} \tag{E.6}$$

F. Shock Strength at $r = 0$

From the definition of U and from equation (A.8), one sees that near $r = 0$ on shock curve A,

$$u = (r/\lambda T)U(\infty). \tag{F.1}$$

Similarly, using equation (A.9) for curve B,

$$u = \left(r/\lambda t\right) U\left(k_B^{-1/b}\ r^{(1-\lambda b)/b}\right). \tag{F.2}$$

In each case, c is given by the same expression with C in place of U.

Substitution into the first of equations (C.2) with $u_0 = c_0 = 0$ and taking limits as r tends to zero shows that

$$C(\eta_{BO}) = \gamma_2 U(\eta_{BO}) \tag{F.3}$$

where

$$\eta_{BO} = \begin{cases} 0 & \text{if } \lambda b < 1 \\ H_B & \text{if } \lambda b = 1, \quad 0 < H_B < \infty \\ \infty & \text{if } \lambda b > 1. \end{cases} \tag{F.4}$$

One sees by equation (C.4) that the shock is infinitely strong at $r = 0$.

By equation (A.9), shock speed w near $r = 0$ is given by

$$w = br/t. \tag{F.5}$$

Substitution in the second of equations (C.2) then gives

$$\lambda b = (\gamma + 1)U(\eta_{BO})/2. \tag{F.6}$$

Further, since the shock moves subsonically with respect to the gas behind it,

$$\lambda b \leq U(\eta_{B0}) + C(\eta_{B0}) \tag{F.7}$$

For $\lambda b < 1$, $\eta_{B0} = 0$. From the work of subsection D above, η can become zero at singular points S_0, S_2, or S_3, or at $U = \sigma_1$ with infinite C. Both the case with infinite C and the case with zero C but nonzero U (point S_2) are ruled out by equation (F.3). Inequality (F.7) rules out S_0 since $\lambda b > 0$. Hence, η becomes zero at S_3.

It was shown in subsection D that η can become infinite only at S_3 or at S_2 with $\lambda < 1$. Since for $\lambda b \geq 1$, one finds η becoming infinite at S_3.

The possibility that $\lambda b < 1$, while $\lambda > 1$, is ruled out since this would require η to become both zero and infinite at S_3. The trajectory would be required to form a loop starting and ending at S_3. But then there would be a focal singular point, necessarily S_4 or S_5 inside the loop. To go around either of these points, the trajectory would cross $D = 0$ twice, at least one of these times at a regular point. But such crossings give double-valued mappings into the physical plane and are not allowed [1, Sect. 162].

For $\lambda b < 1$ and $\lambda < 1$, η becomes zero at S_3 and infinite at S_2.

For $\lambda b \neq 1$, η takes the value η_{B0} at S_3. Equation (F.3) therefore reads as $C_3 = \gamma_2 U_3$. By equation (D.3), one finds that this implies

$$K = 1 - 4U_3 = 1 - 8\lambda/g, \qquad g = 3\gamma - 1, \quad \lambda b \neq 1. \tag{F.8}$$

Also from equations (D.3) and (F.6),

$$b = (\gamma + 1)/g, \qquad \lambda b \neq 1. \tag{F.9}$$

If $\lambda b < 1$ and $\lambda < 1$, then $\eta = 0$ at S_3 and $\eta = \infty$ at S_2. Hence, by equations (E.4) and (E.5),

$$\mu = (3\gamma + \kappa)/(\lambda - 1) - n,$$

$$\nu = 2(\kappa + 3)/(2\gamma - g) - n, \qquad n = 0,1,2. \quad \text{(F.10)}$$

From equations (A.7) and (A.16),

$$\kappa + 3 - n(\lambda - 1) \geq \max \lfloor 0, \lambda\mu, (\lambda b - 1)\nu/b \rceil , \qquad \text{(F.11)}$$

with equality for one of the three values of n. Substitution of values of κ and b from equations (F.8) and (F.9) shows that condition (F.11) is satisfied if $n = 2$; i.e., the finite energy case, $I = K$, is being considered.

For $\lambda b = 1$, η becomes infinite at S_3 so that by equation (E.4),

$$\mu = 2(\kappa + 3)/(2\lambda - g) - n, \qquad n = 0,1,2. \quad \text{(F.12)}$$

Equation (A.14) may be written

$$\kappa + 3 - n(\lambda - 1) \geq \max \lfloor 0, \lambda\mu \rceil. \qquad \text{(F.13)}$$

This condition is satisfied if

$$\kappa = 2\lambda - 5, \quad 2\lambda < g, \quad n = 2, \qquad \lambda b = 1. \quad \text{(F.14)}$$

For $\lambda b > 1$, equation (F.12) gives μ, while κ and b are available in equations (F.8) and (F.9). Condition (A.12) may be written

$$\kappa + 3 - n(\lambda - 1) \geq \max [\lambda\mu, (\lambda b - 1)\mu/b]. \qquad \text{(F.15)}$$

Substitution shows this condition to be satisfied if $n = 2$.

The special case $\lambda = 1$, for which singular points S_1 and S_2 become coincident, is not being considered.

G. Shock at Late Times

Far out in the flow between the shocks, conditions are assumed to become uniform. The shock speed becomes equal to the wave speed. By equation (B.1),

$$\lambda\beta = U(\eta_L) + C(\eta_L) \tag{G.1}$$

where by equation (B.2),

$$\eta_L = \begin{cases} 0 & \text{if } \lambda\beta > 1 \\ H_L & \text{if } \lambda\beta = 1, \quad 0 < H_L < \infty, \\ \infty & \text{if } \lambda\beta < 1. \end{cases} \tag{G.2}$$

If $\lambda < 1$, then $\lambda\beta < 1$, and, since η becomes infinite at S_2,

$$\beta = 1, \qquad \lambda < 1. \tag{G.3}$$

Condition (B.5) reads $j - 1 \leq (\lambda - 1)\mu$ and can be shown by equations (A.7), (F.8), and (F.10) to be satisfied.

If $\lambda > 1$, only cases with $\lambda b \geq 1$ need be considered, as was shown in subsection F. If $\lambda b \geq 1$ and $2\lambda < g$, then η is infinite at S_3 and zero at S_0. The case $\eta_L = 0$ is then contradictory so that either

$$\lambda\beta = U(H_L) + C(H_L) = 1, \qquad \lambda b \geq 1, \; 2\lambda < g, \tag{G.4}$$

or

$$\lambda\beta = U_3 + C_3 < 1, \qquad \lambda b \geq 1, \; 2\lambda < g. \tag{G.5}$$

If $\lambda b \geq 1$ and $g < 2\lambda$, then $U_3 > 1$ so that η is infinite at S_3 and zero at S_2. It follows that the whole trajectory lies in $U > 1$ so that by equation (G.1), $\lambda\beta > 1$, Then by equation (G.2),

$$\beta = 1, \qquad \lambda b \geq 1, \; g < 2\lambda. \tag{G.6}$$

For $\lambda b > 1$, parameter κ is given in equation (F.8) and μ in equation (F.12). Where it is needed, in the case $\lambda\beta > 1$, parameter ν has the form given for μ in equation (F.10). One may then check the conditions given in equations (B.5) and (B.6). One finds that the case given in equation (G.6) satisfies the conditions, that in equation (G.4) does if

$$8g/(8 + g) \leq 2\lambda < g, \qquad \lambda b > 1, \tag{G.7}$$

and the case given in equation (G.5) is contradictory.

For $\lambda b = 1$, parameter κ is given in equation (F.14). Parameters μ and ν are evaluated by the same formulas as for the $\lambda b > 1$ cases. One finds that the case given in equation (G.6) continues to satisfy the conditions in equations (B.5) and (B.6), that in equation (G.4) does so if $\lambda \leq 2$ and that in (G.5) if $\beta g \leq 2$.

Results of subsections F and G are summarized in Table I. In the table $C_B = C(H_B)$ and $C_L = C(H_L)$ and similarly for U_B and U_L. Point S_0 is put in parentheses since no trajectory need reach this point. The "boundary conditions" of Table I are those at $r = 0$ and $t = \infty$.

For large times, particle speed u is given according to equation (B.1) by

$$u = (k_\beta/\lambda)t^{\beta-1}\ U(\eta_L). \tag{G.8}$$

One sees that cases with $\beta = 1$ correspond to flows with positive particle speeds at infinite time, while those for $\beta < 1$ correspond to flows with zero particle speeds. Since sound speed c satisfies an equation of the same form, the sound speed is zero at infinite time in the $\beta < 1$ cases. Since $C(\eta_L) = 0$ for case (5) in Table I, the sound speed is also zero in the case of $\lambda > 1$ with $\beta = 1$.

Table I

CASES SATISFYING THE BOUNDARY CONDITIONS

Case	λ	b	β	κ	η_{BO}	η_L	$\eta = 0$	$\eta = \infty$
1	$\lambda < 1$	$(\gamma+1)/g$	1	$1-8\lambda/g$	0	∞	S_3 $(C_3 = \gamma_2 U_3)$	S_2
2	$1 < \lambda < g/2$ $\lambda \le 2$	$1/\lambda$	$1/\lambda$	$2\lambda-5$	H_B $(C_B = \gamma_2 U_B)$	H_L $(U_L + C_L = 1)$	(S_0)	S_3
3	$1 < \lambda < g/2$ $\lambda \le 2$	$1/\lambda$	$\beta < 1/\lambda$ $\beta < 2/g$	$2\lambda-5$	H_B $(C_B = \gamma_2 U_B)$	∞	(S_0)	S_3 $(U_3 + C_3 = \lambda\beta)$
4	$g/(\gamma+1) < \lambda < g/2$ $4g/(8+g) \le \lambda$	$(\gamma+1)/g$	$1/\lambda$	$1-8\lambda/g$	∞	H_L $(U_L + C_L = 1)$	(S_0)	S_3 $(C_3 = \gamma_2 U_3)$
5	$g/2 < \lambda$	$(\gamma+1)/g$	1	$1-8\lambda/g$	∞	0	S_2	S_3 $(C_3 = \gamma_2 U_3)$

H. Singular Point S_3

In the neighborhood of S_3, differential equation (D.1) may be approximated by

$$\frac{dC}{dU} = \frac{a_5(U - U_3) + a_7(C - C_3)}{a_6(U - U_3) + a_8(C - C_3)} \tag{H.1}$$

where

$$a_5 = \lambda C\bar{B}_U, \qquad a_6 = \lambda(U-1)A_U$$
$$a_7 = \lambda C\bar{B}_C, \qquad a_8 = \lambda(U-1)A_C \tag{H.2}$$

and each of these is evaluated at S_3.

The nature of singular point S_3 is determined by the roots in x of the quadratic

$$x^2 - (a_6 + a_7)x + (a_6a_7 - a_5a_8) = 0. \tag{H.3}$$

If these roots are complex, S_3 is a focus or center. If they are real and of the same sign, S_3 is a node; if they are of opposite signs, S_3 is a saddle point. If the roots are real, there are trajectories through S_3 with slope m, m being a root of

$$a_8m^2 + (a_6 - a_7)m - a_5 = 0, \qquad m = \left.\frac{dC}{dU}\right|_{S_3}. \tag{H.4}$$

Parameters x and m are related by

$$x = a_6 + ma_8. \tag{H.5}$$

On a trajectory of slope m through S_3, one finds that near S_3 equation (D.2) may be approximated by

$$\frac{d(\ln \eta)}{dU} = \frac{k_3}{U - U_3} \tag{H.6}$$

where, after some manipulation,

$$k_3 = \lambda(U_3 - 1)D_3/x, \tag{H.7}$$

and D_3 is function D evaluated at S_3. On a trajectory which enters S_3 with slope m and by equation (H.5) corresponding characteristic root x, variable η becomes zero or infinite depending on whether k_3 is positive or negative.

In the cases listed in Table I where $\lambda b \neq 1$, κ has the value $(1 - 8\lambda/g)$. Evaluation of the various constants gives

$$
\begin{aligned}
\sigma_1 &= [(\gamma+1)U_3 - 1]/\gamma \\
\sigma_2 &= (6 - gU_3)/4, \qquad\qquad g = 3\gamma - 1 \\
\sigma_3 &= 1 - (\gamma - 3)(U_3 - 1)/2\gamma \\
C_3 &= \gamma_2 U_3 \\
a_5 &= (C_3/4)[2U_3^2(2\gamma^2 - 3\gamma + 3) - U_3(3\gamma + 1)(\gamma + 1) + 2(3\gamma - 1)] \\
a_6 &= U_3(U_3 - 1)[-U_3(3\gamma^2 + 3\gamma - 8) + 3\gamma - 5]/2 \\
a_7 &= -3(\gamma - 1)^2 U_3^2 (U_3 - 1)/2 \\
a_8 &= 6C_3(U_3 - 1)^2/\gamma
\end{aligned}
\tag{H.8}
$$

so that equation (H.3) may be rewritten

$$y^2 + a_9 y + a_{10} = 0 \tag{H.9}$$

where

$$
\begin{aligned}
y &= 2x/U_3(U_3 - 1) \\
a_9 &= (6\gamma^2 - 3\gamma - 5)U_3 - (3\gamma - 5) \\
a_{10} &= -6(\gamma - 1)(3\gamma - 1)D_3 \\
D_3 &= (-1/2)[(\gamma + 1)(\gamma - 2)U_3^2 + 4U_3 - 2]
\end{aligned}
$$

In case (1) of Table I, $\eta = 0$ at S_3 so that $k_3 > 0$. Since $0 < U_3 < 1$, by equation (H.7), D_3 and x have opposite signs. Hence by equation (H.9), D_3 and y have the same sign. Since with a_{10} negative the two roots y are real and have opposite signs, one of them will agree with that of D_3. Hence, a_{10} negative is a sufficient condition. This requires a positive D_3. For this

$$U_3 < 1/(1 + \gamma_2) \qquad \text{or} \qquad \lambda < g/2(1 + \gamma_2). \tag{H.10}$$

In cases (4) and (5), $\eta = \infty$ at S_3 so that $k_3 < 0$ is required. A sufficient condition for this is again that D_3 be positive; for then the two roots in y have opposite signs and the proper choice can be made. For case (4), $U_3 < 1$ so that the conditions for D_3 to be positive are found in equation (H.10). For case (5) with $U_3 > 1$, the necessary conditions are

$$\gamma < 2 \quad \text{and} \quad U_3 > 1/(1 - \gamma_2) \quad \text{or} \quad \lambda > g/2(1 - \gamma_2). \tag{H.11}$$

In cases (2) and (3) of Table I, $\lambda b = 1$ and κ equals $2\lambda - 5$. Evaluation of the constants gives

$$\begin{aligned}
\sigma_1 &= 1/\gamma \\
\sigma_2 &= (6 - gU_3)/4, \qquad && g = 3\gamma - 1 \\
\sigma_3 &= (2 + gG)/2\gamma, \qquad && G = \gamma U_3 - 1 \\
C_3^2 &= -\gamma(\gamma - 1)U_3^2(U_3 - 1)/2G \\
a_5 &= (\gamma - 1)[C_3/4\gamma^2 G][6\gamma G^3 - (3\gamma^2 - 3\gamma - 2)G^2 + (3\gamma^2 - 5\gamma + 4)G - 2(\gamma - 1)] \\
a_6 &= -3(\gamma - 1)[U_3(U_3 - 1)/2\gamma G](G^2 + \gamma - 1) \\
a_7 &= -(3/2)(\gamma - 1)^2 U_3^2(U_3 - 1) \\
a_8 &= -(6/\gamma)C_3(U_3 - 1)G.
\end{aligned} \tag{H.12}$$

Equation (H.3) then reads as

$$z^2 + a_{11}z + a_{12} = 0 \tag{H.13}$$

where

$$\begin{aligned}
z &= 2\gamma Gx/(\gamma - 1)U_3(U_3 - 1) \\
a_{11} &= 3[\gamma G^2 + (\gamma - 1)G + \gamma - 1] \\
a_{12} &= 3G[-3(\gamma + 1)G^3 + (3\gamma^2 - 5)G^2 - (\gamma + 1)G + (\gamma - 1)g].
\end{aligned}$$

For both cases (2) and (3), by Table I, $\lambda < g/2$ so that $U_3 < 1$ and C_3^2 is positive only for positive G. Hence for a real singular point S_3, G lies between 0 and $(\gamma - 1)$. For $G = \gamma - 1$, $a_{12} = -6(\gamma - 1)^2 g < 0$.

Hence, for some part of its range at least, a_{12} is negative, the two roots for z have opposite sign, and a root with desired sign will be available.

I. Periodicity

Using subscripts A and B to denote points on shock curves A and B gives

$$\eta_A = t_A/q, \qquad \eta_B = t_B/q, \qquad q = r^\lambda \tag{I.1}$$

where q is a convenient independent variable in place of r. Also

$$\begin{aligned} u_A &= (r/\lambda q)\bar{U}(\eta_A), \qquad & c_A &= (r/\lambda q)\bar{C}(\eta_A) \\ u_B &= (r/\lambda q)\bar{U}(\eta_B), \qquad & c_B &= (r/\lambda q)\bar{C}(\eta_B) \end{aligned} \tag{I.2}$$

where

$$\bar{U}(\eta) = U(\eta)/\eta \ , \qquad \bar{C}(\eta) = C(\eta)/\eta \ . \tag{I.3}$$

The relation connecting corresponding points with the same values of r on the A and B shock curves is $S(u_A, c_A, u_B, c_B) = 0$, given explicitly in the first of equations (C.2). Since S is a homogeneous polynomial of second degree in its arguments, one can multiply through by $(\lambda q/r)^2$ to obtain

$$S[\bar{U}(\eta_A), \ \bar{C}(\eta_A), \ \bar{U}(\eta_B), \ \bar{C}(\eta_B)] = 0. \tag{I.4}$$

Let $[U = U(\eta), \ C = C(\eta)]$ be a trajectory representing a solution in the (U,C) phase plane. This trajectory can be transferred to a trajectory in the $(\bar{U},\bar{C})$ modified phase plane. Equation (I.4) identifies corresponding points A and B that are connected by the shock relations.

Function $V(u_A, c_A, u_B, c_B)$ in the second of equations (C.2) gives the shock speed, w. Since V is of first degree,

$$w = (r/\lambda q)V[\bar{U}(\eta_A), \ \bar{C}(\eta_A), \ \bar{U}(\eta_B), \ \bar{C}(\eta_B)]. \tag{I.5}$$

Using w = dr/dt, one obtains

$$\frac{dt}{dq} = \frac{1}{V} \ . \tag{I.6}$$

By the periodicity assumption, the two shocks are a constant distance apart in time:

$$\frac{dt_A}{dq} = \frac{dt_B}{dq} \; . \qquad (I.7)$$

Replacing t_A and t_B by their equals from equation (I.1) and using equation (I.6) gives

$$\frac{d(q\eta_A)}{dq} = \frac{d(q\eta_B)}{dq} = \frac{1}{V} \; . \qquad (I.8)$$

Solution of equation (I.8) under proper initial conditions gives η_A and η_B as functions of q. These functions determine the locations of curves A and B in the (r,t) physical plane. The curves are shock trajectories provided that equation (I.4) is satisfied along them. That equation is satisfied initially for the cases listed in Table I. It will continue to be satisfied provided that

$$\frac{dS}{dq} = 0. \qquad (I.9)$$

If S_i, i = 1, 2, 3, 4, denotes the partial derivative of S with respect to its i^{th} argument as exhibited in equation (I.4), then equation (I.9) may be expanded to read

$$[S_1\bar{U}'(\eta_A) + S_2\bar{C}(\eta_A)]\eta_A' + [S_3\bar{U}'(\eta_B) + S_4\bar{C}'(\eta_B)]\eta_B' = 0 \qquad (I.10)$$

where the prime indicates differentiation for functions of one variable.

It is difficult to see how condition (I.10) is to be implemented in general. For any particular proposed solution the condition could, of course, be checked by computation.

J. The Modified Phase Plane

After a solution curve $[U = U(\eta), C = C(\eta)]$ has been found by use of the phase plane and its singular points, one may transfer the curve to the modified phase plane with coordinates

$$\bar{U} = U(\eta)/\eta \quad \text{and} \quad \bar{C} = C(\eta)/\eta. \tag{J.1}$$

The modified phase plane retains the advantage that the whole flow is represented by a single curve while it permits the shock condition $S = 0$ of equations (C.2) to be given a geometric interpretation.

The shock condition of equation (I.4) reads

$$S(\bar{U}_A, \bar{C}_A, \bar{U}_B, \bar{C}_B) = 0 \tag{J.2}$$

where $\bar{U}_A = U(\eta_A)/\eta_A$, etc. For a given point $(\bar{U}_B, \bar{C}_B)$, the locus of points $(\bar{U}_A, \bar{C}_A)$, which satisfy equation (J.2), is a curve just as described in subsection C for the hodograph shock curve. It passes through $(\bar{U}_B, \bar{C}_B)$ with slope γ_1, reaches $\bar{C}_A = 0$ with infinite slope and abscissa $\bar{U}_A = \bar{U}_B - \bar{C}_B/\gamma_2$, and has asymptotic slope γ_2 as the point $(\bar{U}_A, \bar{C}_A)$ recedes toward infinity. The slope of the curve is always positive. For point A to be ahead of the shock and B behind it, point A is restricted to the part of the locus with $\bar{C}_A < \bar{C}_B$.

At $r = 0$, $\eta_A = \infty$. Since $U(\eta_A)$ and $C(\eta_A)$ are finite by Table I, $\bar{U}_A = \bar{C}_A = 0$. From the discussion in the preceding paragraph, this implies $\bar{C}_B = \gamma_2 \bar{U}_B$ for the corresponding point on the B curve. This condition is satisfied by all cases in Table I.

For cases (4) and (5) where $\lambda b > 1$, η_B becomes infinite along with η_A at $r = 0$. Near that point, in the notations of subsection H,

$$U \doteq U_3 + k\xi, \quad C \doteq \gamma_2 U_3 + mk\xi, \quad \xi = \eta^{1/k_3}, \quad k_3 < 0 \tag{J.3}$$

Thus

$$\bar{C}/\bar{U} \doteq \gamma_2 + (m - \gamma_2)(k/U_3)\xi \; . \tag{J.4}$$

For the shock curve to join points on the trajectory in the neighborhood of $r = 0$, it is then necessary that

$$(m - \gamma_2)k < 0, \qquad \lambda b > 1 \tag{J.5}$$

since any line joining points on a shock has slope less than γ_2.

For cases (1), (2), and (3) where $\lambda b \geq 1$, $\eta_B = \eta_{B0}$ is finite or zero at $r = 0$. In either case, $\bar{C}_B = \gamma_2 \bar{U}_B$ while $\bar{C}_A = \bar{U}_A = 0$. For the shock curve to connect points on the trajectory as η changes from η_{B0}, it is necessary that

$$\bar{C}_B \leq \gamma_2 \bar{U}_B \quad \text{for } \eta_B \text{ near } \eta_{B0}, \qquad \lambda b \leq 1. \tag{J.6}$$

Corresponding to the region between the shocks at infinite time is a single point of the phase plane trajectory. Since the shocks are weak there, in the modified phase plane the trajectory will have slope γ_1 at that point. In order that the shock locus be able to join nearby points the slope of the trajectory is restricted as follows:

In case (5), $\eta_L = 0$ and the representative point is at infinity in direction $\bar{C} = \gamma_1 \bar{U}$. The slope of the trajectory is required to approach γ_1 from greater values:

$$\bar{C}/\bar{U} \geq \gamma_1 \quad \text{for } \eta \text{ near } \eta_L, \qquad \eta_L = 0. \tag{J.7}$$

However, $\eta = \eta_L$ at singular point S_2. As η approaches η_L, $\bar{C}/\bar{U} = C/U$ approaches $C_2/U_2 = 0$. Condition (J.7) is seen to be violated so that case (5) cannot hold.

In case (1), $\eta_L = \infty$ so that the representative point is at the origin. Here again $\bar{C}/\bar{U} = C/U$ approaches $C_2/U_2 = 0$ near this point. A trajectory with such slopes cannot contain pairs of points on the shock locus.

For case (3), $\eta_L = \infty$ so that the representative point is again at the origin. Near by

$$U \doteq U_3 + k\xi, \qquad C \doteq C_3 + mk\xi \tag{J.8}$$

with ξ defined in equation (J.3). It follows that

$$\bar{C}/\bar{U} = (C_3/U_3) + (m - C_3/U_3)(k\xi/U_3). \tag{J.9}$$

The shock curve gives a weak shock at the origin and is able to properly join pairs of points on the trajectory nearby if

$$C_3/U_3 = \gamma_1 \quad \text{and} \quad (m - \gamma_1)k > 0. \tag{J.10}$$

Using the first of equations (J.10), equation (G.5), and equation (D.3), one may solve for β and find $\beta = (\gamma + 1)/g > 2/g$. This violates a condition given in Table I so that this case (3) cannot hold.

In both remaining cases, (2) and (4), $\eta = H_L$ at infinite time so that the representative point in the modified phase plane has finite, nonzero coordinates. For a weak shock the slope of the trajectory at this point needs to be γ_1.

In summary, conditions in the modified phase plane have eliminated all but cases (2) and (4). For case (2), additional condition (J.6) has been found, while inequality (J.5) applies to case (4).

K. Phase Plane Configurations

For case (2) the solution curve passes through three points in the phase plane: S_3: (U_3,C_3), P_B: (U_B,C_B), and P_L: (U_L,C_L). At these points, η takes the values ∞, η_{BO}, and η_L, respectively. The coordinates of point S_3 are related by a formula given in the set of equations (H.12):

$$C_3^2 = -\gamma(\gamma - 1)U_3^2(U_3 - 1)/2(\gamma U_3 - 1). \tag{K.1}$$

Relations connecting the coordinates of the other points are found in Table I and read

$$C_B = \gamma_2 U_B, \qquad C_L + U_L = 1. \tag{K.2}$$

As pointed out in subsection H, for a real point S_3, $U_3 < 1$. One may readily show that as U_3 varies, point S_3 cannot cross the line $C_3 + U_3 = 1$ as long as $U_3 < 1$ and $\gamma > 1$. Hence, $U_3 + C_3 > 1$. Thus, four subcases arise for case (2), which depend on the order of the points S_3, P_B, and P_L along the phase plane trajectory and which also depend on whether C_3 is greater than $\gamma_2 U_3$ or not.

(2a): (3, B, L) and $C_3 < \gamma_2 U_3$

(2b): (3, B, L) and $C_3 > \gamma_2 U_3$

(2c): (3, L, B) and $C_3 < \gamma_2 U_3$

(2d): (3, L, B) and $C_3 > \gamma_2 U_3$.

In all subcases, S_3 is at the origin of the modified phase plane, P_B is on the ray through that origin of slope γ_2, and the slope of the trajectory at P_L is γ_1. Also, if a point lies on the ray of slope m in the phase plane, the corresponding point lies on a ray of slope m in the modified phase plane. If S_3 lies on a ray of slope m_3 in the phase plane, the trajectory leaves the origin of the modified phase plane tangent to the ray of slope m_3. One may assume that the trajectories cross each ray only once. It follows that in subcase 2(a), $m_3 < m_B = \gamma_2 < m_L$. The point representing shock curve B as it moves from P_B to P_L, moves from a ray of slope γ_2 toward rays of greater slope. This violates condition (J.6). In subcase (2d), $m_B = \gamma_2 < m_L \leq m_3$ so that the same condition is violated. Subcases (2b) and (2c) provide possible configurations.

In case (4), points P_B and S_3 coincide in the phase plane with $C_3 = \gamma_2 U_3$. In the modified phase plane they coincide at the origin. Assuming, as before, that the trajectory crosses each ray through the origin only once implies that the ray through P_L lies on the same side of the ray through S_3 as do points near S_3 on the trajectory. But at these points $C < \gamma_2 U$ according to condition (J.5). It follows that there is just the one configuration in the modified phase plane. The trajectory leaves the origin tangent to the ray of slope γ_2. Its slope then decreases until it reaches the minimum value γ_1 at point P_L.

L. Solutions

A particular problem is posed by specifying the gas, through its ratio of specific heats γ, specifying the amount of energy K in each puff of the siren, and the time T between puffs.

A way of obtaining the solutions is as follows: Guess a value for λ and choose whether case (2) or case (4) is to be followed. Then the parameters will all be known. The family of trajectories in the phase plane will be obtainable as solutions of equation (D.1). The equations of subsection H permit one to check whether singular point S_3 is a saddle point as is now assumed. The one of the two trajectories through S_3 along which η becomes infinite is then chosen and is followed in a direction to intersect the line $U + C = 1$. This determines C as a function of U. Equation (D.2) then gives $\ln \eta$ as a function of U except for an added constant of integration, i.e., $\eta = kF(U)$ with function F known but k an arbitrary constant.

Point P_L is the point where the trajectory crosses $U + C = 1$. Parameter λ must be chosen so that the slope of the trajectory in the modified phase plane takes its required value γ_1 at P_L. Although the variables C and U depend on η, unknown constant k is not needed for the computation of this slope. Important questions not answered here are whether values of λ satisfying the requirement exist and, if so, whether there are solutions for each of cases (2) and (4) or only one of them.

With λ and a definite trajectory known in the phase plane, equations (I.8) may be integrated to obtain the shock trajectories. Although constant k is not needed to perform the integration since it cancels out of the differential equation, it does enter the initial condition $q\eta_A = t_A = T$ at $q = 0$. Alternatively, constant k could be determined from the energy input K by carrying through the integration in equation (A.6).

In summary, specification of γ and K or T may be expected to provide one or a few periodic flows satisfying the various conditions at $r = 0$ and $t = \infty$, the extreme points. Each of these flows determines the unspecified one of K or T. However, these flows are not solutions to the problem unless one can check that shock condition (I.4) is satisfied at intermediate points along the bounding discontinuities as it is at the extreme points.

SECTION IV

NOETHER'S THEOREM

A. J. Penico and C. M. Ablow

A. Introduction

In the classical calculus of variations, the extremal functions, solutions to a given variational problem, are usually found as the solutions to certain differential equations, the Euler equations. Frequently, one may take advantage of certain, apparently fortuitous, circumstances present in the structure of a particular variational problem to effect one or more integrations of the Euler equations, thereby proceeding a part of the way toward obtaining the extremal functions. It is shown in this section that these apparently fortuitous circumstances are actually special cases of a quite important and quite general group-theoretic idea.

The purpose of this section is to develop some proofs and applications of Noether's theorem, which asserts that a variational problem which is invariant under some continuous group of transformations of the variables involved admits of one or more first integrals of the Euler equations. Furthermore, Noether's theorem gives an explicit construction for these first integrals. Since these first integrals frequently include a result in which some differential expression is asserted to be constant, such results are often called conservation laws; many of the familiar physical laws involving conservation of energy, momentum, etc., can be obtained in this way. Dimensional analysis and the similarity transformations, which are often utilized in hydrodynamics, can be viewed as group-theoretic ideas that are closely related to Noether's theorem. It is for this reason that Noether's theorem is being discussed in this report, even though no novel applications have actually been found.

The material in this section is largely based on Gelfond and Fomin [1], although some changes in presentation were motivated by the material in Courant and Hilbert [2] and Funk [3].

B. First Variation of a Functional Involving One Independent Variable

Since the development of Noether's theorem depends on having available a formula for the first variation of a functional, we shall discuss this point fairly extensively. Before proceeding to the general question, we examine some specific problems in detail, since the results in these problems are of interest in their own right. We calculate the first variation of the functional

$$J[u] = \int_{t_0}^{t_1} F(t, u, u', u'')\, dt \tag{B.1}$$

where $u(t)$ is a function of the independent variable t, $u' = du/dt$, $u'' = d^2u/dt^2$, and F is a function that is differentiable in each of its four arguments. It will also be assumed that the function $u(t)$ has a continuous fourth derivative.† The values t_0 and t_1 are assumed fixed. Hence, the domain S of $J[u]$ is the totality of four-times continuously differentiable functions $u(t)$ on the interval (t_0, t_1), and these functions can be viewed as the points of the space S. In the domain S we

† This assumption is actually much too restrictive, and our final conclusions hold under considerably less stringent conditions, but for our purposes, where we merely wish to indicate a mode of derivation of our formulas, the restrictions are made for the sake of simplicity.

assume the existence of a norm or distance function, which enables us to determine whether two functions $u_1(t)$ and $u_2(t)$ are near each other. We shall not specify this norm, but shall merely assume its existence.

Now let $u(t)$ be a fixed point in S and let $u + h \equiv u(t) + h(t)$ be another point near $u(t)$ in S; that is, $h(t)$ is near zero. If we consider $J[u + h]$ and formally expand it in a power series in h about u, and if we retain only first-order terms, we obtain

$$J[u+h] = J[u] + \int_{t_0}^{t_1} [F_u h + F_{u'} h' + F_{u''} h''] \, dt \tag{B.2}$$

where

$$F_{u^{(k)}} \equiv \left[\frac{\partial}{\partial \alpha_k} F(t, \alpha_1, \alpha_2, \alpha_3) \right]_{\alpha_k = u^{(k)}}, \qquad k = 0, 1, 2.$$

Obviously, in this problem the assertion $h(t)$ is near zero entails some specification on $h'(t)$ and $h''(t)$, as well as on $h(t)$. Integration by parts yields

$$\int_{t_0}^{t_1} \left[F_{u'} h' \right] dt = \left[h(t) F_{u'} \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} h \frac{d}{dt} (F_{u'}) \, dt$$

and

$$\int_{t_0}^{t_1} \left[F_{u''} h'' \right] dt = \left[F_{u''} h'(t) - h(t) \frac{d}{dt} (F_{u''}) \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} h \frac{d^2}{dt^2} (F_{u''}) \, dt.$$

Then we can write,

$$\int_{t_0}^{t_1} \left[F_u h + F_{u'} h' + F_{u''} h'' \right] dt = \int_{t_0}^{t_1} \left[F_u - \frac{d}{dt} F_{u'} + \frac{d^2}{dt^2} F_{u''} \right] h(t) \, dt + \left[\left(F_{u'} - \frac{d}{dt} F_{u''} \right) h(t) + F_{u''} h'(t) \right]_{t_0}^{t_1}$$

and from expression (B.2), we can write

$$J[u+h] - J[u] = \int_{t_0}^{t_1} \left[F_u - \frac{d}{dt} F_{u'} + \frac{d^2}{dt^2} F_{u''} \right] h(t) \, dt + \left[\left(F_{u''} - \frac{d}{dt} F_{u''} \right) h(t) + F_{u''} h'(t) \right]_{t_0}^{t_1}. \tag{B.3}$$

If we require $J[u + h] - J[u]$ to vanish for all admissible $h(t)$ so that $h(t_0) = h(t_1) = 0$, we are ultimately able to conclude that

$$F_u - \frac{d}{dt} F_{u'} + \frac{d^2}{dt^2} F_{u''} = 0,$$

which is the Euler-equation for the extremal satisfying the simplest variational problem of this type. Here, we will not have any special interest in this variational problem per se, but will require the formula (B.3) in our next development.

Using the same functional $J[u]$ of (B.1), consider the case in which not only the function $u(t)$ is allowed to vary but the endpoints t_0 and t_1 are allowed to vary as well. Then the points in S, the domain of definition of $J[u]$, will now include all of those arcs satisfying the appropriate differentiability conditions and extending between values of t in some larger interval $a \leq t \leq b$ containing $t_0 \leq t \leq t_1$. Therefore, the integral defining $J[u]$ can be assumed to be extendable, if necessary, over admissible arcs extending outside the interval of immediate interest. We now consider a function $u(t)$ defined over the interval $t_0 \leq t \leq t_1$, an infinitesimal variation in t

$$\delta t = t^*(t) - t, \tag{B.4}$$

and an infinitesimal variation in u

$$\delta u = u^*(t^*) - u(t). \tag{B.5}$$

Keeping in mind that

$$h(t) = u^*(t) - u(t), \tag{B.6}$$

we have, on expanding,

$$u^*(t^*) = u^*(t + \delta t) = u(t + \delta t) + h(t + \delta t)$$

$$= u(t) + u'(t)\delta t + h(t) + h'(t)\delta t.$$

These relations are illustrated in Figure 2.

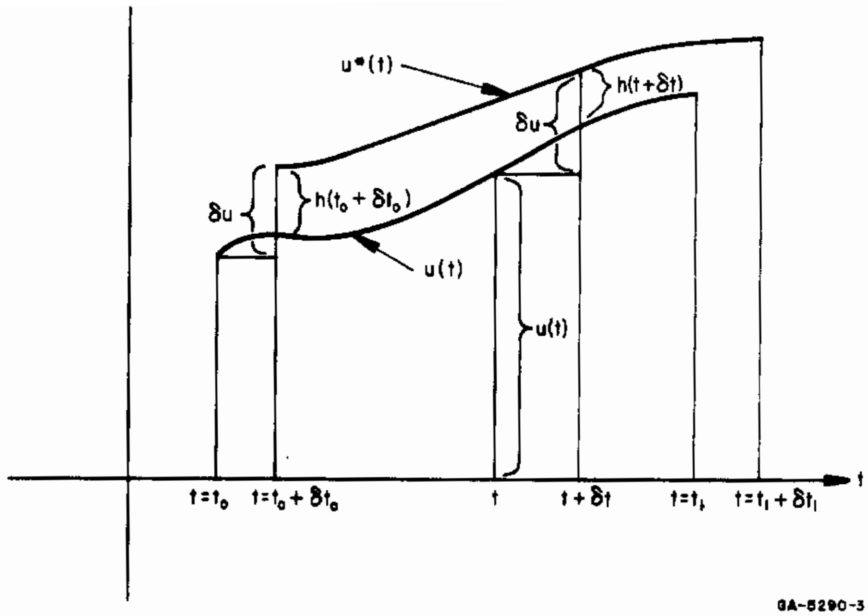

FIG. 2 RELATION BETWEEN A FUNCTION $u(t)$ AND ITS VARIATION $u^*(t)$

Using (B.5) and neglecting the second-order quantity $h'(t)\delta t$, (B.6) now yields

$$\delta u = h(t) + u'(t)\,\delta t. \tag{B.7}$$

In the same way, $\delta u'$, which is defined by

$$\delta u' = \frac{du^*}{dt^*} - \frac{du}{dt}\,, \tag{B.8}$$

may be written

$$\delta u' = h'(t) + u''(t)\,\delta t. \tag{B.9}$$

Similar results would follow for the variation of derivatives of higher order.

We now observe that

$$\begin{aligned} J[u^*] - J[u] &= \int_{t_0+\delta t_0}^{t_1+\delta t_1} F(t,u^*,u^{*\prime},u^{*\prime\prime})\,dt - \int_{t_0}^{t_1} F(t,u,u',u'')\,dt \\ &= \int_{t_0}^{t_1} \Big[F(t,u^*,u^{*\prime},u^{*\prime\prime}) - F(t,u,u',u'') \Big]\,dt \\ &\quad + \int_{t_1}^{t_1+\delta t_1} F(t,u^*,u^{*\prime},u^{*\prime\prime})\,dt - \int_{t_0}^{t_0+\delta t_0} F(t,u,u',u'')\,dt. \end{aligned} \tag{B.10}$$

The first integral in the last expression is, to first order,

$$\int_{t_0}^{t_1} [F_u h + F_{u'} + F_{u''} h''] \, dt, \qquad \text{(B.11)}$$

while the second and third integrals reduce to $F_{t=t_1}\,\delta t_1 - F_{t=t_0}\,\delta t_0$, which we write as

$$\Big[F\delta t\Big]_{t_0}^{t_1} .$$

The integral given in (B.11) is already expressed in alternative form in (B.3), so that we may write, after collecting terms,

$$J[u^*] - J[u] = \int_{t_0}^{t_1} \left[F_u - \frac{d}{dt} F_{u'} + \frac{d^2}{dt^2} F_{u''} \right] h(t) \, dt$$
$$+ \left\{ \left[\left(F_{u'} - \frac{d}{dt} F_{u''} \right) h(t) + h'(t) F_{u''} + F\delta t \right]_{t_0}^{t_1} \right\} . \qquad \text{(B.12)}$$

We now rewrite the term in braces with the use of (B.7) and (B.9) to obtain

$$J[u^*] - J[u] = \int_{t_0}^{t_1} \left[F_u - \frac{d}{dt} F_{u'} + \frac{d^2}{dt^2} F_{u''} \right] h(t) \, dt$$
$$+ \left[\left\{ F - u' \left(F_{u'} - \frac{d}{dt} F_{u''} \right) - u'' F_{u''} \right\} \delta t \right.$$
$$\left. + \left(F_{u'} - \frac{d}{dt} F_{u''} \right) \delta u + F_{u''} \delta u' \right]_{t_0}^{t_1} , \qquad \text{(B.13)}$$

which is the formula for the first variation for the variable-end-point problem. An analogous result can be readily derived when F contains derivatives of u(t) of order higher than the second.

We now state the formula for the first variation for the variable-end-point problem for the functional

$$J[u_1, u_2, \ldots, u_k] \equiv \int_{t_0}^{t_1} F[t; u_1(t), \ldots, u_k(t), u_1'(t), \ldots, u_k'(t)] \, dt, \qquad \text{(B.14)}$$

which involves a single independent variable t, k dependent variables $u_1(t)$, $u_2(t)$, ..., $u_k(t)$, and their first derivatives with respect to t. By analogy with the development of formula (B.13), we assume that the arcs $u_1(t)$, ..., $u_k(t)$ are continuously transformed into the "nearby" arcs $u_1^*(t^*)$, ..., $u_k^*(t^*)$, and we define

$$h_i(t) = u_i^*(t) - u_i(t), \qquad i = 1, \ldots, k, \tag{B.15}$$

and

$$\delta u_i = u_i^*(t^*) - u_i(t) \doteq h_i(t) + u_i'(t)\,\delta t. \tag{B.16}$$

With the use of (B.15) and (B.16), one may obtain the relation [1, pp. 54-59]

$$J[u_1^*, \ldots, u_k^*] - J[u_1, \ldots, u_k] = \int_{t_0}^{t_1} \left\{ \sum_{i=1}^{k} \left(F_{u_i} - \frac{d}{dt} F_{u_i'} \right) h_i(t) \right\} dt + \left[\sum_{i=1}^{k} F_{u_i'} \delta u_i + \left(F - \sum_{i=1}^{k} u_i' F_{u_i'} \right) \delta t \right]_{t=t_0}^{t=t_1}. \tag{B.17}$$

Introducing the symbol

$$H \equiv \sum_{i=1}^{k} u_i' F_{u_i'} - F,$$

we arrive at the formula

$$J[u_1^*, \ldots, u_k^*] - J[u_1, \ldots, u_k] = \int_{t_0}^{t_1} \left\{ \sum_{i=1}^{k} \left(F_{u_i} - \frac{d}{dt} F_{u_i'} \right) h_i(t) \right\} dt + \left[\sum_{i=1}^{k} F_{u_i'} \delta u_i - H \delta t \right]_{t_0}^{t_1} \tag{B.18}$$

Now returning to the u - t space over which the functional J[u] in (B.1) is defined, we consider as acting in this space a continuous group of transformations

$$u^* = \Phi(t^*;\epsilon), \quad t^* = \Psi(t;\epsilon) \tag{B.19}$$

in which ϵ is a continuous parameter specifying the elements of the group, Φ and Ψ are differentiable in ϵ at $\epsilon = 0$, and

$$\Psi(t;0) = t. \tag{B.20}$$

We identify $u(t)$ with $\Phi(t;0)$ by setting

$$\Phi(t;0) = u(t); \tag{B.21}$$

that is, the value $\epsilon = 0$ corresponds to the identity transformation. The operation under which the set of transformations forms a group is composition, the transformation corresponding to the "product" of transformations associated with ϵ_1 and ϵ_2 being given by

$$t^* = \Psi[\Psi(t;\epsilon_1),\epsilon_2], \quad u^* = \Phi(t^*;\epsilon_2) \tag{B.22}$$

Examples of such transformations are the following:

$$\begin{aligned} t^* &= t - \epsilon \\ u^*(t^*) &= u(t), \end{aligned} \tag{B.23}$$

$$\begin{aligned} t^* &= e^{\epsilon} t \\ u^*(t^*) &= e^{k\epsilon} u(t), \end{aligned} \tag{B.24}$$

$$\begin{aligned} t^* &= t \\ u_1^* &= u_1 \cos \epsilon + u_2 \sin \epsilon \\ u_2^* &= -u_1 \sin \epsilon + u_2 \cos \epsilon. \end{aligned} \tag{B.25}$$

Since the functions given in equations (B.19) are continuously differentiable at $\epsilon = 0$, we have the expansions

$$\begin{aligned} t^* &= t + \epsilon\psi(t) + o(\epsilon) \\ u^*(t^*) &= u(t) + \epsilon\varphi(t) + o(\epsilon). \end{aligned} \tag{B.26}$$

It follows from their definitions in equations (B.5) and (B.6) that

$$\begin{aligned} \delta t &= \epsilon\psi(t) \\ \delta u &= \epsilon\left(\frac{\partial\Phi(t,\epsilon)}{\partial t}\right)_{\epsilon=0} = \epsilon\varphi(t). \end{aligned} \tag{B.27}$$

One may also calculate

$$\begin{aligned} \delta u' &= \frac{du^*}{dt^*} - \frac{du}{dt} = \frac{du^*}{dt}\,\frac{dt}{dt^*} - \frac{du}{dt} = \left(\frac{du}{dt} + \epsilon\,\frac{d\varphi}{dt}\right)\left(1 - \epsilon\,\frac{d\psi}{dt}\right) - \frac{du}{dt} \\ &= \epsilon\left(\frac{d\varphi}{dt} - u'\,\frac{d\psi}{dt}\right). \end{aligned} \tag{B.28}$$

For the functional (B.1), the mapping (B.19) will carry the points t_0 and t_1 into t_0^* and t_1^*, respectively, while the function $u(t)$, $t_0 \le t \le t_1$, will be carried into the function $u^*(t^*)$, $t_0^* \le t^* \le t_1^*$. Thus, we may calculate the value of the functional

$$J[u^*] = \int_{t_0^*}^{t_1^*} F\left(t^*,\ u^*(t^*),\ \frac{du^*(t^*)}{dt^*},\ \frac{d^2u^*(t^*)}{dt^{*2}}\right) dt^*.$$

In the present context we are concerned with the <u>invariance</u> of a functional under certain transformation groups. Thus, since each of the transformations in (B.23) and (B.25) is a rigid motion, the functional

$$J[u_1,u_2] = \int_{t_0}^{t_1} \sqrt{[u_1'(t)]^2 + [u_2'(t)]^2}\ dt\ ,$$

which gives the length of the arc, $u_1 = u_1(t)$, $u_2 = u_2(t)$, between t_0 and t_1, will remain unchanged under one of the transformations (B.23) and (B.25). By this we mean that

$$\int_{t_0^*}^{t_1^*} \sqrt{[u_1^{*\prime}(t^*)]^2 + [u_2^{*\prime}(t^*)]^2}\ dt^* = J[u_1,u_2]\ ,$$

using the meanings of u_1^*, u_2^*, t^*, etc., introduced in (B.25). This functional changes value under the similarity transformation (B.24) (unless the arc is one for which $u_1(t)$ and $u_2(t)$ are homogeneous of degree k). In general, we shall say that the functional $J[u]$ in (B.1) is <u>invariant</u> under a transformation group given by (B.19) if

$$J[u^*] = J[u]$$

for every member of the group (B.19), where $J[u^*]$ is defined by (B.29).

At this point we refer all points (and arcs) in the starred coordinate system to the unstarred coordinate system. This means, in this context, that we shift our point of view and regard the curve consisting of points (t,u), $t_0 \le t \le t_1$, as having been transformed into the curve consisting of points (t^*,u^*), $t_0^* \le t^* \le t_1^*$, the (t^*,u^*) being, however, still referred to the (t,u) axes. We then suppose that $J[u]$ in (B.1) is invariant under the group given by (B.19) and suppose, for small ϵ, that the new curve is also in the domain of definition of $J[u]$. For small ϵ, we can calculate the difference, $J[u^*] - J[u]$, to first-order terms in ϵ, and this difference should be obtainable, without any appreciable effort, from the formulas given in (B.13), (B.27), and (B.28) for the several variations. Inserting these results into formula (B.13), we obtain

$$\begin{aligned} J[u^*] - J[u] = \int_{t_0}^{t_1} \left[F_u - \frac{d}{dt} F_{u'} + \frac{d^2}{dt^2} F_{u''} \right] h(t)\, dt \\ + \epsilon \Bigg[\left\{ F - u'\left(F_{u'} - \frac{d}{dt} F_{u''}\right) - u'' F_{u''} \right\} \\ + \left(F_{u'} - \frac{d}{dt} F_{u''}\right) \varphi(t) + F_{u''} \left(\frac{d\varphi}{dt} - u' \frac{d\psi}{dt} \right) \Bigg]_{t_0}^{t_1}. \end{aligned} \quad (B.30)$$

If we now assume

(i) $J[u]$ is invariant under the group of transformations (B.19) so that $J[u^*] - J[u] = 0$, and

(ii) the function $u(t)$, $t_0 \le t \le t_1$, is an extremal for the functional $J[u]$, so that $F_u - \frac{d}{dt} F_{u'} + \frac{d^2}{dt^2} F_{u''} = 0$, then

$$\begin{aligned} \epsilon \Bigg[\left\{ F - u'\left(F_{u'} - \frac{d}{dt} F_{u''}\right) - u'' F_{u''} \right\} \psi(t) \\ + \left(F_{u'} - \frac{d}{dt} F_{u''} \right) \varphi(t) + F_{u''} (\varphi' - u'\psi') \Bigg]_{t_0}^{t_1} = 0 \end{aligned}$$

for every (small) ϵ. Relation (ii) implies that the expression inside the brackets must have the same value at $t = t_1$ and at $t = t_0$. But t_0 and t_1 are arbitrary, so that this same expression must have a constant value; i.e.,

$$\left\{F - u'\left(F_{u'} - \frac{d}{dt} F_{u''}\right) - u'' F_{u''}\right\}\psi(t)$$
$$+ \left(F_{u'} - \frac{d}{dt} F_{u''}\right) \varphi(t) + F_{u'}(\varphi' - u'\psi') = \text{constant}. \qquad \text{(B.31)}$$

As an application of (B.31) we assume that the integrand in the functional J[u], defined in (B.1), does not depend explicitly on t. Then J[u] is invariant under the group of translations given by (B.23). In this case,

$$\psi(t) = -1, \quad \varphi(t) = 0, \qquad \text{(B.32)}$$

whence we obtain

$$F - u'\left(F_{u'} - \frac{d}{dt} F_{u''}\right) - u'' F_{u''} = \text{constant}. \qquad \text{(B.33)}$$

A formula of the form given in (B.33) occurs in its most familiar form in oscillation problems where it connotes conservation of energy. Usually, although not always, in oscillation problems, we have $F_{u''} \equiv 0$.

If we apply the same considerations to the functional given by (B.14) and to the resulting formula (B.18), we can say that if the integrand F in (B.14) does not contain t explicitly, then we can conclude, referring to (B.18), that

$$H = \text{constant}.$$

C. First Variation of a Functional Involving Several Independent Variables

Our purpose here is to explain as simply as possibly the derivation and consequences of Noether's theorem. The examples in subsection B should serve this purpose for variations involving only one independent variable. We now turn to functionals involving functions of several independent variables. Let us first consider the functional

$$K[u] = \int_R F(x,y,t,u,u_x,u_y,u_t)\ dx\ dy\ dt \qquad \text{(C.1)}$$

where R is a region in the space of the independent variables x,y,t, the dependent variable u is a function of x,y,t, and u_x, u_y, u_t are its partial derivatives with respect to x, y, and t, respectively. We first examine the variation of the functional $K[u]$ when R is held fixed. Then, using this result, we allow R to vary also and calculate the general variation of the functional. In this case, however, visualization becomes quite difficult, and it is helpful to be familiar with the calculation for one independent variable. In the calculation for one independent variable, we first stated the question in terms of a transformation of the underlying coordinate frame--the so-called alias viewpoint--so that the original arc under consideration was regarded as held fixed in the plane, while the coordinate frame was moved. Afterward, we essentially restated the problem as one in which the coordinate frame was held fixed--the alibi viewpoint--while the function $u(t)$, $t_0 \leq t \leq t_1$, was (continuously) transformed into the function $u^*(t^*)$, $t_0^* \leq t^* \leq t_1^*$. It was from the alibi viewpoint that the actual calculations were made.

We shall not lay any stress on the analyticity properties to be prescribed for the functions $u(x,y,t)$ in the function space S over which $K[u]$ is defined. It will merely be assumed that the functions $u(x,y,t)$ satisfy properties sufficient for the carrying out of the various differentiation and integration procedures indicated in the derivation of the variational formulas.

For the fixed and bounded region R, we follow the standard procedure and assume

$$u^*(x,y,t) = u(x,y,t) + h(x,y,t) = u + h \ ,$$

whence we quickly arrive at

$$K[u^*] - K[u] = K[u+h] - K[u] = \int_R [F_u h + F_{u_x} h_x + F_{u_y} h_y + F_{u_t} h_t] \, dV$$

where

$$dV = dx \; dy \; dt .$$

Then, using differential identities (analogous to the use of integration by parts in the case of a single independent variable), we obtain finally

$$K[u^*] - K[u] = \int_R \left[F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} - \frac{\partial}{\partial t} F_{u_t} \right] h \, dV$$
$$+ \int_R \left[\frac{\partial}{\partial x} (F_{u_x} h) + \frac{\partial}{\partial y} (F_{u_y} h) + \frac{\partial}{\partial t} (F_{u_t} h) \right] dV \, . \tag{C.2}$$

The second integral in (C.2) is in the form of a divergence, so that we may express it as a surface integral:

$$K[u^*] - K[u] = \int_R \left[F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} - \frac{\partial}{\partial t} F_{u_t} \right] h \, dV$$
$$+ \int_{\partial R} \left[n_x F_{u_x} + n_y F_{u_y} + n_t F_{u_t} \right] h \, dS$$

where (n_x, n_y, n_t) is the unit outward-pointing normal to the region with surface ∂R and surface area element dS.

Let us now regard the coordinate system u, x, y, t as being transformed to a new coordinate system u^*, x^*, y^*, t^*, in which the starred variables are to be regarded as functions of u, x, y, t and a parameter ϵ, with these functions having the property that, for $\epsilon = 0$

$$u^* = u, \quad x^* = x, \quad y^* = y, \quad t^* = t.$$

In other words, we have again prescribed a one-parameter group of transformations (continuous in the parameter) of the coordinates, with this group having the property that the group element corresponding to $\epsilon = 0$ is the identity element of the group. We have

$$\begin{aligned} u^*(x^*, y^*, t^*) &= \Phi(x^*, y^*, t^*, \epsilon) \\ x^* &= G(x, y, t, \epsilon) \\ y^* &= H(x, y, t, \epsilon) \\ t^* &= T(x, y, t, \epsilon) \end{aligned} \tag{C.3}$$

under the conditions

$$
\begin{aligned}
u(x,y,t) &= \Phi(x,y,t,0) \\
x &= G(x,y,t,0) \\
y &= H(x,y,t,0) \\
t &= T(x,y,t,0)
\end{aligned}
\tag{C.4}
$$

The variations are then

$$
\begin{aligned}
\delta x &= \epsilon\xi(x,y,t), \quad \xi = \left(\frac{\partial G}{\partial \epsilon}\right)_{\epsilon=0} \\
\delta y &= \epsilon\eta(x,y,t), \quad \eta = \left(\frac{\partial H}{\partial \epsilon}\right)_{\epsilon=0} \\
\delta t &= \epsilon\tau(x,y,t), \quad \tau = \left(\frac{\partial T}{\partial \epsilon}\right)_{\epsilon=0} \\
h &= \epsilon\left(\frac{\partial \Phi}{\partial \epsilon}\right)_{\epsilon=0}, \quad \Phi = \Phi(x,y,t,\epsilon) \\
\delta u &= h + \epsilon\left(\frac{\partial \Phi}{\partial x}\xi + \frac{\partial \Phi}{\partial y}\eta + \frac{\partial \Phi}{\partial t}\tau\right)_{\epsilon=0} \\
\delta u_x &= h_x + \epsilon\left(\frac{\partial^2 \Phi}{\partial x^2}\xi + \frac{\partial^2 \Phi}{\partial x\partial y}\eta + \frac{\partial^2 \Phi}{\partial x\partial t}\tau\right)_{\epsilon=0} \\
\delta u_y &= h_y + \epsilon\left(\frac{\partial^2 \Phi}{\partial y\partial x}\xi + \frac{\partial^2 \Phi}{\partial y^2}\eta + \frac{\partial^2 \Phi}{\partial y\partial t}\tau\right)_{\epsilon=0} \\
\delta u_t &= h_t + \epsilon\left(\frac{\partial^2 \Phi}{\partial t\partial x}\xi + \frac{\partial^2 \Phi}{\partial t\partial y}\eta + \frac{\partial^2 \Phi}{\partial t^2}\tau\right)_{\epsilon=0}
\end{aligned}
\tag{C.5}
$$

It is then seen that the Jacobian J of the transformation from (x,y,t) to (x^*,y^*,t^*) becomes, to first-order terms in ϵ,

$$
\begin{aligned}
J = \frac{\partial(x^*,y^*,t^*)}{\partial(x,y,t)} &= 1 + \epsilon(\xi_x + \eta_y + \tau_t) \\
&= 1 + (\delta x)_x + (\delta y)_y + (\delta t)_t
\end{aligned}
$$

where a variable in a subscript means partial differentiation with respect to that variable when and only when the variable appears explicitly. Under the transformation given in (C.3), suppose that the region R in (u,x,y,t)-space is taken into the region R* in (u^*,x^*,y^*,t^*)-space. Now, starting from the definition (C.1), we examine the functional

$$K[u^*] = \int_{R^*} F\left[x^*, y^*, t^*, u^*, \frac{\partial u^*}{\partial x^*}, \frac{\partial u^*}{\partial y^*}, \frac{\partial u^*}{\partial t^*}\right] dx^*\, dy^*\, dt^*. \quad (C.6)$$

If we now take the view that the manifold $u^*(x^*,y^*,t^*)$ is merely obtained by a continuous, infinitesimal transformation of the manifold $u(x,y,t)$ in the (u,x,y,t)-space, then the former manifold can be referred back to the unstarred coordinate system, and we can write

$$K[u^*] = \int_R F\big[G,\ H,\ T,\ \Phi(G,H,T,\epsilon),\ \Phi_1(G,H,T,\epsilon),\ \Phi_2(G,H,T,\epsilon),\ \Phi_3(G,H,T,\epsilon)\big]\, J\, dV \qquad (C.7)$$

where the subscript i denotes the partial derivative of a function with respect to its i^{th} argument. To the first order in ϵ

$$K[u^*] - K[u] = \int_R \big[\epsilon(F_1\xi + F_2\eta + F_3\tau) + F_4\,\delta u + F_5\,\delta u_x + F_6\,\delta u_y + F_7\,\delta u_t + \epsilon F(\xi_x + \eta_y + \tau_t)\big]\, dV\ .$$

This may be rearranged to read

$$\begin{aligned} K[u^*] - K[u] &= \int_R \Big[\Big(F_4 - \frac{\partial}{\partial x}F_5 - \frac{\partial}{\partial y}F_6 - \frac{\partial}{\partial t}F_7\Big) h \\ &\qquad + \frac{\partial}{\partial x}(F_5 h + \epsilon F\xi) + \frac{\partial}{\partial y}(F_6 h + \epsilon F\eta) + \frac{\partial}{\partial t}(F_7 h + \epsilon F\tau)\Big]\, dV \\ &= \int_R \Big(F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y} - \frac{\partial}{\partial t}F_{u_t}\Big) h\, dV \\ &\qquad + \int_{\partial R}\big[n_x(F_{u_x}h + F\,\delta x) + n_y(F_{u_y}h + F\,\delta y) \\ &\qquad\qquad + n_t(F_{u_t}h + F\,\delta t)\big]\, dS \end{aligned}$$

where the usual variational notations have been introduced as: $F_u = F_4$, $F_{u_x} = F_5$, $\delta x = \epsilon\xi$, etc.

If $K[u]$ is invariant under the group of transformations for an arbitrary region R and u is an extremal of $K[u]$, then

$$\frac{\partial}{\partial x}(F_{u_x}h + F\,\delta x) + \frac{\partial}{\partial y}(F_{u_y}h + F\,\delta y) + \frac{\partial}{\partial t}(F_{u_t}h + F\,\delta t) = 0 \qquad (C.9)$$

SECTION V

A NUMERICAL METHOD FOR CALCULATING CONTINUOUS PERIODIC FLOWS WITH SPHERICAL SYMMETRY UNDER NEAR-ACOUSTIC CONDITIONS

G. M. Muller

A. Introduction

Under a previous contract, a preliminary investigation was made of a numerical method for calculating continuous, periodic flows of a perfect gas under the conditions of spherical symmetry and constant entropy. Our main purpose in this section is to present a completely revised version of the original method and to attempt some assessment of its capabilities and limitations. Although the emphasis will be on underlying principles, we shall give a general description and complete listing of SPHERE, a FORTRAN code embodying these principles, and discuss several sample calculations.†

The idea of a periodic flow without shocks entails certain assumptions that we now examine. Consider a sphere immersed in an infinite mass of fluid initially at rest. The radius of the sphere is taken to be the unit of distance and the fluid is assumed to be a perfect gas. Denote time by t and distance from the center of the sphere by r. Starting at $t = 0$, we prescribe, say, the mass flow through $r = 1$ as a continuous periodic function, $f(t)$, with $f(t + T) = f(t)$. Because of the periodicity,

† SPHERE is designed specifically on the assumption that the perfect gas is air, with the ratio of specific heats, γ, equal to 1.4.

the surface $r = 1$ is a source of alternating compression and rarefaction waves, with each compression wave eventually steepening into a shockfront. If a strictly periodic regime of period T exists in the limit $t \to \infty$, it must involve an inner region, $1 < r < R$, without shocks and an outer region, $r > R$, containing an infinite sequence of shockfronts S_n. The shockfronts may be labeled S_{-m}, $S_{-(m-1)}$, ..., S_0, S_1, ... at some particular instant t. After the lapse of time interval of length T, a new shockfront, $S_{-(m+1)}$, has been created at $r = R$, and each S_n has moved into the position previously occupied by S_{n+1}. Since any shock increases the entropy of the gas behind it, conditions at $r = R$ will be periodic if and only if each new shock is formed in gas of constant entropy. It is plausible to assume (without rigorous proof) that this condition is at least approximately equivalent to the requirement that the d.c. component of the local Mach number at R (i.e., the local Mach number averaged over the period T), be greater than one. Ignoring rather extreme flows of this sort, we seek a boundary condition to be imposed at $r = R$ that, although permitting an adiabatic periodic flow of period T in the region $1 < r < R$, is reasonably realistic in terms of the physical model underlying our considerations for a near-acoustic range of conditions. (In this new context, R may be less than the shock-formation distance.) The imposition of this boundary condition then allows us to confine our calculations to the region $1 < r < R$. It should be noted that not all possible boundary conditions are consistent with adiabatic flow in $1 < r < R$. Thus, if we prescribe zero particle velocity at R, we have effectively enclosed the flow field in a rigid spherical shell and this will generally lead to shocks.

The condition actually incorporated in SPHERE is that the pressure and particle velocity at R satisfy the same relation as that in an outgoing spherical wave obeying the laws of linear acoustics for a lossless medium. We shall call this the acoustic boundary condition. There is at the present time no existence or uniqueness theory for the problem we have just formulated. However, on the basis of our computing experience with the finite-difference analog of this problem, it seems safe to assume that for sufficiently small amplitudes of the prescribed motion at $r = 1$, a continuous solution does exist and is unique.†

We now try to determine whether the acoustic boundary condition is realistic. We first discuss the case of slab geometry (i.e., motion depending on a single cartesian space coordinate x) where the situation is relatively simple. Here it is known that for outgoing waves the shocks do not influence the flow behind them if one can neglect the change in entropy across each shock. Under this assumption, if the gas is at rest at $t = 0$, and we prescribe a periodic motion at $x = 0$ for $t > 0$, the motion at any point $x > 0$ is periodic for $t > x/c_0$, c_0 being the ambient sound velocity. Hence, in slab geometry, it is not necessary to consider the periodic motion as the limiting case $t \to \infty$ of a motion originally starting from rest. For sufficiently small amplitudes, the entropy change resulting from each new shock, being of the third order in the shock strength, will produce only an insignificant effect. However, these changes are cumulative so that for sufficiently large values of t,

† See subsection G, paragraph 3, concerning the possible existence of subharmonic regimes.

any volume of gas located beyond X, the shock-formation distance, will have undergone some shock heating. Moreover, for $x > X$, the temperature of the gas decreases as x increases because of the decreasing number of shocks that have passed x. Consequently, at $x = X$ the outgoing waves enter a region of variable mean density and sound velocity so that, even in the limit of linear acoustics, the proper boundary condition to be imposed at X (or at some distance less than X) is not that there be no ingoing waves but rather that the ingoing and outgoing waves be related according to the effective impedance mismatch at X. Although this new boundary condition is time dependent, in many cases of interest this dependence will be negligible over a period of the motion so that in the calculation of the flow in $0 < x < X$, time may be regarded as affecting the boundary condition only as a parameter.

For motion with spherical symmetry, the problem becomes somewhat more complicated because the creation of a discontinuity, such as a shock, will produce a mechanical disturbance traveling backward into the region $1 < r < R$, even if the entropy change across the shock can be ignored. However, in this case we can introduce real-gas effects to reason as follows: Viscosity and thermal conductivity predominate over nonlinear effects for sufficiently small amplitudes and thus prevent the formation of shocks. On the other hand, the disturbances produced by the real-gas effects themselves are attenuated exponentially; therefore, for a suitable range of conditions, they do not penetrate significantly into the part of the flow field in which we wish to carry out our computations under the assumption that the gas is perfect. These considerations are applicable only if the amplitude of the acoustic pressure at R is sufficiently small, of course. Under suitable conditions, therefore, we again have nonuniform heating of the gas outside R, which may be approximately accounted for by a boundary condition at R. (It is likely, in fact, that an approximately correct boundary condition can be found even if shocks do occur beyond R, provided merely that the amplitude of the acoustic pressure at R is sufficiently small.)

As we have already mentioned, the boundary condition actually incorporated in SPHERE is that of no reflection at R; it corresponds to the physical situation at a time when a (nearly) periodic regime has already been established but before significant heating has occurred beyond R. It will be apparent later that the machine program can readily accommodate other boundary conditions.

The basic element of the method of computation is the integration of a single partial differential equation of the form $w_x = F(w, w_\eta, \eta, x)$ by a predictor-corrector scheme. This is presented in subsection B, together with a summary of numerical results for a particular equation, $w_x = ww_\eta$, for which the exact solution is known. Subsection C contains the basic formulation for the spherical-wave problem; the choice of variables employed in this formulation is ultimately connected with the necessity of solving what is essentially a boundary- rather than initial-value problem for a hyperbolic system of partial differential equations. In subsection D, we derive the exact form of the boundary conditions and determine the form of the solution in the limit of linear acoustics. In subsection E, we describe in considerable detail the method of computation for obtaining the solution of the problem formulated in subsections C and D. Subsection F contains instructions for the use of the SPHERE code in its present form, a discussion of its limitations, and a survey of results obtained so far. In subsection G, we recommend specific additional calculations, suggest certain modifications of the SPHERE code, and indicate the importance of some fundamental problems connected with the possibility of subharmonic oscillations.

B. A Predictor-Corrector Method for Integrating $w_x = F(w, w_\eta, \eta, x)$

1. The General Method

In this subsection, we consider a finite-difference method for solving partial differential equations of the form

$$w_x = F(w, w_\eta, \eta, x), \tag{B.1}$$

subject to the initial condition

$$w(\eta, 0) = f(\eta) \tag{B.2}$$

and the periodicity condition

$$w(\eta + T, x) = w(\eta, x). \tag{B.3}$$

We shall assume that F and f are analytic in their respective arguments, and confine our attention to a range of x such that $w(\eta, x)$ is analytic; note that (B.2) and (B.3) require that $f(\eta + T) = f(\eta)$. We shall carry out our computations in $0 \leq \eta < T$, and use (B.3) to obtain the required approximation for w_η at the endpoints.

For the finite-difference method, the η-range is divided into N intervals, of length $\Delta\eta = N/T$; integration in the x-direction is performed in steps of size Δx. For any function $\varphi(\eta, x)$ we write

$$\varphi_{i,j} \equiv \varphi(i\Delta\eta,\ j\Delta x); \tag{B.4}$$

since we are only interested in functions of period T,

$$\varphi_{N,j} = \varphi_{0,j}; \quad \varphi_{-1,j} = \varphi_{N-1,j} . \tag{B.5}$$

To obtain a finite-difference approximation for (B.1), we first integrate (at constant η) from $j\Delta x$ to $(j+1)\Delta x$:

$$w_{i,j+1} - w_{i,j} = \int_{j\Delta x}^{(j+1)\Delta x} F(w, w_\eta, \eta, x)\ dx . \tag{B.6}$$

We next approximate the integral by the trapezoidal rule and replace the derivative w_η by the corresponding central-difference quotient to obtain

$$w_{i,j+1} = w_{i,j} + \frac{\Delta x}{2}\left[F_{i,j+1} + F_{i,j}\right] , \tag{B.7}$$

where

$$F_{i,j} \equiv F\left(w_{i,j}, \frac{w_{i+1,j} - w_{i-1,j}}{2\Delta\eta}, i\Delta\eta, j\Delta x\right) . \tag{B.8}$$

Suppose now that we have calculated $w_{i,j}$ for $0 \le i \le N-1$ and wish to carry the computation to $j + 1$. We first obtain $w_{N,j}$ and $w_{-1,j}$ from (B.5); this enables us to calculate $F_{i,j}$ for $0 \le i \le N-1$. If we cannot solve (B.7) explicitly for the $w_{i,j+1}$, we may proceed by an iteration method. Let $w^{(0)}_{i,j+1}$ for $0 \le i \le N-1$ be a first guess at the value of $w_{i,j+1}$. We obtain $w^{(0)}_{N,j}$ and $w^{(0)}_{-1,j}$ from (B.5) and are then in a position to calculate a corresponding first approximation $F^{(0)}_{i,j+1}$ to $F_{i,j+1}$. We may use this approximation to calculate a new approximation $w^{(1)}_{i,j+1}$ from (B.7), and in general, we have the iterative formula

$$w^{(n)}_{i,j+1} = w_{i,j} + \frac{\Delta x}{2}\left[F^{(n-1)}_{i,j+1} + F_{i,j}\right], \qquad 0 \le i \le N-1. \tag{B.9}$$

We evaluate

$$\varepsilon^{(n)}_{j+1} \equiv \max_i \left| w^{(n)}_{i,j+1} - w^{(n-1)}_{i,j+1} \right| \tag{B.10}$$

in the course of each iteration, and discontinue the computation (assuming the process converges) after this quantity becomes less than some preassigned limit; we then set $w_{i,j+1} = w^{(n)}_{i,j+1}$ and $F_{i,j+1} = F^{(n-1)}_{i,j+1}$ and are now ready to carry the computation to $j + 2$.

Whether the iterative process does, in fact, converge, and if so, how rapidly, will depend on the functions F and f, and on the values of $\Delta\eta$ and Δx; evidently, convergence will improve with decreasing Δx. If we choose $w^{(0)}_{i,j+1} = w_{i,j}$, then unless Δx is excessively small, it will usually take several iterations to satisfy the criterion on $\epsilon^{(n)}_{j+1}$. The computation may often be greatly speeded up by using the following scheme. First, observe that with the choice $w^{(0)}_{i,j+1} = w_{i,j}$, we also have $F^{(0)}_{i,j+1} = F_{i,j}$; hence to find $w^{(1)}_{i,j+1}$, we need not reevaluate F. Generally, the length of the entire computation will be largely determined by the number of times we have to calculate F; it will therefore pay us to have saved the F's for, say, $j - 1$ and $j - 2$ [$F_{i,j}$ is needed in any case, in (B.7)] and extrapolate to $j + 1$. We write down the relevant expressions:

(a) No extrapolation:

$$F^{(0)}_{i,j+1} = F_{i,j}$$

(b) Linear extrapolation:

$$F^{(0)}_{i,j+1} = 2\,F_{i,j} - F_{i,j-1} \tag{B.11}$$

(c) Quadratic extrapolation:

$$F^{(0)}_{i,j+1} = 3\,F_{i,j} - 3\,F_{i,j-1} + F_{i,j-2}\,.$$

In the machine program, the appropriate expression from (B.11) is combined with (B.9) to yield $w^{(1)}_{i,j+1}$ directly. For $j+1 = 1$, we have to use (a); for $j+1 \geq 2$, we may use (b); and for $j+1 \geq 3$, we may use (c). A slight modification is required if we wish to change the step-size Δx during the course of the computation. Suppose that the second subscript on F refers to the value of x identified by the same subscript. Then, for quadratic extrapolation,

$$F^{(0)}_{i,j+1} = R_2 F_{i,j} + R_1 F_{i,j-1} + R_0 F_{i,j-2} \tag{B.12}$$

where

$$R_0 \equiv Q_1 Q_2, \qquad R_1 \equiv -Q_2 Q_0, \qquad R_2 \equiv Q_0 Q_1,$$

$$Q_0 \equiv \frac{x_{j+1} - x_{j-2}}{x_j - x_{j-1}}, \quad Q_1 \equiv \frac{x_{j+1} - x_{j-1}}{x_j - x_{j-2}}, \quad Q_2 \equiv \frac{x_{j+1} - x_j}{x_{j-1} - x_{j-2}} \tag{B.13}$$

Equations (B.11) and (B.9) employed together to yield $w^{(1)}_{i,j+1}$ may be called a predictor, and the subsequent application of (B.9) alone may be called a corrector, in the standard terminology used for the discussion of numerical integration schemes for initial-value problems in ordinary differential equations [6, p. 186]. Although predictor-corrector methods are not commonly used for solving initial-value problems in partial differential equations, they can be quite effective as we shall show later.

2. Preliminary Remarks About the Simple-Wave Equation, $w_x = ww_\eta$

We consider the equation

$$w_x = ww_\eta \tag{B.14}$$

subject to the initial condition

$$w(\eta, 0) = \sin \eta \tag{B.15}$$

and the periodicity condition

$$w(\eta + 2\pi, x) = w(\eta, x). \tag{B.16}$$

It is easy to show that $w(2\pi - \eta, x) = -w(\eta, x)$ so that we need only consider the range $0 \leq \eta \leq \pi$; moreover, (B.15) and (B.14) together imply

$$w(0,x) = w(\pi,x) = 0, \text{ all } x, \tag{B.17}$$

which makes the explicit use of (B.16) unnecessary. The solution of this problem is well known; it is provided in parametric form by

$$w(\eta,x) = \sin\tau \tag{B.18}$$

where τ satisfies Kepler's equation,

$$\tau - x\sin\tau = \eta. \tag{B.19}$$

For any x, $0 \le x \le 1$, as τ increases from 0 to π, so does η; hence, we may obtain τ as a single-valued function of η, $0 \le \eta \le \pi$, by reverse interpolation and thus calculate $w(\eta,x)$ to the desired accuracy.

Equations (B.18) and (B.19) together define a simple wave [1; 4, Chaps. 2 and 3]. For $x > 1$, τ is no longer a single-valued function of η; in a gas-dynamic context, $x = 1$ may be interpreted as the shock-formation distance. At the point $x = 1$, $\eta = 0$ ($\eta = 0$ corresponds to $\tau = 0$), $\partial\tau/\partial\eta$ is infinite; this point is the beginning of the shockfront. To determine the behavior of $w(\eta,1)$ in the vicinity of the incipient shockfront, we expand (B.19) for small τ,

$$\tau - (\tau - \frac{1}{6}\tau^3 + \ldots) = \eta, \tag{B.20}$$

and obtain $\tau \simeq (6\eta)^{1/3}$, hence

$$w(\eta,1) \simeq (6\eta)^{1/3}, \qquad \eta \ll 1. \tag{B.21}$$

3. Application of the Predictor-Corrector Method to the Equation $w_x = ww_\eta$

The numerical method described in paragraph 1 is suitable for the problem defined by equations (B.14), (B.15), and (B.17). The range $0 \le \eta \le \pi$ is divided into N intervals, of length $\Delta\eta = \pi/N$. In view of the discussion in paragraph B.2, the appropriate range for x is $0 \le x \le 1$;

it is divided into K intervals, of length $\Delta x = 1/K$. Because of condition (B.17), the calculations need only be carried out for i = 1, ..., N-1; setting

$$w_{0,j}^{(n)} = w_{N,j}^{(n)} = 0 \qquad \text{(B.22)}$$

for all n and j replaces the procedure described immediately following equation (B.8).

To specify the computation completely, we must provide numbers ϵ_j such that the iterative procedure for a particular j is discontinued when $|\epsilon_j^{(n)}| < \epsilon_j$, where $\epsilon_j^{(n)}$ is the number defined in (B.10). Since the error resulting from using a finite number of iterations at a particular value of j affects the computed values of w for all larger j, it seems reasonable to use a variable ϵ_j, with $\epsilon_j < \epsilon_{j+1}$. In the machine program for carrying out these computations, ϵ_j was given by

$$\epsilon_j = \mathrm{Lim}_1 + \frac{\mathrm{Lim}_2 - \mathrm{Lim}_1}{K} \cdot j \qquad \text{(B.23)}$$

where Lim_2 and Lim_1 are input data. The program was written in ALGOL for the Burroughs 5500 computer (with 32K core memory), originally without provision for extrapolation. Both linear and quadratic extrapolation were then investigated; after the superiority of quadratic extrapolation had become apparent, the program was optimized from the standpoint of running efficiency.

4. Discussion of Results

The number of iterations† required at selected values of $x(= j\Delta x)$ is given in Table II. Numbers 0, 1, and 2 in the column headed "predictor"

† This is the number of times that the values of the $F_{i,j}$ were computed (for a particular j) from the expression (B.8); not included in the count is the initial estimate, (B.11).

Table II

COMPUTATIONAL DETAILS FOR SEVERAL
NUMERICAL SOLUTIONS OF THE SIMPLE-WAVE PROBLEM

Case	K	N	Pre-dictor (Degree)	$-\log_{10} Lim_1$	$-\log_{10} Lim_2$	Number of Iterations at x =										Computation Time (B5500) (Seconds)
						0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	
1	100	200	0	8	6	4	4	4	4	4	4	4	4	5	7	
2	100	100	0	8	6	4	4	4	4	4	4	4	4	5	6	
3	100	100	1	8	6	2	2	2	2	2	2	3	3	3	4	
4	100	100	2	8	6	1	1	1	1	1	1	2	2	3	3	17.1
5	50	50	0	6	5	4	4	4	4	4	4	4	4	5	5	
6	50	50	2	6	5	1	1	1	1	1	1	2	2	3	3	4.7
7	20	20	2	6	5	3	3	3	3	3	3	3	3	3	3	2.6
8	40	100	0	8	6	5	5	5	5	5	5	5	6	7	9	
9	20	100	0	8	6	6	6	6	6	6	7	8	9	14	20	

refer, respectively, to (a), (b), and (c) of (B.11). "Computation time" is the time elapsed between the reading of the DATA card and the execution of the first WRITE instruction; it includes all preliminary computations such as the calculation of sin ($i\Delta\eta$). (Computation times are quoted only for the optimized program.)

The validity of the following observations is apparent from an inspection of Table II:

(a) There is a general tendency for the number of iterations to increase as $x \rightarrow 1.0$, i.e., as the singularity in the exact solution is approached.

(b) A quadratic predictor is superior to a linear predictor; a linear predictor is superior to a zero-degree predictor. This superiority is more significant when the exact solution is smooth (small x) than in the neighborhood of the singularity (large x) (cases 2,3,4; 5,6).

(c) Increasing Δx with $\Delta\eta$ fixed increases the number of iterations required, especially for large x (cases 2,8,9).

(d) Increasing Δx and $\Delta\eta$ with $\Delta x/\Delta\eta$ fixed may leave the number of iterations substantially unchanged (cases 2,5; 4,6) or produce a moderate increase (cases 6,7).

These remarks are based on a limited range of experience, and some caution should be used in drawing general conclusions.

The numerical results obtained depend on the finite-difference net, i.e., on K and N; as one would expect, however, they are substantially independent of the degree of the predictor used. Figure 3 shows contour

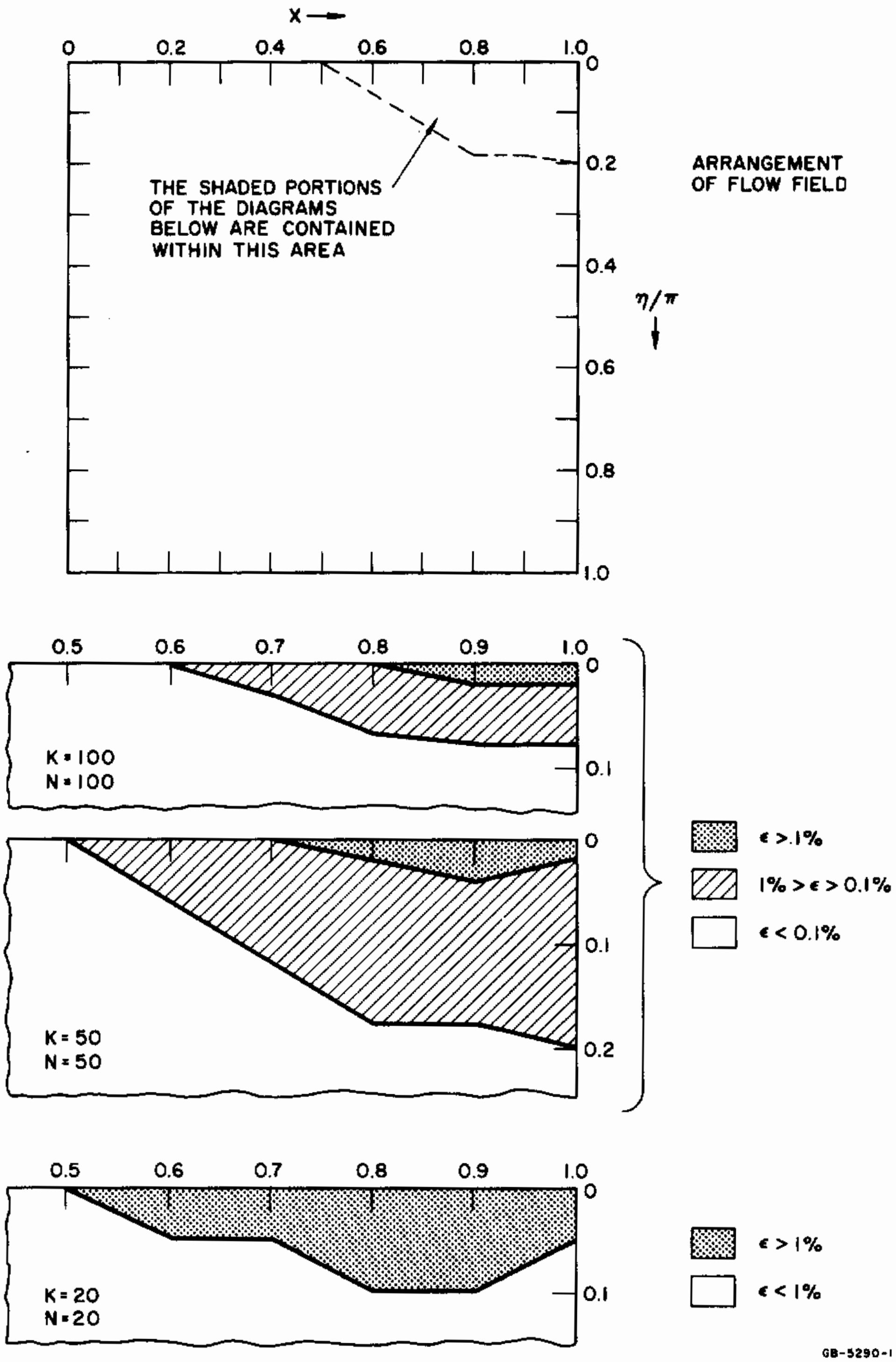

FIG. 3 PERCENTAGE ERROR (ε) RELATIVE TO EXACT SOLUTION OF THREE SIMPLE-WAVE CALCULATIONS

maps of the percentage error relative to the exact solution for three finite-difference computations.† Large errors are confined to the immediate neighborhood of the singularity; a 50 X 50 net seems satisfactory for many purposes.

To give a more precise idea of the behavior of the finite-difference approximation near the singularity, the computed values of $w(\eta,1)$ are shown in Figure 4 for $0 \le \eta/\pi \le 0.2$ and $N = 200, 100, 50$, and 20, along with the exact solution. Note that in each case, $w_{1,1}$ shows a large overshoot, strongly reminiscent of the behavior of the Fourier approximation to a discontinuous function--the so-called Gibbs phenomenon [3, Chap. 9]. In view of the close connection between finite-difference methods and finite Fourier representations in the case of linear partial differential equations [11, p. 10], this is not too surprising.

C. Formulation of the Spherical Wave Problem

1. The Differential Equations

The equations of gas dynamics for the spherically symmetric, homentropic flow of a perfect fluid obeying a polytropic equation of state may be written in the form

$$\begin{aligned} \frac{\partial q}{\partial t} + \frac{2}{\gamma - 1}\frac{\partial c}{\partial t} + (q + c)\left[\frac{\partial q}{\partial r} + \frac{2}{\gamma - 1}\frac{\partial c}{\partial r}\right] &= -2\,\frac{cq}{r}\,, \\ \frac{\partial q}{\partial t} - \frac{2}{\gamma - 1}\frac{\partial c}{\partial t} + (q - c)\left[\frac{\partial q}{\partial r} - \frac{2}{\gamma - 1}\frac{\partial c}{\partial r}\right] &= 2\,\frac{cq}{r}\,, \end{aligned} \tag{C.1}$$

where the various symbols have the following meanings:

t: time
r: Eulerian radial coordinate
q: particle velocity

† The maps are based on the values of w at $x = 0, 0.1, \ldots, 0.9, 1.0$ even though the computations themselves also involved intermediate net-points.

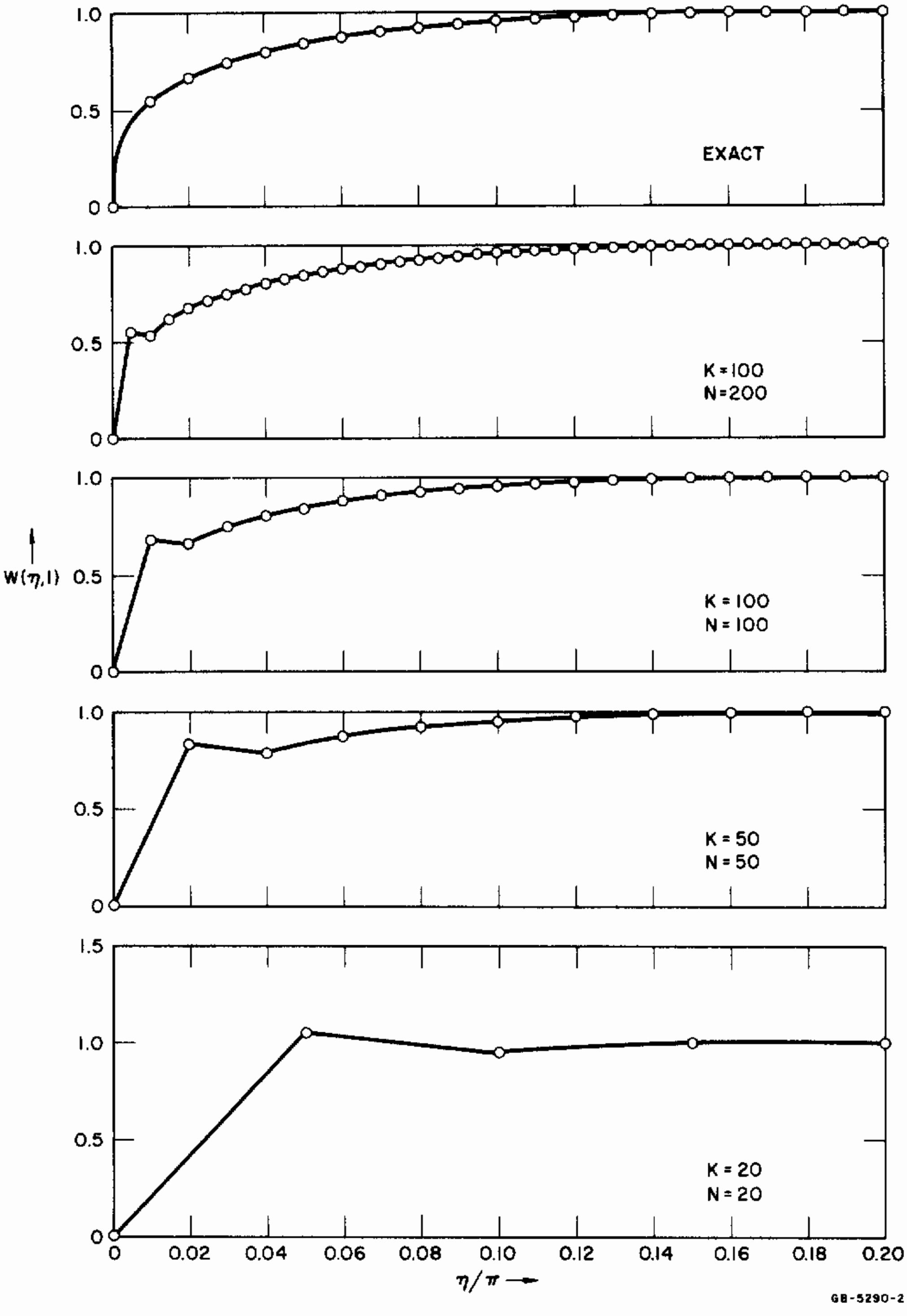

FIG. 4 EXACT AND FINITE-DIFFERENCE SOLUTIONS OF THE SIMPLE-WAVE EQUATION FOR $x = 1$, $0 \leq \eta/\pi \leq 0.2$. Note the Gibbs phenomenon near the incipient shock front, $\eta = 0$.

c: local sound velocity

γ: ratio of specific heats.

In the case of slab geometry, the right-hand sides of equations (C.1) vanish, and it is customary to introduce new dependent variables $q \pm 2c/(\gamma - 1)$, the Riemann invariants. These variables are also useful in the spherical case; however, since our concern is with relatively small deviations from an ambient state ($q = 0$, $c = c_0$), it is convenient to define as dependent variables the fractional deviation of $q \pm 2c/(\gamma - 1)$ from the respective ambient value:

$$
\begin{aligned}
\rho &= \frac{c - c_0}{c_0} + \frac{\gamma - 1}{2} \frac{q}{c_0} , \\
\sigma &= \frac{c - c_0}{c_0} - \frac{\gamma - 1}{2} \frac{q}{c_0} .
\end{aligned}
\tag{C.2}
$$

Additionally, we introduce a time variable τ measured in units of the time taken by a signal to travel unit distance at the ambient sound speed, and define two constants a and b:

$$
\begin{aligned}
\tau &= c_0 t , \\
a &= \frac{\gamma + 1}{2(\gamma - 1)} = 3 , \\
b &= \frac{3 - \gamma}{2(\gamma - 1)} = 2 ,
\end{aligned}
\tag{C.3}
$$

where the numerical values for a and b are appropriate for air, $\gamma = 1.4$. With these definitions, (C.1) becomes

$$
\begin{aligned}
\frac{\partial \rho}{\partial \tau} + (1 + a\rho - b\sigma) \frac{\partial \rho}{\partial r} &= - \frac{1}{r} (\rho - \sigma + \frac{1}{2} \rho^2 - \frac{1}{2} \sigma^2), \\
\frac{\partial \sigma}{\partial \tau} - (1 + a\sigma - b\rho) \frac{\partial \sigma}{\partial r} &= - \frac{1}{r} (\rho - \sigma + \frac{1}{2} \rho^2 - \frac{1}{2} \sigma^2).
\end{aligned}
\tag{C.4}
$$

The characteristics [4, Chaps. 2 and 3] of this system are described by the differential equations

$$C_+: \quad \frac{dr}{dt} = +(1 + a\rho - b\sigma),$$
$$C_-: \quad \frac{dr}{dt} = -(1 + a\sigma - b\rho). \tag{C.5}$$

As we shall deal with problems for which ρ and σ are small, we introduce characteristic coordinates for the corresponding linearized system:

$$\eta = \tau - r + 1,$$
$$\zeta = \tau + r - 1; \tag{C.6}$$

the additive constants are chosen so that for $r = 1$, $\zeta = \eta = \tau$. Because both boundary conditions will be prescribed along lines r = constant, it turns out to be useful to keep three independent variables, η, ζ, and r, and to use them in pairs, (η, r) and (ζ, r). Also, for reasons that will become apparent later, we introduce $u = r\rho$ and $v = r\sigma$ as new dependent variables. To indicate that u and v are to be regarded as functions of η and r, we shall use lower case letters; if u and v are to be considered as functions of ζ and r, we shall denote them by U and V.† Accordingly, our definitions are as follows:

$$u = U = r\rho; \quad u(\eta, r) \equiv U(\zeta, r);$$
$$v = V = r\sigma; \quad v(\eta, r) \equiv V(\zeta, r). \tag{C.7}$$

With these conventions, we obtain the following pair of partial differential equations:

$$\frac{\partial u}{\partial r} = g(\eta, r) \equiv \frac{(au - bv)\left(\frac{\partial u}{\partial \eta} + ur^{-1}\right) + v - \frac{1}{2}(u^2 - v^2)r^{-1}}{r[1 + (au - bv)r^{-1}]},$$
$$\frac{\partial V}{\partial r} = h(\zeta, r) \equiv \frac{(aV - bU)\left(-\frac{\partial V}{\partial \zeta} + Vr^{-1}\right) + U - \frac{1}{2}(V^2 - U^2)r^{-1}}{r[1 + (aV - bU)r^{-1}]}. \tag{C.8}$$

† The convention regarding the use of lower and upper case letters applies only to u and v.

2. Boundary and Periodicity Conditions

We shall wish to impose boundary conditions at $r = 1$ and $r = R$. For the moment assume that these may be put in the form

$$u(\eta,1) = L_1[v(\eta',1)], \qquad V(\zeta,R) = L_2[U(\zeta',R)], \tag{C.9}$$

where L_1 and L_2 may be functional operators; the exact expressions will be derived in subsection D.

It will be convenient to assume that the period T of the flow refers to the variable τ (= $c_0 t$) rather than t. We see from (C.6) that, for fixed r, both η and ζ equal τ plus a constant. Therefore, the periodicity conditions for u and V are simply

$$u(\eta + T,r) = u(\eta,r), \qquad V(\zeta + T,r) = V(\zeta,r). \tag{C.10}$$

3. Comments on the Formulation--The Quasi-Cartesian Approximation

The problem defined by (C.8), (C.9), and (C.10) lends itself to an iterative method of solution that will be described in subsection E. For the moment, we note that if $v(\eta,r)$ is prescribed as an arbitrary analytic function, with $v(\eta + T,r) = v(\eta,r)$, the first members of each pair of equations (C.8), (C.9), and (C.10) define a problem substantially identical with that discussed in subsection B if u and r are identified, respectively, with w and x. A very similar remark applies to the three equations involving U and V.

As a consequence of the coordinate system used, the right-hand sides of (C.8) involve derivatives only as second-order terms; in fact, if we set $v = 0$ and discard terms of order r^{-2}, then the first equation (C.8) becomes simply

$$\frac{\partial u}{\partial r} = r^{-1} a u \frac{\partial u}{\partial \eta} . \tag{C.11}$$

If we now define

$$x = a \log r$$

$$w(\eta, x) \equiv u(\eta, r), \tag{C.12}$$

then (C.11) becomes

$$w_x = w w_\eta, \tag{C.13}$$

which is identical with equation (B.14). Although the solution of this equation, combined with $v = 0$, does not satisfy (C.8) exactly, this particular approximation--we shall call it the quasi-cartesian approximation--is essentially equivalent to an approximation that has been proposed as having some validity for large r [2; 8].

4. Acoustic Flow Variables in Terms of u and v

The variables u and v (or U and V) are convenient for computation; however, we shall need to relate them to more common physical variables. For the latter we shall use nondimensional acoustic quantities, viz., the reduced velocity $\tilde{q} = q/c_0$, the reduced acoustic pressure $\tilde{p} = (p - p_0)/p_0$, and the reduced mass flow $\tilde{\theta} = q\bar{\rho}r^2/(\bar{\rho}_0 c_0)$. Here p and $\bar{\rho}$ are pressure and density, respectively, and a subscript 0 denotes ambient values.†
From (C.2) and (C.7) we obtain

$$\frac{q}{c_0} = \frac{1}{\gamma - 1} \frac{u - v}{r},$$

$$\frac{c - c_0}{c_0} = \frac{1}{2} \frac{u + v}{r}. \tag{C.11}$$

Furthermore, we have the thermodynamic relations

$$\frac{p}{p_0} = \left(\frac{c}{c_0}\right)^{2\gamma/(\gamma-1)},$$

$$\frac{\bar{\rho}}{\bar{\rho}_0} = \left(\frac{c}{c_0}\right)^{2/(\gamma-1)}. \tag{C.12}$$

† We use the notation $\bar{\rho}$ to avoid confusion with ρ as defined in (C.2).

From (C.11) and (C.12) we finally obtain the following expressions:

$$\tilde{q} = \frac{q}{c_0} = \frac{1}{\gamma - 1}\,\frac{u - v}{r} = 2.5\,\frac{u - v}{r}\,,$$

$$\tilde{p} = \frac{p - p_0}{p_0} = \left(\frac{1}{2}\,\frac{u + v}{r} + 1\right)^{2\gamma/(\gamma-1)} - 1 = \left(\frac{1}{2}\,\frac{u + v}{r} + 1\right)^{7} - 1, \qquad \text{(C.13)}$$

$$\tilde{\theta} = \frac{\bar{\rho} q r^2}{\bar{\rho}_0 c_0} = \frac{r}{\gamma - 1}\,(u - v)\left(\frac{1}{2}\,\frac{u + v}{r} + 1\right)^{2/(\gamma-1)} = 2.5\,r(u - v)\left(\frac{1}{2}\,\frac{u + v}{r} + 1\right)^{5}.$$

The last expression in each case is appropriate for air, $\gamma = 1.4$.

D. Boundary Conditions and an Initial Approximation

1. Outgoing Spherical Waves in Classical (Linear) Acoustics

In classical acoustics, the pressure and particle velocity in an outgoing spherical wave moving through a medium of constant properties can be derived from a single function $F(z)$:

$$\begin{aligned} p(t,r) - p_0 &= \frac{F'(c_0 t - r + 1)}{r}\,, \\ c_0 \bar{\rho}_0 q(t,r) &= \frac{F'(c_0 t - r + 1)}{r} + \frac{F(c_0 t - r + 1)}{r^2}\,. \end{aligned} \qquad \text{(D.1)}$$

These formulas are obtained from the fact that p satisfies the wave equation, and from the relation $\partial p/\partial r = \bar{\rho}_0 \partial q/\partial t$ [10, p. 242]. If we specify $q(t,1)$ as a given function of t, the second of equations (D.1) becomes an ordinary differential equation for $F(c_0 t)$. The constant of integration gives rise to a term Ar^{-2} in the expression for velocity; hence, even if $p = p_0$, there may be a velocity gradient, corresponding to a constant mass flow $4\pi A$. Now by Bernoulli's law, $\frac{1}{2}\,\bar{\rho}_0 q^2 + p = \text{constant}$;† the

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† More precisely, for a stationary adiabatic flow vanishing at infinity, $\frac{1}{2}\,q^2 = -\int_{p_0}^{p} dp/\bar{\rho} = (p - p_0)/\bar{\rho}_0 + O[(p - p_0)^2]/(p_0\bar{\rho}_0)$. For $p \ll p_0$, the neglected term is $O(q^4)$.

velocity gradient, $\partial q/\partial r = -2Ar^{-3}$, therefore requires a pressure gradient

$$\frac{\partial p}{\partial r} = -\bar{\rho}_0 q \frac{\partial q}{\partial r} = 2\bar{\rho}_0 A^2 r^{-5}.$$

The reason that this pressure gradient does not show up in classical acoustics is that it is of the second order in the velocity. We mention the point to reassure ourselves that the presence of a nonvanishing mean mass flow does not invalidate the use of the acoustic boundary condition at $r = R$.

2. The Differential Equation for $V(\zeta, R)$

Let us find the relation between the variables u and v in the limit of classical acoustics, under the hypothesis that p and q are related by (D.1). First, we note that for $\tilde{p} \ll 1$, the second equation in (C.13) becomes

$$\tilde{p} = \frac{\gamma}{\gamma - 1} \frac{u + v}{r} . \tag{D.2}$$

If we also use the first equation in (C.13), and the definition of η, we find

$$\begin{aligned} u + v &= \frac{\gamma - 1}{\gamma p_0} F'(\eta), \\ u - v &= \frac{\gamma - 1}{\bar{\rho}_0 c_0^2} [F'(\eta) + r^{-1}F(\eta)]. \end{aligned} \tag{D.3}$$

We now make use of the thermodynamic relation $c_0^2 = \gamma p_0/\bar{\rho}_0$ to write the second equation in the form

$$u - v = \frac{\gamma - 1}{\gamma p_0} [F'(\eta) + r^{-1}F(\eta)]; \tag{D.4}$$

hence

$$\begin{aligned} u &= \frac{\gamma - 1}{2\gamma p_0} [2F'(\eta) + r^{-1}F(\eta)], \\ v &= \frac{\gamma - 1}{2\gamma p_0} r^{-1}F(\eta). \end{aligned} \tag{D.5}$$

By differentiating this last equation with respect to η, we can eliminate $F(\eta)$, and have, finally,

$$\frac{\partial v}{\partial \eta} + \frac{v}{2r} = -\frac{u}{2r} \, . \tag{D.6}$$

To determine the form of the boundary condition at $r = R$, we have to put (D.6) in terms of U and V. From (C.6) it is clear that, for constant r, $d\zeta = d\eta$ since each is equal to $d\tau$. Accordingly, from the definition (C.7), $\partial v/\partial \eta = \partial V/\partial \zeta$. Since r enters (D.6) only as a parameter, the partial derivative may be treated as an ordinary derivative, and we obtain

$$\frac{dV(\zeta,R)}{d\zeta} + \frac{V(\zeta,R)}{2R} = -\frac{U(\zeta,R)}{2R} \, . \tag{D.7}$$

We defer the solution of this equation until paragraph D.4.

3. The Differential Equation for the Initial Approximation

Suppose that the reduced velocity, $\tilde{q}$, at $r = 1$ is prescribed as a function of τ, $Q(\tau)$. For $r = 1$, $\tau = \eta$; hence we obtain from (C.13) that

$$u(\eta,1) - v(\eta,1) = (\gamma - 1)Q(\eta). \tag{D.8}$$

Let us define now

$$\psi(\eta) \equiv \frac{1}{2\gamma p_0} F(\eta) ; \tag{D.9}$$

then (D.4) and (D.8) yield the ordinary differential equation,

$$\psi'(\eta) + \psi(\eta) = Q(\eta). \tag{D.10}$$

Once we have calculated $\psi(\eta)$, u and v may be obtained from (D.5), which now takes the form (we use (D.10) to eliminate ψ'):

$$\begin{aligned} u(\eta,r) &= (\gamma - 1)[2Q(\eta) - (2 + r^{-1})\psi(\eta)], \\ v(\eta,r) &= -(\gamma - 1)r^{-1}\psi(\eta). \end{aligned} \tag{D.11}$$

These expressions will serve as a first approximation in computing solutions for the nonlinear problem.

4. Solution of the Differential Equations

Equations (D.7) and (D.10) are of the elementary type

$$y'(\eta) + \beta y(\eta) = \Phi(\eta) \tag{D.12}$$

where β is a constant; the general solution of this equation is

$$y(\eta) = Ce^{-\beta\eta} + \int_0^{\eta} \Phi(t)e^{\beta\tau}\,d\tau. \tag{D.13}$$

Suppose now that $\Phi(\eta + T) = \Phi(\eta)$; if we require that the solution $y(\eta)$ have the same periodicity, i.e., that $y(\eta + T) = y(\eta)$, the constant C may be evaluated. Setting $y(0) = y(T)$, we obtain

$$C = Ce^{-\beta T} + e^{-\beta T}\int_0^T \Phi(\tau)e^{\beta\tau}\,d\tau\,; \tag{D.14}$$

hence

$$y(\eta) = e^{-\beta\eta}\left[(e^{\beta T} - 1)^{-1}\int_0^T \Phi(\tau)e^{\beta\tau}\,d\tau + \int_0^{\eta} \Phi(\tau)e^{\beta\tau}\,d\tau\right]. \tag{D.15}$$

Since it is elementary but slightly tedious to verify that with this choice of C, $y(\eta + T) = y(\eta)$ for all η, we omit the details.

5. The Boundary Conditions

To obtain the acoustic boundary condition at $r = R$ for the nonlinear computation, we use the results of paragraph D.4 on equation (D.7). This yields

$$V(\zeta,R) = -\beta e^{-\beta\zeta}\left[(e^{\beta T}-1)^{-1}\int_0^T U(\tau,R)e^{\beta\tau}\,d\tau + \int_0^{\zeta} U(\tau,R)e^{\beta\tau}\,d\tau\right], \qquad \beta \equiv \frac{1}{2R}, \tag{D.16}$$

a relation that expresses $V(\zeta,R)$ as a linear functional of $U(\zeta,R)$.

We shall consider two possible boundary conditions at $r = 1$. If the reduced velocity is prescribed, then (C.13) gives

$$u(\eta,1) = v(\eta,1) + (\gamma - 1)\tilde{q}(\eta,1)\,; \tag{D.17}$$

if the reduced mass flow is prescribed, then

$$u(\eta,1) = v(\eta,1) + (\gamma - 1)\tilde{\theta}(\eta,1)\left[1 + \frac{u(\eta,1) + v(\eta,1)}{2}\right]^{-2/(\gamma-1)} \tag{D.18}$$

Although (D.18) is an implicit equation for $u(\eta,1)$, it lends itself to a rapidly convergent iterative method of solution. For each required value of η, an iterative formula is obtained by writing $u^{(n)}(\eta,1)$ for $u(\eta,1)$ on the right-hand side, and $u^{(n+1)}(\eta,1)$ on the left-hand side of (D.18); together with $u^{(0)}(\eta,1) = 0$, this defines the iterative process.

Note that the boundary conditions presented in this paragraph conform with (C.9). For numerical purposes, they must be supplemented, in the case of (D.18), by the iterative procedure just given, and, in the case of (D.16), by a suitable numerical quadrature formula.

6. The Initial Approximation

In the limit of classical acoustics, the distinction between $\widetilde{q}(\eta,1)$ and $\widetilde{\theta}(\eta,1)$ vanishes; consequently, the considerations of paragraph D.3 are equally valid if $Q(\tau)$ is the prescribed reduced mass flow at $r = 1$. Applying the results of paragraph D.4, we have

$$\psi(\eta) = e^{-\eta}\left[(e^{T} - 1)^{-1}\int_0^T Q(\tau)e^{\tau}d\tau + \int_0^{\eta} Q(\tau)e^{\tau}d\tau\right] \tag{D.19}$$

from which the classical values of u and v may be obtained by (D.11). A numerical procedure for evaluating (D.16) and (D.19) will be given in subsection E.

E. Numerical Solution of the Boundary-Value Problem

1. Specification of the Problem

The problem we are considering is basically the following: Find the solution of the pair of partial differential equations (C.8), subject to the periodicity conditions (C.10), the "outer" boundary condition (D.15) at $r = R$, and one or the other of the "inner" boundary conditions (D.16) or (D.17) at $r = 1$. The detailed specification of the problem may be conveniently divided into several levels.

a. Basic [OPTBDY, T, NOINFC, INDC, INC, INS]†

OPTBDY specifies whether boundary condition (D.16) or (D.17) will be used: OPTBDY = 0 corresponds to (D.16) (reduced velocity prescribed at $r = 1$) and OPTBDY = 1 corresponds to (D.17) (reduced mass flow prescribed at $r = 1$).

T is the assumed period of the flow, in terms of τ.

Designate by $Q(\tau)$ either $\tilde{q}(\tau,1)$ or $\tilde{\theta}(\tau,1)$. (Remember that for $r = 1$, $\tau = \eta$.) $Q(\tau)$ is specified in terms of a finite number of Fourier coefficients. More precisely, if we write

$$Q(\tau) = a_0 + \sum_{j=1}^{M} a_j \cos\left(\frac{2\pi j\tau}{T}\right) + \sum_{j=1}^{M} b_j \sin\left(\frac{2\pi j\tau}{T}\right) , \qquad (E.1)$$

then a_0 = INDC, M = NOINFC, a_j = INC(J), and b_j = INS(J).

b. Finite-difference net [N, K, K0, K1, K2, X, R, SINE, LOGY]

The range of the two variables η and ζ is $[0,T]$; it is divided into N equal intervals of length N/T. The interval size in the r-direction is not arbitrary; if $r_0 = 1$, then if every net-point in (η, r) coordinates is to be a net-point in (ζ, r) coordinates, it is necessary and sufficient that the r-coordinate of a net-point be of the form $r_j = 1 + 1/2\ M\delta$ where M is an integer depending on j and $\delta = T/N$.†† The r_j are determined by K, K0, K1, K2 in the following manner: If K1 × K2 = K, there are K2 zones,

† Expressions appearing in square brackets after a heading are the FORTRAN names of SPHERE variables whose functions are described under that heading.

†† This follows readily from (C.6).

each consisting of K1 intervals. In the first zone (i.e., the zone containing $r = 1$), each interval is of length $1/2\ \delta \times K0$; in the m^{th} zone, each interval is of length $2^{m-1}(1/2\ \delta \times K0)$. If $K1 \times K2 < K$, there is an additional zone, containing $K - K1 \times K2$ intervals of the same length as those in the $K2^{th}$ zone. The reason for this type of zoning is to allow for the possibility that we may wish to assign relatively less importance to the calculation for large r than for small r. The zoning is roughly equivalent to replacing r by a constant times log r as the distance variable; see (C.12) and the discussion in paragraph E.2a below.

The array X contains the values of r as a function of the index;† i.e., $X(J) = r_j$ where $J = j + 1$; in particular, R, the position of the outer boundary of the flow field, is found in X(K). Perhaps somewhat unfortunately, SPHERE also uses an array R; however, no confusion should result. The array R contains integers required for shifting between η and ζ coordinates:

$$R(J) = (2N/T)[X(J) - 1] \pmod N; \tag{E.2}$$

its use will be explained in paragraph E.2.

The array SINE contains $(\gamma - 1)Q(\tau)$ for appropriate values of τ; specifically,

$$SINE(I) = 0.4Q(iT/N), \qquad I = i + 1. \tag{E.3}$$

The array LOGY contains factors that enter the finite-difference equations to be described in paragraph E.2; viz.,

$$LOGY(1) = 0$$

$$LOGY(J) = 0.25(\log r_j - \log r_{j-1}), \qquad j \geq 1,\ J = j + 1. \tag{E.4}$$

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† Because FORTRAN arrays are limited to positive indices, a FORTRAN index usually equals the corresponding conventional index plus one.

c. Amplitude [A, AI, LIMA, CASNUM, CASECO]

SPHERE is designed, in effect, to solve a whole sequence of problems rather than a single problem. This offers certain computational advantages, and obviates the need for making a preliminary estimate of the shock-formation distance.

The idea is the following: Once we have picked T and N, the value of R, the position of the outer boundary, is determined by K0, K1, K2, and K. Naturally, we would like to make R as large as possible, but we have to balance this desideratum against such considerations as the need for a finite-difference net of reasonably fine mesh, memory capacity of the machine, and availability of machine time. In any case, suppose that we have specified the finite-difference net, and hence R. We now prescribe a function $Q(\tau)$ of low amplitude at $r = 1$. By this we mean, with reference to (E.1), that

$$a_0 \ll 1, \qquad A \equiv \left[\sum_{j=1}^{M} (a_j^2 + b_j^2)^{1/2} \right] \ll 1. \tag{E.5}$$

The program then calculates u and v for this case, using the acoustic solution--equations (D.19) and (D.11)--as an initial approximation. After printing the results, the program goes on to the next case, which differs from the preceding case only in the amplitude of $Q(\tau)$:

$$\text{new } Q(\tau) = [\text{old } Q(\tau)] \times \text{AI}; \tag{E.6}$$

the program now multiplies the values of u and v from the previous calculation by AI to provide an initial approximation for the current case. This process continues until A > LIMA, or the case number, CASNUM, exceeds CASECO.†

† The functions of the two limits, LIMA and CASECO, are redundant.

The program will terminate earlier if the number of iterations in one of the several iterative loops in the program exceeds some preset limit; this condition will generally be a signal that an amplitude has been reached such that a shock is about to appear in the flow field.

2. Some Preliminaries

a. The Logarithmic Trapezoidal Rule

In subsection B, paragraph 1, we obtained an implicit formula for $w_{i,j+1}$ by using the trapezoidal rule to evaluate the integral in (B.6). The resulting formula, (B.7), would appear to be suitable for each of equations (C.8), assuming that v and U, respectively, are given. We pointed out in subsection C, paragraph 3, that in a certain approximation, the first of equations (C.8) can be reduced to the simple-wave equation by the substitution $x = a \log r$. From the results of subsection B, paragraphs 3 and 4, we know that the finite-difference approximation derived from the use of the trapezoidal rule leads to quite accurate results in that case. However, before applying this observation to the more general case, it is important to note that by substituting x for r as the variable of integration, we remove a factor r^{-1} from g and h in (C.8). This factor changes rapidly for small r; hence the approximation derived from the direct application of the trapezoidal rule is not very good. Because of the restriction on the values of r_j mentioned in paragraph E.1b, it is not feasible to replace r explicitly by log r as the independent variable for the more general case; however, we may do so *effectively* between adjacent net-points. Suppose we set

$$\xi = \log r; \tag{E.7}$$

then

$$\int_{r_1}^{r_2} r^{-1}\ f(r)\ dr = \int_{\log r_1}^{\log r_2} f(e^{\xi})\ d\xi. \tag{E.8}$$

We now apply the trapezoidal rule to the second member of (E.8) to obtain the logarithmic trapezoidal rule,

$$\int_{r_1}^{r_2} r^{-1}\ f(r)\ dr \simeq \frac{1}{2}\ (\log r_2 - \log r_1)[f(r_2) + f(r_1)]. \tag{E.9}$$

This rule is exact if $f(e^{\xi}) = a + b\xi$, i.e., if $f(r) = a + b \log r$.

b. Notation

We introduce a convention for subscripts that reflects the definitions (C.7). Let

$$\begin{aligned}
&\delta \equiv T/N, \quad \Delta r_j \equiv r_j - r_{j-1}, \quad \Delta\xi_j \equiv \frac{1}{2}\ (\log r_j - \log r_{j-1}) \\
&u_{i,j} \equiv u(i\delta, r_j), \qquad v_{i,j} \equiv v(i\delta, r_j) \\
&U_{i,j} \equiv U(i\delta, r_j), \qquad V_{i,j} \equiv V(i\delta, r_j) \\
&\nabla u_{i,j} \equiv \frac{u_{i+1,j} - u_{i-1,j}}{2\delta}, \qquad \nabla V_{i,j} \equiv \frac{V_{i+1,j} - V_{i-1,j}}{2\delta};
\end{aligned} \tag{E.10}$$

we further set

$$\begin{aligned}
G[u;\ v] &\equiv \frac{(au - bv)\ (\nabla u + ur^{-1}) + v - \frac{1}{2}\ (u^2 - v^2)r^{-1}}{1 + (au - bv)r^{-1}}, \\
H[V;\ U] &\equiv \frac{(aV - bU)\ (-\nabla V + Vr^{-1}) + U - \frac{1}{2}\ (V^2 - U^2)r^{-1}}{1 + (aV - bU)r^{-1}}.
\end{aligned} \tag{E.11}$$

It is convenient to introduce yet another convention: a single subscript on u, v, U, and V will indicate a vector of N components. Specifically,

$$u_j \equiv \{u_{0,j},\ u_{1,j},\ \ldots,\ u_{i,j},\ \ldots,\ u_{N-1,j}\} \tag{E.12}$$

with the corresponding definitions for v_j, U_j, and V_j. Consider now an expression such as $G[u_j; v_j]$. This is the vector whose i^{th} component is obtained by setting $u = u_{i,j}$, $v = v_{i,j}$, and $r = r_j$ (not a vector!) in the expression for $G[u; v]$.† Because of the occurrence of $\nabla u_{i,j}$, we require the value of $u_{-1,j}$ to compute $G[u_{0,j}; v_{0,j}]$ and the value of $u_{N,j}$ to compute $G[u_{N,j}; v_{N,j}]$. These values are of course supplied by the periodicity conditions,

$$u_{-1,j} = u_{N-1,j}; \qquad u_{N,j} = u_{0,j}; \tag{E.13}$$

corresponding considerations apply to H and V. There is a simple operational implementation of (E.13): Whenever a vector u appears as the first argument in $G[u; v]$, it shall be an extended vector of N + 2 components, the components corresponding to $i = -1$ and $i = N$ being given by (E.13); and similarly with $H[V; U]$ and V. In SPHERE, extended vectors are always stored in the array W2.

In connection with the various convergence tests used in SPHERE, we also define a vector norm by

$$\|w_j\| \equiv \max_{0 \le i \le N-1} |w_{i,j}|$$

where w_j is any vector.

c. Shifting Coordinates

The integration scheme of SPHERE requires us to evaluate $v_{i,j}$, given the vector V_j, and to evaluate $U_{i,j}$, given the vector u_j. From (C.6), we have, for fixed r,

$$\begin{aligned} \eta &= \zeta - 2(r - 1), \\ \zeta &= \eta + 2(r - 1); \end{aligned} \tag{E.14}$$

† All expressions involving vectors are to be interpreted in this way.

hence

$$v(\eta,r) = V(\zeta,r) = V(\eta + 2r - 2,r),$$
$$U(\zeta,r) = u(\eta,r) = u(\zeta - 2r + 2,r). \tag{E.15}$$

As we already mentioned (paragraph E.1b), $r_j - 1$ is some integer, say s, times $\delta/2$. Hence

$$v_{i,j} = V(i\delta + s\delta, r_j),$$
$$U_{i,j} = u(i\delta - s\delta, r_j). \tag{E.16}$$

Because of the periodicity conditions, the functional values remain unchanged if s is replaced by s_j,

$$s_j \equiv s \pmod N, \qquad 0 \leq s_j < N. \tag{E.17}$$

Since $s = (2/\delta)(r_j - 1)$, we may write

$$s_j \equiv (2N/T)(r_j - 1) \pmod N, \qquad 0 \leq s_j < N, \tag{E.18}$$

which is the quantity referred to in (E.2). The full prescription for shifting then becomes

$$\begin{aligned} v_{i,j} &= V_{\ell,j} \quad \text{where } \ell = i + s_j \text{ if } i + s_j < N \\ &\qquad\qquad\qquad\;\; = i + s_j - N \text{ otherwise;} \\ U_{i,j} &= u_{m,j} \quad \text{where } m = i - s_j \text{ if } i - s_j \geq 0 \\ &\qquad\qquad\qquad\;\; = i - s_j + N \text{ otherwise.} \end{aligned} \tag{E.19}$$

d. <u>The Finite-Difference Approximations for Equations (C.8)</u>

We now replace $\partial u/\partial\eta$ and $\partial V/\partial\zeta$ in (C.8) by the corresponding central difference quotients and employ the logarithmic trapezoidal rule between r_{j-1} and r_j to obtain the finite-difference approximations

$$u_j = u_{j-1} + \Delta\xi_j\left\{G[u_{j-1};\ v_{j-1}] + G[u_j;\ v_j]\right\} \tag{E.20}$$

and

$$V_{j-1} = V_j - \Delta\xi_j\left\{H[V_j;\ U_j] + H[V_{j-1};\ U_{j-1}]\right\} \tag{E.21}$$

Equations (E.20) and (E.21) have been deliberately arranged so as to suggest that the evaluation of u is carried on in the direction of increasing j, that of V in the direction of decreasing j.

e. A Finite-Difference Procedure for Evaluating (D.15)

To make use of (D.16) and (D.19), we require a procedure for evaluating expressions of the type (D.15). Let

$$\lambda \equiv e^{\beta\delta}, \qquad y_i \equiv y(i\delta), \qquad \Phi_i \equiv \Phi(i\delta),$$
$$\Delta I_i \equiv \lambda^{-i} \int_{i\delta}^{i\delta+\delta} \Phi(\tau) e^{\beta\tau} d\tau. \tag{E.22}$$

(Note that y_i, Φ_i, and ΔI_i are not vectors but components of vectors.) Then (D.15) becomes, for $\eta = i\delta$,

$$y_i = \lambda^{-1}\left[(\lambda^N - 1)^{-1} \sum_{k=0}^{N-1} \lambda^k \Delta I_k + \sum_{k=0}^{i-1} \lambda^k \Delta I_k\right], \tag{E.23}$$

from which we obtain

$$y_i = \lambda^{-1}(y_{i-1} + \Delta I_{i-1}),$$
$$y_0 = (\lambda^N - 1)^{-1} \sum_{k=0}^{N-1} \lambda^k \Delta I_k, \qquad 1 \le i \le N-1. \tag{E.24}$$

To approximate ΔI_k, we use the 4-point integration formula [6, p. 167]

$$\int_{\eta_i}^{\eta_{i+1}} f(\tau) d\tau \simeq \frac{\delta}{24} \left[-f(\eta_{i-1}) + 13f(\eta_i) + 13f(\eta_{i+1}) - f(\eta_{i+2})\right] \tag{E.25}$$

where $\eta_{k+1} - \eta_k = \delta$. Application of this formula leads to

$$\Delta I_i = C_1\Phi_{i-1} + C_2\Phi_i + C_3\Phi_{i+1} + C_4\Phi_{i+2},$$
$$C_1 \equiv \frac{-\delta}{24\lambda}, \quad C_2 \equiv \frac{13\delta}{24}, \quad C_3 \equiv \frac{13\delta\lambda}{24}, \quad C_4 \equiv \frac{-\delta\lambda^2}{24}. \tag{E.26}$$

If Φ is regarded as an extended vector (i.e., as a vector with additional components $\Phi_{-1} = \Phi_{N-1}$, $\Phi_N = \Phi_0$), (E.26) and (E.24) allow the computation of the vector y. It will be convenient to write symbolically

$$y = L[\beta; \Phi] \tag{E.27}$$

to indicate the operation of computing a vector y from a vector Φ by the procedure just outlined.

3. The Iterative Loops

a. The Inner Iterative Loop [INTE0, INTE1, INTE2, W0, W1, W2, WU, WV, R0, R1, R2, Q0, Q1, Q2, AL, AL1, MAXIT, UONE, UTWO, VONE, VTWO, MMAX; Subroutines JLOOP, FLOOP]

For reasons to be discussed presently, we introduce two intermediate variables, u^* and V^*; u^*_j is computed from (E.20) and V^*_j from (E.21). Let us consider the calculation of u^*_j. At the beginning of this calculation, the following vectors are available:

$G[u^*_{j-3};\ v_{j-3}]$ in INTE0

$G[u^*_{j-2};\ v_{j-2}]$ in INTE1

$G[u^*_{j-1};\ v_{j-1}]$ in INTE2

u^*_{j-1} in W0

V_j in row $J = j+1$ of the two dimensional array WV.

We compute the first approximation, $u_j^{(1)}$, to u^*_j by extrapolating the values of $G[u^*_k;\ v_k]$ for $k = j-3,\ j-2,\ j-1$ to $k = j$; hence

$$u_j^{(1)} = u^*_{j-1} + \Delta\xi_j\Big\{(R_2+1)G[u^*_{j-1};\ v_{j-1}] + R_1 G[u^*_{j-2};\ v_{j-2}] + R_0 G[u^*_{j-3};\ v_{j-3}]\Big\} \tag{E.29}$$

Here R_0, R_1, and R_2 are the extrapolation coefficients defined in (B.13), with $x_k = r_{k-1}$ for $k = j+1,\ j,\ j-1,\ j-2$. (To account for the shifted index recall that (B.13) is appropriate for stepping from j to $j+1$ rather than from $j-1$ to j.) Incidentally, in SPHERE, R2 has the meaning $R_2 + 1$; R0, R1, Q0, Q1, and Q2 have the same meaning as in (B.13).

As the components of $u_j^{(1)}$ are computed, they are stored in W2. We next put INTE1 into INTE0 and INTE2 into INTE1 so that we may use INTE2 for storing $G[u_j^{(1)};\ v_j]$; in calculating the components of this vector, we find $v_{j,i}$ from V_j according to (E.19). Next we compute

$$u_j^{(2)} = u_{j-1} + \Delta\xi_j\left\{G[u^*_{j-1};\ v_{j-1}] + G[u_j^{(1)};\ v_2]\right\} \tag{E.30}$$

which is stored in W1. This process may be repeated: $u_j^{(m-1)}$ is put into W2, $G[u_j^{(m-1)};\ v_j]$ is computed and stored in INTE2, and $u_j^{(m)}$ is computed from

$$u_j^{(m)} = u_{j-1} + \Delta\xi_j\left\{G[u^*_{j-1};\ v_{j-1}] + G[u_j^{(m-1)};\ v_j]\right\} \tag{E.31}$$

and stored in W1. In the course of computing each $u_j^{(m)}$, we also evaluate the quantity $\epsilon_j^{(m)} = \|u_j^{(m)} - u_j^{(m-1)}\|$. Under normal circumstances, the iteration is discontinued when

$$\epsilon_j^{(m)} < \frac{\mathrm{Lim}_2}{K}\cdot j;^\dagger \tag{E.32}$$

this is condition (B.23) with $\mathrm{Lim}_1 = 0$. (For the determination of Lim_2, see subsection F, paragraph 1b, Card 6.) The current (vector) value $u_j^{(m)}$ is then accepted as the correct approximation to u^*_j and $G[u_j^{(m-1)};\ v_j]$ as the correct approximation to $G[u^*_j;\ v_j]$. Finally, u^*_j is moved from W1 to W0, and we arrive at exactly the situation (E.28), with j replaced by j + 1. The entire procedure of getting from u^*_{j-1} to u^*_j is called an <u>inner iterative loop</u>.

The extrapolation of G at the beginning of an inner iterative loop has to be modified if j = 1 or 2. The nature of the modification follows from the discussion following equation (B.11) and does not require further comment.††

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† Iteration is also discontinued if m − 1 > MAXIT; under this condition, the entire program is terminated.

†† The vector u^*_0, required to compute u^*_1, is supplied by the inner boundary condition (see paragraph E.3b).

The subroutine JLOOP is designed to calculate the vectors u_j^* from $j = 1$ to $j = K$ by a succession of inner iterative loops. In general, we shall go through JLOOP several times in the course of a given computation. Let us designate the vectors V_j stored in WV at the beginning of a particular JLOOP by $V_j^{[n]}$ where n is the iteration number referring to the outer iterative loop, which we shall discuss in paragraph E.3b. Similarly, the array WU contains the vectors $u_j^{[n]}$. In the course of JLOOP, the contents of the array WV remain unchanged; however, as each u_j^* is computed, the vector $u_j^{[n]}$ in the appropriate row of WU is replaced by

$$u_j^{[n+1]} = \alpha u_j^* + (1-\alpha)u_j^{[n]} \tag{E.33}$$

where α is the relaxation factor. (In SPHERE, α = AL and $1 - \alpha$ = AL1.) If we were to use $\alpha = 1$, the introduction of the intermediate vector u_j^* would be unnecessary. The reason for (E.33) is that although the outer iterative loop fails to converge for $\alpha = 1$, it does appear to converge for $\alpha = 0.5$; this will be discussed further in paragraph E.3c below.

The computation of V can be made completely analogous to that of u. If we replace j by j + 1 in (E.21), introduce a new index f, and write

$$\widetilde{V}_f = V_j, \qquad j = K - f, \tag{E.34}$$

we obtain

$$\widetilde{V}_f = \widetilde{V}_{f-1} - \Delta\xi_{K-(f-1)}\left\{H[\widetilde{V}_{f-1};\ U_{k-(f-1)}] + H[\widetilde{V}_f;\ U_{K-f}]\right\} \tag{E.35}$$

Again, we introduce an intermediate variable $\widetilde{V}^*$. Consider the computation of $\widetilde{V}_f^*$. At the beginning of the calculation, the following vectors are available:

$H\left[\widetilde{V}^*_{f-3};\ u_{K-(f-3)}\right]$ in INTE0

$H\left[\widetilde{V}^*_{f-2};\ u_{K-(f-2)}\right]$ in INTE1

$H\left[\widetilde{V}^*_{f-1};\ u_{K-(f-1)}\right]$ in INTE2

$\widetilde{V}^*_{f-1}$ in WO

u_{K-f} in row $J = j+1 = K-f+1$ of the two-dimensional array WU.

An inner iterative loop then allows us to compute $\widetilde{V}^*_f$. The extrapolation coefficients R_0, R_1, and R_2 are again obtained from (B.13), with $x_k = r_{K-(k-1)}$ for $k = f+1$, f, $f-1$, $f-2$; and j is replaced by f in the convergence test (E.32).

The subroutine FLOOP calculates the vectors $\widetilde{V}^*_f$ from $f = 1$ to $f = K$. FLOOP follows JLOOP, and hence, at the beginning of FLOOP, the array WU contains the vectors $u_j^{[n+1]}$ whereas WV still contains $V_j^{[n]}$. As the computation of each $\widetilde{V}^*_f$ is completed, the vector $V_{K-f}^{[n]}$ in WV is replaced by

$$V_{K-f}^{[n+1]} = \alpha\widetilde{V}^*_f + (1-\alpha)V_{K-f}^{[n]} , \tag{E.37}$$

exactly as in JLOOP except for the order in which replacement occurs.

For simplicity of exposition, it has been convenient to treat the two-dimensional arrays WU and WV as though they were directly accessible during the course of computation. In earlier versions of SPHERE, WU and WV were, in fact, stored in core; however, with a 32K core, this put an unrealistically low limit on the number of net-points that could be used in the computation. In the present version of SPHERE, this limitation has been overcome by the use of four binary tapes (or disk segments), addressed as logical units UONE, VONE, UTWO, VTWO. In JLOOP, $u^{[n]}$ and $V^{[n]}$ are read from UONE and VONE into the actual core arrays WU and WV;

as $u^{[n+1]}$ is calculated, both $u^{[n+1]}$ and $v^{[n]}$ are written on UTWO and VTWO. Each of the arrays WU and WV contains 2 MMAX vectors; the computation is interrupted only for the transfer of blocks of MMAX vectors to or from tape buffers, and the comparatively slow transfer from tape buffer to tape or vice versa takes place while computation involving the other blocks is in progress. in FLOOP, the roles of the logical units are interchanged: $u^{[n+1]}$ and $v^{[n]}$ are read from UTWO and VTWO, and $u^{[n+1]}$ and $v^{[n+1]}$ are read out onto UONE and VONE.

b. <u>The Outer Iterative Loop</u> [SOMANY, MKU, MKV]

The subroutines JLOOP and FLOOP are connected by the boundary conditions to form the <u>outer iterative loop</u>. Suppose that we have just finished FLOOP; then the arrays WU and WV contain the vectors $u_j^{[n]}$ and $v_j^{[n]}$. We now apply the boundary conditions (D.17) or (D.18) to obtain the vector u_0^* with which to start a new JLOOP. By (E.18), $s_j = 0$; hence $v_{i,0} = V_{i,0}$. Let us denote by $\widetilde{Q}$ the vector stored in the array SINE (see paragraph E.1a and equation (E.3)). Then we have either

$$u_0^* = v_0^{[n]} + \widetilde{Q} \tag{E.38}$$

or

$$u_0^* = v_0^{[n]} + \widetilde{Q}\left[1 + \frac{u_0^* + v_0^{[n]}}{2}\right]^{-2/(\gamma-1)} \tag{E.39}$$

Equation (E.38) yields u_0^* explicitly; if the prescribed boundary condition is (E.39), we calculate u_0^* by the iterative procedure described in connection with (D.18). Having computed u_0^*, we calculate $G\left[u_0^*;\ v_0^{[n]}\right]$ and $u_0^{[n+1]} = \alpha u_0^* + (1-\alpha)u_0^{[n]}$ and are then ready to go through JLOOP.

After completing JLOOP, we take the vector $U_K^{[n+1]}$ (obtained from $u_K^{[n+1]}$ by (E.19)) and calculate V_K^* according to the scheme of paragraph

E.2e; i.e.,

$$V_K^* = L\left[(2R)^{-1};\quad U_K^{[n+1]}\right] \tag{E.40}$$

in the notation of (E.27). This corresponds to the application of the boundary condition (D.16). We then calculate $H\left[V_K^*;\ U_K^{[n+1]}\right]$ and $V_K^{[n+1]} = \alpha V_K^* + (1-\alpha)V_K^{[n]}$ and are ready to go through FLOOP.

We require a convergence test to determine when to discontinue the outer iterative loop during normal operation of the program.† Ideally, we should base our test on the two quantities

$$\max_j \left\| u_j^{[n]} - u_j^{[n-1]} \right\|, \qquad \max_j \left\| v_j^{[n]} - v_j^{[n-1]} \right\|; \tag{E.41}$$

however, because of the nature of JLOOP and FLOOP, these quantities are not likely to differ significantly (if at all) from

$$MKU_1 = \left\| u_K^{[n]} - u_K^{[n-1]} \right\|, \qquad MKV_1 = \left\| v_0^{[n]} - v_0^{[n-1]} \right\|. \tag{E.42}$$

In SPHERE, these two expressions are computed, respectively, at the end of JLOOP and FLOOP. The outer iterative loop is discontinued if, after at least two iterations, both $(MKU_1)/\alpha$ and $(MKV_1)/\alpha$ are less than Lim_2. The reason for the factor α^{-1} is the following: From (E.33) and (E.37) we have

$$MKU \equiv (MKU_1)/\alpha = \left\| u_K^* - u_K^{[n-1]} \right\|, \qquad MKV \equiv (MKV_1)/\alpha = \left\| v_0^* - v_0^{[n-1]} \right\|, \tag{E.43}$$

and these are more appropriate measures of how close we have come to the exact solution of the system of finite-difference equations than are the expressions (E.42).

† The outer iterative loop is also discontinued if the number of iterations exceeds the preset number SOMANY; under this condition, the entire program is terminated.

c. The Relaxation Factor

The nature of the outer iterative loop and the role of the relaxation factor α can best be understood by first considering the corresponding linear case. Neglecting second-order terms in (E.11), we have simply

$$G[u;\ v] = v; \qquad H[U;\ V] = U. \tag{E.44}$$

Because G is independent of u, and H is independent of V, equations (E.20) and (E.21) yield u_j and V_{j-1} explicitly:

$$\begin{aligned} u_j &= u_{j-1} + \Delta\xi_j[v_{j-1} + v_j], \\ V_{j-1} &= V_j - \Delta\xi_{j-1}[U_j + U_{j-1}]. \end{aligned} \tag{E.45}$$

It is now convenient to introduce some matrix notation. First, we note that the vector v_j is obtained from V_j by multiplying V_j by a permutation matrix P_j (a permutation matrix has one element equal to one in every row and column, and zeros elsewhere), and U_j is obtained from u_j by multiplication by the inverse of P_j:

$$v_j = P_j V_j, \qquad U_j = P_j^{-1} u_j. \tag{E.46}$$

(For present purposes, we shall regard vectors as column vectors and use the convention that whenever a matrix appears in front of a vector, matrix multiplication is implied.) We may then write for (E.44)

$$\begin{aligned} u_j &= u_{j-1} + \Delta\xi_j P_{j-1} V_{j-1} + \Delta\xi_j P_j V_j, \\ V_{j-1} &= V_j - \Delta\xi_{j-1} P_j^{-1} u_j - \Delta\xi_{j-1} P_{j-1}^{-1} u_{j-1}, \end{aligned} \tag{E.47}$$

and repeated application of (E.47) leads to

$$\begin{aligned} u_j &= u_0 + \Delta\xi_1 P_0 V_0 + 2\sum_{k=1}^{j-1} \Delta\xi_k P_k V_k + \Delta\xi_j P_j V_j, \\ V_j &= V_K - \Delta\xi_{K-1} P_K^{-1} u_K - 2\sum_{k=j+1}^{K-1} \Delta\xi_{k-1} P_k^{-1} u_k - \Delta\xi_j P_j^{-1} u_j. \end{aligned} \tag{E.48}$$

We may now express V_K in terms of $U_K = P_K^{-1} u_K$ analogously to (E.40); since the operation on the vector y symbolized by $L[\beta; y]$ is linear, it may be represented as a matrix which, for $\beta = (2R)^{-1}$, we denote by L_B. Hence

$$V_K = L_B P_K^{-1} u_K, \tag{E.49}$$

and similarly, from the linear boundary condition (D.17) at $j = 0$,

$$u_0 = V_0 + \widetilde{Q} \tag{E.50}$$

where $\widetilde{Q}$ is the vector defined in connection with (E.38). Inserting these expressions into (E.48) and remembering that $P_0 = I$, the $N \times N$ identity matrix, we find, after slight rearrangement,

$$\begin{aligned} u_j &= (1 + \Delta\xi_1) I V_0 + \sum_{k=1}^{j-1} (2\Delta\xi_k P_k) V_k + \Delta\xi_j P_j V_j + \widetilde{Q}, \\ V_j &= -\Delta\xi_j P_j^{-1} u_j + \sum_{k=j+1}^{K-1} (-2\Delta\xi_{k-1} P_k^{-1}) u_k + (I + L_B P_K^{-1}) u_K. \end{aligned} \tag{E.51}$$

If we define matrices $A_{j,k}$ and $B_{j,k}$ for $0 \le j \le K$ by

$$\begin{aligned} A_{j,0} &\equiv (1 + \Delta\xi_1) I \\ A_{j,k} &\equiv 2\Delta\xi_k P_k, && 0 < k < j \\ A_{j,j} &\equiv \Delta\xi_j P_j \\ A_{j,k} &\equiv 0, && j < k < K \\ B_{j,k} &\equiv 0, && 0 \le k < j \\ B_{j,j} &\equiv -\Delta\xi_j P_j^{-1} \\ B_{j,k} &\equiv -2\Delta\xi_{k-1} P_k^{-1}, && j < k < K \\ B_{j,K} &\equiv I + L_B P_K^{-1}, \end{aligned} \tag{E.52}$$

we may write (E.51) in the form

$$\begin{aligned} u_j &= \sum_{k=0}^{K} A_{j,k} V_k + \widetilde{Q}, \\ V_j &= \sum_{k=0}^{K} B_{j,k} u_k. \end{aligned} \tag{E.53}$$

Equations (E.53) are a system of 2K + 2 equations for the 2K + 2 vectors u_j and V_j for $0 \le j \le K$. We may put this system into the form of the two matrix equations

$$\bar{u} = A\bar{V} + \bar{Q}$$
$$\bar{B} = B\bar{u} \tag{E.54}$$

where u, V, and Q are vectors of N(K+1) components, and A and B are N(K+1) × N(K+1) matrices. The vectors $\bar{u}$ and $\bar{Q}$ have the forms

$$\bar{u} = \begin{bmatrix} (u_0)_0 \\ (u_0)_1 \\ \vdots \\ (u_0)_{N-1} \\ (u_1)_0 \\ (u_1)_1 \\ \vdots \\ \vdots \\ (u_K)_{N-1} \end{bmatrix}, \qquad \bar{Q} = \begin{bmatrix} (\tilde{Q})_0 \\ (\tilde{Q})_1 \\ \vdots \\ (\tilde{Q})_{N-1} \\ (\tilde{Q})_0 \\ (\tilde{Q})_1 \\ \vdots \\ \vdots \\ (\tilde{Q})_{N-1} \end{bmatrix}, \tag{E.55}$$

and $\bar{V}$ is defined analogous to $\bar{u}$. (We have written $(u_j)_i$ for the i^{th} component of the N-dimensional vector u_j to avoid confusion with our earlier usage, $u_{i,j}$, which is appropriate to row vectors.) The matrices A and B have a natural partition into $(K+1)^2$ N × N submatrices, $A_{j,k}$ and $B_{j,k}$, as defined in (E.52). With respect to this partition, A is block-lower diagonal and B is block-upper diagonal.

From these definitions it follows that if $\bar{V}$ is the particular vector $\bar{V}^{[n-1]}$, the operation $A\bar{V}^{[n-1]} + \bar{Q}$ produces just the vector $\bar{u}^*$ whose N(K+1) components consist of the components of all the vectors u_j^* generated by the application of the boundary condition at j = 0 followed by the subroutine JLOOP. Thus, by (E.33),

$$\bar{u}^{[n]} = \alpha_1(A\bar{V}^{[n-1]} + \bar{Q}) + (1-\alpha_1)\bar{u}^{[n-1]}. \qquad \text{(E.56)}$$

Similarly, the application of the outer boundary condition, followed by FLOOP, is equivalent to

$$\bar{V}^{[n]} = \alpha_2 B\bar{u}^{[n]} + (1-\alpha_2)\bar{V}^{[n-1]} \qquad \text{(E.57)}$$

In SPHERE, the two relaxation factors α_1 and α_2 are taken to be equal; however, a trivial modification of the program would permit the use of the more general relaxation scheme that we are considering at the moment.

In order to study the convergence of the iterative method defined by (E.56) and (E.57), we add to these equations the two equations obtained by substituting $n-1$ for n; we may then eliminate $\bar{V}^{[n]}$, $\bar{V}^{[n-1]}$, and $\bar{V}^{[n-2]}$ to obtain

$$\bar{u}^{[n]} = [\alpha_1\alpha_2\mathcal{L} + (2-\alpha_1-\alpha_2)\bar{I}]\bar{u}^{[n-1]} - (1-\alpha_1)(1-\alpha_2)\bar{u}^{[n-2]} + \alpha_1\alpha_2\bar{Q} \qquad \text{(E.58)}$$

where $\bar{I}$ is the $(K+1)N \times (K+1)N$ identity matrix and

$$\mathcal{L} = AB. \qquad \text{(E.59)}$$

We first note, provided $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, that (E.59) may be regarded as a scheme for solving the equation

$$\bar{u} = [\alpha_1\alpha_2\mathcal{L} + (2-\alpha_1-\alpha_2)I]\,\bar{u} - (1-\alpha_1)(1-\alpha_2)\bar{u} + \alpha_1\alpha_2\bar{Q} \qquad \text{(E.60)}$$

and that this equation is equivalent to

$$\bar{u} = \mathcal{L}\bar{u} + \bar{Q}\ , \qquad \text{(E.61)}$$

the equation obtained from (E.54) by the elimination of $\bar{V}$. We have thus verified that if (E.58) converges, it converges to the correct solution. (This point is, in fact, rather obvious if we go back to the system of equations (E.56) and (E.57).)

If we let $\alpha_1 = \alpha_2 = 1$, we obtain the simplest scheme for solving (E.61), viz.,

$$\bar{u}^{[n]} = \mathcal{L}\bar{u}^{[n-1]} + \bar{Q} \qquad \text{(E.62)}$$

It is well known [12, Chap. 2] that the sequence $\{\bar{u}^{[n]}\}$ converges for every initial vector $\bar{u}^{[0]}$ if and only if

$$\rho(\mathcal{L}) < 1; \qquad \text{(E.63)}$$

here $\rho(\mathcal{L})$ is the <u>spectral</u> <u>radius</u> of $\mathcal{L}$, defined by

$$\rho(\mathcal{L}) = \max_i |\lambda_i| , \qquad \text{(E.64)}$$

where the λ_i are the eigenvalues of $\mathcal{L}$.

Let us now define the matrix

$$\mathcal{L}_\alpha \equiv \alpha\mathcal{L} + (1-\alpha)\bar{I} . \qquad \text{(E.65)}$$

If we set $\alpha_1 = \alpha$ and $\alpha_2 = 1$, (E.58) becomes

$$\bar{u}^{[n]} = \mathcal{L}_\alpha\bar{u}^{[n-1]} + \bar{Q} ; \qquad \text{(E.66)}$$

this scheme converges if and only if

$$\rho(\mathcal{L}_\alpha) < 1, \qquad \text{(E.67)}$$

i.e., if and only if the eigenvalues of $\mathcal{L}_\alpha$ lie in the open unit disk in the complex plane. It is easily shown that this is equivalent to the requirement that the eigenvalues of $\mathcal{L}$ lie inside the circle whose equation (in the complex z-plane) is

$$| \alpha z + 1 - \alpha | = 1, \qquad \text{(E.68)}$$

i.e., the circle of radius α^{-1} and center $1 - \alpha^{-1}$. Let us call the interior of this circle $\mathcal{R}_\alpha$; it is evident that $\alpha < \beta$ implies $\mathcal{R}_\alpha \supset \mathcal{R}_\beta$, and that if S is any bounded, closed set in the half-plane Re $z < 1$, there is a value of α such that $S \subset \mathcal{R}_{\alpha'}$ for $\alpha' < \alpha$. Hence, if the eigenvalues of $\mathcal{L}$ all lie in the half-plane Re $z < 1$, we can choose an α such that the scheme (E.58), or equivalently, the scheme consisting of (E.56) and (E.57) converges. The scheme (E.66) is thus far more powerful than the scheme (E.62); in fact, as we shall establish in great generality in Section VI, there is an intimate connection between pairs of iterative methods such as (E.62) and (E.58) on the one hand, and various concepts of classical summability theory on the other hand. In the terminology of Varga [12, Chap. 5] the method (E.58) is called semi-iterative with respect to the iterative method (E.62).

As mentioned earlier, the method actually employed in SPHERE corresponds to setting $\alpha_1 = \alpha_2 = \alpha$ in (E.58); i.e.,

$$\bar{u}^{[n]} = [\alpha^2 \mathcal{L} + 2(1-\alpha)\bar{I}]\bar{u}^{[n-1]} - (1-\alpha)^2 \bar{u}^{[n-2]} + \alpha^2 \bar{Q}. \qquad \text{(E.69)}$$

In subsection E of Section VI we shall show how to find the boundary of the region in which the eigenvalues of $\mathcal{L}$ must lie if a scheme of this type is to converge. Application of that discussion to (E.69) shows that the equation of the boundary is given parametrically by

$$e^{i\vartheta} = [\alpha^2 z + 2(1-\alpha)]e^{i\theta} - (1-\alpha)^2; \qquad \text{(E.70)}$$

i.e.,

$$z = \frac{2 - 2\alpha + \alpha^2}{\alpha^2} \cos\vartheta - \frac{2-2\alpha}{\alpha^2} + i\,\frac{2-\alpha}{\alpha} \sin\theta. \qquad \text{(E.71)}$$

For $0 < \alpha < 2$, this equation represents an ellipse whose major axis is the line segment $[-(2\alpha^{-1} - 1)^2, 1]$ and whose minor axis is the line segment $[2\alpha^{-2} - 2\alpha^{-1} - i(2\alpha^{-1} - 1),\ 2\alpha^{-2} - 2\alpha^{-1} + i(2\alpha^{-1} - 1)]$. If we designate the interior of this ellipse by $\mathfrak{R}^*_\alpha$, then we have again that $\alpha < \beta$ implies $\mathfrak{R}^*_\alpha \supset \mathfrak{R}^*_\beta$. However, unlike the circles $\mathfrak{R}_\alpha$ connected with the method (E.66), the ellipses $\mathfrak{R}^*_\alpha$ do not fill the half-plane Re $z < 1$. By writing $z = x + iy$ and taking the real and imaginary parts of (E.71), and then letting $\alpha \to 0$ for fixed $x < 1$, we can show that the $\mathfrak{R}^*_\alpha$ cover the infinite region bounded by the parabola $y^2 = 4(1-x)$ and containing the real axis from $-\infty$ to $+1$.

If some precise information about the location of the eigenvalues of $\mathcal{L}$ is available (e.g., if it is known that all the eigenvalues are real and lie in the closed interval [a,b], b < 1), there are various standard semi-iterative methods designed to maximize the rate of convergence [5, Chap. 9; 12, Chap. 5]. Over the range of cases investigated so far, the method incorporated in SPHERE, (E.69), seems to converge quite well with a value $\alpha = 0.5$ determined by numerical experimentation. There is little doubt that the rate of convergence could be significantly improved with further analysis and numerical experimentation.

The analysis given so far in this subparagraph refers to the convergence of the outer iterative loop under the assumption that the relations (E.44) are valid. In any actual computation with SPHERE, this condition can be approached arbitrarily closely by prescribing a sufficiently small amplitude for the function $Q(\tau)$ (see (E.1)). The value $\alpha = 0.5$ was chosen on the basis of computations with $Q(\tau) = A \sin(2\pi\tau/T)$, and values of A ranging between 0.001 and 0.1. It appears therefore that the failure of the "simple-minded" scheme ($\alpha = 1$) is connected with the linear part of the problem, i.e., with the location of the eigenvalues of the matrix $\mathcal{L}$.

4. Output Calculations

a. The Evaluation of Fourier Coefficients

Suppose $f(\eta)$ is continuous and of period T, with Fourier coefficients

$$a_0 = \frac{1}{T}\int_0^T f(\eta)\, d\eta \ ,$$

$$a_m = \frac{2}{T}\int_0^T f(\eta) \cos(\omega_m\eta)\, d\eta \ , \qquad \text{(E.72)}$$

$$b_m = \frac{2}{T}\int_0^T f(\eta) \sin(\omega_m\eta)\, d\eta \ ,$$

where
$$\omega_m = 2\pi m/T. \tag{E.73}$$

Suppose next that $f(\eta)$ is known only at the points

$$\eta_i = i\delta, \quad \delta = T/N, \quad i = 0, 1, \ldots, N-1;^{\dagger} \tag{E.74}$$

how should we evaluate the integrals in (E.72)? It is known [9, p. 375] that if the trapezoidal rule is applied directly, the resulting finite Fourier series

$$a_0 + \sum_{m=1}^{M} a_m \cos \omega_m \eta + \sum_{m=1}^{M} b_m \sin \omega_m \eta \tag{E.75}$$

gives, for $M < N/2$, the best mean-square approximation to the function $f(\eta_i)$ whose domain consists of the N points η_i. The appropriate expressions for the coefficients are, in this case,

$$\begin{aligned} a_0 &= \frac{\delta}{T} \sum_{i=0}^{N-1} y_i \\ a_m &= \frac{2\delta}{T} \sum_{i=0}^{N-1} y_i \, C_i^{(m)} \\ b_m &= \frac{2\delta}{T} \sum_{i=0}^{N-1} y_i \, S_i^{(m)} \end{aligned} \tag{E.76}$$

where we have set

$$y_i \equiv f(\eta_i), \quad C_i^{(m)} \equiv \cos \omega_m \eta_i, \quad S_i^{(m)} \equiv \sin \omega_m \eta_i. \tag{E.77}$$

On the other hand, for the purposes of SPHERE, we wish to evaluate the Fourier coefficients in such a way that the resulting finite Fourier series is the best mean-square approximation to the function $f(\eta)$ whose domain is the interval $[0,T]$. One way to do this is to approximate $f(\eta)$ between η_{i-1} and η_i by a polynomial for each i, and then calculate the integrals exactly.

---

† Because of the periodicity of f, we regard points congruent modulo T as identical.

The simplest fit is by a set of constants:

$$f(\eta) = \frac{1}{2}\,(y_{i-1} + y_i), \qquad \eta_{i-1} < \eta < \eta_i . \tag{E.78}$$

Then

$$\int_{\eta_{i-1}}^{\eta_i} f(\eta)\,\cos\omega_m\eta\,d\eta = \frac{1}{2\omega_m}\left(S_i^{(m)} - S_{i-1}^{(m)}\right)(y_{i-1} + y_i); \tag{E.79}$$

summing from $i = 1$ to $i = N$ and making use of the periodicity condition at the end-points, we find

$$\int_0^T f(\eta)\,\cos\omega_m\eta\,d\eta = \frac{1}{2\omega_m}\sum_{i=0}^{N-1}\left(S_{i+1}^{(m)} - S_{i-1}^{(m)}\right)y_i . \tag{E.80}$$

Now by elementary trigonometry,

$$S_{i+1} - S_{i-1} = 2C_i\,\sin\omega_m\delta, \tag{E.81}$$

so that we have

$$a_m = \frac{\sin\omega_m\delta}{\omega_m\delta}\,\frac{2\delta}{T}\sum_{i=0}^{N-1} y_i C_i^{(m)}, \tag{E.82}$$

and similarly

$$b_m = \frac{\sin\omega_m\delta}{\omega_m\delta}\,\frac{2\delta}{T}\sum_{i=0}^{N-1} y_i S_i^{(m)}; \tag{E.83}$$

the expression for a_0 is identical with that in (E.76). Thus, we see that the only difference between (E.82) and (E.83) on the one hand, and (E.76) on the other hand, is the factor

$$\frac{\sin\omega_m\delta}{\omega_m\delta} \simeq 1 - \frac{1}{6}\,(\omega_m\delta)^2. \tag{E.84}$$

Next, we fit $f(\eta)$ by a set of linear polynomials:

$$f(\eta) = y_{i-1} + \frac{y_i - y_{i-1}}{\delta}\,(\eta - \eta_{i-1}), \qquad \eta_{i-1} < \eta < \eta_i . \tag{E.85}$$

The same kind of analysis, though somewhat tedious, then leads to the expressions

$$a_0 = \frac{\delta}{T} \sum_{i=0}^{N-1} y_i ,$$

$$a_m = 2 \frac{1 - \cos(\omega_m \delta)}{(\omega_m \delta)^2} \frac{2\delta}{T} \sum_{i=0}^{N-1} y_i C_i^{(m)} , \tag{E.86}$$

$$b_m = 2 \frac{1 - \cos(\omega_m \delta)}{(\omega_m \delta)^2} \frac{2\delta}{T} \sum_{i=0}^{N-1} y_i S_i^{(m)} .$$

These are the formulas actually used in SPHERE. The correction factor

$$2 \frac{1 - \cos(\omega_m \delta)}{(\omega_m \delta)^2} \simeq 1 - \frac{1}{12} (\omega_m \delta)^2 \tag{E.87}$$

will generally not differ much from unity. We have

$$\frac{1}{12} (\omega_m \delta)^2 = \frac{\pi^2}{3} \left(\frac{m}{N} \right)^2 ; \tag{E.88}$$

with N = 100, this expression is less than 0.01 for the first five harmonics ($m \leq 5$).

b. The Flow Variables

The principal output of SPHERE is the set of leading Fourier coefficients of $r\tilde{p}$, $r\tilde{q}$, and $\tilde{\theta}$ at selected values of the distance r; we recall that $\tilde{p}$, $\tilde{q}$, and $\tilde{\theta}$ are, respectively, the reduced acoustic pressure, the reduced velocity, and the reduced mass flow. Output of the functional values themselves is optional. The flow variables are calculated in (η, r) coordinates by means of (C.13), with

$$u(\eta_i, r_j) = u_{i,j}^{[n]}, \qquad v(\eta_i, r_j) = v_{i,j}^{[n]} \tag{E.89}$$

where n is the final iteration number of the outer iterative loop, η_i is defined in (E.74), and r_j is one of the selected values of r. $v_{i,j}^{[n]}$ is obtained from $V_{i,j}^{[n]}$ according to (E.19). With an obvious notation, we write $\tilde{p}_{i,j}$, $\tilde{q}_{i,j}$, and $\tilde{\theta}_{i,j}$ for the values of the expressions (C.13) when u and v are replaced by $u_{i,j}^{[n]}$ and $v_{i,j}^{[n]}$. The Fourier coefficients are then calculated from (E.86), with $y_i = r_j\tilde{p}_{i,j}$, $r_j\tilde{q}_{i,j}$, or $\tilde{\theta}_{i,j}$, as required. The program actually prints out the amplitude and phase of the complex Fourier coefficient $a_m + ib_m$, i.e.,

$$\left(a_m^2 + b_m^2\right)^{1/2} \qquad \text{and} \qquad \tan^{-1}(b_m/a_m). \tag{E.90}$$

For each flow variable, the Fourier amplitudes are normalized so that the amplitude of the fundamental, $\left(a_1^2 + b_1^2\right)^{1/2}$, at r = 1 is unity; phases are given in units of $\pi/2$.

5. General Plan of SPHERE [FC, FS; subroutines INIT, MAIN2]

SPHERE consists of the main program and four subroutines: INIT, JLOOP, FLOOP, and MAIN2. After reading the data cards, the program calls INIT. The function of this subroutine is to calculate those variables and arrays that will remain unchanged during the entire run; typical arrays that are filled in INIT are those mentioned in paragraph E.1b, and the arrays FC and FS which contain the sines and cosines required by (E.1) and (E.86). Finally, INIT calculates the acoustic approximation to u and V and stores it on logical units UONE and VONE. The program then goes through the outer iterative loop until the convergence criterion is satisfied. The sequence is: MAIN (Inner Boundary Condition) ...JLOOP...MAIN (Outer Boundary Condition)...FLOOP...MAIN (Inner Boundary Condition)... . At the conclusion of each outer iteration, a line is printed with information relevant to the convergence of the calculation. After completion of the outer iterative loop, the program calls MAIN2, which performs the output calculations described in paragraph E.4, and prints the results. Finally, MAIN2 carries out the preliminaries for going to the next case, as described in paragraph E.1c; the initial approximation to u and V for the next case is stored on UONE and VONE. A new heading is printed, and the program once again goes into the outer iterative loop, with a new function $Q(\tau)$ in the inner boundary condition. This sequence continues until it is terminated according to either of the criteria described in paragraph E.1c, or for exceeding the limit on the permissible number of iterations in one of the iterative loops.

We supplement this description with a complete listing of SPHERE.

LISTING OF "SPHERE"

```
$IBFTC MAIN     NODECK
      COMMON /TEMPAL/  OPTVEL,OPTMAV,OPTPRE
      COMMON A,AC,ACC,AI,AL,AL1,AMP,AMPFAC,AMPS,AVIT,B,C,CASNUM,CC1,CC2,
     1CC3,CC4,CI,CO,C1,C2,C3,C4,C5,D,DCCOMP,DE,DER,DER1,DMAX,DUM,EE1,EE2
     1,EPS,E1,E2,F,FC,FM,FS,G0,G1,G2,G3,HMAX,H0,H1,H2,H3,I,IC,IFOUR,INC,
     1INDC,INS,INTE0,INTE1,INTE2,ITHR,ITU,ITV,J,K,KA,KK,K0,K1,K2,K3,K4,L
     1,LIM,LIMA,LIM2,LL,LOGY,LZ,M,MAV,MAXIT,MF,MKU,MKV,MMAX,MR,MS,MU,MUA
     1,MV,MVA,M2,N,NOINFC,NOIT,NU,N1,P,PC,PHA,PI,PMAX,PR,PRINTT,P0,Q,Q0,
     1Q1,Q2,R,RJ,R0,R1,R2,SI,SINE,SI1,SOMANY,S0,S1,S2,S3,S4,S5,S6,S7,S8,
     1T,UONE,UTWO,VONE,VTWO,W,WU,WV,W0,W1,W2,X,XI,Z,Z1,Z2
      REAL DCCOMP(21)
   10 REAL A,ACC,AI,AL,AL1,KK,  B,C1,C2,C3,C4,C5,DE,DER,DER1,EPS,E1,E2,
     CG0,G1,G2,G3,H0,H1,H2,H3,LIMA,LIM,LIM2,MKU,MKV,MU,MV,M2,Q0,Q1,
     CQ2,R0,R1,R2,S1,S2,S3,S4,S5,S6,S7,S8,T,W0(101),W1(101),
     CW2(102),SINE(101),LOGY(402),X(401),FC(11,101),FS(11,101),
     CINTE0(101),INTE1(101),INTE2(101),WU(101,36),WV(101,36),
     CPR( 6,101),AMP(21),PHA(21),PI,P0,D,S0,C0,CI,SI1,SI,EE1,EE2,
     CCC1,CC2,CC3,CC4,MUA,MVA,AMPS(21),W( 6,101),AMPFAC(3)
   80 INTEGER C,F,FM,I,IC,ITU,ITV,J,K,K0,K1,K2,L,LZ,MAXIT,N,NOIT,N1,P,
     CPC,Q,RJ,SOMANY,Z,Z1,Z2,K3,K4,XI(401),R(401),AVIT,NU,KA
     C,DUM, PMAX, DMAX, HMAX,PRINTT,UONE,VONE,UTWO,VTWO
C 100 MAX N IS 100,K IS 400, FM IS 20, FOR LARGER N,K,OR FM, THE
C 110 DIMENSIONS MUST BE CHANGED
C 120 ALL ARRAY SUBSCRIPTS HAVE BEEN INCREMENTED BY ONE SINCE THERE IS
C 130 NO ZEROTH ELEMENT IN FORTRAN
      INTEGER OPTMAV,OPTVEL,OPTPRE,OPTFOU,OPTBDY
      INTEGER CASNUM
      INTEGER CASECO
      REAL MAV(6,101)
      REAL INS(10),INC(10),INDC
  150 FORMAT(25H0PERIODIC SPHERICAL WAVES//3H A=F8.5,10X,5H ACC=F9.6,
     C10X,7H ALPHA= F7.4,10X,3H T=F12.8/ 3H K= I4,20X,4H K0= I4,
     C20X,4H K1= I4,20X,4H K2= I4/ 3H N= I4//22H OUTER BOUNDARY AT X
     C= F10.6 )
 9010 FORMAT(11H0INPUT DATA //8H OPTBDY= I2,20X,9H OPTVEL = I2,20X,
     C9H OPTPRE = I2,20X,9H OPTMAV = I2 // 6H AVIT= I5,10X,
     C4H FM=  I5,10X,7H MAXIT=, I5,10X,8H NOINFC= I5,10X,8H SOMANY=
     CI5,10X,3H P= I5,// 4H AI= F10.5, 4H AL= F10.5, 6H LIMA=
     CF10.5 //6H MMAX= I5)
 8014 FORMAT (1H0,10X,I6,9I12)
  195 FORMAT (5F15.8/)
 9001 FORMAT(4I5)
 9002 FORMAT(6I5)
 9003 FORMAT(6 I10)
 9004 FORMAT(6F10.5)
 9005 FORMAT (I5)
 9006 FORMAT (26H1PRINTOUT WILL OCCUR EVERY,I5,7H CYCLES)
 9007 FORMAT (/////6H0UONE=,I5/6H VONE=,I5/6H UTWO=,I5/6H VTWO=,I5/1H1//
     C///////)
  192 FORMAT(44H FOURIER COMPONENTS OF THE MASS FLOW (AMPLIT   ,
     C52HUDE RELATIVE TO FUNDAMENTAL AT X=1,PHASE IN UNITS OF,
     C12H 90 DEGREES))
 8005 FORMAT(63H0FOURIER COMPONENTS OF VELOCITY PRESCRIBED AT INNER BOUN
     CDARY... )
 8015 FORMAT (1H0,5H SINE  ,5X,10E12.5)
```

```
 8016 FORMAT (8H0 COSINE,3X,10E12.5)
 8017 FORMAT (4H0 DC    7X,E12.5///)
 8010 FORMAT(64H0FOURIER COMPONENTS OF MASS FLOW PRESCRIBED AT INNER ROU
     CNDARY...  )
  151 FORMAT (7H0 CASE=  I3//)
  270 FORMAT(52HITERATION ITU      MU             MKU       ITV       MV,
     C12H          MKV)
  299 FORMAT(1H1)
  320 FORMAT(2X,I3,5X,2(I4,1X,2E13.5))
10320 DO 350 I =1,10
      INS(I) = 0.
  350 INC(I) = 0.
      READ (5,9005) PRINTT
      WRITE (6,9006) PRINTT
      READ (5,9001) UONE,VONE,UTWO,VTWO
      WRITE (6,9007) UONE,VONE,UTWO,VTWO
      REWIND UONE
      REWIND VONE
      REWIND UTWO
      REWIND VTWO
      ITHR=UONE+VONE
      IFOUR =UTWO+VTWO
      READ (5,9001) OPTBDY,OPTVEL,OPTPRE,OPTMAV
      READ (5,9002) AVIT,  FM,  MAXIT,  MMAX,  NOINFC,  SOMANY
C     MMAX MUST BE LESS THAN  K/2 +1 AND EVEN
      READ (5,9003) K,K0,K1,K2,N,P
      READ (5,9004) ACC,  AI,  AL,  INDC,  LIMA,  T
      READ (5,195) (INC(I),I=1,NOINFC),(INS(I),I=1,NOINFC)
      READ (5,9001) CASECO
      WRITE (6,9010) OPTBDY, OPTVEL, OPTPRE, OPTMAV, AVIT, FM, MAXIT, NOINF
     CINFC, SOMANY, P, AI, AL, LIMA, MMAX
      DO 341 I =1,101
 8001 W1(I) = 0.0
  341 SINE(I) =0.
      ICASE = 1
 9340 CALL INIT
 1330 IF(L - SOMANY) 1331,1331,4200
 1331 HMAX = PMAX
      MS = PMAX
      MF = DMAX
 1326 BACKSPACE UONE
      IF(OPTBDY) 4200,1340,1352
 1340 DO 1350  I=1,N1
 1350 W1(I) = SINE(I) + WV(I,M)
      IF (CASNUM.NE.ICASE) GO TO 1351
      ICASE=ICASE+1
      WRITE (6,8005)
      WRITE (6,8014) ( I ,I=1,NOINFC)
      WRITE (6,8015) (INS(I), I = 1, NOINFC)
      WRITE (6,8016) (INC(I), I = 1, NOINFC)
      WRITE (6,8017) INDC
      WRITE (6,151) CASNUM
      WRITE (6,270)
 1351 GO TO 1360
 1352 DO 1359 I =1,N1
      IC = 0
 1353 S2 = W1(I)
      W1(I) = SINE(I)/(1. + .5 * (W1(I) + WV(I,M)))**5+WV(I,M)
      IF (ABS(W1(I) - S2).LE.A     * ACC *.02) GO TO 1359
      IC = IC + 1
```

```
      IF(IC.GT.15) GO TO 11359
      GO TO 1353
11359 WRITE(6, 9696) IC
 9696 FORMAT(1H0  I5)
      GO TO 4200
 1359 CONTINUE
      IF (CASNUM.NE.ICASE)  GO TO 1360
      ICASE=ICASE+1
      WRITE (6,8010)
      WRITE (6,8014) ( I ,I=1,NOINFC)
      WRITE (6,8015) (INS(I), I = 1, NOINFC)
      WRITE (6,8016) (INC(I), I = 1, NOINFC)
      WRITE (6,8017) INDC
      WRITE (6,151) CASNUM
  590 WRITE (6,270)
 1360 DO 1361 I=2,N
 1361 W2(I) = W1(I-1)
 1370 W2(1) =W2(N)
 1380 W2(N+1)= W2(2)
 1390 DO 1460 I=1,N1
 1400 S2 = W2(I+1)
 1410 S3 = WV(I,M)
 1420 S4 = (W2(I+2) -W2(I))*DER
 1430 S5 =  S2 - S3
 1440 S6 =  S5 + S5 + S2
 1450 INTE1(I) =((S4+S2+S2)*S6-(S2+S3)*S5+S3+S3)/(S6+1.0)
 1460 WU(I,M) = WU(I,M) * AL1 + W1(I) * AL
      IF(M.EQ.1) GO TO 1470
 1332 DUM = M - 1
 1333 DO 1336 MR =1,DUM
 1334 DO 1336 I =1,N1
 1335 WU(I,MR) = 0.0
 1336 WV(I,MR) = 0.0
 1337 IF (M.NE.MMAX) GO TO 1470
      BACKSPACE VONE
      MS = 1
      MF = MMAX
      WRITE (VTWO) ((WV(I,MR),I=1,N1),MR=MS,MF)
      WRITE(UTWO)((WU(I,MR),I=1,N1),MR=MS,MF)
      READ(VONE)((WV(I,MR),I = 1,N1),MR=MS,MF)
      READ(UONE)((WU(I,MR),I = 1,N1),MR=MS,MF)
      HMAX = DMAX +1
      BACKSPACE UONE
      BACKSPACE VONE
      BACKSPACE UONE
 1470 G3 = 1.0
 1480 LIM =0.0
 1484 LL= 0
 1486 CALL JLOOP
 2215 RJ = R(K)
 2216 LIM = 0.0
 2217 LL=ITHR-LL
 2218 BACKSPACE LL
 2220 G3 =X(K)
 2230 Q = N-RJ
 2240 H1 = WU(Q-1,M)
 2250 IF (Q-N) 2251, 2260, 2260
 2251 H2 = WU(Q,M)
 2252 GO TO 2269
 2260 H2 = WU(1,M)
```

```
2269 NU = 2 - RJ
2270 IF (Q+1-N) 2271, 2280, 2280
2271 H3 = WU(Q+1,M)
2272 GO TO 2290
2280 H3 = WU(NU,M)
2290 S1 = E1
2300 S3 =0.0
2310 DO 2390 I=1,N1
2320 S1 = S1*E2
2330 H0 =H1
2340 H1 =H2
2350 H2 =H3
2360 NU = MOD(Q+I,N1)
2370 H3 = WU(NU +1,M)
2380 S2=(C1*H0 +C2*H1+C3*H2 +C4*H3)*S1
2381 INTE1(I) = S2
2390 S3 = S3+S2
2400 S1= S1*E2
2410 W1(1)=-C5*S3/(1.0 -S1)
2420 S1=1.0
2430 DO 2450 I=1,N1
2440 S1=S1*E1
2450 W1(I+1)=W1(I)*E1 +INTE1(I)*S1*C5
2460 DO 2461 I=2,N
2461 W2(I) = W1(I-1)
2462 W2(1) = W2(N)
2463 W2(N+1) =W2(2)
2470 DO 2560 I=1,N1
2480 S2=W2(I+1)
2490 IF (I-1-RJ) 2491, 2493, 2493
2491 NU = Q+I-1
2492 GO TO 2500
2493 NU = I-RJ
2500 S3 = WU(NU,M)
2510 S4=(W2(I+2)-W2(I))*DER1
2520 S5= S2-S3
2530 S6= S5+S5+S2
2540 S7= G3+G3
2550 INTE1(I)=((-S4+S2+S2)*S6-(S2+S3)*S5+S7*S3)/(S6+G3)
2560 WV(I,M) = WV(I,M) * AL1 + W1(I) * AL
     BACKSPACE VTWO
2561 LL = 0
2562 IF(M.NE.DMAX) GO TO 2567
2563 HMAX = MMAX
2564 MF = DMAX
2565 MS = PMAX
2566 GO TO 2569
2567 HMAX = 0
     MF = MMAX
2568 MS = 1
2569 KA = K-1
     CALL FLOOP
3291 LL=IFOUR -LL
3292 BACKSPACE LL
3297 L=L+1
3298 MUA = MKU/AL
3299 MVA = MKV/AL
3300 WRITE (6,320) L,ITU,MU,MUA    ,ITV,MV,MVA
3310 IF (MKU-MKV) 3320,3320,3340
3320 S1=MKV
```

```
3330 GO TO 3350
3340 S1=MKU
3350 S1 =S1/AL
3360 S4 = ITU +ITV
3370 ITU =0
3380 MU =0.0
3390 MKU=0.0
3391 MUA = 0.0
3392 MVA = 0.0
3400 ITV=0
3410 MV=0.0
3420 MKV=0.0
3430 IF (L.EQ.1.OR.S1.GT.LIM2) GO TO 1330
     CALL MAIN2
4160 WRITE (6,299)
4168 J = K-1
     M = DUM +1
4169 NU = FM - 1
     WRITE (6,150) A,  ACC,  AL,  T,  J,  K0,  K1,  K2, N1,  X(K)
     CASNUM = CASNUM + 1
     IF (CASNUM.GT.CASECO)  GO TO 10320
4190 GO TO 1330
4200 CONTINUE
     RETURN
     END



     SUBROUTINE MAIN2
     COMMON /TEMPAL/  OPTVEL,OPTMAV,OPTPRE
     COMMON A,AC,ACC,AI,AL,AL1,AMP,AMPFAC,AMPS,AVIT,B,C,CASNUM,CC1,CC2,
    1CC3,CC4,CI,C0,C1,C2,C3,C4,C5,D,DCCOMP,DE,DER,DER1,DMAX,DUM,EE1,EE2
    1,EPS,E1,E2,F,FC,FM,FS,G0,G1,G2,G3,HMAX,H0,H1,H2,H3,I,IC,IFOUR,INC,
    1INDC,INS,INTE0,INTE1,INTE2,ITHR,ITU,ITV,J,K,KA,KK,K0,K1,K2,K3,K4,L
    1,LIM,LIMA,LIM2,LL,LOGY,LZ,M,MAV,MAXIT,MF,MKU,MKV,MMAX,MR,MS,MU,MUA
    1,MV,MVA,M2,N,NOINFC,NOIT,NU,N1,P,PC,PHA,PI,PMAX,PR,PRINTT,P0,Q,Q0,
    1Q1,Q2,R,RJ,R0,R1,R2,SI,SINE,SI1,SOMANY,S0,S1,S2,S3,S4,S5,S6,S7,S8,
    1T,UONE,UTWO,VONE,VTWO,W,WU,WV,W0,W1,W2,X,XI,Z,Z1,Z2
     REAL DCCOMP(21)
  10 REAL A,ACC,AI,AL,AL1,KK,  B,C1,C2,C3,C4,C5,DE,DER,DER1,EPS,E1,E2,
    CG0,G1,G2,G3,H0,H1,H2,H3,LIMA,LIM,LIM2,MKU,MKV,MU,MV,M2,Q0,Q1,
    CQ2,R0,R1,R2,S1,S2,S3,S4,S5,S6,S7,S8,T,W0(101),W1(101),
    CW2(102),SINE(101),LOGY(402),X(401),FC(11,101),FS(11,101),
    CINTE0(101),INTE1(101),INTE2(101),WU(101,36),WV(101,36),
    CPR( 6,101),AMP(21),PHA(21),PI,P0,D,S0,C0,CI,SI1,SI,EE1,EE2,
    CCC1,CC2,CC3,CC4,MUA,MVA,AMPS(21),W( 6,101),AMPFAC(3)
  80 INTEGER C,F,FM,I,IC,ITU,ITV,J,K,K0,K1,K2,L,LZ,MAXIT,N,NOIT,N1,P,
    CPC,Q,RJ,SOMANY,Z,Z1,Z2,K3,K4,XI(401),R(401),AVIT,NU,KA
    C,DUM, PMAX, DMAX, HMAX,PRINTT,UONE,VONE,UTWO,VTWO
     INTEGER OPTMAV,OPTVEL,OPTPRE,OPTFOU,OPTBDY
     INTEGER CASNUM
     INTEGER CASECO
     REAL MAV(6,101)
     REAL INS(10),INC(10),INDC
 190 FORMAT(///52HFOURIER COMPONENTS OF THE ACCOUSTIC PRESSURE (AMPLIT,
    C52HUDE RELATIVE TO FUNDAMENTAL AT X=1,PHASE IN UNITS OF,
    C12H 90 DEGREES))
 192 FORMAT(44H FOURIER COMPONENTS OF THE MASS FLOW (AMPLIT  ,
    C52HUDE RELATIVE TO FUNDAMENTAL AT X=1,PHASE IN UNITS OF,
    C12H 90 DEGREES))
```

```
 8002 FORMAT(13H0MASS FLOW...   )
  220 FORMAT(7H   ORDER)
  230 FORMAT(5H   (A),I2,9X,6(F8.5,8X))
  240 FORMAT(5H   (P),I2,9X,6(F8.5,8X))
  250 FORMAT(53H ACCOUSTIC PRESSURE(IN UNITS OF AMBIENT PRESSURE) * X)
  260 FORMAT(45H VELOCITY(IN UNITS OF AMBIENT SOUNDSPEED) * X)
  290 FORMAT(/7H      X=,9X,6(F8.4,8X))
  299 FORMAT(1H1)
  300 FORMAT(4H    I)
  310 FORMAT(I4,9X,6(F11.8,5X))
  193 FORMAT(////43H FOURIER COMPONENTS OF THE VELOCITY (AMPLIT,
     C52HUDE RELATIVE TO FUNDAMENTAL AT X=1,PHASE IN UNITS OF,
     C12H 90 DEGREES))
      WRITE(UTWO)((WU(I,MR),I=1,N1),MR=1,DMAX)
      WRITE(VTWO)((WV(I,MR),I=1,N1),MR=1,DMAX)
 3439 PC = PC +1
      KKK =1
      MS = 1
      MF = MMAX
      GO TO 3440
3440  DO 13596 Z1=1,PC
 3450 Z =(Z1-1)*6*P+1
 3460 IF (Z+5*P-K) 3470,3470,3490
 3470 LZ = Z+5*P
 3480 GO TO 3491
 3490 LZ = K
 3491 IF (MOD(CASNUM-1,PRINTT).NE.0)GO TO 3539
 3495 IF(OPTVEL)4200,3539,3500
 3500 WRITE (6,299)
 3510 WRITE (6,260)
 3520 WRITE (6,290)(X(Z2), Z2=Z,LZ,P)
 3530 WRITE (6,300)
 3539 IC = 0
 3540 DO 3588 Z2 = Z,LZ,P
      IF (Z2.EQ.1) GO TO 3546
      MMM = M+P
      MDIV = (MMM-1)/MMAX
      MDIVI =MDIV - KKK/2
      IF (MDIVI.EQ.0)GO TO 1
      IF (K-Z2.GE.MMAX) GO TO 5
      MDIVI = MDIVI - 1
      KKK = 3 - KKK
      IF (MDIVI.LE.0) GO TO 4
    5 DO 2 IK = 1,MDIVI
      BACKSPACE UONE
      BACKSPACE VONE
      READ(UONE)((WU(I,MR),I=1,N1),MR=MS,MF)
      READ(VONE)((WV(I,MR),I=1,N1),MR=MS,MF)
      WRITE(VTWO)((WV(I,MR),I=1,N1),MR=MS,MF)
      WRITE(UTWO)((WU(I,MR),I=1,N1),MR=MS,MF)
      MS = PMAX + 1 -MS
      MF = DMAX+MMAX-MF
      BACKSPACE UONE
      BACKSPACE VONE
    2 KKK = 3 - KKK
    4 IF(KKK.EQ.1) GO TO 3
      M =MMM - MMAX*(MDIV -1)
      GO TO 3546
    3 M = MMM - MMAX * MDIV
      GO TO 3546
```

```
    1 M = M+P
 3546 IC = IC + 1
 3550 DO 3582 I = 1,N1
 3560 IF (I-N+R(Z2)) 3569,3579,3579
 3569 NU = R(Z2) +I
 3570 GO TO 3580
 3579 NU = I - N + R(Z2) +1
 3580 W(IC,I) = (WU(I,M) - WV(NU,M))*2.5
 3581 PR(IC,I) = (((WU(I,M) + WV(NU,M)) * 0.5
     C/ X(Z2) + 1.0)**7 - 1.0) * X(Z2)
3582  MAV(IC,I)=(2.5*(WU(I,M)-WV(NU,M))*(1.0+0.5*(WU(I,M)+WV(NU,M))/
     1X(Z2))**5)*X(Z2)
 3584 IF (Z2 .GE. LZ) GO TO 3588
 3588 CONTINUE
      IF (MOD(CASNUM-1,PRINTT).NE.0)GO TO 13596
 3589 IF(OPTVEL)4200,3593,3590
 3590 DO 3592 I = 1, N1
 3591 NU=I-1
 3592 WRITE (6,310) NU, (W(Z2,I),Z2=1,IC)
 3593 IF(OPTPRE)4200,3815,3660
 3660 WRITE (6,299)
 3670 WRITE (6,250)
 3680 WRITE (6,290)(X(Z2),Z2=Z,LZ,P)
 3690 WRITE (6,300)
 3700 DO 3810 I=1,N1
 3809 NU = I-1
 3810 WRITE (6,310)NU,(PR(Z2,I), Z2=1,IC)
 3815 IF(OPTMAV)4200,3859,3820
 3820 WRITE (6,299)
 3830 WRITE (6,8002)
 3840 WRITE (6,290)(X(Z2),Z2=Z,LZ,P)
 3850 WRITE (6,220)
 3851 DO 3858 I =1,N1
      NU = I - 1
 3858 WRITE (6,310)  NU, (MAV(Z2,I),Z2 = 1,IC)
 3859 CONTINUE
      DO  3596  JUMPIN = 1,3
      WRITE (6,299)
      IF (JUMPIN - 2) 3861,3862,3863
 3861 WRITE (6,193)
      GO TO 3864
 3862 WRITE(6,190)
      GO TO  3864
 3863 WRITE (6,192)
 3864 CONTINUE
      WRITE (6,290) (X(Z2),Z2 = Z,LZ,P)
      WRITE (6,220)
 3860 DO 3980 J=2,FM
 3870 DO 3961 Z2=1,IC
 3880 S1=0.0
 3890 S2=0.0
      S3=0.0
 3900 IF(JUMPIN.GT.1) GO TO 3902
      DO 3901 I = 1,N1
      IF (J.EQ.2) S3=S3+W(Z2,I)
      S1 = S1 + FS(J,I) * W(Z2,I)
 3901 S2 = S2 + FC(J,I) * W(Z2,I)
      GO TO 3930
 3902 IF (JUMPIN .EQ.3) GO TO 3904
      DO  3903  I =1,N1
```

```
      IF (J.EQ.2) S3=S3+PR(Z2,I)
      S1 = S1 + FS(J,I) * PR(Z2,I)
 3903 S2 = S2 + FC(J,I) *  PR(Z2,I)
      GO TO 3930
 3904 DO 3905  I =1,N1
      IF (J.EQ.2) S3=S3+MAV(Z2,I)
      S1 = S1 + FS(J,I) * MAV(Z2,I)
 3905 S2 = S2 + FC(J,I) * MAV(Z2,I)
 3930 AMP(Z2)=  SQRT(S1*S1+S2*S2)*FC(J,N)
 3940 IF (Z1.EQ.1.AND.Z2.EQ.1.AND.J.EQ.2) AMPFAC(JUMPIN)=
     C1.0/AMP(1)
 3950 PHA(Z2)=0.9999
 3960 IF (S2.NE.0.0)  PHA(Z2)=0.63662 * ATAN(S1/S2)
      IF (J.EQ.2) DCCOMP(Z2)=S3*AMPFAC(JUMPIN)*0.5
 3961 AMPS(Z2) = AMP(Z2)*AMPFAC(JUMPIN)
 3969 NU = J-1
 3970 WRITE (6,230)NU,(AMPS(Z2)   ,Z2=1,IC)
 3978 WRITE (6,240)NU,(PHA(Z2),Z2 = 1,IC)
 3980 CONTINUE
 4039 NU = 0
4040  WRITE (6,230) NU,(DCCOMP(Z2),Z2=1,IC)
 3596 CONTINUE
13596 CONTINUE
      KKOUNT=K-LZ-MMAX
      IF (KKOUNT.LE.0) GO TO 3600
      KKOUNT = (KKOUNT-1)/MMAX + 1
      DO 3599 J = 1,KKOUNT
      BACKSPACE UONE
      BACKSPACE VONE
      READ(UONE)((WU(I,MR),I=1,N1),MR=1,MMAX)
      READ(VONE)((WV(I,MR),I=1,N1),MR=1,MMAX)
      BACKSPACE UONE
      BACKSPACE VONE
      WRITE(VTWO)((WV(I,MR),I=1,N1),MR=1,MMAX)
 3599 WRITE(UTWO)((WU(I,MR),I=1,N1),MR=1,MMAX)
 3600 CONTINUE
 4041 PC = PC - 1
 4050 L=0
 4060 LIM2 =LIM2*AI
      DO 4070 I = 1, NOINFC
      INS(I) = INS(I) * AI
4070  INC(I) = INC(I) * AI
      A=A*AI
 4080 B=B*AI
 4089 NU = S4
 4090 IF (A.GT.LIMA.OR.NU.GT.NOIT) GO TO 4200
 4100 DO 4110 I=1,N1
 4110 SINE(I) =SINE(I)*AI
      M = PMAX
      BACKSPACE VTWO
      BACKSPACE UTWO
      READ (UTWO) ((WU(I,MR),I=1,N1),MR=1,MMAX)
      READ (VTWO) ((WV(I,MR),I=1,N1),MR=1,MMAX)
      BACKSPACE UTWO
      BACKSPACE VTWO
 4120 DO 4156 F=1,K
      IF (M.NE.1) GO TO 4130
      BACKSPACE UTWO
      BACKSPACE VTWO
      WRITE(UONE)((WU(I,MR),I=1,N1),MR=1,MMAX)
```

```
      WRITE(VONE)((WV(I,MR),I=1,N1),MR=1,MMAX)
      IF (J.LE.PMAX +MMAX) GO TO 12
      READ(UTWO)((WU(I,MR),I=1,N1),MR=1,MMAX)
      READ(VTWO)((WV(I,MR),I=1,N1),MR=1,MMAX)
      M = PMAX
      BACKSPACE UTWO
      BACKSPACE VTWO
      GO TO 4130
   12 M = DMAX +1
      READ(UTWO)((WU(I,MR),I=1,N1),MR=1,DMAX)
      READ(VTWO)((WV(I,MR),I=1,N1),MR=1,DMAX)
      BACKSPACE UTWO
      BACKSPACE VTWO
 4130 M = M-1
      J = K+1-F
 4135 DO 4150 I=1,N1
 4140 WU(I,M) = WU(I,M) * AI
 4150 WV(I,M) = WV(I,M) * AI
 4156 CONTINUE
      RETURN
4200  CALL EXIT
      RETURN
      END



      SUBROUTINE INIT
      COMMON A,AC,ACC,AI,AL,AL1,AMP,AMPFAC,AMPS,AVIT,B,C,CASNUM,CC1,CC2,
     1CC3,CC4,CI,CO,C1,C2,C3,C4,C5,D,DCCOMP,DE,DER,DER1,DMAX,DUM,EE1,EE2
     1,EPS,E1,E2,F,FC,FM,FS,G0,G1,G2,G3,HMAX,H0,H1,H2,H3,I,IC,IFOUR,INC,
     1INDC,INS,INTE0,INTE1,INTE2,ITHR,ITU,ITV,J,K,KA,KK,K0,K1,K2,K3,K4,L
     1,LIM,LIMA,LIM2,LL,LOGY,LZ,M,MAV,MAXIT,MF,MKU,MKV,MMAX,MR,MS,MU,MUA
     1,MV,MVA,M2,N,NOINFC,NOIT,NU,N1,P,PC,PHA,PI,PMAX,PR,PRINTT,P0,Q,Q0,
     1Q1,Q2,R,RJ,R0,R1,R2,SI,SINE,SI1,SOMANY,S0,S1,S2,S3,S4,S5,S6,S7,S8,
     1T,UONE,UTWO,VONE,VTWO,W,WU,WV,W0,W1,W2,X,XI,Z,Z1,Z2
      REAL DCCOMP(21)
      INTEGER CASNUM
      REAL INS, INC, INDC, MAV,AMPFAC(3)
   10 REAL A,ACC,AI,AL,AL1,KK,  B,C1,C2,C3,C4,C5,DE,DER,DER1,EPS,E1,E2,
     CG0,G1,G2,G3,H0,H1,H2,H3,LIMA,LIM,LIM2,MKU,MKV,MU,MV,M2,Q0,Q1,
     CQ2,R0,R1,R2,S1,S2,S3,S4,S5,S6,S7,S8,T,W0(101),W1(101),
     CW2(102),SINE(101),LOGY(402),X(401),FC(11,101),FS(11,101),
     CINTE0(101),INTE1(101),INTE2(101),WU(101,36),WV(101,36),
     CPR( 6,101),AMP(21),PHA(21),PI,P0,D,S0,C0,CI,SI1,SI,EE1,EE2,
     CCC1,CC2,CC3,CC4,MUA,MVA,AMPS(21),W( 6,101)
   80 INTEGER C,F,FM,I,IC,ITU,ITV,J,K,K0,K1,K2,L,LZ,MAXIT,N,NOIT,N1,P,
     CPC,Q,RJ,SOMANY,Z,Z1,Z2,K3,K4,XI(401),R(401),AVIT,NU,KA
     C,DUM, PMAX, DMAX, HMAX,PRINTT,UONE,VONE,UTWO,VTWO
  150 FORMAT(25H0PERIODIC SPHERICAL WAVES//3H A=F8.5,10X,5H ACC=F9.6,
     C10X,7H ALPHA= F7.4,10X,3H T=F12.8/ 3H K= I4,20X,4H K0= I4,
     C20X,4H K1= I4,20X,4H K2= I4/ 3H N= I4//22H OUTER BOUNDARY AT X
     C= F10.6 )
  299 FORMAT(1H1)
      SINE(1) = 0.0
      DO 341 I=1,NOINFC
341   SINE(1)=SINE(1)+INC(I)
      SINE(1)=(SINE(1)+INDC)*0.4
      DO 342  J=1,100
      FS(2,J) = 0.0
  342 FC(2,J) = 0.0
```

```
 343 L = 0
 344 IC = 0
 345 ITU = 0
 346 ITV = 0
 350 C = K/P
 351 MU = 0.0
 352 MV = 0.0
 353 MKU = 0.0
 354 MKV = 0.0
 360 N1 =N
 370 AL1 = 1.0 - AL
     A = 0.
     DO 7040 I = 1, NOINFC
7040 A = A + INS(I) ** 2 + INC(I) ** 2
     A = SQRT(A)
 380 NOIT = AVIT*K*2
 390 LIM2 = A*ACC*0.1
 399 KK = K
 400 B =  LIM2/KK
 410 PC =C/6
 420 PI = 3.14159265359
 429 KK=N1
 430 P0 = PI*2.0/KK
 440 DER =KK/T


 450 D = .5/DER
 461 N = N+1
 462 K = K+1
 470 K3 = K1*K2+1
 479 XI(1) = 0
 480 DO 490 J=2,K3
 490 XI(J) = 2**((J-2)/K1)*K0 +XI(J-1)
 500 K4 =XI(K3) -XI(K3-1)
 510 K3 = K3+1
 520 DO 530 J=K3,K
 530 XI(J) =XI(J-1) +K4
 540 DO 560 J=2,K
 549 R(J) =XI(J)/N1
 550 R(J) = XI(J)-N1*R(J)
 559 KK = XI(J)
 560 X(J) = 1.0 + KK*D
 561 R(1) = 0
 570 X(1) =1.0
 571 WRITE(6,299)
 579 NU = K-1
     CASNUM = 1
     WRITE (6,150) A,  ACC,  AL,  T, NU,  K0,  K1,  K2, N1,  X(K)
 600 K3 =K +1
 610 DO 620 J=2,K3
 619 KK = X(J-1)
 620 LOGY(J) =-ALOG(X(J-1))*0.25
 630 DO 640 J=2,K
 640 LOGY(J) = LOGY(J) - LOGY(J+1)
 649 FM = FM+1
 650 DO 820 J=2,FM
 660 IF (J-2)700,670,700
 670 S0 = SIN(P0)
 680 C0 = COS(P0)
 690 GO TO 720
```

```
 700 S0 = FS(2,J)
 710 C0 = FC(2,J)
 720 SI = 0.0
 730 FC(J ,1) =1.0
 740 CI  = 1.0
 748 NU = J-1
 749 KK = NU
 750 FC(J,N)=2.0*(1.0-C0)/(P0*P0*KK*KK)
 760 DO 820 I=2,N1
 770 SI1 =SI
 780 SI = SI*C0 + CI*S0
 790 FS(J,I) = SI
 800 CI = CI*C0 - SI1*S0
 810 FC(J,I) = CI
     IF (J.GT.NOINFC+1) GO TO 820
     SINE(I) = INS(J-1) * FS(J,I) + SINE(I)+INC(J-1)*FC(J,I)
     IF (J.EQ.NOINFC+1)  SINE(I)=(SINE(I)+INDC)*0.4
820  CONTINUE
 830 S1 =-D/X(K)
 840 E1 = EXP(S1)
 850 E2 = 1.0/E1
 860 C1 = -E1*D/12.0
 870 C2 = -E2*C1*13.0
 880 C3 = E2*C2
 890 C4 = -E2*C3/13.0
 900 C5 = -0.5/X(K)
 910 S1 = -D*2.0
 920 EE1 = EXP(S1)
 930 EE2 = 1.0/EE1
 940 CC1 = -EE1*D/12.0
 950 CC2 = -EE2*CC1*13.0
 960 CC3 = EE2*CC2
 970 CC4 = -EE2*CC3/13.0
 980 H1 =SINE(N-1)
 990 H2 =SINE(1)
1000 H3 =SINE(2)
1010 S1 =EE1
1020 S3 =0.0
1030 DO 1140 I=1,N1
1040 S1 =S1*EE2
1050 H0 =H1
1060 H1 = H2
1070 H2 = H3
1080 IF (I+2-N) 1090,1109,1109
1090 H3 = SINE(I+2)
1100 GO TO 1120
1109 NU = I - N +3
1110 H3 = SINE(NU)
1120 S2 = ((CC1*H0)+(CC2*H1)+(CC3*H2)+(CC4*H3))*S1
1130 INTE1(I) = S2
1140 S3 = S3+S2
1150 S1 = S1*EE2
1160 W1(1) = S3/(S1-1.0)
1170 S1 =1.0
1180 DO 1210 I=1,N1
1190 S1=S1*EE1
1200 W1(I+1)=(W1(I)*EE1)+(INTE1(I)*S1)
1210 W0(I) = SINE(I) -W1(I)
1211 DMAX = 2 * MMAX
1212 PMAX = MMAX + 1
```

```
1213 M = PMAX
1220 DO 1325 F = 1,K
1221 IF (M.NE.1) GO TO 1228
1222 IF (J.GT.PMAX + MMAX) GO TO 1225
1223 M = PMAX + MMAX
1224 GO TO 1226
1225 M = PMAX
1226 WRITE(UONE)((WU(I,MR),I =1,N1),MR =1,MMAX)
1227 WRITE(VONE)((WV(I,MR),I =1,N1),MR =1,MMAX)
1228 M = M - 1
1229 J = K +1 - F
1230 S1 =0.5/X(J)
1240 RJ = R(J)
1250 Q = N-RJ
1260 DO 1321 I=1,N1
1270 S2 = W1(I)*S1
1280 WU(I,M) = W0(I)    + S2
1290 IF (I-Q) 1299,1319,1319
1299 NU = RJ +I
1300 WV(NU,M) =-S2
1310 GO TO 1321
1319 NU = I - Q + 1
1320 WV(NU,M) =-S2
1321 CONTINUE
1325 CONTINUE
1327  RETURN
      END


1489 SUBROUTINE JLOOP
      COMMON A,AC,ACC,AI,AL,AL1,AMP,AMPFAC,AMPS,AVIT,B,C,CASNUM,CC1,CC2,
     1CC3,CC4,CI,C0,C1,C2,C3,C4,C5,D,DCCOMP,DE,DER,DER1,DMAX,DUM,EE1,EE2
     1,EPS,E1,E2,F,FC,FM,FS,G0,G1,G2,G3,HMAX,H0,H1,H2,H3,I,IC,IFOUR,INC,
     1INDC,INS,INTE0,INTE1,INTE2,ITHR,ITU,ITV,J,K,KA,KK,K0,K1,K2,K3,K4,L
     1,LIM,LIMA,LIM2,LL,LOGY,LZ,M,MAV,MAXIT,MF,MKU,MKV,MMAX,MR,MS,MU,MUA
     1,MV,MVA,M2,N,NOINFC,NOIT,NU,N1,P,PC,PHA,PI,PMAX,PR,PRINTT,P0,Q,Q0,
     1Q1,Q2,R,RJ,R0,R1,R2,SI,SINE,SI1,SOMANY,S0,S1,S2,S3,S4,S5,S6,S7,S8,
     1T,UONE,UTWO,VONE,VTWO,W,WU,WV,W0,W1,W2,X,XI,Z,Z1,Z2
      REAL DCCOMP(21)
   10 REAL A,ACC,AI,AL,AL1,KK,   B,C1,C2,C3,C4,C5,DE,DER,DER1,EPS,E1,E2,
     CG0,G1,G2,G3,H0,H1,H2,H3,LIMA,LIM,LIM2,MKU,MKV,MU,MV,M2,Q0,Q1,
     CQ2,R0,R1,R2,S1,S2,S3,S4,S5,S6,S7,S8,T,W0(101),W1(101),
     CW2(102),SINE(101),LOGY(402),X(401),FC(11,101),FS(11,101),
     CINTE0(101),INTE1(101),INTE2(101),WU(101,36),WV(101,36),
     CPR( 6,101),AMP(21),PHA(21),PI,P0,D,S0,C0,CI,SI1,SI,EE1,EE2,
     CCC1,CC2,CC3,CC4,MUA,MVA,AMPS(21),W( 6,101),AMPFAC(3)
   80 INTEGER C,F,FM,I,IC,ITU,ITV,J,K,K0,K1,K2,L,LZ,MAXIT,N,NOIT,N1,P,
     CPC,Q,RJ,SOMANY,Z,Z1,Z2,K3,K4,XI(401),R(401),AVIT,NU,KA
     C,DUM, PMAX, DMAX, HMAX,PRINTT,UONE,VONE,UTWO,VTWO
1490 DO 2211 J = 2,K
1491 M = M + 1
1492 IF (M.NE.HMAX) GO TO 1502
1493 IF(K-J.LE.MMAX) GO TO 1502
1494 IF (LL.EQ.VONE) GO TO 1500
1495 READ(VONE)((WV(I,MR),I=1,N1),MR=MS,MF)
1496 MF = 3*MMAX - MF
1497 MS = PMAX - MS +1
1499 GO TO 1501
1500 READ(UONE)((WU(I,MR),I =1,N1),MR=MS,MF)
```

```
1501 HMAX = HMAX + MMAX/2
1502 IC = 0
1510 LIM = LIM +B
1520 RJ = R(J)
1530 S1 = LOGY(J)
1540 Q = N-RJ
1550 DO 1551 I=1,N1
1551 W0(I) = W1(I)
1560 G0 =G1
1570 G1 =G2
1580 G2 =G3
1590 G3 =X(J)
1600 DER1 = G3*DER
1610 IF (J-2) 1640,1620,1640
1620 S7 = S1*2.0
1630 GO TO 1710
1640 IF (J.EQ.3) GO TO 1710
1650 Q0 =(G3-G0)/(G2-G1)
1660 Q1 =(G3-G1)/(G2-G0)
1670 Q2 =(G3-G2)/(G1-G0)
1680 R0 = Q1*Q2
1690 R1 = -Q2*Q0
1700 R2 = Q0*Q1 +1.0
1710 DO 1751 I=1,N1
1720 IF (J.EQ.2) W2(I+1)= INTE1(I)*S7 +W1(I)
1730 IF (J.EQ.3) W2(I+1)=(INTE2(I)*3.0 -INTE1(I))*S1+W1(I)
1740 IF (J.NE.2 .AND.J.NE.3)  W2(I+1)=(INTE2(I)*R2+INTE1(I)*R1
    C+INTE0(I)*R0)*S1 + W1(I)
1751 CONTINUE
1752 IF(M.NE.HMAX - MMAX/4)     GO TO 1760
1753 IF(LL.EQ.0) GO TO 1760
1754 BACKSPACE LL
1760 W2(1) = W2(N)
1770 W2(N+1) =W2(2)
1780 IF (J.EQ.2) GO TO 1810
1790 DO 1791 I=1,N1
1791 INTE0(I) = INTE1(I)
1800 DO 1801 I=1,N1
1801 INTE1(I) = INTE2(I)
1810 M2 = 0.0
1820 IC =IC +1
1830 IF (IC-MAXIT) 1840,1840,9211
1840 IF (IC.EQ.1) GO TO 1880
1850 DO 1851 I=2,N
1851 W2(I) = W1(I-1)
1860 W2(1)=W2(N)
1870 W2(N+1)=W2(2)
1880 DO 2050 I=1,N1
1890 S2=W2(I+1)
1900 IF (I-Q) 1909,1929,1929
1909 NU = RJ + I
1910 S3 = WV(NU,M)
1920 GO TO 1940
1929 NU = I - Q  + 1
1930 S3 = WV(NU,M)
1940 S4 = (W2(I+2)-W2(I))*DER1
1950 S5 = S2-S3
1960 S6 = S5+S5+S2
1970 S7 = G3+G3
1980 INTE2(I) =((S4+S2+S2)*S6-(S2+S3)*S5+S7*S3)/(S6+G3)
```

```
1990 S8 =(INTE1(I)+INTE2(I))*S1 +W0(I)
2000 W1(I) =S8
2010 EPS = ABS(S8-S2)
2050 IF (EPS.GT.M2) M2=EPS
2060 IF (M2.GT.LIM) GO TO 1810
2070 ITU =ITU+IC
2080 IF (M2.GT.MU) MU=M2
2090 DO 2190 I=1,N1
2100 S2 = WU(I,M)
2110 S3=W1(I)
2120 S4 = S2*AL1 + S3*AL
2130 IF (J.NE.K)  GO TO 2190
2140 S5 = ABS(S4-S2)
2180 IF (S5.GT.MKU) MKU =S5
2190 WU(I,M) = S4
     IF(M.NE.HMAX -1) GO TO 2211
2191 IF (K-J.LE.PMAX) GO TO 2206
2192 IF (LL.NE.0) GO TO 2198
2193 LL=VONE
2194 MS = PMAX +1 - MS
2196 MF = 3*MMAX - MF
2197 GO TO 2202
2198 LL=ITHR-LL
2199 BACKSPACE LL
2202 IF(LL.EQ.VONE) GO TO 2205
2203 WRITE(VTWO)((WV(I,MR),I=1,N1),MR =MS,MF)
2204 GO TO 2206
2205 WRITE(UTWO)((WU(I,MR),I=1,N1),MR =MS,MF)
2206 IF(M.EQ.DMAX) GO TO 2209
2208 GO TO 2211
2209 M = 0
2210 HMAX = 1
2211 CONTINUE
9211  RETURN
      END


      SUBROUTINE FLOOP
      COMMON A,AC,ACC,AI,AL,AL1,AMP,AMPFAC,AMPS,AVIT,B,C,CASNUM,CC1,CC2,
     1CC3,CC4,CI,C0,C1,C2,C3,C4,C5,D,DCCOMP,DE,DER,DER1,DMAX,DUM,EE1,EE2
     1,EPS,E1,E2,F,FC,FM,FS,G0,G1,G2,G3,HMAX,H0,H1,H2,H3,I,IC,IFOUR,INC,
     1INDC,INS,INTE0,INTE1,INTE2,ITHR,ITU,ITV,J,K,KA,KK,K0,K1,K2,K3,K4,L
     1,LIM,LIMA,LIM2,LL,LOGY,LZ,M,MAV,MAXIT,MF,MKU,MKV,MMAX,MR,MS,MU,MUA
     1,MV,MVA,M2,N,NOINFC,NOIT,NU,N1,P,PC,PHA,PI,PMAX,PR,PRINTT,P0,Q,Q0,
     1Q1,Q2,R,RJ,R0,R1,R2,SI,SINE,SI1,SOMANY,S0,S1,S2,S3,S4,S5,S6,S7,S8,
     1T,UONE,UTWO,VONE,VTWO,W,WU,WV,W0,W1,W2,X,XI,Z,Z1,Z2
      REAL DCCOMP(21)
   10 REAL A,ACC,AI,AL,AL1,KK,  B,C1,C2,C3,C4,C5,DE,DER,DER1,EPS,E1,E2,
     CG0,G1,G2,G3,H0,H1,H2,H3,LIMA,LIM,LIM2,MKU,MKV,MU,MV,M2,Q0,Q1,
     CQ2,R0,R1,R2,S1,S2,S3,S4,S5,S6,S7,S8,T,W0(101),W1(101),
     CW2(102),SINE(101),LOGY(402),X(401),FC(11,101),FS(11,101),
     CINTE0(101),INTE1(101),INTE2(101),WU(101,36),WV(101,36),
     CPR( 6,101),AMP(21),PHA(21),PI,P0,D,S0,C0,CI,SI1,SI,EE1,EE2,
     CCC1,CC2,CC3,CC4,MUA,MVA,AMPS(21),W( 6,101) ,AMPFAC(3)
   80 INTEGER C,F,FM,I,IC,ITU,ITV,J,K,K0,K1,K2,L,LZ,MAXIT,N,NOIT,N1,P,
     CPC,Q,RJ,SOMANY,Z,Z1,Z2,K3,K4,XI(401),R(401),AVIT,NU,KA
     C,DUM, PMAX, DMAX, HMAX,PRINTT,UONE,VONE,UTWO,VTWO
 2570 DO 3290 F = 1,KA
 2575 M = M - 1
```

```
2580 J =K-F
2581 IF(M.NE.HMAX) GO TO 2590
2582 IF (J+DUM.LE.MMAX)    GO TO 2590
2583 IF (LL.EQ.VTWO) GO TO 2586
2584 READ(VTWO)((WV(I,MR),I=1,N1),MR=MS,MF)
2585 GO TO 2589
2586 READ(UTWO)((WU(I,MR),I=1,N1),MR =MS,MF)
2587 MF = 3*MMAX - MF
2588 MS = PMAX - MS + 1
2589 HMAX = HMAX - MMAX/2
2590 IC=0
2600 LIM= LIM+B
2610 RJ=R(J)
2620 S1=-LOGY(J+1)
2630 Q= N-RJ
2640 DO 2641 I=1,N1
2641 W0(I) = W1(I)
2650 G0=G1
2660 G1=G2
2670 G2=G3
2680 G3=X(J)
2690 DER1 =G3*DER
2700 IF (F-1) 2730,2710,2730
2710 S7=S1*2.0
2720 GO TO 2800
2730 IF (F.EQ.2) GO TO 2800
2740 Q0=(G3-G0)/(G2-G1)
2750 Q1=(G3-G1)/(G2-G0)
2760 Q2=(G3-G2)/(G1-G0)
2770 R0=Q1*Q2
2780 R1=-Q2*Q0
2790 R2= Q0*Q1 +1.0
2800 CONTINUE
2810 DO 2851 I=1,N1
2820 IF (F.EQ.1) W2(I+1)=INTE1(I)*S7+W1(I)
2830 IF (F.EQ.2) W2(I+1)=(INTE2(I)*3.0-INTE1(I))*S1+W1(I)
2840 IF (F.GT.2) W2(I+1)=(INTE2(I)*R2+INTE1(I)*R1+INTE0(I)*R0)
    C*S1 +W1(I)
2851 CONTINUE
2860 W2(1)=W2(N)
2870 W2(N+1)=W2(2)
2880 IF (F.EQ.1) GO TO 2910
2890 DO 2891 NU=1,N1
2891 INTE0(NU) = INTE1(NU)
2900 DO 2901 NU=1,N1
2901 INTE1(NU) = INTE2(NU)
2910 CONTINUE
2912 IF (M.NE.HMAX + MMAX/4) GO TO 2920
2913 IF(LL.EQ.0) GO TO 2920
2914 BACKSPACE LL
2920 M2=0.0
2930 IC=IC+1
2940 IF (IC.GT.MAXIT) GO TO 7000
2950 IF (IC.EQ.1) GO TO 2990
2960 DO 2961 I=2,N
2961 W2(I) = W1(I-1)
2970 W2(1)=W2(N)
2980 W2(N+1)=W2(2)
2990 DO 3140 I=1,N1
3000 S2=W2(I+1)
```

```
 3010 IF (I-1-RJ)  3011, 3013, 3013
 3011 NU = Q+I-1
 3012 GO TO 3020
 3013 NU = I-RJ
 3020 S3 = WU(NU,M)
 3030 S4=(W2(I+2)-W2(I))*DER1
 3040 S5= S2-S3
 3050 S6= S5+S5+S2
 3060 S7 = G3+G3
 3070 INTE2(I)=((S2+S2-S4)*S6-(S2+S3)*S5+S7*S3)/(S6+G3)
 3080 S8=(INTE1(I)+INTE2(I))*S1 +W0(I)
 3090 W1(I) =S8
 3100 EPS = ABS(S8-S2)
 3140 IF (EPS.GT.M2) M2=EPS
 3150 IF (M2.GT.LIM) GO TO 2920
 3160 ITV = ITV + IC
 3170 IF (M2.GT.MV) MV=M2
 3180 DO 3275 I = 1,N1
 3190 S2 = WV(I,M)
 3200 S3 =W1(I)
 3210 S4 =S2*AL1 +S3*AL
 3220 IF (F.NE.KA)  GO TO 3275
 3230 S5 = ABS(S4-S2)
 3270 IF (S5.GT.MKV) MKV=S5
 3275 WV(I,M) = S4
 3276 IF (M.NE.HMAX+1) GO TO 3290
 3277 IF (J + DUM .LE. PMAX) GO TO 3287
      IF (LL.NE.0) GO TO 3279
      LL=UTWO
      GO TO 3283
 3279 LL=IFOUR -LL
 3280 BACKSPACE LL
 3283 IF (LL.EQ.UTWO) GO TO 3286
 3284 WRITE(UONE)((WU(I,MR),I=1,N1),MR=MS,MF)
 3285 GO TO 3287
 3286 WRITE(VONE)((WV(I,MR),I =1,N1),MR = MS,MF)
 3287 IF (M.NE.1) GO TO 3290
 3288 M = DMAX + 1
 3289 HMAX = DMAX
 3290 CONTINUE
7000  RETURN
      END
```

F. Operation of SPHERE

1. Input

a. Arrangement of Data Deck

We list here the names of the variables to be punched into each data card. A variable name followed by a single number, say M, in parentheses indicates that the variable is an integer whose value should be entered right-justified in column M. Thus, suppose N = 100; then N(50) indicates that "1" should be punched in column 48 and "0" in columns 49 and 50. A variable name followed by two numbers (hyphenated) in parentheses, "(L-M)," indicates that the value of the variable in decimal notation should be punched anywhere in the field bounded (inclusively) by columns L and M.

Card 1: PRINTT(5)

Card 2: UONE(5) VONE(10) UTWO(15) VTWO(20)

Card 3: OBTBDY(5) OPTVEL(10) OPTPRE(15) OPTMAV(20)

Card 4: AVIT(5) FM(10) MAXIT(15) MMAX(20) NOINFC(25) SOMANY(30)

Card 5: K(10) K0(20) K1(30) K2(40) N(50) P(60)

Card 6: ACC(1-10) AI(11-20) AL(21-30) INDC(31-40) LIMA(41-50) T(51-60)

Card 7 (additional cards if necessary): The number of entries in this card or group of cards depends on the value of NOINFC in card 4. Numbers in decimal notation should be entered sequentially in fields (1-15)(16-30)(31-45)(46-60)(61-75) of each card for a total of 2 X NOINFC entries; each card of this group except the last must have 5 entries.

Card 8: CASECO(5)

b. Definitions and Discussion of Data Variables

Card 1: As mentioned in subsection E, paragraph 1c, SPHERE goes through a whole sequence of cases that differ only in the amplitude of the function $Q(\tau)$, using the final values of u and V of each case, after multiplication by AI, to provide an initial approximation for the next case. Under some circumstances, one may wish to examine the values of the flow variables or their Fourier components only for a subsequence of cases. SPHERE prints out a heading and convergence data for each case; print-out of the flow variables, however, occurs only for cases 1, 1 + PRINTT, 1 + 2 X PRINTT, etc.

Card 2: This card contains the values assigned to the names of the four logical units required for storage. In systems where specific tape drives are permanently declared as logical units A, B, C, and D, these numbers are entered in card 2. If the processor has two independent I/O channels, SPHERE will operate most effectively if UONE and UTWO are on one channel, VONE and VTWO on the other. In systems without such permanent assignment, any integers A, B, C, and D not reserved by the system may be entered into card 2; the data deck must then be preceded by a MAP subroutine, or equivalent, in which certain storage units (tape or disk) are declared as logical units A, B, C, and D.

Card 3: This card specifies various options. For OPTBDY, see subsection E, paragraph 1a. The other variables specify print-out of functional values of the flow variables. For the subsequence of cases determined by the value of PRINTT (see card 1), a specified number of the Fourier components of $r\tilde{q}$, $r\tilde{p}$, and $\tilde{\theta}$ at selected space positions r_j

are printed. The values of $r\tilde{q}$, $r\tilde{p}$, and $\tilde{\theta}$ as functions of η are printed or not according to the following scheme:

OPTVEL = 0: Do not print $r\tilde{q}$

OPTVEL = 1: Print $r\tilde{q}$

OPTPRE = 0: Do not print $r\tilde{p}$

OPTPRE = 1: Print $r\tilde{p}$

OPTMAV = 0: Do not print $\tilde{\theta}$

OPTMAV = 1: Print $\tilde{\theta}$.

Card 4: The program will terminate if

(i) the number of iterations in an inner iterative loop exceeds MAXIT;

(ii) the number of iterations in the outer iterative loop exceeds SOMANY;

(iii) the average number of iterations per inner iterative loop in the case just completed exceeds AVIT. (This average is based on the final outer iteration--JLOOP plus FLOOP.)

The limits MAXIT and SOMANY are set to keep the program from going into a permanent loop. If the average number of iterations for the inner iterative loops becomes large, this will generally be an indication that the next case will fail to converge; this is the reason for the limit AVIT. (In this connection, it should be mentioned that if the number of iterations required to evaluate u_0^* in (E.39) exceeds 15 for any one component of this vector, the program is also terminated.) Reasonable values for these variables are: AVIT = 4, MAXIT = 6, SOMANY = 30.

The variable FM determines how many Fourier components of $r\tilde{q}$, $r\tilde{p}$, and $\tilde{\theta}$ are calculated and printed. NOINFC determines the number of Fourier components that will be used to describe the function $Q(\tau)$; see (E.1). The following limitations must be observed: $1 \leq$ NOINFC $\leq$ FM ≤ 10.

MMAX determines the number of vectors transferred in one operation to or from tape or disk. The following limitations apply: MMAX must be even, $2 \leq$ MMAX ≤ 18, MMAX $\leq 1/2\, K + 1$. As a rule of thumb, MMAX should be chosen as large as possible, subject to the above limitations; thus, for $K \geq 34$, set MMAX = 18.

Card 5: The role of all variables punched in this card, except P, has been explained in subsection E, paragraph 1b. The following limitations apply: $K1 \times K2 \leq K \leq 400$, $K1 \geq 2$, $N \leq 100$.

The integer P determines the positions r_j at which the flow variables $r\tilde{q}$, $r\tilde{p}$, and $\tilde{\theta}$ are computed: r_0 (inner boundary), r_P, r_{2P}, etc. If the outer boundary is to be included in this set, P must be a divisor of K.

Card 6: ACC is an approximate measure of the maximum error with which the program solves the complete system of finite-difference equations, in units of A, the amplitude of the oscillatory component of $Q(\tau)$--see (E.5). The convergence limits for the inner and outer iterative loops depend on the quantity Lim_2 (see the discussions preceding (E.32) and (E.42)), and the program sets $\mathrm{Lim}_2 = A \times \mathrm{ACC} \times 0.1$. The factor 0.1 is introduced partly because, by (D.17),† the amplitude of

---

† The situation in the case of (D.18) is essentially the same.

$u(\eta,1) - v(\eta,1)$ is only 0.4A, and partly to yield an estimate of accuracy somewhat on the conservative side. The limit for the iterations involved in applying the boundary condition (D.18) is set in a similar manner. A reasonable value for ACC is 0.001.

The variables AI and LIMA were discussed in subsection E, paragraph 1c. Reasonable values are AI = 2 and LIMA = 1. AL is α, the relaxation factor; AL = 0.5 appears to work quite well, but if the outer iterative loop fails to converge, a slightly smaller value of AL should be tried. INDC is the d.c. component of $Q(\tau)$, i.e., the coefficient a_0 in (E.1). T is the period of the flow, in units of the time taken by a sound wave under ambient conditions to travel a distance equal to the radius of the sphere on whose surface the inner boundary condition is prescribed.

Card 7: The 2 X NOINFC entries on this card or group of cards are read sequentially as the values of a_1, a_2, ..., a_{NOINFC}, b_1, b_2, ..., b_{NOINFC} that will be used in the first case calculated by the program; see (E.1). For the most effective operation of SPHERE, it is probably best to impose the limitation $A \leq 0.01$--see (E.5). In addition, because of the normalization of the Fourier coefficients during the output calculation, it is necessary that $a_1^2 + b_1^2 \neq 0$.

Card 8: CASECO has been discussed in subsection E, paragraph 1c.

2. Output

a. Beginning of a Run

At the beginning of a run, the program prints the values of all data variables in cards 1, 2, 3, and 4, and also the values of P, AI, AL, LIMA.

b. Case Heading

The output for each case begins with a heading of the type shown in Figure 5. Most of the entries are self-explanatory; note, however, that the distance variable r is referred to as "X". The line beginning "FOURIER COMPONENTS" states whether the function $Q(\tau)$ prescribed at the inner boundary is the reduced velocity $\tilde{q}$ or the reduced mass flow $\tilde{\theta}$. Following this line are printed the values of the sine and cosine coefficients of $Q(\tau)$, i.e., the b's and a's of (E.1). The line labeled "DC" contains a single entry, the value of a_0. (The column heading "1" refers only to the sine and cosine coefficients.) The values of all these coefficients (and of A) are those appropriate to the current case; only for CASE = 1 will they be identical with the values read in from card 6 (INDC) and card 7.

c. Iterations

Following the heading, a line is printed every time the program has completed one cycle through the outer iterative loop. ITU is the total number of iterations in the K inner iterative loops of the subroutine JLOOP; in the particular case whose output we see in Figure 5. we observe that ITU = K + 1, whence it follows that each inner iterative loop, except one, required only a single iteration to meet the convergence criterion (E.32). If we let $\bar{\epsilon}_j = \epsilon_j^{(m)}$ (see the definition just preceding (E.32)), with m the final iteration number of the j^{th} inner iterative loop, MU is defined as $\max_j \bar{\epsilon}_j$. We may think of this number as the worst error made in any inner iterative loop of JLOOP. MKU pertains to the convergence of the outer iterative loop and is defined following (E.42). The quantities ITV, MV, and MKV have the same meaning for FLOOP as do ITU, MU, and MKU for JLOOP.

d. Flow Variables

A portion of the format for tabulating the values of the variables $r\tilde{q}$, $r\tilde{p}$, or $\tilde{\theta}$ is shown in Figure 6. The line beginning with "X =" refers to the r-coordinate. The column headed "I" refers to the η-coordinate, with η = IT/N; I ranges from 0 to N - 1. In the case shown, A = 0.001, and the pressure should be very nearly given by the first equation of (D.1); we verify this equation, written in the form

$$r\tilde{p} = F'(\eta)/p_0, \tag{F.1}$$

by noting that in the numerical calculation, $r\tilde{p}$ is, in fact, substantially a function of η only.

e. Fourier Components

The format for printing the amplitudes and phases of the complex Fourier coefficients is shown in Figure 7. (Properly speaking, the heading should read "FOURIER COMPONENTS OF THE ACOUSTIC PRESSURE × X"; a similar remark applies to the heading for the Fourier components of the velocity.) It should be mentioned that the calculated phase of a Fourier coefficient of very small amplitude has no significance.

3. Limitations

a. Amplitude

The form of the denominators in the two equations (C.8) imposes a basic limitation on the amplitude of the function $Q(\tau)$ prescribed at $r = 1$. If we set $\tilde{c} = c/c_0$, then for $r = 1$, these denominators equal

PERIODIC SPHERICAL WAVES

A= 0.00400 ACC= 0.005000 ALPHA= 0.5000 T= 10.00000000
K= 20 K0= 1 K1= 10 K2= 2
N= 80

OUTER BOUNDARY AT X= 2.875000

FOURIER COMPONENTS OF VELOCITY PRESCRIBED AT INNER BOUNDARY...

	1
SINE	0.40000E-02
COSINE	0.
DC	0.

CASE= 3

ITERATION	ITU	MU	MKU	ITV	MV	MKV
1	21	0.13046E-06	0.16475E-05	22	0.79297E-07	0.16071E-05
2	21	0.13007E-06	0.13575E-05	22	0.79410E-07	0.39696E-06

FIG. 5 TYPICAL CASE HEADING AND ITERATION COUNT

ACCOUSTIC PRESSURE(IN UNITS OF AMBIENT PRESSURE) * X

X= / I	1.0000	1.1250	1.2500	1.3750	1.5000	1.6250
0	0.00063047	0.00063048	0.00063051	0.00063055	0.00063061	0.00063078
1	0.00065967	0.00065970	0.00065971	0.00065983	0.00065987	0.00065994
2	0.00068483	0.00068486	0.00068489	0.00068495	0.00068506	0.00068519
3	0.00070570	0.00070587	0.00070588	0.00070589	0.00070605	0.00070604
4	0.00072230	0.00072242	0.00072245	0.00072253	0.00072263	0.00072265
5	0.00073440	0.00073450	0.00073457	0.00073458	0.00073468	0.00073469
6	0.00074202	0.00074213	0.00074213	0.00074217	0.00074237	0.00074232
7	0.00074504	0.00074508	0.00074513	0.00074521	0.00074534	0.00074536
8	0.00074337	0.00074354	0.00074357	0.00074363	0.00074378	0.00074367
9	0.00073722	0.00073732	0.00073731	0.00073744	0.00073750	0.00073757
10	0.00072648	0.00072652	0.00072662	0.00072668	0.00072670	0.00072689
11	0.00071123	0.00071139	0.00071149	0.00071148	0.00071152	0.00071163
12	0.00069161	0.00069179	0.00069180	0.00069183	0.00069194	0.00069197
13	0.00066772	0.00066780	0.00066792	0.00066800	0.00066800	0.00066807
14	0.00063975	0.00063987	0.00063989	0.00064001	0.00064000	0.00064010
15	0.00060773	0.00060782	0.00060793	0.00060803	0.00060808	0.00060804

FIG. 6 TYPICAL PRINT-OUT OF FUNCTIONAL VALUES

FOURIER COMPONENTS OF THE ACCOUSTIC PRESSURE (AMPLITUDE RELATIVE TO FUNDAMENTAL AT X=1,PHASE IN UNITS OF 90 DEGREES)

X= / ORDER	1.0000	1.1250	1.2500	1.3750	1.5000	1.6250
(A) 1	1.00000	1.00009	1.00016	1.00023	1.00030	1.00036
(P) 1	0.35764	0.35764	0.35763	0.35760	0.35756	0.35753
(A) 2	0.00042	0.00037	0.00032	0.00031	0.00031	0.00032
(P) 2	0.62928	0.72572	0.82050	0.89352	0.89480	0.93503
(A) 3	0.00001	0.00000	0.00001	0.00001	0.00002	0.00002
(P) 3	-0.35408	-0.59269	-0.30037	-0.05730	0.79851	-0.78432
(A) 4	0.00001	0.00002	0.00003	0.00000	0.00002	0.00001
(P) 4	-0.09928	-0.82340	0.40999	0.13705	-0.62159	-0.27533
(A) 0	-0.00038	-0.00028	-0.00025	-0.00022	-0.00019	-0.00018

FIG. 7 TYPICAL PRINT-OUT OF FOURIER COMPONENTS

$\tilde{c} + \tilde{q}$ and $\tilde{c} - \tilde{q}$; this follows readily from (C.3) and (C.11). Consequently, $|\tilde{q}|$ must be somewhat less than unity; i.e., we must limit A to a value somewhat less than 1. Essentially, this limitation arises from our choice of coordinate system; if either $\tilde{c} + \tilde{q}$ or $\tilde{c} - \tilde{q}$ is zero, one or the other of the characteristic directions of the system (C.8) coincides with the r-direction.

b. Size and Type of Problem

By size we mean the number of net-points that the program can handle. Unlike the amplitude limitation, the size limitation is machine-dependent. SPHERE is presently set up to run on machines with 32K core memory and four available external memory units; specifically, it has been checked out on the Stanford University IBM 7090 and the Air Force Systems Command IBM 7094-7044 at Wright-Patterson Air Force Base. Within this frame of reference, the limit on N, $N \leq 100$, is relatively firm; however, the limit on K, $K \leq 400$, could readily be extended to, say, $K \leq 2000$ with only trivial modifications of the program. The principal reason for restricting the value of K is the time taken for the computation. This will be discussed further in subsection G; for the moment we merely note that $K \leq 400$ is a practical restriction that eventually may be considerably lightened, and discuss the type of problem that may be solved with this restriction.

In the system of units employed, the wavelength corresponding (under ambient conditions) to the fundamental frequency equals T, the fundamental period. It is therefore reasonably consistent to choose the spacing of net-points in the r-direction equal to that in the η- (or τ-) direction, at least for the first few wavelengths. This spacing is obtained by setting K0 = 2, K2 = 1, K1 = K; the position of the outer boundary is then given by $R = TK/N + 1$. If the calculation is to yield significant values of several harmonics of the flow variables, N should probably be chosen to be not less than 50, and preferably 100; the maximum possible values of R are then, approximately, 8T and 4T, respectively. The amplitude of the reduced acoustic pressure $\tilde{p}$ at $r = R$ is approximately A/R, and this quantity should be small if the use of the acoustic boundary condition is to be justified (see subsection A). Although it is not quite clear what exact meaning should be attached to "small," let us arbitrarily put a limit of 0.01 on A/R. Then we obtain the following rules:

$$N = 50,\ K = 400:\quad A \le 0.08T,$$

$$N = 100,\ K = 400:\quad A \le 0.04T.$$

Therefore, if we are interested in moderately high amplitudes at the source, say A = 0.4, we have the following restrictions:

$$N = 50,\ K = 400:\quad T \ge 5,$$

$$N = 100,\ K = 400:\quad T \ge 10.$$

Essentially, then, SPHERE is limited to the calculation of low-frequency nonlinear periodic flows, i.e., flows whose fundamental wavelength is at least several source diameters. However, as larger and faster computers become available, and with the modifications suggested in subsection G, paragraph 2, the calculation of a periodic flow whose fundamental wavelength is of the same order as the source diameter should become feasible. This point is important because on general grounds, one would expect this to be the kind of calculation least amenable to analytic techniques.

4. Typical Results

a. The Runs SU and WP

The computations actually carried out with SPHERE and its several forerunners were primarily intended to check the operation of the program and to develop some feeling for the proper choice of data variables. We shall discuss in some detail a one-hour run (SU) on the Stanford University IBM 7090 (May 1966) and a two-hour run (WP) on the Wright-Patterson Air Force Base IBM 7094-7044 (September 1966).

The finite-difference net and period were identical for the two runs: N = 100, K0 = 2, K2 = 1, K = 100, T = 10. This corresponds to a uniform spacing $\Delta r = \Delta\eta = 0.1$, with the position of the outer boundary, R, equal to 11. In both runs, the inner-boundary function $Q(\tau)$ was of the form $A \sin(2\pi\tau T^{-1})$; this function specified the reduced velocity $\tilde{q}$ in SU and the reduced mass flow $\tilde{\theta}$ in WP. For small A, the difference between these two conditions affects the numerical values of u and V very little, and thus has no effect on the required number of iterations. The iteration count is, however, affected by the value of ACC, and this variable was set at 10^{-4} in SU and 10^{-3} in WP. With respect to the number of inner iterations, the effect is relatively minor: In the 8 completed cases of SU, ITU was 112-113, and ITV was 128-129; in the 17 completed cases of WP, ITU was 106-107, and ITV was 101-102. In contrast, because of the slower rate of convergence of the outer iterative loop, the number of outer iterations depends significantly on the value of ACC. However, from the printed values of MKU, MKV, and A, we can always determine the number of outer iterations that would have taken place if ACC had been set to a larger value than that actually used in the computation;

we need only recall that after two iterations, the outer iterative loop is discontinued when max (MKU, MKV) $<$ A $\times$ ACC $\times$ 0.1.

b. Iteration Counts

Although the initial value of A was 10^{-3} in both runs, the value of AI was 2 in SU and $2^{1/3}$ in WP. The purpose of using two quite different values of AI was to get some idea of how the total number of outer iterations required to carry the calculation from one amplitude to another depends on the rate of increase in amplitude per case. This is of interest because to a rough approximation the total computation time per run is proportional to the total number of outer iterations during the run, regardless of the number of cases.

Table III presents the outer iteration count for the completed cases of WP and SU. For WP, we have added a column of 3-case counts, i.e., the total number of iterations for the current case and the two previous cases. For SU, we give the actual count (ACC = 10^{-4}), and the counts that would have been obtained with ACC = 10^{-3} and ACC = 10^{-2}. In the light of the remarks above, the relative efficiency of the two choices of AI may be obtained by comparing the column of 3-case counts for WP with the ACC = 10^{-3} column of SU; for the range covered, it is clear that 2 is a much better value for AI than $2^{1/3}$.

By comparing the "actual" and ACC = 10^{-3} columns of SU, we also see that it takes from 4 to 6 iterations to reduce the error by a factor of 10. Generally, in a given case, the values of MKU may undergo a slight oscillation for the first few iterations; after that, MKU and MKV both decrease more or less regularly with increasing iteration number. The

Table III

COMPARATIVE OUTER ITERATION COUNTS
FOR THE RUNS WP AND SU

WP: ACC = 10^{-3}, AI = $2^{1/3}$

A	Case	Iteration Count	3-Case Total
0.001	1	5	(5)
	2	2	
	3	2	
0.002	4	2	6
	5	3	
	6	2	
0.004	7	4	9
	8	4	
	9	4	
0.008	10	5	13
	11	5	
	12	5	
0.016	13	6	16
	14	6	
	15	6	
0.032	16	7	19
	17	7	

SU: ACC = 10^{-4}, AI = 2

		Iteration Count		
A	Case	Actual	ACC = 10^{-3}	ACC = 10^{-2}
0.001	1	11	5	2
0.002	2	9	5	2
0.004	3	11	6	2
0.008	4	13	7	3

initial oscillation is, as one would expect, more pronounced if the initial guess for the solution is a poor one. On the basis of several earlier computations with A = 0.1 and various net-spacings, it also appears that the rate of convergence of the outer iterative loop slowly decreases with amplitude; in the computations just mentioned, it generally required 7 or 8 iterations to reduce the error by a factor of 10.

c. Fourier Components

Table IV shows the amplitude of the Fourier components of the reduced acoustic pressure $\tilde{p} \times r$ for two cases of SU and one case of WP. Two kinds of nonlinear effects are evident: One is the nonlinear relation between $\tilde{p}$ and $\tilde{q}$ and between $\tilde{p}$ and $\tilde{\theta}$. Although $Q(\tau)$ is a pure sine wave, at r = 1 the reduced pressure has significant second harmonic and d.c. components, even for A = 0.001. Note that these components are approximately proportional to A. The other nonlinear effect is the growth of higher harmonics with radial distance. This effect, well known in the plane case, is modified in the spherical case by an initial dip that becomes more pronounced with increasing A.

It seems clear that extensive further computations are required to obtain results that may be regarded as significant. Suggestions for such computations will be given in subsection G.

Table IV

AMPLITUDE OF FOURIER COMPONENTS OF $\widetilde{rp}$ (RUNS SU AND WP)†

$Q(\tau) = A \sin (2\pi\tau T^{-1})$

Run	A	Order	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8	r = 9	r = 10	r = 11
SU	0.001	1	1.0000	1.0000	1.0000	1.0001	1.0001	1.0000	0.9999	0.9998	1.0000	0.9999	0.9999
$Q:\tilde{q}$		2	0.0006	0.0005	0.0006	0.0006	0.0007	0.0007	0.0006	0.0007	0.0007	0.0007	0.0008
		0 (d.c.)	-0.0004	-0.0002	-0.0002	-0.0003	-0.0003	-0.0003	-0.0004	-0.0005	-0.0005	-0.0005	-0.0006
SU	0.008	1	1.0000	1.0000	1.0000	1.0001	1.0001	1.0000	0.9999	1.0000	1.0001	1.0001	1.0000
$Q:\tilde{q}$		2	0.0053	0.0044	0.0048	0.0051	0.0051	0.0055	0.0056	0.0057	0.0058	0.0059	0.0060
		0 (d.c.)	-0.0027	-0.0003	-0.0001	0	0	0	0	0	0	0	0
WP	0.032	1	1.0000	0.9971	0.9997	0.9998	0.9998	0.9997	0.9996	0.9996	0.9997	0.9997	0.9996
$Q:\tilde{\theta}$		2	0.0144	0.0083	0.0098	0.0113	0.0125	0.0135	0.0144	0.0151	0.0158	0.0164	0.0169
		3	0.0002	0.0001	0.0001	0.0002	0.0002	0.0003	0.0003	0.0004	0.0004	0.0004	0.0004
		0 (d.c.)	-0.0108	-0.0012	-0.0030	0	0	0.0001	0.0002	0.0003	0.0003	0.0003	0.0004

† In each of the three cases, the Fourier amplitudes are normalized so that the amplitude of the fundamental at r = 1 is unity.

G. Suggestions for Further Work

1. Computations with the Present Version of SPHERE

Although it appears desirable to modify SPHERE with a view to enhancing its efficiency, there are certain computations of interest that may reasonably be carried out with the present version. Three two-hour runs (7094-7044) in particular are suggested:

(i) A repetition of WP with AI = 2 instead of $2^{1/3}$. This should lead to the calculation of cases well in the nonlinear range.

(ii) The same as (i), but with K0 = 1, K1 = K = 200, K2 = 1. This change leaves the position of the outer boundary at r = 11 but decreases the net-spacing in the r-direction by one-half. Computation time per case will be approximately doubled, but from a comparison of cases covered by both (i) and (ii), the truncation error arising from the use of the logarithmic trapezoidal rule in constructing the finite-difference approximations (E.20) and (E.21) may be estimated.

(iii) The same as (i), but with K0 = 2, K1 = K = 200, K2 = 1. This change leaves the net-spacing as in (i) but moves the outer boundary to r = 21. From a comparison of cases covered by both (i) and (iii), it should be possible to say something about the extent to which the position of the outer boundary influences the calculated flow field. An understanding of this question is important in providing a reasonably definitive validation of the use of the acoustic boundary condition, and also in justifying the patching of an approximate analytic solution, valid for large r, onto the numerical solution.

2. Modifications of SPHERE

Before undertaking computations of much larger scope than those just outlined, it would probably be wise to attempt to reduce the running time of SPHERE. We have already indicated that the rate of convergence of the outer iterative loop can probably be increased if the location of the eigenvalues of the matrix $\mathcal{L}$ can be estimated reasonably accurately (paragraph 3.c of subsection E); this problem appears quite capable of solution. Once this estimate is available, separate relaxation factors for JLOOP and FLOOP can easily be determined for optimum rate of convergence; the change in the SPHERE code required to accommodate this modification of the underlying numerical method is trivial.

Another way in which the computation may be speeded up is to provide better initial approximations for each case (after the second) of a run by making greater use of the solutions of previous cases. There is an obvious extrapolation method--analogous to that now used in going from one inner iterative loop to the next--that should provide quite accurate initial approximations even for fairly large values of AI. To accommodate this scheme will require two additional external memory units, and a moderate amount of additional programming.

There is, of course, no reason why these two modifications should not be combined; the resulting gain in efficiency should be very considerable.

3. Investigation of Possible Subharmonic Regimes

So far, only periodic solutions having the periodicity of the inner-boundary function $Q(\tau)$ have been considered. Although the point has not been investigated, it appears intuitively certain (a rigorous proof would be of some interest) that within reasonable restrictions, if such a solution exists, it is unique. However, it is well known that a nonlinear system with one degree of freedom (i.e., a system governed by a second-order nonlinear ordinary differential equation) may have a stable oscillation whose fundamental period is a multiple of the period of the forcing function [7, Chap. 7]. It seems possible therefore that if we modify our basic problem (see subsection E, paragraph 1) by relaxing the periodicity conditions (C.10) on the functions $u(\eta, r)$ and $V(\zeta, r)$, and require only that

$$u(\eta + nT, r) = u(\eta, r), \qquad V(\zeta + nT, r) = V(\zeta, r), \tag{G.1}$$

where n is some integer greater than 1, we may find subharmonic solutions that satisfy (G.1) but not (C.10). Since any solution satisfying (C.10) also satisfies (G.1), the solution of the modified problem is not unique. The question then arises as to which of the several solutions are stable; if more than one stable solution exists, which of these is actually reached as $t \to \infty$ will depend on the initial conditions. These problems are quite well understood for very simple types of nonlinear ordinary differential equations; their investigation in connection with nonlinear acoustics would appear to be both very difficult and rather important, and would have to depend a good deal on numerical experimentation.

SECTION VI

SEMI-ITERATIVE METHODS AND SUMMABILITY

G. M. Muller

A. Introduction

Systems of linear equations encountered in numerical analysis often appear rather naturally in the form

$$y = My + f \tag{A.1}$$

where M is a complex n X n matrix, f a given n-dimensional vector, and y the unknown n-dimensional solution vector. One of the "obvious" ways of attempting to solve (A.1) is to set up the iterative scheme

$$y_n = My_{n-1} + f \tag{A.2}$$

with some initial vector y_0. It is, of course, well known that y_n will converge to y for arbitrary f and y_0 if and only if all the eigenvalues of M are less than one in absolute value. Suppose now that the eigenvalues of M are real and lie in the open interval $(1 - \mu, 1)$, $\mu > 2$. Then the eigenvalues of I - M (where I is the n X n identity matrix) lie in the open interval $(0, \mu)$, so that I - M is nonsingular and hence the solution of (A.1) exists; on the other hand, the scheme (A.2) will fail to converge. The common remedy [6, p. 188] is to observe that we may multiply (A.1) by any nonsingular matrix H and add to both sides the term (I - H)y to obtain the equivalent system

$$y = (HM + I - H)y + Hf, \tag{A.3}$$

but with a different associated iterative scheme,

$$v_n = (HM + I - H)v_{n-1} + Hf. \tag{A.4}$$

Here we have used the symbol v_n to denote the successive iterates, to distinguish them from the y's produced by (A.2). In the case at hand, we choose H to be a constant α to obtain the scheme

$$v_n = (\alpha M + 1 - \alpha) v_{n-1} + \alpha f. \tag{A.5}$$

It is easily verified that the eigenvalues of $\alpha M + (1-\alpha)I$ will lie in the interval (-1,1) if and only if the constant α satisfies the condition

$$0 \leq \alpha < 2/\mu < 1. \tag{A.6}$$

With any such choice of α, the scheme (A.5) will converge to the unique solution of (A.3), which, for $\alpha \neq 0$, is identical with the solution of equation (A.1). Thus, whereas the sequence y_n of (A.2) diverges, we have produced another sequence v_n which converges, and to the "right" limit y. This result suggests that $\{v_n\}$ may be related to $\{y_n\}$ by a linear sequence-to-sequence transformation of the type studied in connection with the theory of divergent series. In fact, if we set $v_0 = y_0$, it is easily shown by induction that v_n is explicitly given by

$$v_n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} y_k \tag{A.7}$$

whence it follows that $\{v_n\}$ is the well-known Euler-Knopp transform [1; 7, Chap. 8] of order α of $\{y_n\}$.

A related problem concerns the situation where the scheme (A.2) converges and it is desired to form a sequence of linear combinations v_n of the vectors y_n such that $\{v_n\}$ converges faster than $\{y_n\}$. Methods which do this are called semi-iterative by Varga [15, Chap. 5]. For the best-known such methods, it again turns out that it is unnecessary to form the linear combinations explicitly from the y's; rather, the v's may be obtained directly from a modification of the original iterative scheme.

We mention, in particular, Richardson's first- and second-order methods, of the form (respectively),

$$v_n = (\alpha_n M + 1 - \alpha_n)v_{n-1} + \alpha_n f \ , \tag{A.8}$$

and

$$v_n = \alpha_n M v_{n-1} + (1 - \alpha_n)v_{n-2} + \alpha_n f. \tag{A.9}$$

A third method, the Chebyshev semi-iterative method, is of the same form as (A.9); in each case, the α's are determined according to certain prescriptions that need not concern us here. (For details and references to the original literature, see [15, Chap. 5].)

We may ask about the algebraic relation of the v's of (A.8) to the y's of (A.2), allowing the α_n's to have arbitrary nonzero complex values and putting aside, for the moment, any questions of convergence. If we set $d_n = (1 - \alpha_n)/\alpha_n$, then, as we shall see later, $\{v_n\}$ is the $[F,d_n]$, or generalized Lototsky, transform of $\{y_n\}$, provided we choose $v_0 = y_0$; the Euler-Knopp transform mentioned earlier is a particular case, with $d_n = (1 - \alpha)/\alpha$ for all n. It is perhaps remarkable that, although the scheme (A.7) dates back to 1910 [17], the $[F,d_n]$ transform was first defined in 1959, by Jakimovski [9].

Equations (A.2) and (A.9) together define a sequence-to-sequence transform that is as yet nameless. Our purpose in this section is to define and study a class of sequence-to-sequence transformations which can be put into one-to-one correspondence with semi-iterative schemes of the sort we have been discussing. In subsection B we shall define this class of $\mathcal{P}$-<u>transformations</u>, basing our definition on sets of polynomials satisfying linear recurrence relations. We shall then establish the one-to-one correspondence between $\mathcal{P}$-transformations and "reasonable" semi-iterative schemes; the results obtained are valid in arbitrary vector spaces. In subsection C we shall study a more general transformation, the <u>normalized transformation</u>, in complex euclidean n-space C^n and prove several variants of the basic result that, if a given transformation sums the geometric series to its correct sum in an open set O of the complex plane, it will correctly sum the matrix analogue of the geometric series if the eigenvalues of the matrix lie in O. In subsection D we shall show how the open set O may be determined from the consideration of an associated homogeneous linear difference equation if the normalized transformation is a $\mathfrak{P}$-transformation. Finally, in subsection E we shall study a number of specific summability methods, based on particular classes of $\mathfrak{P}$-transformations, and determine their open sets of summability; here we shall make considerable use of results from the theory of linear difference equations.

Completely omitted from the present account is any consideration of the conditions under which a particular type of $\mathfrak{P}$-transformation is regular, i.e., transforms every convergent sequence into a sequence converging to the same limit. For the $[F,d_n]$-transformation, this problem has been extensively treated [5; 9; 11]; a general investigation, however, is probably quite difficult.

B. <u>Normalized and $\mathfrak{P}$-Transforms in a General Vector Space</u>

In this subsection, let $\mathfrak{F}$ be an arbitrary but fixed field; polynomials and vector spaces will always be over $\mathfrak{F}$.

<u>Definition B.1</u>. For a given positive integer k, let $P_m(x)$, $m = 0, 1, \ldots, k-1$ be given polynomials (in an indeterminate x), at least one

of which is not identically zero. For $m = k, k+1, \ldots,$ and $\ell = 1, \ldots, k$, let $Q^{(m)}(x)$ be given polynomials; all but $Q_k^{(m)}(x)$ may be identically zero. For $m = k, k+1, \ldots,$ let

$$P_m(x) = \sum_{\ell=1}^{k} Q_\ell^{(m)}(x)\, P_{m-\ell}(x). \tag{B.1}$$

The set $\{P_m(x)\}$ of polynomials defined in this way for all nonnegative integers m will be called a recursive polynomial set of order k. We shall call the Q's the generators and $P_0, \ldots, P_{k-1}$ the initial polynomials of the set.

<u>Definition B.2</u>. Let $\{P_m(x)\}$ be a recursive polynomial set of order k, and let $N(m)$ denote the degree of $P_m(x)$. $\{P_m(x)\}$ will be called a $\mathcal{P}_k$-set if

$$P_m(1) = 1, \quad m = 0, 1, \ldots, k-1 \tag{B.2}$$

$$\sum_{\ell=1}^{k} Q_\ell^{(m)}(1) = 1, \quad m = k, k+1, \ldots \tag{B.3}$$

$$\sum_{\ell=1}^{k} Q_\ell^{(m)}(x) - 1 \neq 0 \text{ identically}, \quad m = k, k+1, \ldots \tag{B.4}$$

A $\mathcal{P}_k$-set will be called proper if, additionally,

$$N(m) \to \infty \text{ as } m \to \infty\,. \tag{B.5}$$

We propose the unproved conjecture that every $\mathcal{P}_k$-set is, in fact, proper; however, we shall not need to make use of (B.5) in establishing the results of this section.

<u>Definition B.3</u>. Let $\{P_m(x)\}$ be a $\mathcal{P}_k$-set, and define the coefficients a_{ij} by

$$P_i(x) = \sum_{j=0}^{N(i)} a_{ij}\, x^j \tag{B.6}$$

for $i = 0, 1, \ldots,$ and $j = 0, 1, \ldots, N(i)$; and by $a_{ij} = 0$ for $j > N(i)$. The row-finite matrix (a_{ij}) will be called the $\mathcal{P}_k$-matrix associated with

$\{P_m(x)\}$. If $\{s_n\}$ is an arbitrary sequence of elements in a vector space V over $\mathfrak{F}$, the sequence $\{t_n\}$ defined by

$$t_i = \sum_{j=0}^{N(i)} a_{ij} s_j \tag{B.7}$$

will be called the $\mathcal{P}_k$-transform of $\{s_n\}$ associated with $\{P_m(x)\}$.

We note that (B.1), (B.2), and (B.3) imply the important property

$$P_i(1) = \sum_{j=0}^{N(i)} a_{ij} = 1, \text{ all } i, \tag{B.8}$$

so that the t's are weighted means of the s's.

<u>Definition B.4</u>. A polynomial set (transform, matrix) is defined to be a (proper) $\mathcal{P}$-set (transform, matrix) if it is a (proper) $\mathcal{P}_k$-set (transform, matrix) for some positive integer k.

<u>Definition B.5</u>. Let Q(x) be an arbitrary polynomial. We shall designate by $\tilde{Q}(x)$ the polynomial obtained from Q(x) by the substitution

$$\begin{aligned} x^p &\to \sum_{n=0}^{p-1} x^n, && p \geq 1; \\ &\to 0, && p = 0. \end{aligned}$$

The following relation is easily verified:

$$Q(1) - Q(x) = (1 - x)\,\tilde{Q}(x); \tag{B.9}$$

in view of the division algorithm for polynomials, this relation can be used as an alternative definition of $\tilde{Q}(x)$.

We are now ready to state and prove the principal theorem of this subsection.

<u>Theorem B.1</u>. Let V be a vector space, L a linear operator in V, and let f, $s_0 \in$ V be arbitrary. Let $\{P_m(x)\}$ be a $\mathcal{P}_k$-set, $A = (a_{ij})$ its associated matrix, and $Q_\ell^{(m)}$ its generators. Let $\{s_m\}$ be the sequence determined by

$$s_m = Ls_{m-1} + f, \quad m = 1, 2, \ldots . \tag{B.10}$$

Then the sequence $\{t_m\}$ determined by the initial polynomials through equation (B.7) for m = 0, 1, ..., k-1 and by

$$t_m = \sum_{\ell=1}^{k} Q_\ell^{(m)}(L)\, t_{m-\ell} + \sum_{\ell=1}^{k} \tilde{Q}_\ell^{(m)}(L) f \tag{B.11}$$

for m = k, k+1, ..., is the ρ_k-transform of $\{s_m\}$.

<u>Proof</u>: From (B.10) it is easily shown by induction that

$$s_{m+p} = L^p s_m + \sum_{n=0}^{p-1} L^n f, \qquad p \geq 1. \tag{B.12}$$

Now consider an auxiliary vector space, V', whose elements are formal finite sums $\Sigma\, c_j s_j'$; the undefined elements s_j' are assumed to form a basis for V'. We define the shift operator E by

$$E\left(\sum_{j=0}^{m} c_j s_j'\right) = \sum_{j=0}^{m} c_j s_{j+1}' \tag{B.13}$$

and observe that E is a linear operator in V'. We note in particular that

$$s_{m+p}' = E^p s_m' . \tag{B.14}$$

Polynomials in linear operators obey the same rules of multiplication and addition as do polynomials in an indeterminate x. Accordingly, if we denote by $\{t_m'\}$ the ρ_k-transform of $\{s_m'\}$, it follows from (B.6), (B.7), (B.14), and (B.1) that, for m = k, k+1, ...,

$$\begin{aligned} t_m' = P^{(m)}(E) s_0' &= \sum_{\ell=1}^{k} Q_\ell^{(m)}(E)\; P^{(m-\ell)}(E) s_0' \\ &= \sum_{\ell=1}^{k} Q_\ell^{(m)}(E)\, t_{m-\ell}' . \end{aligned} \tag{B.15}$$

We next define a mapping H from V' to V by

$$\Sigma\, c_j s_j' \underset{H}{\rightarrow} \Sigma\, c_j s_j ; \tag{B.16}$$

clearly, H is a homomorphism, and (B.12) is the image of $E^p s_m'$ under H. Since H preserves multiplication by scalars and the formation of finite sums, for any polynomial $Q(x)$ we have, by the definition of $\tilde{Q}(x)$,

$$Q(E)s'_j \underset{H}{\rightarrow} Q(L)s_j + \tilde{Q}(L)f \ . \tag{B.17}$$

Again making use of the fact that H is a homomorphism, we multiply both sides of (B.17) by a_{mj} and sum over j to obtain

$$Q(E) \sum_{j=0}^{N(m)} a_{mj}s'_j \underset{H}{\rightarrow} Q(L) \sum_{j=0}^{N(m)} a_{mj}s_j + \tilde{Q}(L) \sum_{j=0}^{N(m)} a_{mj}f \tag{B.18}$$

or, by virtue of (B.8),

$$Q(E)\ t'_n \underset{H}{\rightarrow} Q(L)\ t_n + \tilde{Q}(L)f \ . \tag{B.19}$$

We now apply H to the first and last members of (B.15) and make use of (B.19) to obtain (B.11), thus proving the theorem.

So far, we have been concerned with the relation between two sequences, $\{s_n\}$ and $\{t_n\}$. We now consider the relation between the corresponding underlying linear equations, i.e., the equations obtained from (B.10) and (B.11) by setting $s_m = s$, all m, and $t_m = t$, $m > k$. If I denotes the identity operator, we obtain

$$(I - L)s = f \tag{B.20}$$

and

$$\left[I - \sum_{\ell=1}^{k} Q_\ell^{(m)}(L) \right] t = \sum_{\ell=1}^{k} \tilde{Q}_\ell^{(m)}(L)f \ . \tag{B.21}$$

Suppose we define

$$\sum_{\ell=1}^{k} Q_\ell^{(m)}(x) = Q^{[m]}(x) \ ; \tag{B.22}$$

then

$$Q^{[m]}(1) = 1 \tag{B.23}$$

because of (B.3), and

$$\sum_{\ell=1}^{k} \tilde{Q}_\ell^{(m)}(x) = \tilde{Q}^{[m]}(x) \tag{B.24}$$

because the operation "~" is distributive with respect to addition. Accordingly, we may use (B.9) to rewrite (B.21) in the form

$$\tilde{Q}^{[m]}(L)\ (I-L)t = \tilde{Q}^{[m]}(L)f \tag{B.25}$$

from which it is clear that if s satisfies (B.20), then t = s will satisfy (B.25). (The converse is not necessarily true.)

We may also write (B.25) in the form

$$t = \left[\tilde{Q}^{[m]}(L)\ L + I - \tilde{Q}^{[m]}(L)\right] t + \tilde{Q}^{[m]}(L)f\,; \tag{B.26}$$

this equation is related to (B.20) in exactly the same way as (A.3) is to (A.1), with $\tilde{Q}^{[m]}$ taking the place of H. Thus, any $\mathcal{P}_k$-set induces a semi-iterative scheme (B.11) with a set of "reasonable" underlying linear equations (B.26). We note that condition (B.4) ensures that none of the $\tilde{Q}^{[m]}(x)$ vanish identically.

We are now ready to define the class of semi-iterative schemes that stand in one-to-one correspondence with the class of $\mathcal{P}_k$-transforms. We require the following theorem:

Theorem B.2. For a given positive integer k, let $\left\{\tilde{Q}^{[m]}(x)\right\}$, $m \geq k$, be a given sequence of nonzero polynomials. Let

$$Q^{[m]}(x) = (x-1)\ \tilde{Q}^{[m]}(x) + 1\ , \tag{B.27}$$

and write each $Q^{[m]}(x)$ as the sum of k polynomials $Q_\ell^{(m)}(x)$, $1 \leq \ell \leq k$; all but $Q_k^{(m)}(x)$ may be identically zero. The the $Q_k^{(m)}(x)$ are the generators of a $\mathcal{P}_k$-set.

Proof: (B.27) implies (B.3) and (B.3).

Suppose, then, that we are given a set $\left\{\tilde{Q}^{[m]}(x)\right\}$, $m \geq k$. We use this set to construct an infinite set of linear equations of the type (B.26). With the definition of $Q^{[m]}(x)$ as given by (B.27), these equations may be written in the form

$$t = Q^{[m]}(L)t + \tilde{Q}^{[m]}(L)f \quad . \tag{B.28}$$

By writing $Q^{[m]}(L)$ as a sum of terms $Q_{\ell}^{(m)}(L)$, we obtain from (B.28) a k^{th}-order recursion relation of type (B.11); if we further specify t_m, $m = 0, 1, \ldots, k-1$, as weighted means of the s_m, we have constructed a semi-iterative scheme with respect to the iterative scheme (B.10), and at the same time uniquely specified a $\wp_k$-transform to which the semi-iterative scheme corresponds. Moreover, according to our discussion following the proof of Theorem B.1, any $\wp_k$-transform may be specified in this way; hence, the class of k^{th}-order semi-iterative schemes constructed as above corresponds one-to-one to the class of $\wp_k$-transforms. We may, therefore, unambiguously speak of a $\wp_k$-semi-iterative scheme.

It is convenient at this point to define a more general sequence-to-sequence transformation than the $\wp$-transformation.

Definition B.6. An arbitrary infinite set of polynomials $\{P_m(x)\}$ will be called normalized if $P_m(1) = 1$, all m. The matrix (a_{ij}) derived from the set via (B.6) will be called a normalized matrix, and the corresponding transformation defined by (B.7) a normalized transformation.

The coefficients a_{ij} of a normalized matrix satisfy (B.8); it follows that any constant sequence $\{g, g, g, \ldots\}$ is transformed into itself by a normalized transformation. A $\wp$-transformation is obviously a normalized transformation.

Now let s_0, $\hat{s}_0$ be different elements of V, and consider the two sequences $\{s_m\}$, $\{\hat{s}_m\}$ generated by (B.10). Let $\{P_m(x)\}$ be normalized, and $\{t_m\}$, $\{\hat{t}_m\}$ the corresponding transforms of $\{s_m\}$, $\{\hat{s}_m\}$. Define

$$\hat{\varepsilon}_m = t_m - \hat{t}_m . \tag{B.29}$$

Applying (B.12) to s_0 and $\hat{s}_0$, we obtain

$$s_p - \hat{s}_p = L^p s_0 - L^p \hat{s}_0 \tag{B.30}$$

from which it readily follows that

$$\epsilon_m = P_m(L)(s_0 - \hat{s}_0) \ . \tag{B.31}$$

Suppose now that s satisfies (B.20). If we put $\hat{s}_0 = s$, then $\hat{s}_m = s$, all m, and hence also $\hat{t}_m = s$. This gives us the following result:

Theorem B.3. Let $\{P_m(x)\}$ be normalized, and let $\{t_m\}$ be the associated transform of the sequence $\{s_m\}$ defined by (B.10). If s satisfies (B.20) and we define

$$\epsilon_m = t_m - s, \tag{B.32}$$

then

$$\epsilon_m = P_m(L)(s_0 - s). \tag{B.33}$$

C. Summability in C^n

The relations established in the last subsection were purely algebraic; we now turn to consideration of convergence. If V is a complete normed linear space, we can define summability in an obvious way. With reference to a particular sequence-to-sequence transformation A, we shall say that a sequence $\{s_m\}$ is summable (A) to the vector t if the A-transformed sequence $\{t_m\}$ converges to t, and we shall speak in this context of the A-method of summability. Although summability for more general spaces may be of interest, we shall confine ourselves here to deriving some results valid in the complex euclidean n-space C^n. A polynomial in an indeterminate x may now be identified with the corresponding complex polynomial, and a linear operator with the appropriate $n \times n$ matrix; I will denote the identity matrix. Convergence in the euclidean norm is equivalent to component-wise convergence.

Let L be a matrix such that I - L is nonsingular. Then (B.20) has the unique solution

$$s = (I - L)^{-1} f \ , \tag{C.1}$$

and the quantities ϵ_m defined by (B.32) (with reference to a specific normalized transformation) represent the error in using t_m as an approximation to s. It is possible for t_m to converge to some vector $t \neq s$, in which case $\epsilon_m \to \epsilon \neq 0$; this may happen, e.g., with a $\mathcal{P}$-transform such that for the polynomials $\tilde{Q}^{[m]}(x)$ of Theorem B.2, some or all of the eigenvalues of $\tilde{Q}^{[m]}(L)$ either vanish or tend to zero as $m \to \infty$. In the present subsection, we develop some preliminary tools for investigating this type of problem.

<u>Theorem C.1</u>. Let O be an open set of the complex plane. Suppose a linear sequence-to-sequence transformation A sums the partial sums of the geometric series,

$$\sigma_m = 1 + z + \ldots + z^m \tag{C.2}$$

to $(1 - z)^{-1}$ for $z \in O$, uniformly on compact subsets of O. If the eigenvalues of L lie in O, then A sums the sequence

$$s_m = (I + L + \ldots + L^m) f \ , \tag{C.3}$$

$f \in C^n$, to $(I - L)^{-1} f$.

<u>Proof</u>: Following Faddeev and Faddeeva [6, p. 532], we observe that if ζ is a complex number, the components of the vector $(1 - \zeta L)^{-1} f$ are rational functions of ζ with poles $1/\lambda_i$, where the λ_i are the eigenvalues of L. Accordingly, each component may be written as a finite sum of partial fractions of the form

$$a_{i\nu}(1 - \lambda_i \zeta)^{-\nu} \tag{C.4}$$

where $a_{i\nu}$ is a constant and ν is a nonnegative integer not exceeding the multiplicity of λ_i.

On the other hand, $(I - \zeta L)^{-1}f$ may be expanded in the form

$$If + \zeta Lf + \zeta^2 L^2 f + \dots \tag{C.5}$$

and this expansion converges to the correct value for sufficiently small values of $|\zeta|$. The vector components of the partial sums of (C.5) may therefore be identified with the partial sums obtained by adding the contributions from the expansions of the several terms (C.4). Now because of the hypothesis about uniform convergence, the A-transform sums all derivatives of the geometric series to their correct values, for $z \in O$. The expansion of each term (C.4) is either a geometric series or its $(\nu-1)$st derivative, with argument $z = \lambda_i \zeta$. Hence, for $\zeta = 1$ and $\lambda_i \in O$, the A-transformation sums each vector component of (C.5) to its correct value. This proves the theorem.

It is a well-known phenomenon in summability theory that convergence of the transform of a sequence $\{\sigma_0, \sigma_1, \dots\}$ to a limit σ does not necessarily imply the convergence of the transform of the sequence $\{0, \sigma_0, \sigma_1, \dots\}$ to σ. However, if the σ's are given by (C.2), and we define the shifted sequence $\{\hat{\sigma}_m\}$ by

$$\hat{\sigma}_0 = 0; \quad \hat{\sigma}_m = \sigma_{m-1}, \quad m = 1, 2, \dots \tag{C.6}$$

we can prove the following lemma:

Lemma C.1. Let the transformation A of Theorem C.1 be normalized. Then A sums the complex sequence $\{\hat{\sigma}_m\}$ defined by (C.2) and (C.6) to $(1 - z)^{-1}$ for $z \in O$, uniformly on compact subsets of O.

Proof: The case $z = 0$ is trivial. For $z \neq 0$, we can write

$$\hat{\sigma}_m = \frac{1}{z} (\sigma_m - 1), \quad m = 0, 1, \dots . \tag{C.7}$$

For fixed z, the linearity of A and the fact that A is normalized imply that A sums $\{\hat{\sigma}_m\}$ to

$$\hat{\sigma} = \frac{1}{z}\left(\frac{1}{1-z} - 1\right) = \frac{1}{1-z}\ , \tag{C.8}$$

as asserted; the uniformity is obvious.

We can now prove a somewhat stronger theorem for normalized transformations.

Theorem C.2. If, in addition to the hypotheses of Theorem C.1, it is supposed that A is normalized, then A sums the sequence

$$s_m = L^m s_0 + (I + L + \ldots + L^{m-1})f\ , \tag{C.9}$$

$f,\ s_0 \in C^n$, to $(I - L)^{-1}f$.

Proof: By the lemma just proved, the sequence $\hat{\sigma}_m$ is summed to $(1 - z)^{-1}$. An obvious adaptation of the proof of Theorem C.1 shows that the theorem remains valid if σ_m in (C.2) is replaced by $\hat{\sigma}_m$, and s_m in (C.3) by $\hat{s}_m$, the latter being defined by

$$\hat{s}_0 = 0;\quad \hat{s}_m = s_{m-1},\quad m = 1,\ 2,\ \ldots\ . \tag{C.10}$$

Accordingly, for any f,

$$L^m f = s_m - \hat{s}_m \tag{C.11}$$

is summed to zero. In particular, $L^m s_0$ is summed to zero. If we write

$$S_m = L^m s_0 + \hat{s}_m \tag{C.12}$$

the conclusion of the present theorem follows.

Remark C.1. The sequence S_m defined by (C.9) is just the sequence generated by (B.10).

We now derive a simple corollary of Theorem B.3 that allows us to restate in a convenient form the hypotheses of the theorem just proved.

We note that Theorem B.3 applies, in particular, if $V = C$; under the obvious isomorphism, vectors and matrices may be identified with complex numbers, and vector and matrix norms with absolute values. If we choose $s_0 = f = 1$, $L = z$, then $s = (1 - z)^{-1}$, and the sequence $\{s_m\}$ generated by (B.10) consists of the partial sums of the geometric series, i.e., $s_m = \sigma_m$, in the notation of (C.2). Therefore, if $\{\tau_m\}$ is a normalized transform of $\{\sigma_m\}$, Theorem B.3 gives us

$$\tau_m - \frac{1}{1 - z} = \frac{-z}{1 - z} P_m(z), \tag{C.13}$$

and hence the following result:

<u>Lemma C.2</u>. For any set O of the complex plane, the partial sums of the geometric series are (uniformly) summable to $(1 - z)^{-1}$ for $z \in O$ by the transformation associated with the normalized set $\{P_m(z)\}$ if and only if, for $z \in O$, $P_m(z) \to 0$ (uniformly) as $m \to \infty$.

Theorem C.2 may now be restated as follows:

<u>Theorem C.3</u>. Let $\{P_m(z)\}$ be normalized, and let O be an open set of the complex plane such that for $z \in O$, $P_m(z) \to 0$, uniformly on compact subsets of O. Let L be a matrix whose eigenvalues lie in O. Then the transformation associated with $\{P_m(z)\}$ sums the sequence

$$S_m = L^m s_0 + (I + L + \ldots + L^{m-1})f, \tag{C.14}$$

f, $s_0 \in C^n$, to $(I - L)^{-1}f$.

This theorem, together with the preceding lemma, is analogous to a result of Okada [13] on analytic continuation of functions analytic at the origin.

D. The Associated Linear Difference Equation

The recursion relation (B.1) may be regarded as a linear difference equation. To emphasize this, we write

$$P(m,z) \equiv P_m(z); \quad Q_\ell(m,z) \equiv Q_\ell^{(m)}(z) \tag{D.1}$$

and

$$P(m,z) = \sum_{\ell=1}^{k} Q_\ell(m,z)\, P(m-\ell,z). \tag{D.2}$$

Whereas the emphasis so far has been on the fact that the P's are polynomials in z, Equation (D.2), regarded as a linear difference equation, relegates to z the role of a parameter. In the light of Theorem C.3, the important question is whether, for a given value of z, P(m,z) tends asymptotically to zero. The problem is not uniquely defined until we specify the k initial values P(0,z), ..., P(k-1,z). However, for many purposes it is enough to find the largest open set O in C for which every solution of (D.2) tends to zero, uniformly on compact subsets of O. Since it is known from the theory of linear difference equations [12, p. 356] that any solution of (D.2) can be expressed as a linear combination of k fundamental solutions, we can combine Theorems B.1 and C.3 to obtain Theorem D.1.

Theorem D.1. For some positive integer k, let $\{Q_\ell^{(m)}(z)\}$ be an infinite set of complex polynomials, defined for $\ell = 1, \ldots, k$ and all integers $m \geq k$. Let these polynomials be subject to the following conditions:

(a) $\sum_{\ell=1}^{k} Q_\ell^{(m)}(1) = 1;$

(b) $\sum_{\ell=1}^{k} Q_\ell^{(m)}(z) - 1$ is not identically zero.

Define the polynomials $\tilde{Q}^{[m]}(z)$ by

$$\tilde{Q}^{[m]}(z) = \left[\sum_{\ell=1}^{k} Q_\ell^{(m)}(z) - 1\right](z-1)^{-1}.$$

Let O be an open set of the complex plane, and for each $z \in O$, let $P_m^{(1)}(z), \ldots, P_m^{(k)}(z)$ be a fundamental system of solutions of the linear difference equation

$$P_m(z) = \sum_{\ell=1}^{k} Q_\ell^{(m)}(z)\, P_{m-\ell}(z),$$

such that, for each j, $P_m^{(j)}(z) \to 0$ as $m \to \infty$, uniformly on compact subsets of O. Let L be an $n \times n$ matrix with eigenvalues in O. Let f be an arbitrary vector, and define the vectors t_m by

$$t_m = \sum_{\ell=1}^{k} Q_\ell^{(m)}(L)\, t_{m-\ell} + \tilde{Q}^{[m]}(L)f, \qquad m \geq k,$$

with arbitrary vectors $t_0, \ldots, t_{k-1}$. Then $t_m \to (I - L)^{-1}f$ as $m \to \infty$.

The largest set O satisfying the hypotheses of the theorem will be called the <u>open set of summability</u> for the difference equation and its associated $\mathcal{P}_k$-methods.

E. Some Particular Classes of $\mathcal{P}$-Methods

1. The $\mathcal{P}_1$-Methods

The question of several linearly independent solutions to (D.2) does not even arise for the $\mathcal{P}_1$-methods; we discuss these separately because they correspond to summability methods that have been extensively investigated in recent years. Reverting to the notation of (B.1), we have (omitting the unnecessary subscript on Q)

$$P_m(z) = Q^{(m)}(z)\, P_{m-1}(z) \tag{E.1}$$

with $P_0(1) = 1$, $Q^{(m)}(1) = 1$, and $Q^{(m)}(z) \not\equiv 1$ identically. Clearly, every $\mathcal{P}_1$-set is proper in the sense of (B.5). The solution of (E.1) is given explicitly by

$$P_m(z) = \prod_{k=1}^{m} Q^{(k)}(z)\, P_0(z); \tag{E.2}$$

if we choose $P_0(z) = 1$, the sequence-to-sequence transformation defined by $\{P_m(z)\}$ is identical with the $[F^*, P_n]$-transformation of Meir [10]. [In Meir's notation, our $Q^{(k)}(z)$ is written as $P_k(z)$.] Meir's method is a further generalization of the $[F, d_n]$- or generalized Lototsky transformation [5; 9; 11] in which the Q's are linear polynomials, written in the form

$$[F, d_n]\text{-method:} \qquad Q^{(k)}(z) = \frac{z + d_k}{1 + d_k}\ ; \quad d_k \neq -1; \quad P_0 = 1.$$

If we define $\alpha_k = (1 + d_k)^{-1}$, then $Q^k(z) = \alpha_k z + 1 - \alpha_k$, and the corresponding semi-iterative scheme is (A.8). We mention two particular $[F, d_n]$-methods. If we set $\alpha_k = \alpha$, then $P_m(z) = (\alpha z + 1 - \alpha)^m$. The corresponding classical summarility method is the Euler-Knopp method of order α [1; 9]; the open set of summability, O, is the open disk $|z - (\alpha^{-1} - 1)| < |\alpha^{-1}|$ and the corresponding semi-iterative scheme is Equation (A.5)

If we set $d_k = k - 1$, then $\alpha_k = k^{-1}$, $Q^{(k)}(z) = k^{-1}z + (1 - k^{-1})$, and it is known [2] that $O = \{z | \text{Re}\, z < 1\}$. The corresponding classical summability method is the Lototsky method [2; 3], a special case of the somewhat more general Karamata-Stirling methods [16]. It is worthwhile to state Theorem D.1 for this particular case:

Theorem E.1. Let L be an $n \times n$ matrix whose eigenvalues lie in the half-plane $\text{Re}\, z < 1$. If f and t_0 are arbitrary vectors, and

$$t_m = (m^{-1} L + 1 - m^{-1}) t_{m-1} + m^{-1} f, \qquad m \geq 1,$$

then

$$t_m \to (I - L)^{-1} f \quad \text{as} \quad m \to \infty .$$

Meir [10] has given an explicit construction for finding a sequence $\{d_n\}$ such that the corresponding $[F, d_n]$-transformation sums the geometric series in $\{z | \text{Re } P(z) < 0\}$, where $P(z)$ is a given polynomial with Re $P(1) = 0$. It will be convenient to consider this method in paragraph 3.

2. Methods of Euler-Knopp (E-K) Type

We use this designation for $\mathcal{P}$-methods in which the polynomials $Q_\ell^{(m)}(z)$ are independent of m. The associated difference equation then reads

$$P_m(z) = \sum_{\ell=1}^{k} Q_\ell(z)\, P_{m-\ell}(z); \tag{E.3}$$

for a particular value of z, a fundamental system of solutions is obtained from the characteristic equation

$$\mu^k - \sum_{\ell=1}^{k} Q_\ell(z)\, \mu^{k-\ell} = 0. \tag{E.4}$$

To each simple root μ of (E.4) there corresponds a solution μ^m of (E.3); for a root μ_j of multiplicity $r(j)$ there is a set of solutions μ_j^m, $m\mu_j^m$, ..., $m^{r-1}\mu_j^m$ [12, p. 386]. Each root will, in general, be a function of z, and every solution of (E.3) may be written in the form

$$P_m(z) = \sum_j \sum_{i=1}^{r(j)} f_{ji}(z)\, m^{i-1}\, \mu_j^m(z), \tag{E.5}$$

where the j-summation extends over the distinct roots of (E.4). As far as the difference equation (E.3) is concerned, the coefficient functions $f_{ji}(z)$ are arbitrary; in connection with the $\mathcal{P}$-transformation, however, we are interested in choosing the f's so as to make (E.5) a polynomial, with the additional requirement that $P_m(1) = 1$. This is always possible; there are k arbitrary functions f_{ji} which can be chosen so that the expression (E.5) is identical, for m = 0, 1, ..., k-1, with k given polymials $P_m(z)$, $P_m(1) = 1$. It then follows from the fact that (E.5) satisfies (E.4), and from (B.3), that (E.5) represents polynomials for all m, with $P_m(1) = 1$.

Unless one or more roots of (E.4) have a multiplicity greater than one for all values of z [as would happen if (E.4) contains a factor $[F(\mu,z)]^p$, $F(\mu,z)$ being a polynomial in μ and z], the expression (E.5) will generally be of the particular form

$$P_m(z) = \sum_j f_{j1}(z)\, \mu_j^m(z). \tag{E.6}$$

It can easily be shown that the $f_{j1}(z)$ involve the reciprocal of the Vandermonde determinant [12, p. 9] of the μ's, i.e., of the product $\Pi(\mu_j - \mu_k)$ taken over all j and k such that $j > k$. If two or more roots coalesce for an isolated point z_0, the singularity produced by the coalescence must be removable since (E.6) is analytic and bounded in every deleted neighborhood of z_0. Evaluating the indeterminate form (E.6) as $z \to z_0$ by L'Hospital's rule leads to the more general expression (E.5).

The open set of summability is $O = \{z \mid \max_j |\mu_j| < 1\}$. In some cases, the boundary of O may be found as follows. Suppose we can solve (E.4) explicitly for z; then setting $\mu = e^{i\theta}$, θ real, yields a parametric representation of the locus C of points on which $|\mu| = 1$. C will either be the boundary of a single unbounded region R_∞, or it will divide the plane into two or more disjoint regions R_j. Since each of the several roots μ of (E.4) depends continuously on z, it follows that R_∞ or R_j belongs to O if and only if for some point z_0 in R_∞ or R_j, $|\mu| < 1$ for all roots. Note that since the pair of values $\mu = 1$, $z = 1$ satisfies (E.4), C always goes through $z = 1$. As a simple illustration, consider the $\mathcal{P}_2$-methods associated with the difference equation

$$P_m(z) = \alpha z P_{m-1}(z) + (1-\alpha) P_{m-2}(z). \tag{E.7}$$

The characteristic equation reads

$$\mu^2 - \alpha z\mu - (1-\alpha) = 0, \tag{E.8}$$

and C has the representation

$$z = \frac{1}{\alpha}\, e^{i\theta} - \frac{1-\alpha}{\alpha}\, e^{i\theta}. \tag{E.9}$$

The pair of values $\mu = -1$, $z = -1$ satisfies (E.8), hence C also passes through the point $z = -1$. The curve described by (E.9) is an ellipse for any $\alpha \neq 0$. From (E.8) we see that for $z = 0$, $|\mu| = |1-\alpha|$. Hence the interior of the ellipse belongs to O if and only if $|1-\alpha| < 1$. For any $\alpha \neq 0$, (E.8) has one root $\mu \sim \alpha z$ for $|z| \to \infty$; hence the exterior of the ellipse does not belong to O. In particular, if α is real, we may write (E.9) in the form

$$z = \cos\theta + i\left(\frac{2}{\alpha} - 1\right)\sin\theta, \tag{E.10}$$

which represents an ellipse through the points 1, $i(2\alpha^{-1} - 1)$, -1, $-i(2\alpha^{-1} - 1)$. In the special case $\alpha = 2$, the ellipse degenerates into the line segment $[-1, 1]$ which bounds a single region R_∞.

For the methods associated with

$$P_m(z) = \frac{1}{2}\, z\, [P_{m-1}(z) + P_{m-2}(z)],$$

C has the representation

$$z = \frac{\cos 2\theta + \cos\theta}{\cos\theta + 1} + i\,\frac{\sin 2\theta + \sin\theta}{\cos\theta + 1}\,. \tag{E.11}$$

C has a double point at $z = -2$, and divides the plane into three disjoint regions. Two of these are unbounded; the third coincides with O. O contains the open unit disk and the line segment $(-2, 1)$; its boundary is tangent to the unit circle at $z = 1$, and has a cusp at $z = -2$.

If one or more of the $Q_\ell(z)$ are quadratic in z, we can still solve (E.4) explicitly for z; there may be two distinct curves, C_1 and C_2, for which $|\mu| = 1$, corresponding to the two branches of z considered as a function of μ. For Q's of higher degree, probably the only case which is reasonably tractable is the $\mathcal{P}_1$-method of E-K type, with the difference equation

$$P_m(z) = Q(z)\ P_{m-1}(z); \tag{E.12}$$

O is the set $|Q(z)| < 1$, consisting of the interiors of the one or several branches of the lemniscate $|Q(z)| = 1$; for $Q(z) = \alpha z + 1 - \alpha$ we have the classical E-K method. (For the properties of lemniscates, see [8, Vol. 2, p. 264].)

We conclude this paragraph with a few remarks on what may be called parasitic extensions of $\mathcal{P}_k$-methods of E-K type. Let us write $\Psi(\mu,z)$, a polynomial in μ and z, for the lefthand side of (E.4). Suppose

$$\Psi(\mu,z) = F(\mu)\ \Psi_0(\mu,z), \tag{E.13}$$

$F(\mu)$ being a polynomial of degree p in μ whose coefficients are independent of z. Consider the $\mathcal{P}_{k-p}$-methods associated with the difference equation whose characteristic equation is

$$\Psi_0(\mu,z) = 0. \tag{E.14}$$

If we call the open set of summability of the methods derived from (E.14) O_0 and that of the original group of methods O_1, then clearly $O_1 = O_0$ if for all of the roots of $F(\mu) = 0$, $|\mu| < 1$; and O_1 is empty if for at least one of the roots of $F(\mu) = 0$, $|\mu| \geq 1$. Therefore, the original $\mathcal{P}_k$-methods may be called parasitic extensions of the $\mathcal{P}_{k-p}$-methods associated with the difference equation derived from (E.14). Very similar considerations apply if

$$\Psi(\mu,z) = [F(\mu,z)]^p\ \Psi_0(\mu,z), \tag{E.15}$$

where $F(\mu,z)$ is a polynomial in μ and z; here it suffices to consider the simpler methods derived from $F(\mu,z)\ \Psi_0(\mu,z) = 0$.

3. Asymptotically Singular $\mathcal{P}_k$-Methods

If the coefficients of the difference equation (D.2) satisfy the conditions

$$Q_\ell(m,z) \to Q_\ell(z) \quad \text{as } m \to \infty, \tag{E.16}$$

the equation is of a type first studied by Poincaré. His original work on the asymptotic behavior of the solutions was subsequently extended by many investigators, notably Perron; a summary and extensive bibliography is given by Schäfke [14]. Milne-Thomson [12, p. 523] presents the theorems of Poincaré and Perron in detail. We shall apply some of the results to the study of <u>asymptotically singular</u> $\mathcal{P}_k$-methods; by this we shall understand methods associated with difference equations for which $Q_\ell(z)$ in (E.16) is a constant a_ℓ, independent of z. The solutions $P_\infty(m,z)$ of the asymptotic form of (D.2) then do not depend on z; together with the requirement $P(m,1) = 1$, this implies $P_\infty(m,z) = 1$.

As an illustration, we consider the class of difference equations

$$P_m(z) = \sum_{\ell=1}^{k} a_\ell \, P_{m-\ell}(z) + m^{-\alpha} \sum_{\ell=1}^{k} q_\ell(z) \, P_{m-\ell}(z), \tag{E.17}$$

where α and the a's are constants and the q(z)'s are polynomials, subject to the conditions,

$$\begin{aligned} &\sum_{\ell=1}^{k} a_\ell = 1, \quad \sum_{\ell=1}^{k} q_\ell(1) = 0; \\ &\sum_{\ell=1}^{k} q_\ell(z) \text{ not identically } 0; \\ &\alpha > 0. \end{aligned} \tag{E.18}$$

These conditions ensure that (E.17) is, in fact, the difference equation associated with a class of asymptotically singular $\mathcal{P}_k$-methods.

Let

$$\theta(\mu) = \mu^k - \sum_{\ell=1}^{k} a_\ell \, \mu^{k-\ell} \tag{E.19}$$

be the characteristic polynomial of the asymptotic form of (E.17). Consider the characteristic equation

$$\theta(\mu) = 0. \tag{E.20}$$

Evidently, it has a root $\mu_1 = 1$. We restrict our attention to cases where the remaining roots μ_i all have moduli less than 1, with $|\mu_i| \neq |\mu_j|$, for $i \neq j$. A theorem of Perron [12, p. 531] then asserts that there exists a fundamental set of solutions of (E.17), $P_m^{(i)}(z)$, $i = 1, \ldots, k$, such that (for fixed z),

$$\lim_{m \to \infty} \frac{P_{m+1}^{(i)}(z)}{P_m^{(i)}(z)} = \mu_i ; \tag{E.21}$$

clearly, for $i \neq 1$, $P_m^{(i)}(z) \to 0$ as $m \to \infty$. The behavior of $P_m^{(1)}(z)$ for $m \to \infty$ is obtained as a special case of a theorem of Aljancić [4]:

$$\frac{P_{m+1}^{(1)}(z)}{P_m^{(1)}(z)} = 1 + m^{-\alpha}[\theta'(1)]^{-1} \sum_{\ell=1}^{k} q_\ell(z) + o(m^{-\alpha}). \tag{E.22}$$

It follows from well-known properties of infinite products that for any z such that $P_m^{(1)}(z) \neq 0$ for some finite m, $P_m^{(1)}(z) \to$ constant $\neq 0$ for $\alpha > 1$, and that for $0 < \alpha \leq 1$, $P_m^{(1)}(z) \to 0$ if and only if

$$\mathrm{Re}\left[\theta'(1) \sum_{\ell=1}^{k} q_\ell(z)\right] < 0. \tag{E.23}$$

We have not established uniformity (with respect to z on compact sets) either in (E.21) or for $P_m^{(1)} \to 0$. This could be obtained from a reconsideration of the proofs of the cited theorems of Perron and Aljancić; however, since the criterion (E.23) depends only on $\Sigma\, q_\ell(z)$, the open set of summability cannot, in any case, exceed that of the corresponding $\mathcal{P}_1$-method for which convergence on any open set does indeed imply uniform convergence on compact subsets. For k = 1, equation (E.17) takes the simple form

$$P_m(z) = [1 + m^{-\alpha} q(z)]\, P_{m-1}(z), \tag{E.24}$$

with $q(0) = 0$; except for some trivial modifications, the choice $\alpha = 1$ gives Meir's method, referred to at the end of paragraph 1, with the open set of summability $\mathrm{Re}\, q(z) < 0$.

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13. ABSTRACT

The possibility that nonlinear acoustic flows may be represented by spherical progressive waves (in the sense of Courant and Friedrichs) was examined and found to be unlikely. An iterative finite-difference method for the calculation of continuous periodic spherical flows was developed together with a FORTRAN code, SPHERE, that implements the method. Sample calculations have shown that the code is effective but slow, and several ways for reducing the computation time are suggested. Convergence difficulties in one of the iterative loops were overcome by the use of a semi-iterative underrelaxation scheme. When applied to linear systems, such semi-iterative schemes were found to be equivalent to a class of summability methods that may be regarded as generalizations of Euler summation.

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