FOREWORD

This study was initiated by the Biophysics Laboratory of the 6570th Aerospace Medical Research Laboratories, Aerospace Medical Division, Wright-Patterson Air Force Base, Ohio. The research was conducted by the Electrical Engineering Laboratory, University of Illinois, Urbana, Illinois, under Contract No. AF 33(616)-6428. Prof. Heinz Von Foerster was the principal investigator. Major J. E. Steele of the Mathematics and Analysis Branch, Biodynamics and Bionics Division, was the contract monitor for the 6570th Aerospace Medical Research Laboratories. The work was performed in support of Project No. 7232, "Logical Structure and Function of the Nervous System," and Task No. 723205, "Mathematical Theory of Neural and Mental Processes." The research sponsored by this contract was started in March 1959 and completed in December 1961.

ABSTRACT

Systems capable of identifying objects must be able to detect the properties (invariants) these objects possess. It is shown that property detecting systems consisting of networks of linear or non-linear elements in parallel may be constructed without accurate point to point connections as long as certain constraints on the distribution of connectivity among elements in the network are satisfied.

The report discusses in three parts a general approach toward "property detection" and considers in particular networks capable of detecting acoustical invariants.

While the first part "Computation of Invariants in Linearly Interacting Continua" considers networks whose elements constitute a continuum, the second part "Kinds of Interaction in Sets of Discrete Linear Elements" treats discrete networks. In both papers the distinction is made between interaction phenomena within a network and action phenomena from one network to another network. It is shown that under fairly general conditions action and interaction systems are equivalent. The third part "Simulation of Interaction Functions on PACE Analog Computer" is concerned with the simulation of some action and interaction phenomena on an analog computer.

PUBLICATION REVIEW

This technical documentary report has been reviewed and is approved.

Jos M Quashnock
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INTRODUCTION

When in 1947 McCulloch and Pitts published their paper "How We Know Universals"[1] the ideas presented in this article were generally considered to be very interesting from a theoretical point of view but highly speculative from a neurophysiological standpoint. It was not easy to believe that nerve nets which grow apparently haphazardly should form well defined connectivities which are capable of complex parallel computations which would render with high reliability in the set of all stimuli certain "abstractions" which are "meaningful" - i.e. have higher survival value - for the living organism endowed with such nerve nets. After a series of brilliant experiments Lettvin et al. were able to present unmistakable proof of the existence of physiological nets which extract certain visual invariants in the set of all optical stimuli in the visual field of the frog and reported their findings in the now celebrated article "What the Frog's Eye Tells the Frog's Brain"[2]. At about the same time at the occasion of the winter conference of the AIEE in New York, Von Foerster presented the nucleus of a theory[3] which allows the calculation of those logical or arithmetic operations to be carried out by topological neighbor elements which compute certain desired invariants. As can be shown, accurate point to point connection is in many cases not necessary in order to obtain reliable operation of such nets, only a certain "tendency" of growth is required to guarantee reliable functioning of such structures[4].

Although in many special cases the powerful approach of parallel computation led to the discovery of a series of "properties" which could be detected by such networks, a more general approach to their structure was still lacking. The following three papers which constitute this report represent an attempt to expand the idea of "neighborhood interaction" and to put it on a more general basis. The first paper by Inselberg and Von Foerster dismisses cellular distinguishability and treats "neighborhood" from an analytic point of view. Second, Yeh, presents an interesting theorem of equivalence which holds between discrete action and interaction systems. Some examples of discrete action and interaction functions are given by Cheng who programmed these functions into a PACE analog computer.

We are today not yet in the possession of a general theory of linear, as well as non-linear property detector fields and nets. However, it is hoped that the work presented on the following pages may be of some help to the designer as well as to the theoretician who wants to construct nets which perform certain desired operations.

PART I

Computation of Invariants in Linearly Interacting Continua

by

Alfred Inselberg and Heinz von Foerster

1. INTRODUCTION

It is generally accepted practice to consider the phenomenon of hearing as the result of two kinds of processes, one mechanical, and the other electro-chemical, with the cochlea as the locus - so to say - of the important transformation from the one kind into the other.

Since we shall be mostly concerned with problems connected with the transfer of information from environmental sound until its perception and with the possibilities of the realization of this information transfer in a series of electronic systems, we shall forego this important distinction which is justified by the anatomy and physiology of living organisms and shall adopt instead a distinction which is based on whether the processes under consideration takes place in a given unalterable structure (pre-organization), or the structure will change as a result of these processes (self-organization).

In the following chapter we will restrict ourselves mainly to some deterministic features of such systems, drawing analogies to physiological systems whenever such analogies may serve as clues for the construction of the corresponding electronic device.

In the second chapter we will discuss the notion of a sensory layer and shall distinguish between layers composed of discrete elements and layers where the stimulus and response of the layer are given as point functions. We shall furthermore distinguish between action phenomena resulting from the stimulus activity of one layer upon another layer and interaction phenomena where the stimulus to an element in the layer may come from an element located in the same layer.

The subsequent chapters will deal with some special sensory layers; first, a frequency sensitive layer composed of a series of resonators each tuned to a different frequency, and secondly, with computational nets which perform sharpening transformations on the response of the frequency sensitive layers.

2. DETERMINISTIC SYSTEM

The most general representation of a deterministic system is by an asynchronous sequential machine.[5] In the particular case of our interest, however, a satisfactory description can be given in terms of a series of synchronous sequential machines.

Each machine will have a finite number of input states $s_o, \ldots, s_m$, a finite number of output states $s_{m+1}, \ldots, s_{m+p}$ and a finite number of internal states $q_o, \ldots, q_n$. A discrete time scale is assumed. The set of all input states to the machine will be denoted by Σ_1 and the set of output states by Σ_2. The output state s_s, at any given time, is determined by the pair (s_i, q_r) where $s_i \in \Sigma_1$ and q_r is the internal state of the machine at that time. Furthermore, the internal state of the machine, at a given time, depends only on its internal state at the previous time and the previous input symbol (see Figure 1).

In analogy to the physiological apparatus, our system will be partitioned into three parts as shown in Figure 2. The first part is supposed to represent the outer ear and middle ear, having a set of input states Σ_{11}, a set of output states Σ_{12} and a set of internal states $q_{11}, \ldots, q_{i\nu_1}$. The second sequential machine, representing the cochlea has Σ_{21} as its set of inputs, Σ_{22} as its set of outputs with internal $q_{21}, \ldots, q_{2\nu_2}$. The third part represents the nerve connections between the cochlea and the "auditory cortex". Finally there is a box labelled "Brain" indicating an analyzer for the stimuli from other sensory organs. Provision has been made for feedback loops from the "Brain" to the three lumped systems, as described above, as well as between these systems themselves. These loops may effectively change the internal states of one or all of the three parts.

In order to describe the environment of this system we shall postulate the existence of a number of external states Q_j, where j belongs to some index set Λ. An event e_i will be defined as a combination of the Q_j's:

$$e_i = (Q_{i_1}, \ldots , Q_{i_n}, \ldots)$$

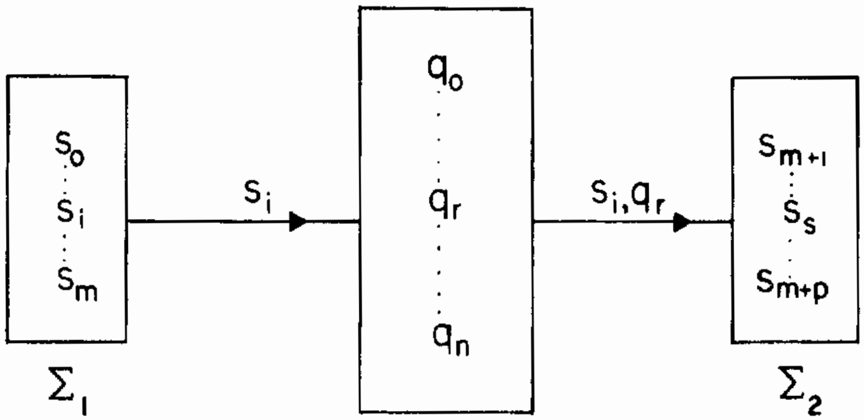


FIGURE 1. SEQUENTIAL MACHINE. WHEN THE INTERNAL STATE IS q_r AND THE INPUT IS s_i THE OUTPUT WILL BE s_s.

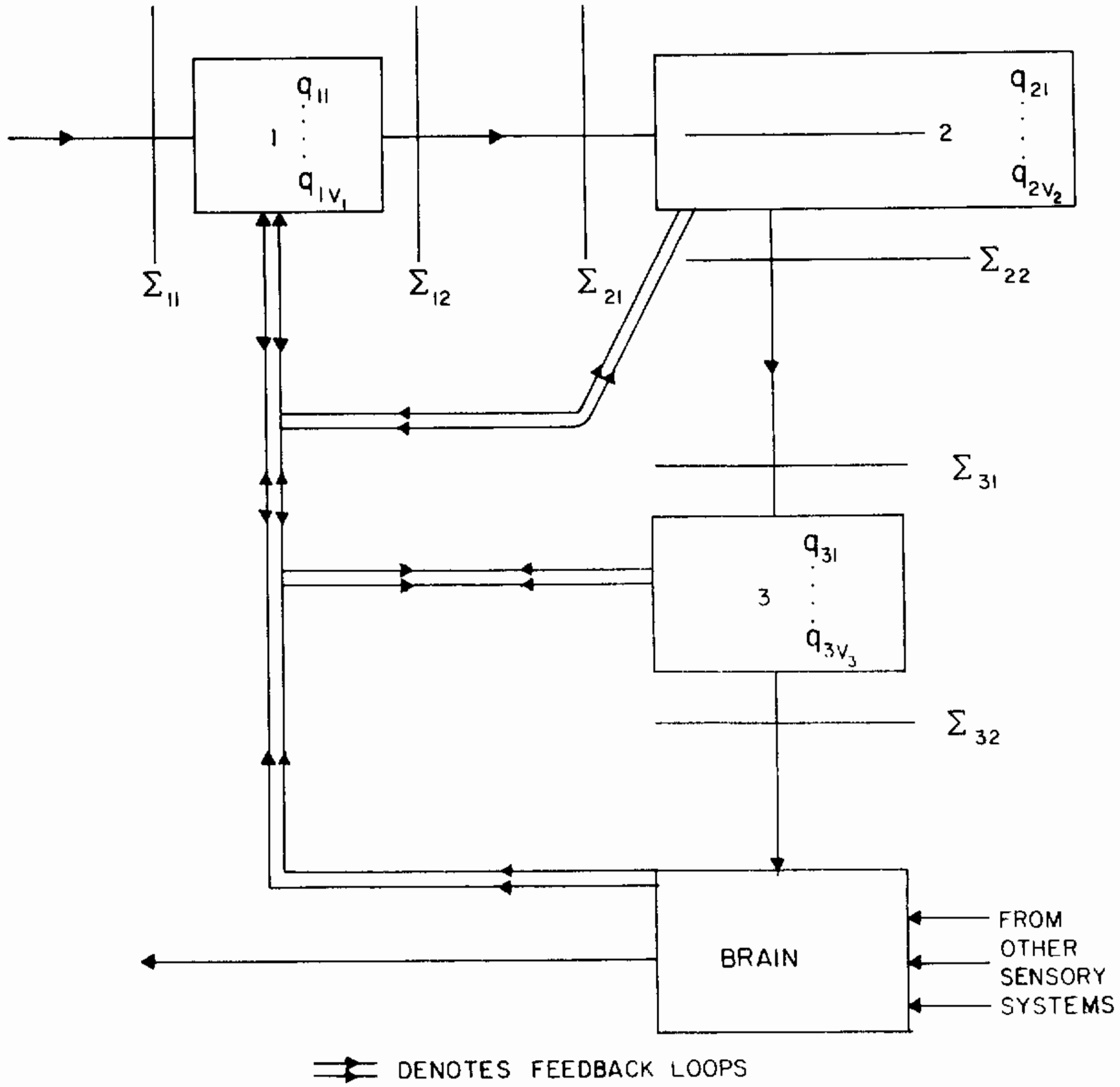


FIGURE 2. SCHEMATIC OF ACOUSTICAL INFORMATION FLOW

while environment will be defined by the set

$$E = \left\{ e_i \,|\, e_i = (Q_{i_1}, \ldots, Q_{i_n}, \ldots) \right\}$$

The event e_i contains a sound signal when there exists at least one i_n, where Q_{i_n} is one of the formative states of e_i, such that $Q_{i_n} \in \Sigma_{11}$.

An event e_i may be interpreted as a time sequence if its states are arranged so that the state Q_{i_n} was reached at time t_{i_n} and the state $Q_{i_{n+1}}$ at time $t_{i_{n+1}}$ where $t_{i_n} \leq t_{i_{n+1}}$. If e_i is a time sequence and if it contains a sound signal, each of the states of the sound signal will act as an input state to the system in the order in which these states appear in the time sequence.

Drawing an analogy from the mammalian acoustical organ where some data-reduction is believed to occur over the information flow, we introduce the following assumptions which also make our system data-reducing

(1) $\Sigma_{21} \supset \Sigma_{1,\ i+1}$ and $\Sigma_{21} \neq \Sigma_{1,i+1}$

(2) There exists an element θ in Σ_{1i} and Σ_{2i} such that

(a) $(\theta, q_{ij}) \rightarrow \theta$ for any j, where $1 \leq j \leq \nu_i$

(b) There exists s_{1ij} and $q_{i\alpha}$, where $1 \leq \alpha \leq \nu_i$

$s_{1ij} \neq \theta$ and s_{1ij} is the j-th input state of the i-th machine, such that

$$(s_{1ij}, q_{i\alpha}) \rightarrow \theta$$

(c) If $s_{2ij} \in \Sigma_{21}$ but $s_{2ij} \notin \Sigma_{2,i+1}$ then

$$(s_{2ij}, q_{i+1,\beta}) \rightarrow \theta \text{ for any } \beta \text{ where } 1 \leq \beta \leq \nu_{i+1}$$

(d) If at time t_{i_n} the internal state of the i-th machine is q_{ij} and the input state is θ then the internal state at the next discrete time interval $t_{i_{n+1}}$ will be still q_{ij}.

The element θ will be called the null element because of property 2(d).

The description of our system in terms of sequential machines is advantageous for the following reasons:

(1) It points out that the manner of operation of our system, whether it is mechanical or electrochemical, is immaterial. It will be sufficient to know that there exists a well-specified response to a well-defined stimulus.

(2) The description is flexible enough to permit the introduction of data-reduction. In addition, even though only successive single inputs are treated the system can accommodate simultaneous inputs if the need arises. This may be done by redefining the input and output states as n-tuples with state parameters as elements of the n-tuples.

(3) Parts of the system can be isolated and studied much as a "free body" is studied in mechanics.

3. LINEAR INTERACTIONS IN CONTINUA

Let a field act upon a continuum of elements which perform linear operations on their input stimulations. Within the continuum L, each element has a specified connectivity with all other elements in L.

The input stimulations from the field acting upon an element $p \in L$ will be denoted by $\sigma(p)$ and the response of the element p will be called $\rho(p)$. In general, an element in L may receive stimulations from other elements in L and may furthermore provide stimulations to other elements in L including itself (see Figure 3).

We shall assume that L is a measurable set over some measure function μ. The interaction function $K_1(p,q)$ where $p,q \in L$ is defined such that

$$K_1(q,p)\,\rho(q) \tag{2.1}$$

is the amount of stimulation received by p from q. If p is a fixed element of L and q is permitted to range over all of L, then the total amount of stimulation received by p will equal its response, therefore

$$\rho(p) = \kappa\sigma(p) + \lambda \int_{q\in L} K_1(q,p)\,\rho(q)\,d\mu \tag{2.2}$$

where κ is an amplifying factor for $\sigma(p)$ and λ an amplifying factor for the stimulation received by p from all other elements in q.

A word of caution is in order concerning Equation (2.2).

1) It is tacitly assumed that $K_1(q,p)\rho(q)$ is integrable

2) Equation (2.2) implies that the elements of L are non-conservative.[5]

When L is in an n-dimensional vector space and μ is the ordinary Lebesgue measure (2.2) becomes:

$$\rho(\vec{x}_o) = \kappa\,\sigma(\vec{x}_o) + \lambda \int_L K_1(\vec{x}, \vec{x}_o)\,\rho(\vec{x})\,d\vec{x} \tag{2.3}$$

where $\vec{x}_o = (x_1^{(o)}, \dots, x_n^{(o)})$ are the coordinates of p and $\vec{x} = (x_1, \dots, x_n)$ are the coordinates of q.

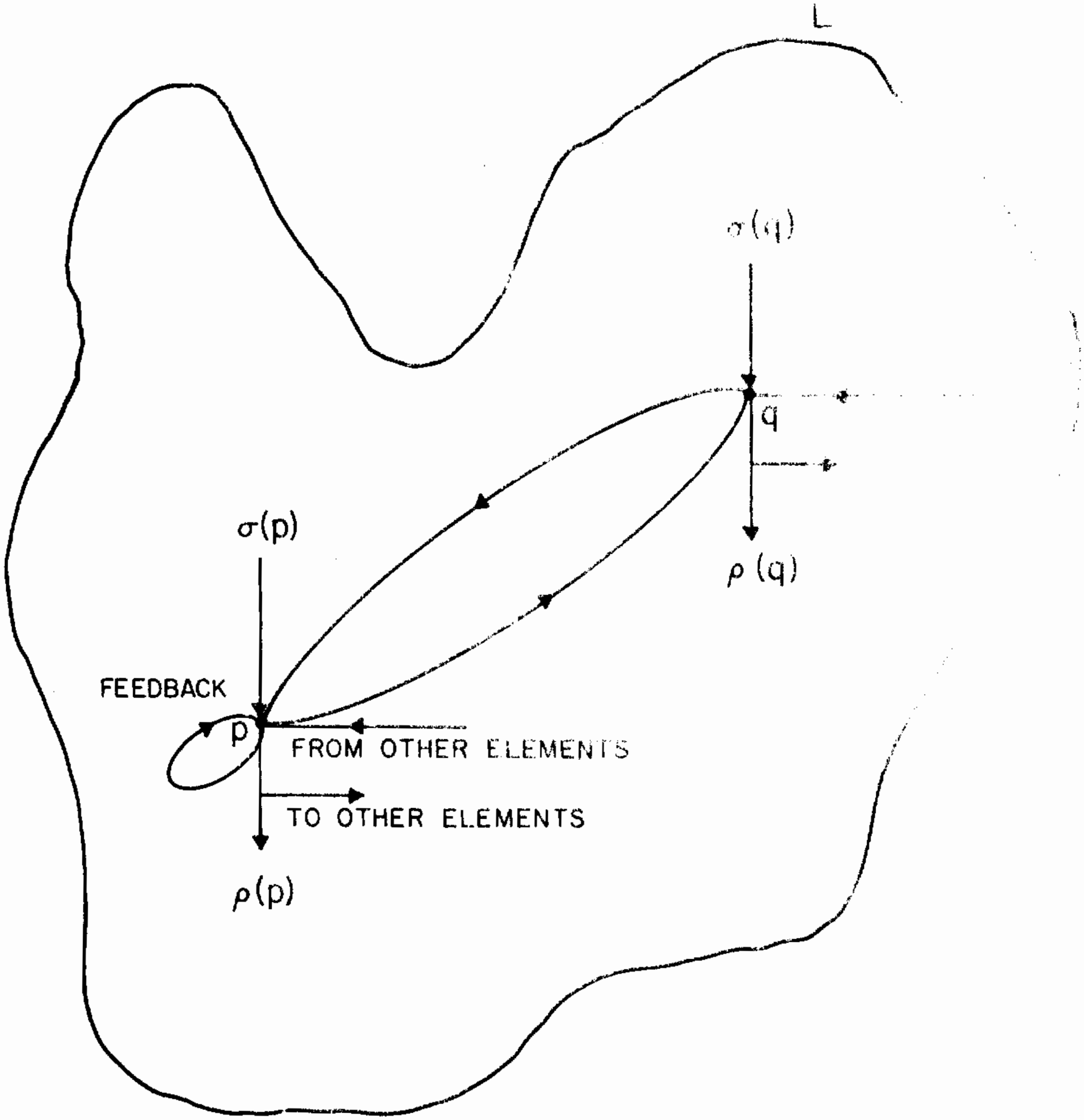


FIGURE 3. INTERACTION IN AN N-DIMENSIONAL CONTINUUM

Equation (2-2) is an integral equation of the second kind and may be solved for some specific choice of kernels $K_1(\vec{x}, \vec{x}_o)$. We shall solve this equation for $\rho(x)$ for an infinite one-dimensional layer where $K_1(\vec{x}, \vec{x}_o) = k_1(x_o - x)$. The choice of $k_1(x_o - x)$ includes the cases where the kernel is a function of the distance between the points x and x_o and furthermore makes the problem amenable to a mathematical treatment. With these assumptions Equation (2-3) becomes

$$\rho(x_o) = \kappa\, \sigma(x_o) + \lambda \int_{-\infty}^{\infty} k_1(x_o - x)\rho(x)\, dx \; ; \qquad (2\text{-}4)$$

the solution of this equation is given by:

$$\rho(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\kappa\, \Sigma(u)}{1 - \sqrt{2\pi}\, \lambda\, K_1(u)}\, e^{-ixu}\, du \, , \qquad (2\text{-}5)$$

where $\Sigma(u)$ and $K_1(u)$ are the Fourier integral transforms of $\sigma(x)$ and $k(t)$, $t = x_o - x$, respectively, i.e., $\Sigma(u) = \int_{-\infty}^{\infty} \sigma(x)\, e^{ixu}\, dx$ and $i^2 = -1$. The simplest conditions for which this solution is valid are given by the following theorem[6].

Theorem 1.

Let $\kappa\, \sigma(x) \in \mathcal{L}^2(-\infty, \infty)$ and $k(x) \in \mathcal{L}(-\infty, \infty)$ and let the upper bound of $\lambda\, K_1(u)$ be less than $1/\sqrt{2\pi}$. Then (2-5) gives a solution of (2-4) of the class $\mathcal{L}^2$, and any other solution of $\mathcal{L}^2$ is equal to it almost everywhere. Note: We say $f(x) \in \mathcal{L}^p(a, b)$ if f(x) is measurable and

$$\int_a^b |f(x)|^p\, dx < \infty \, .$$

As an example of a one-dimensional layer of sensory elements one may consider the basilar membrane with its lining of hair cells. Under the assumption that the essential information with respect to the acoustical environment is coded in terms of displacement as a function of distance along the basilar membrane's lengthwise extension from the basal to the apical end, neglecting displacements perpendicular to this direction, the problem of analysis becomes a one-dimensional one.

We shall now return to our "amorphous" continuum L and we will introduce a slightly less general structure to it.

Let L be separated into two subsets L_1 and L_2 such that

(1) $L = L_1 \cup L_2$,

(2) Each element $q \in L_1$ receives an excitation $\sigma_1(q)$ from a field acting on L_1. Furthermore the elements of L_1 are not interconnected and the response $\rho_1(q)$ of $q \in L_1$ is solely a function of $\sigma_1(q)$.

(3) Each element $p \in L_2$ is connected to every element in L_1 and receives from an element $q \in L_1$ an excitation equal to $K_1(q,p)\ \rho_1(q)$, where $K_2(q,p)$ is a specified function. (See Figure 4).

(4) (i) L_1 is a measurable set with respect to a measure function μ_1

(ii) $K_2\ (q,p)\ \rho_1(q)$ is an integrable function relative to μ_1.

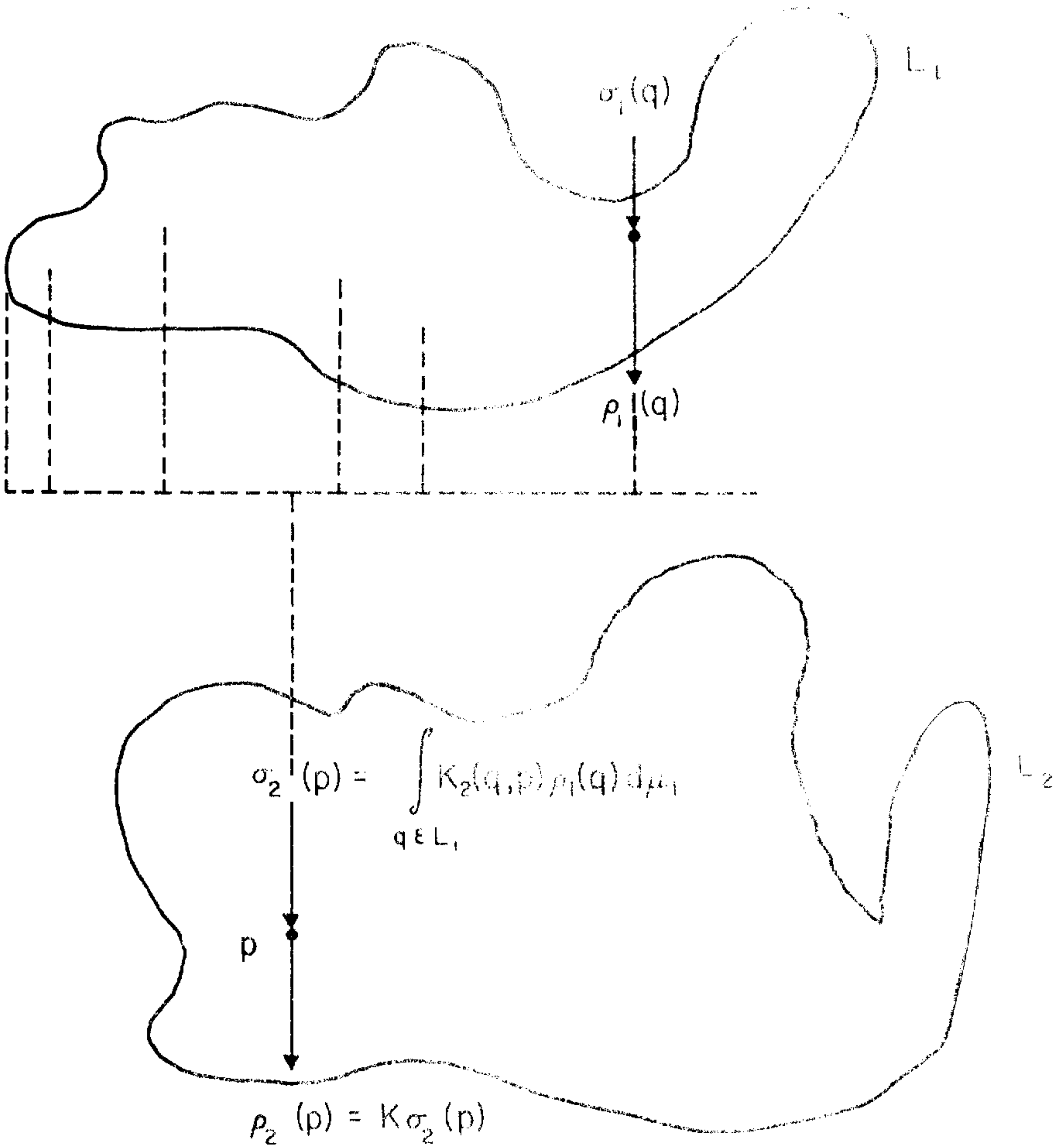
Under these conditions the excitation which $p \in L_2$ receives is given by

$$\sigma_2(p) = \int_{q \in L_1} K_2(q,p)\ \rho_1(q)\ d\mu_1\ . \tag{2-6}$$

If p simply amplifies by a factor κ its incoming stimulus we have:

$$\rho_2(p) = \kappa\, \sigma_2(p) = \kappa \int_{q \in L_1} K_2(q,p)\ \rho_1(q)\ d\mu_1\ , \tag{2-7}$$

Again if L is immersed in an n-dimensional vector space (2-7) becomes



NOTE: L_1 AND L_2 ARE NOT PHYSICALLY SEPARATED

FIGURE 4. DIRECTED ACTION IN AN N-DIMENSIONAL CONTINUUM

$$\rho_2(\vec{x}_o) = \kappa \int_{L_1} K_2(\vec{x}, \vec{x}_o)\, \rho_1(\vec{x})\, d\vec{x} , \tag{2-8}$$

$\vec{x} = (x_1, x_2, \ldots , x_n)$ and $\vec{x}_o = (x_1^{(o)}, x_2^{(o)}, \ldots , x_n^{(o)})$ being the coordinate systems of L_1 and L_2 respectively.

We shall refer to the cases where (2-7) holds as action phenomena in order to distinguish them from the general interaction phenomena governed by Equation (2-2). The function $K_2(q,p)$ will be called the action function between L_1 and L_2.

The reader will have no difficulty visualizing the partition of L into ν subsets where two consecutive subsets L_i and L_{i+1}, $i \leq \nu-1$, having properties analogous to L_1 and L_2 described above. Then,

$$\rho_{i+1}(\vec{x}_{i+1}) = \kappa_{i+1} \int_{L_i} K_{i,i+1}(\vec{x}_i, \vec{x}_{i+1})\, \rho_i(\vec{x}_i)\, d\vec{x}i , \tag{2-9}$$

where $\rho_{i+1}(\vec{x}_{i+1})$ is the response of an element in L_{i+1} with coordinates x_{i+1}, $\rho_i(\vec{x}_i)$ is the response of an element in L_i with coordinates x_i, κ_{i+1} a constant and $K_{i,i+1}(\vec{x}_i, \vec{x}_{i+1})$ the action function from L_i to L_{i+1}.

Using (2-9) we may derive $\rho_i(x_i)$ as a function of $\rho_j(\vec{x}_j)$, $j < i$ and $K_{ij}(\vec{x}_i, \vec{x}_j)$, the action function from L_i to L_j, to be defined below. Proceeding by

$$\rho_i(\vec{x}_i) = \kappa_i \int_{L_{i-1}} K_{i,i-1}(\vec{x}_i, \vec{x}_{i-1})\, \rho_{i-1}(\vec{x}_{i-1})\, d\vec{x}_{i-1} =$$

$$= \kappa_i \int_{L_{i-1}} K_{i,i-1}(\vec{x}_i, \vec{x}_{i-1}) \left\{ \kappa_{i-1} \int_{L_{i-2}} K_{i-1,i-2}(\vec{x}_{i-1}, \vec{x}_{i-2})\, \rho_{i-2}(\vec{x}_{i-2})\, d\vec{x}_{i-2}\, d\vec{x}_{i-1} \right\} =$$

$$= \ldots = \kappa_i\, \kappa_{i-1} \ldots \kappa_j \int_{L_{i-1}} \int_{L_{i-2}} \ldots_{L_j} K_{ij}(\vec{x}_i, \vec{x}_j)\, \rho_j(\vec{x}_j)\, d\vec{x}_{i-1} \ldots d\vec{x}_j \tag{2-10}$$

where

$$K_{ij}(\vec{x}_i, \vec{x}_j) = K_{i,i-1}(\vec{x}, \vec{x}_{i-1}) \ldots K_{j+1,j}(\vec{x}_{j+1}, \vec{x}_j) , \qquad (2\text{-}11)$$

Returning to Equation (2-8) this equation may be solved for $\rho_1(x)$ if $\rho_2(x_o)$ is known and for some specific cases of the kernel $K_2(x,x_o)$. We shall again work in an infinite one-dimensional layer with $K_2(x,x_o) = k(x_o - x)$. Equation (2-8) becomes:

$$\rho_2(x_o) = \kappa \int_{-\infty}^{\infty} k_2(x_o - x)\, \rho_1(x)\, dx ; \qquad (2\text{-}12)$$

the solution of this equation is given by:

$$\rho_1(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{P_2(u)}{k\, K_2(u)} e^{ixu}\, du , \qquad (2\text{-}13)$$

where $P_2(u)$ and $K_2(u)$ are the Fourier integral transforms of $\rho_2(x_o)$ and $k(x-x_o)$ respectively.

We also have the theorem :

Theorem 2[2']

Let $\rho_2(x) \in \mathcal{L}^2(-\infty, \infty)$ and $k(x) \in \mathcal{L}(-\infty, \infty)$. Then in order that there should be a solution $\rho_1(x)$ of $\mathcal{L}^2(-\infty, \infty)$, it is necessary and sufficient that $P_2(u)/_\kappa K_2(u)$ should belong to $\mathcal{L}^2(-\infty, \infty)$.

Having defined action and interaction phenomena it is instructive to see when interaction and action are equivalent. By equivalence we mean that given $\sigma(q)$ and $K_1(q,p)$ in Equation (2-2) and $\rho_1(q)$ and $K_2(q,p)$ in Equation (2-7) how should $K_1(q,p)$ and $K_2(q,p)$, the structural properties of the two systems, be related in order that $\rho_1(p)$ obtained from (2-2) and $\rho_2(P)$ obtained from

Equation (2-7) are equal.

In order to facilitate the mathematics involved we shall again work in an infinite one-dimensional layer with $K_1(x, x_o) = k_1(x_o - x)$ and $K_2(x, x_o) = k_2(x_o - x)$. We shall also suppress the constants κ and λ which have no importance in the desired result. For easy reference we restate the equations

$$\rho_1(x_o) = \sigma(x_o) + \int_{-\infty}^{\infty} k_1(x_o - x)\, \rho\,(x)\, dx \tag{2-14}$$

$$\rho_2(x_o) = \int_{-\infty}^{\infty} k_2(x_o - x)\, \rho_1(x)\, dx \tag{2-15}$$

Taking Fourier transforms of both sides of Equation (2-14) we obtain, where the capitalized letters stand for the transforms of the respective functions,

$$P(u) = \frac{\Sigma(u)}{1 - \sqrt{2\pi}\, K_1(u)} , \tag{2-16}$$

Similarly for (2-15)

$$K_2(u) = \frac{1}{\sqrt{2\pi}} \frac{P_2\ (u)}{P_1\ (u)} , \tag{2-17}$$

When $\sigma(x) = \rho_1(x)$ and $\rho_2(x) = \rho(x)$ Equation (2-17) yields

$$K_2(u) = \frac{1}{\sqrt{2\pi}} \frac{P\ (u)}{\Sigma\ (u)} . \tag{2-18}$$

Substituting (2-16) into (2-18) we have:

$$K_2(u) = \frac{1}{\sqrt{2\pi}} \frac{1}{1 - \sqrt{2\pi}\, K_1(u)} . \tag{2-19}$$

This establishes the following theorem:

<u>Theorem 3:</u> (Equivalence)

Under the conditions of Theorems 1 and 2 the interaction phenomenon defined by (2-14) and the action phenomenon defined by (2-15) are equivalent if and only if (2-19) holds.

While the conditions for which equivalence was established are rather restrictive, it is comforting to know that equivalence is possible for the aforementioned special, but important, case. At the present an effort was being made to find more general conditions for equivalence.

It is pertinent at this point to discuss the use of the derived results. It has been pointed out[4] that action and interaction fields may be used either by themselves or in conjunction with other devices (i.e. thresholds) to discover some properties of a stimulus activity pattern. It is therefore desirable to examine action and interaction fields from a general point of view. In addition by establishing equivalence, in certain cases, a particular property detector field may be built either utilizing the interaction principles or action principles depending on which is easier and more economical

In the subsequent chapters we will apply the results of this chapter to some problems connected with the analysis of sound.

4. COMPUTATIONAL FIELDS

By computational fields we shall mean continua which perform sharpening transformations upon a given stimulus pattern. We shall begin our search for such fields by defining the notion of sharpening.

Let $C_i^{(1)}$ be the class of all real valued functions f(x) defined on a subset of an n-dimensional vector space which have a first partial derivative in the variable x_i in a region R.

Definition 1 (Sharpening):

Let $f(\vec{x}) \in C_i^{(1)}$ let T be a transformation of $C_i^{(1)}$ into itself, e.g. T: $C_i^{(1)} \rightarrow C_i^{(1)}$ and let $f_T(\vec{x})$ be the image of $f(\vec{x})$ under T. We say that $f_T(\vec{x})$ is an i-sharpening of $f(\vec{x})$ in the set $a_i \leq x_i \leq b_i$ if

$$\left|\frac{\partial f_T(\vec{x})}{\partial x_i}\right| \geq \left|\frac{\partial f(\vec{x})}{\partial x_i}\right| \qquad a_i \leq x_i \leq b_i$$

with equality holding only on a set of Lebesgue measure zero.

We immediately see from this definition that sharpening is a local phenomenon. To illustrate this point consider the two curves in Figure 5. Within [a,b], $f_T(x)$ is sharper than f(x) while outside [a,b] the opposite is true.

One disadvantage of our definition is that we can only discuss functions which have a first derivative in some interval while seemingly we can say nothing about functions for which this is not true. We can enlarge the class of functions under consideration for at least the one dimensional case by using the terminology of Schwartz Distributions.

Definition 2 (Testing Function):

The functions $\phi(x)$ which are continuous and have continuous derivatives of all orders and vanish identically outside some finite interval will be called testing functions.

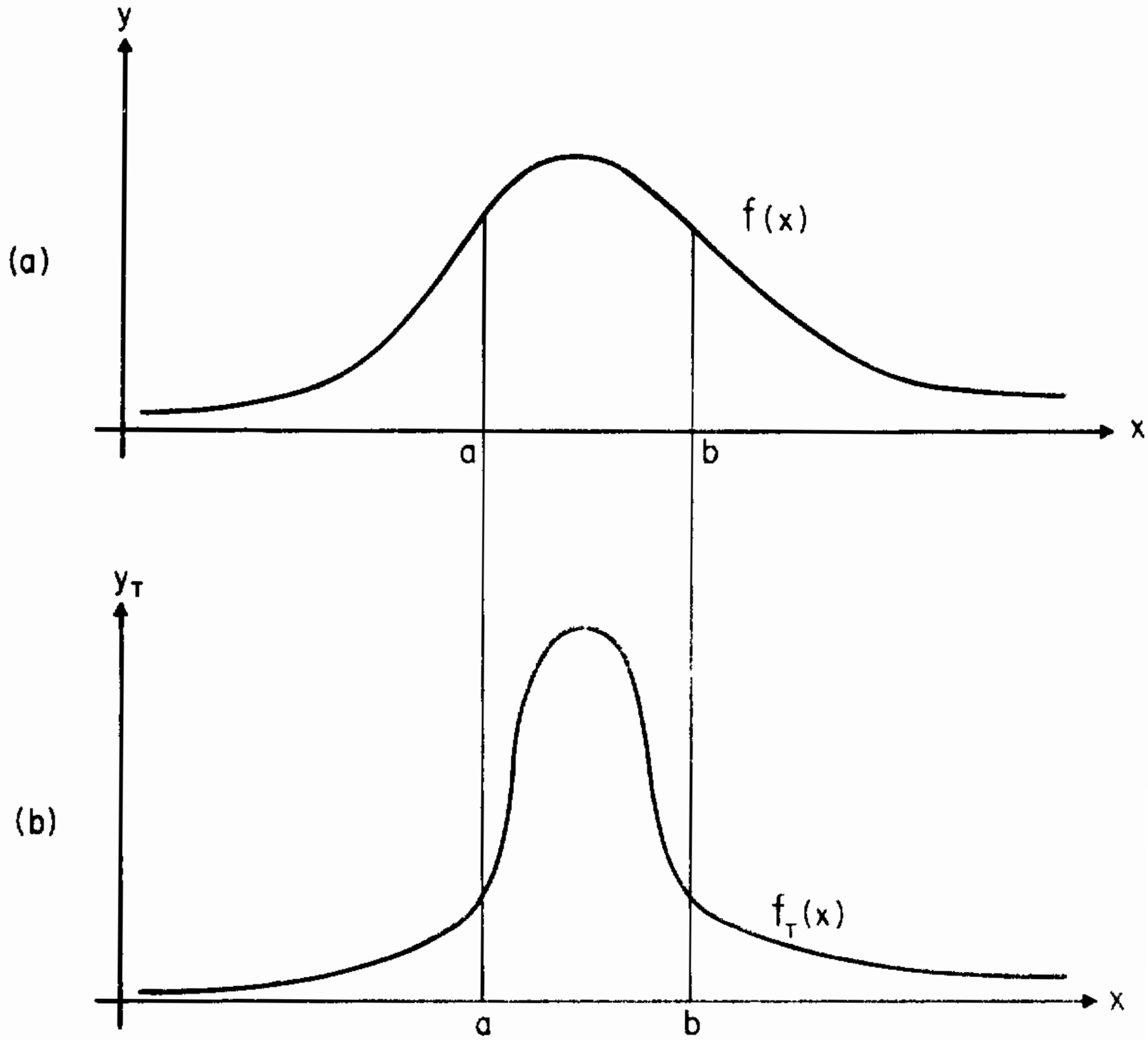


FIGURE 5. SHARPENING IS A LOCAL PHENOMENON

Definition 3 (Linear Functional):

$F(\phi)$ is a linear functional if to every testing function $\phi(x)$ a real or complex number $F(\phi)$ is assigned such that

$$F(\phi_1 + \phi_2) = F(\phi_1) + F(\phi_2)$$

$$F(c\phi) = c\,F(\phi)$$

where c is any scalar.

We say a sequence of testing functions $\phi_n(x)$ converges to zero if the functions $\phi_n(x)$ and all their derivatives converge uniformly to zero and if all the functions $\phi_n(x)$ vanish identically outside the same finite interval. A functional $F(\phi)$ is said to be continuous if the sequence of numbers $F(\phi_n)$ converges to zero whenever the sequence of testing functions $\phi_n(x)$ converges to zero.

Given any continuous linear functional $F(\phi)$ on the space of testing functions we shall introduce a functional symbol, say s(x), and put

$$\int_{-\infty}^{\infty} s(x)\,\phi(x)\,dx = F(\phi)$$

We shall say s(x) is a symbolic function.

The first derivative of s(x) will be defined by

$$\int_{-\infty}^{\infty} s'(x)\,\phi(x)\,dx = -\int_{-\infty}^{\infty} s(x)\,\phi'(x)\,dx \tag{3-1}$$

The advantage of this definition is that in case s(x) has a derivative in the ordinary sense it may be obtained from (3-1) while for certain cases where s(x) does not have an ordinary derivative, Equation (3-1) will yield a "derivative" for s(x). We shall call such derivative a symbolic derivative.

We shall now return to our primary objective, that of finding sharpening transformations. The discussion will be restricted to the one-dimensional case for most of what we will say carries an obvious generalization to n-dimensions.

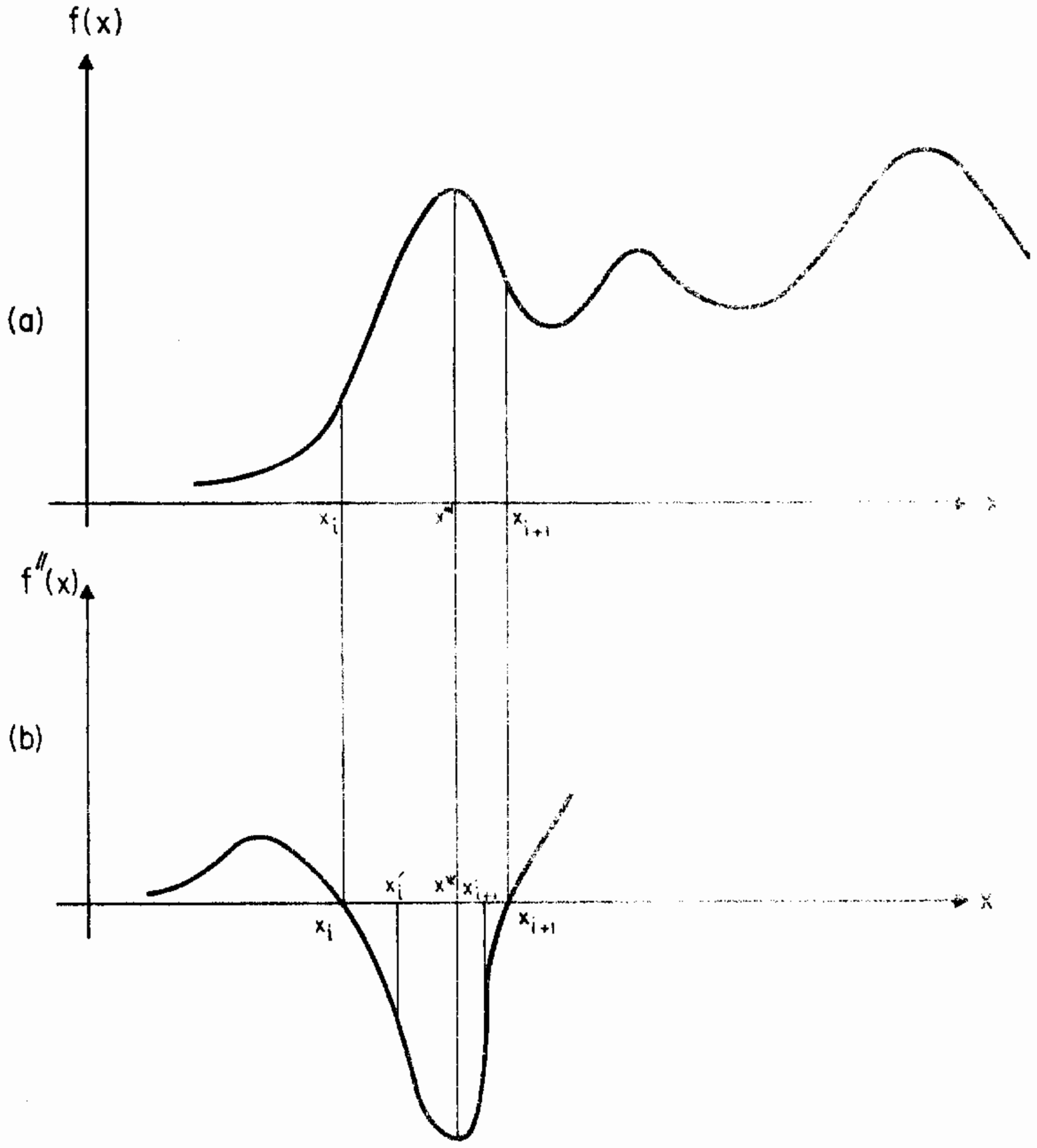


FIGURE 6. "SHARPENING" BY DOUBLE DIFFERENTIATION

The transformation, T, such that $f_T(x) = cf(x)$ with $| c | > |$, is perhaps the simplest way of sharpening a function. For easy reference we will call such a transformation amplification.

Another interesting suggestion for a sharpening transformation is one by Huggins and Licklider[7]. They indicated that some curves may be sharpened by a process of differentiation or double differentiation. In particular they used the polynomial approximation

$$f''(x) \Big|_{x=i} \approx 2\, f(i) - [\, f(i+1) + f(i-1)\,]$$

for the second derivative. Independently H. Von Foerster arrived at similar conclusions and applied these results to some problems relating to optical property filters[4].

If we combine amplification together with the operation of taking even derivatives e.g. $f_T(x) = c\, f(2n)(x)$, some $|c| > 1$ we obtain a sharpening around the maxima and minima of a certain class of functions. If $n = 1$ this class of functions is composed of all the continuous functions having $\xi \geq 2$ inflection points and possessing second derivatives everywhere. This is seen by observing that the maxima and minima of such functions are in between pairs of inflection points. Furthermore the second derivative of such functions will be zero at the inflection points and hence amplification of the second derivative will leave $f''(x_i)$ unchanged, where $(x_i, f(x_i))$ is an inflection of $f(x)$, while enlarging all other $f''(x)$.

For $n = 2$, let $(x^*, f[x]^*)$ be a local maximum or local minimum of a function $f(x)$ having $\xi \geq 2$ inflection points and possessing a fourth derivative everywhere. Let $(x_i, f(x_i))$ be the inflection point immediately to the left and $(x_{i+1}, f(x_{i+1}))$ the inflection points immediately to the right of

$(x^*, F(x^*))$ (See Figure 6). Let $(x'_i, f''(x_i))$ and $(x'_{i+1}, f''(x'_{i+1}))$ be the inflection points of $f''(x)$ immediately adjacent to $(x^*, f''(x^*))$. If $[x'_i, x'_{i+1}] \subset [x_i, x_{i+1}]$ and $[x'_i, x'_{i+1}] \neq [x_i, x_{i+1}]$ then $c\, f^{(4)}(x)$ will be a sharpening of $f(x)$ in $[x^*_1, x'_{i+1}]$ where c is such that $|cf^{(4)}(x^*)| > |f''(x^*)|$.

In general then if $[(x^*_1, f(x^*_1)), \ldots, (x^*_\mu, f(x^*_\mu)]$ are the local maxima and local minima of a curve having 2n-th derivative everywhere, let $(x_i^{(j)}, f^{(2j)}(x_i^{(j)}))$ and $(x_{i+1}^{(j)}, f^{(2j)}(x_{i+1}^{(j)})$ be the inflection points adjacent on the left and the right respectively of $(x^*_i, f^{(2j)}(x^*_i))$ where $1 \leq j \leq \nu \leq n$ and $1 \leq i \leq \mu$. If $[x_i^{(j)}, x_{i+1}^{(j)}] \subset [x_i^{(j-1)}, x_i^{(j-1)}]$ with proper inclusion for all $1 \leq j \leq \nu$ then for an appropriate choice of c, $cf^{(2\nu)}_{(x)}$ will be a sharpening of $f(x)$ in $[x_i^{(\nu)}, x_{i+1}^{(\nu)}]$.

In order to judge the effectiveness of amplification together with double differentiation the reader is referred to Appendix C.

It remains to discuss for what kernel an action layer will compute even derivatives of a given stimulus. Hence we will have to solve the equation:

$$\sigma^{(2n)}(x_o) = \int_{-\infty}^{\infty} K(x, x_o)\, \sigma(x)\, dx \tag{3-2}$$

for $K(x, x_o)$.

Equation (3-2) defines the symbolic function[8]

$$K(x, x_o) = \delta^{(2n)}(x - x_o) \tag{3-3}$$

where $\delta(x - x_o)$ is the Dirac delta function such that

$$\int_{-\infty}^{\infty} \varphi(x)\, \delta(x - x_o)\, dx = \varphi(x_o)$$

for an arbitrary continuous function $\varphi(x)$. The symbolic function $\overset{(2n)}{\delta}(x - x_o)$ is the 2n-th derivative of $\delta(x - x_o)$.

If we would like to find an interaction field which computes the 2n-th derivative of a given continuous function $\sigma(x)$ we will have to solve the equation:

$$\sigma(x_o) - \sigma^{(2n)}(x_o) = \int_{-\infty}^{\infty} K(x, x_o)\, \sigma^{(2n)}(x)\, dx \qquad (3\text{-}4)$$

for $K(x, x_o)$. It is well to point out at this point that we cannot use the equivalence relation (2-19) because we do not yet know if $K(x, x_o)$ satisfies the conditions of Theorem 1.

If $K(x, x_o)\, \sigma^{(2\nu)}(x)$, $1 \leq \nu \leq n$, vanishes at the endpoints we can apply integration by parts to (3-4) and obtain

$$\sigma(x_o) - \sigma^{(2n)}(x_o) = \int_{-\infty}^{\infty} K^{(2n)}(x, x_o)\, \sigma(x)\, dx \qquad (3\text{-}5)$$

By inspection one solution of (3-5) is

$$K^{(2n)}(x, x_o) = \overset{(2n)}{\delta}(x - x_o) - \delta(x - x_o) \qquad (3\text{-}6)$$

Since (3-6) involves one more term than (3-3) we can see that it will be more economical to build the action field.

5. FREQUENCY SENSITIVE FIELD

An array of electrical resonating circuits each tuned to a different frequency was described elsewhere[9]. Here we shall interpret such an array as an action field.

If ω stands for the frequency of an element q in a continuum L_1 and ω_o stands for the frequency of a resonating element p in a continuum L_2 (see Figure 4) then the kernel of the action field will be:

$$K(q,p) = \frac{1}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} \right)^2}} \tag{4-1}$$

where Q is the damping constant of element p.

Because (4-1) is somewhat unwieldy it is desirable to approximate it by a different expression. One such expression is

$$K(q,p) \approx \frac{1}{\sqrt{1 + b^2 \left(\frac{\omega}{\omega_o} \right)^2}} \tag{4-2}$$

where b is a function of Q.

We may obtain b=b(Q) by equating the energies corresponding to the two kernels, in particular:

$$\int_1^\infty \frac{dx}{1 + b^2 (x-1)^2} = \int_1^\infty \frac{dx}{1 + Q^2 \left(x - \frac{1}{x} \right)^2} \tag{4-3}$$

where $x = \frac{\omega}{\omega_o}$ and the integration is carried out from the maxima of both curves to infinity.

Equation (4-3) gives

$$b = \frac{2\ Q}{1 + \frac{\log 4\ Q}{\pi\ Q}} \tag{4-4}$$

In Figures 8 to 10, (4-1) and (4-2) are compared for different values of Q.

The general equation for the action field will be:

$$\rho(\omega_o) = \int_0^\infty \frac{\sigma(\omega)\, d\omega}{\sqrt{1 + b^2 \left(\frac{\omega}{\omega_o}\right)^2}} \tag{4-5}$$

with b as defined by (4-4).

We shall now expand $\sigma(\omega)$ by a Fourier series and compute the terms arising out of the sine and cosine components of $\sigma(\omega)$, e.g.

$$\int_0^\infty \frac{\cos n k\omega}{\sqrt{1 + b^2 \left(\frac{\omega}{\omega_o}\right)^2}}\, d\omega \tag{4-6}$$

$$\int_0^\infty \frac{\sin n k\omega}{\sqrt{1 + b^2 \left(\frac{\omega}{\omega_o}\right)^2}}\, d\omega \tag{4-7}$$

where k is a constant and $n = 1, 2, \ldots$, let $y = \frac{\omega}{\omega_o}$. Upon integration of (4-6) and (4-7) (See Appendix B), we obtain:

$$\int_0^\infty \frac{\cos n \omega_o ky}{\sqrt{1 + b^2 y^2}}\, dy = -\frac{\pi}{2b} U_o\left(\frac{n k\omega_o}{b}\right) \tag{4-8}$$

where $U_o(x)$ is the real part of $N_o(ix)$ and $N_o(x)$ is Neumann's function of order zero

$$\int_0^\infty \frac{\sin n k\omega_o y}{\sqrt{1 + b^2 y^2}}\, dy = \frac{\pi}{2b}\left\{ V_o\left(\frac{n k\omega_o}{b}\right) - L_o\left(\frac{n k\omega_o}{b}\right)\right\} \tag{4-9}$$

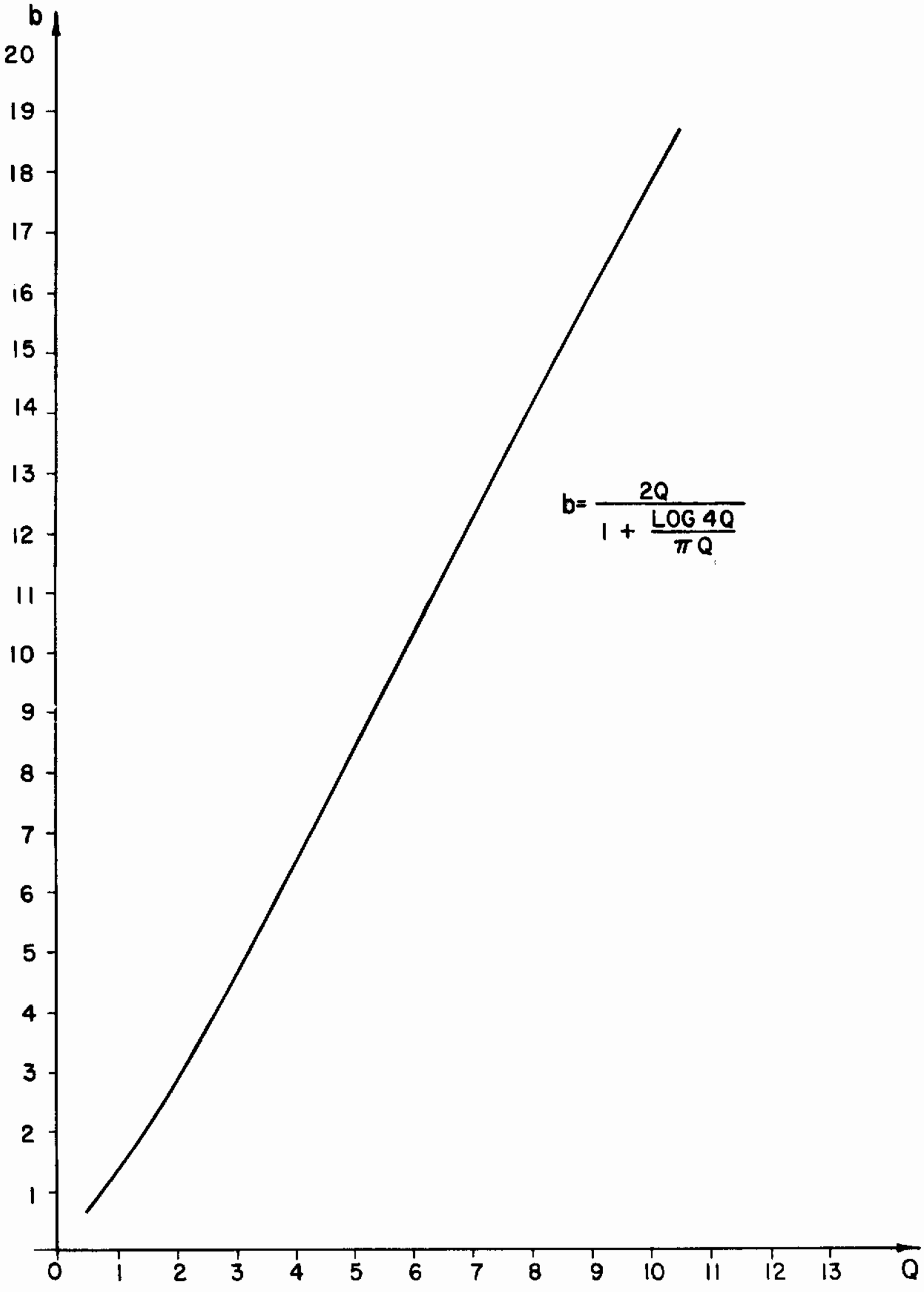


FIGURE 7: Equation (4-4)

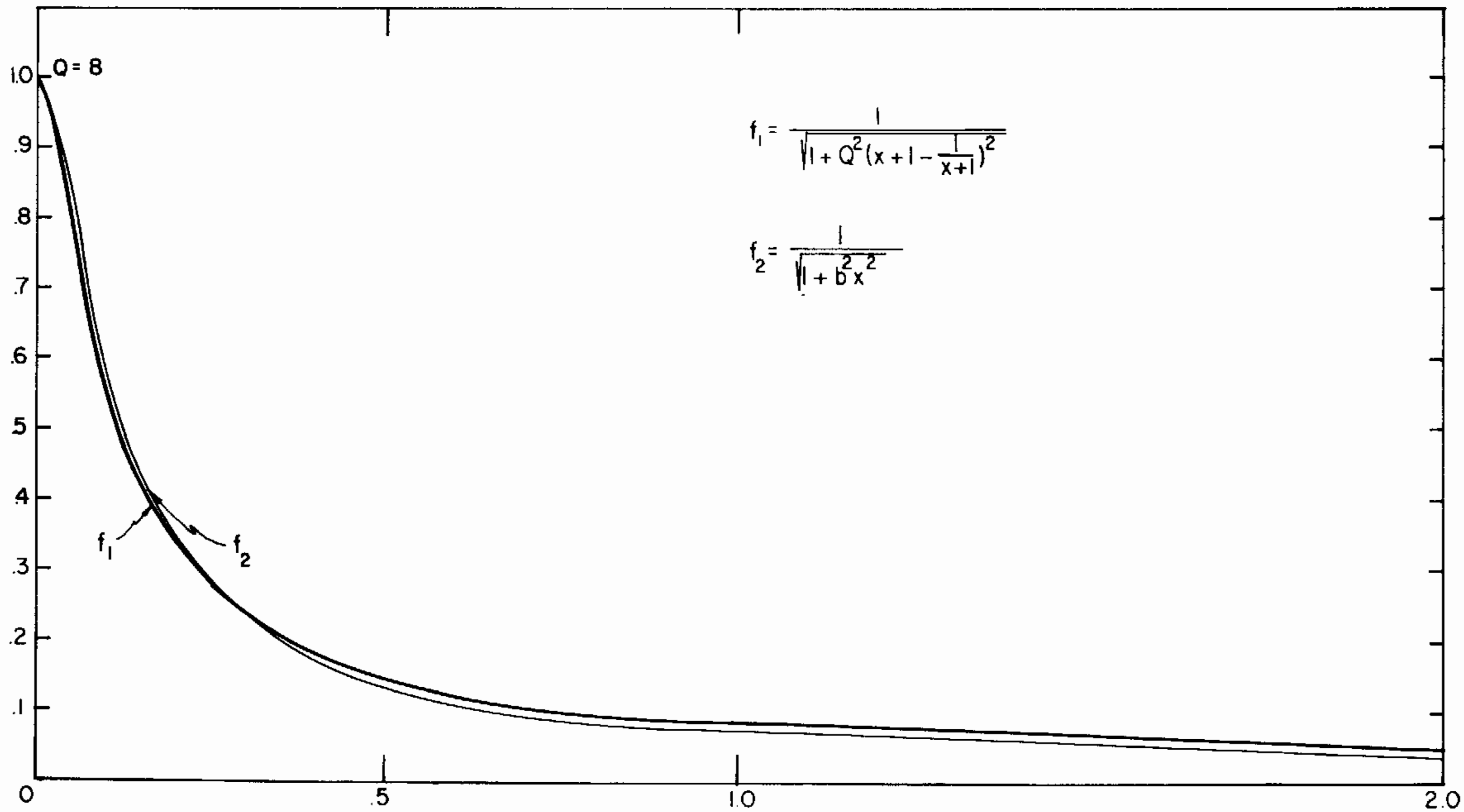


FIGURE 8. COMPARISON OF THE TWO KERNELS f_1 AND f_2 FOR Q = 4

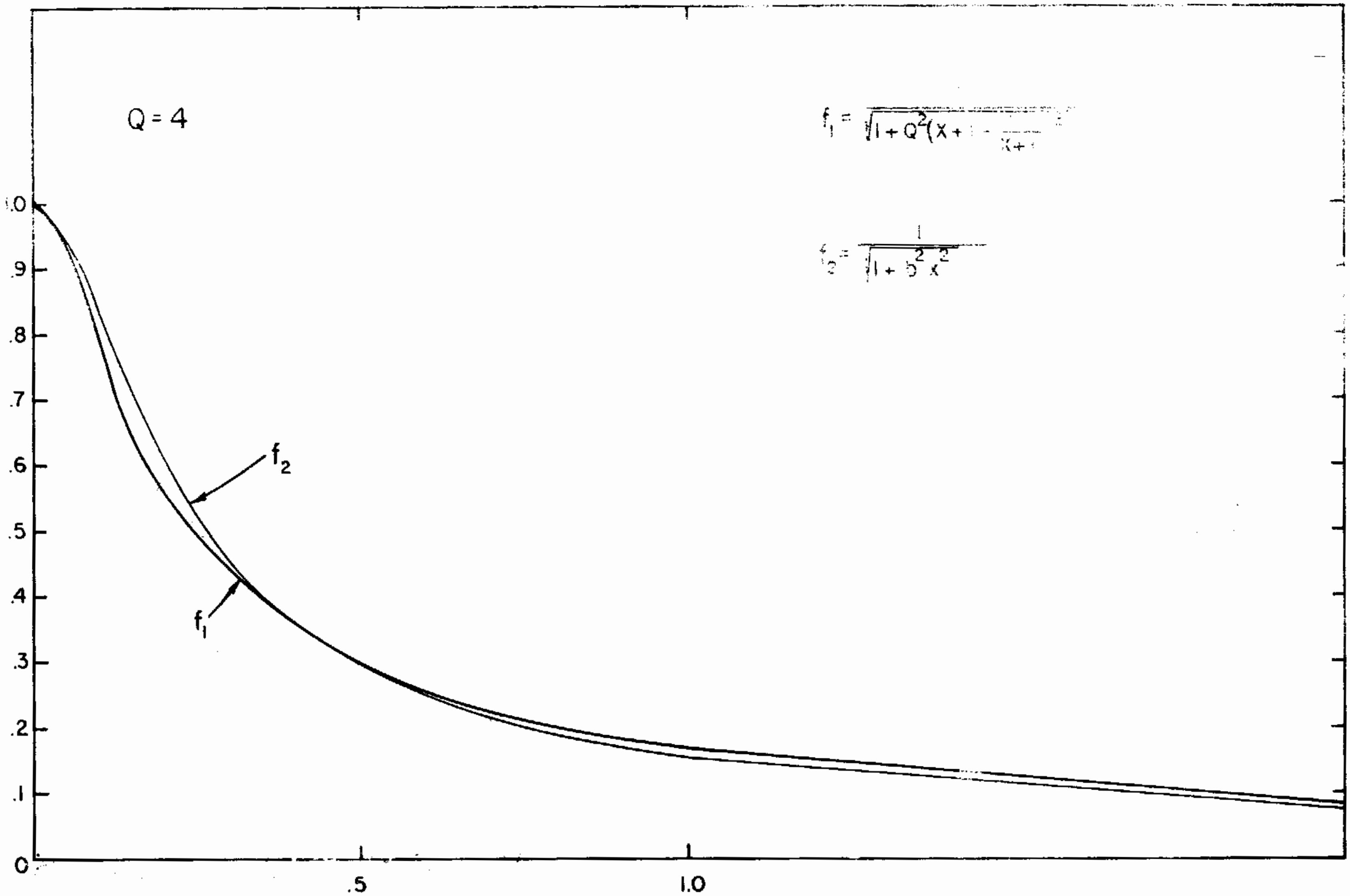


FIGURE 9. COMPARISON OF THE TWO KERNELS f_1 AND f_2 FOR Q = 6

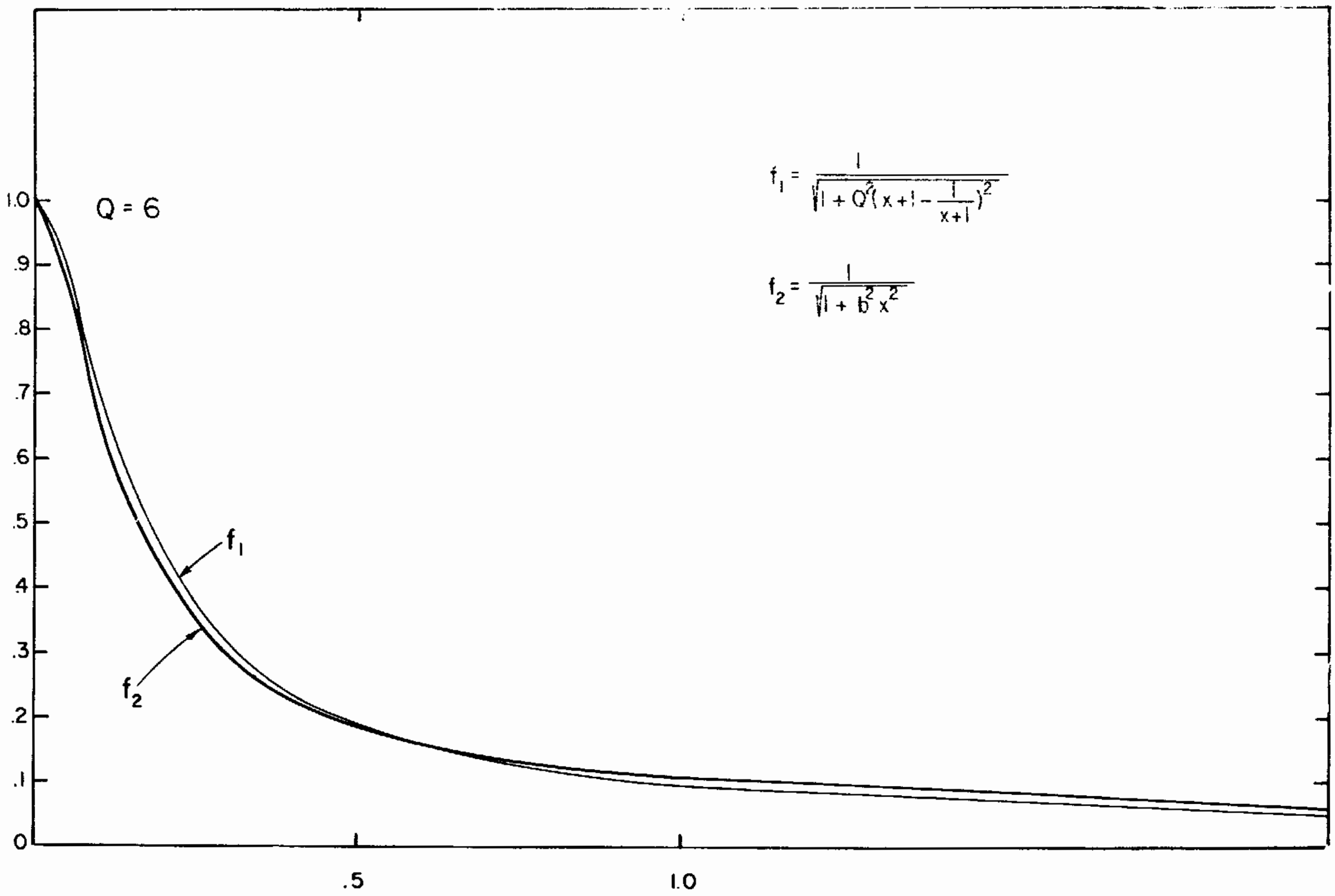


FIGURE 10. COMPARISON OF THE TWO KERNELS f_1 AND f_2 FOR Q = 8

where V_o is the imaginary part of $N_o(ix)$ and

$$L_o(x) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2} x)^{2m+1}}{\left\{\Gamma(m + \frac{3}{2})\right\}^2}$$

with $\Gamma(x)$ being the gamma function.

The integrals (4-8) and (4-9) are plotted for $k\omega_o = 1$ in Figures 11 and 12.

Equations (4-8) and (4-9) give the response for each component of the Fourier Series at the point p. This example illustrates the relative ease with which a fairly general problem can be handled using the techniques outlined in Section 3.

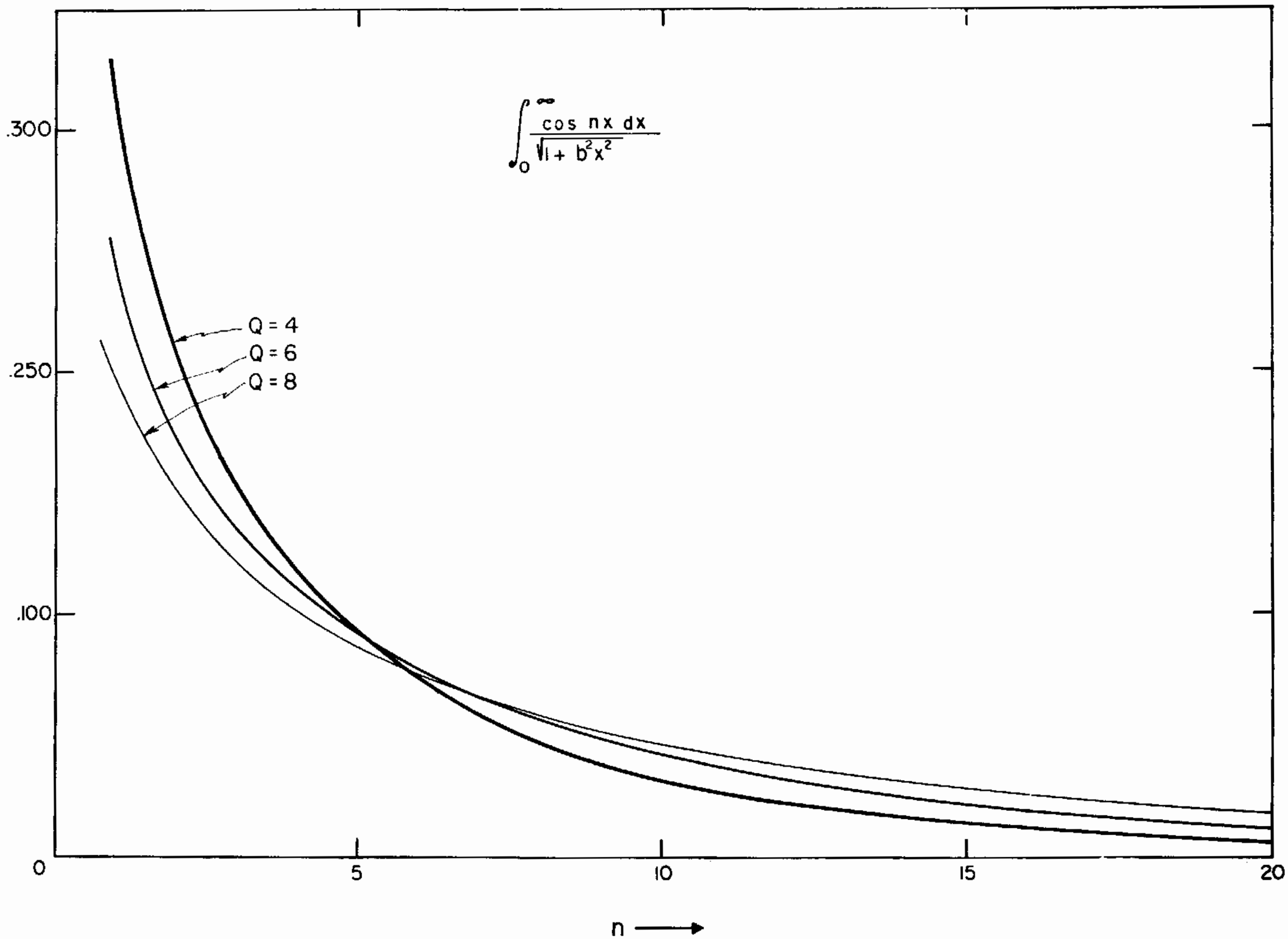


FIGURE 11. RESPONSE OF FREQUENCY SENSITIVE LAYER DUE TO COSINE COMPONENTS OF $\sigma(\omega)$

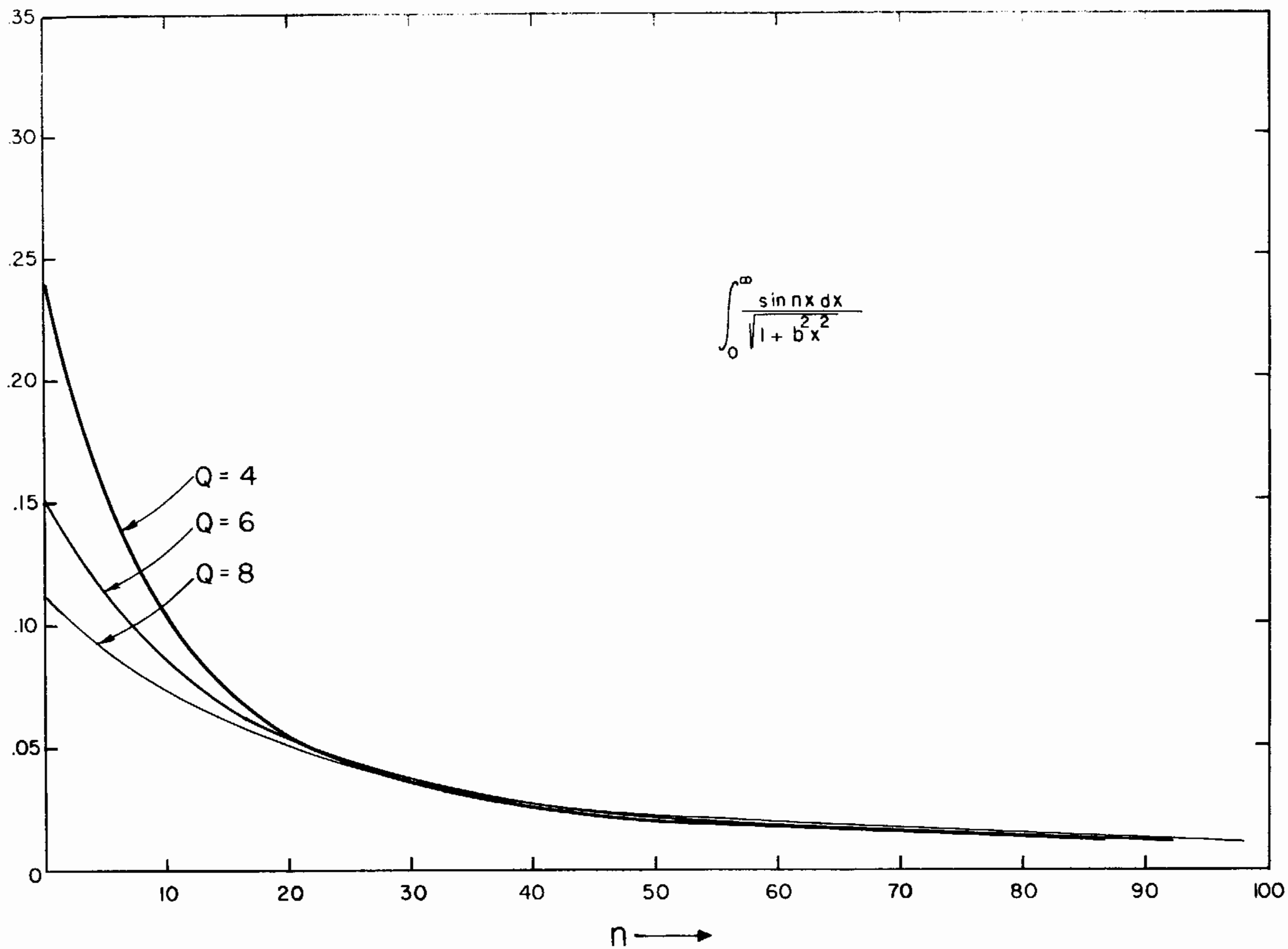


FIGURE 12. RESPONSE DUE TO SINE COMPONENTS OF $\sigma(\omega)$

APPENDIX A

INTERACTIONS IN CONSERVATIVE SYSTEMS

Let p be an element of a continuum L and let $\mathcal{E}$ be an energy unit

Then,
$$\frac{\text{energy input at } p}{\mathcal{E}} = \sigma(p) + \int_{q \in L} K(q,p)\; p(q)\; d\mu \qquad \text{(A-1)}$$

$$\frac{\text{energy output from } p}{\mathcal{E}} = p(p) \int_{q \in L} K(q,p)\; d\mu \qquad \text{(A-2)}$$

replacing p by x_o and q by x and equating (A-1) to (A-2) we have

$$\sigma(x_o) + \int_{x \in L} K(x_o, x)\, \rho(x)\, dx = \rho(x_o) \int_{x \in L} K(x_o, x)\, dx \qquad \text{(A-3)}$$

or

$$\sigma(x_o) - \rho(x_o)\, G(x_o) = - \int_{x \in L} K(x_o, x)\, \rho(x)\, dx \qquad \text{(A-4)}$$

where

$$\int_{x \in L} K(x_o, x)\, dx = G(x_o)$$

if $G(x_o) \neq 0$, (A-4) may be transformed into

$$\rho(x_o) = -\sigma_1(x_o) + \int_{x \in L} K_1(x_o, x)\, \rho(x)\, dx \qquad \text{(A-5)}$$

where

$$\sigma_1(x_o) = \frac{\sigma(x_o)}{G(x_o)}$$

and

$$K_1(x_o, x) = \frac{K(x_o, x)}{G(x_o)}$$

Equation (A-5) is of the type discussed in Part II and can therefore be handled by the same method.

APPENDIX B

INTEGRATION OF EQUATIONS (4-8) AND (4-9)

Integration of $\int_0^\infty \frac{\sin n k_1 y}{\sqrt{1 + b^2 y^2}} dy$

and

$$\int_0^\infty \frac{\cos n k_1 y}{\sqrt{1 + b^2 y^2}} dy, \quad k_1 = k \omega_o$$

$$\int_0^\infty \frac{\sin n k_1 y}{\sqrt{1 + b^2 y^2}} = \frac{1}{2i} \int_0^\infty \frac{e^{ink_1 y}}{\sqrt{1 + b^2 y^2}} dy - \frac{1}{2i} \int_0^\infty \frac{e^{-ink_1 y}}{\sqrt{1 + b^2 y^2}} dy \qquad \text{(B-1)}$$

Now[10]

$$\int_0^\infty \frac{e^{-st}}{\sqrt{1 + t^2}} dt = \frac{\pi}{2} \left\{ H_o(s) - N_o(s) \right\} \qquad \text{(B-2)}$$

where $H_o(s)$ is Struve's function of 0-th order and $N_o(s)$ Neumann's function of zero order. Letting by = t in (B-1) we have

$$\int_0^\infty \frac{\sin n k_1 y}{\sqrt{1 + b^2 y^2}} dy = \frac{1}{2bi} \int_0^\infty \frac{e^{\frac{in k_1 t}{b}}}{\sqrt{1 + t^2}} dt - \frac{1}{2bi} \int_0^\infty \frac{e^{-\frac{in k_1 t}{b}}}{\sqrt{1 + t^2}} dt$$

$$= \frac{\pi}{4bi} \left\{ H_o\left(- \frac{ink_1}{b} \right) - N_o \left(- \frac{ink_1}{b} \right) - H_o \left(\frac{ink_1}{b}\right) + N_o \left(\frac{ink_1}{b} \right)\right\} \qquad \text{(B-3)}$$

$$H_o\left(\frac{ink_1}{b} \right) = i\, L_o \left(\frac{n k_1}{b} \right)$$

where

$$L_o(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}x\right)^{2m+1}}{\left\{\Gamma\left(m + \frac{3}{2}\right)\right\}^2}$$

Letting $N_o(ix) = U_o(x) + i\, V_o(x)$ (B-3) yields

$$\int_0^{\infty} \frac{\sin n k_1 y}{\sqrt{1 + b^2 y^2}}\, dy = \frac{\pi}{2b} \left\{ V_o\left(\frac{n k_1}{b}\right) - L_o\left(\frac{n k_1}{b}\right)\right\} \qquad \text{(B-2)}$$

APPENDIX C
UNIVERSITY OF ILLINOIS DYNAMIC SIGNAL ANALYZER

The Dynamic Signal Analyzer (DSA)[9] built at the Biological Computer Laboratory at the University of Illinois contains a band of 96 resonating filters each tuned to a different frequency. There are eight octaves each divided into twelve equal semitones according to the well-tempered scale[11].

If f_o is the frequency of the fundamental then the frequency, f_1, of the first semitone after the fundamental is given by:

$$f_1 = \sqrt[12]{2} \cdot f_o = 1.05946\ f_o$$

Similarly:

$$f_2 = \sqrt[12]{2}\ f_1 = (\sqrt[12]{2})^2\ f_o = 1.12246\ f_o$$

and in general

$$f_i = \sqrt[12]{2}\ f_{i-1} = (\sqrt[12]{2})^i\ f_o \qquad \text{(C-1)}$$

In the well tempered scale the deviations from the Pythagorean system are distributed uniformly along the octave and although no interval other than the octave is according to the Pythagorean system our ear has become accustomed to this "error". In musical string instruments the advantages of this system far outweigh its flaws and this system is used in the sound spectrum analyzer section of the DSA.

The magnitude of the response of the ith filter is given by*:

$$\left|\frac{E}{E_i}\right| = [1 + Q^2 (\frac{f}{f_i} - \frac{f_i}{f})^2]^{-1/2} \qquad \text{(C-2)}$$

where

E - is the output sinusoidal voltage when the filter is driven with a constant amplitude, A, sinusoidal source at a frequency f.

E_i - is the output sinusoidal voltage when the filter is driven with a constant amplitude, A, sinusoidal source at a frequency f_i.

---

* Reference 9, p. 60, Equation (C-3).

f_i - is the design center frequency of the filter and is the resonant frequency of the prototype filter.

Q - is the equivalent quality factor of the filter and driver circuit combination.

The function defined by Equation (C-2) possesses a rather flat maximum at $f = f_i$. Hence the problem of resolving a given sound signal in terms of its frequency components, f_i, 0 = 1, 2, ... , 96, becomes rather insoluble unless a sharpening can be applied to the responses of each of the individual filters. We apply the sharpening discussed in Part 3.

For the purposes of differentiation three different approximations for the second derivative are used.

$$- f''(x)\Big|_{x=i} \approx 2f(i) - (f(i + 1) + f(i - 1)) \qquad \text{(C-3)}$$

$$- f''(x)\Big|_{x=i} \approx \frac{1}{2^2}\,[2f(i) - (f(i + 2) + f(i - 2))] \qquad \text{(C-4)}$$

$$- f''(x)\Big|_{x=i} \approx \frac{1}{3^2}\,[2f(i) - (f(i + 3) + f(i - 3))] \qquad \text{(C-5)}$$

It is seen that in all three cases a central difference, around the point where the derivative is sought, is employed.

Expanding by means of a Taylor series, where $\Delta x > 0$

$$f(i+\Delta x) = f(i) + f'(i)\frac{\Delta x}{j!} + f''(i)\frac{(\Delta x)^2}{2!} + f'''(i)\frac{(\Delta x)^3}{3!} + f^{(iv)}_{(i)}\frac{(\Delta x)^4}{4!} + $$

$$+ f^{(v)}_{(i)}\frac{(\Delta x)^5}{5!} + \ldots$$

$$f(i - \Delta x) = f(i) - f'(i)\frac{\Delta x}{1!} + f''(i)\frac{(\Delta x)^2}{2!} - f'''(i)\frac{(\Delta x)^3}{3!} + f^{(iv)}_{(i)}\frac{(\Delta x)^4}{4!} - $$

$$- f^{(v)}_{(i)}\frac{(\Delta x)^5}{5!} + \ldots$$

$$\therefore \; - f''(i) \quad \frac{1}{(\Delta x)^2} \; [2f(i) - (f(i + \Delta x) + f(i - \Delta x)] =$$

$$= \frac{1}{(\Delta x)^2} \left[- f''(i) \; (\Delta x)^2 - 2f^{(iv)}_{(i)} \frac{(\Delta x)^4}{4!} - 2f^{(vi)}_{(i)} \frac{(\Delta x)^6}{6!} - \ldots \right]$$

hence

$$- f''(i) \approx - f''(i) - \frac{2}{4!} f^{(iv)}_{(i)} \; (\Delta x)^2 - \frac{2}{6!} f^{(vi)}_{(i)} \; (\Delta x)^4 - \ldots \qquad \text{(C-6)}$$

Therefore our approximations imply that we are neglecting higher order terms with (C-5) approaching

$$- f''(i) = \frac{1}{(\Delta x)^2} \; [2f(i) - (f(i + \Delta x) + f(i - \Delta x)] \qquad \text{(C-7)}$$

As Δx tends to zero.

Using Equation (C-7) we obtain for the fourth derivative

$$- f''''(i) \approx \frac{1}{(\Delta x)^4} \; [6f(i) - 4(f(i + \Delta x) + f(i - \Delta x) + (f(i + 2\Delta x) + f(i - 2\Delta x))] \qquad \text{(C-8)}$$

which for $\Delta x = 1$ yields:

$$- f''''(i) \approx 6f(i) - 4(f(i + 1) + f(i - 1) + (f(i + 2) + f(i - 2) \qquad \text{(C-9)}$$

for $\Delta x = 2$

$$- f''''(i) \approx \frac{1}{2^4} \; [6f(i) - 4(f(i + 2) + f(i - 2)) + (f(i + 4) + f(i - 4))] \qquad \text{(C-10)}$$

and for $\Delta x = 3$

$$- f''''(i) \approx \frac{1}{3^4} \; [6f(i) - 4(f(i + 3) + f(i - 3)) + (f(i + 6) + f(i - 6))] \qquad \text{(C-11)}$$

In Figure 13 a typical resonance curve, Equation (C-2), is approximated by a step function and amplified by a factor of 20. In Figure 14a the second derivative using (C-3) is amplified by a factor of 5. Figure 14b shows the second derivative obtained by (C-4) with an amplification of 5. Equation (C-5) is used for the second derivative in Figure 14c. The same procedure on $f''(x)$ is employed in order to obtain the fourth derivative, using (C-9), (C-10) and (C-11) and this is shown in Figures 15a, 15b and 15c.

Equation (C-3), and it's "twin" Equation (C-9), is clearly superior, to the others, for sharpening purposes. From Figure 14a we see that the ordinate at the origin is highly discernible from all other points and we need not compute derivatives of higher order in order to achieve further sharpening.

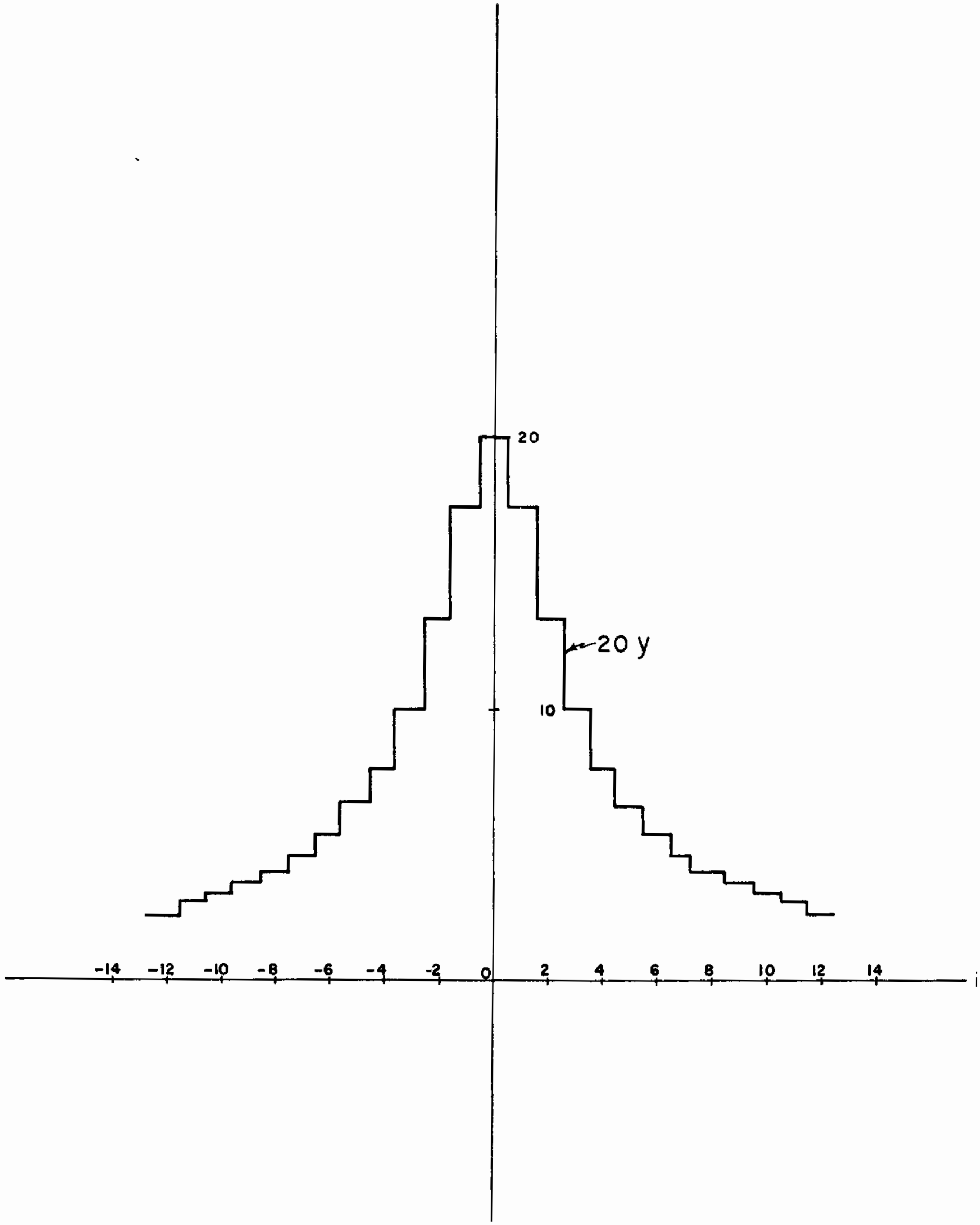


FIGURE 13. AMPLIFIED (20X) STEP APPROXIMATION TO A TYPICAL RESONANCE CURVE

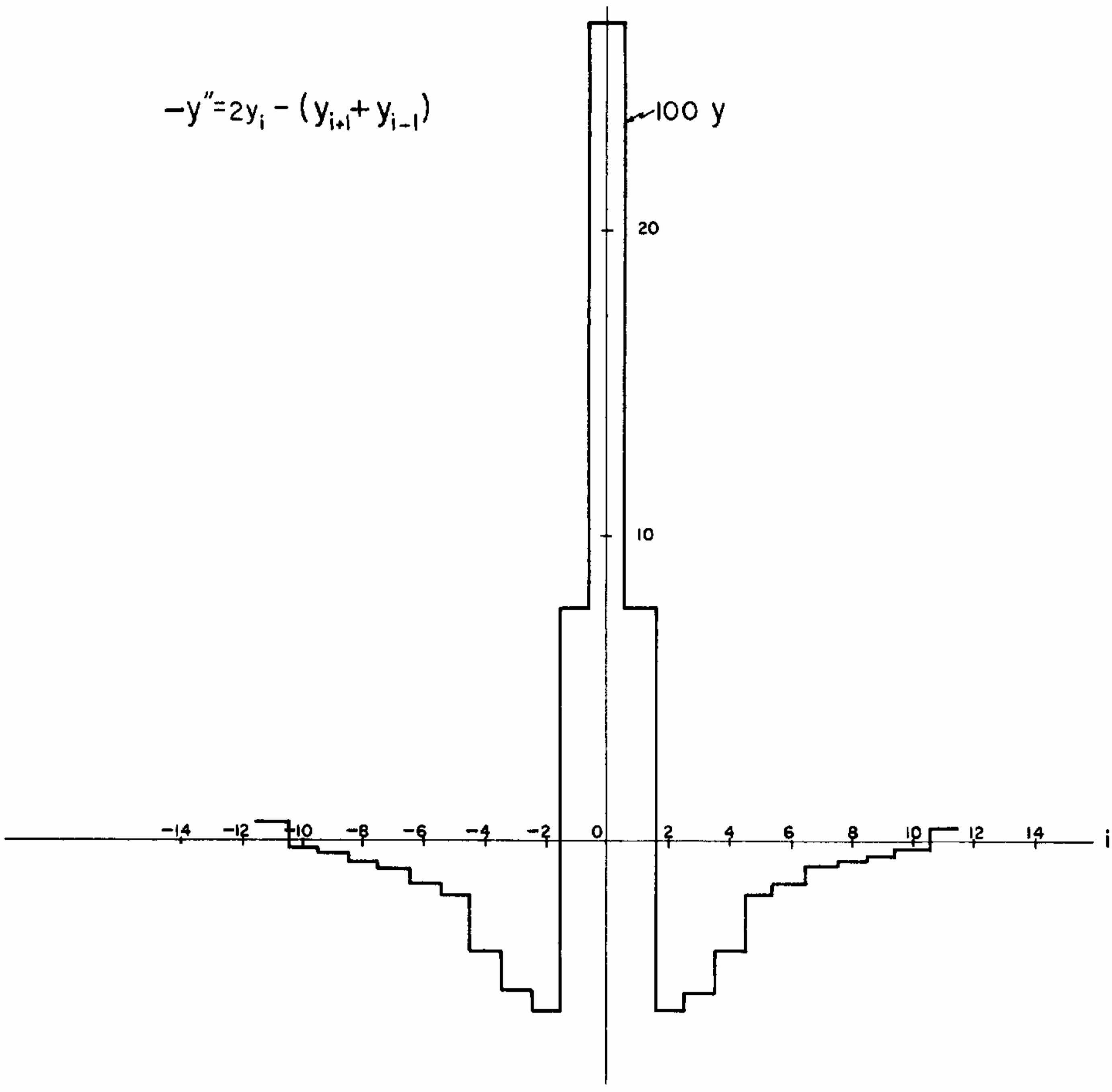


FIGURE 14a. SECOND DERIVITIVE BY APPROXIMATION (C-3)

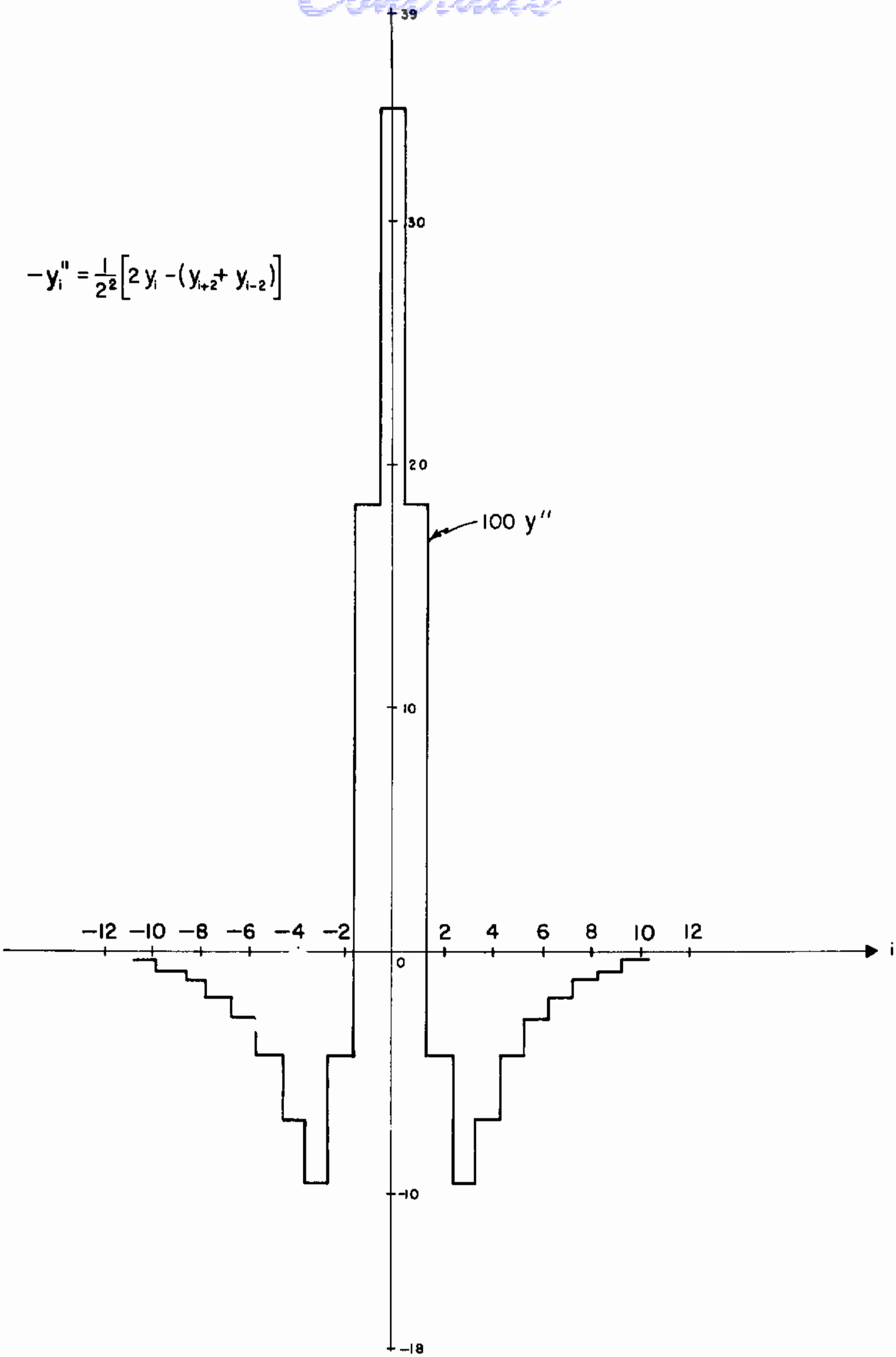


FIGURE 14b. SECOND DERIVITIVE BY APPROXIMATION (C-4)

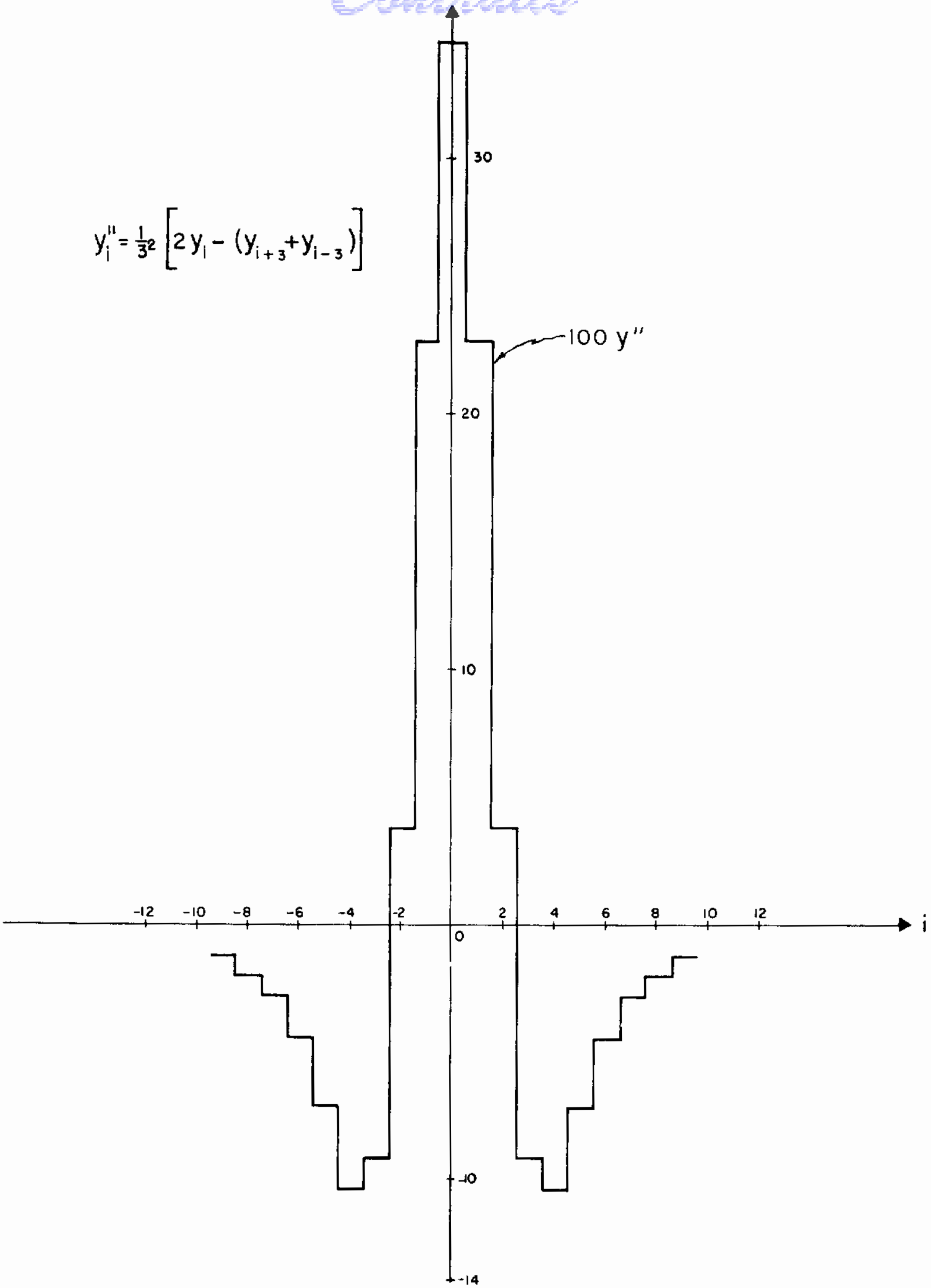


FIGURE 14c. SECOND DERIVITIVE BY APPROXIMATION (C-5)

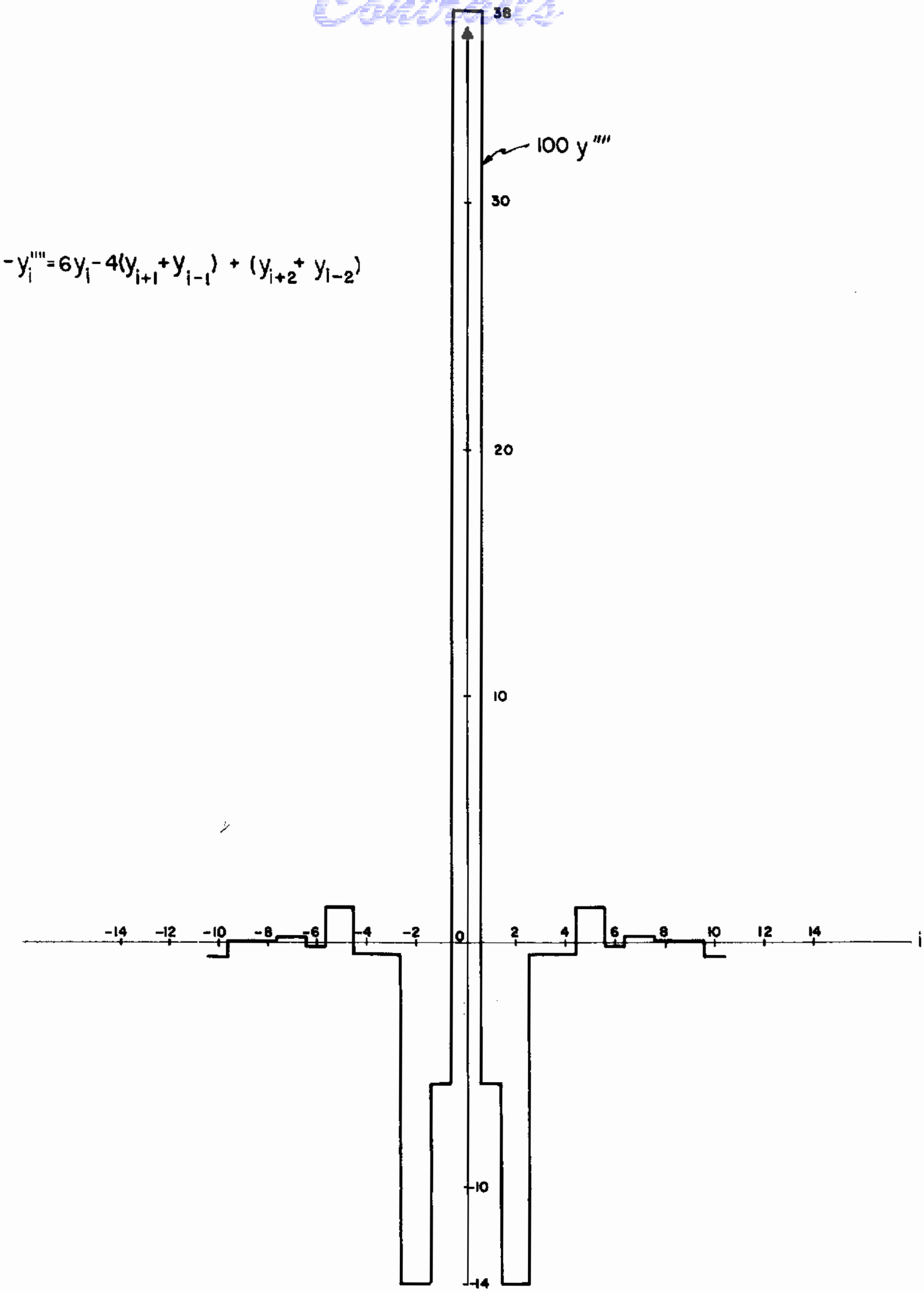


FIGURE 15a. FOURTH DERIVITIVE BY APPROXIMATION (C-9)

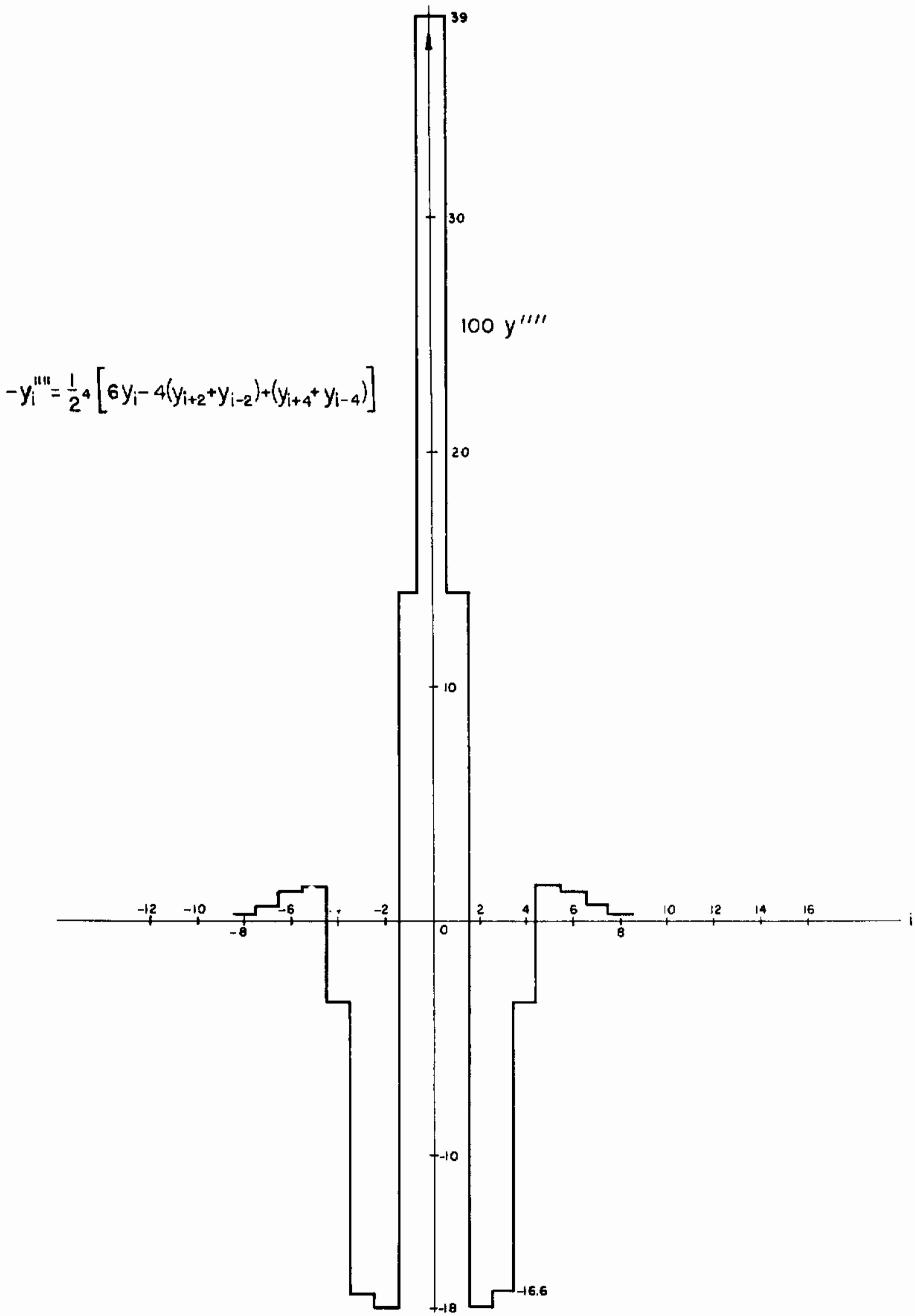


FIGURE 15b. FOURTH DERIVITIVE BY APPROXIMATION (C-10)

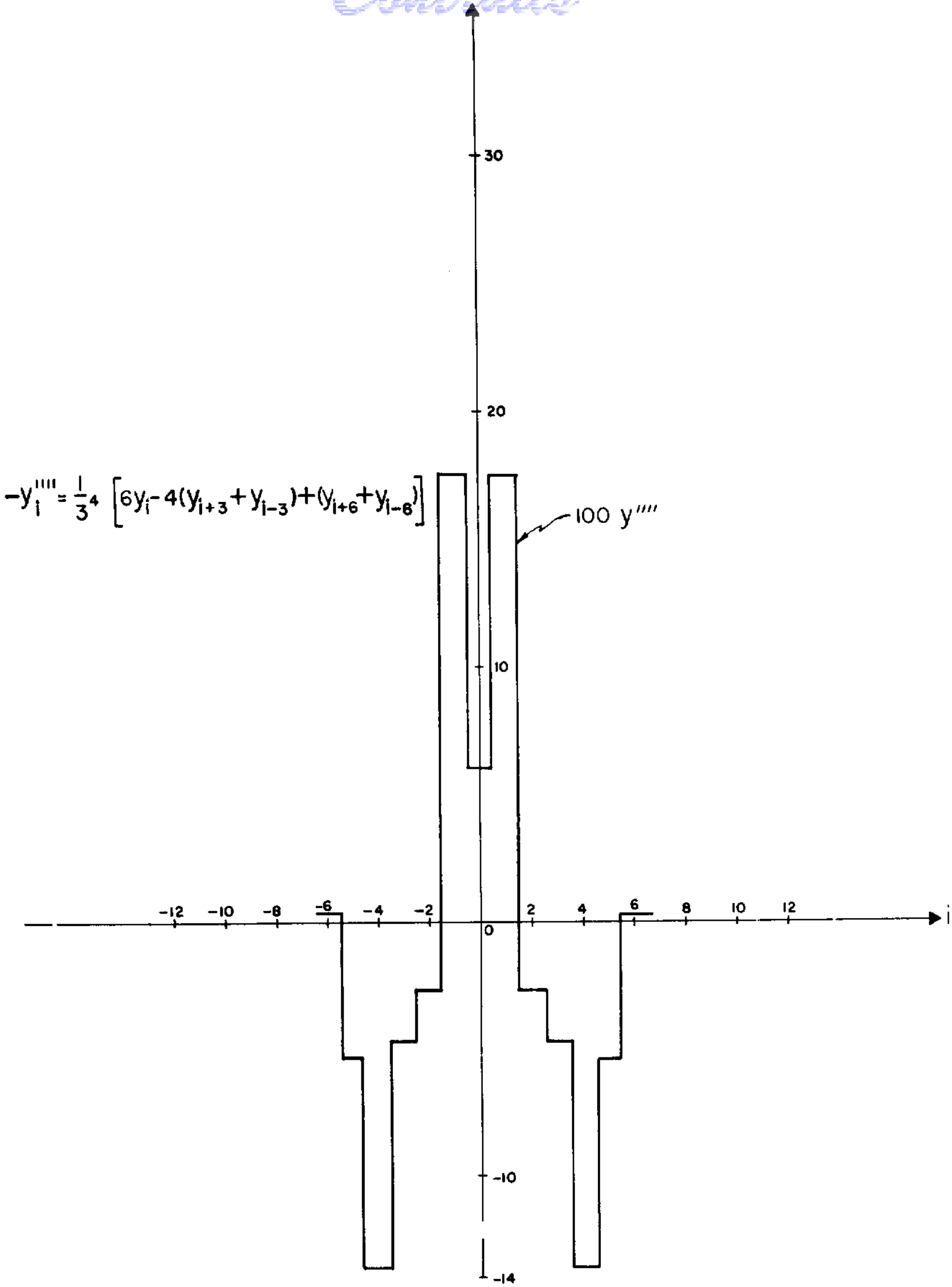


FIGURE 15c. FOURTH DERIVITIVE BY APPROXIMATION (C-11)

PART II

KINDS OF INTERACTIONS IN SETS OF DISCRETE, LINEAR ELEMENTS

by

Raymond Yeh

1. FINITE INTERACTION NETWORKS

Consider a set S of a finite number of elements each of which has a specific connectivity with other elements in S and each performs linear operations upon its input stimulations when S is acted upon by a stimulus field. S is called a linear network. The input stimulation of the i^{th} element of S will be denoted by σ^i while the response of the same element is represented by ρ^i. If elements in S are connected such that each is receiving stimulations from other elements and also to provide stimulations to other elements including itself, then S is called a finite linear interaction network.

Let p and q be the i^{th} and the j^{th} elements in S, and k^i_j, k^j_i denote the interaction coefficients such that k^s_t specifies the amount of stimulations being transferred from the s^{th} element to the t^{th} element. Figure 16 is a sample representation of the connectivities between just two elements of an interaction network. Depending on the specific interaction function between elements, the response of any element in S can be expressed in the following form

$$\rho^i = k^i_o \rho^o + k^i_1 \rho' + \ldots + k^i_i \rho^i + \ldots + k^i_n \rho^n \qquad (1)$$

$$i = 1, 2, \ldots, n$$

where ρ^o is defined to be the stimulation of the particular element under consideration, in this case, $\rho^o = \sigma^i$. The n-equations may be expressed in matrix form below

$$\rho = K' \sigma + K\rho \qquad (2)$$

where $K' = \| k^i_o \|$ and $K = \| k^i_j \|$ ρ and σ here denote column matrices. We may rewrite (2) below

$$\sigma = M\rho \qquad (3)$$

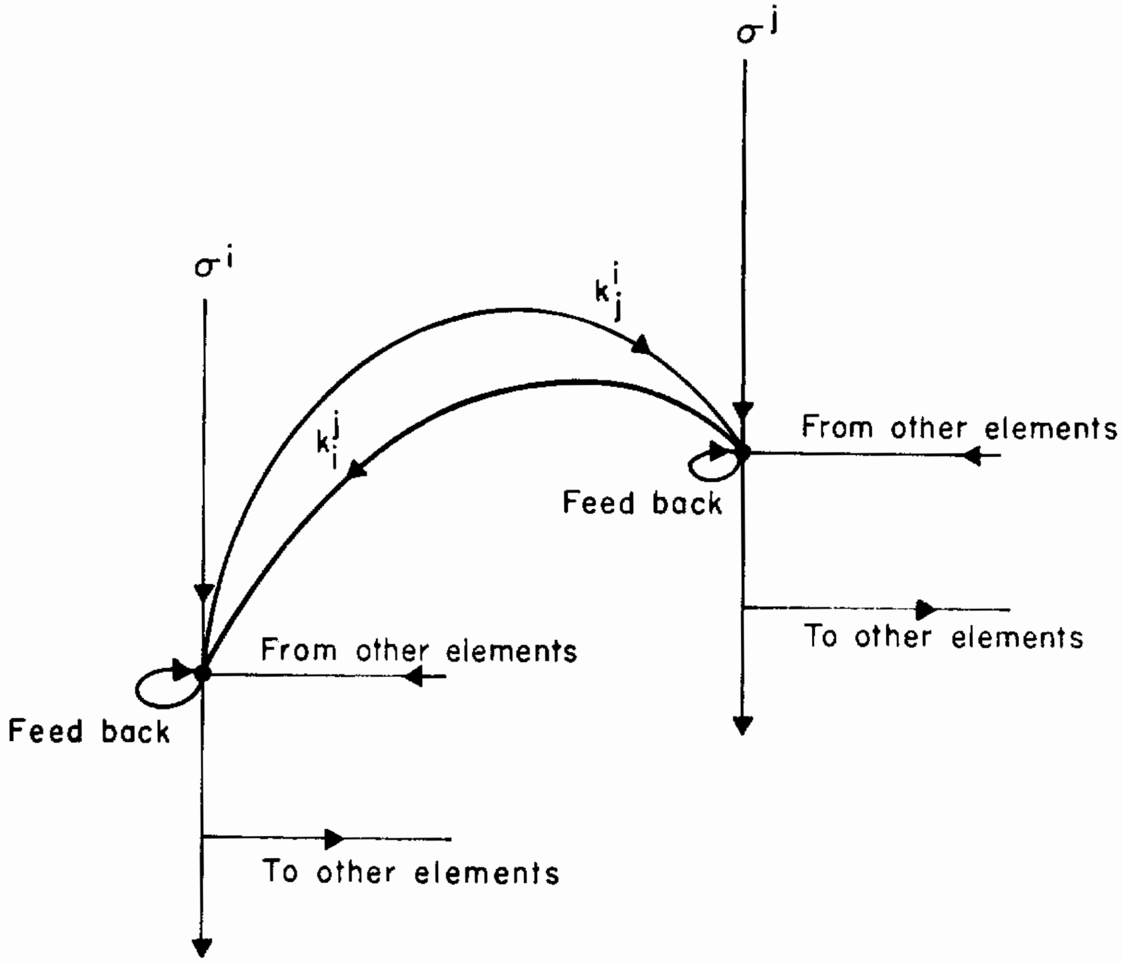


FIGURE 16. CONNECTIVITY BETWEEN TWO ARBITRARY ELEMENTS IN AN INTERACTION NETWORK.

where $M = || m_j^i ||$, $i, j = 1, 2, \ldots, n$.

$$m_j^i = \begin{cases} \dfrac{-k_j^i}{k_o^i} & \text{if } i \neq j \\[2ex] \dfrac{-k_i^i + 1}{k_o^i} & \text{if } i = j \end{cases}$$

If M above is non-singular, we may express the responses in terms of stimulations by simply inverting the coefficient matrix M. Thus

$$\rho = M^{-1}\,\sigma = R\,\sigma \text{ if } |\, m_j^i \,| \neq 0 \tag{4}$$

where
$$M^{-1} = R = \frac{||\, M_i^j \,||}{|\, m_j^i \,|} = ||\gamma_j^i||$$

M_i^j = cofactor of the j^{th} row and i^{th} column of M

$|m_j^i|$ = determinant of M.

If M is singular, ($|m_j^i| = 0$), then M^{-1} is unstable and ρ cannot be expressed in terms of σ under this condition.

2. FINITE ACTION NETWORK

Consider the case where S is partitioned into m subsets $S_1, S_2, \ldots, S_m$ arranged in order so that

(1) $S = \bigcup_{k=1}^{m} S_k$

(2) Each element $P_1 \in S_1$ receives a stimulation from the stimulus field acts upon S.

(3) Each element $P_2 \in S_2$ receives stimulations only from elements of S_1 and gives stimulations to S_3. In general, each element $P_i \in S_i$ receives stimulations from elements of S_{i-1} and its responses provide stimulations to the elements of S_{i+1}.

(4) No interaction within each subset is allowed. S is called a finite action network under the above conditions and is sketched in Figure 17.

Figure 18 indicates the action relations between the elements of the adjacent subsets, S_i and S_j . S_i acts upon S_j. An arbitrary element is picked in S_i as a sample representation of the action relation from the element of S_i to that of S_j, while the symbol a_j^i represents the action coefficients which define the activities transferred from the elements of S_i to that of S_j. Before going further, we shall adopt the following summation convention:

$$a_k \rho^k = a_1 \rho^1 + a_2 \rho^2 + \ldots + a_n \rho^n \quad \text{for } k = 1,2,\ldots,n.$$

If both, S_i and S_j above, contain n elements, the action relation can be expressed as

$$\rho^j = a_j^i \sigma^j \qquad i, j = 1,2, \ldots, n \tag{5}$$

the n-equation expressed in matrix form is

$$\rho = A \sigma$$

where

$$A = || a_j^i || \tag{6}$$

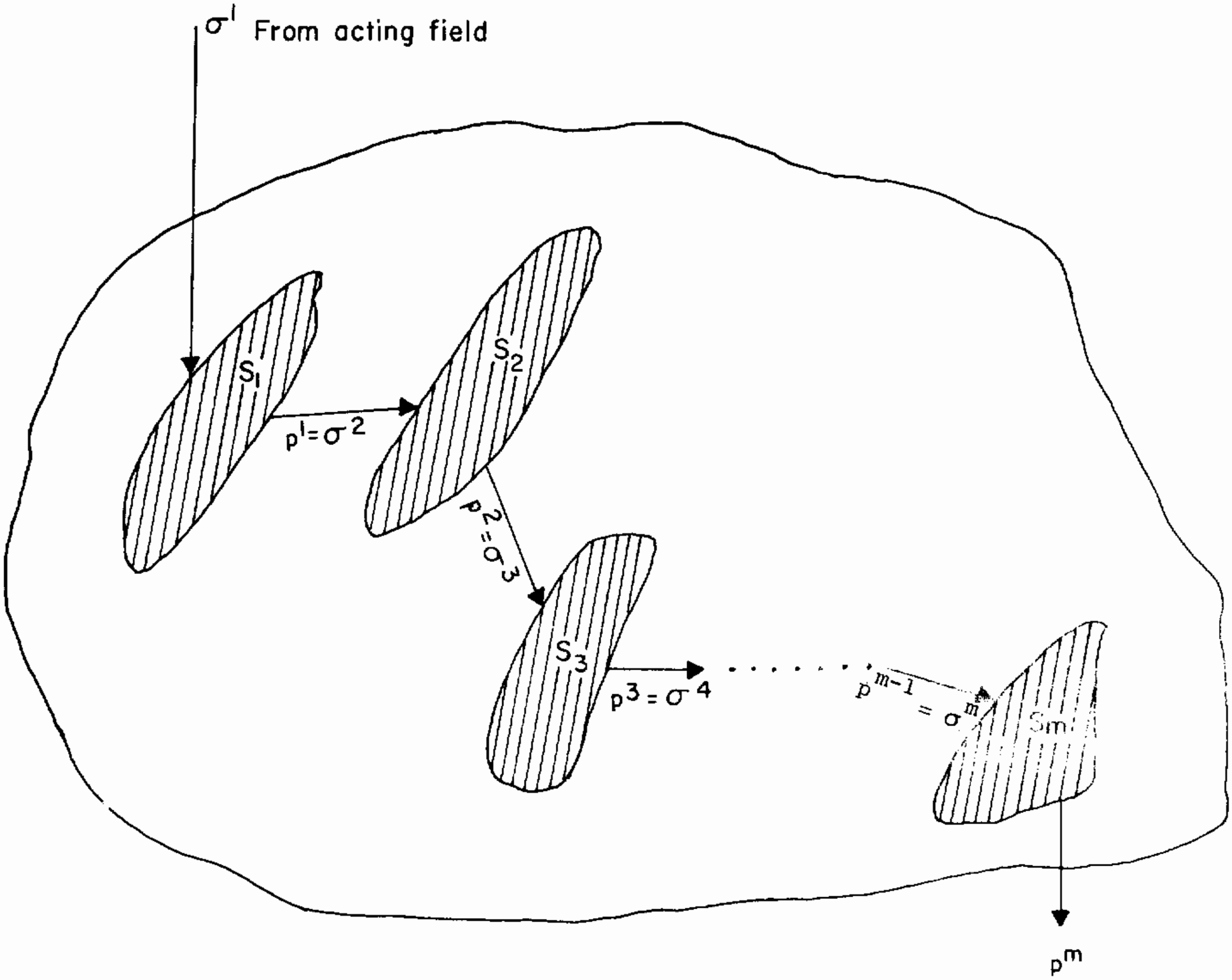


FIGURE 17. FINITE ACTION NETWORK

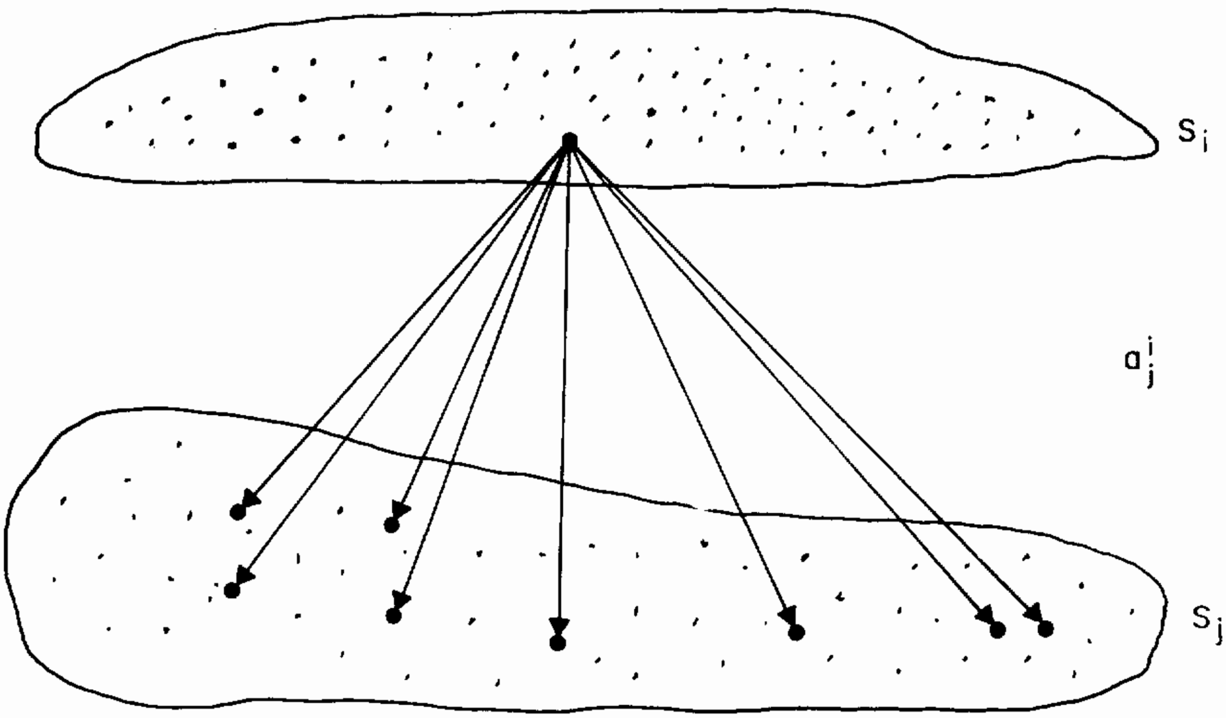


FIGURE 18. ACTION RELATIONS BETWEEN TWO SETS

For A non-singular, then σ can be expressed in terms of ρ by simply inverting the coefficient matrix A:

$$\sigma = A^{-1}\rho = B\rho \qquad (7)$$

Note that instability will result in the inversion if A is singular

3. EQUIVALENCE OF STABLE ACTION AND INTERACTION NETWORKS

From the above discussion, we see that the action networks involve at least two sets of linear elements while the interaction networks only involve one set of linear elements. Though the structure of the two kinds of networks are different, they perform equivalent functions with respect to stimulations and responses. Let us call the functions performed by action and interaction networks the action and the interaction functions respectively. The following theorem is in order.

Theorem I

In finite linear networks, for each stable action function, there exists an equivalent interaction function and vice versa.

Proof

From Equations (3), (4), (6), and (7) we have

$$\sigma = M\rho \Longrightarrow \rho = R\sigma$$

$$\rho = A\sigma \Longrightarrow \sigma = B\rho$$

The above two equations show that a stable action function implies an interaction function and vice versa.

4. INFINITE PERIODIC NETWORKS

A set S which contains infinite number of elements is called an infinite set. If elements of S are being grouped into a countable number of subsets, such that $S = \bigcup_{n=1}^{\infty} S_n$, each contains N elements and all elements in each subset S_n possess the same operational characteristics, then S is called an infinite, periodic network. Let π denote the periodicity, then the above network is said to have periodicity $\pi = N$.

4.1 One-Dimensional Symmetric Action Network

Considering $\bar{S} = S_i \cup S_j$ such that S_i acts upon S_j, each is an infinite periodic set with $\pi = 1$ and satisfying the following conditions.

(1) Elements of either set are arranged in a one dimensional layer such that each element is $\triangle$ distance apart from its adjacent elements.

(2) The action coefficients which define the activities transferred from elements of S_i to those of S_j fulfill the conditions below

(a) $$a^{i_m}_{j_n} = a^{i_k}_{j_{(n + |k - m|)}}$$

(b) $$a^{i_{m+k}}_{j_n} = a^{i_{m-k}}_{j_n}$$

(3) Stimulations and responses of the elements in either set are functions of distance denoted by $\sigma(x)$ and $\rho(x)$ respectively.

$\bar{S}$ is called a one dimensional symmetrical action network which may be illustrated with the aid of Figure 19. Let a^i_j be represented by a_K such that $K \in [\rho]$ where $[\rho]$ is the set of non-negative integers, then the following stimulus-response relation of the one-dimensional action network can be obtained:

$$\rho(x) = a_o\, \sigma(x) + a_1[\sigma(x + \triangle) + \sigma(x-\triangle)]$$

$$+ a_2 [\, \sigma(x + 2\triangle) + \sigma(x-2\triangle)] + \ldots$$

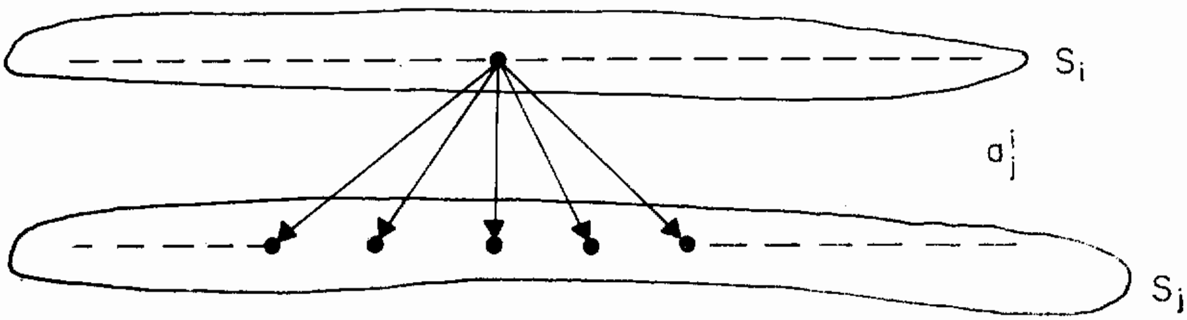


FIGURE 19. ONE-DIMENSIONAL ACTION NETWORK

By use of Taylor's expansion, we have

$$\rho(x) = a_o\,\sigma(x)$$

$$+ a_1\,[\sigma(x) + \sigma'(x)\Delta + \sigma''(x)\frac{\Delta^2}{2!} + \sigma'''(x)\frac{\Delta^3}{3!} + \ldots$$

$$+ \sigma(x) - \sigma'(x)\Delta + \sigma''(x)\frac{\Delta^2}{2!} - \sigma'''(x)\frac{\Delta^3}{3!} + \ldots]$$

$$+ a_2\,[\sigma(x) + 2\sigma'(x)\Delta + \frac{2^2 b}{2!}\sigma''(x)\Delta^2 + \frac{2^3}{3!}\sigma'''(x)\Delta^3 + \ldots$$

$$+ \sigma(x) - 2\sigma'(x)\Delta + \frac{2^2}{2!}\sigma''(x)\Delta^2 - \frac{2^3}{3!}\sigma'''(x)\Delta^3 + \ldots]$$

$$+ \ldots$$

$$= \sigma(x)\frac{2\Delta^o}{0!}[1^o a_1 + 2^o a_2 + 3^o a_3 + \ldots + \frac{1}{2}a_o]$$

$$+ \sigma''(x)\frac{2\Delta^2}{2!}[1^2 a_1 + 2^2 a_2 + 3^2 a_3 + \ldots]$$

$$+ \sigma''''\frac{2\Delta^4}{4!}[1^4 a_1 + 2^4 a_2 + 3^4 a_3 + \ldots]$$

$$\rho(x) = \sigma(x)\cdot 2\beta_o\frac{\Delta^o}{0!} + \sigma''(x)\cdot 2\beta_2\frac{\Delta^2}{2!} + \sigma'''(x)\cdot 2\beta_4\frac{\Delta^4}{4!} + \ldots$$

$$= 2\sum_{o}^{\infty}\sigma^{(2i)}(x)\frac{\Delta^{2i}}{2i!}\beta_{2i} \qquad (8)$$

where

$$\beta_j = \begin{cases} \sum_{i=1}^{\infty} i^j a_i & j \neq o \\ \frac{1}{2}a_o + \sum_{i=1}^{\infty} a_i & j = o \end{cases}$$

Equation (8) represents stable action if and only if the series converges. Assuming $\sigma(x)$ has finite derivatives up to 2^{ith} order and $\frac{\Delta^{2i}}{2i!}$ is finite, then the series converges if β_{2i} represents a converging series, i.e., if β_{2i} is a constant as expressed by

$$\beta(n) = \sum_{x=0}^{\infty} x^n f(x) = c_n \qquad (n = 0, 1, \ldots) \tag{9}$$

where c_n is a constant depending on n. If one can find a solution of f(x) such that (9) is satisfied, then Equation (8) is a stable action case. We shall approximate (9) by integration:

$$\beta(n) \approx \int_0^{\infty} x^n f(x)\, dx = c_n \qquad (n = 0, 1, \ldots) \tag{10}$$

A solution of f(x) in (10) exists[12]:

$$\xi f(\xi^2) = \frac{1}{\pi} \lim_{\lambda \to \infty} \int_0^{\lambda} (1 - \frac{S}{\lambda})\, \phi(S) \cos \xi\, d_S \tag{11}$$

As a special case, if $C_n = 0$, then one has

$$f(x) = e^{-x^{\mu} \cos \mu \pi} \sin (x^{\mu} \sin^{\mu} \pi)$$

$$\text{for all } \mu < 1/2 \tag{12}$$

The above equation assumes the convergence of $\beta(n)$ and hence the stability of $\rho(x)$ in (8). The response in (8) is observed to be expressed in terms of even derivatives of the stimulations. Therefore, $\bar{S}$ may also be called a differentiating action network. If Δ is small, then higher derivatives are negligible in magnitude in the Taylor's expansion, i.e. only the second derivative is the dominant factor. Consider the case where Δ is small and the action coefficients is such that

$$a^{i_m}_{j_n} = \begin{cases} 2 & \text{if } m = n \\ -1 & \text{if } /m-n/ = 1 \\ 0 & \text{all other cases} \end{cases} \tag{13}$$

then the stimulus-response relation for a particular element is

$$\rho(x) = -\,\sigma(x-\Delta) + 2\sigma(x) - \sigma\,(x-\Delta)$$

$$= -\sigma(x) + \sigma'(x)\Delta - \sigma''\,(x)\,\frac{\Delta^2}{2!} + 2\sigma(x) - \sigma^9(x) - \sigma^1(x)\Delta$$

$$-\,\sigma''\,(x)\,\frac{\Delta^2}{2!}$$

$$= -\sigma''(x)\;\Delta^2$$

with the motivation, the theorem of Binomial connectivity comes in order.

Theorem II

If in a finite ordered set of infinite, periodic, symmetrical sets with periodicity one such that each set is acted upon by the preceding one with action coefficients satisfying (13), then the responses of the elements of the last set to be acted upon are 2x (number of sets-1)th derivative of the stimulus function and these responses can be expressed in terms of the stimulus function with its coefficients according to binomial expansion in the following form

$$y^{(2n)}(x) = \sum_{r=0}^{2n} (-1)^{2^r+n} \binom{2n}{r}\, y\,[x-(n-r)\Delta] \tag{14}$$

where y(x) is the stimulus function of a particular element and Δ is assumed to be small.

Proof

$$y^{(2)}(x) = -y(x-\Delta) + 2y(x) - y(x+\Delta)$$

$$= \sum_{r=0}^{2} (-1)^{1+r} \binom{2}{r}\, y\,[x-(1-r)\Delta]$$

$$y^{(4)}(x) = -y^{(2)}(x-\Delta) + 2y^{(2)}(x) - y^{(2)}(x+\Delta)$$

$$= -[-y\,(x-2\Delta) + 2y(x-\Delta) - y(x)]$$

$$+2[-y\,(x-\Delta) + 2y(x) - y(x+\Delta)]$$

$$-[-y(x) + 2y(x+\Delta) - y(x+2\Delta)]$$

$$= \sum_{r=0}^{4} (-1)^{2+r} \binom{4}{r}\, y\,[x-(2-r)\,\Delta]$$

Now assume

$$y^{(2m)}(x) = \sum_{r=0}^{2m} (-1)^{r+m} \binom{2m}{r} y[x-(m-r)\Delta]$$

then

$$y^{2(m+1)}(x) = -y^{(2m)}(x-\Delta) + 2y^{(2m)}(x) - y^{(2m)}(x+\Delta)$$

$$= \sum_{r=0}^{2m} (-1)^{r+m+1} \binom{2m}{r} y[x-(m-r+1)\Delta]$$

$$+ 2\sum_{r=0}^{2m} (-1)^{r+m} \binom{2m}{r} y[x-(m-r)\Delta]$$

$$+ \sum_{r=0}^{2m} (-1)^{r+m+1} \binom{2m}{r} y[x-(m-r-1)\Delta]$$

$$= \sum_{r=0}^{2m} \Big[(-1)^{m+r+1} \binom{2m}{r} + 2(-1)^{m+r+1} \binom{2m}{r-1}$$

$$+ (-1)^{m+r+1} \binom{2m}{r-2} \Big] . \; y[x-(m-r-1)\Delta]$$

$$y^{2(m+1)}(x) = \sum_{r=0}^{2m} (-1)^{m+r+1} \binom{2m}{r} + 2\binom{2m}{r-1} + \binom{2m}{r-2} \; . y[x-(m-r-1)\Delta]$$

Where

$$\binom{2m}{r} + 2\binom{2m}{r-1} + \binom{2m}{r-2} = \frac{2m!}{(2m-r)!(r!)} + \frac{2(2m!)}{(2m+1-1)!(r-1)!} + \frac{2m!}{(2m+2-1)!(r-2)!}$$

$$= \frac{(2m+2)!}{(2m+2-r)!\,r!} \frac{1}{(2m+2)(2m+1)} \Big[(2m+2-r)(2m+1-1) + 2(2m+2-1)\;r + r(r-1)\Big]$$

$$= \frac{(2m+2)!}{(2m+2)!\,r!} = \binom{2m+2}{r}$$

$$y^{2(m+1)}(x) = \sum_{r=0}^{2m} (-1)^{m+r+1} \binom{2m+2}{r} y[x-(m-r-1)\Delta] \qquad \text{Q.E.D.}$$

in terms of ρ's and σ's, the action relation between any two consecutive sets can be expressed in the following form

$$\rho^{(m-1)}_{k} = \sum_{r=0}^{2m} (-1)^{r+m} \binom{2m}{r} \sigma^{(m-1)}_{(r+k-m)} \qquad (15)$$

where the superscripts represent the number of the set while the subscripts denote the location of a particular element in the set concerned.

4.2 One-Dimensional Symmetrical Interaction Networks

Consider an infinite, periodic set S with periodicity "1" such that

(1) Elements in S are located in a one-dimensional layer and each element is Δ distance apart from its two adjacent elements.

(2) The interaction coefficients which define the activities transferred from one element to another satisfy the following conditions

(a) $a^{m+k}_{n} = a^{m-k}_{n}$

(b) $a^{m}_{n} = a^{k}_{m+k-n}$

S is called a one-dimensional symmetrical interaction network. The stimulation response relation can be expressed in the following way:

$$\rho(x) = a^{o}\sigma(x) + \alpha_{o}\,\rho(x) + \alpha_{1}[\rho(x+\Delta) + \rho(x-\Delta)] + \alpha_{2}[\rho(x+2\Delta) + \rho(x-2\Delta)]$$

$$+ \alpha_{3}[\rho(x+3\Delta) + \rho(x-3\Delta)] + \ldots$$

where

$$\alpha_{o} = a^{m}_{n} \quad \text{for } m = n$$

$$\alpha_{n} = a^{m}_{n} \quad \text{for } |m-n| = K$$

Solving for $\sigma(x)$, one has

$$\sigma(x) = a_o' \; P(x) + a_1'[P(x+\Delta) + P(x-\Delta)] + a_2' \; [P(x+2\Delta) + P(x-2\Delta)] + \ldots$$

where

$$a_i' = \frac{a_i}{a^o} \qquad i \neq 0 \qquad \text{and } a^o \neq 0$$

$$a_o' = \frac{(a_o - 1)}{a^o}$$

By Taylor's expansion, the above equation can be written as,

$$\begin{aligned}
\sigma(x) = a_o' \; P(x) + a_1'[\; & \rho(x) + \rho'(x)\Delta + \rho''(x)\frac{\Delta^2}{2!} + \rho'''(x)\frac{\Delta^3}{3!} + \ldots \\
& + \rho(x) - \rho'(x)\Delta + \rho''(x)\frac{\Delta^2}{2!} - \rho'''(x)\frac{\Delta^3}{3!} + \ldots] \\
& + a_2'[\rho(x) + 2\rho'(x)\;\Delta + a^2\rho''(x)\frac{\Delta^2}{2!} + 2^3\rho'''\frac{\Delta^3}{3!} + \ldots \\
& + \rho(x) - 2\rho'(x)\;\Delta\; 2^2\rho''(x)\frac{\Delta^2}{2!} - 2^3\rho'''\frac{\Delta^3}{3!} + \ldots] \\
& + \ldots
\end{aligned} \tag{16}$$

$$\begin{aligned}
&= \rho(x)\frac{2\;\Delta^o}{0!}[\;1^o a_1' + 2^o a_2' + \ldots + \frac{1}{2}a_o'\;] \\
&+ \rho''(x)\frac{2\;\Delta^2}{2\;!}[\;1^2 a_1' + 2^2\; a_2' + \ldots\;] \\
&+ \rho''''(x)\frac{2\;\Delta^4}{4!}[\;1^4 a_1' + 2^4 a_2' + 3^4 a_3' + \ldots] + \ldots
\end{aligned}$$

$$\sigma(x) = \rho(x)\;\; 2\beta_o'\;\frac{\Delta^o}{0!} + \rho''(x)\;\; 2\beta_2'\;\frac{\Delta^2}{2!} + \rho''''(x)\;\; 2\beta_4'\;\frac{\Delta^4}{4!} + \ldots$$

$$= 2\sum_o^\infty \; \rho^{(2i)}(x)\;\frac{\Delta^{2i}}{2i!}\;\beta_{2i}' \tag{17}$$

where

$$\beta_i' = \begin{cases} \sum_o^\infty \; i^{\,j} a_i & \text{when } j \neq 0 \\ \frac{1}{2}a_o' + \sum_{i=1}^\infty \; a_i' & \text{when } j = 0 \end{cases}$$

Equation (17) represents a stable interaction if the series converges. Notice that Equation (17) is identical in form with Equation (8). Thus all the conditions which make (8) a converging series also apply to (17), i.e. Equation (10) and (11) may be used to approximate $\beta'(n)$.

We have shown that in finite linear networks a function is stable if the coefficient matrix is nonsingular. In the infinite networks, we define that the function of a network is stable if the stimulations of the network can be expressed in terms of its responses and vice versa.

Lemma: Stability Criterion

In infinite linear networks, an action or interaction function is stable if and only if its transfer coefficients together with that of the corresponding interaction or action functions satisfy the following conditions.

$$\alpha_{jm}^{im} \ \sigma_{jm} = \alpha_{m}^{\prime m} \rho_m + \alpha_{o}^{\prime m} \ \sigma_m$$

$$\alpha_{jn}^{im} \ \sigma_{jn} = \alpha_{n}^{\prime m} \rho_n \qquad (18)$$

Proof:

Assume the function is stable. For finite networks (18) is satisfied as we have shown in previous sections. Since we are dealing with discrete elements, elements of an infinite network can be put in 1-1 correspondence with positive integers. Hence, by the second law of mathematical induction, (18) is satisfied Now, suppose (18) is satisfied, by the definition of stability, the function must be stable.

Theorem III:

In infinite linear networks, for each stable action function, there exists an equivalent interaction function and vice versa.

Proof:

By the stability criterion, it is clear that in infinite linear networks, stable action and interaction functions are functionally equivalent.

Corollary:

For each stable infinite symmetrical linear action network, there exists an equivalent infinite symmetrical linear interaction network and vice versa.

Proof:

Recall equations (8) and (17) where

$$\rho(x) = 2 \sum_{o}^{\infty} \sigma^{(2i)}(x) \frac{\Delta^{2i}}{2i!} \beta_{2i} \qquad (8)$$

$$\sigma(x) = 2 \sum_{o}^{\infty} \rho^{(2i)}(x) \frac{\Delta^{2i}}{2i!} \beta_{2i} \qquad (17)$$

With Δ's the same in both equations and stability assumed then

$$\beta_{2i}\, \sigma^{(2i)} \neq \beta'_{2i}\, \rho^{(2i)}$$

Thus $\rho(x)$ and $\sigma(x)$ in either equation can switch places under the assumption of stability.

5. FINITE PERIODIC NETWORKS

If a set S of a finite number of linear elements are grouped into m subsets S_1, S_2, ..., S_m such that $S = \bigcup_{k=1}^{m} S_k$ and each S_k contains M elements which possess some operational characteristics, then S is called a finite periodic network with periodicity $\pi = M$.

5.1 One Dimensional Symmetrical Action and Interaction Networks

Symmetrical networks in the finite case are defined as the infinite case. From Equations (8) and (17), the following equations are obtained

$$\rho(x) = 2 \sum_{o}^{n} \sigma^{(2i)}(x) \frac{\Delta^{2i}}{2i!} \beta_{2i}$$

$$\beta_0 = \begin{cases} \sum_{i=1}^{n} i^j \alpha_i & j \neq 0 \\ 1/2\, \alpha_o + \sum_{i=1}^{n} \alpha_i & j = 0 \end{cases} \tag{20}$$

$$\sigma(x) = 2 \sum_{o}^{n} \rho^{(2i)}(x) \frac{\Delta^{2i}}{2i!} \beta'_{2i}$$

$$\beta_j = \begin{cases} \sum_{o}^{n} i^j \alpha'_i & j \neq 0 \\ 1/2\, \alpha_o + \sum_{i=1}^{n} \alpha'_i & j = 0 \end{cases} \tag{21}$$

Consider two sets S_i and S_j such that S_i acts upon S_j, if an action coefficient a_j^i satisfies a binomial connectivity theorem, then the stimulus-response relation between the two sets can be expressed in matrix form below

$$\begin{bmatrix} P_1^j \\ \vdots \\ p_m^j \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & .0 \\ \dots & & & & \\ 0 & \dots & & & +2 \end{bmatrix} \begin{bmatrix} \sigma_1^j \\ \vdots \\ \sigma_m^j \end{bmatrix} \tag{22}$$

$$\text{or } ||P|| = ||A||\ ||\sigma|| \tag{23}$$

As was mentioned in the previous section under the binomial connectivity, the response of the last set can be expressed in terms of the stimulation of the first set in a multi-set action network. The general expression in the finite case is given below

$$\phi_k^{(2m)} = \sum_{r=0}^{2m} (-1)^{r+m} \binom{2m}{r} \sigma_{r+k-m} \qquad \sigma = 0 \text{ for } \begin{cases} k \leq 0 \\ k > m \end{cases} \tag{24}$$

where

n = order of the action coefficient matrix

2m = order of the derivative

K = location of a particular row in the coefficient matrix

The equivalent interaction function corresponding to the action function in (22) can be obtained by the direct inversion of A as shown below

$$||A|| = \frac{1}{n+1} \begin{bmatrix} n & (n-1) & (n-2) & 1 \\ (n-1) & 2(n-1) & 2(n-2) & 2 \\ \vdots & \vdots & 3(n-2) & \vdots \\ \cdots & \cdots & \cdots & \cdots \\ 2 & & & (n-1) \\ 1 & 2 & (n-1) & n \end{bmatrix} \tag{25}$$

Since $||A||^{-1} = \dfrac{||A_i^j||}{|a_j^i|}$ we have

$$A_i^j = \begin{cases} j(n-j+1) & \text{if } j = i \\ i(n-j+1) & \text{if } j > i \\ j(n-i+1) & \text{if } i > j \end{cases} \tag{26}$$

A general interaction expression for the 2nd derivative is given below

$$\sigma^{(2)}_{K(n)} = \frac{1}{n+1}\left[\sum_{i=1}^{K} (i)\ (n-K+1)\ \rho_i + \sum_{j=0}^{(n-K+1)} K(n-K-j)\ \rho_{K+j+1}\right] \qquad (27)$$

where n = rank of the coefficient matrix.

As an example for n = 9, m = 2, k = 3

$$\sigma^{(2)}_{3(9)} = \frac{1}{10}\left[7\rho_1 + 14\rho_2 + 21\rho_3 + 18\rho_4 + 15\rho_5 + 12\rho_6 + 9\rho_7 + 6\rho_8 + 3\rho_9\right]$$

Three dimensional models of 11 × 11 action and interaction matrices were built for the purpose of visualizing the transformation. Figure 20 shows the model of action matrix while Figure 21 shows interaction model.

5.2 One Dimensional Asymmetrical Network

A one-dimensional asymmetrical action network was proposed by Milner* in order to account for the so-called "sharpening effect"[7] performed by the nerve net associated with the basilar membrane. The place theory requires high accuracy in determining the maximum amplitude of the oscillation of the basilar membrane in order to account for the observed high pitch discrimination.

According to Békésy[13], however, we know that only a relative flat maximum is observed in the basilar membrane (Figures 22 and 23). According to Milner, the cutoff is much more strongly pronounced if the asymmetrical action network were used as a sharpening device.

The proposed action function along with its interaction function in matrix form are shown below

$$||A|| = \begin{bmatrix} 1 & 0 & 0 & \cdots\cdots\cdots & 0 \\ 1 & 2 & 0 & \cdots\cdots\cdots & 0 \\ 1 & 2 & 3 & 0\cdots\cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ 1 & 2 & & \cdots\cdots\cdots & n \end{bmatrix} \qquad (28)$$

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* Milner, P. M., Oral Communication with Professor H. von Foerster of the University of Illinois

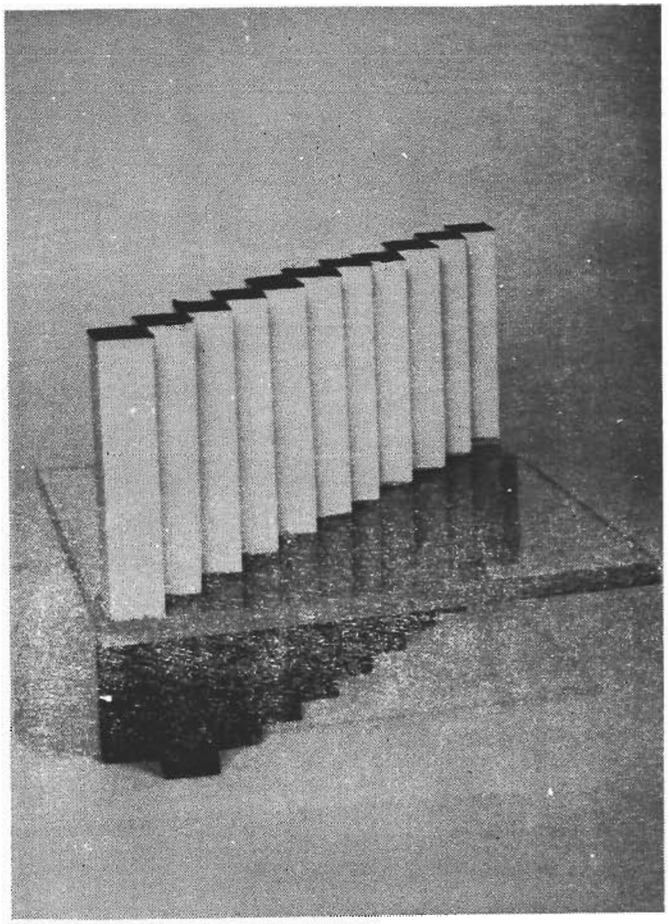

FIGURE 20

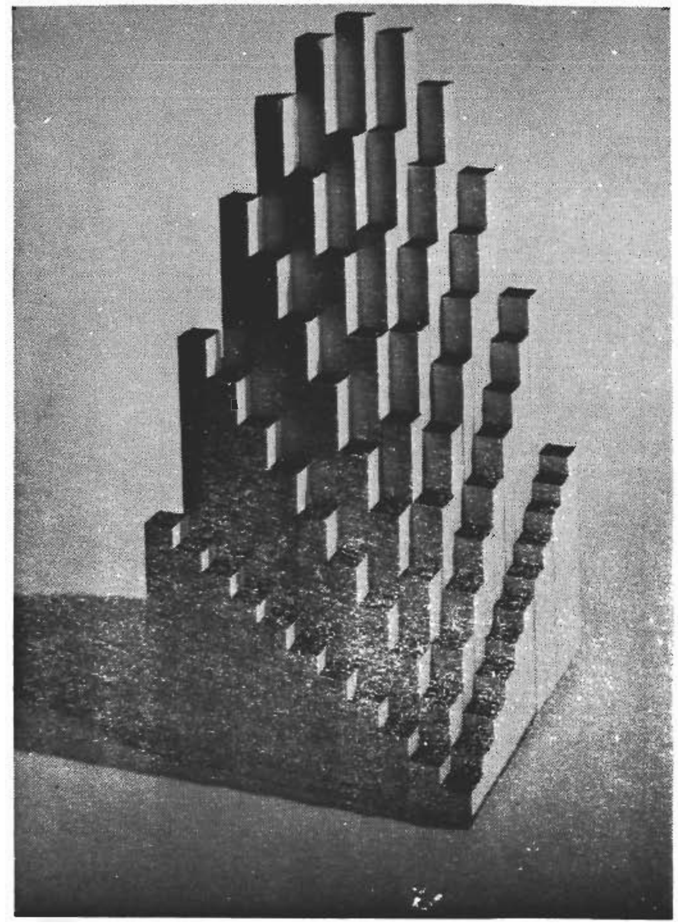

FIGURE 21

FIGURE 20 SHOWS THREE-DIMENSIONAL MODEL OF AN 11 × 11 SYMMETRICAL ACTION FUNCTION AND FIGURE 21 SHOWS THE CORRESPONDING INTERACTION FUNCTION OBTAINED FROM THE ACTION FUNCTION THROUGH PROPER LINEAR TRANSFORMATION. THE WHITE BLOCKS IN FIGURE 20 DENOTE POSITIVE FUNCTIONAL VALUES (+2) WHILE THE BLACK ONES DENOTE NEGATIVE FUNCTIONAL VALUES (-1). NOTE THAT ALL FUNCTIONAL VALUES OF THE INTERACTION FUNCTION ARE POSITIVE AS SHOWN IN FIGURE 21.

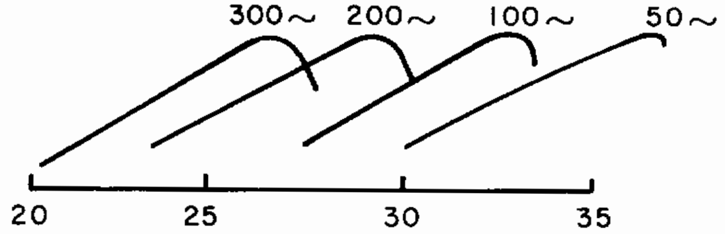

FIGURE 22. AMPLITUDE AT VARIOUS DISTANCE ALONG THE COCHLEAR PARTITION. REPRODUCED FROM BEKESY.

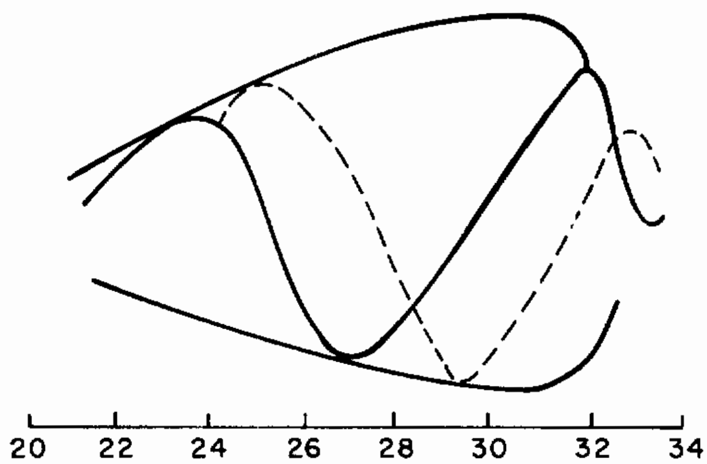

FIGURE 23. THE MEASURED LONGITUDINAL BENDING OF THE COCHLEAR PARTITION FOR A TONE OF 200 cps FOR TWO MOMENTS IN TIME SEPARATED BY A QUARTER TIME. REPRODUCED FROM BEKESY.

$$||A||^{-1} = \begin{bmatrix} 1 & 0 & \cdots\cdots\cdots & 0 \\ 1/2 & 1/2 & 0 \cdots\cdots & 0 \\ 0 & -1/3 & 1/3 \quad 0 \cdots & 0 \\ 0 \cdots\cdots & \cdots & -1/n & 1/n \end{bmatrix} \qquad (29)$$

The sharpening effect of this asymmetrical action function is demonstrated in Figure 24. Three dimensional models of the two matrices given by Equations (28) and (29) were built and are shown in Figure 25. A similar structure was operated on a Pace analog computer by Cheng* on the latter part of this report.

5.3 Two Dimensional Networks

Consider a set S of linear elements being partitioned into M ordered subsets $S_1, S_2, \ldots, S_m$ such that

(1) $S = \bigcup_{k=1}^{n} S_k$

(2) Each subset S_k contains an equal number of elements and all elements in each S_k are located in a plane.

(3) Each subset is acted upon by the preceding one. S is called a two dimensional finite action network. We will assume that each element in a certain subset is located by coordinate system. Action relation between two consecutive subsets of S is sketched in Figure 26.

In Figure 26 one element in S_i and six elements in S_{i+1} are taken as sample representations of the action relation between two sets. K_{i+1} is action coefficient from the ith set to the (i+1)th set. If each set contains n elements then the coefficient matrix is an $n \times n$ matrix. In order to find the equivalent interaction, one can simply invert the coefficient matrix. However, if n is large, the inversion becomes a tedious task. Although inversion of matrices by partition has been mentioned in many places in the literature[14], a simple method derived by Hohn[15] is given below.

Let $||A||$ be the action coefficient matrix. Then $||A||$ can be written in partitioned form as

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* Cheng, Shih-mei, "Simulation of Interaction Function on a Pace Analog Computer", Figure 33.

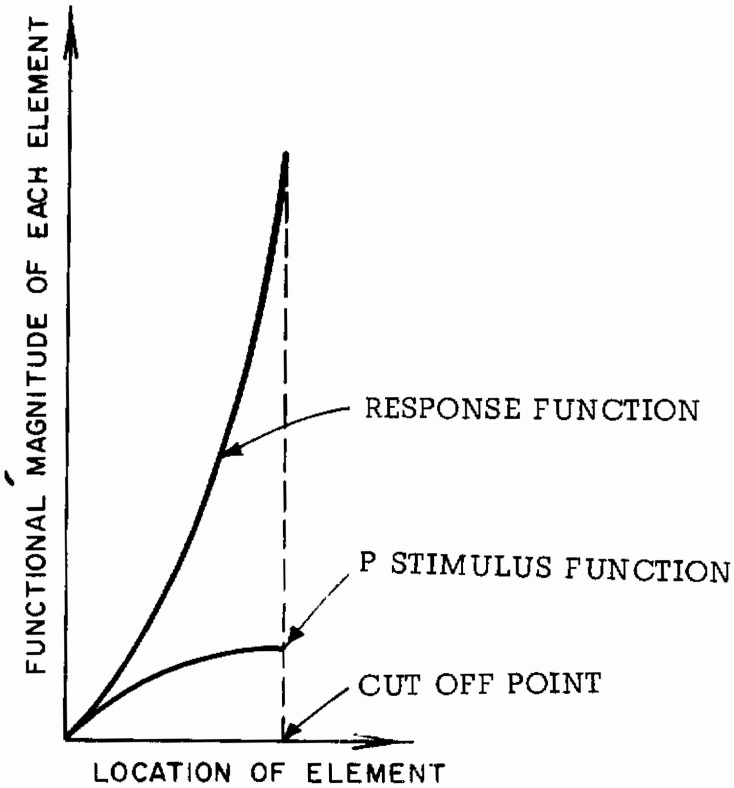

Figure 24. Stimulus-Response Relation for ar Asymmetrical Action Network

FIGURE 24. STIMULUS-RESPONSE RELATION FOR AN ASYMMETRICAL ACTION NETWORK

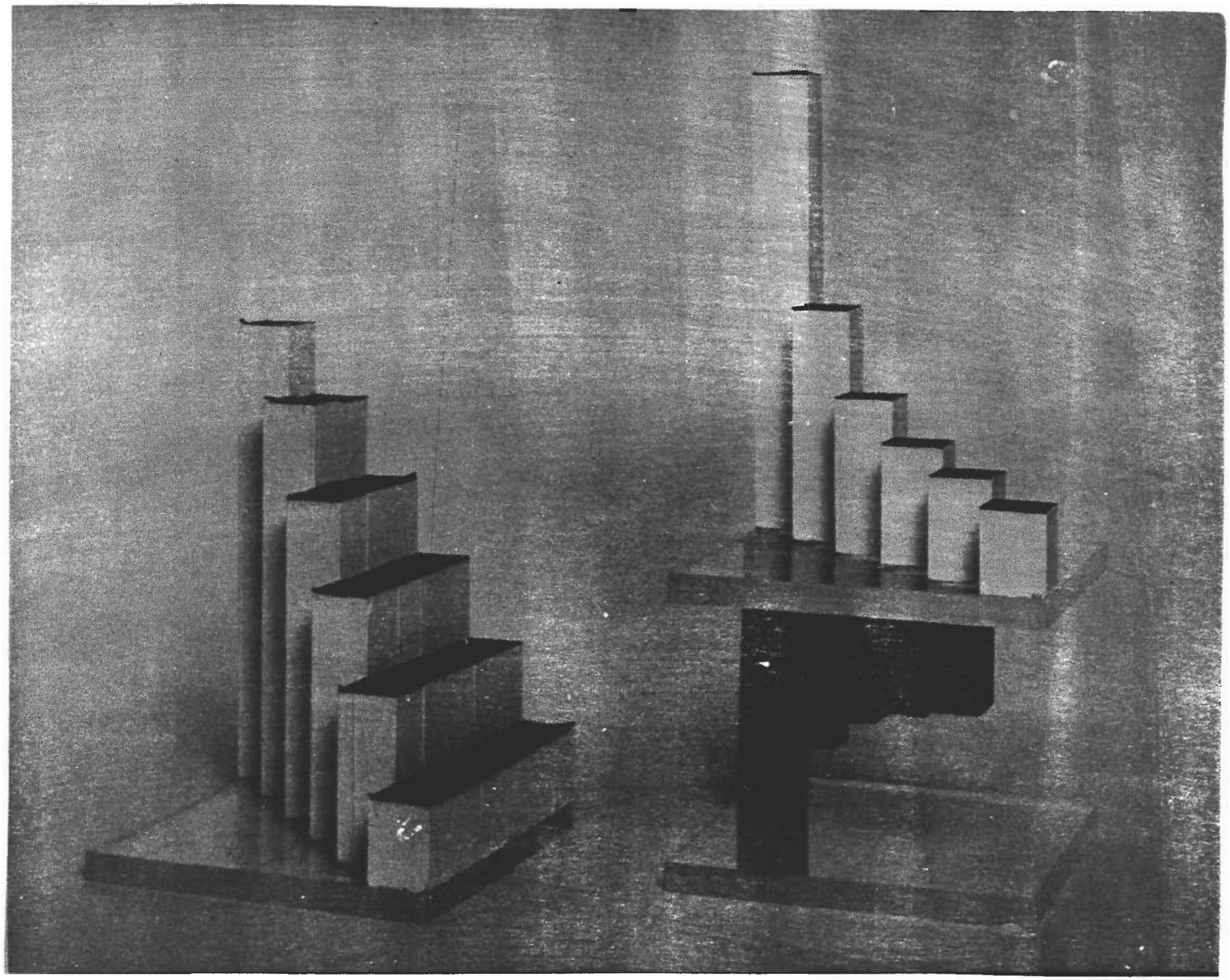

FIGURE 25. THREE-DIMENSIONAL MODELS OF ASYMMETRICAL ACTION AND INTERACTION FUNCTIONS.

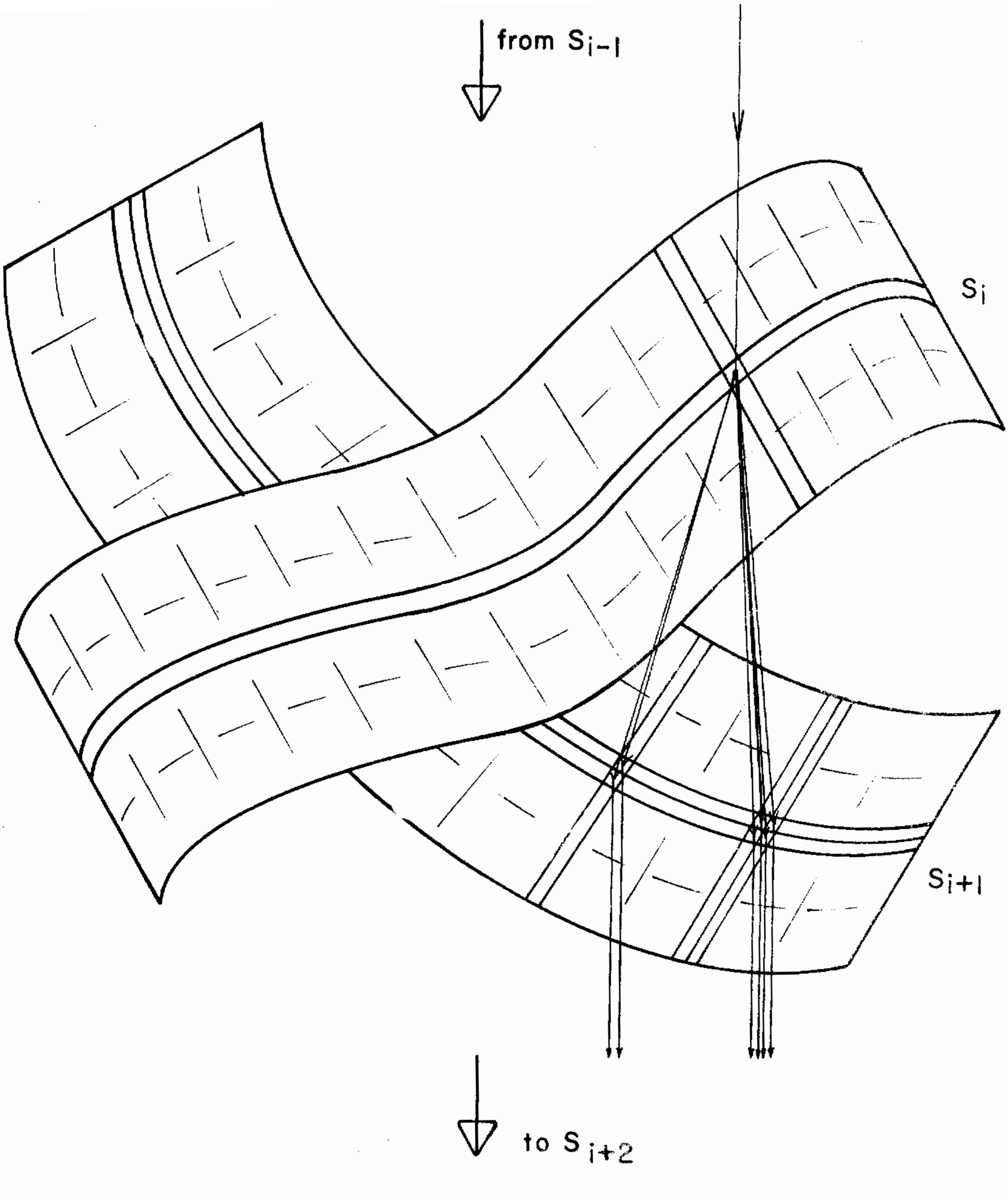

FIGURE 26. ACTION RELATION BETWEEN TWO SETS

$$||A|| = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (30)$$

$$||A||^{-1} \begin{bmatrix} A_{11}^{-1} + BC^{-1}B^T & -BC^{-1} \\ (-BC^{-1})^T & C^{-1} \end{bmatrix} \quad (31)$$

where $B = A_{11}^{-1} A_{12}$, $B^T = A_{21} A_{11}^{-1}$

$$C = A_{22} - A_{21} A_{11}^{-1} A_{12} = A_{22} - A_{21} B$$

If n is large in Equation (30), then in order to obtain A_{11}^{-1}, a partitive process is involved. Tensor notations are employed in order to reduce labor.

Suppose two planes $S_1 \in S$ and $S_2 \in S$ such that S_1 precedes S_2 and the cartesian coordinate system in each plane is employed to locate the elements the action relation in tensor notation is:

$$\rho^{\beta}_{j} = A^{\beta\, i}_{j\, \alpha} \; \rho^{\alpha}_{i} \; (i,\ j,\ \alpha,\ \beta = 1,\ 2,\ \ldots,\ n) \quad (32)$$

where ρ^{β}_{j}, ρ^{α}_{i} are the responses of the elements in S_1 and S_2 respectively. Symbols α, β represent rows and i, j the columns of S_1 and S_2. Note that α, i precede β, j, alphabetically.

Consider now a set S' of linear elements such that

(1) Each element receives a stimulation from a field acting upon S'

(2) The responses of each element in S' provide stimulation to all elements of S', including itself.

(3) All elements of S' lie in the same plane.

S' is called a two-dimensional interaction network and the interaction is expressed in the following

$$
\begin{aligned}
\rho^{i_o} &= K\,\sigma^{i_o} + B^{i_o}_{\alpha_o}{}^{\alpha}\;\rho^{i}_{\alpha} \qquad (i,\ \alpha = 1,\ 2,\ \ldots\ n) \\
\sigma^{i_o}_{\alpha_o} &= C^{i_o}_{\alpha_o}{}^{\alpha}_{i}\;\rho^{i}_{\alpha} \qquad (i,\ \alpha = 1,\ 2,\ \ldots,\ n)
\end{aligned}
\tag{33}
$$

$$
C^{i_o\alpha}_{\alpha_o i} = \begin{cases} -\,B^{i_o}_{\alpha_o}{}^{\alpha}_{i} & \text{for } \begin{cases} i_o \neq i_o \\ \alpha \neq \alpha_o \end{cases} \\[2ex] \dfrac{-(B^{i}_{\alpha}{}^{\alpha}_{i} - 1)}{K} & \text{for } \begin{cases} i = i_o \\ \alpha = \alpha_o \end{cases} \end{cases}
$$

6. COMPARISON OF ONE DIMENSIONAL FINITE AND INFINITE PERIODIC NETWORKS AND OTHER RELATED NETWORKS

Finite and infinite networks not only differ at the structure but the stimulus-response relations are quite different due to the presence of boundary elements in the finite network.

Consider a one-dimensional infinite set S being acted upon by a stimulus field A. If obstructions were placed on the stimulus pathway such that only a finite number of elements, say $P_1, P_2, \ldots, P_n$ are able to receive stimulations from the acting field, then the set $\bar{S} = \{ x \mid x \in P_k, \ k = 1, 2, \ldots, n \}$ is a one-dimensional finite set. Figure 27 indicates this situation. If elements in Figure 27 are interconnected such that

$$a_n^m = \begin{cases} -1 & \text{if } |m-n| = 1 \\ 0 & \text{all other cases} \end{cases}$$

and

$$\sigma_K = 2, \ K \in [P]$$

then for uniform input stimulations with obstructions removed, zero response occurs for each element (since a + (-1) + (-1) = 0). However, if obstructions are assumed, then all elements, except the two boundary elements, have zero responses. The boundary elements, henceforth called edges, have responses "1" since 2 + (-1) + 0 = 1. This situation is termed "edge effect" which is demonstrated graphically by Cheng.

In general, distortions of input stimulations for infinite symmetrical networks cause similar effect as edge effect. However, edge effect occurs inherently in finite networks due to the existence of boundary elements. If a finite action network contains more than two sets, the distortion will propagate from one set to another, and beyond a certain set, stimulation to each element of the sets following are distorted. This situation is demonstrated in Figure 28 by a one-dimensional finite symmetrical network with its elements connected binomially. Dotted lines in Figure 28 represent the stimulation provided by

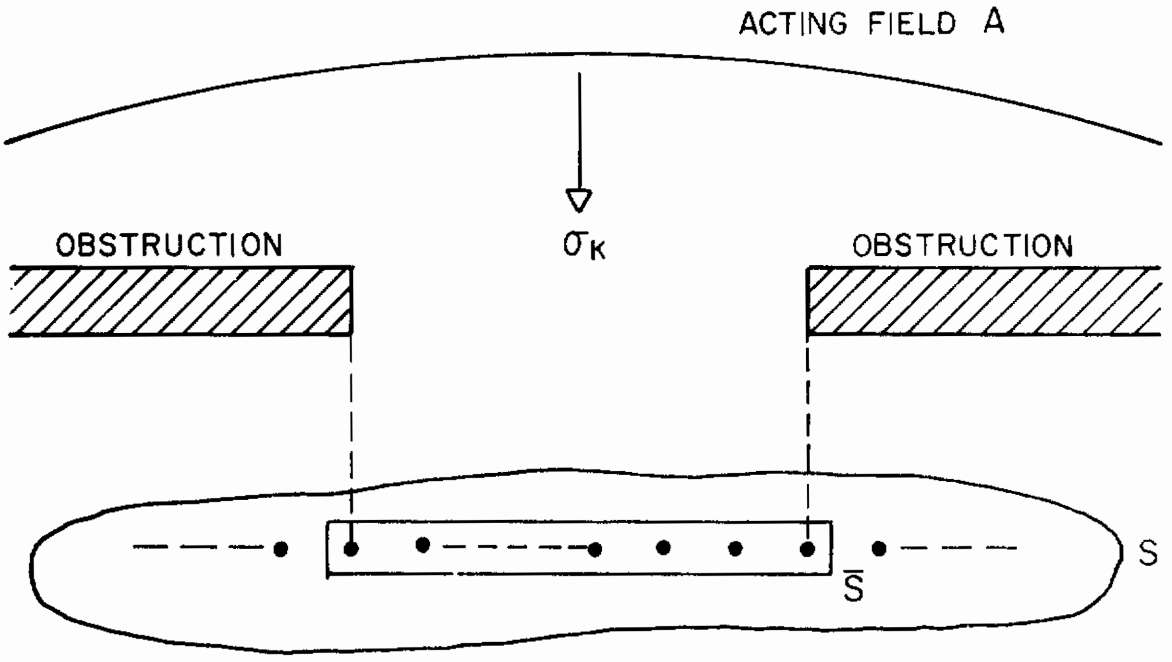

FIGURE 27. FINITE SET AS A SPECIAL CASE OF INFINITE SET

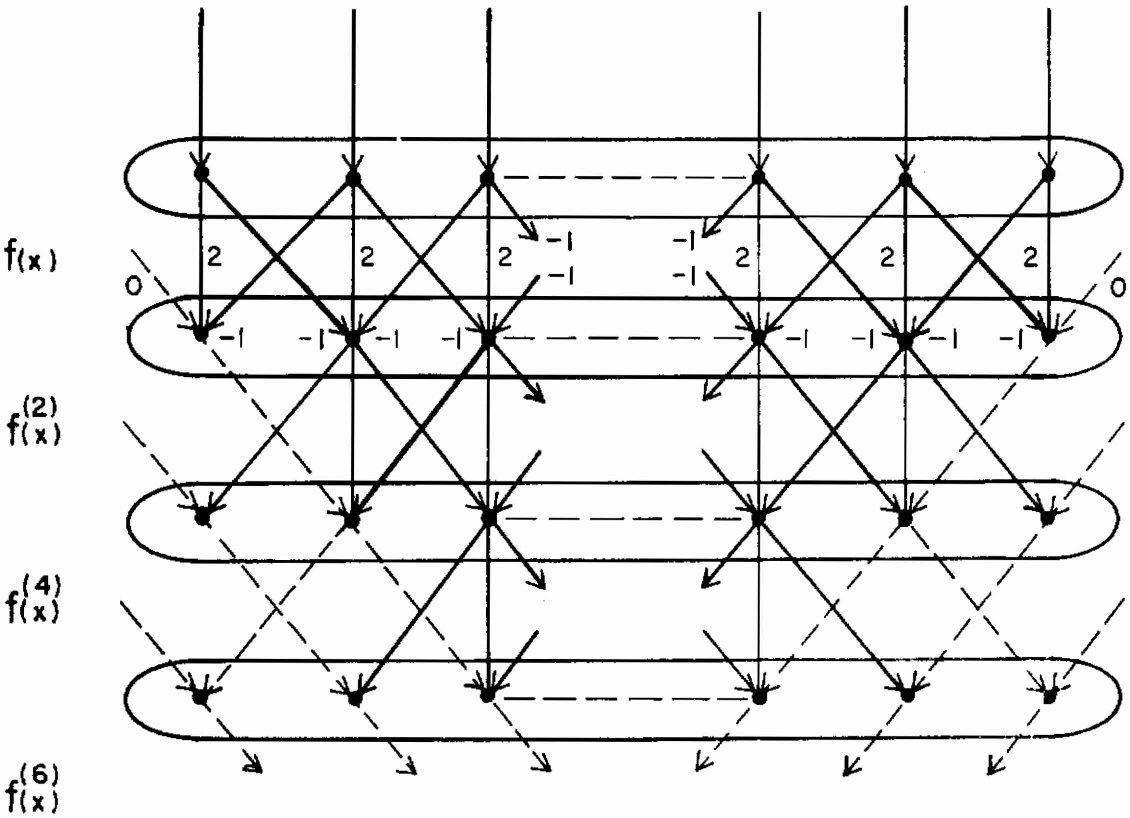

FIGURE 28. EDGE EFFECT OF A FINITE LINEAR NETWORK.

a particular element in a set to other elements of the next set is disturbed. With the aid of Figure 28, we may conclude that for a finite, symmetrical action network S such that $S = \bigcup_{k=1}^{m} S_k$, each S_k consists of n linear elements and the action coefficient satisfies Equation (14), then the input of each element is distorted for every set beyond $S_{\frac{m+1}{2}}$ if m is odd and $S_{\frac{m}{2}}$ if m is even for $n \geq \frac{m}{2}$.

Due to the propagation of distortion, some elements in each set of a finite action network will not receive the amount of stimulations expected according to connectivities, but rather the distorted stimulation. The stimulus-response relations of infinite and finite networks are compared below.

(Infinite) $$\rho_K^{(2m)} = \sum_{r=0}^{2m} (-1)^{r+m} \binom{2m}{r} \sigma_{r+K-m} \qquad (35)$$

(Finite) $\sigma_K = 0$ for $K < 0$ $\qquad \rho_K^{(2m)} = 0$ for $K < 0$

$$\rho_K^{(2m)} = \sum_{r=m+k}^{2m} (-1)^{r+m} \left[\binom{2m}{r} - \binom{2m}{2(m-K-1)-r} \right] \sigma_{r-K} \qquad (36)$$

The two equations above are demonstrated by examples below

$$\rho_o^{(*)} = \sigma_{-4} - 8\sigma_{-3} + 28\sigma_{-2} - 56\sigma_{-1} + 70\sigma_o - 56\sigma_1 + 28\sigma_2 - 8\sigma_3 + \sigma_4$$

(Infinite case)

$$\rho_o^{(*)} = 42\sigma_o - 48\sigma_1 + 27\sigma_2 - 8\sigma_3 + \sigma_4$$

(Finite case)

In order to resemble the stimulus-response relation of infinite network with a finite number of elements, a "ring" is examined. A ring R is a system of finite symmetrical sets such that the boundary elements in each set are connected the same way as other elements of the set. Consider a set R' such that $R' = R_1' \cup R_2'$, R_1' precedes R_2' and binomial connectivity between R_1' and R_2'

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* Cheng, Shih-mei, Figures 33 and 38, (pp. 93 and 101).

is assumed. Then for uniform input stimulation, zero response is obtained for each element of R_2'. If we examine the action coefficient matrix $||A'||$ of R', we see that the determinant of $||A'||$ vanishes as shown below

$$|A'| = \begin{vmatrix} 2 & -1 & 0 & \dots & -1 \\ -2 & 2 & -1 & 0 \;.. & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 \;\dots\dots & & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & \dots & -1 & 2 \end{vmatrix} = 0 \qquad (37)$$

Equation (37) implies that under binomial connectivity there exists no corresponding interaction function for the action function of a ring In order to secure stability, the connectivity between boundary elements is made variable. This structure is called a pseudo-ring. Let the boundary connectivity be x, then the determinant of the action coefficient matrix is a function of x and we have

$$D_m^{(2)} = -(m-1)\left[x - \frac{1}{m-1}\right]^2 + \frac{m^2}{m-1} \qquad (38)$$

where m is the rank of the action coefficient matrix.

Since inversion would have the following form

$$[\sigma] = \frac{[A_j^i]}{D}[\rho] \qquad (39)$$

where

$$D = \left| a_j^i \right|$$

we see that for a desired response in a particular pseudo-ring, minimum input stimulation possible can be obtained if $D = D_{max}$. Figures 14, 15, and 16 illustrate the relationships between x, D and m. Different stimulus functions applied to the ring are also shown graphically by Cheng*.

The discussions thus far are concerned mainly with the transformation of activities between sets of linear elements; If activities of one set are transferred to another, we say the first set acts upon the second and the totality of these two sets is called a linear action network and its transfer function, the action function. If activities are transferred from elements of a set to

* Cheng, Shih-mei, Figures 32 and 34, (pp. 93 and 95).

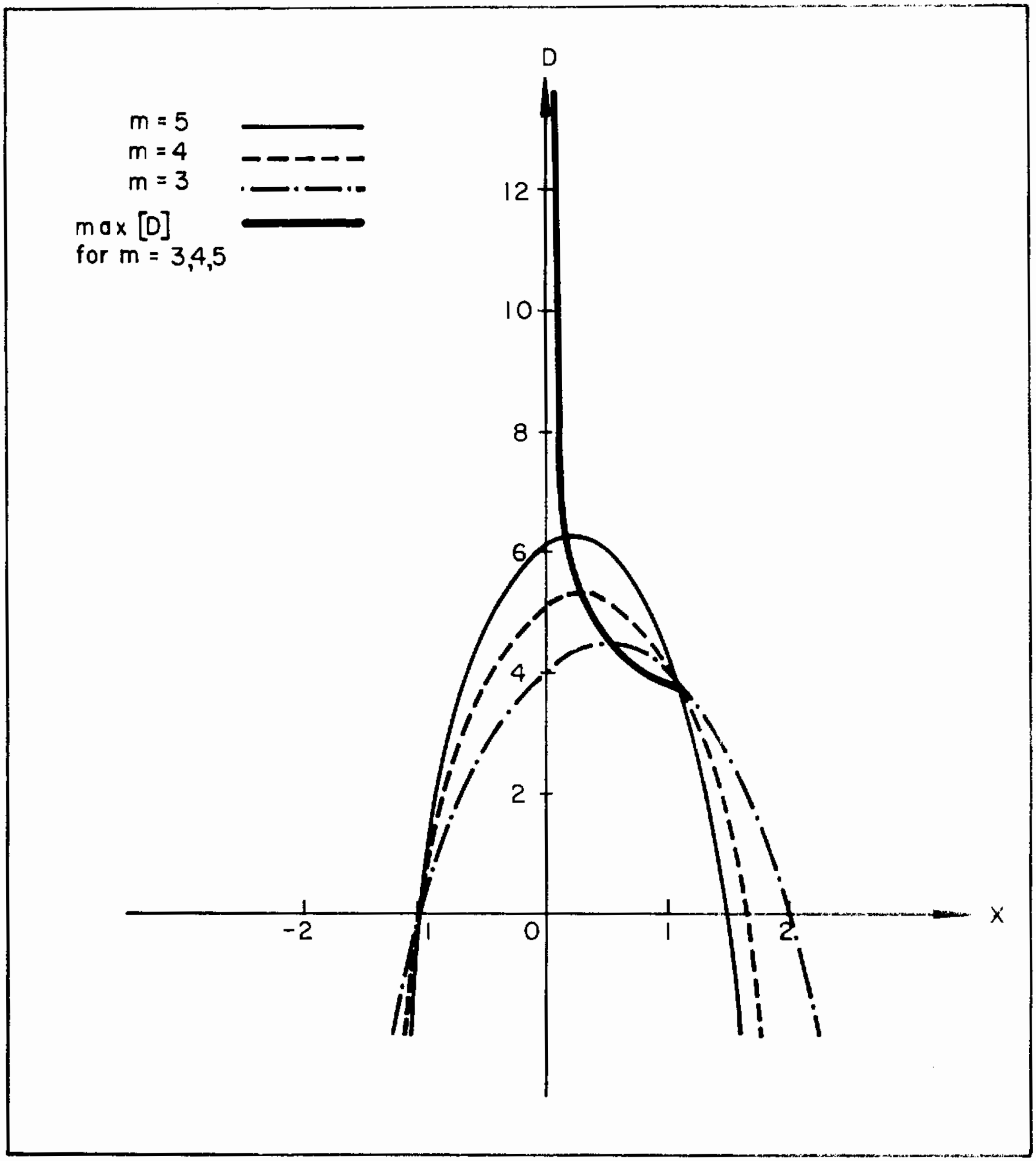


FIGURE 29. ASYMPTOTIC BEHAVIOR OF D_{MAX} WITH VARYING BOUNDARY CONNECTIVITY X

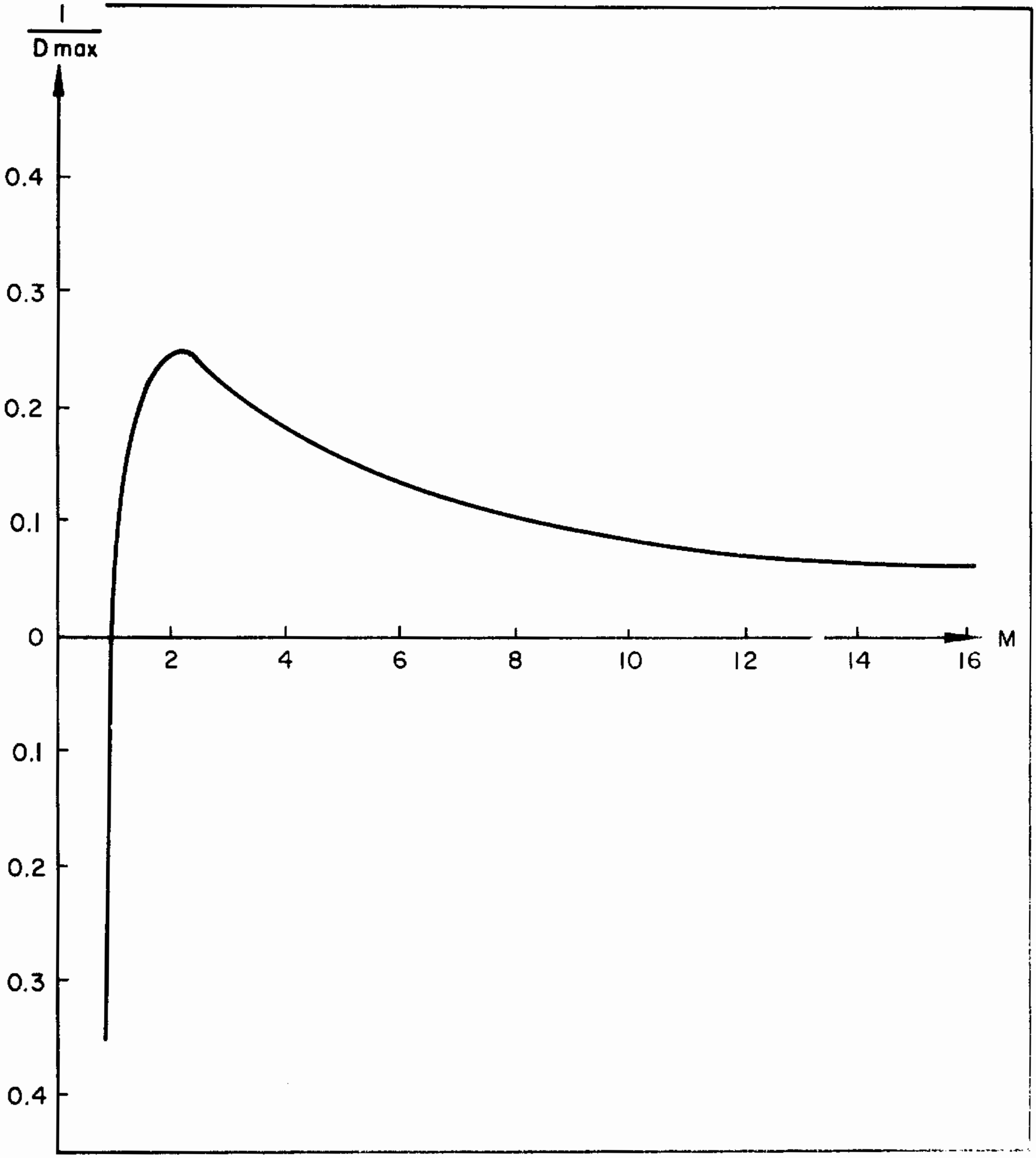


FIGURE 30. INVERSE D_{MAX} AS A FUNCTION OF M

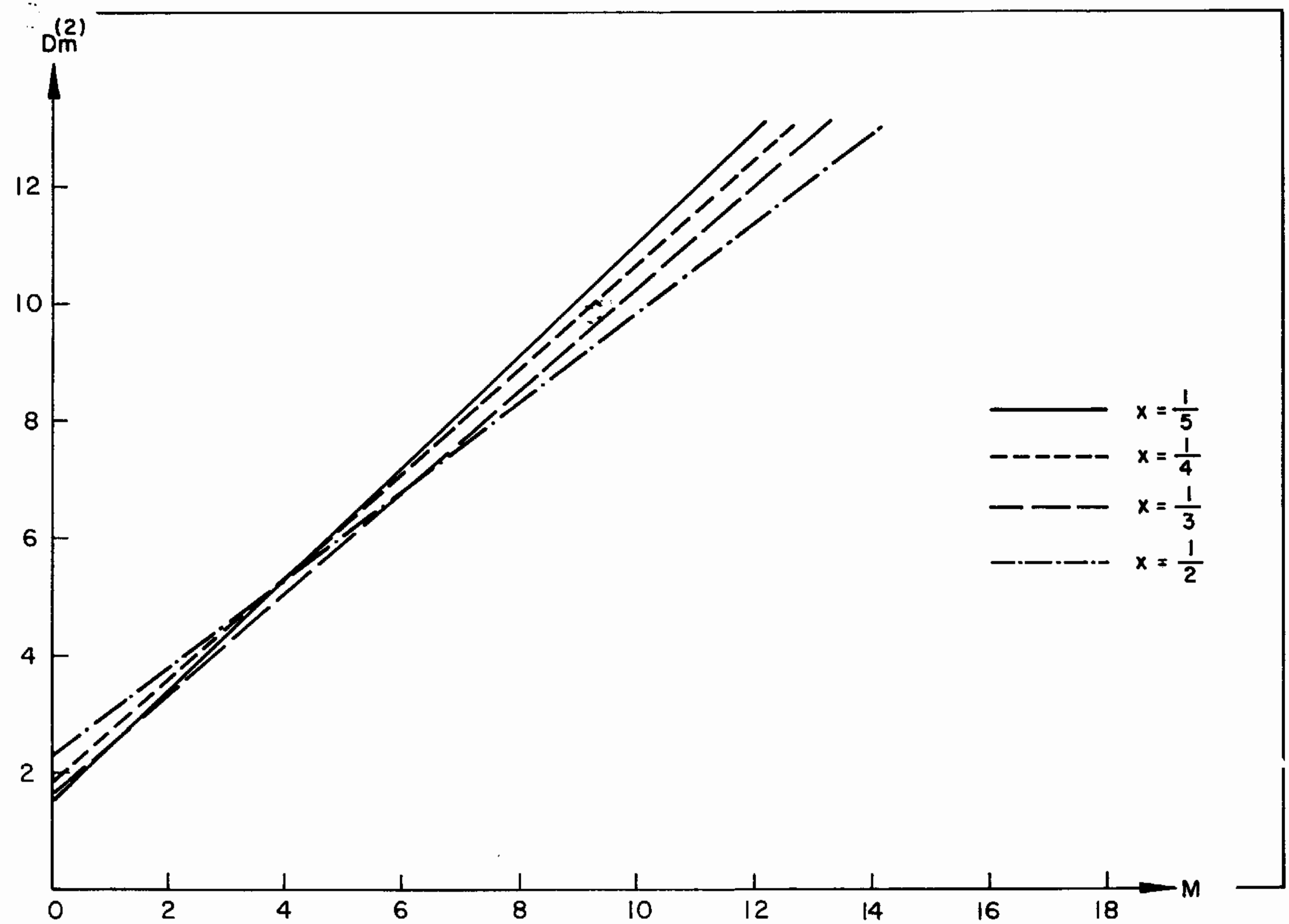


FIGURE 31. D AS A FUNCTION OF M

that of the same set, then the set is called a linear interaction network and its transfer function, the interaction function. The most important character of these functions is the functional equivalence of the two under stable states

We have concentrated mostly on one-dimensional networks, though two-dimensional networks have also been considered. Problems arise due to the inversion of huge matrices in more dimensional sets. Tensor notations have been suggested as a means of simplification; however, further investigations are needed in dealing with multi-dimensional networks.

PART III

SIMULATION OF INTERACTION FUNCTIONS ON PACE ANALOG COMPUTER

by
Shih-mei Cheng

1. INTRODUCTION

With a view to reduce the labour involved in the solution of simultaneous equations as a means to obtain approximate solution of corresponding integral equations arising from the study of interaction function, an analog system is built on the PACE computer of the Electrical Engineering Department of the University of Illinois. Furthermore, an understanding of the behavior of the interaction function can be facilitated by examining the variation of actual physical quantities. But the use of an analog computer for algebraic equations would often lead to instability, which imposes a limitation to the loop gain. Various conditions of loop gain have been considered in the literature* and if we simply employ Equation (1) in the computer,

$$[A][\rho] = [\sigma] \tag{1}$$

where A is interaction matrix

ρ is response

σ is stimulus

it has been found that the coefficients cannot be set in the range of our interest. To improve the stability, a system

$$[\rho'] + [A][\rho] = [\sigma] \tag{2}$$

is built instead. But this system is still subject to the condition that [A] is positive definite so that it would yield $[\rho'] = 0$ when a steady state is reached. In cases where [A] is not positive definite, it has been shown* that

$$[\rho'] + [A]^t [A] [\rho] = [A]^t [\sigma]$$

may bring this system into a stable region but manual operation of multiplication by $[A]^t$ is involved.

L. Gephart+ has built a system

* Rogers & Connolly "Analog Computation in Engineering Design"

\+ Landix Gephart, "Linear Algebraic Systems and the REAC", Math. Tables and Other Aids to Computations, July, 1952, National Research Council.

$$\sum_{j=1}^{n} a_{ij} X_j - B_i = \xi_i \qquad i = 1, 2, \ldots n$$

$$e_i = K\int \xi\, i dt$$

$$X_j = -\sum_{i=1}^{n} a_{ij} \quad e_i$$

which he refers to as "general method". This setup assures freedom from instability but requires approximately more than twice of the equipment used in system employed.

The usual "rule of thumb" of determining positive definiteness is to see if the diagonal elements of the matrix are much larger than the neighboring ones. Fortunately, the matrices of our interest fall into this category and to exercise efficient use of available elements on PACE, (2) is chosen as our machine equation.

Since there are 48 operational amplifiers and 100 coefficient potentiometers in the new PACE installation, and each amplifier has seven inputs as well as outputs, our matrix is limited to that with seven non-zero coefficients in each column and each row, and the dimension of the matrix may be as high as 12. If special interest dictates a matrix beyond this limitation, boosters and special arrangements may be employed to meet the requirement.

2. COMPUTER SETUP

A word of special importance which should be said about the matrix of our interest is that, since we are primarily interested in a system where every element has the same interaction with its neighbors, the matrix has similar rows except laterally displaced. So, we have

$$y_i = a_i X_i + a_{i,i-1} y_{i-1} + a_{i,i+1} y_{i+1} + \ldots$$

$$= a_i X_i + \sum_{j \neq i}^{n} a_{ij}, \quad y_j$$

Since the system is iterative, the setup for ith element will be clear enough to show the complete structure. According to Equation (2), we have the following setup: (See Figure 32.)

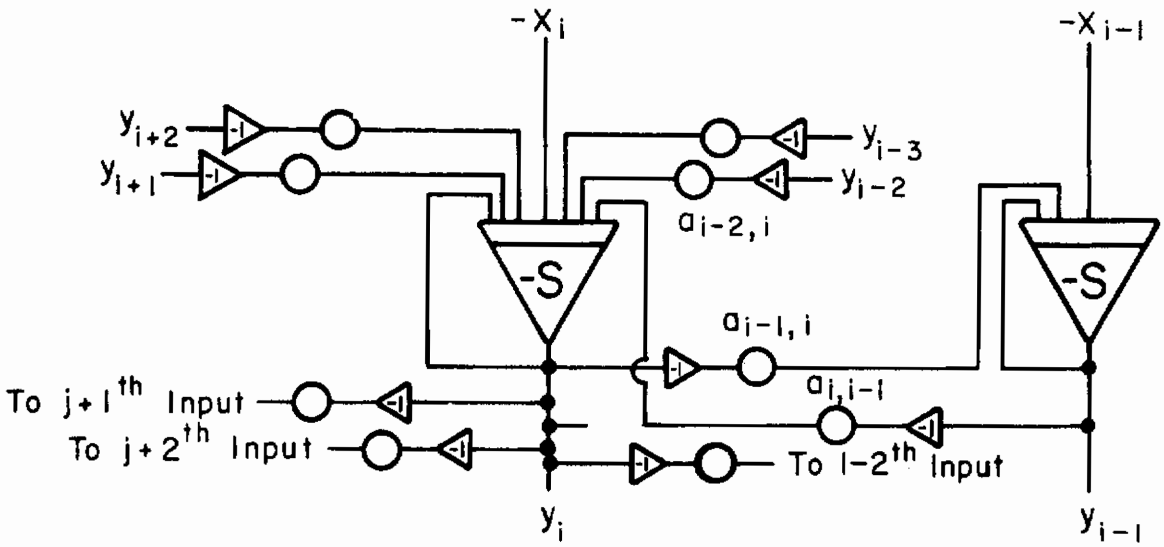


FIGURE 32. y_i THE CONNECTION of i^{th} ELEMENT WITH ITS NEIGHBORS.

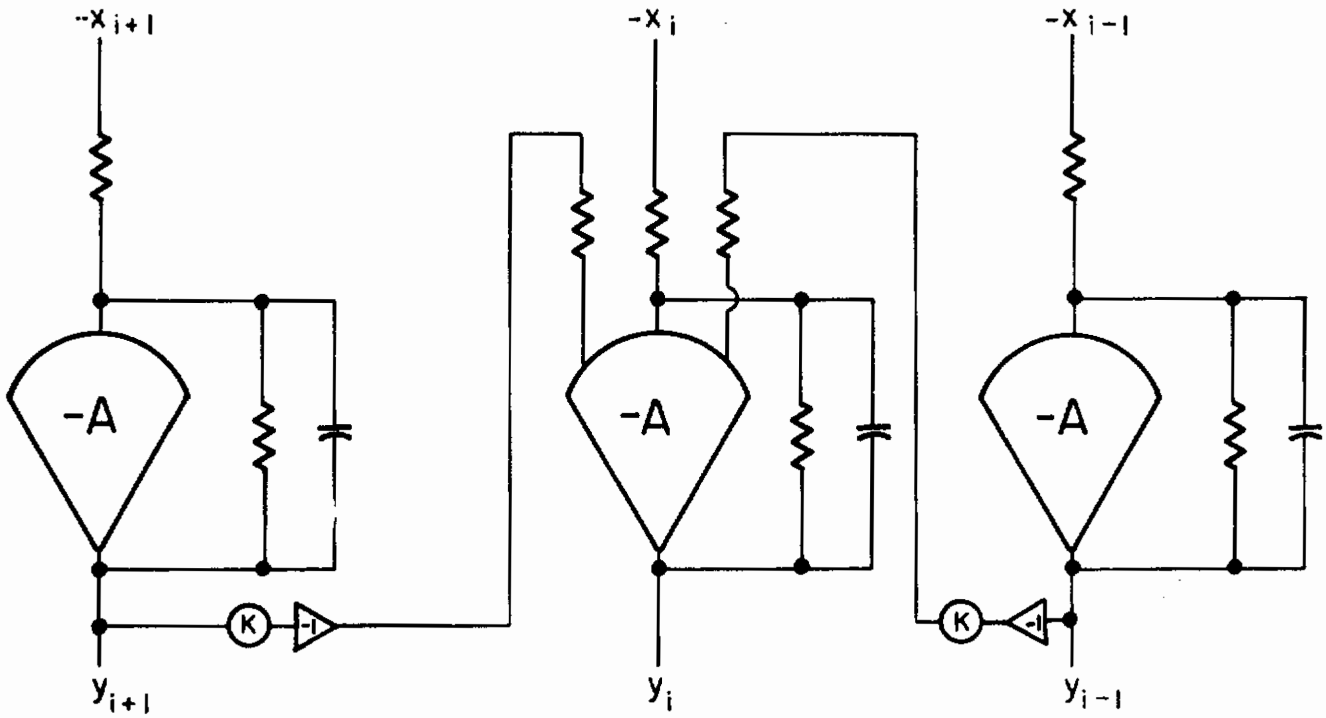


FIGURE 33. THE CONNECTION OF $y_i = + a_i X_i + K(y_{i-1} + y_{i+1})$

3. RESULTS AND DISCUSSIONS

A system of 10 × 10 matrix with each element interacting with six of its neighbors has been built on PACE. By varying the coefficient potentiometers, thus varying the interaction function, responses to different inputs have been examined as the follows:

1. Positive coefficients in the matrix correspond to inhibition in action function. Negative coefficients correspond to facilitation. This important fact can be explained directly from the computer setup as follows:

Let

$$y_i = a X_i + K(y_{i=1} + y_{i+1})$$

then

$$-K y_{i-1} + y_i - Ky_{i+1} = a X_i$$

If $K > 0$ then a phase inverter should be inserted in series with pot K, therefore, x_{i-1} goes to the input of the ith element without phase inversion, which means facilitation. (see Figure 33.)

Similarly, if $K < 0$, no phase inverter is inserted and x_{i-1} flows to the input of y_i with opposite polarity, which shows inhibition.

This fact has been verified by comparing Figure 34 with 35 where inhibition has sharpening effect at the edge and facilitation gives a round off response.

2. The effect of inhibition increases with the increase of a_{ij} or K, as we may find in Figure 34.

3. Chain structure differs from ring structure only at the ends, and so is the response. This is checked by Figures 34 and 36.

4. In the case of ring structure, if

$$y_i = X_i - K_1(y_{i-1} + y_{i+1}) - K_2 (y_{i-2} + y_{i+2}) \quad \ldots$$

$$= X_i - \sum_{j \neq i}^{n} K_j(y_{i-j} + y_{i+j})$$

Then for uniform stimulus $x_i = x =$ constant

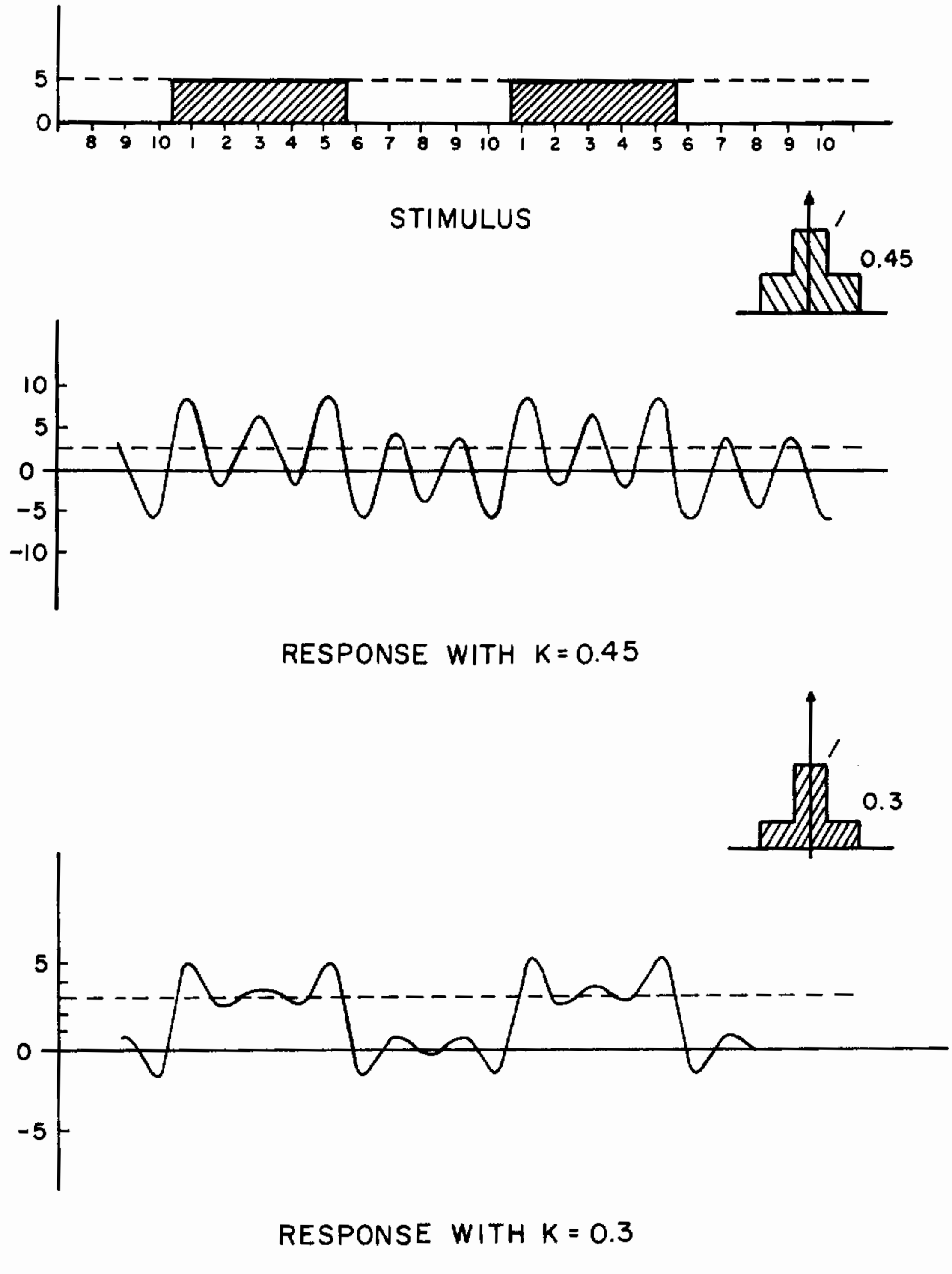


FIGURE 34. BEHAVIOR OF $y_i = x_i - K(y_{i-1} + y_{i+1})$ OF RING STRUCTURE. DOTTED LINE SHOWS UNIFORM STIMULUS AND RESPONSE.

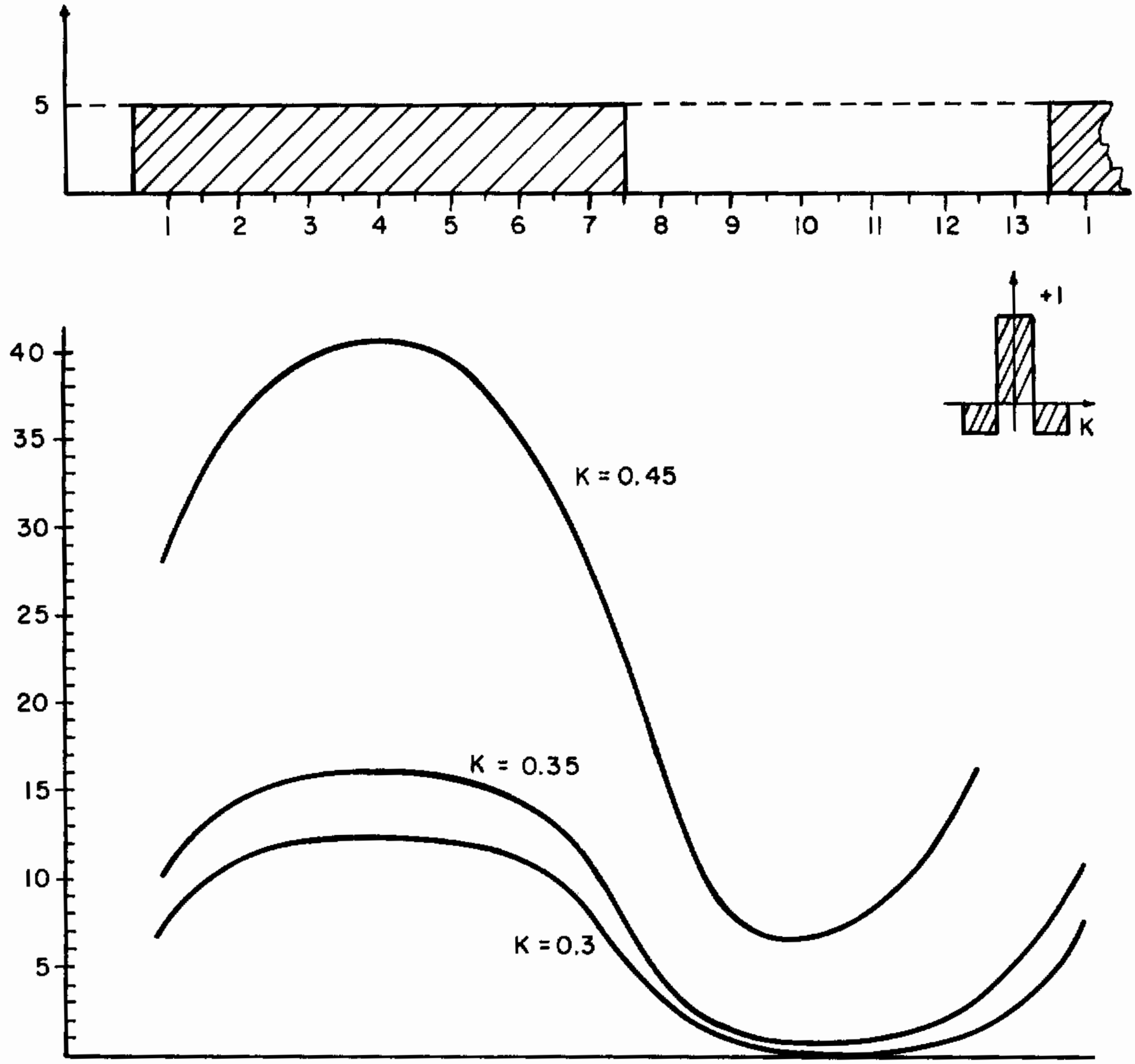


FIGURE 35. BEHAVIOR OF $y_i = X_i + K(y_{i+1} + y_{i-1})$ RING STRUCTURE.

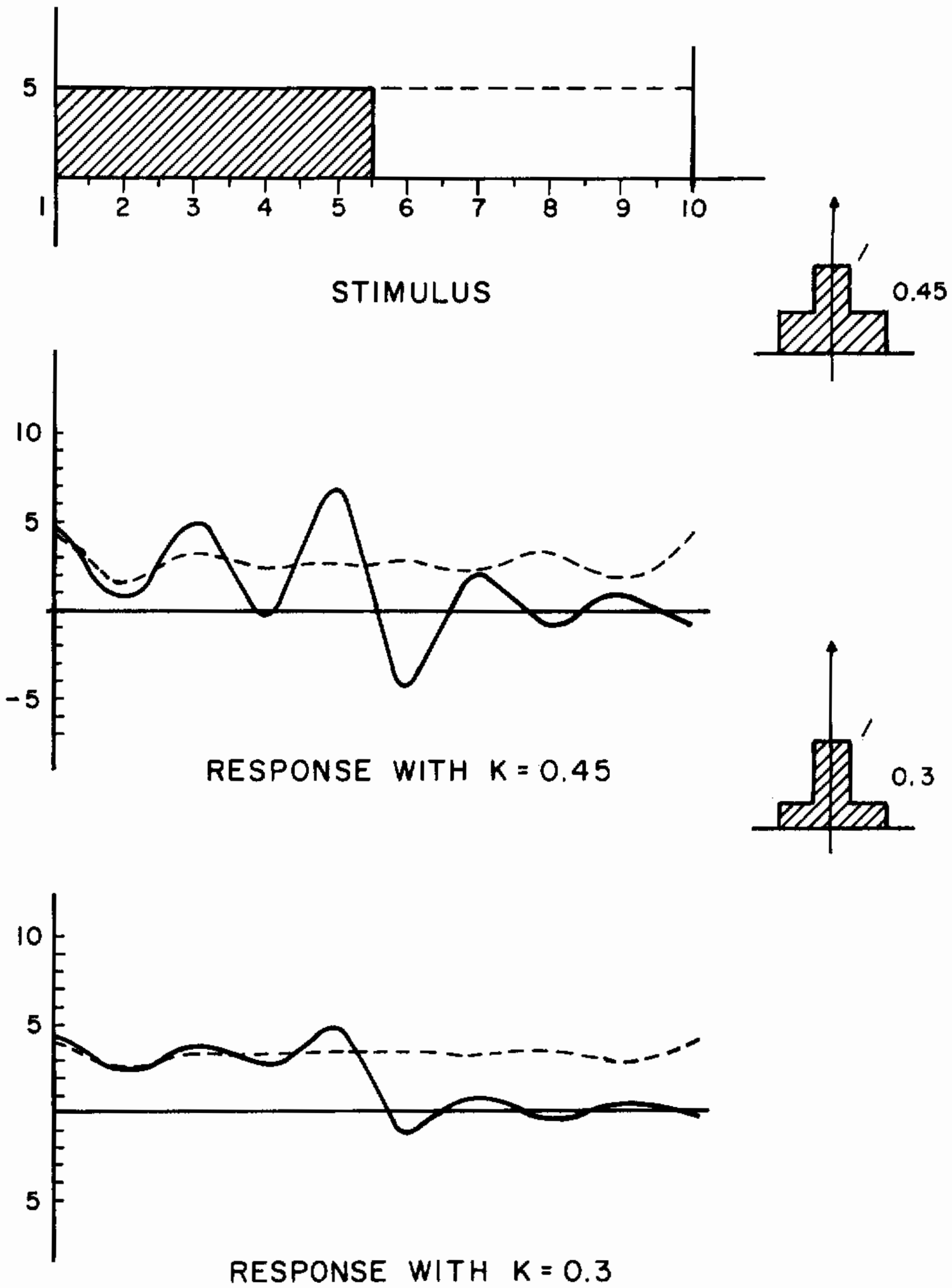


FIGURE 36. BEHAVIOR OF $y_i = X_i - K(y_{i-1} + y_{i+1})$ OF CHAIN STRUCTURE. DOTTED LINE SHOWS UNIFORM STIMULUS AND RESPONSE.

$$y_i = \bar{y} = \frac{x}{1 + 2 \sum_{j=1}^{n} K_j} \qquad i = 1, 2, 2, \ldots n$$

This useful as a check in the setting of potentiometers. If there is any irregularity in the structure, the system would give non-uniform response even if the stimulus is uniform, and this irregularity transmits a disturbance that averages to the y calculated above, as we may see from Figure 37.

5. In the case of asymmetric interaction function, an enhancement at the right edge results if the interaction is left-sided and vice versa. This is also clear from the structure of the simulator, for example, if

$$y_i = ax_i - Ky_{i-1}$$

then it is constructed as shown in Figure 38.

6. A triangular interaction function is examined (Figure 39). Since the term ΣK_j is large, the response to the uniform stimulus is a small constant level. Also, the rich inhibition pronounces the response at the edge. It is inferred that with strong inhibition, the behavior of an infinite system is like the following: (Figure 40).

The region D will extend to a rather large distance (which increases with the increase of interaction). If we add another stage of average or density detector with the action function:

$$y_i = \frac{(x_i + x_{i-1})}{2}$$

Then the response is of the shape shown in Figure 40. This is clear from the observation of Figure 34 that the response excited by the edge averages approximately to the uniform response. The region where the stimulus is x = constant, the response average is approximately

$$\bar{y} = \frac{x}{1 + 2 \sum_{j=1}^{n} K_j}$$

and in the region where x = 0, the output averages to zero. Still another advantage of the average detector stage is that the effect of imperfection in the structure is greatly eliminated, as we have found in Paragraph 4 that the defect in structure produces a response that almost averages to $\bar{y}$ of perfect structure.

7. Asymmetry and imperfection tend to reduce the violent swing of output, as we may conclude from the comparison of Figure 34 with 41; the former has smaller symmetric interaction but produces higher amplitude variation than the latter, which is asymmetric. For the case of imperfect structure, compare Figure 37 with Figure 38.

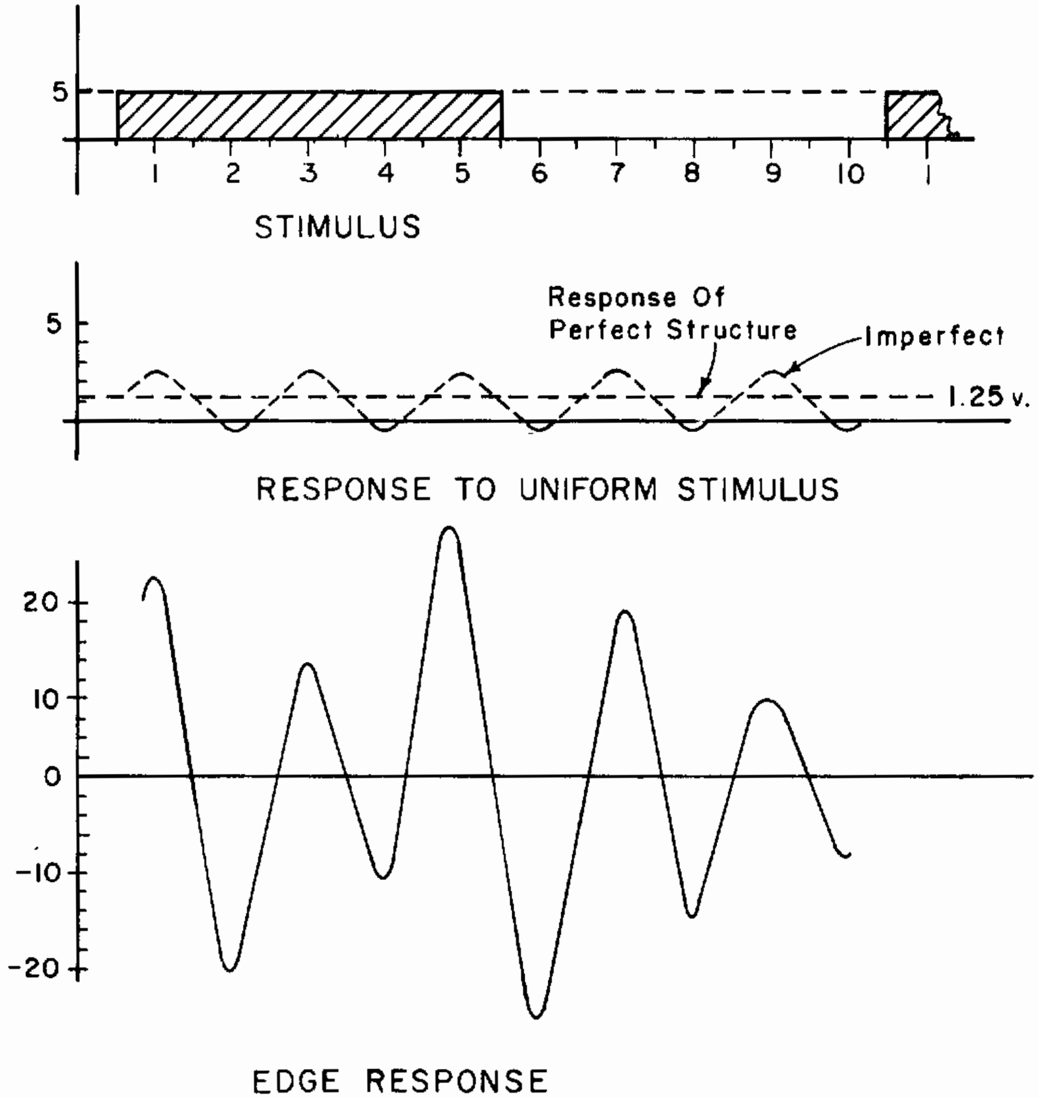
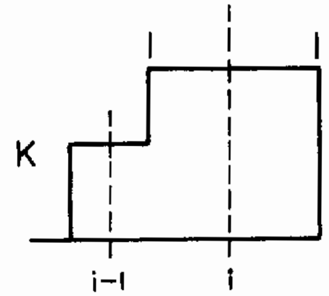


FIGURE 37. EFFECT OF STRUCTURAL IRREGULARITY. $y_i = X_i - 0.75(y_{i-1}+y_{i+1}) -0.5\ (y_{i-2}+ y_{i+2}) - 0.25\ (y_{i-3} + y_{i+3})$ Except one 0.5 pot. is set to 0.75.



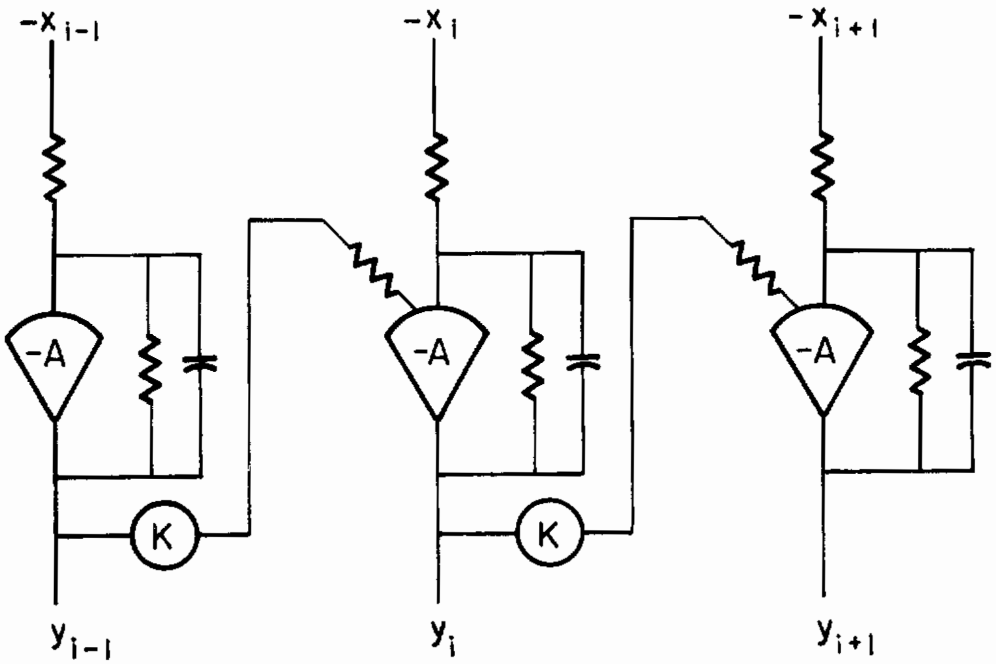


FIGURE 38. THE STRUCTURE ASYMMETRIC INTERACTION FUNCTION. y_i TRANSMITS INTERACTION TO ITS LEFT NEIGHBOR IF THE INTERACTION FUNCTION IS RIGHT-SIDED.

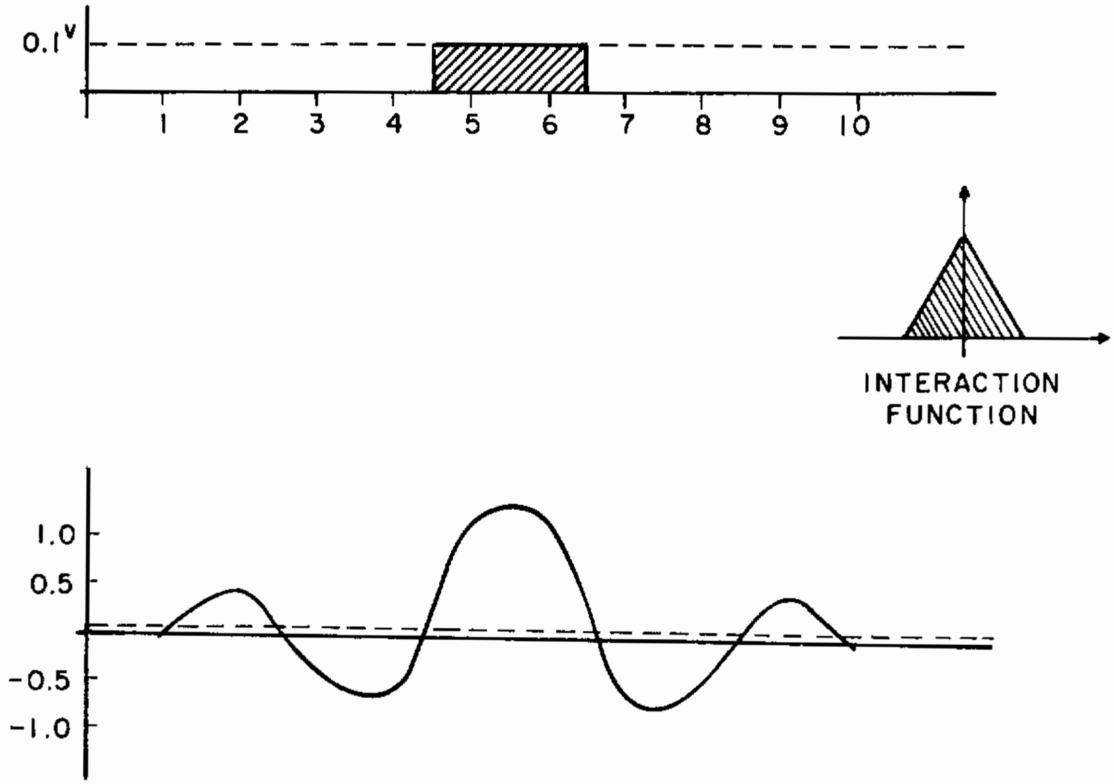


FIGURE 39. BEHAVIOR OF $y_i = X_i - 0.75(y_{i-1} + y_{i+1}) - 0.5(y_{i-2} + y_{i+2}) - 0.25(y_{i-3} + y_{i+3})$ THE RESPONSE TO EDGE STIMULUS IS VERY LARGE.

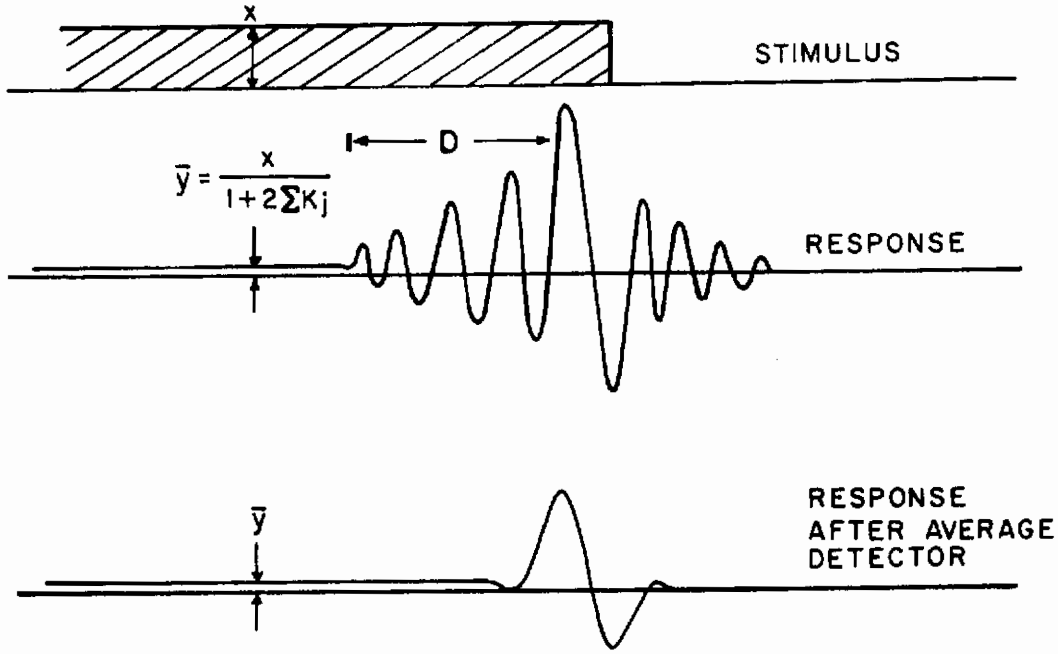


FIGURE 40. BEHAVIOR OF INFINITE SYSTEM WITH RICH INTERACTION

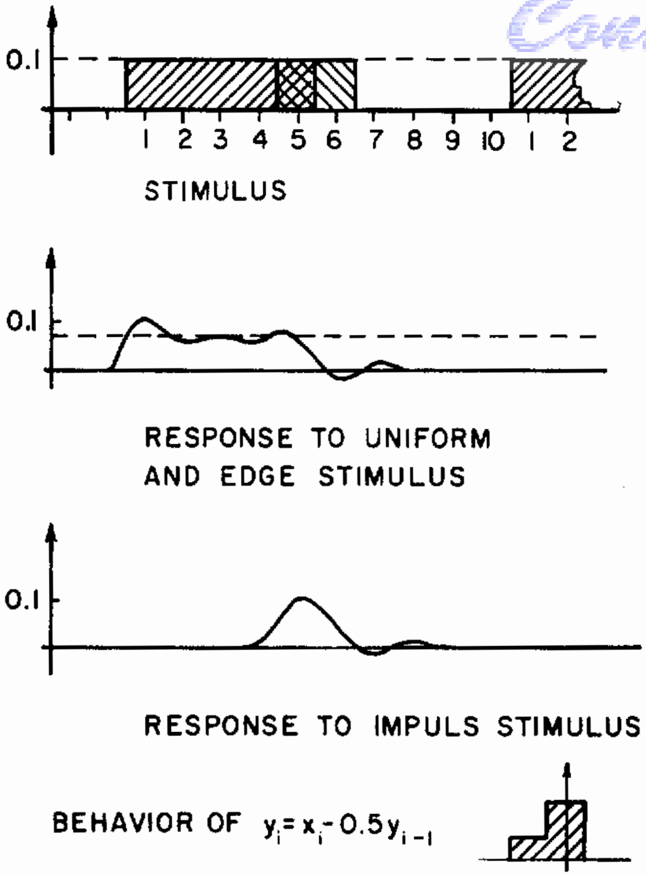


FIGURE 41a

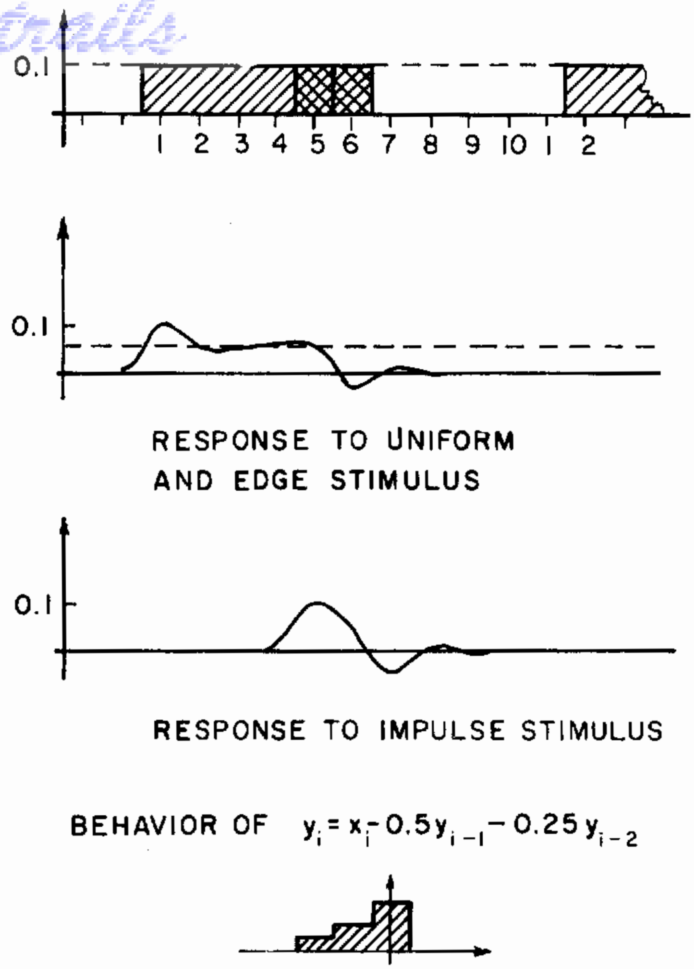


FIGURE 41b

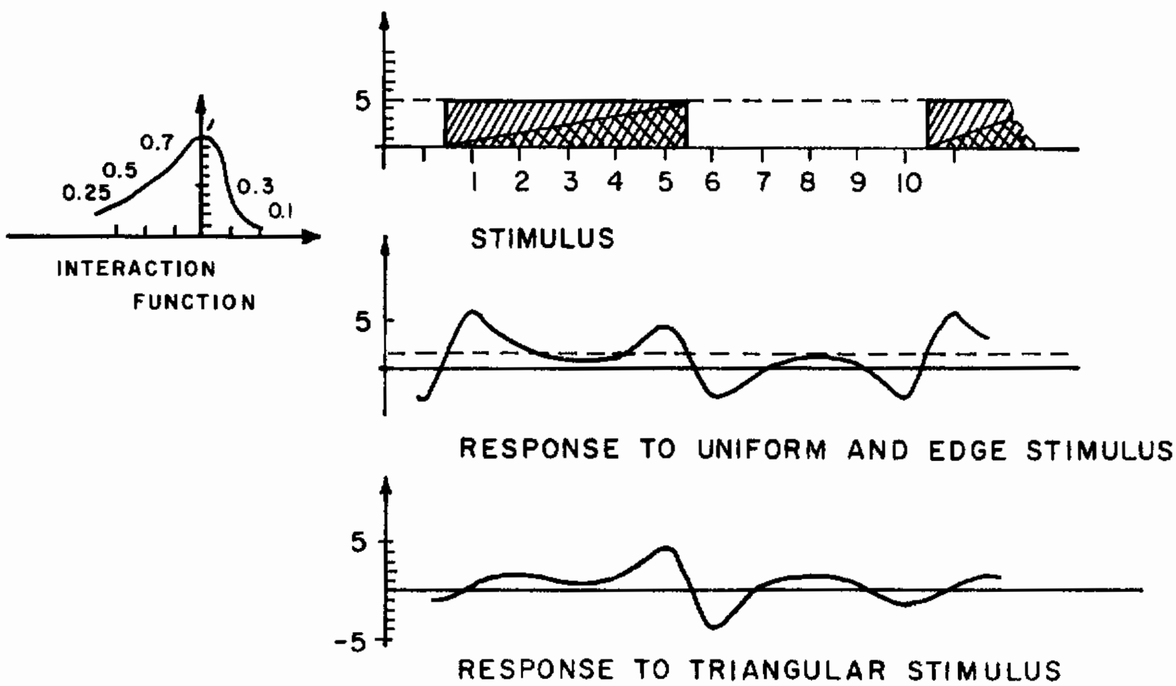


FIGURE 41c. BEHAVIOR OF $y_i = X_i - 0.7\ y_{i-1} - 0.5\ y_{i-2} - 0.25\ y_{i-3} - 0.3\ y_{i+1} - 0.1\ y_{i+2}$

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