STRUCTURAL BEHAVIOR OF TAPERED INFLATED FABRIC CYLINDERS UNDER VARIOUS LOADING CONDITIONS

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The purpose of this paper is to find approximate relations for the design of inflatable members of linearly variable (tapered) cross sections under various loadings. Considered in the analysis are axial loads (tension, compression, Euler effect), torsional moment, and internal pressure. Inflatable members are usually constructed of impregnated fabrics such as Dacron, but at the present time "exact" solutions for such materials are inadequate. Consequently, approximate procedures are needed to determine stresses in this type of member. This paper discusses not only the degree of approximation but also the procedural limitations. It includes, in addition, the effect of fiber inclination on rigidity of the member.

INTRODUCTION

Because of the rapid development of applications for inflatable structures, it is becoming increasingly important to develop a systematic theory for design work and for predicting structural behavior. The active interest in space stations, inflatable recovery vehicles, inflatable shelters, and similar structures has greatly accelerated the theoretical and experimental efforts to develop analysis methods for inflatable structures.

The theory of elasticity, which has been used with success in the analysis of conventional structures, does not adequately predict the behavior of inflatable fabric structures due to the nonlinear behavior of the composite material. Characteristic large deformations under ultimate load, composite material behavior, and hysteresis effects under cyclical loadings preclude using the conventional linear theory to accurately predict the stresses, deformations, and strength of the inflatable structure. As long as these important nonlinear effects are not well known, there is very little reason to develop an "exact" linear solution. For no matter how comprehensive the linear approach may be, the resulting equations will not adequately describe or predict the state of stresses in the inflatable structure. Consequently, in view of the urgent need for criteria for evaluating inflatable members, simplified approaches seem to be more logical. It is for this reason that a difference of 15 percent between test results and developed theory may be considered a very satisfactory correlation at the present time.

The methods described in this paper, which is a continuation of an earlier investigation[1], are intended for preliminary design work. It is also hoped that they will suggest characteristics of inflatable structures which have not been considered previously, such as composite behavior and the effects of orientation of fibers.

CIRCULAR INFLATED CYLINDERS

The circular inflated cylinder is the most convenient model by which to study the effects of changes in the modulus of elasticity and stresses due to the orientation of the fibers at an angle to the principal directions. Such a configuration will seldom arise in practice, but from a theoretical point of view this study will lead to important conclusions which can be used in the development of an analytical procedure for tapered columns in which the fibers are not parallel to the principal directions.

Figure 1 represents the envelope of a cylinder in which the fibers are oriented at an angle α to the circumferential and longitudinal directions.

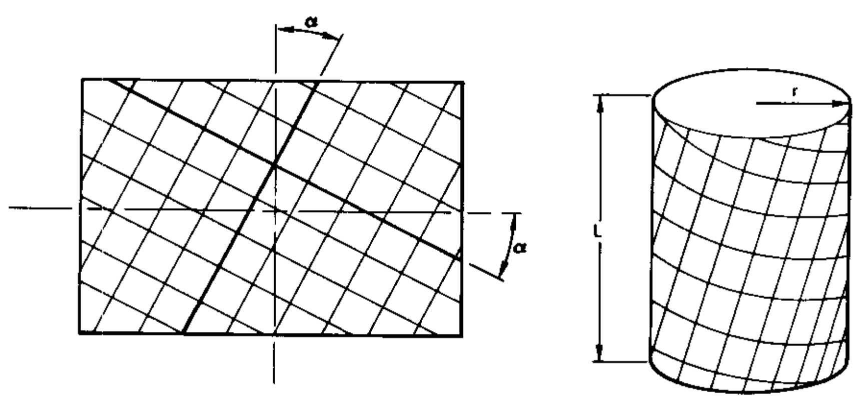

Figure 1. Cylinder With Inclined Fibers

Stresses due to internal pressure are given by the well-known relations, based on equilibrium, in the longitudinal and circumferential directions, respectively.

$$q_l = \frac{pr}{2}, \; q_c = pr$$

If the fibers are inclinded to the principal directions, the shear stress on the inclined element must be considered in the analysis.

Mohr's circle (Figure 2) is used to determine stresses in the direction of the fibers and the shear stress, yielding the following equations for the tensile and shear stresses:

$$\left.\begin{aligned} q_y &= 0.50\ pr\ (2 - \cos^2 \alpha) \\ q_x &= 0.50\ pr\ (1 + \cos^2 \alpha) \\ q_\tau &= 0.50\ pr\ \sin\alpha \cos\alpha \end{aligned}\right\} \qquad (1)$$

Generally, q_y and q_x are taken by the fibers and elastomer. The shear q_τ is taken exclusively by the elastomer.

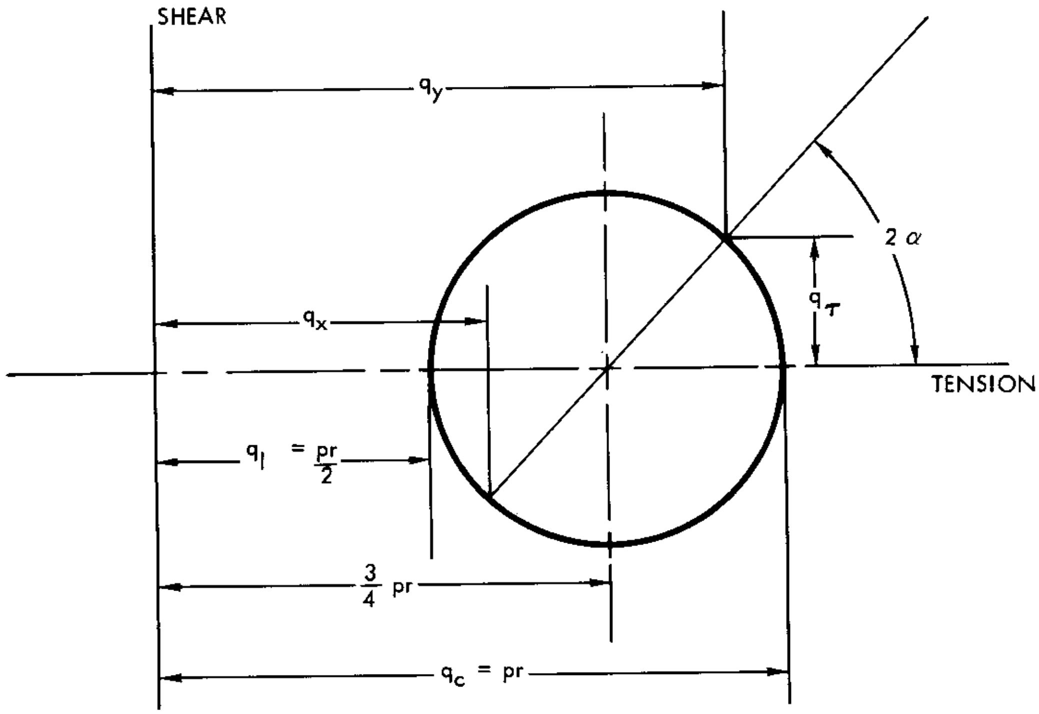

Figure 2. Mohr's Circle

The tensile stress in the elastomer is proportional to the relative stiffness of the elastomer:

$$\sigma_y = \frac{q_y}{A_r} \frac{A_r E_r}{A_r E_r + A_f E_f} = \frac{q_y}{A_r} \left(\frac{1}{1 + \frac{A_f E_f}{A_r E_r}} \right)$$

$$\sigma_x = \frac{q_x}{A_r} \frac{A_r E_r}{A_r E_r + A_f E_f} = \frac{q_x}{A_r} \left(\frac{1}{1 + \frac{A_f E_f}{A_r E_r}} \right)$$

where

A_r, A_f = cross section of elastomer and fibers, respectively

E_r, E_f = modulus of elasticity of elastomer and fibers, respectively

The equation for the shear modulus of Dacron has been derived[1], in which the elasticity of the elastomer, influence of tension on fibers (friction), and the concentration of fibers were considered. Consequently, G was the variable value, and the relation, which is used to determine deformation due to shear, depends on this variable value:

$$\gamma = \frac{\tau}{G} \tag{2}$$

Dacron can be considered a typical material for inflatable structures. For the material tested at S&ID[1], the following moduli of elasticity were noted:

$$\text{fibers, } E_f \approx 840,000 \text{ psi}$$
$$\text{elastomer, } E_r \approx 4,000 \text{ psi}$$

This leads to the conclusion that the tension stresses in the elastomer, σ_x and σ_y, can be neglected and total tension may be carried by the fibers only. The shear, however, is assumed to be taken exclusively by the elastomer since the Dacron fibers behave as a mechanism under shear loads.

The shear deformation, γ, induces a rotation of the cylinder by an amount

$$\theta = \frac{\gamma L}{r}$$

Substituting for γ in Equations 1 and 2, we obtain the following expression for rotation of the cylinder:

$$\theta = \frac{0.50\, pL \sin\alpha \cos\alpha}{tG}$$

Since the shear modulus, G, is relatively small, the deformation angle (Equation 2) is significant; and angle θ can be observed visually in tests because of large cylinder length. Consequently, under internal pressure, the bulkheads rotate with respect to one another until the position of equilibrium is reached. The fibers then will not be inclined at the angle, α, but at another angle, $\alpha - \gamma$. It is assumed that the angle, γ, defined by Equation 2, is approximately equal to the angle γ determined by Equation 2a, since the inclination of fibers is relatively small.

The same phenomenon occurs if a torsional moment is applied to the end of the cylinder with the fibers parallel to the center line of the cylinder:

$$T \approx 2\pi r^2 q_\tau = \pi p r^3 \sin\alpha \cos\alpha$$

where q_τ is taken from Equation 1, and the effects of the small inclination of q_τ to the principal directions of the cylinder are neglected.

The angle of rotation of the cylinder is also of interest, since it is simple to measure during tests:

$$\theta = \frac{TL}{JG}$$

where

θ = angle of rotation of the upper bulkhead with respect to the lower

The required angle, γ, (Equation 2) can now be rewritten:

$$\gamma = \theta \frac{r}{L} \qquad (2a)$$

where

$J = 2\pi r^3 t$ or $J^* = 2\pi r^3$, depending on dimensionality of G^1

L = length of cylinder

Since γ is not too small, a change of geometry occurs and the cylinder fibers no longer intersect at 90 degrees, and the longitudinal fibers, when equilibrium is reached, are no longer inclined at the angle α, but at $\alpha - \gamma$.

Due to the change of geometry, the new values for q_x and q_y must be determined, using the new angle, $\alpha - \gamma$, rather than the inclination angle, α. Similarly, the stresses in the deformed structure are

$$\left.\begin{aligned} q_y &= \frac{1}{2}\, pr\,[2 - \cos^2(\alpha - \gamma)] \\ q_x &= \frac{1}{2}\, pr\,[1 + \cos^2(\alpha - \gamma)] \\ q_\tau &= \frac{1}{2}\, pr\,(\sin\alpha \cos\alpha) \end{aligned}\right\} \qquad (3)$$

The equation for q_τ is not affected by the new angle because q_τ is the stress which already corresponds to the angle $\alpha - \gamma$.

If an axial load P is applied to the column, the column will rotate and α will be either increased or decreased, depending on whether the load is one of compression or tension. Before the column can carry the load, it must be pressurized; therefore, P will always be accompanied by internal pressure. However, the effect of P can be studied separately, using Mohr's circle as before. For the case of the compression load

$$q_1 = -\frac{P}{2\pi r}\,;\; q_c = 0 \tag{4}$$

In a manner similar to the preceding approach, expressions can be obtained for q_x, q_y and q_τ, $\phi = (\alpha - \gamma)$. For a compression load P

$$\left.\begin{aligned} q_y &= -\frac{P}{2\pi r}\cos^2\phi \\ q_x &= -\frac{P}{2\pi r}\sin^2\phi \\ q_\tau &= \mp\frac{P}{2\pi r}\sin\phi\cos\phi \end{aligned}\right\} \tag{5}$$

Again, q_τ introduces a rotation equivalent to that produced by the torsional moment:

$$T \approx 2\pi r^2 q_\tau = 2\pi r^2 \left| -\frac{P}{2\pi r} \right| \sin\phi\cos\phi = \left| -P \right| r \sin\phi\cos\phi$$

and this leads to an additional rotation of the column, θ_P, in accordance with the equation

$$\theta_P = \frac{TL}{JG}\,;\; \gamma_P = \theta_P \frac{r}{L}$$

where G is given by the equations in Reference 1.

θ_P will affect the first two stresses (Equation 5), as in the case of pressurization. Therefore, Equation 5 may be rewritten:

$$\left.\begin{aligned} q_y &= -\frac{P}{2\pi r}\cos^2(\phi + \gamma_P) \\ q_x &= -\frac{P}{2\pi r}\sin^2(\phi + \gamma_P) \\ q_\tau &= \mp\frac{P}{2\pi r}\sin\phi\cos\phi \end{aligned}\right\} \tag{5a}$$

Since compression is always accompanied by internal pressurization, Equations 3 and 5a can be superimposed, and

$$\left.\begin{aligned} q_y &= \frac{1}{2}\,pr\,[2 - \cos^2(\alpha - \gamma)] - \frac{P}{2\pi r}\cos^2(\alpha - \gamma + \gamma_P) \\ q_x &= \frac{1}{2}\,pr\,[1 + \cos^2(\alpha - \gamma)] - \frac{P}{2\pi r}\sin^2(\alpha - \gamma - \gamma_P) \\ q_T &= \frac{1}{2}\,pr\,\sin\alpha\cos\alpha - \frac{P}{2\pi r}\sin(\alpha - \gamma)\cos(\alpha - \gamma) \end{aligned}\right\} \qquad (6)$$

If P is a tension load, the signs for P and γ_P will be reversed.

EFFECTIVE MODULUS OF ELASTICITY IN LONGITUDINAL DIRECTION

The inclination of fibers does not affect I, but may affect E. It is simpler to study the change of the elasticity modulus by assuming a tensile load on the cylinder; therefore, Equation 6 is an effective starting point. The tensile load, P, which is applied to the column and varies from an initial value of zero to the final value, tends to decrease the angle ϕ between the longitudinal fibers and the longitudinal axis of the cylinder; the angle ϕ is already a function of the following two factors:

original inclination of fibers
internal pressure, which will decrease α by angle γ

Therefore

$$\phi = \alpha - \gamma \qquad (7)$$

To straighten the fiber, a load, P_o, must be applied, which will cancel ϕ. This means that P_o will cause a torsional moment, T_o, which will cause a rotation of the cylinder through an angle, $-\phi$:

$$-\phi = \frac{T_o L}{JG} \cdot \frac{r}{L}$$

$$T_o = \phi\,\frac{JG}{L} \cdot \frac{L}{r}$$

where

G = shear modulus

We also know that

$$T \approx q_T 2\pi r^2$$

Combining the above expressions, we obtain

$$q_{T_o} = \frac{r\phi Gt}{r} = + \frac{P_o}{2\pi r} \sin \phi \cos \phi$$

Thus the load, P_o, can be determined by

$$P_o = \phi \frac{2\pi r\ Gt}{\sin \phi \cos \phi} \tag{8}$$

Under this load, the fibers will become parallel to the cylinder axis. If this force is increased, the cylinder will be elongated but no rotation will occur. The elongation for any $P > P_o$ will be in accordance with the known elasticity modulus, E_ℓ, for the cylindrical skin in the longitudinal direction, if the fibers are oriented in the same direction. However, for any load $P < P_o$, longitudinal elongation will not occur at the same rate. The corresponding modulus of elasticity will be designated by E_{eff}, which depends on not only the elongation but also the cylinder rotation.

To determine E_{eff}, assume the cylinder to be loaded with a tensile load, $P' < P_o$, causing a rotation $\phi' < \phi$. Then

$$\phi' = \phi \frac{P'}{P_o}$$

In accordance with Figure 3, elongation in the principal direction can be expressed by

$$\delta = y_1 + y_2$$

where

$$y_1 = \frac{q_P tL}{E_\ell A} \cos(\phi - \phi') = \frac{q_P L}{E \cdot I} \cos(\phi - \phi')$$

and

$$q_P = q_y \text{ (from Equation 5a)}$$

$$y_2 = L \cos(\phi - \phi') - L \cos \phi$$

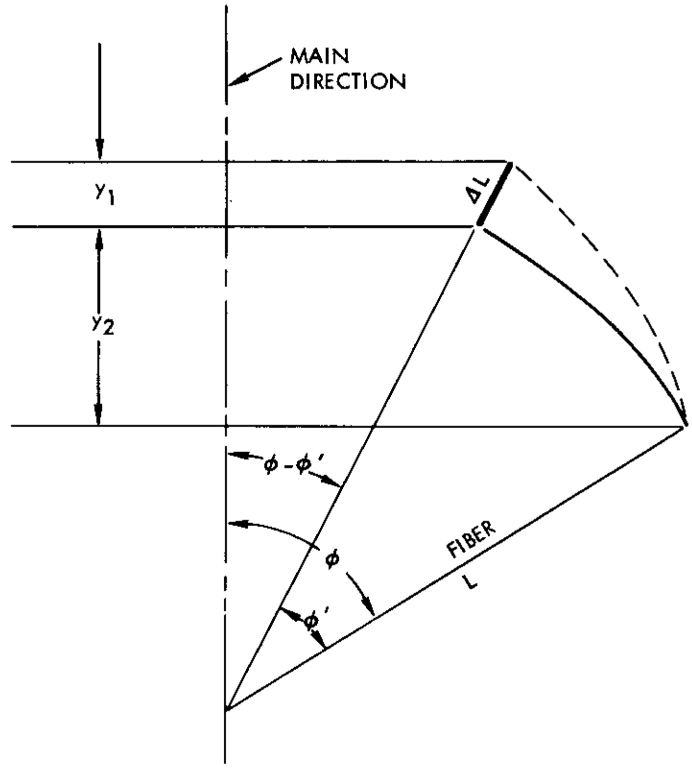

Figure 3. Nonlinear Elongation

Therefore

$$\delta = y_1 + y_2 = \frac{q_P L \cos(\phi - \phi') + E_\ell L [\cos(\phi - \phi') - \cos\phi]}{E_\ell}$$

Now consider a cylinder with the fibers oriented in the principal directions and whose length is

$$\ell = L \cos\phi$$

The corresponding elongation will then be

$$\delta_v = \frac{{}_P q_1 \; L \cos\phi}{E}$$

where ${}_P q_1$ is the value of q_1 in Equation 4. Since the elongation is inversely proportional to the modulus of elasticity, the effective modulus in the principal direction for the cylinder with inclined fibers will be

$$\left.\begin{aligned} E_{eff} &= E_{\ell} n = E_1 \frac{\delta}{\delta_v} \\ \text{where} \qquad n &= \frac{q_P \cos(\phi - \phi') + E_{\ell}\,[\cos(\phi - \phi') - \cos\phi]}{P^{q_1} \cos\phi} \end{aligned}\right\} \tag{9}$$

TAPERED CYLINDERS

Inflatable cylinders of constant cross section usually can be made from material with fibers oriented parallel to the principal directions. Unfortunately, this is not the case with conical structures because they must be assembled from one or more sectoral pieces of material. The fibers cannot be oriented circumferentially and longitudinally. Figure 4 illustrates the typical sector, which has the following properties:

1. Only along the center line of the sector are the fibers located in the direction of the principal stresses.
2. Along any line passing through the vertex (i.e., $\overline{oa}$), the inclination of fibers is the same everywhere.
3. Inclination of the fibers varies along any circumferential line, $\overline{bb}$.
4. Maximum inclination of fibers to the principal directions is at the boundaries $\overline{cc}$ and $\overline{dd}$ of the sector.

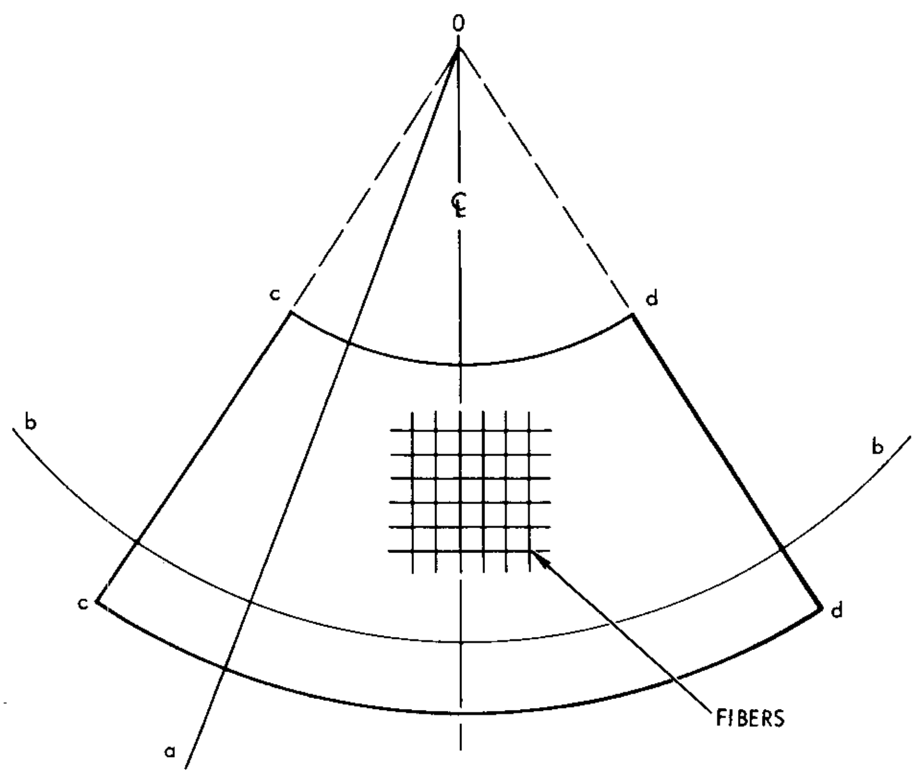

Figure 4. Typical Sector

To prevent the occurence of too large an angle between the fibers and principal directions, the cone may be made of more than one sector. In order to fabricate the cone, a sector as shown in Figure 5 is required, with a radius of ℓ and a central angle ϕ.

$$\phi = \frac{r}{\ell} 360 \text{ (in degrees)}$$

where

r = the radius of the base of the cone

If the cone is composed of n sectors, each sector will have a radius ℓ and central angle ϕ/n (obviously, allowance must be made for joints).

If the fibers are placed so that at the center line $\overline{aa}$ of the sector they are oriented in the principal direction, then

$$\max \alpha = \frac{\phi}{2n}$$

where

α = the inclination of the fibers from the principal directions

n = number of sectors

For any line $\overline{ab}$ defined by angle α (Figure 5), the fibers will be inclined an angle α to the principal directions of stress.

To locate any point of interest, it is necessary to choose a system of coordinates. As a reference take $\overline{aa}$, the line whose fiber is in the direction of the principal stresses. Any other line $\overline{ab}$, relative to line $\overline{aa}$, can be defined by the angle α. Any point on line $\overline{ab}$ is defined by the distances from the vertex. The angle α can be easily measured on the sector but not on the cone, and it is reasonable to replace α with the length u. Therefore, any point on the conical surface is defined by two numbers, u and S (Figure 5).

The relationship between α, u, and S is

$$u = \pi S \frac{\alpha}{180} \text{ or } = S\,\alpha \ (\alpha \text{ in radians})$$

$$\alpha = \frac{u\,180}{\pi S} \text{ or } \alpha = \frac{u}{S}$$

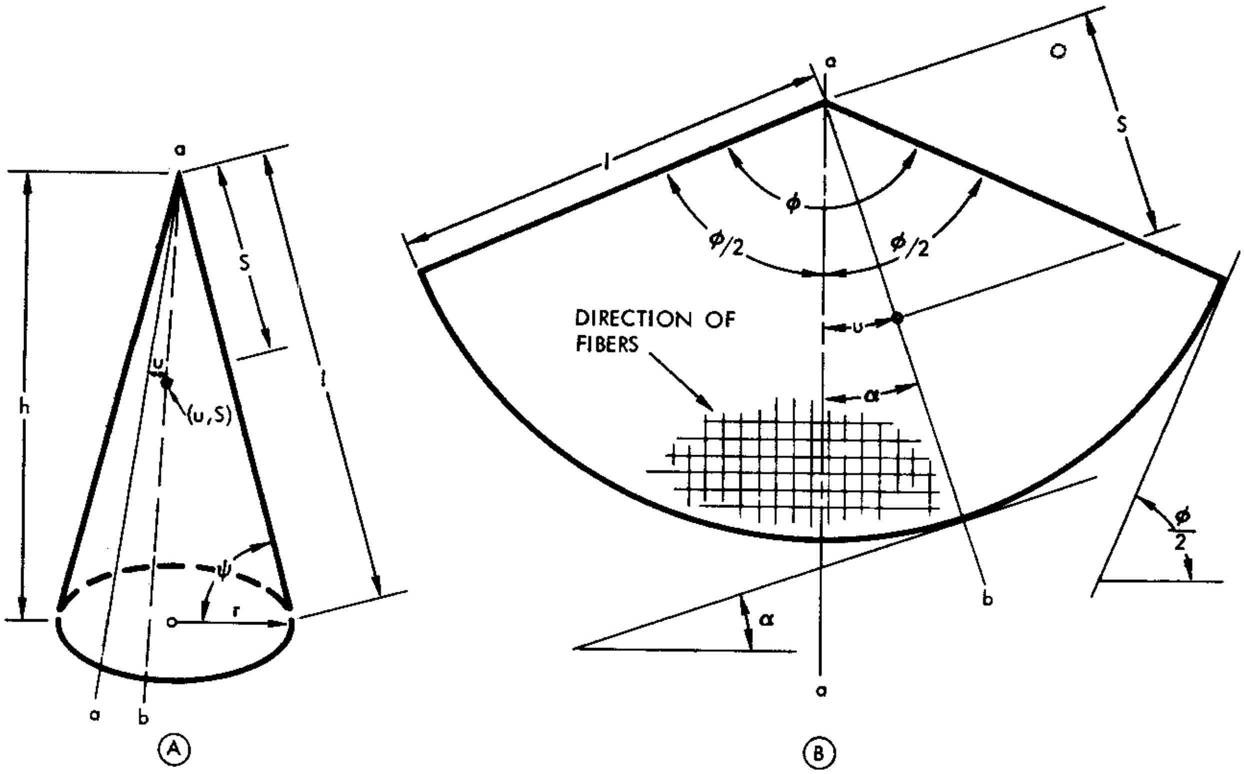

Figure 5. Inclination of Fibers From Principal Direction and Cone Nomenclature

Therefore, the point which is defined by u, S will correspond to the angle α:

$$(u,\ S) \rightarrow \alpha = \frac{u}{S}$$

For $\alpha_{max} = \frac{\phi}{2}$

$$(S\,\frac{\phi}{2},\ S) \rightarrow \alpha_{max} = \frac{\phi}{2}$$

If the cone is made of n sectors, the above relation will be modified:

$$(u = S\,\frac{\phi}{2n},\ S) \rightarrow \alpha_{max} = \frac{\phi}{2n}$$

where

ϕ = the total central angle of all sectors together

SKIN STRESSES OF INFLATED CONE

Internal Pressure

The stresses in the principal direction of a cone which are due to pressurization are given by the following equations:

$$\left.\begin{aligned} q_c &= ps \cot \psi \\ q_\ell &= \frac{1}{2} ps \cot \psi \end{aligned}\right\} \qquad (10)$$

where

c and ℓ subscripts indicate the circumferential and longitudinal directions

The angle ψ can be easily determined as a function of r and ℓ:

$$\psi = \arc\tan \frac{\sqrt{\ell^2 - r^2}}{r} \qquad (11)$$

The stresses (Equation 10) can be taken by fibers only along the line $\overline{aa}$, where the fibers are oriented in the same directions as those stresses. At any other line $\overline{ab}$, the fibers are inclined at an angle α to the principal directions. Therefore, the above stresses will be split into components and the component stresses in the direction of the fibers will be determined. The induced shear will be taken in accordance with the shear modulus, G, of the material. The stress in the direction of the circumferential fiber will be denoted by q_x, and in the direction of the longitudinal fiber by q_y. The corresponding shear will be denoted by q_τ. Therefore, to known stresses q_c and q_ℓ, q_x, q_y, and q_τ will correspond at any point of the shell:

$$(q_c, q_\ell) \rightarrow (q_x, q_y, q_\tau)$$

The following expressions are based on Mohr's circle:

$$\left.\begin{aligned} q_y &= \frac{1}{2} ps \cot \psi (1 + \sin^2 \alpha) \\ q_x &= \frac{1}{2} ps \cot \psi (1 + \cos^2 \alpha) \\ q_\tau &= \pm \frac{1}{2} ps \cot \psi \sin \alpha \cos \alpha \end{aligned}\right\} \qquad (12)$$

To prevent wrinkling of the skins, the following conditions must be satisfied:

$$q_x q_y \geq q_\tau^2$$

From this relationship can be found the limiting angle α for pressurization loading only:

$$\max q_\tau = \frac{1}{2} \text{ ps } \cot \psi \sqrt{2 + \sin^2 \alpha \cos^2 \alpha}$$

$\mathrm{Sin}^2 \alpha$ and $\cos^2 \alpha$ are always positive; therefore, the radical is positive too. The max q_τ condition will be dictated by the maximum value of

$$y = \left[\sin^2 \alpha \cos^2 \alpha\right]_{\max}$$

The minimization process leads to the conclusion that the permissible max α is 45 degrees. Then

$$\max q_\tau = \frac{1}{2} \text{ ps } \cot \psi \sqrt{2.25} = 0.75 \text{ ps } \cot \psi$$

Axial Compression

If an inflated conical member is loaded by axial compression (Figure 6), the following stresses in the circumferential and longitudinal directions will be induced (due to compression only):

$$q_\ell = -\frac{P}{2\pi\xi \cos\psi};\ q_c = 0$$

Substituting $S \cdot \sin \psi$ for ξ, we obtain

$$\left.\begin{aligned} q_\ell &= -\frac{P}{2\pi S \sin \psi \cos \psi} \\ q_c &= 0 \end{aligned}\right\} \qquad (13)$$

Actually, stresses due to axial loading cannot act alone and will always be accompanied by pressurization stresses. However, these two effects can be handled separately, and the results superimposed.

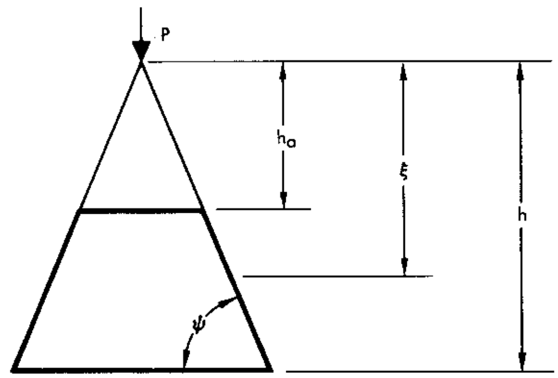

Figure 6. Section of Cone Along Axis of Rotation

Mohr's circle can again be used to determine q_x, q_y, and q_τ. And reasoning similar to that employed before will lead to similar equations. Therefore, for axial compression:

$$\left.\begin{aligned} q_y &= \frac{-P}{2\pi S \sin\psi \cos\psi}\cos^2\phi \\ q_x &= \frac{-P}{2\pi S \sin\psi \cos\psi}\sin^2\phi \\ q_\tau &= \pm\frac{-P}{2\pi S \sin\psi \cos\psi}\sin\phi \cos\phi. \end{aligned}\right\} \tag{14}$$

To modify Equation 14 for the case of tensile load P, we have only to change the sign before P to a plus.

Now, to carry the analysis any further, we need to know the expressions for the rotational angles of the variable section. If the longitudinal fibers are parallel to the center line of each equal sector, no rotation of the cross section will occur, since the average deformation due to q_τ is zero. However, if the fibers are not parallel to the center line of the sector (arrangement of the sector fibers is not symmetrical), the rotation due to q_τ wiii not be zero. This type of section tends to rotate.

Torsion

Assume that an inflatable tapered cantilever (Figure 7) is loaded on the end with the torsional moment T. The distance x locates the section under consideration. The torsional moment of inertia is variable:

$$J_1 > J_x > J_o$$

$$J_x = 2\pi r_x^3 t$$

The angle of rotation at x is

$$\theta_x = \frac{\mathcal{A}}{G} \tag{15}$$

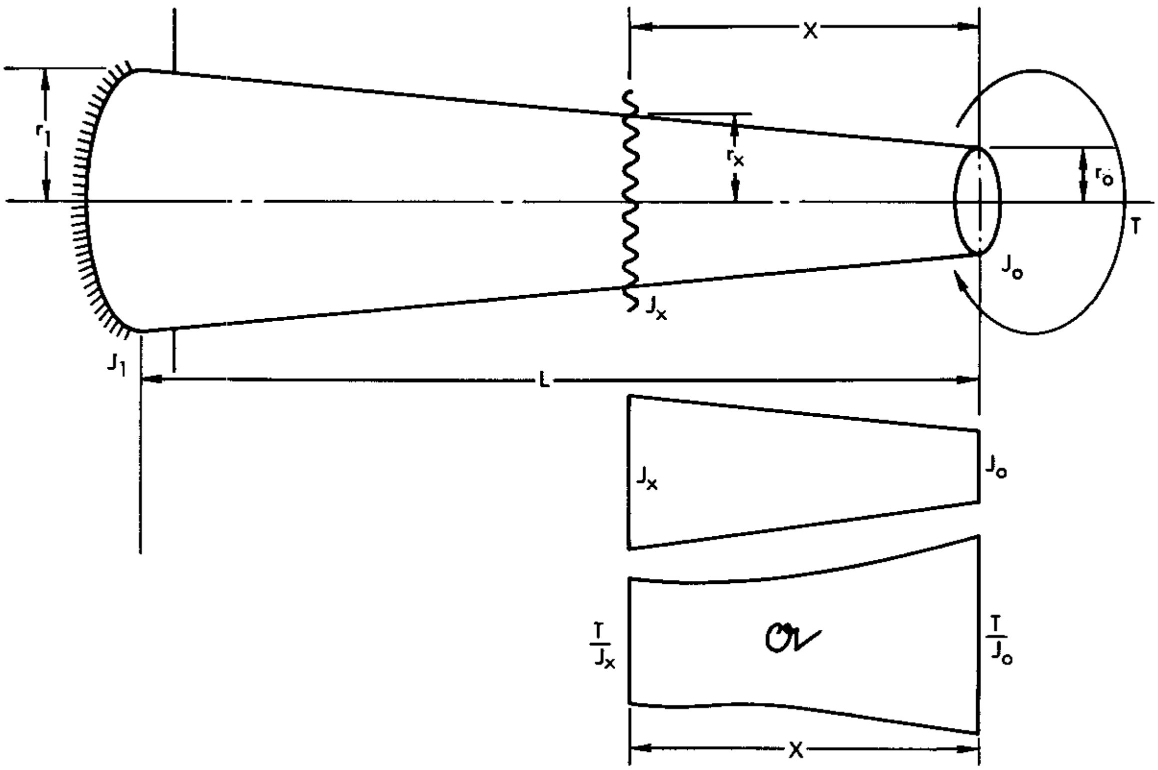

Figure 7. Inflatable Tapered Cantilever Subjected to Torsion

Some average value must be assumed for G, based on the previously derived formula for G as a function of the inclination of fibers.

$$\mathcal{A} = \int_o^x \frac{T}{J_x} dx = T \int_o^x \frac{dx}{2\pi r_x^3 t} = \frac{T}{2\pi t} \int_o^x \frac{dx}{r_x^3}$$

$$r_x = x\left(\frac{r_1 - r_o}{L}\right) + r_o$$

If we let

$$\frac{r_1 - r_o}{L} = a$$

then

$$r_x = ax + r_o$$

Therefore

$$\mathcal{N} = \frac{T}{2\pi t} \int_o^x \frac{dx}{(ax + r_o)^3} = \frac{-T}{4\pi ta\,(ax + r_o)^2}$$

$$\theta = \frac{-T}{4\pi ta\,(ax + r_o)^2\,G} \qquad (16)$$

where

$$a = \frac{r_1 - r_o}{L}$$

To obtain γ at x, we could first form the differential element and integrate along the length, which would lead to a relatively complicated equation for γ. Actually, the cone is only slightly tapered; therefore, it seems reasonable to use an approximate value for γ.

$$\gamma \approx \theta \frac{r_o}{x} \qquad (17)$$

Final Expressions for Stresses

Now we can return to Equation 12. The shear stresses create a torsional moment

$$T \approx \bar{q}_T \, 2\pi r_x^2$$

where

$$\bar{q}_T = \frac{\int_o^{2\pi r} q_T \, dx}{2\pi r}$$

which will cause, in accordance with Equations 16 and 17, an additional twisting of the fibers:

$$\gamma = \frac{\bar{q}_T}{4\pi ta\,(ax + r_o)^2\,G} \cdot \frac{r_o}{x} \qquad (18)$$

Taking into account this additional change of angle, we may rewrite Equation 12:

$$\left.\begin{aligned} q_y &= \frac{1}{2}\,ps\cot\psi\,[1+\sin^2(\alpha-\gamma)] \\ q_x &= \frac{1}{2}\,ps\cot\psi\,[1+\cos^2(\alpha-\gamma)] \\ q_T &= \pm\frac{1}{2}\,ps\cot\psi\sin\alpha\cos\alpha \end{aligned}\right\} \tag{19}$$

Similarly, axial compression will cause an additional torsional moment, distributed locally in accordance with Equation 14:

$$T = \bar{\bar{q}}_T\, 2\pi r_x^2$$

where

$$\bar{\bar{q}}_T = \frac{\int_0^{2\pi r} q_T\, dx}{2\pi r}$$

which, in accordance with Equations 16 and 17, results in the angle of rotation, γ_P:

$$\gamma_P = \frac{\bar{\bar{q}}_T}{4\pi ta\,(ax+r_o)^2\, G}\cdot\frac{r_o}{x} \tag{20}$$

Now we may rewrite the expression for axial compression, Equation 14.

$$\left.\begin{aligned} q_y &= \frac{\mp P}{2\pi S\sin\psi\cos\psi}\cos^2(\phi\pm\gamma_P) \\ q_x &= \frac{\mp P}{2\pi S\sin\psi\cos\psi}\sin^2(\phi\pm\gamma_P) \\ q_T &= \pm\frac{\mp P}{2\pi S\sin\psi\cos\psi}\sin\phi\cos\phi \end{aligned}\right\} \tag{21}$$

The upper signs for P and γ_P are for the case of compression; the lower signs, for tension.

For cases in which an axial load acts simultaneously with pressurization loads, the stresses (Equations 12 and 14) are superimposed:

$$\left.\begin{aligned} q_y &= \frac{1}{2} ps \cot \psi \left[1 + \sin^2 (\alpha - \gamma) \right] \mp \frac{P \cos^2 (\phi \pm \gamma_P)}{2\pi S \sin \psi \cos \psi} \\ q_x &= \frac{1}{2} ps \cot \psi \left[1 + \cos^2 (\alpha - \gamma) \right] \mp \frac{P \sin^2 (\phi \pm \gamma_P)}{2\pi S \sin \psi \cos \psi} \\ q_\tau &= \frac{1}{2} ps \cot \psi \sin \alpha \cos \alpha \mp \frac{P \sin \phi \cos \phi}{2\pi S \sin \psi \cos \psi} \end{aligned}\right\} \quad (22)$$

Effect of Variable Modulus of Elasticity

Inclination of fibers was defined by the angle α, which, along the circumferential direction, varies from $\alpha = 0$ to $\alpha = \alpha_{max}$. The modulus of elasticity E will decrease with an increasing angle α (derived in a study of an idealized cylinder with inclined fibers). Since α is constant along any line $\overline{ab}$, E is constant along $\overline{ab}$; but it varies along the circumference.

The stiffness of the section depends on EI; this parameter is used in bending and stability expressions.

If a section is divided into small elements, we can write

$$EI = E \Sigma I_i$$

where

I_i represents the moment of inertia of the ith small element. But E is variable too; therefore, the above relation can be rewritten:

$$EI = \Sigma E_i I_i$$

where

E_i is the reduced modulus of elasticity on the ith element to account for inclination of the fibers.

However, it is simpler to work with a constant elasticity modulus E instead of variable E_i. Therefore, let us define I_i^*:

$$I_i^* = I_i \frac{E_i}{E}$$

If y is the distance of the ith element (whose area is A) from the neutral axis, we can write

$$I_i = y^2 A \rightarrow I_i^* = (y^*)^2 A$$

which leads to

$$y^* = y \sqrt{\frac{E_i}{E}} \tag{23}$$

where

y^* = the distance of an element (corresponding to the ith element) with moment of inertia I_i^* and area A

For such a section

$$EI^* = E\Sigma (y^*)^2 A \tag{24}$$

The moment of inertia can be so written because the section is usually symmetrical about the neutral axis. This means that we are working with a transformed section which remains symmetrical about the neutral axis, which has a constant elasticity modulus E, but which is no longer circular. The group EI^* is less than EI. Such transformed sections are represented in Figure 8 for two cases; the section is composed of two and four sectors. It is usually easy to predict which directions will be most affected by the transformation; and this may be employed for determining the effective I^* to be used in calculating the Euler effect because the column will tend to be bent in the weakest direction.

The same reasoning may be used in the study of the bending of a tapered beam, and it helps one understand where to place the joints of the material in order to have a symmetrical section about the neutral axis. Generally, the transformed section will be determined at two end points of the tapered beam, because any section in between is obtained by linear interpolation.

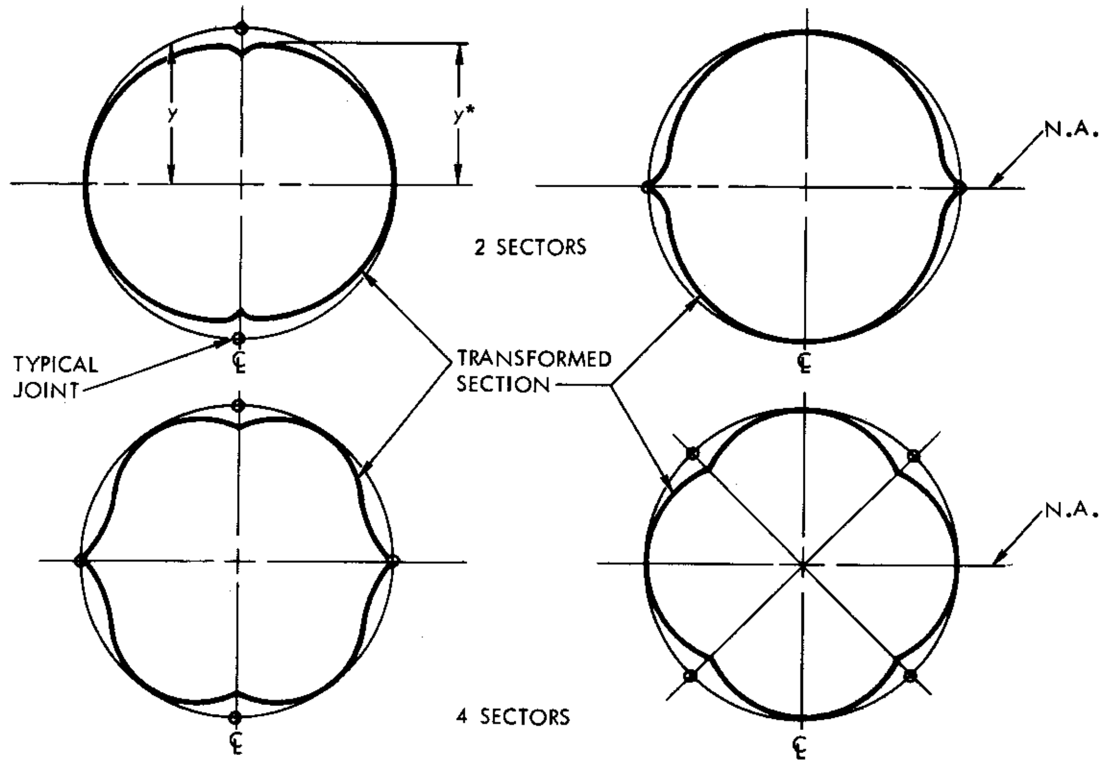

Figure 8. Transformed Sections

This is one way to determine EI. The modulus E_ℓ was unchanged but the moment of inertia was modified. This is a more or less accurate way since no unjustified averaging is performed.

It is possible to approach the problem differently. Assume I to be as it is, but instead of a variable E, we use some average value. This means that the circumference is divided into segments, and the elastic modulus E_i corresponding to each segment is to be determined. Then

$$E_{eff} = \frac{\sum_{1}^{n} E_i}{n}$$

where

n = the number of segments

This approximation is less justifiable than the previous one, but it may lead to a faster determination of results.

Euler Buckling Load

In the derivation of an expression for the buckling load, the variation of the geometry along with the length must be considered. A linear relation is assumed for the variation of column depth along the length. For a circular section, the depth is the diameter; for an elliptical section, the diameter of the minor axis; and for a square section, the side. For any transformed section (described in previous paragraph), the depth is the distance between the most distant points from the neutral axis under consideration (Figure 9).

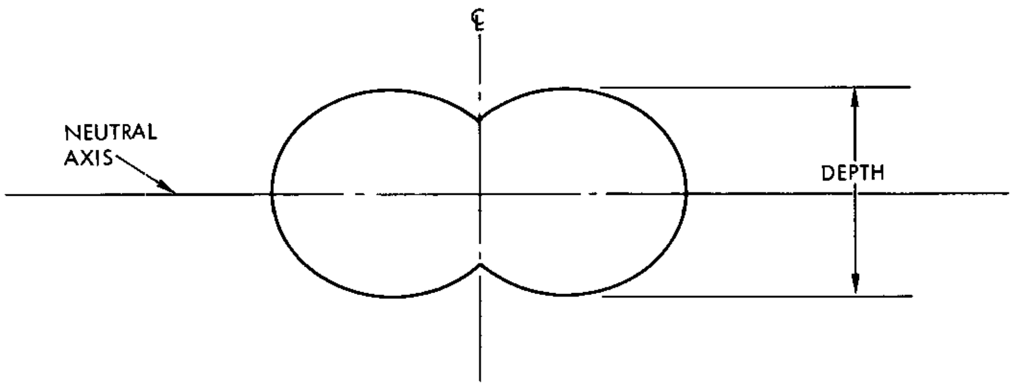

Figure 9. Depth of a Transformed Section

Consider a tapered inflated tube with rigid ends (Figure 10), where

d_A = upper, smaller depth of section

d_B = lower, larger depth of section

x = distance from upper end along the axis of the column

L = length of column

Depth of the section[2] corresponding to any x can be expressed by

$$d_x = d_A \left[1 + \left(\frac{d_B}{d_A} - 1 \right) \frac{x}{L} \right] \tag{25}$$

The following known equation[2] can be used for determining I at any point of column x:

$$I_x = I_A \left[1 + \left(\frac{d_B}{d_A} - 1 \right) \frac{x}{L} \right]^n \tag{26}$$

where

I_x = moment of inertia at distance x from upper end

I_A = moment of inertia at upper end

n = shape factor

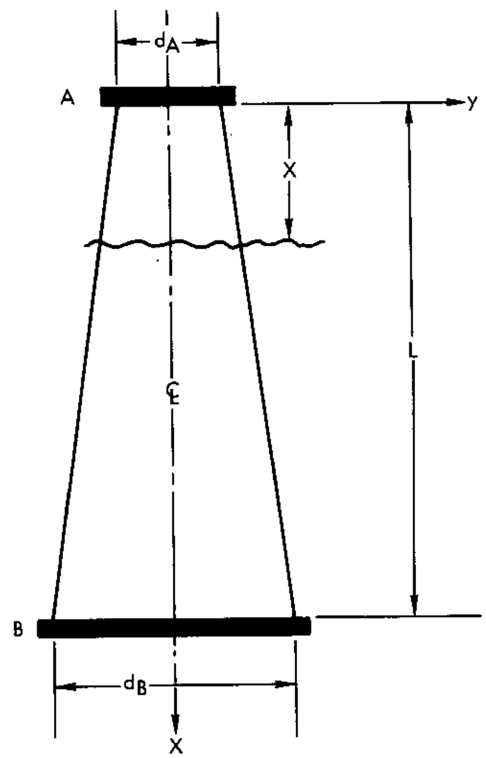

Figure 10. Tapered Inflated Cylinder

The shape factor[2] can be evolved by observing that Equation 26 must give $I_x = I_B$ when $x = L$. This leads to

$$n = \frac{\log \frac{I_B}{I_A}}{\log \frac{d_B}{d_A}} \tag{27}$$

For the conical case, n varies between 2 and 4.

In Reference 2 a useful equation is presented for critical loads:

$$P_{cr} = P^* \left[\frac{\pi^2 EI_A}{L^2}\right] \tag{28}$$

P^* is given in graphs for various n and d_B/d_A (Figure 11). Equation 28 corresponds to pinned supports on both ends. In the same reference similar graphs were obtained for different supporting conditions. However, for the inflatable fabric column another requirement must be considered. From a derivation of the Euler load for a cylindrical column, we know that the Euler load is limited by the limiting load (Reference 1), which is a function of the internal pressure. The limiting load will be derived in the following section.

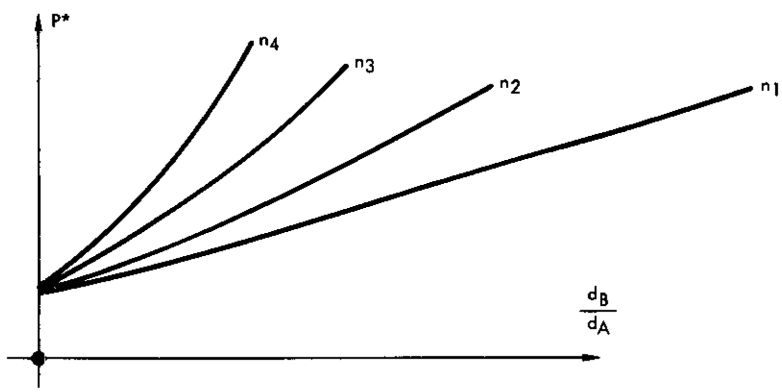

Figure 11. Determination of Critical Load

The Limiting Load P_L

The limiting load is the function of σ_ℓ or, in our case, q_y, which was presented in Equation 20. It is evident that, with reference to any section, the limiting load is variable and can be determined from the following equation:

$$q_y = \frac{1}{2} ps \cot \psi \left[1 + \sin^2 (\alpha - \gamma)\right] - \frac{P_L \cos^2 (\phi + \gamma_P)}{2\pi S \sin \psi \cos \psi} = 0$$

which leads to

$$P_L = \frac{\pi p s^2 \left[1 + \sin^2 (\alpha - \gamma)\right] \cos^2 \psi}{\cos^2 (\phi + \gamma_P)} \tag{29}$$

The weakest material will be at $\alpha = \alpha_{max}$; consequently, if this value or some average value

$$\frac{\alpha_{min} + \alpha_{max}}{2} = \alpha_{av}$$

is introduced into Equation 29, there remains only one variable, S, which can be used to determine the variation of P_L along the length of the tapered beam.

CONCLUSIONS

The material presented in this paper may be regarded as a contribution to the preliminary design of inflatable structures. However, the concepts should be extended to include memory effects and hysteresis, when these phenomena have been better defined for impregnated fabric materials. Moreover, considerable research in the area of materials behavior is needed. In the meantime, it is hoped that the approaches treated here will be of use to the structural analyst.

NOMENCLATURE

q_ℓ	Stress in ℓ direction (lb/in.)
q_c	Stress in c direction (lb/in.)
q_x	Stress in x direction (lb/in.)
q_y	Stress in y direction (lb/in.)
q_τ	Shear stress (lb/in.)
σ_x	Stress in x direction (lb/in.^{2})
σ_y	Stress in y direction (lb/in.^{2})
τ	Shear stress (lb/in.^{2})
E, E_r, E_f	Young's modulus in general, of elastomer, of fibers, respectively (lb/in.^{2})

E_{eff}	Effective modulus of elasticity (lb/in.^{2})
G, G*	Shear modulus in lb/in.^{2}, lb/in., respectively
p	Internal pressure (lb/in.^{2})
P	Axial load (lb)
P_E	Euler load (lb)
P_o	Load at which inclined fibers of cylinder will be straightened (lb)
P_L	Limiting load (lb)
T	Torque (lb-in.)
I	Moment of inertia (in.^{4})
I_i	Moment of inertia of small element (in.^{4})
J	Polar moment of inertia (in.^{4})
J_o, J_1	Polar moment of inertia at locations o and 1
$\mathcal{A}$	Area of diagram of length x and ordinates $\frac{T}{J}$, which also represents reaction of fictive loading (lb/in.^{2})
α	Angle of inclination of fibers
γ	Angle of distortion due to shear stress
θ	Angle of rotation of end bulkheads with respect to each other
θ_P	Rotation angle θ due to P
γ_P	Angle γ due to P
ϕ	Total inclination of fibers, $\phi = \alpha - \gamma$; also central angle of a sector of fibers
ψ	Angle between directrix of cone and radius of base
Y_1, Y_2,	Elongation of fibers (in.)
$\delta = y_1 + y_2$	(in.)
A, A_r, A_f	Total area, area of elastomer, area of fibers, respectively (in.^{2})
t	Thickness (in.)
ℓ, L	Length of cylinder (in.)
r	Radius of cylinder (in.)
d_A, d_B, d_x	Diameter of tapered cylinder at A, B, x locations (in.)
S	Distance from vertex of cone to any circumference (in.)
u	Circumferential coordinate (in.)
*	Modification
N. A.	Neutral axis

REFERENCES

1. Kovalevsky, L., and F. L. Rish. On the Structural Behavior of Inflated Fabric Cylinders Under Various Loading Conditions, presented at the First Symposium on Inflatable Structures (1963).

2. Gere, James M., and Winfred O. Carter. "Critical Buckling Loads for Tapered Columns," Journal of the Structural Division of ASCE, Vol. 88 (Feb. 1962).