SECTION 5

OPTIMUM CONTROL OF A CLASS OF MIXED DISTRIBUTED AND LUMPED PARAMETER SYSTEMS

by

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ABSTRACT: The problem of optimum control of a class of mixed distributed and lumped parameter systems are considered. The mathematical model for the system is in the form of a coupled system of equations consisting of a scalar linear partial differential equation and a first-order vector linear ordinary differential equation. Classical variational techniques are used to derive the optimum control equations. The problems associated with the optimum control of systems whose distributed and lumped parameter subsystems are weakly coupled to each other are also considered. The meaning of some of the results is interpreted in the framework of two physical problems pertaining to the control of temperature in a solid and the pitching motion of a simplified flexible aerodynamic vehicle.

5.1 INTRODUCTION

The problems associated with the optimum control of distributed parameter dynamical systems governed by a set of partial differential equations or integral equations have been investigated recently.[1-6] In many physical situations, the system can be considered as a composite of a distributed parameter and several lumped parameter subsystems. The dynamical behavior of the former is described by a partial differential equation and the dynamical behavior of the lumped parameter subsystems is described by a set of ordinary differential equations. The interaction between these subsystems can be expressed mathematically in the form of a set of boundary conditions. An example of such a system is an aerodynamic vehicle with a flexible body. The rigid body mode of the vehicle's motion can be described by a set of ordinary differential equations, whereas the deforming motion of the flexible body can be described by a partial differential equation. Also, the mathematical model of many distributed parameter systems with moving spatial boundaries can be cast into the same framework by appropriate transformations.[6]

This section considers the problem of optimum control of a particular class of mixed distributed and lumped parameter systems which has special bearing on the control of elastic bodies such as flexible aerodynamic vehicles. Classical variational techniques will be used to derive the optimum control equations, and the general form of their solutions will be discussed. The problems associated with the optimum control of systems whose distributed and lumped parameter subsystems are weakly coupled to each other will be considered. Finally, the meaning of some of the results will be interpreted in the framework of two physical problems pertaining to the control of temperature in a solid and the pitching motion of a simplified flexible aerodynamic vehicle.

5.2 STATEMENT OF PROBLEM

A linear mixed distributed and lumped parameter dynamical system will be considered. It is assumed that the distributed portion of the system is governed by a scalar partial differential equation

$$a_o(t,x)\,\frac{\partial^2 u(t,x)}{\partial t^2} + a_1(t,x)\,\frac{\partial u(t,x)}{\partial t} = \mathcal{L}\, u(t,x) \qquad (5.2\text{-}1)$$

defined for $t > t_o$ on a one-dimensional spatial domain $(0,1)$, where a_o and a_1 are specified continuous functions of their arguments. $\mathcal{L}$ is a linear spatial differential operator of the form

$$\mathcal{L} = \sum_{n=0}^{N} b_n(t,x)\,\frac{\partial^n}{\partial x^n} \quad . \qquad (5.2\text{-}2)$$

The lumped portion of the system is governed by a vector linear ordinary differential equation

$$\frac{dW(t)}{dt} = A(t)W(t) + (\mathcal{L}_o + \mathcal{L}_1)\, u(t,x) + (\mathcal{L}_{t0} + \mathcal{L}_{t1}) \frac{\partial u(t,x)}{\partial t} + G(t)\, f(t) \tag{5.2-3}$$

where W is a M-vector, A and G are $M \times M$ matrix-valued and M-vector-valued functions, respectively, and f is a scalar control function. $\mathcal{L}_i$ and $\mathcal{L}_{ti}$, $i = 0,1$, are vector linear spatial differential boundary operators of the form

$$\mathcal{L}_i = \mathrm{Col}\left[\sum_{j=0}^{S_{i1}} c_{ij}^{(1)}(t,x) \frac{\partial^j}{\partial x^j}\bigg|_{x=i}, \ \ldots, \ \sum_{j=0}^{S_{iM}} c_{ij}^{(M)}(t,x) \frac{\partial^j}{\partial x^j}\bigg|_{x=i} \right] \tag{5.2-4}$$

$$S_{ik} < N, \ i = 0,\ 1,\ k = 1, \ldots, M .$$

Furthermore, there are given a set of boundary conditions

$$\tilde{\mathcal{L}}_o u(t,x) + \tilde{\mathcal{L}}_{t0} \frac{\partial u(t,x)}{\partial t} + P_o(t)\, W(t) + Q_o(t)\, f(t) = 0 \tag{5.2-5}$$

$$\tilde{\mathcal{L}}_1 u(t,x) + \tilde{\mathcal{L}}_{t1} \frac{\partial u(t,x)}{\partial t} + P_1(t)\, W(t) + Q_1(t)\, f(t) = 0 \tag{5.2-6}$$

and initial conditions at time t_o

$$u(t_o, x) = u_o(x), \quad \frac{\partial u(t,x)}{\partial t}\bigg|_{t_o} = \dot{u}_o(x), \quad W(t_o) = W_o, \tag{5.2-7}$$

where P_o and P_1 are $K \times M$ and $(N-K) \times M$ matrix-valued functions of t respectively. Q_o and Q_1 are vector-valued functions having K and $(N-K)$ components respectively. $\tilde{\mathcal{L}}_i$ and $\tilde{\mathcal{L}}_{ti}$, $i = 0,1$ are vector linear spatial differential boundary operators of a form similar to (5.2-4). Note that (4.2-5) and (4.2-6) form a set of N boundary conditions where N corresponds to the order of the highest derivative in $\mathcal{L}$. Also, although it is possible to combine (4.2-3), (4.2-5), and (4.2-6) into a more compact form, their present form will be preserved for clarifying the subsequent development.

For the system under consideration, the state S_t at any time t can be specified by the functions $u(t,x)$ and $\partial u(t,x)/\partial t$ defined on $(0,1)$ and the vector $W(t)$. Hence, the state space Γ can be assumed to be the Cartesian product of a function space and a finite dimensional vector space. In this case, Γ may be taken to be a subset of a Hilbert space with a norm defined by

$$\| S_t \| = \left[\int_0^1 \left\{ \left(\frac{\partial u(t,x)}{\partial t} \right)^2 + \sum_{n=0}^{N} \left(\frac{\partial^n u(t,x)}{\partial x^n} \right)^2 \right\} dx + W'(t) W(t) \right]^{1/2} \quad (5.2\text{-}8)$$

where $(\ )'$ denotes transpose. Also, two states S_t and $\hat{S}_t$ will be considered to be identical when $\| S_t - \hat{S}_t \| = 0$.

Note that the solutions to the initial and boundary-value problem corresponding to (5.2-1) - (5.2-7) - if they exist - can have a wide range of behaviors depending on the nature of the coefficients a_0, a_1, and b_n, $n = 1, \ldots, N$, particularly when they degenerate on certain subsets of the $t - x$ plane. For example, in the case where $a_1 > 0$ for all (t,x) and $\mathcal{L}$ is a second-order elliptic operator, if $a_0 > 0$ everywhere except it vanishes on a certain subset of the $t - x$ plane, then (5.2-1) becomes a mixed equation of parabolic and hyperbolic types. To avoid the mathematical difficulties which may arise in the above situations, it is assumed here that certain conditions are satisfied so that the initial and boundary-value problem corresponding to (5.2-1) - (5.2-7) with any initial state $S_{t_0} \in \Gamma$ and any control f belonging to a certain admissible class of functions is well posed; i.e., the solutions exist and are unique in addition to depending continuously on the initial data. This assumption is certainly justifiable from the physical standpoint.

The optimum control problem is to find a control law (f as a function of the instantaneous state S_t) which minimizes a quadratic performance index

$$\mathscr{P} = \frac{1}{2} \int_{t_0}^{t_1} \left\{ \int_0^1 \int_0^1 u(t,x_1) q(t,x_1,x_2) u(t,x_2)\, dx_1\, dx_2 + W'(t) R(t) W(t) + \beta(t) f^2(t) \right\} dt \quad (5.2\text{-}9)$$

where the kernel q is symmetric with respect to x_1 and x_2, R is a $M \times M$ symmetric matrix whose elements are continuous functions of t, and β is a specified function which is continuous in t. Note that for simplicity, $\mathscr{P}$ has been defined with respect to the state variables u and W, and the control variable f only. In more general situations, $\mathscr{P}$ may be chosen to depend on various derivatives of u with respect to t and x. In addition, a terminal performance index may be added.

5.3 OPTIMUM CONTROL EQUATIONS

Necessary and sufficient conditions for the existence of optimum controls will be derived. Here, it will be assumed that no magnitude constraint is imposed on the control function f, and the initial data are sufficiently smooth so that the resulting solutions to the system equations are continuous with respect to t and x.

Consider the equivalent problem of minimizing a modified functional:

$$\mathcal{P}' = \mathcal{P} + \int_{t_o}^{t_1} \int_o^1 \lambda_0(t,x) \left[a_o(t,x) \frac{\partial^2 u(t,x)}{\partial t^2} + a_1 t,x) \frac{\partial u(t,x)}{\partial t} - \mathcal{L} u(t,x) \right] dx \; dt$$

$$+ \int_{t_o}^{t_1} \Lambda'(t) \left[\frac{dW(t)}{dt} - A(t)W(t) - (\mathcal{L}_0 + \mathcal{L}_1) u(t,x) - (\mathcal{L}_{t0} + \mathcal{L}_{t1}) \frac{\partial u(t,x)}{\partial t} - G(t)f(t) \right] dt$$

(5.3-1)

where λ_0 and Λ are Lagrange multipliers, and $\Lambda = \mathrm{Col}\,(\lambda_1, \ldots, \lambda_M)$.

A necessary condition for extremum is that the first variation of $\mathcal{P}'$ vanishes, i.e.,

$$\delta \mathcal{P}' = \int_{t_0}^{t_1} \int_0^1 \left[\frac{\partial^2 (a_0(t,x)\lambda_0(t,x))}{\partial t^2} - \frac{\partial (a_1(t,x)\lambda_0(t,x))}{\partial t} \right.$$

$$\left. - \sum_{n=0}^{N} (-1)^n \frac{\partial^n (b_n(t,x)\lambda_0(t,x))}{\partial x^n} + \int_0^1 q(t,x,x_1) u(t,x_1)\, dx_1 \right] \delta u(t,x)\, dx\, dt$$

$$+ \int_{t_0}^{t_1} \left[\left(- \frac{d\Lambda'(t)}{dt} - \Lambda'(t)\, A(t) + W'(t) R(t) \right) \delta W(t) + (\beta(t) f(t) - G'(t)\Lambda(t))\; \delta f(t) \right] dt$$

$$+ \eta_1 + \eta_2 = 0 \;, \qquad (5.3\text{-}2)$$

where

$$\eta_1 = \int_0^1 \left[\lambda_0(t,x)\, a_0(t,x) \frac{\partial \delta u(t,x)}{\partial t} + a_1(t,x)\lambda_0(t,x) - \frac{\partial (a_0(t,x)\lambda_0(t,x))}{\partial t} \delta u(t,x) \right] \Bigg|_{t_0}^{t_1} dx$$

(5.3-3)

$$+ \Lambda'(t)\, \delta\, W(t) \Big|_{t_0}^{t_1}$$

and

$$\eta_2 = \int_{t_0}^{t_1} \left\{ \left[\sum_{j=0}^{N-1} \sum_{n=1+j}^{N} (-1)^j \frac{\partial^j (b_n(t,x)\lambda_0(t,x))}{\partial x^j} \frac{\partial^{n-(1+j)} \delta u(t,x)}{\partial x^{n-(1+j)}} \right] \Bigg|_0^1 \right.$$

$$\left. - \Lambda'(t) \left[(\mathcal{L}_0 + \mathcal{L}_1)\, \delta u(t,x) + (\mathcal{L}_{t0} + \mathcal{L}_{t1})\, \delta \left(\frac{\partial u(t,x)}{\partial t} \right) \right] \right\} dt \qquad (5.3\text{-}4)$$

For the free-end problem, the terminal conditions are

$$\lambda_0(t_1, x) = \frac{\partial \lambda_0(t,x)}{\partial t} \Bigg|_{t=t_1} = 0 \quad \text{for almost all} \quad x \in (0,1) \,, \quad \Lambda(t_1) = 0$$

(5.3-5)

and for the fixed-end problem:

$$u(t_1,x) = u_1(x) \ , \quad \left.\frac{\partial u(t,x)}{\partial t}\right|_{t=t_1} = \dot{u}_1(x) \ , \quad W(t_1) = W_1 \qquad (5.3\text{-}6)$$

while λ_0, $\partial \lambda_0/\partial t$ and Λ at t_1 are free. Finally, for the problem where only the state variables of the lumped parameter subsystem at t_1 are specified, and u, $\partial u/\partial t$ at t_1 are left free, the terminal conditions are:

$$\lambda_0(t_1,x) = \left.\frac{\partial \lambda_0(t,x)}{\partial t}\right|_{t=t_1} = 0 \ \text{ for almost all } \ x \in (0,1), \ W(t_1) = W_1 \qquad (5.3\text{-}7)$$

In all the above problems, $\eta_1 = 0$.

Due to the presence of imposed boundary conditions (5.2-5) and (5.2-6), it is necessary to incorporate them into η_2. Here it is assumed that (5.2-5) and (5.2-6) can be substituted into (5.3-4) and the resulting η_2 is reducible to the form:

$$\eta_2 = \int_{t_0}^{t_1} \Big\{ \sum_{i=0,\,1} \Big[\mathcal{A}_i \ \lambda_0(t,x) + B_i(t)\ \Lambda(t))' \ \mathcal{B}_i \ \delta u(t,x) \Big]$$

$$+ \ ((\mathcal{C}_1 + \mathcal{C}_0)\ \lambda_0(t,x) + D(t)\ \Lambda(t))' \delta W(t)$$

$$+ \ ((\hat{\mathcal{C}}_1 + \hat{\mathcal{C}}_0)\ \lambda_0(t,x) + \hat{D}'(t)\Lambda(t))\ \delta f(t) \Big\} \ dt \ , \qquad (5.3\text{-}8)$$

where $\mathcal{A}_i$, $\mathcal{B}_j$, $\mathcal{C}_k$, i,j,k = 0, 1 are vector linear spatial differential boundary operators; $\hat{\mathcal{C}}_i$, i = 0, 1 are scalar linear spatial differential boundary operators; D, B_i, C_i, i = 0, 1 are matrix-valued functions of t; D is a M-vector-valued function of t.

Substituting (5.3-8) into (5.3-2) and setting the coefficients of the independent variations $\delta u(t,x)$, $\delta W(t)$, and $\delta f(t)$ to zero lead to the following necessary conditions for an extremum:

$$\frac{\partial^2(a_0 t,x)\,\lambda_0(t,x))}{\partial t^2} - \frac{\partial(a_1(t,x)\lambda_0(t,x))}{\partial t} = \mathcal{D}^* \lambda_0(t,x) - \int_0^1 q(t,x,x_1)u(t,x_1)dx_1 \qquad (5.3\text{-}9)$$

where

$$\mathcal{L}^* = \sum_{n=0}^{N} (-1)^n \frac{\partial^n (b_n(t,x) \cdot \ )}{\partial x^n} , \qquad (5.3\text{-}10)$$

and

$$\frac{d\Lambda(t)}{dt} = (A'(t) - D(t))\,\Lambda(t) + (\mathcal{C}_1 + \mathcal{C}_0)\,\lambda_0(t,x) + R(t)\,W(t) \qquad (5.3\text{-}11)$$

$$f(t) = \beta^{-1} [(G(t) - \hat{D}(t))'\Lambda(t) - (\hat{\mathcal{C}}_1 + \hat{\mathcal{C}}_0)\,\lambda_0(t,x)] \qquad (5.3\text{-}12)$$

Also, the following boundary conditions associated with (5.3-9) have been imposed:

$$\mathcal{A}_i \lambda_0(t,x) + B_i(t)\,\Lambda(t) = 0 \ ; \quad i = 0,\ 1 \qquad (5.3\text{-}13)$$

Note that if $\mathcal{A}_1$ has K' components and B_1 is a $K' \times M$ matrix, then in order to have N boundary conditions, $\mathcal{A}_0$ must have (N-K') components and B_0 is a $(N-K') \times M$ matrix.

Equations (5.2-3) and (5.3-11), in view of (5.3-12), can be combined into the following form:

$$\frac{d}{dt}\begin{bmatrix} W(t) \\ \Lambda(t) \end{bmatrix} = \begin{bmatrix} A(t) & \beta^{-1}(t)G(t)\,(G(t) - \hat{D}(t))' \\ R(t) & -(A'(t) - D(t)) \end{bmatrix} \begin{bmatrix} W(t) \\ \Lambda(t) \end{bmatrix} +$$

$$\begin{bmatrix} (\mathcal{L}_0 + \mathcal{L}_1) & (\mathcal{L}_{t0} + \mathcal{L}_{t1}) & -\beta^{-1}(t)\,(\hat{\mathcal{C}}_0 + \hat{\mathcal{C}}_1) \\ 0 & 0 & (\mathcal{C}_0 + \mathcal{C}_1) \end{bmatrix} \begin{bmatrix} u(t,x) \\ \dfrac{\partial u(t,x)}{\partial t} \\ \lambda_0(t,x) \end{bmatrix}$$

(5.3-14)

Thus, the pertinent equations for determining the optimum control law consist of (5.2-1), (5.3-9), and (5.3-14) along with boundary conditions (5.2-5), (5.2-6), and (5.3-13), initial condition (5.2-7), and appropriate terminal conditions ((5.3-5), (5.3-6), or (5.3-7)). Although the pertinent equations are linear and homogeneous, the determination of their solutions satisfying the given initial, terminal, and boundary conditions is generally a difficult task. However, assuming the existence of a Green's function, their solutions at time t_1 corresponding to initial conditions given at time t can be symbolically written in the form:

$$\tilde{S}_{t_1} = \Phi(t_1, t)\, \tilde{S}_t$$

where $\tilde{S}_t = \mathrm{Col}(\tilde{s}_1(t), \ldots, \tilde{s}_6(t)) = \mathrm{Col}(W(t), u(t,x)\, \partial u(t,x)/\partial t,\ \Lambda(t),\ \lambda_0(t,x),$ $\partial \lambda_0(t,x)/\partial t)$, and Φ is a matrix linear operator whose elements ϕ_{ij} may be a matrix operator or a spatial integral operator depending on (i,j).

By imposing the specified terminal condition on $\tilde{S}_{t_1}$, a relation between $S^*_{t_0} = \mathrm{Col}(\Lambda(t), \lambda_0(t,x)\, \partial \lambda_0(t,x)/\partial t)$, and $S_t = \mathrm{Col}(W(t),\ u(t,x),\ \partial u(t,x)/\partial t)$ can be established. For the terminal conditions given by (5.3-5) - (5.3-7) with $u_1(x) = 0$ and $W_1 = 0$, the above relation can be expressed as:

$$\hat{\Phi}_1(t_1, t)\, S^*_t = -\hat{\Phi}_2(t_1, t) S_t$$

For the particular case where the terminal conditions are given by (5.3-7), $\hat{\Phi}_1$ and $\hat{\Phi}_2$ have the form:

$$\hat{\Phi}_1(t_1 t) = \begin{bmatrix} \phi_{11}(t_1,t) & \phi_{12}(t_1,t) & \phi_{13}(t_1,t) \\ \phi_{51}(t_1,t) & \phi_{52}(t_1,t) & \phi_{53}(t_1,t) \\ \phi_{61}(t_1,t) & \phi_{62}(t_1,t) & \phi_{63}(t_1,t) \end{bmatrix}$$

$$\hat{\Phi}_2(t_1,t) = \begin{bmatrix} \phi_{14}(t_1 t) & \phi_{15}(t_1,t) & \phi_{16}(t_1,t) \\ \phi_{54}(t_1,t) & \phi_{55}(t_1,t) & \phi_{56}(t_1,t) \\ \phi_{64}(t_1,t) & \phi_{65}(t_1,t) & \phi_{66}(t_1,t) \end{bmatrix}$$

By considering the second variation of $\mathcal{P}'$, it can be deduced that a sufficient condition for the existence of an optimum control law is that for any fixed $t \in [t_0, t_1]$, both $R(t)$ and the kernel $q(t, x_1, x_2)$ are positive definite and $\beta(t) > 0$, and the matrix linear operator $\hat{\Phi}_1(t_1, t)$ has an inverse for all $t \in [t_0, t_1]$. Assuming that the above condition is satisfied, the optimum control law has the form:

$$f(S_t) = -\beta^{-1}(t)\, \Psi(t)\, \hat{\Phi}_1^{-1}(t_1,t)\, \hat{\Phi}_2(t_1,t)$$

where Ψ is a row vector given by

$$\Psi(t) = [G(t) - D(t))',\ -(\hat{\mathcal{E}}_1 + \hat{\mathcal{E}}_0),\ 0]$$

It is evident that the optimum control law is a function of the state of the complete system.*

Note that the variational problem considered here differs somewhat from the usual ones involving multiple integrals due to the presence of imposed nonhomogeneous boundary conditions. The derivation of conditions for extremum has been based on the assumption that the boundary term η_2 is reducible to the form given by (5.3-8). Otherwise, certain compatibility conditions have to be satisfied.

5.4 WEAKLY COUPLED SYSTEMS

First, the manner in which weakly coupled systems can arise in various physical situations will be discussed.

(i) In many practical instances, emphasis is placed on controlling the motion of the lumped parameter subsystem only. Here, one may choose a performance index defined by (5.2-9) with $q = 0$. Also, the effect of the distributed subsystem on the motion of the lumped parameter subsystem may be regarded as a disturbance. If one proceeds the analytical design of the optimum controller on the basis of the mathematical model of the complete system, the implementation of the resulting control law generally requires a knowledge of the state of both the lumped and distributed parameter subsystems. Since the state of the distributed subsystem is specified by a set of functions, it is difficult to measure physically. On the other hand, measurements of physical variables defined at a fixed spatial "point" are relatively easy to perform.

In view of the fact that the distributed subsystem is coupled to the lumped parameter subsystem through boundary terms only, one may attempt in practice to use "point" sensors to cancel the disturbance effect of the distributed subsystem, and design the optimum controller solely on the basis of the lumped parameter mathematical model. However, precise cancellation is never possible in practice. The effect of the residual must be considered. The resulting mathematical model corresponding to (5.2-3) can usually be expressed in the form:

$$\frac{dW(t)}{dt} = A(t)W(t) + \mu\left[(\mathcal{L}_0 + \mathcal{L}_1)\, u(t,x) + (\mathcal{L}_{t0} + \mathcal{L}_{t1})\, \frac{\partial u(t,x)}{\partial t}\right] + G(t)f(t) \qquad (5.2\text{-}3')$$

where μ is a small parameter.

* It should be emphasized here that the results are expressed in terms of abstract operators. They are far from being satisfactory solutions from the practical standpoint. Since approximation in one form or another is imperative in obtaining numerical solutions, a possible starting point is to select a suitable approximation for the equations derived from the optimization process.

(ii) In certain physical situations, the distributed and lumped parameter subsystems are weakly coupled to each other as a result of their natural physical structure. In this case, one can often express the mathematical model in the form of (5.2-1) and (5.2-3') with boundary conditions

$$\tilde{\mathcal{L}}_0 u(t,x) + \tilde{\mathcal{L}}_{t0} \frac{\partial u(t,x)}{\partial t} + \mu[P_0(t)\, W(t) + Q_0(t) f(t)] = 0 \qquad (5.2\text{-}5')$$

$$\tilde{\mathcal{L}}_1 u(t,x) + \tilde{\mathcal{L}}_{t1} \frac{\partial u(t,x)}{\partial t} + \mu[P_1(t)\, W(t) + Q_1(t) f(t)] = 0 \qquad (5.2\text{-}6')$$

In the first case, the optimum control law, determined by using only the lumped parameter model; (i.e., Eq. (5.2-3') with $\mu = 0$), has the form

$$f(W(t)) = \mathcal{K}_0(t_1, t)\, W(t) \qquad (5.4\text{-}1)$$

where $\mathcal{K}_0$ is a row vector whose explicit form is well known.[7]

In the second case, if the state of the distributed subsystem can be measured, one may use a perturbation approach to the control problem. In particular, if the performance index is defined in terms of W and f only, one may seek a first-order approximation to the optimum control law of the form:

$$f(S_t) = \mathcal{K}_0(t_1, t)\, W(t) + \mu \mathcal{K}_1(t_1, t)\, U(t,x)$$

where $U(t,x) = \text{Col}\,(u,(t,x)\ ,\ \partial u(t,x)/\partial t)$, and $\mathcal{K}_1$ is a row vector.

In the sequel, discussions will be confined to a weakly coupled system which is describable by (5.2-1) and (5.2-3') with boundary conditions (5.2-5') and (5.2-6') and with control law (5.4-1) which minimizes the performance index (5.2-9) with $q = 0$ and $t_1 = \infty$ will be regarded as the optimum one. For this system, the following questions having practical importance may be posed:

(1) Is the complete system with control law (5.4-1) asymptotically stable?

(2) If the answer to (1) is affirmative, what is the performance loss induced by the disturbance effect of the distributed subsystem?

The first question is difficult to answer in general terms. However, in many practical situations, it is meaningful to require the lumped parameter subsystem to be asymptotically stable while the distributed subsystem is only stable. In the subsequent development, a sufficient condition for which the above condition is satisfied will be derived. Also, an approximate expression for the loss of performance will be given.

5.4.1 Stability

Assuming the existence of various Green's functions associated with (5.2-1) along with boundary conditions (5.2-5') and (5.2-6'), the following integral equation can be written for u :

$$u(t,x) = \int_0^1 [k_0(t,t_0,x,\xi)\, u_0(\xi) + k_1(t,t_0,x,\xi)\, \dot{u}_0(\xi)]\, d\xi$$

$$+ \mu \int_{t_0}^{t} \sum_{i=0,1} K_i'(t,\tau,x)\,(P_i(\tau) + Q_i(\tau)\,\mathcal{K}_0(\tau))\, W(\tau)\, d\tau \qquad (5.4\text{-}2)$$

where k_0 and k_1 are scalar functions of their arguments; K_0 and K_1 are vector-valued functions having K and (N-K) components respectively.

Similarly, an integral equation can be written for W:

$$W(t) = \Phi_c(t,t_0)W_0 + \mu \int_{t_0}^{t} \Phi_c(t,\tau)\left[(\mathcal{L}_0 + \mathcal{L}_1)\, u(\tau,x) + (\mathcal{L}_{t0} + \mathcal{L}_{t1})\frac{\partial u(\tau,x)}{\partial \tau}\right] d\tau \qquad (5.4\text{-}3)$$

where Φ_c is the transition matrix corresponding to (5.2-3') with $\mu = 0$ and control law (5.4-1). In the following, a sufficient condition for asymptotic stability (in the sense of Lyapunov) of W(t) will be established.

Substituting (5.4-2) directly into (5.4-3), the integral equation for W(t) can be rewritten as:

$$W(t) = \Phi_c(t,t_0)\, W_0 + \mu \int_{t_0}^{t} \Phi_c(t,\tau) \int_0^1 Z(\tau,\xi)\, U_0(\xi)\, d\xi\, d\tau$$

$$+ \mu^2 \int_{t_0}^{t} \Phi_c(t,\tau)\left[\int_{t_0}^{\tau} (\mathcal{L}_0 + \mathcal{L}_1)\, Y(\tau,\tau_1,x)\, W(\tau_1)\, d\tau_1\right.$$

$$\left. + (\mathcal{L}_{t0} + \mathcal{L}_{t1})\frac{\partial}{\partial \tau} \int_{t_0}^{\tau} Y(\tau,\tau_1,x)\, W(\tau_1)\, d\tau_1\right] d\tau \qquad (5.4\text{-}3')$$

where

$$Z(\tau,t_0,x) = \left[(\mathcal{L}_0 + \mathcal{L}_1)\, k_0(\tau,t_0,x,\xi) + (\mathcal{L}_{t0} + \mathcal{L}_{t1})\frac{\partial k_0(\tau,t_0,x,\xi)}{\partial \tau} \;\vdots\; (\mathcal{L}_0 + \mathcal{L}_1)\, k_1(\tau,t_0,x,\xi) + (\mathcal{L}_{t0} + \mathcal{L}_{t1})\frac{\partial k_1(\tau,t_0,x,\xi)}{\partial \tau}\right] \qquad (5.4\text{-}4)$$

$$Y(\tau, \tau_1, x) = \sum_{i=0,1} K_i'(\tau, \tau_1, x)\,(P_i(\tau_1) + Q_i(\tau_1)\mathcal{K}_0(\tau_1)) \qquad (5.4\text{-}5)$$

Note that Z is a $M \times 2$ matrix, and Y is a row vector having M components.

It will be assumed that the transition matrix Φ_c satisfies an estimate of the form:

$$\| \Phi_c(t, \tau) \| \leq \exp\left[\int_{\tau}^{t} \sigma(\eta)\, d\eta \right] \quad ; \qquad t \geq \tau \geq t_0 \qquad (5.4\text{-}6)$$

where $\| \Phi_c \|$ denotes the norm of the matrix operator Φ_c induced by the chosen norm for the vector W, and σ is a function depending on the form of $\| W \|$. In particular, if the norm for the vector W is the usual Euclidean norm

$$\| W \| = \left\{ \sum_{i=1}^{M} |w_i|^2 \right\}^{1/2} ,$$

then σ can be taken to be the maximum eigenvalue of the symmetric matrix

$$(A(t) + G(t)\mathcal{K}_0(t)) \;+\; (A(t) + G(t)\mathcal{K}_0(t))'$$

for any fixed $t \geq t_0$. Alternatively, if $\|W\| = \underset{i}{\text{Max}} |w_i|$, then σ can be taken to be

$$\sigma(t) = \underset{i}{\text{Max}} \left\{ \tilde{a}_{ii}(t) + \sum_{i \neq j} |\tilde{a}_{ij}(t)| \right\} \qquad (5.4\text{-}7)$$

where $\tilde{a}_{ij}$ are the elements of $A(t) + G(t)\mathcal{K}_0(t)$. The expression (5.4-7) coincides with an estimate of the maximum value of the real parts of the eigenvalues of $A(t) + G(t)\mathcal{K}_0(t)$ for any fixed t using the well-known "Gersgorin's circle theorem".

To derive an estimate for $\|W\|$, use will be made of the following differential inequality:

<u>Lemma</u>:

Let w be a scalar function of t defined for $t \geq t_0$, and having a continuous first derivative. If w satisfies the differential inequality $dw(t)/dt - a(t)w(t) \leq b(t)$ and $w(t_0) \leq \delta$, where a and b are specified continuous functions of t, and δ is a constant, then

$$w(t) \leq \left\{ \delta + \int_{t_0}^{t} \exp\left[-\int_{t_0}^{\tau} a(\eta)\, d\eta \right] b(\tau)\, d\tau \right\} \exp\left(\int_{t_0}^{t} a(\eta)\, d\eta \right). \qquad (5.4\text{-}8)$$

<u>Proof</u>:

Let $dw(t)/dt - a(t)\, w(t) = c(t)$. Solving this differential equation and making use of the hypothesis that $c(t) \leq b(t)$ and $w(t_0) \leq \delta$ lead directly to (5.4-8).

<u>Theorem</u>:

Suppose that

(i) there exists a continuous function ν such that

$$\frac{\partial \zeta(t,\tau)}{\partial t} \leq \nu(t)\, \zeta(t,\tau) \quad \text{for all } t \geq t_0 \ , \tag{5.4-9}$$

where

$$\zeta(t,\tau) = \left| (\mathcal{L}_0 + \mathcal{L}_1)\, Y(t,\tau,x) + (\mathcal{L}_{t0} + \mathcal{L}_{t1}) \frac{\partial Y(t,\tau,x)}{\partial t} \right| \tag{5.4-10}$$

(ii) there exists a finite time $t_1 > t_0$ such that

$$1 + \nu(t) \int_{t_0}^{t} h(t,\tau)\, d\tau \leq 0 \quad \text{for all } t \geq t_1 \ ; \tag{5.4-11}$$

(iii) the Green's function Y has the property

$$Y(t,t,x) = 0 \ , \quad \left. \frac{\partial Y(t,\tau,x)}{\partial t} \right|_{\tau = t} = 0 \quad ;$$

(iv) $\| Z(t,t_0,\xi) \| \leq \mathscr{K}(t,t_0)\, \chi(\xi)$, where $\mathscr{K}$ and χ are positive functions of their arguments ;

(v) the functions

$$h(t,t_0) \to 0 \quad \text{as} \quad t \to \infty \ , \tag{5.4-12}$$

$$h(t,t_0) \int_{t_0}^{t} h^{-1}(\tau,t_0)\, \mathscr{K}(\tau, t)\, d\tau$$

$$= \int_{t_0}^{t} h(t,\tau)\, \mathscr{K}(\tau,t_0)\, d\tau \to 0 \quad \text{as} \quad t \to \infty \ ,$$

where

$$h(t,\tau) = \exp\left(\int_{\tau}^{t} \sigma(\eta)\, d\eta \right) \ , \quad t \geq \tau \geq t_0 \ ; \tag{5.4-13}$$

then $W(t)$ is asymptotically stable.

Proof:

The norm of W, in view of (5.4-6), can be bounded by

$$\|W(t)\| \le h(t,t_0)\|W_0\| + |\mu| \int_{t_0}^{t} h(t,\tau) \int_0^1 \|Z(\tau,t_0,\xi)\| \, d\xi \, d\tau$$

$$+ \mu^2 \int_{t_0}^{t} h(t,\tau) \left\{ \|(\mathcal{L}_{t0} + \mathcal{L}_{t1}) \, Y(\tau,\tau,x)\| \; \|W(\tau)\| \right.$$

$$\left. + \int_{t_0}^{\tau} \zeta(\tau,\tau_1) \, \|W(\tau_1)\| \, d\tau_1 \right\} d\tau . \qquad (5.4\text{-}14)$$

where ζ is a scalar-valued function defined by (5.4-10).

In view of condition (iii), the first term of the $\{ \ . \ . \ . \}$ expression in (5.4-14) vanishes. Also, since ζ is a non-negative function, (5.4-14) remains valid when the upper limit τ of the integral in $\{ \ . \ . \ . \}$ is replaced by t.

Let the right-hand side of (5.4-14) be denoted by $\omega(t)$. By a straightforward computation, the derivative of ω with respect to t is given by

$$\frac{d\,\omega(t)}{dt} = \sigma(t)\,\omega(t) + |\mu| \int_0^1 \|Z(t,t_0,\xi)\,U_0(\xi)\| \, d\xi$$

$$+ \mu^2 \int_{t_0}^{t} \left[\zeta(t,\tau) + \left(\int_{t_0}^{t} h(t,\tau_1)\, d\tau_1 \right) \frac{\partial \zeta(t,\tau)}{\partial t} \right] \|W(\tau)\| \, d\tau$$

(5.4-15)

In view of conditions (i) and (ii) in the theorem, the last integral in (5.4-15) is ≤ 0 for all $t \ge t_1$. Hence,

$$\frac{d\,\omega(t)}{dt} \le \sigma(t)\,\omega(t) + |\mu| \int_0^1 \|Z(t,t_0,\xi)\,U_0(\xi)\| \, d\xi \quad \text{for } t \ge t_1 .$$

(5.4-15')

It follows from the lemma that $W(t)$ satisfies the estimate

$$\|W(t)\| \le \omega(t) \le \left\{ \|W_0\| + |\mu| \int_{t_0}^{t} h^{-1}(\tau,t_0) \int_0^1 \|Z(\tau,t_0,\xi)U_0(\xi)\| \, d\xi \, d\tau \right\} h(t,t_0)$$

$$\text{for } t \ge t_1 \qquad (5.4\text{-}16)$$

From conditions (iv) and (v), it is evident that $\|W(t)\| \to 0$ as $t \to \infty$. Also, it can be readily verified by applying Gronwall's inequality [8] to (5.4-14) with the upper limit τ of the integral in $\{\ \dots\ \}$ replaced by t, that under conditions (i) and (iv), $\|W(t)\|$ remains bounded in any finite interval $[t_0, t_1]$. Hence, the proof is complete.

Remark 1:

The asymptotic stability conditions given in the theorem are independent of the coupling parameter μ. This suggests that the restrictions imposed on the system parameters as implied by these conditions are too stringent. Sharper sufficient conditions for asymptotic stability may be obtained when more detailed information on the system equations is given. On the other hand, one may interpret the conditions in the theorem as restrictions on the form rather than on the "strength" of coupling. In particular, conditions (5.4-9) and (5.4-11) imply that $\nu(t)$ must be < 0 and ζ is bounded above by a decaying exponential function of t. Since ζ is related only to the back coupling effect of the lumped parameter subsystem on the distributed subsystem, (5.4-9) and (5.4-11), respectively, represent restrictions on the form of this back coupling and on the behavior of the back coupling relative to that of the natural (uncoupled) lumped parameter subsystem. Condition (iv), however, represents a restriction on the form of the forward coupling of the distributed subsystem with the lumped parameter subsystem, and the second condition in (5.4-12) represents a restriction on the behavior of this coupling relative to that of the natural lumped parameter system. Finally, condition (iii) implies that an instantaneous change in the coupling terms in the boundary conditions (5.2-5') and (5.2-6') does not cause instantaneous changes in the solutions $u(t,x)$ and $\partial u(t,x)/\partial t$.

Remark 2:

For the special case where $\sigma(t) = -\alpha_0$, $\nu(t) = -\alpha_1$ and $\mathscr{K}(t,t_0) = \exp(-\alpha_2(t-t_0))$, where α_i are positive constants, it can be readily shown that (5.4-12) is automatically satisfied, and (5.4-11) is satisfied if

$$\alpha_1 \geq \alpha_0 \tag{5.4-17}$$

Remark 3:

Making use of estimate (5.4-16), it can be shown that under the conditions imposed by the theorem, the solution $U(t,x)$ is bounded in the sense of L_2 norm; i.e., there exists a constant M depending on t_0 such that

$$\|U(t,x)\| = \left\{ \int_0^1 \left[u^2(t,x) + \left(\frac{\partial u(t,x)}{\partial t} \right)^2 \right] dx \right\}^{1/2} \leq M(t_0) \text{ for all } t \geq t_0.$$

5.4.2 Performance Loss

Let the performance index evaluated along the optimum motion of the lumped parameter subsystem with control law (5.4-1) be denoted by $\mathscr{P}_0$:

$$\mathscr{P}_0 = \int_{t_0}^{\infty} W'(t)(R(t) + \beta(t)\mathcal{K}_0(t))\, W(t)\, dt \tag{5.4-18}$$

where

$$W(t) = \Phi_c(t,t_0)\, W_0 \, .$$

Also, let the performance index evaluated along any non-optimum motion resulting from the disturbance effect induced by the distributed subsystem be denoted by:

$$\hat{\mathscr{P}} = \int_{t_0}^{\infty} \hat{W}'(t)(R(t) + \beta(t)\mathcal{K}_0'(t)\mathcal{K}(t))\, \hat{W}(t)\, dt \quad . \tag{5.4-19}$$

Hence, the loss of performance $\Delta\mathscr{P}$ can be defined as:

$$\Delta\mathscr{P} = \mathscr{P}_0 - \hat{\mathscr{P}}. \tag{5.4-20}$$

If the motion of the lumped parameter subsystem is asymptotically stable, then, for sufficiently small coupling parameter μ , the disturbed motion $\hat{W}(t)$ can be expressed approximately by

$$\hat{W}(t) \approx \Phi_c(t,t_0)W_0 + \mu \int_{t_0}^{t} \Phi_c(t,\tau) \int_{o}^{1} Z(\tau,\xi)U_0(\xi)\, d\xi\, d\tau \tag{5.4-21}$$

where Z is given by (5.4-4).

Direct computation leads to the following approximate expression for the performance loss $\Delta\mathscr{P}$:

$$\Delta\mathscr{P} = 2\mu \int_{t_0}^{\infty} (\Phi_c(\tau,t_0)\, W_0)'\, (R(\tau) + \beta(\tau)\, \mathcal{K}_0'(\tau)\, \mathcal{K}(\tau)) \int_{t_0}^{\tau} \Phi_c(\tau,\tau_1) \int_{o}^{1} Z(\tau_1,\xi)U_0(\xi)d\xi\, d\tau_1\, d\tau. \tag{5.4-22}$$

5.5 EXAMPLES OF PHYSICAL PROBLEMS

In this section, examples of two physical problems pertaining to the control of temperature in a solid and the pitching motion of a simplified flexible aerodynamic vehicle will be discussed.

5.5.1 Temperature Control

Consider the problem of heating a thick homogeneous solid in a furnace having finite heat capacity. An idealized mathematical model for this system is given by the following set of equations:

$$\frac{\partial u(t,x)}{\partial t} = k_0 \frac{\partial^2 u(t,x)}{\partial x^2} \quad ; \quad x \in (0,1) \tag{5.5-1}$$

$$\frac{dw(t)}{dt} = k_1 \left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} + f(t) \ , \tag{5.5-2}$$

where u and w correspond to the temperatures of the solid and the furnace, respectively; f is the control variable which is proportional to the rate of supply of heat to the furnace; x is the normalized spatial coordinate variable; and k_0, k_1 are positive proportionality constants.

Assuming that one face of the slab is insulated and the heat flux at the other face is proportional to the difference between the furnace temperature and that of the slab surface, the boundary conditions for (5.5-1) take on the form:

$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=1} = 0 \ , \quad \left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = k_2(u(t,0) - w(t)) \tag{5.5-3}$$

where k_2 is a positive constant. Also, there are given a set of initial conditions:

$$w(0) = w_0 \ , \qquad u(0,x) = u_0(x) \tag{5.5-4}$$

Here, a possible problem is to determine an optimum control law for regulating the temperature in the slab on the basis of minimizing a quadratic performance index:

$$\mathscr{P} = 1/2 \int_0^{t_1} \left[\int_0^1 \int_0^1 (u_d(x_1) - u(t,x_1)) q(x_1,x_2) \cdot (u_d(x_2) - u(t,x_2))\, dx_1 dx_2 + \beta f^2(t) \right] dt \tag{5.5-5}$$

where u_d is the desired temperature distribution in the slab; q is a positive definite symmetric kernel; and β is a positive constant.

Following the same approach outlined in Section 5.3, it can be readily shown that the equations for determining the optimum control law consist of (5.5-1), (5.5-3), (5.5-4), and

$$\frac{d}{dt}\begin{bmatrix} w(t) \\ \lambda_1(t) \end{bmatrix} = \begin{bmatrix} -k_1k_2 & \beta^{-1} \\ 0 & k_1k_2 \end{bmatrix}\begin{bmatrix} w(t) \\ \lambda_1(t) \end{bmatrix} + \begin{bmatrix} k_1k_2(\cdot)\Big|_{x=0} & 0 \\ 0 & -k_0k_2(\cdot)\Big|_{x=0} \end{bmatrix}\begin{bmatrix} u(t,x) \\ \lambda_0(t,x) \end{bmatrix} \tag{5.5-6}$$

$$\frac{\partial\lambda_0(t,x)}{\partial t} = -k_0\frac{\partial^2\lambda_0(t,x)}{\partial x^2} - \int_0^1 q(x,x_1)(u_d(x_1) - u(t,x_1))dx_1 \quad ; \quad x \in (0,1) \tag{5.5-7}$$

with boundary conditions:

$$\left.\begin{aligned} &k_0\frac{\partial\lambda_0(t,x)}{\partial x}\Big|_{x=0} - k_0k_2\,\lambda_0(t,0) + k_1k_2\,\lambda_1(t) = 0 \;, \\ &\frac{\partial\lambda_0(t,x)}{\partial x}\Big|_{x=1} = 0 \;, \end{aligned}\right\} \tag{5.5-8}$$

and

$$f(t) = \beta^{-1}\lambda_1(t) \tag{5.5-9}$$

For the case where both w and u at t_1 are free, the terminal condition for λ_0 and λ_1 is

$$\lambda_0(t_1x) = 0 \;, \qquad \lambda_1(t_1) = 0 \,. \tag{5.5-10}$$

The foregoing set of equations cannot be readily solved explicitly. For the special case where $t_1 = \infty$, it can be shown that the optimum control law has the form:

$$f(w,u) = c\,w(t) + \int_0^1 h_1(x)\,(h_2(x)u_d(x) - u(t,x))\,dx$$

where c is a constant, and h_i are spatial weighting functions.

Now, consider the special case where $u_d(x) = u_{do}$ = constant, and only one stationary "point" sensor is used for measuring the slab temperature. Here, one may use a simple linear control law of the form:

$$f(w,u) = -c_o w(t) + h_0 u(t,x_1) \; ; \quad x \in [0,1] \tag{5.5-11}$$

where all temperatures are referenced with respect to u_{do}, c_o, and h_0 are constants. In what follows, a sufficient condition for asymptotic stability of the complete system (Eqs. (5.5-1) - (5.5-3)) with control law (5.5-11) will be derived by applying the theorem given in Section 5.4.1.

For this system, the function defined by (5.4-12) takes the form:

$$h(t,\tau) = \exp((\tau - t)(k_1 k_2 + c)) \tag{5.4-12'}$$

Also, it can be shown by straightforward computations that other pertinent parameters in the theorem can be taken to be

$$\left.\begin{aligned} \nu(t) &= k_0 \gamma_0^2 \\ \mathscr{K}(t,t_0) &= \exp(-k_0(t-t_0)\,\mathrm{Min}(\tan^{-1} k_2)^2) \end{aligned}\right\} \tag{5.5-12}$$

where γ_0 is the smallest positive root of

$$\tan\gamma = k_2 \gamma^{-1} \tag{5.5-13}$$

In view of Remark 2 in Section 5.4.1, a sufficient condition for asymptotic stability of $w(t)$ is:

$$k_0 \gamma_0^2 \geq (k_1 k_2 + c) \tag{5.5-14}$$

The fact that $u(t,x)$ must $\to 0$ as $t \to \infty$ when $w(t)$ is asymptotically stable is evident from the form of boundary conditions (5.5-3).

5.5.2 Flexible Aerodynamic Vehicle Control

The problem of controlling the pitching motion of an aerodynamic vehicle with a flexible fuselage (See Figure 5-1) will be considered. The control surface is taken to be a flat plate whose angle-of-attack can be varied. It is assumed that the vehicle can be represented by a non-uniform cantilever beam with a rigid main lifting surface rigidly attached to the fuselage. The fixed end of the beam is taken to be coinciding with the vehicle's center of mass. With the motion of the fuselage restricted to plane bending, its deflection can be defined with respect to a rotating coordinate system with the origin at the center of mass and one of the axis coinciding with the undeformed elastic axis.

By making additional assumptions similar to those stated in Reference 9, a highly simplified mathematical model for the vehicle can be described by the following equations:

For the bending motion:

$$m(x)v_0^2 \ell^2 \frac{\partial^2 u(t,x)}{\partial t^2} + v_0 \ell^3 k_d(t,x) \frac{\partial u(t,x)}{\partial t} = - \frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 u(t,x)}{\partial x^2} \qquad (5.5\text{-}15)$$

For the rigid-body pitching motion:

$$I_o \frac{d\dot{\theta}(t)}{dt} = 2\pi \rho_a ba \ell^3 \left[\dot{\theta}(t) + \theta(t) - \left(\frac{\partial u(t,x)}{\partial x} + \frac{\partial u(t,x)}{\partial t} \right|_{x=1} + \delta_c \right], \qquad (5,5\text{-}16)$$

where both the deflection u and the spatial coordinate x have been normalized with respect to the length of the tail section ℓ ; t is the dimensionless time normalized with respect to the quantity ℓ/v_0; m, k_d, and EI are linear mass density, distributed damping coefficient, and bending rigidity, respectively.

The boundary conditions for (5.5-15) are:

$$u(t,0) = \left. \frac{\partial u(t,x)}{\partial x} \right|_{x=0} = EI(x) \left. \frac{\partial^2 u(t,x)}{\partial x^2} \right|_{x=1} = 0 \ , \qquad (5.5\text{-}17)$$

$$\frac{\partial}{\partial x} EI(x) \left. \frac{\partial^2 u(t,x)}{\partial x^2} \right|_{x+1} =$$

$$2\pi \rho_a v_0^2 \ell^2 ba \left[\theta(t) + \dot{\theta}(t) - \left(\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} \right) \Big|_{x=1} + \delta_c \right] \qquad (5.5\text{-}18)$$

where ρ_a is the mass density of the undisturbed air, a and $2b$ are the total length and width of the control surface, respectively, and $\dot{\theta}(t) = d\theta(t)/dt$. The above mathematical model has a form identical to that given in Section 5.2

Suppose that the pitch autopilot is designed on the basis of the rigid-body model by minimizing a quadratic performance index

$$\mathscr{P} = \int_0^\infty (W'(t)\, R\, W(t) + \beta\, \delta_c^2(t))\, dt \qquad (5.5\text{-}19)$$

where $W = \text{Col}(\theta, \dot{\theta})$, R is a positive definite symmetric matrix, and β is a positive constant, then the resulting control law has the form:

$$\delta_c = -c_0\theta(t) - c_1\theta(t) \qquad (5.5\text{-}20)$$

where c_0 and c_1 are positive constants depending on R, β, and the parameters of the rigid-body model. Now, a question of practical importance is if control law (5.5-20) is used, will the equilibrium state of the vehicle be asymptotically stable? This question cannot be readily answered by conventional analysis. On the other hand, assuming that the equilibrium state of the uncontrolled vehicle is exponentially asymptotically stable, and the Green's function corresponding to (5.5-15) with boundary conditions (5.5-17) and

$$\left[\frac{\partial}{\partial x} EI(x) \frac{\partial^2 u(t,x)}{\partial x^2} - 2\pi \rho_a v_0^2 \ell^2 ba \left(\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x}\right)\right]\Bigg|_{x=1} = \delta(t-\tau) \qquad (5.5\text{-}21)$$

where δ is the Dirac delta function, satisfies inequality (5.4-9) with $\nu(t) = -\alpha_1$, then it follows from the theorem in Section IV-A that a sufficient condition for asymptotic stability of $W(t)$ and boundedness of $u(t,x)$ is

$$\alpha_1 \geq \pi\rho_a ba\ell^3 I_0^{-1}(c_1 - 1) > 0 \;\; ; \;\; c_0 > 1 \;. \qquad (5.5\text{-}22)$$

Although the above condition is weak, it may serve as a preliminary guide in the analytical design of the pitch autopilot.

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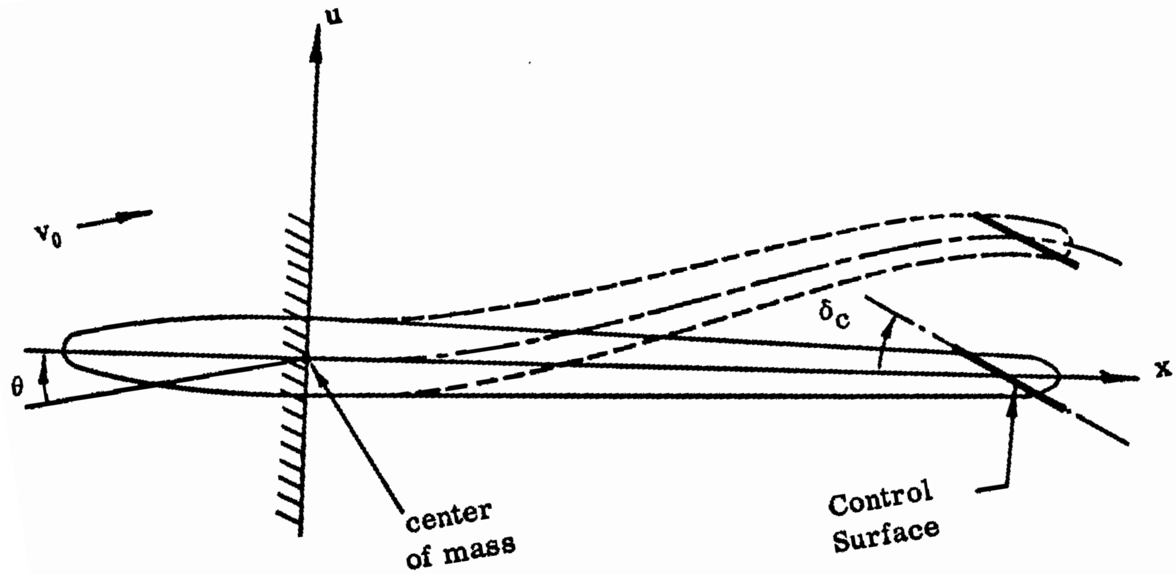

Figure 5.1