A PERTURBATION METHOD FOR THE ANALYSIS OF FREE-LAYER DAMPING TREATMENTS

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ABSTRACT
The feasibility of using perturbation techniques to determine the effect of added viscoelastic damping treatments on the modal properties of a system is investigated. Linear perturbation equations for the changes introduced into system eigenproperties are derived and applied to several examples involving the flexural vibration of beams with varying degrees of damping treatment. Both large and small perturbations are considered. Comparison of the results with those obtained by direct solution of the corresponding complex eigenproblem shows the procedure to be accurate. The perturbation approach described can accommodate frequency-dependent material properties, and the procedures involved are illustrated in an example. The perturbation approach appears to be particularly well-suited for design situations where a number of damping configurations must be investigated.

INTRODUCTION

Addition of a viscoelastic damping treatment to a structure alters its mass and stiffness and introduces damping effects. If these changes are relatively small, addition of the damping material constitutes a small perturbation to the existing structure. This raises the possibility of using structural modification (perturbation) techniques[1-6] to analyze the effect of added damping treatments.

Perturbation techniques have the advantage that the changes in the modal properties can be expressed entirely in terms of the eigenproperties of the original system and the changes in system mass and stiffness. In the case of added damping treatments, this means that the natural frequencies, loss factors and mode shapes of the damped system can be obtained directly, without the need to re-solve the eigenvalue problem. From a computational point of view, this feature of the perturbation approach is highly attractive. Viscoelastic damping treatments often lead to nonproportional damping, with complex eigenvalues and eigenvectors. Solution of large-order complex eigenvalue problems is time consuming and costly. This is particularly true for damping treatment design, which may require consideration of a number of different damping configurations.

In this paper we explore the feasibility of using perturbation techniques to determine the effects of added viscoelastic damping treatments. Attention is restricted to free-layer treatments applied to systems whose vibratory response is described by discretized equations of motion. The basic perturbation equations are derived and applied to several examples involving the flexural vibration of an elastic cantilever beam with varying degrees of damping treatment over its length. This configuration was chosen because of its simple geometry and because of the existence of other solutions with which to compare the results of the perturbation approach.

Values of the natural frequencies, loss factors and mode shapes for the damped beam are presented for varying degrees of damping treatment. These results are shown to be in very good agreement with those obtained by direct solution of the corresponding complex eigenvalue problem. Both small and large perturbations are considered. Large perturbations are treated as a series of smaller changes. Results showing the rates of convergence of this sequential approach are presented. Also presented are results showing the optimum locations along the beam for placement of partial damping treatments. Use of the perturbation approach to account for the frequency dependence of the damping material properties also is discussed and illustrated in an example problem.

GENERAL CONSIDERATIONS

Consider a conservative vibratory system with symmetric mass and stiffness matrices [M] and [K]. The corresponding eigenvalue problem is of the form[7]

$$\lambda_i^2 [M] \{\psi\}_i = [K] \{\psi\}_i \tag{1}$$

where λ_i^2 is the eigenvalue for the ith mode of vibration and $\{\psi\}_i$ is the corresponding eigenvector (mode shape). It is assumed that the system eigenproperties are known.

Suppose, now, that a linearly viscoelastic damping treatment is added to the system. Since Eq. (1) is expressed in the frequency domain, the complex modulus is the proper representation for the properties of the damping material. Consequently, addition of the damping treatment produces a real-valued change $[\Delta M]$ in the mass matrix and a complex-valued change $[\Delta K(\omega)]$ in stiffness. Since the properties of viscoelastic materials are frequency dependent, the change in stiffness also depends upon the frequency, ω.

These changes in the system parameters give rise to a new set of eigenvalues, $\bar{\lambda}_i^2$, and eigenvectors, $\{\bar{\psi}\}_i$:

$$\bar{\lambda}_i^2 = \lambda_i^2 + \Delta\lambda_i^2 \tag{2}$$

$$\{\bar{\psi}\}_i = \{\psi\}_i + \{\Delta\psi\}_i \tag{3}$$

Except for simple structures with uniform damping treatments over the entire surface, addition of viscoelastic layers usually gives rise to a system with nonproportional damping. In this case, the eigenvectors are complex-valued and the eigenvalues are of the form[7]

$$\bar{\lambda}_i^2 = \omega_i^2\,(1 + i\eta_i) \tag{4}$$

Here, ω_i is the damped natural frequency and η_i is the corresponding modal loss factor for the system.

One possibility for determining the eigenvalues and eigenvectors of the damped system is to re-solve Eq. (1) using $[M+\Delta M]$ and $[K+\Delta K(\omega)]$ as the mass and stiffness matrices. This is not an attractive proposition for large order systems, particularly if a number of different damping configurations are to be investigated. Solution of large-order, complex-valued eigenproblems is time consuming and costly.

Another possibility is to use approximate methods of analysis, such as the modal strain energy approach of Johnson and Kienholz[8] or the Rayleigh Quotient approach of Stevens et al[9]. These approaches are relatively simple to apply and give results that are useful in many situations. However, they do not always provide all the information needed. The modal strain energy approach is restricted to problems with proportional damping and provides estimates only of the modal loss factors; the Rayleigh Quotient approach gives estimates of both the loss factors and damped natural frequencies and applies to arbitrary damping configurations. Both methods are based upon the mode shapes of the undamped system, and neither provides information about changes in the mode shapes.

The alternative considered herein is to use linear perturbation techniques to express the changes $\Delta\lambda_i^2$ and $\{\Delta\psi\}_i$ in the eigenvalues and eigenvectors directly in terms of the changes $[\Delta M]$ and $[\Delta K(\omega)]$ and the eigenproperties of the original system. This approach has the obvious advantage that information about all the eigenproperties can be obtained for a variety of damping configurations, while the eigenproblem need be solved only once. It should be noted that the process need not start with a mathematical model of a conservative system, as assumed in the preceding discussion. The original system can be damped. Systems whose natural frequencies and mode shapes are obtained experimentally via modal testing techniques[7] also can be handled, provided an appropriate set of mode shapes is available.

There is one potential problem with the use of linear perturbation techniques. The density of common damping materials is of the same order of magnitude as common metals, so the changes in the mass matrix can be relatively large. A higher-order perturbation theory[10] could be used, but the resulting equations are lengthy and the computations time consuming. The alternative, used in this paper, is to treat large modifications as a series of smaller ones. Changes in system stiffness usually are relatively small and cause no particular difficulties. This is because the modulus of damping materials typically in several orders of magnitude less than that of the structure to which they are applied.

PERTURBATION EQUATIONS

First-order perturbation equations for the linear eigenvalue problem can be derived in a variety of ways, and are available in various references[1-5]. These derivations hold in the current case provided proper care is taken in handling the change in stiffness, which is now complex-valued. Suffice it to say that, for a system with distinct eigenvalues, the first-order approximations for the changes in the eigenproperties are:

$$\Delta\lambda_i^2 = \frac{1}{M_i}\{\psi\}_i^T \left[[\Delta K] - \lambda_i^2 [\Delta M] \right] \{\psi_i\} \tag{5}$$

and

$$\Delta\{\psi\}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_{ij} \{\psi\}_j \tag{6}$$

where

$$\alpha_{ij} = \frac{1}{M_i(\lambda_i^2 - \lambda_j^2)} \{\psi\}_j^T \left[[\Delta K] - \lambda_i^2 [\Delta M] \right] \{\psi\}_i \qquad (7)$$

Here the superscript T denotes a vector or matrix transpose, and M_i is the modal mass:

$$M_i = \{\psi\}_i^T [M] \{\psi\}_i \qquad (8)$$

Expressions for [ΔM] and [ΔK] are given in the following Section (for a beam). Once these are known, the step-by-step procedure for applying the perturbation equations is as follows:

1. Solve the eigenvalue problem, Eq. (1), for the system without damping treatment. This gives λ_i^2 and $\{\psi\}_i$, and is the starting point for the modification steps.

2. Determine [ΔM] and [ΔK] for the damping treatment of interest. If either change in relatively large, divide it into a member of smaller changes.

3. Solve Eqs. (5) and (6) for an increment of [ΔK] and [ΔM]. This determines the changes in the modal parameters.

4. Update the modal parameters using Eqs. (2) and (3).

5. Repeat steps 2 through 4 until the desired modification [ΔM] and [ΔK) is achieved.

6. Solve for the damped natural frequencies and modal loss factors using Eq. (4).

Frequency dependent material properties can be handled in a similar way[11]. First, the eigenproperties of the original system are determined using values of the material properties at some convenient reference value of frequency. The resulting values of the natural frequencies are then used to determine updated values for the material properties and the corresponding changes in stiffness [ΔK(ω)]. Application of the perturbation equations then provides a new estimate for the natural frequencies, and the process is repeated. This iterative process is carried out mode-by-mode. Since the material properties usually are slowly varying functions over the frequency interval of interest, convergence is rapid.

Updating the mode shapes at each step of the perturbation process is a labor-intensive operation. If the mode shapes are not updated at each step, i. e., if the mode shapes of the undamped structure are used throughout, the perturbation method yields essentially the same results as the Rayleigh energy approach[12-14]. These approximate results may be accurate enough in many instances. If so, the costly mode shape updating process can be avoided. For damping treatment configurations that result in proportional damping, the mode shapes are the same as those of the undamped structure[7]. Updating of the mode shapes is not required in this case.

It also is possible to minimize the calculations needed to update the mode shapes. As can be seen from Eqs. (6) and (7), α_{ij} will be small for those modes for which the values of $|i-j|$ is large. Thus, these modes contribute little to the mode shape changes and can be ignored in the updating process. This feature of updating only certain mode shapes is useful, especially for large system models. Use of the perturbation approach with condensed dynamic system models is not considered in this paper.

APPLICATIONS TO A BEAM

In this section, the perturbation approach described is applied to four examples involving the flexural vibration of an elastic cantilever beam with varying lengths of viscoelastic damping treatment on one side (Figure 1). These examples also illustrate the capabilities of the perturbation approach. The computations are based upon a finite element model of the beam, which will be discussed in more detail later. Expressions for the incremental mass and stiffness matrices [ΔM] and [ΔK] are given in the Appendix, and the dimensions and material properties used in the examples are listed in Table 1. Except where noted, all examples are for an aluminum beam with a commercially-available damping layer. Computer programs were written to perform the necessary computations.

Example 1: Accuracy of the Method

The objective of this example was to assess the accuracy of the perturbation equations. To this end, they were used to compute the damped natural frequencies and modal loss factors for a beam with damping treatments ranging in length from $x/L = 0$ to $x/L = 1$ and with thickness ratios $t_V/t = 0.2$, 0.6 and 1.0 (see Figure 1). Here, and in the following, the subscript V denotes the viscoelastic damping layer. The perturbations were carried out in ten, fifteen or thirty equal-sized increments, depending upon the thickness ratio, and the material properties were assumed to be independent of frequency.

In order to provide a basis of comparison for the perturbation solutions, the complex eigenvalue problem associated with the finite element model was solved directly using the IMSL subroutine EIGZC. The finite element model was verified by comparing results for the damped natural frequencies and modal loss factors for different numbers of elements. Results for an undamped beam also were compared with theoretical values of the natural frequencies and mode shapes. It was found that the use of ten elements gave very accurate results for the first five modes, with a variety of damping treatment lengths and thicknesses. Validation of the finite element model was essential because the main purpose of this example was to establish the accuracy of the perturbation method. Significant finite element discretization errors would have confused the issue.

The natural frequencies and loss factors of the damped beam for the first and second modes are presented in Figures (2) and (3). These results are indicative of those for all the lower flexural modes. In these figures, ω_0 is the natural frequency of the undamped beam and FEM refers to finite element results obtained by direct solution of the complex eigenvalue problem. As mentioned previously, these results were validated carefully and can be considered to be "exact". As can be seen, the perturbation approach gives excellent results; they are indiscernible from the finite element solutions. Shear effects may be important for the thicker damping layers considered, but were neglected in this investigation.

Example 2: Rate of Convergence

Relative errors in the perturbation solutions for the damped natural frequencies and modal loss factors can be expressed as

$$e_\omega = \left| \frac{\omega_p - \omega_{fem}}{\omega_{fem}} \right| \tag{9}$$

$$e_\eta = \left| \frac{\eta_p - \eta_{fem}}{\eta_{fem}} \right| \tag{10}$$

Here, the subscript p refers to the perturbation solution and fem denotes finite element results. These errors depend upon the step size used in the perturbation solution, or, alternatively, upon the number of steps used to implement the total perturbation.

Figure (4) shows the variation of e_ω and e_η with the number of perturbation steps for the first mode. The total modification was a complete damping treatment with thickness ratio $t_v/t = 1.0$, which corresponds to a 21% increase in element mass. Again, the material properties were assumed to be frequency independent. The eigenvalues, eigenvectors, and modal mass were updated at each step.

For the particular case considered, the relative error in the loss factor is always greater than the error in the damped natural frequency. As can be seen from Figure (4), convergence is relatively rapid. With ten steps, the error in the loss factor is approximately 2%, while the error in the damped natural frequency is about 0.05%. Almost identical results were obtained for the first five flexural modes. Although the results are not presented here, the eigenvectors also were found to converge rapidly.

Example 3 Partial Damping Treatment

This example was designed to illustrate the effect of the location of a partial damping treatment along the length of the beam. The beam was divided into thirty elements of equal length, and the damping material was added to one element at a time. The thickness ratio was $t_v/t = 0.6$, and the modification was carried out in ten steps. As before, the material properties were assumed to be frequency independent.

Figure (5) shows the results for the first and second modes. As can be seen, the highest loss factors are achieved when the damping material is placed at the nodes of the undamped member. The natural frequencies decrease the most when the damping layer is located at the anti-nodes (results not shown here). These results are as expected. At the nodes, bending strains in the viscoelastic coating are maximum. Thus, damping material located near the nodes is most effective. Placing it at the anti-nodes dissipates little energy, but adds mass to the system and lowers the damped natural frequency.

Example 4 Frequency Dependent Material Properties

To illustrate the capability to handle frequency dependent material properties, Example 1 was repeated (for $t_v/t = 1$) using the hypothetical material properties shown in Figures (6) and (7). Note that damping in the aluminum beam is now included. The dashed curves indicate the assumed variation in the material properties, while the solid lines define the reference values used in the initial calculations. The frequency dependence has been exaggerated so that its effect can be more readily observed.

Figure (8) shows the system loss factors for the first two modes. The perturbation solutions account for the frequency dependence of the material properties, while the finite element results do not. They are based upon the reference values of the material properties. Effects of the frequency dependence on the system natural frequencies were negligible for this example.

CONCLUDING REMARKS

First-order matrix perturbation methods can be an efficient means for predicting the dynamic characteristics of modified structural systems. Viscoelastic coating modifications are particularly suitable for this technique. The modifications need not be small, but, if they are not, they must be built up by a series of small modification steps. Because this technique works with discretized systems, it can be applied to structures of general shape and can be implemented along with finite element codes.

The first-order stepwise perturbation technique used in this investigation gave close approximations to the damped natural frequencies and loss factors for a beam with various configurations of complete and partial damping treatments. Relative errors in the loss factors were found to be greater than those in the damped natural frequencies. If the mode shapes are not updated at each modification step, the perturbation solution produces an approximation which is comparable to that of the Rayleigh energy approach[9]. Application to problems with frequency-dependent material properties was described and illustrated in an example problem. These same procedures also can be used for material properties that are temperature dependent.

An important factor not addressed in this paper is the question of the efficiency of the perturbation approach. Does it require more or less computational effort than a re-solution of the complex eigenvalue problem? The answer to this question depends very much upon the algorithms used to implement the perturbation equations. For example, for partial damping treatments covering only a small portion of a structure, the incremental mass and stiffness matrices consist almost entirely of zero elements. Computational time can be reduced by taking advantage of such facts. For small modifications requiring only a single solution step, the perturbation technique appears to be very effective.

REFERENCES

1. Kato, T., Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1966.

2. Rellich, F., Perturbation Theory of Eigenvalue Problems, Gordon Breach Science Publishers, Inc., New York, 1969.

3. Huag, E. G., Choi, K. K., and Komkov, V., Design Sensitivity Analysis of Structural Systems, Academic Press, Inc., Orlando, 1986.

4. Chen, S., Liu, Y., and Huang, D., "A Matrix Perturbation of Vibration Modal Analysis," Proceedings IMAC 3, Orlando, FL, 1985, pp. 757-62.

5. Ma, Z., Chen, S., and Lu, Y., "Perturbation Method for Complex Eigenvalues of Damped System and its Associated Iterative Algorithm," Proceedings IMAC 4, Los Angeles, CA, 1986, pp. 1060-6.

6. Snyder,V. W., "Structural Modification and Modal Analysis - A Survey," Int. J. Analytical and Experimental Modal Analysis, Vol. 1, Jan., 1986, pp. 45-52.

7. Ewins, D. J., Modal Testing: Theory and Practice, Wiley & Sons, New York, 1984.

8. Johnson, C. D. and Kienholz, D. A., "Finite Element Prediction of Damping in Structures with Constrained Viscoelastic Layers," AIAA Journal, Vol. 20, No. 9, 1982, pp. 1284-1290.

9. Stevens, K. K., Kung, C. H., and Dunn, S. E., "Damping of Plates by Partial Viscoelastic Coatings-Part 1-Analysis," Proc. Noise-Con 81, Rayleigh, NC, pp. 445-448, June, 1981.

10. Ma, Z., Chen, S. and Lu, Y., "Perturbation Method for Complex Eigenvalues of Damped System and Its Associate Iterative Algorithm," Proceedings IMAC III, Orlando, FL, 1985, pp. 1060-1066.

11. Cao, X. S., Dai, D. P. and Ku, Ch. H., "Perturbation Method for Dynamic Analysis of Structures with Constrained Viscoelastic Layer," Proceeding IMAC IV, 1986, pp. 826-830.

12. Stevens, K. K., Martinez, M. E., and Kelly, W. J., "A Comparison of Exact and Approximate Methods of Analysis for Added Viscoelastic Damping Treatments," Proceedings IMAC VI, Orlando, FL, 1988.

13. Kelly, W. J., Jr., "An Application of Perturbation Techniques to the Vibration Analysis of Structures with Viscoelastic Coating Modifications," M.S. Thesis, Florida Atlantic University, Boca Raton, FL, 1988.

14. Kelly, W. J., Jr., and Stevens, K. K., "Application of Perturbation Techniques to the Modal Analysis of a Shaft with Added Viscoelastic Damping," Proceedings IMAC VII, Las Vegas, Nevada, 1989.

15. Weaver, W., Jr. and Johnston, P. R., Structural Dynamics by Finite Elements, Prentice-Hall, 1987.

TABLE 1. PROPERTIES FOR BEAM EXAMPLE

Beam (Aluminum)	$E = 70 \times 10^9$ N/m^2
	$\rho = 2.7 \times 10^3$ kg/m^3
Damping Layer (Commercially Available)	$E_v = 0.69 \times 10^9$ N/m^2
	$\eta_v = 0.64$
	$\rho_v = 0.58 \times 10^3$ kg/m^3

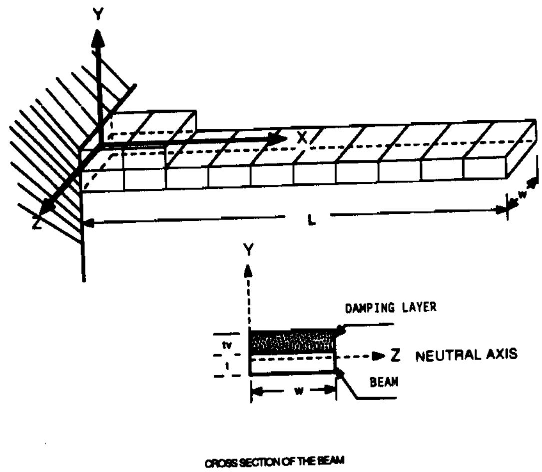

Figure 1. Cantilever Beam with a Partial Damping Treatment.

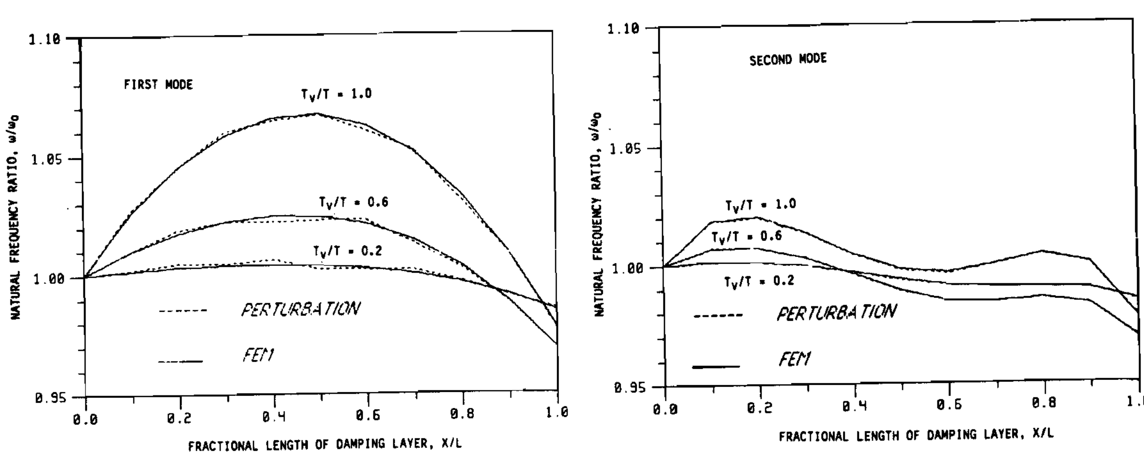

Figure 2. Effect of a Partial Damping Treatment on Natural Frequencies of a Cantilever Beam.

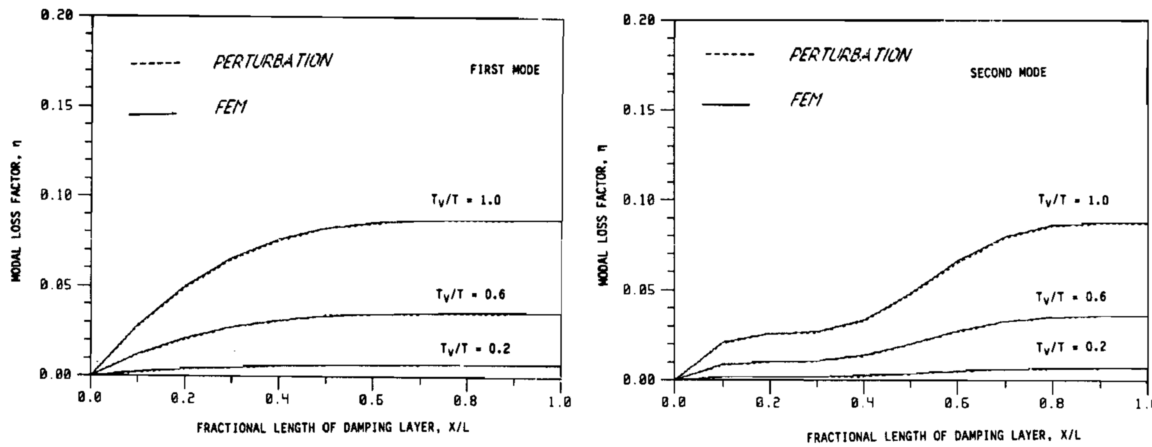

Figure 3. Effect of a Partial Damping Treatment on Modal Loss Factors of a Cantilever Beam.

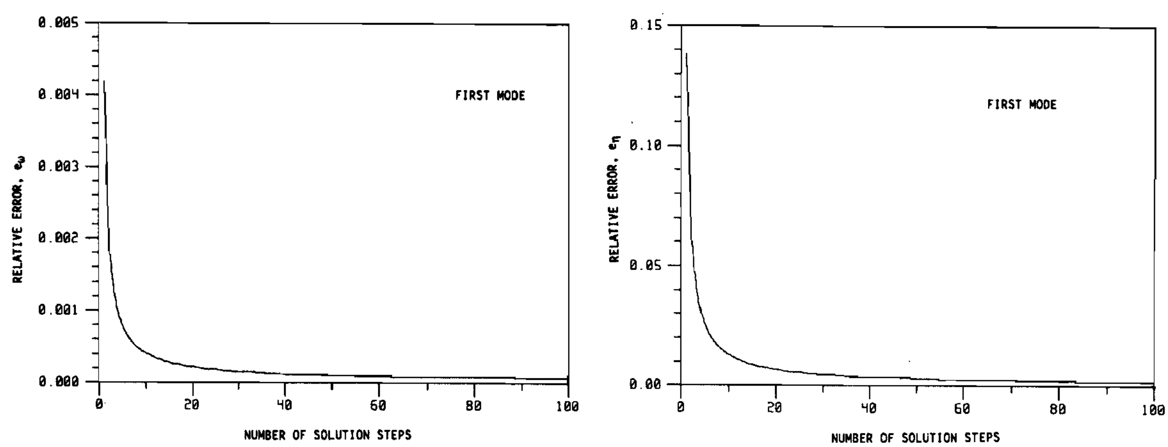

Figure 4. Rate of Convergence of Perturbation Solution.

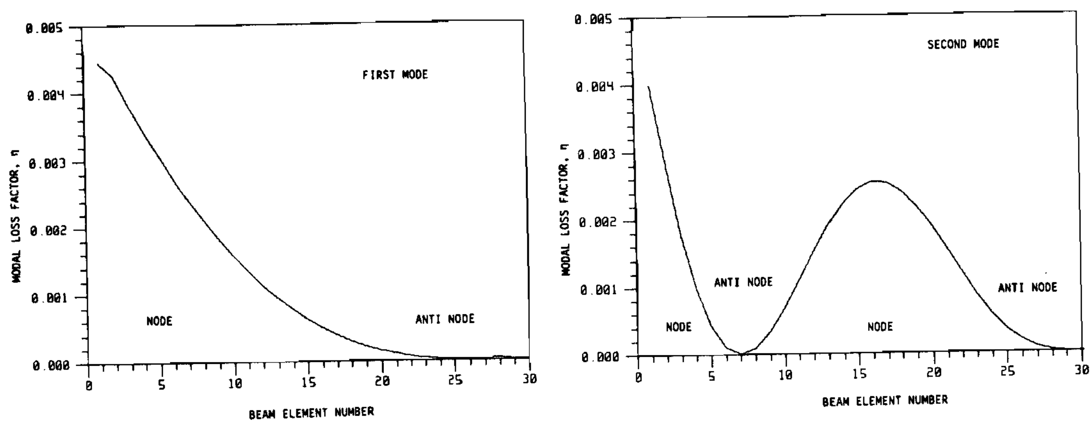

Figure 5. Loss Factor of a Cantilever Beam with Damping Treatment applied to a Single Beam Element (30 total elements).

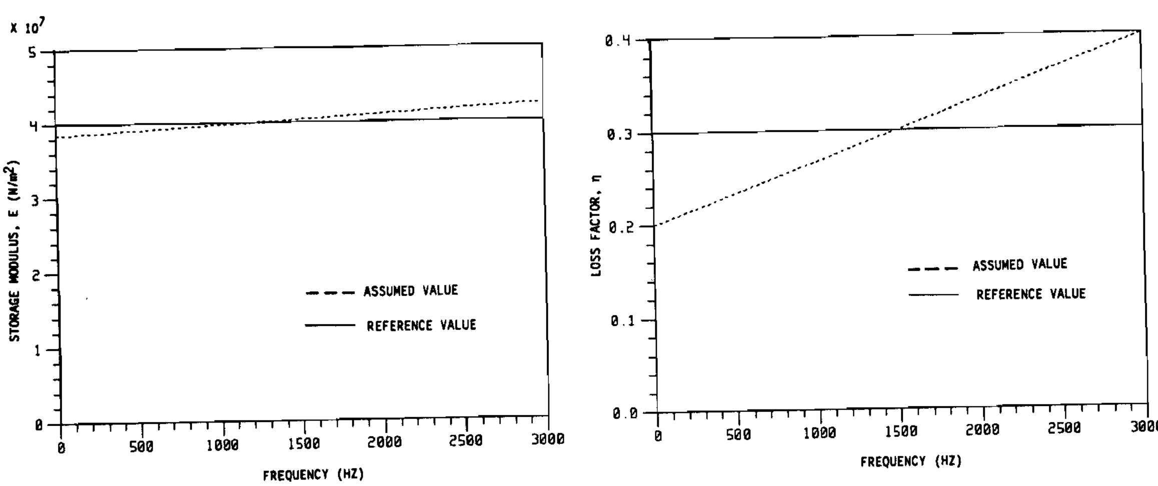

Figure 6. Properties of a Hypothetical Damping Material.

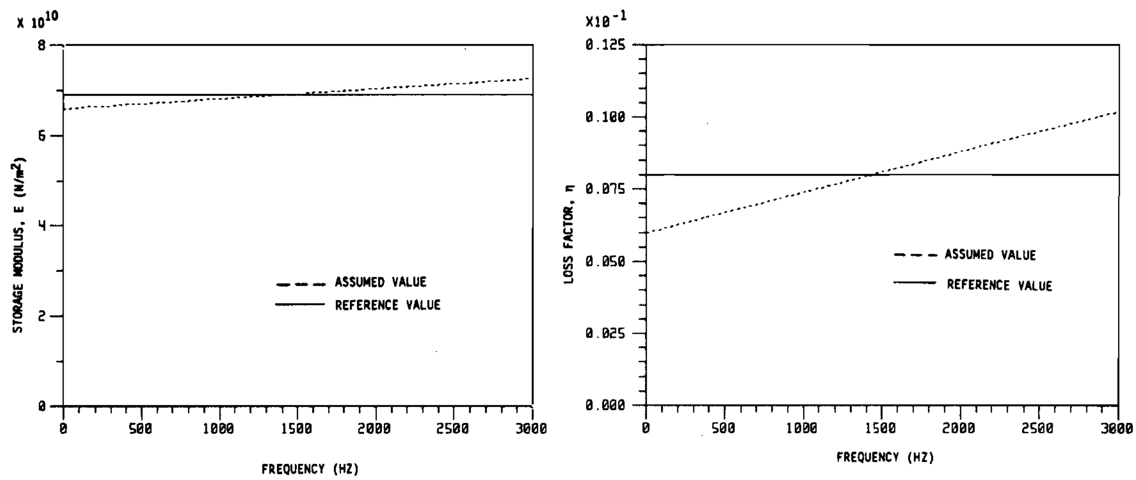

Figure 7. Hypothetical Frequency-Dependent Properties of Aluminum.

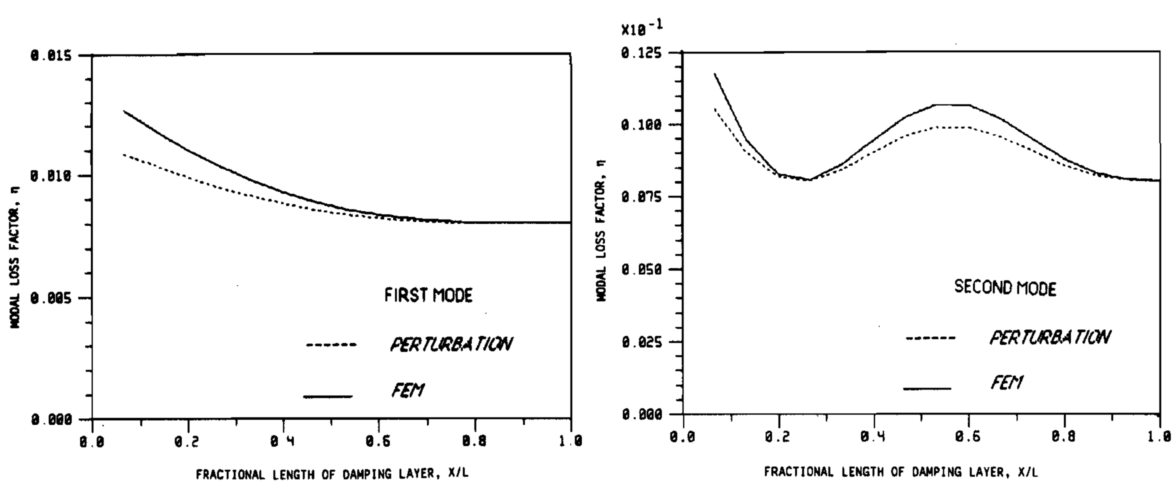

Figure 8. Effect of Frequency-Dependent Material Properties on the Loss Factor of a Cantilever Beam with a Partial Damping Treatment. (Frequency dependence not included in FEM results).

APPENDIX-BEAM EQUATIONS

Consider an elastic cantilever beam with thickness t, flexure rigidity EI, length L and elastic modulus E, with a viscoelastic damping layer of thickness t_v bonded to it over a portion of its length (Figure 1). The properties of the damping layer are described by the complex modulus

$$E^* = E_v\,(1+i\eta_v) \tag{A1}$$

where E_v is the storage modulus and η_v is the material loss factor. Here, and in the following, the subscript v denotes the viscoelastic damping material.

Addition of a damping layer to one side of a beam causes a shift in the neutral axis of the cross-section. Using simple beam theory, this shift can be shown to be

$$\bar{y} = y^*/t = \frac{\bar{E}\bar{t}(1+\bar{t})}{2(1+\bar{E}\bar{t})} \tag{A2}$$

where y^* is the distance between the neutral axis of the composite cross section and the midplane of the beam and

$$\bar{E} = E_v/E \qquad \bar{t} = t_v/t \tag{A3}$$

Note that $\bar{y}$ usually is small, since $\bar{E}$ is typically small, and often can be neglected.

The mass and stiffness matrices for a beam element are available in the literature[15]:

$$[K] = \frac{EI}{a^3}\begin{bmatrix} 12 & 6a & -12 & 6a \\ & 4a^2 & -6a & 2a^2 \\ & & 12 & -6a \\ \text{SYM} & & & 4a \end{bmatrix} = k\,[\bar{K}] \tag{A4}$$

$$[M] = \frac{\rho A t a}{420} \begin{bmatrix} 156 & 22a & 54 & -13a \\ & 4a^2 & 13a & -3a^2 \\ & & 156 & -22a \\ \text{SYM} & & & 4a^2 \end{bmatrix} = m\,[\bar{M}] \tag{A5}$$

Here, ρ is the mass density, A is the cross-sectional area and a is the element length.

The incremental mass matrix [ΔM] due to the viscoelastic layer is given by Eq. (A5) with the beam dimensions and properties replaced by those of the layer. The incremental stiffness matrix is given by Eq. (A4), except with k replaced by k_v. Using standard finite element procedures and simple beam theory (shear effects neglected), it can be shown that

$$k_v = \frac{EI}{a^3}\left[\overline{EI} + 12\bar{y}^2 + 12\bar{t}\,[\tfrac{1}{2}(1+\bar{t}) - \bar{y}]^2\right] \tag{A6}$$

where

$$\bar{I} = I_v/I \tag{A7}$$