ANALYSIS OF GENERAL COUPLED THERMOELASTICITY PROBLEMS BY THE FINITE ELEMENT METHOD

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This paper treats the formulation of discrete models of the general thermoelasticity problem. It is concerned with the development of consistent finite element formulations of general dynamic problems of coupled thermoelasticity and to application of the finite element models to the analysis of representative problems. Numerical examples are included.

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SECTION I

INTRODUCTION

The theory of thermoelasticity is concerned with predicting the thermomechanical behavior of elastic solids. As such, it represents a generalization of both the theory of elasticity and the theory of heat conduction in solids. The development of the modern theory of thermoelasticity began with Duhamel's celebrated memoir of 1837 Reference 1, and has been the subject of intense research in recent years. Extensive references to previous work can be found in the survey articles in References 2, 3, and 4, and in the books of Boley and Weiner, Reference 5, Nowacki, Reference 6, and Parkus, Reference 7.

Despite over a century of research on thermoelasticity, many problems of current interest are intractable when solutions are attempted by classical methods. This fact has led some investigators to consider numerical procedures for solving thermoelasticity problems, and some attempts at finite element formulations of various problems of thermal deformation have been made. Early applications of the technique considered the temperature distribution of a certain body as a known function of time and the material properties of the body as being given by empirical functions of temperature. Then incremental loading procedures were used to predict a step by step response of the body due to successive temperature and load increments. All coupling between thermal and mechanical effects was ignored in these analyses. Finite element formulations of the heat conduction problem, without mechanical coupling, are given in References 8 and 9 and finite-element solutions of a class of thermoelastic problems are considered by Visser, Reference 10, who developed equations based on Gurtin's variational principles for linear initial value problems, Reference 11. Visser confined his attention to uncoupled thermoelasticity. The first finite element formulation of the coupled thermoelastic problem is apparently that given recently by Nickell and Sackman, References 12 and 13, who, following Gurtin's work, Reference 11, developed appropriate variational principles for a class of linear thermoelastic materials. Nickell and Sackman considered the one-dimensional initial boundary-value problem of the thermoelastic half-space with boundary layer thermal conductance and the problem of a half-space subjected to ramp heating at the surface boundary. We also consider this problem in Section IX of this paper.

The present paper considers the development of general finite-element models for the analysis of coupled thermoelasticity problems. The general dynamical theory is considered, and the resulting equations are sufficiently general to include those previously presented

as special cases. A significant feature of the analysis is that the finite element equations are derived from general energy balances, thereby freeing the development of finite element models from a dependency on the availability of suitable variational principles.

Following this introduction, the general field equations for thermoelasticity continua are reviewed, and local and global forms of the first law of thermodynamics and the Clausius-Duhem inequality are presented. It is assumed, as is customary, that the Helmholtz free energy is a differentiable function of the instantaneous strain and absolute temperature; but specific forms of this function are not introduced until later. Attention is then given to the development of general finite-element representations of the displacement, velocity, and temperature fields which are used in energy balances for typical finite elements. This results in coupled equations of motion and heat conduction for such elements. It is then shown that the notions of generalized nodal heat fluxes and entropy fluxes come about naturally in the analysis. Several types of boundary conditions, including specified boundary temperatures, heat fluxes, and those for convective heat transfer, are examined. Finite element equations for general forms of the free energy are presented, but special emphasis is given to the classical quadratic form for linear anisotropic solids. Numerical examples are included.

SECTION II

THERMOMECHANICAL PRELIMINARIES

Consider a continuous body B under the action of a general system of external forces and prescribed temperatures. To trace the motion of the body and the time variation of its temperature, we introduce the time parameter τ which is assigned the value zero in some reference configuration C_o of the body. Preferably, we select some natural, unstrained state under a uniform temperature T_o to correspond with the reference configuration. In the reference configuration, we establish an intrinsic (embedded) coordinate system x^i (i = 1,2,3) which is etched onto the body and which, for simplicity, is assumed to be rectangular carte-sian at $\tau = 0$. At some later time $\tau = t$, the body occupies another configuration C and the rectangular cartesian coordinates of a particle in C which was initially x^i in C_o, relative to a fixed rectangular coordinate system in C_o, are denoted z_i. The set of three functions, $z_i(x^1, x^2, x^3, t) = z_i(\underset{\sim}{x}, t)$ defines the motion of the body relative to the reference configuration.

We postulate that the behavior of a volume v of the body of surface area A is governed by the following fundamental physical laws:

1. Balance of Linear Momentum

$$\int_{v_o} \rho_o \bar{u}_i \, dv_o = \int_{v_o} \rho_o F_i \, dv_o + \int_{A_o} S_i \, dA_o \tag{1}$$

2. Balance of Angular Momentum

$$\int_{v_o} \rho_o \epsilon_{ijk} z_i \dot{u}_j \, dv_o = \int_{v_o} \rho_o \epsilon_{ijk} z_i F_j \, dv_o + \int_{A_o} \epsilon_{ijk} z_i S_j \, dA_o \tag{2}$$

3. Conservation of Mass

$$\int_{v_o} \rho_o \, dv_o = \int_{v} \rho \, dv \tag{3}$$

4. Conservation of Energy

$$\int_{v_o} \rho_o \ddot{u}_i \dot{u}_i \, dv_o + \int_{v_o} \rho_o \dot{\epsilon} \, dv_o = \int_{v_o} \rho_o F_i \dot{u}_i \, dv_o + \int_{A_o} S_i \dot{u}_i \, dA_o$$

$$+ \int_{v_o} \rho_o h \, dv_o + \int_{A_o} q_i n_i \, dA_o \tag{4}$$

5. Clausius-Duhem Inequality

$$\int_{V_0} \rho_0 \dot{\eta} \, dv_0 - \int_{V_0} \rho_0 \frac{h}{\theta} \, dv_0 - \int_{A_0} \frac{q_i}{\theta} \, n_i \, dA_0 \geq 0 \tag{5}$$

The quantities appearing in these equations are defined as follows:

$\rho_0, \; \rho$ = mass densities in configurations C_0 and C, respectively

$v_0, \; v$ = volume of the body in C_0 and C

$A_0, \; A$ = surface areas of the body in C_0 and C

u_i = $z_i(\underset{\sim}{x}, t) - x_i$ = components of displacement relative to x_i in C_0

F_i = components of body force per unit mass in C_0 referred to x_i

S_i = components of surface traction per unit "undeformed area" (area in C_0) referred to x_i

n_i = components of a unit vector normal to the surface A_0

ϵ = internal energy per unit mass in the "undeformed configuration" C_0

h = heat supplied per unit mass in C_0 from internal sources

q_i = components of heat flux per unit area in C_0 referred to x_i

η = entropy per unit mass in C_0

θ = absolute temperature

The superposed dots ($\cdot$) in these equations indicate time rates and the entropy flux and entropy source supply are assumed to be given by q_i/θ and h/θ respectively. If σ^{ij} are the components of the stress tensor per unit undeformed area referred to the convected coordinate lines x^i in C, then

$$S_i = \sigma^{mj} z_{i,j} \, n_m \tag{6}$$

where here and in the following the comma denotes partial differentiation with respect to the x^i (i.e $\partial z_i / \partial x^j = z_{i,j}$).

Under suitable smoothness assumptions, the following local forms of Equations 1 through 5 can be obtained

$$\rho_0 \ddot{u}_i = (\sigma^{mj} z_{i,j})_{,m} + \rho_0 F_i \tag{7}$$

$$\sigma^{ij} = \sigma^{ji} \tag{8}$$

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$$\rho_0 = \rho \sqrt{G} \tag{9}$$

$$\rho_0 \, \dot{\epsilon} = \sigma^{ij} \dot{\gamma}_{ij} + q_{i,i} + \rho_0 h \tag{10}$$

$$\rho_0 \theta \dot{\eta} \geq q_{i,i} + \rho_0 h - \frac{1}{\theta} q_i \, \theta_{,i} \tag{11}$$

Here

$$\gamma_{ij} = \frac{1}{2} \left[u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} \right] \tag{12}$$

is the strain tensor and

$$G = \det \left(\delta_{ij} + 2\gamma_{ij} \right) \tag{13}$$

We now introduce the specific free energy ϕ and the internal dissipation σ defined by

$$\phi = \epsilon - \eta \theta \tag{14}$$

$$\sigma = \sigma^{ij} \dot{\gamma}_{ij} - \rho_0 (\dot{\phi} + \eta \dot{\theta}) \tag{15}$$

so that we have the dissipation inequality

$$\sigma - \frac{1}{\theta} q_i \, \theta_{,i} \geq 0 \tag{16}$$

and Equations 10 and 11 become

$$\rho_0 \dot{\phi} = \sigma^{ij} \dot{\gamma}_{ij} - \rho_0 \eta \, \dot{\theta} - \sigma \tag{17}$$

$$\rho_0 \theta \dot{\eta} = q_{i,i} + \rho h + \sigma \tag{18}$$

SECTION III

EQUATIONS OF THERMOELASTICITY

Fundamental assumptions of the theory of thermoelasticity are that the internal dissipation is zero and that the specific free-energy is a differentiable function of only the instantaneous strain and absolute temperature:

$$\sigma = 0 \tag{19}$$

$$\phi = \hat{\phi}(\gamma_{ij}, \theta) \tag{20}$$

Thus

$$\dot{\phi} = \frac{\partial \hat{\phi}}{\partial \gamma_{ij}} \dot{\gamma}_{ij} + \frac{\partial \hat{\phi}}{\partial \theta} \dot{\theta} \tag{21}$$

and from Equations 17 and 18 we have

$$\sigma_{ij} = \rho_0 \frac{\partial \hat{\phi}}{\partial \gamma_{ij}} \tag{22}$$

$$\eta = -\frac{\partial \hat{\phi}}{\partial \theta} \tag{23}$$

$$\rho \theta \dot{\eta} = q_{i,i} + \rho h \tag{24}$$

Equations 22 and 23 are constitutive equations for the stress and entropy. To these must be added an independent constitutive equation for the heat flux q_i. It is customary to take the linear Fourier law of heat conduction as a first-order approximation:

$$q_i = k_{ij} \theta_{,j} \tag{25}$$

where the k_{ij} are coefficients of thermal conductivity.

The basic equations of motion and heat conduction of coupled thermoelasticity are obtained by introducing (22) into (7) and (23) into (24):

$$\left(\frac{\partial \phi}{\partial \gamma_{mj}} z_{i,j} \right)_{,m} + \rho_0 F_i = \rho_0 \ddot{u}_i \tag{26}$$

$$-\rho \theta \frac{d}{dt} \left(\frac{\partial \hat{\phi}}{\partial \theta} \right) = (k_{ij} \theta_{,j})_{,i} + \rho h \tag{27}$$

The classical linear equations are obtained by assuming further that

$$\theta(\underset{\sim}{x},t) = T_0 + T(x,t) \tag{28}$$

where T_0 is a uniform reference temperature and $T(x,t)$ is a temperature increase such that $|T(x,t)| << T_0$; and the displacement gradients $u_{i,j}$ are sufficiently small that their products with themselves and $T(\underset{\sim}{x},t)$ can be neglected in comparison with terms linear in $u_{i,j}$ and T.

SECTION IV

FINITE-ELEMENT APPROXIMATIONS

We now undertake the construction of finite-element models of appropriate fields appearing in the basic thermoelasticity equations. In the present analysis, these are taken to be the displacement field $u_i(\underset{\sim}{x},t)$ and the temperature field $\theta(\underset{\sim}{x},t)$ (or the temperature increment field $T(\underset{\sim}{x},t)$).

Following the usual procedure, we consider the body to be approximated by an assembly of a finite number E of discrete elements of relatively simple geometric shapes, connected together at various nodal points in such a way that the collection of elements forms a continuous body, with no internal gaps and discontinuities other than those which appear in the actual body it represents (Figure 1). Then, confining our attention to a typical finite element e, we approximate the local displacement and temperature increment field over that element by functions of the form

$$u_i(\underset{\sim}{x},t) = \psi_N(\underset{\sim}{x})u_{Ni}(t) \tag{29}$$

and

$$T(\underset{\sim}{x},t) = \psi_N(\underset{\sim}{x})T_N(t) \tag{30}$$

where the repeated nodal indices N are to be summed from 1 to N_e, N_e being the total number of nodes belonging to element e. In these equations, u_{Ni} are the components of displacement
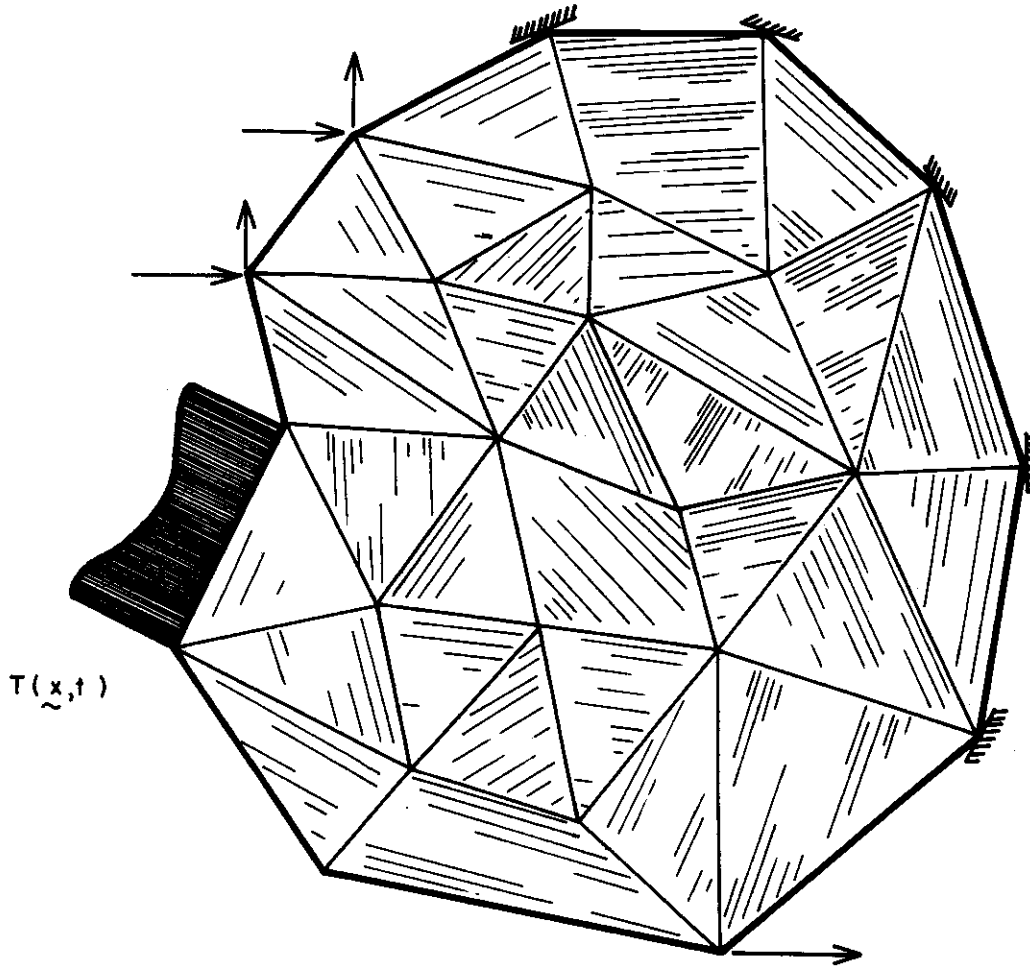
Figure 1

of node, N, T_N is the temperature (temperature change) at node N, and the interpolating functions $\psi_N(\underset{\sim}{x}) = \psi_{N(e)}(\underset{\sim}{x})$ for element e are defined as follows:

$$\psi_{N(e)}(\underset{\sim}{x}) \equiv 0 \quad \text{if } \underset{\sim}{x} = x^i \text{ does not belong to element } e$$

$$\psi_{N(e)}(x_M) = \begin{cases} 1 & \text{if node M of element e with coordinates} \\ & a_M = x^i_M \text{ is identically node N of the} \\ & \text{element} \\ 0 & \text{if otherwise} \end{cases} \tag{31}$$

These properties, of course, are recognized as those of generalized lagrange interpolation functions (generally taken to be polynomials) defined locally over each finite element. Later, we shall affix labels (e) to the local fields u_i and T of Equations 29 and 30 to distinguish those associated with one finite element from those associated with another.

The absolute temperature over an element is then

$$\theta(x,t) = T_0 + \psi_N(x) T_N \tag{32}$$

where dependence of T_N on time is understood. Moreover, from Equation 12, the components of strain in the element are

$$\gamma_{ij} = \frac{1}{2}\left[\psi_{N,i} u_{Nj} + \psi_{N,j} u_{Ni} + \psi_{N,k} \psi_{M,k} u_{Ni} u_{Mj}\right] \tag{33}$$

where i, j, k = 1, 2, 3 and M, N = 1, 2, ..., N_e. The last term on the right side of Equation 33 is surpressed in the case of small strain gradients.

The formal assembly of local displacement and temperature fields is accomplished by means of the Transformations 14 through 17.

$$u_{Ni(e)} = \Omega_{N\Delta}^{(e)} U_{\Delta i} \tag{34}$$

$$T_{N(e)} = \Omega_{N\Delta}^{(e)} T_\Delta \tag{35}$$

where $U_{\Delta i}$ and T_Δ are global values of displacement components and temperatures at node Δ of the assembled system of elements and

$$\Omega_{N\Delta}^{(e)} = \begin{cases} 1 & \text{if node N of element e coincides} \\ & \text{with node } \Delta \text{ of the assembly of elements} \\ 0 & \text{if otherwise} \end{cases} \tag{36}$$

In (34), the repeated global node index Δ is summed from 1 to G, G being the total number of nodes in the assembled system of elements. Equations 34 and 35 of course, represent a mathematically formal procedure for assembling the local fields. In applications to linear problems, simpler devices can be used.

The complete finite-element representations of the fields $u_i(x,t)$ and $T(x,t)$ over the body can now be written in the forms

$$u_i(x,t) = \sum_{e=1}^{E} \Omega_{N\Delta}^{(e)} \psi_{N(e)}(x) U_{\Delta i} \tag{37}$$

$$T(x,t) = \sum_{e=1}^{E} \Omega_{N\Delta}^{(e)} \psi_{N(e)}(x) T_\Delta \tag{38}$$

SECTION V

EQUATIONS OF MOTION AND HEAT CONDUCTION
OF A THERMOELASTIC FINITE ELEMENT

To develop finite element models of various physical problems, it is necessary that means be available to translate a set of relations that hold at a point (local equations) into relations that hold for a finite subregion (global equations). Customarily variational principles are devised for this purpose, but any means of converting a local relation into a global one, such as the local and global forms of the energy balances, can also be used. Indeed, a point relation is to a continuum as the equations governing a finite element are to the discrete model of the continuum.

With these observations in mind, we temporarily confine our attention to a typical finite element of the body with local displacement and temperature fields given by Equations 29 and 30, and we return to the global form of the conservation of energy given in Equation 4. Introducing Equations 14 and 29 into Equation 4

$$m_{NM} \ddot{u}_{Mi} \dot{u}_{Ni} + \int_{v_e} \rho_0 (\dot{\phi} + \eta \dot{\theta}) \, dv_0$$

$$= P_{Ni} \dot{u}_{Ni} + \int_{v_e} (q_{i,i} + \rho h - \rho \theta \dot{\eta}) \, dv_0 \tag{39}$$

where

$$m_{NM} = \int_{v_e} \rho_0 \psi_N(\underset{\sim}{x}) \psi_M(\underset{\sim}{x}) \, dv_0 \tag{40}$$

$$P_{Ni} = \int_{v_e} \rho_0 F_i \psi_N(\underset{\sim}{x}) \, dv_0 + \int_{A_e} S_i \psi_N(\underset{\sim}{x}) \, dA_0 \tag{41}$$

Here v_e is the volume of the finite element A_e is its surface area, m_{NM} (N, M = 1, 2, ..., N_e) is the consistent mass matrix for the element, and P_{Ni} are the components of "consistent" generalized force at node N of the element. The repeated index N in Equation 39 is to be summed from 1 to N_e.

According to the thermoelastic law of entropy production Equation 24, the last integral in Equation 39 vanishes. Moreover, in view of Equation 23, the integral on the left side of the equality sign can be reduced as follows:

$$\int_{V_e} \rho_0 (\dot{\phi} + \eta \dot{\theta}) dv_0 = \int_{V_e} \left[\rho_0 \frac{\partial \hat{\phi}}{\partial \gamma_{ij}} \dot{\gamma}_{ij} + \rho_0 (\eta + \frac{\partial \hat{\phi}}{\partial \theta}) \dot{\theta} \right] dv_0$$

$$= \int_{V_e} \rho_0 \frac{\partial \hat{\phi}}{\partial \gamma_{ij}} \dot{\gamma}_{ij} dv_0$$

$$= \int_{V_e} \rho_0 \frac{\partial \hat{\phi}}{\partial \gamma_{ij}} \psi_{N,k} (\delta_{ik} + \psi_{M,k} u_{Mi}) \dot{u}_{Nj} dv_0 \tag{42}$$

The term $\rho_0 \, \partial \hat{\phi} / \partial \gamma_{ij}$ is recognized as the stress tensor σ^{ij} of Equation 22. Thus, the general equation of energy balance for a finite element of a thermoelastic continuum is

$$\left[m_{NM} \ddot{u}_{Mi} + \int_{V_e} \rho_0 \frac{\partial \hat{\phi}}{\partial \gamma_{ij}} \psi_{N,k} (\delta_{jk} + \psi_{M,k} u_{Mj}) dv_0 - P_{Ni} \right] \dot{u}_{Ni} = 0 \tag{43}$$

Since this equation must hold for arbitrary nodal velocities $\dot{u}_{Ni}$, we have the general equations of motion for a thermoelastic finite element

$$m_{NM} \ddot{u}_{Mi} + \int_{V_e} \rho_0 \frac{\partial \hat{\phi}}{\partial \gamma_{ij}} \psi_{N,k} (\delta_{jk} + \psi_{M,k} u_{Mj}) dv = P_{Ni} \tag{44}$$

Notice that no restrictions have as yet been placed on the form of the free energy function $\hat{\phi}$ or on the order of magnitude of the displacements or displacement gradients.

Turning now to the total entropy product of the element, we observe that if Equation 24 is multiplied through by the temperature increment T and use is made of the identity

$$(T q_i)_{,i} = T_{,i} q_i + T q_{i,i} \tag{45}$$

we arrive at the local law

$$\rho T \theta \dot{\eta} = (T q_i)_{,i} - T_{,i} q_i + \rho T h \tag{46}$$

To obtain the comparable global form of this law, we integrate both sides over the volume of the element and use the Green-Gauss theorem to convert the integral of the first term on the right side of the equation into a surface integral. Recalling Equations 19 and 28, we get

$$-\int_{V_e} \rho_0 T(T_0 + T)\frac{d}{dt}(\frac{\partial \hat{\phi}}{\partial \theta})dv_0 + \int_{V_e} T_{,i} q_i dv_0$$

$$= \int_{V_e} \rho_0 Th\, dv_0 + \int_{A_e} Tq_i n_i\, dA_0 \tag{47}$$

For simplicity, we assume that $|T| << T_0$. Noting also that $\partial \hat{\phi}/\partial \theta = \partial \hat{\phi}/\partial T$, we introduce Equation 30 into Equation 46 and obtain

$$\left[-\int_{V_e} \rho_0 \psi_N(\underset{\sim}{x})T_0 \frac{d}{dt}(\frac{\partial \hat{\phi}}{\partial T})dv_0 + \int_{V_e} \psi_{N,i}(\underset{\sim}{x})q_i\, dv_0 - q_N \right] T_N = 0 \tag{48}$$

where

$$q_N = \int_{V_e} \rho_0 h\, \psi_N(\underset{\sim}{x})\, dv_0 + \int_{A_e} \psi_N(\underset{\sim}{x})\, q_i n_i\, dA_0 \tag{49}$$

Since (47) must hold for arbitrary T_N,

$$\int_{V_e} \psi_{N,i}(\underset{\sim}{x})q_i\, dv_0 - \int_{V_e} \rho_0 \psi_N(\underset{\sim}{x})T_0 \frac{d}{dt}(\frac{\partial \hat{\phi}}{\partial T})dv_0 = q_N \tag{50}$$

This result is a general equation of heat conduction for a finite element of a thermoelastic continuum.

SECTION VI

LINEAR COUPLED THERMOELASTICITY

Equations 44 and 50 represent the general coupled equations of motion and heat conduction for any type of thermoelastic media. No restrictions have as yet been imposed on the form of the free energy function, the order of magnitude of the strains or displacements, or the constitutive equation for heat flux. Thus, these equations apply to the general, nonlinear, coupled thermoelasticity problem. However, we shall postpone an exploration of various nonlinear thermoelasticity problems to future investigations. In the present study, we shall henceforth confine our attention to linear thermoelastic theory, which is characterized by a free energy function which is a quadratic form in the strains and the temperature increments:

$$\rho_0 \hat{\phi} = \frac{1}{2} E^{ijkl} \gamma_{ij} \gamma_{kl} + B^{ij} \gamma_{ij} T + \frac{1}{2} \frac{\alpha}{T_0} T^2 \tag{51}$$

where E^{ijkl} and B^{ij} are arrays of material parameters which may be functions of T but which are usually assumed to be constants for homogeneous bodies, and which have the symmetries

$$E^{ijkl} = E^{jikl} = E^{ijlk} = E^{klij} \tag{52}$$

$$B^{ij} = B^{ji} \tag{53}$$

In Equation 50, α is a thermoelastic constant (see Section 9) and $|\gamma_{ij}| = 0 \, (\epsilon)$, $\epsilon^2 \ll 1$. As an additional assumption, we suppose that γ_{ij} are given by the linearized strain-displacement relations

$$\gamma_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \tag{54}$$

According to Equations 22 and 23, the stress tensor and the specific entropy are given by

$$\sigma^{ij} = \rho_0 \frac{\partial \hat{\phi}}{\partial \gamma_{ij}} = E^{ijkl} \gamma_{kl} + B^{ij} T \tag{55}$$

$$\eta = - \frac{\partial \hat{\phi}}{\partial T} = - \frac{1}{\rho_0} (B^{ij} \gamma_{ij} + \frac{\alpha}{T_0} T) \tag{56}$$

Substituting Equation 55 into 44 and noting that because of the restriction to small displacement gradients implied in Equation 54 the term $(\delta_{jk} + \psi_{M,k} u_{Mj})$ in Equation 44 reduces to simply δ_{jk}, we arrive at general equations of motion of a linear thermoelastic finite element

$$m_{NM} \ddot{u}_{Mi} + a_{MN}^{im} u_{Mn} + b_{MN}^i T_M = p_{Ni} \tag{57}$$

where

$$a_{NM}^{im} = \int_{v_e} E^{ikmn} \psi_{N,k} \psi_{M,n} dv_0 \tag{58}$$

$$b_{MN}^i = \int_{v_e} B^{ik} \psi_M \psi_{N,k} dv_0 \tag{59}$$

To obtain the accompanying coupled heat conduction equations for the finite element, we assume that the linear Fourier law Equation 25 holds with $k_{ij} \theta_{,j} = k_{ij} T_{,j}$ and introduce Equations 25, 30, and 56 into Equation 50

$$k_{Nm} T_M - T_0 b_{MN}^i \dot{u}_{Mi} - c_{NM} \dot{T}_M = q_N \tag{60}$$

where

$$k_{NM} = \int_{v_e} k_{ij} \psi_{N,i} \psi_{M,j} dv_0 \tag{61}$$

$$c_{NM} = \int_{v_e} a \psi_N \psi_M dv_0 \tag{62}$$

If we denote by $P_{\Delta i}$ and Q_Δ the global values of generalized force and heat flux corresponding to (Equations 34 and 35), then

$$P_{\Delta i} = \sum_e \Omega_{N\Delta}^{(e)} p_{Ni(e)} \tag{63}$$

$$Q_\Delta = \sum_e \Omega_{N\Delta}^{(e)} q_{N(e)} \tag{64}$$

We therefore have for the general coupled equations of motion and heat conduction for an assembly of linear thermoelastic finite elements,

$$M_{\Delta\Gamma} \ddot{U}_{\Gamma i} + A_{\Delta\Gamma}^{im} U_{\Gamma m} + B_{\Delta\Gamma}^i T_\Gamma = P_{\Delta i} \tag{65}$$

$$K_{\Delta\Gamma} T_\Gamma - T_0 B_{\Delta\Gamma}^i \dot{U}_{\Gamma i} - C_{\Delta\Gamma} \dot{T}_\Gamma = Q_\Delta \tag{66}$$

in which

$$M_{\Delta\Gamma} = \sum_e \Omega^{(e)}_{N\Delta} m^{(e)}_{NM} \Omega^{(e)}_{M\Gamma} \tag{67}$$

$$A^{im}_{\Delta\Gamma} = \sum_e \Omega^{(e)}_{N\Delta} a^{im(e)}_{NM} \Omega^{(e)}_{M\Gamma} \tag{68}$$

$$B^i_{\Delta\Gamma} = \sum_e \Omega^{(e)}_{N\Delta} b^{i(e)}_{MN} \Omega^{(e)}_{M\Gamma} \tag{69}$$

$$K_{\Delta\Gamma} = \sum_e \Omega^{(e)}_{N\Delta} k^{(e)}_{NM} \Omega^{(e)}_{M\Gamma} \tag{70}$$

and

$$C_{\Delta\Gamma} = \sum_e \Omega^{(e)}_{N\Delta} c^{(e)}_{NM} \Omega^{(e)}_{M\Gamma} \tag{71}$$

SECTION VII

BOUNDARY CONDITIONS

Following the plan of Becker and Parr (Reference 9), we point out three possible boundary conditions involving the thermal variables q_N and T_N. Mechanical boundary conditions involve simply prescribing $U_{\Delta i}$ or $P_{\Delta i}$ at various boundary nodes.

1. Convective Heat Transfer

If T^* is the temperature of the media surrounding the body, and T_o^* is the temperature across a surface boundary layer, then the convective heat flux is given by

$$q_i = f_i(T^* - T_o^*) \tag{72}$$

where f_i are film coefficients.

2. Specified Heat Flux

The heat flux may be specified on the boundary

$$q_i = \overline{q_i} \tag{73}$$

An important special case of Equation 73 is that of adiabatic behavior for which $q_i = 0$.

3. Specified Temperature

Temperatures are prescribed on the boundary by simply prescribing the nodal values T_N at appropriate boundary nodes.

Combining Equations 72 and 73, we have

$$q_i = \overline{q_i} + f_i(T - T_o^*) \tag{74}$$

so that Equation 49 becomes

$$q_N = q_N^* + f_{NM}T_M \tag{75}$$

where

$$q^*_N = \int_{v_e} \rho_0 h \psi_N dv_0 + \int_{A_e} \psi_N n_i \bar{q}_i dA_0 + \int_{A_e} \psi_N n_i f_i T^*_0 dA_0 \qquad (76)$$

and

$$f_{NM} = \int_{A_e} \psi_N \psi_M f_i n_i dA_0 \qquad (77)$$

Thus, instead of Equation 60, we may use

$$(k_{NM} - f_{NM}) T_M - T_0 b^i_{MN} \dot{u}_{Mi} - c_{NM} \dot{T}_M = q^*_N \qquad (78)$$

Likewise Equation 66 becomes

$$(K_{\Delta\Gamma} - F_{\Delta\Gamma}) T_\Delta + T_0 B^i_{\Delta\Gamma} \hat{U}_{\Gamma i} + C_{\Delta\Gamma} \dot{T}_\Gamma = Q^*_\Delta \qquad (79)$$

where

$$F_{\Delta\Gamma} = \sum_e \Omega^{(e)}_{N\Delta} f^{(e)}_{NM} \Omega^{(e)}_{M\Gamma} \qquad (80)$$

$$Q^*_\Delta = \sum_e \Omega^{(e)}_{N\Delta} q^*_{N(e)} \qquad (81)$$

SECTION VIII

SOME SPECIAL CASES

The above finite element equations are valid for any type of finite-element representation for which the general, local interpolation Equations 29 and 30 are used. The equations are considerably simplified, however, if the familiar simplex models for the local fields are used. Then tetrahedral, triangular, or lineal elements with four, three, or two nodes are used in three-, two-, or one-dimensional problems, respectively. Then the interpolating functions $\psi_N(x)$ of Equation 31 are given by the linear forms

$$\psi_N(x) = \alpha_N + x^i \beta_{iN} \tag{82}$$

where α_N and β_{iN} are constants which depend only on the geometry of the finite element. Physically, Equation 82 depicts the deformation of each element as being homogeneous and the local temperature field of each element as being homogeneous (Reference 17).

In this case, the elemental mass, stiffness, coupling, and conductivity matrices of Equations 40, 58, 59, 61, and 62 reduce to

$$m_{NM} = \alpha_N \alpha_M m_e + 2\alpha_M \beta_{iM} s^i + \beta_{iN} \beta_{jN} i_{ij} = \frac{\rho_o}{a} c_{NM} \tag{83}$$

$$a_{NM}^{im} = v_e \beta_{kN} \beta_{NM} E^{ikmn} \tag{84}$$

$$k_{NM} = v_e k_{ij} \beta_{iN} \beta_{jM} \tag{85}$$

$$b_{MN}^i = \frac{1}{\rho_o} B^{ik} \beta_{kN} (\alpha_M m_e + \beta_{jM} s^j) \tag{86}$$

where m_e and v_e are the mass and volume of the element, s^j are the mass moments with respect to the system x^i, and i_{ij} is the mass-moment-of-inertia tensor of the element with respect to the x^i.

Equations for the coefficients α_N and β_{iN} for the one-, two-, and three-dimensional cases can be found in, for example, (Reference 17). In the two-dimensional case, for example,

$$\alpha_N = \frac{1}{2A_o} \begin{bmatrix} x_2^1 x_3^2 - x_3^1 x_2^2 \\ x_1^2 x_3^1 - x_1^1 x_3^2 \\ x_1^1 x_2^2 - x_1^2 x_2^1 \end{bmatrix} \tag{87}$$

$$\beta_{iN} = \frac{1}{2A_o} \begin{bmatrix} x_2^2 - x_3^2 & x_3^2 - x_1^2 & x_1^2 - x_2^2 \\ x_3^1 - x_2^1 & x_1^1 - x_3^1 & x_2^1 - x_1^1 \end{bmatrix} \tag{88}$$

where A_o is the area of the element and x_N^α ($N = 1, 2, 3$; $\alpha = 1, 2$) are the coordinates of node N. For one-dimensional elements,

$$\alpha_1 = x_2/L \qquad \alpha_2 = -x_1/L \qquad \beta_{11} = -\beta_{21} = -1/L \tag{89}$$

where x_1 and x_2 are the x-coordinates of nodes 1 and 2 and $L = x_2 - x_1$.

It is interesting to note that if Equation 57 is ignored, the term $-T_o b_{MN}^i \dot{u}_{Mi}$ is deleted, and the values in Equations 87 and 88 are introduced in Equations 83 and 85 for the purpose of calculating k_{NM} and c_{NM}, then Equation 78 reduces to the heat conduction model of Becker and Parr (Reference 9). If the two-dimensional simplex values of Equations 87 and 88 are used in Equations 83 through 86, and if the inertia term $m_{NM}\ddot{u}_{Mi}$ and the coupling term $T_o b_{MN}^i \dot{u}_{Mi}$ are deleted from Equations 57 and 60 or 78, then uncoupled quasi-static equations of the type considered by Visser (Reference 10) are obtained. Finally, if the one-dimensional model defined by Equation 89 is introduced into Equations 83 through 86, 57, and 60, the finite-element formulation of Nickell and Sackman (Reference 12) is obtained.

SECTION IX

NUMERICAL RESULTS

In order to demonstrate the merits of the finite element representation developed in this paper, several numerical examples are presented in this section. Solutions of the finite element differential equations were obtained using a standard Rung-Kutta-Gill integration scheme. The first examples consider a linear thermoelastic half-space subjected to ramp heating on its stress-free bounding surface. This problem was first investigated by Sternberg and Chakravorty (Reference 19) for the thermoelastically uncoupled case. The special case of sudden step heating of the bounding plane had been studied earlier by Danilovskaya (Reference 20) for the thermoelastically coupled case and is included in this paper. Nickell and Sackman (References 12 and 13) also investigated this problem using a one-dimensional finite element model obtained through special variational principles.

It is customary to use the following nondimensional parameters:

$$\xi = \frac{a}{\kappa} x_1 \qquad \tau = \frac{a^2}{\kappa} t$$

$$\bar{\theta} = \frac{T}{T_0} \qquad \bar{u} = \frac{a(\lambda + 2\mu)}{\kappa \beta T_0} u \qquad (90)$$

where

$$\kappa = \frac{K}{\rho C_v} \qquad a^2 = \frac{(\lambda + 2\mu)}{\rho}$$

$$\beta = \bar{a}(3\lambda + 2\mu) \qquad \delta = \frac{\beta^2 T_0}{\rho C_v (\lambda + 2\mu)} \qquad (91)$$

In the above equations x_1 is a characteristic length, t the real time, K the thermal conductivity, C_v the specific heat at constant volume, λ and μ are the isothermal Lamé constants, and $\bar{a}$ is the linear coefficient of thermal expansion. The quantity δ in (Equation 91) is a thermomechanical coupling parameter.

Now consider a linear elastic half-space $(x_i > 0)$ with bounding surface at $x_1 = 0$ assumed to be stress free. Let the bounding plane be subjected to a ramp surface heating of the form

$$T_1 = \frac{T_F}{t_0} t \qquad 0 \leq t \leq t_0$$

$$T_1 = T_F \qquad t_0 \leq t \qquad (92)$$

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where T_1 is the surface temperature, T_F is the final surface temperature, and t_o is the boundary temperature rise time.

Define the nondimensional rise time τ_o by

$$\tau_o = \frac{a^2}{\kappa} t_o \qquad (93)$$

The dimensionless boundary condition is

$$\theta_1 = \frac{\tau}{\tau_o} \qquad 0 \leq \tau \leq \tau_o$$

$$\theta_1 = 1 \qquad \tau_o \leq \tau \qquad (94)$$

The quantity $\dfrac{T_F}{T_o}$ has been set equal to unity in Equation (93) for convenience.

The numerical results for the half-space are presented in Figures 2 through 6. Figures 2 and 3 contain the dimensionless temperature $\overline{\theta}$ and displacement $\overline{u}$ at $\xi = 1.0$ with coupling parameters δ of 0.0 and 1.0 as a function of dimensionless time τ, for the case when $\tau_o = 1.0$, whereas Figures 4 and 5 are for the case when $\tau_o = 0.25$. In these figures, the number of elements between the free surface and $\xi = 1.0$ is denoted by NE, whereas the total number of elements used in the analysis is denoted as TNE. The "exact" solutions presented in these figures were obtained by Nickell and Sackman (Reference 12), using a numerical Laplace Transform inversion procedure.

Excellent agreement between the "exact" solution and the finite element solution for the temperature is seen in Figure 2. It is noted in Figure 3 that the displacement solution diverged when NE = 2 and TNE = 10. Increasing the number of elements, however, gave improved results. The finite element solution for the temperature in the case of the higher sloped ramp heating in Figure 4 also yielded good agreement with the "exact" solution. An illustration of the rates of convergence characteristics of the finite element solution is illustrated in Figure 5 for the uncoupled case.

Figure 6 shows the temperature distribution through a portion of the half-space for the case of a suddenly applied step temperature rise on the bonding surface. Exact solutions were obtained by Boley and Tolins (Reference 21). It is noted that, as the time increases, the finite element solution is lower than the exact solution for significant distances into the half-space ($\xi > 4.0$). Since the total number of elements for this analysis is 48 and the
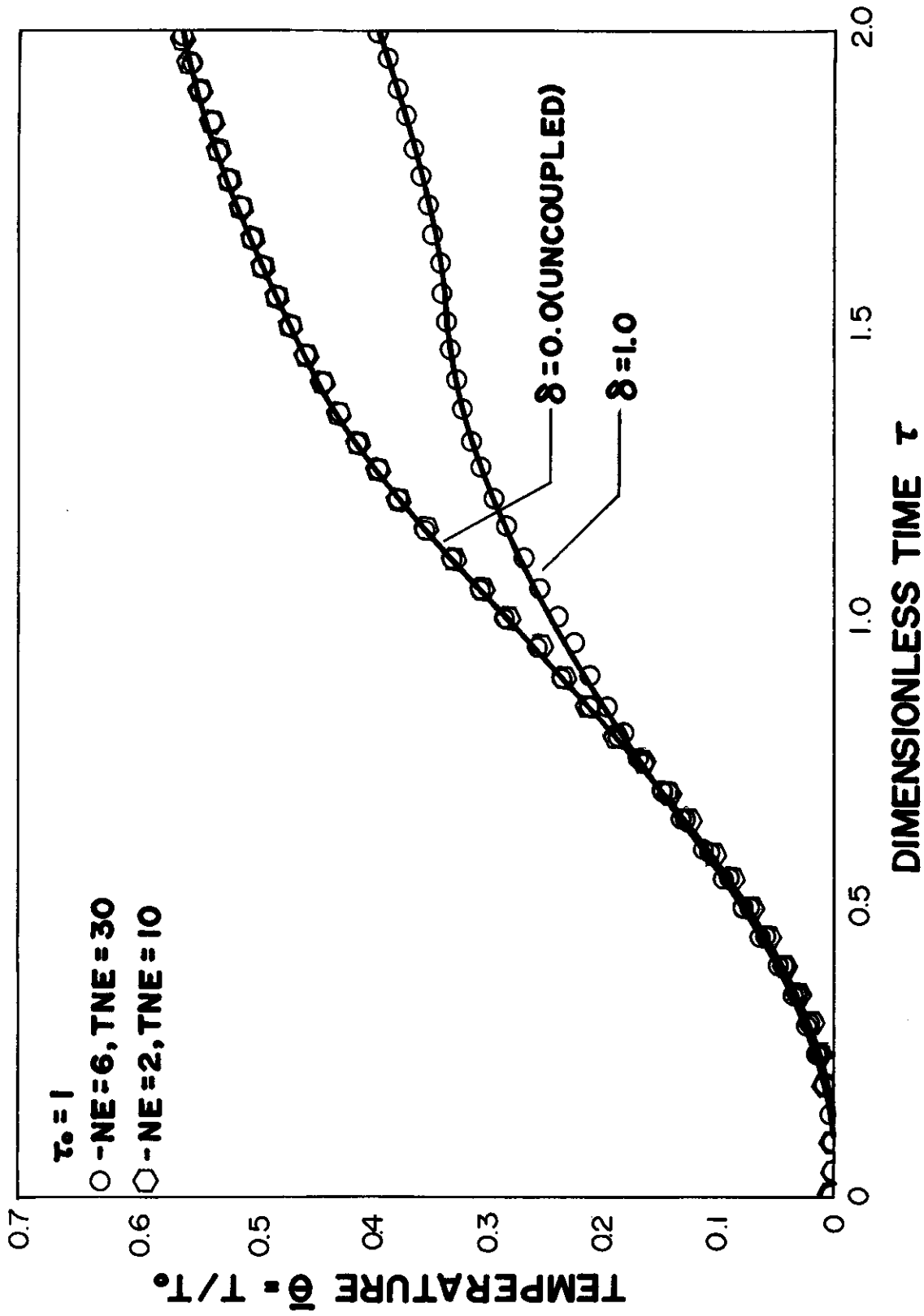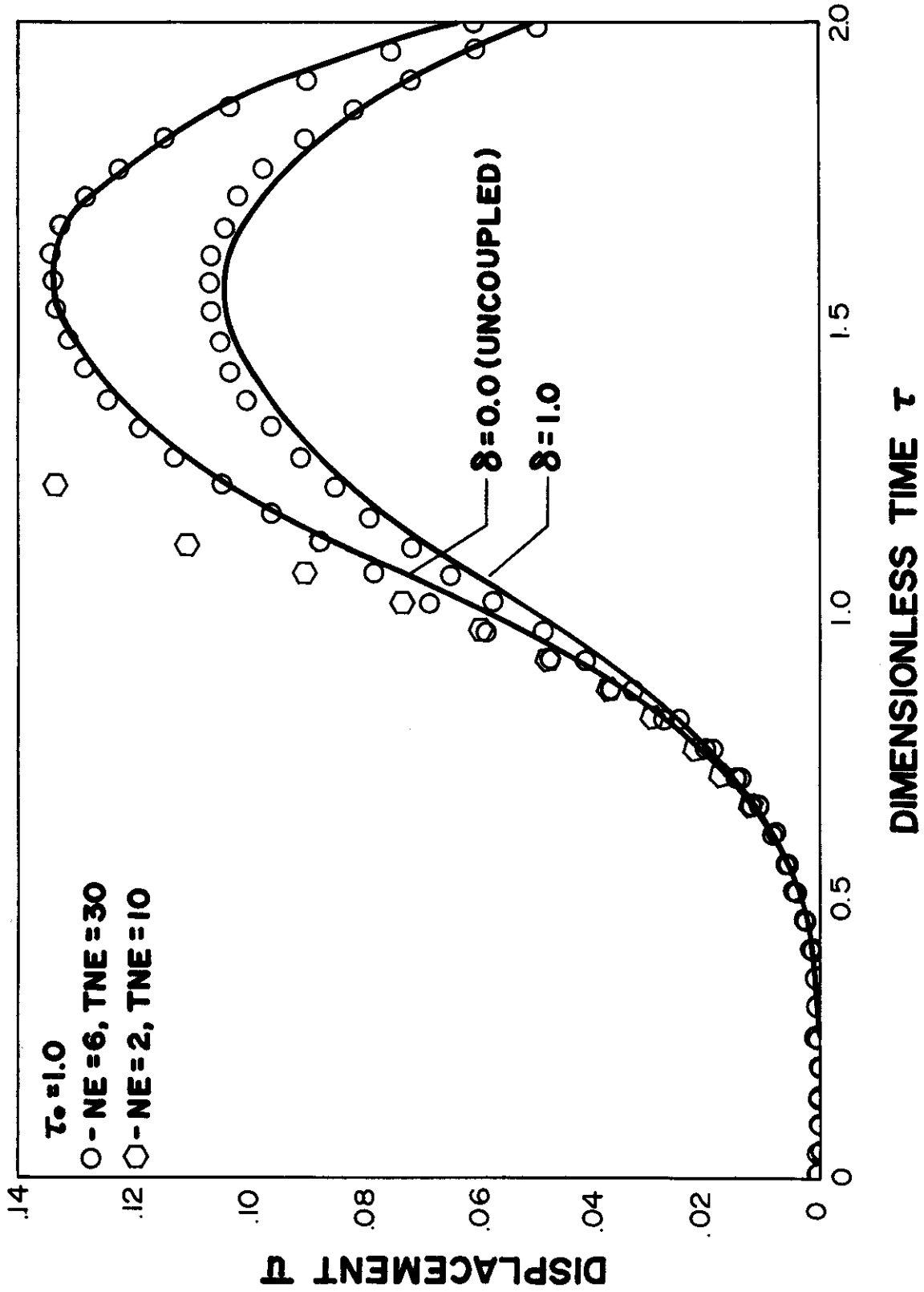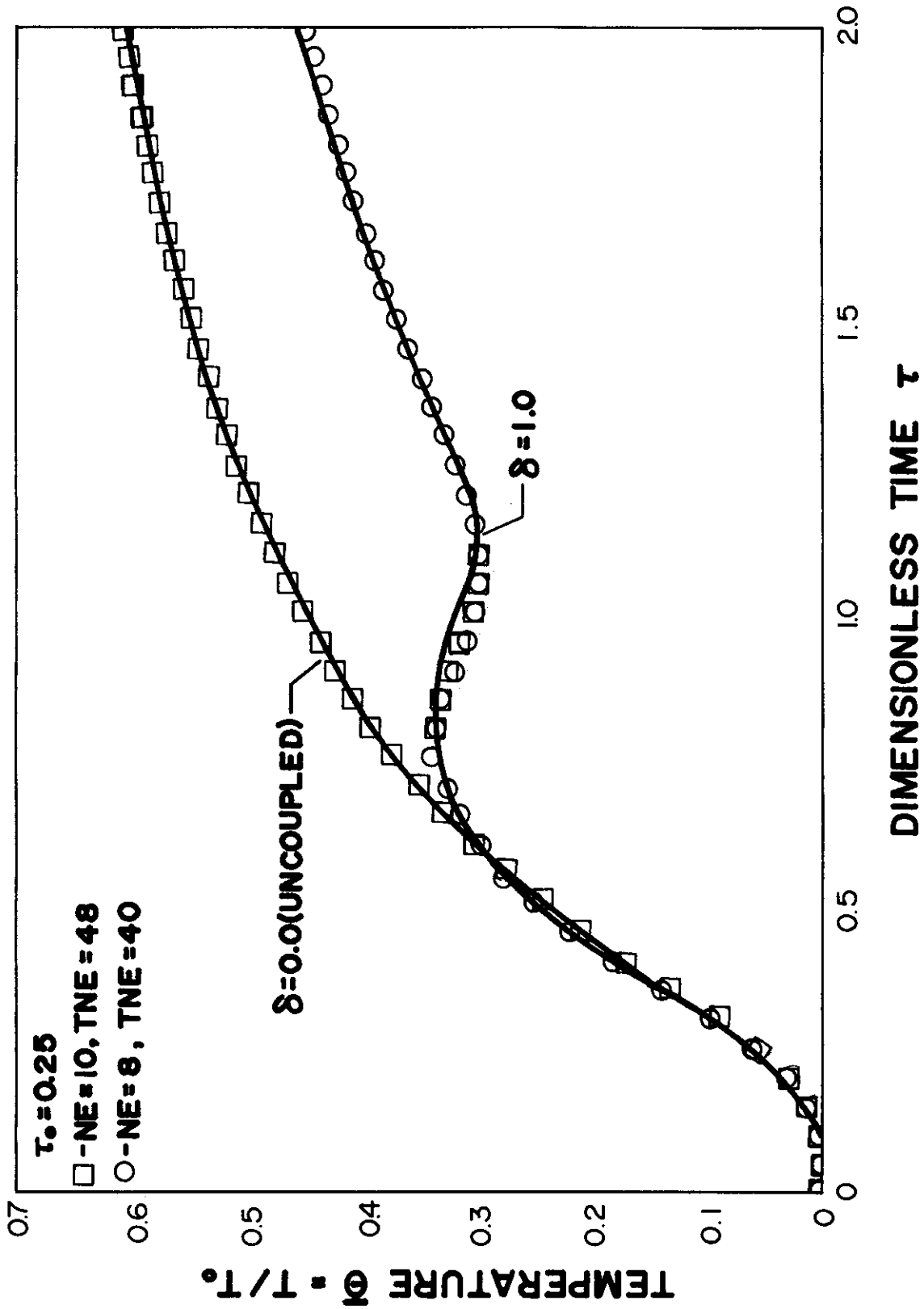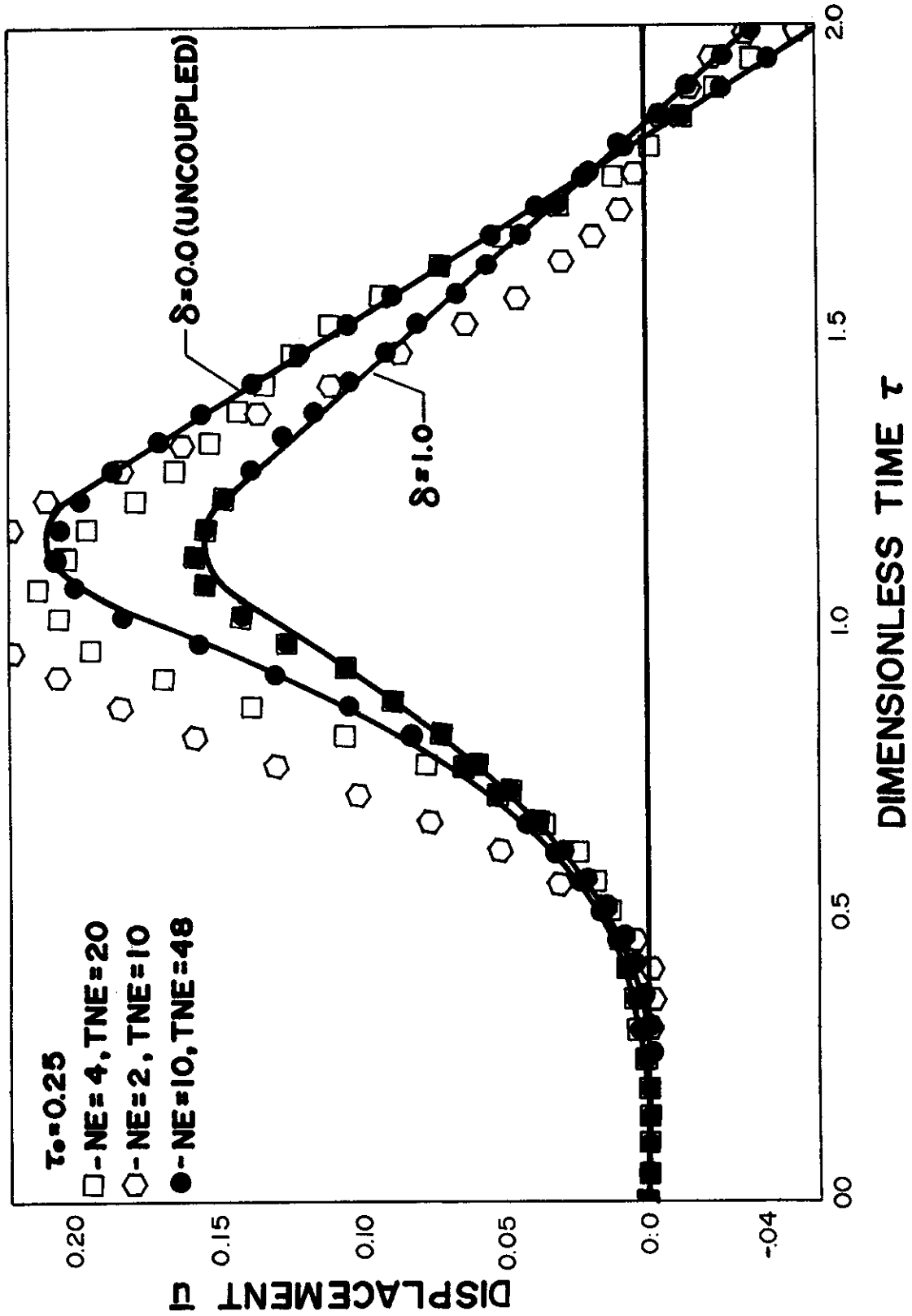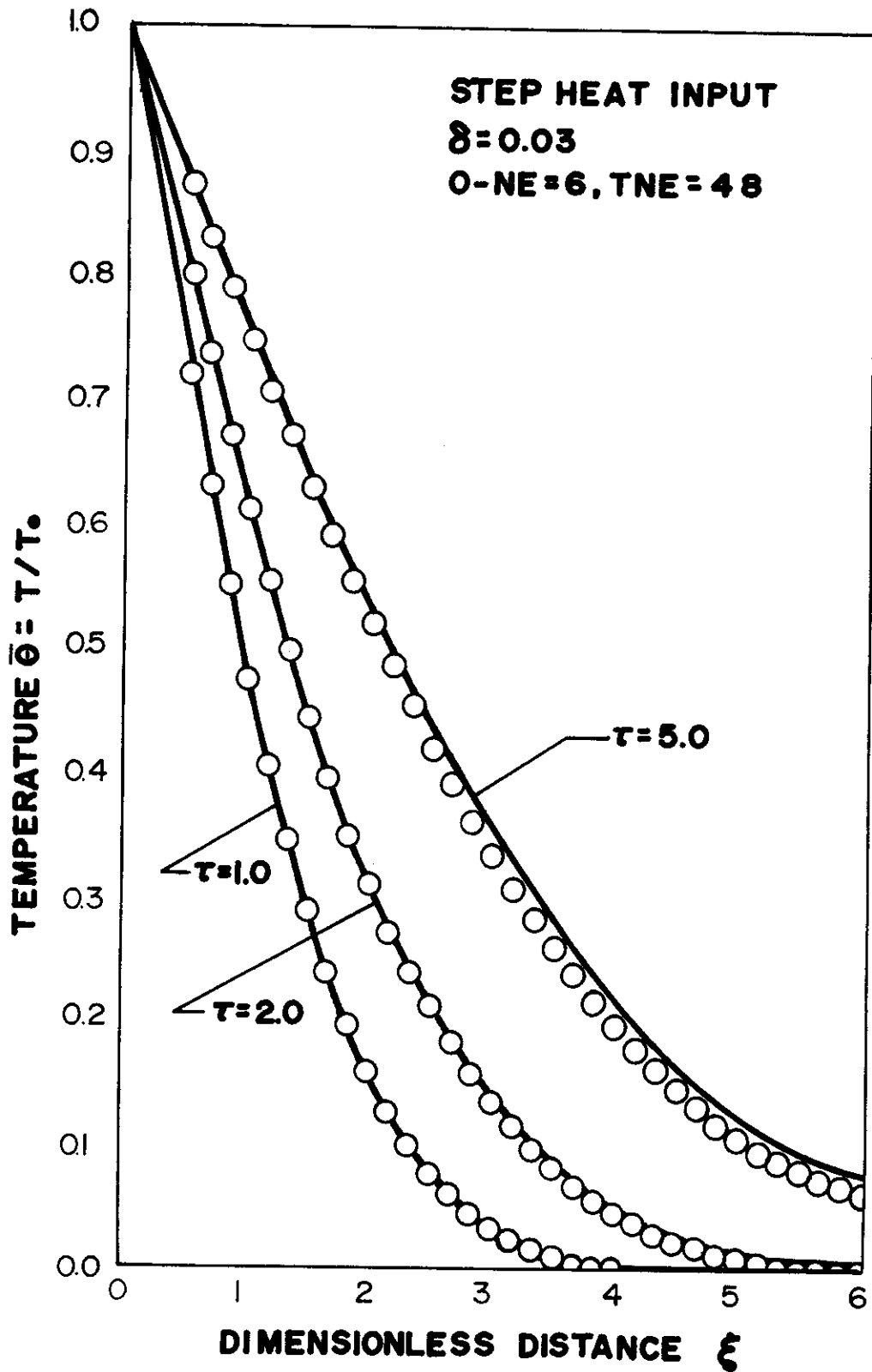
Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

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number of elements from $\xi = 1.0$ to the bounding surface ($\xi = 0.0$) is 6, the "infinite" half-space extends only to $\xi = 8.0$. At the "infinity" point, the temperature and displacement are prescribed as zero. The finite element solution, therefore, yields results which are low in regions near the "infinity" point. An improved solution can be obtained by simply increasing the total number of finite elements while maintaining the same number between $\xi = 1.0$ and the surface.

Acknowledgment

SECTION X

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