A BENDING ANALYSIS FOR PLATES

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In this paper a general, approximate solution method applicable to the bending analysis of structural plates is presented and illustrated. The analysis includes the effects of shearing deformations and, as a consequence, is applicable to both thin and moderately thick plates. The plate bending equations (including shear deformation effects), written as functions of the transverse deflection, and the bending moments are expressed by means of a variational theorem. The plate to be analyzed is represented by a series of finite elements (triangles). Forms of the primary dependent variables (transverse deflection and moments) are assumed within each element and are related to their values at the element nodes (i.e., at the triangle vertices). The approximate solution is obtained by taking the variational of the function with respect to the node values of the unknowns, thus generating a set of linear algebraic equations which define these node values. The rotations and shears are calculated from the values of a deflection and moments. The solution technique is utilized to analyze three significant problems, for which exact solutions are available, with excellent accuracy.

INTRODUCTION

Structural plates have a multitude of applications in the aerospace and construction industries. The analysis of plates in flexure is, therefore, of considerable importance and has received the attention of a great number of investigators. As a result, a large number of papers have been published giving approximate or exact solutions to particular plate problems. Unfortunately, these solutions have in general, been limited to constant thickness homogeneous plates of relatively simple shapes. It would be most desirable if a general solution scheme applicable to plates of arbitrary shapes (including variable thickness) and subjected to arbitrary loading conditions were available. Such a versatile solution method would find many applications in both the aerospace and the construction industries, e.g., the analysis of impeller vanes, airplane wings, building floors, etc.

The finite element procedure is ideally suited for such a general analysis scheme as any plate may be approximately represented as a series of simple shapes. There is, however, a wide range of possible forms that the finite element analysis may have and, therefore, some preliminary considerations must be given to the type of finite element procedure to be employed. The shape of the element should be sufficiently arbitrary to permit representation of very complicated bodies. The assumed variation, within each element, of the primary variables should be relatively simple so as not to unduly complicate the computer programming of the solution. The resulting set of simultaneous equations must be of a form that lends itself to solution techniques which are applicable to very large systems. The plate analysis must yield not only an accurate prediction of the deflection, but also accurate values of moments and shears. In particular, it is desirable to avoid analyses where moments

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and shears are obtained as second and third derivatives of the approximate deflection profile because such procedures are especially prone to error. Formulations which utilize stress functions as unknowns are undesirable, since it is difficult to apply them to plates that have displacement boundary conditions and/or to plates that are multiply connected. Lastly, it is desirable to base the approximate solution upon a plate theory which permits the application of three boundary conditions at each boundary point instead of two as in the case of the classical thin plate theory. Thus, along free and simply supported edges of plates, the actual boundary conditions could be applied instead of the rather artificial boundary conditions which make use of the "effective" transverse force as demanded by thin plate theory (Reference 1). In order to be able to specify three boundary conditions, the effects of transverse shearing deformations must be considered (Reference 2).

In seeking a finite element analysis for plates which would satisfy the above requirements for an acceptable solution technique, the following criteria were employed: 1) the basic element had to be a triangle, 2) the assumed variation, within each element, of the unknowns had to be linear, 3) all desired quantities (deflection, rotation, moments and shears) had to be calculated directly or from first derivatives of the primary variables, 4) the formulation had to lend itself to both stress and displacement boundary conditions, 5) the analysis had to be based upon a plate theory which included transverse shearing deformations.

There does not appear to be any finite element plate analyses reported in the literature which includes transverse shear deformations. Several finite element analyses based upon classical plate theory are reported in the literature (Reference 3 for a summary of several of these analyses) however, the convergence of the majority of these solutions is questionable as the assumed variation of the transverse deflection does not satisfy the requirements placed upon an admissible form of the function by the theorem of minimum potential energy (Reference 3). The many plate analyses in which the plate is replaced by an indeterminate beam system suffers from the same shortcomings. Two analyses have been reported which do appear to satisfy the necessary continuity conditions (References 3 and 4). However, neither of these analyses satisfy all of the criteria specified above for a desirable finite element plate analysis and, in addition, neither of the analyses has been illustrated with actual examples and thus, their utility still needs to be demonstrated.

As the literature did not appear to offer any plate analysis which fulfilled all of the previously mentioned criteria, it became necessary to develop a finite element analysis which satisfies these conditions.

PLATE THEORY

The notation used in this report is illustrated in Figure 1. The symbols q, h, w, M_x, M_{xy}, Q_x and α, respectively, denote normal load, plate thickness, transverse deflection, normal moment acting on the x face, twisting moment, shearing force on the x face and the angle between the normal $\bar{n}$ and the x axis. The elastic properties of the plate, Young's modulus and Poisson's ratio, are denoted by E and ν. The significant section properties of the plate are defined by

$$S = \frac{12}{Eh^3} \tag{1}$$

$$L = \frac{6(1+\nu)}{5\,Eh} \tag{2}$$

The symbol S_o denotes some reference value of S. A function W, referred to as the scaled transverse deflection, is related to the transverse deflection w by the following equation:

$$W = \frac{1}{S_o} w \tag{3}$$

The thermal moment due to a temperature change of ΔT is denoted by

$$m_T = E \int_{-h/2}^{h/2} \alpha_T \Delta T \, z \, dz \tag{4}$$

where α_T is the linear coefficient of thermal expansion.

The interface conditions between elements require the continuity of transverse deflection (w), normal rotation (R_n), tangential rotation (R_s), normal moment (M_n), twisting moment (M_{ns}) and normal shear force (Q_n). To allow the assumption of linear variation within each element for the primary variables, they must be related to these interface quantities by derivatives no higher than the first. This was accomplished by selecting as primary variables, the scaled transverse deflection (W) and the rectangular Cartesian components of the moments (M_x, M_y and M_{xy}). The equations which govern the flexural behavior of plates (including transverse shear deformations Reference 2), expressed in terms of these variables, are

$$W_{,xy} + \frac{S(1+\upsilon)}{S_o} M_{xy} - \frac{L}{S_o}(M_{x,xy} + M_{y,xy} + M_{xy,xx} + M_{xy,yy}) = 0 \tag{5}$$

$$W_{,xx} + \frac{S}{S_o}(M_x - \upsilon M_y) - \frac{2L}{S_o}(M_{x,xx} + M_{xy,xy})$$

$$- \frac{\upsilon L}{S_o(1+\upsilon)} q + \frac{S}{S_o} m_T = 0 \tag{6}$$

$$W_{,yy} + \frac{S}{S_o}(M_y - \upsilon M_x) - \frac{2L}{S_o}(M_{xy,xy} + M_{y,yy})$$

$$- \frac{\upsilon L}{S_o(1+\upsilon)} q + \frac{S}{S_o} m_T = 0 \tag{7}$$

$$M_{x,xx} + M_{y,yy} + 2M_{xy,xy} + q = 0 \tag{8}$$

(partial differentiation is denoted by a comma).

The first three equations are moment-strain relationships and the fourth is an equilibrium equation. The derived quantities R_x, R_y, Q_x and Q_y are related to the primary variables by the following equations:

$$Q_x = M_{x,x} + M_{xy,y} \tag{9}$$

$$Q_y = M_{y,y} + M_{xy,x} \tag{10}$$

$$R_x = \left[S_o W_{,x} - 2L Q_x \right] \tag{11}$$

$$R_y = \left[S_o W_{,y} - 2L Q_y \right] \tag{12}$$

The first two equations are equilibrium equations; the last two follow from the strain displacement assumptions. The rotations have been denoted by R_x and R_y and are related to the displacements as follows:

$$u_x = - zR_x \tag{3}$$

$$u_y = - zR_y \tag{14}$$

If the constant L is set equal to zero, the above set of equations reduce to the governing set of equations for the classical thin plate theory. The inclusion of the effects of transverse shear deformations does not change the order of the set of equations, when written in this form, and only slightly complicates their algebraic form. In addition, the effect of thickness deformations may be included without materially increasing the complexity of the governing equations (Reference 5).

In order to develop a finite element plate analysis based upon this formulation, it is necessary to express the plate problem by an equivalent variational equation.

The plate bending problem defined by Equations 5 through 8 with auxiliary relations 9 through 12 is expressed by the variational equation

$$\delta V = 0 \tag{15}$$

where V is the function whose variation yields as Euler Equations, Equations 5 through 8 and as natural boundary conditions Equations 9 through 12. The function V is determined to have the form

$$V = \iint_B \Big\{ - qW + \Big[\frac{\nu L q}{S_o(1+\nu)} - \frac{S\, m_T}{S_o} \Big] (M_x + M_y) + M_{x,x} W_{,x} + M_{y,y} W_{,y}$$

$$+ M_{xy,y} W_{,x} + M_{xy,x} W_{,y} - \frac{S}{S_o} \Big[(1+\nu) M_{xy}^2 + \frac{1}{2} M_x^2 + \frac{1}{2} M_y^2 - \nu M_x M_y \Big] \tag{16}$$

$$- \frac{L}{S_o} \Big[(M_{x,x})^2 + 2 M_{xy,y} M_{x,x} + (M_{y,y})^2 + 2 M_{xy,x} M_{y,y} + (M_{xy,x})^2$$

$$+ (M_{xy,y})^2 \Big] \Big\} dx\, dy - \int_{S_1} \Big[Q_n^a W + \frac{1}{S_o} (R_n^a M_n + R_s^a M_{ns}) \Big] ds$$

The surface integral is to be evaluated over the entire plate B and the line integral is to be evaluated over that portion of the plate boundary where R_n^a and/or R_s^a and/or Q_n^a are specified. The symbols Q_n^a, R_n^a and R_s^a, respectively, denote specified values of edge shear, normal edge rotation and tangential edge rotation. The values of the moments along a boundary whose outward normal is denoted by $\hat{n}$ (Figure 1), are

$$M_n = M_x (\cos \alpha)^2 + M_y (\sin \alpha)^2 + M_{xy} \sin 2a \tag{17}$$

$$M_{ns} = \frac{1}{2} (M_y - M_x) \sin 2\alpha + M_{xy} \cos 2\alpha \tag{18}$$

$$Q_n = Q_x \cos \alpha + Q_y \sin \alpha \tag{19}$$

An admissible state for the primary variables is one which a) satisfies the prescribed moment and transverse deflection boundary conditions, b) has continuous second derivatives within each element and c) is continuous across all element boundaries.

The proof of the variational equation is the fact that its Euler equations are the desired governing relationships, Equations 5 through 8, and that the natural boundary conditions require the continuity of the shear force and the rotations between elements (the restrictions placed upon an admissible form of W, M_x, M_y and M_{xy} insure that w, M_n and M_{ns} are continuous between elements). This is shown by taking the independent variations δW, δM_x, δM_y and δM_{xy}; after suitable manipulation and integration by parts, the variational equation becomes

$$\delta V = \iint_B \Big\{ -\Big[q + M_{x,xx} + M_{y,yy} + 2 M_{xy,xy}\Big] \delta W - \Big[W_{,xx} + \frac{S}{S_o}(M_x - \nu M_y) - \frac{2L}{S_o}(M_{x,xx} + M_{xy,xy}) - \frac{\nu Lq}{S_o(1+\nu)} + \frac{Sm_T}{S_o}\Big] \delta M_x - \Big[W_{,yy} + \frac{S}{S_o}(M_y - \nu M_x) - \frac{2L}{S_o}(M_{y,yy} + M_{xy,xy}) - \frac{\nu Lq}{S_o(1+\nu)} + \frac{Sm_T}{S_o}\Big] \delta M_y - 2\Big[W_{,xy} + \frac{S(1+\nu)}{S_o} M_{xy} - \frac{L}{S_o}(M_{x,xy} + M_{y,xy} + M_{xy,xx} + M_{xy,yy})\Big] \delta M_{xy} \Big\} dx\, dy + \int_{S_1} \Big\{ \Big[Q_n - Q_n^a\Big] \delta W + \frac{1}{S_o}\Big[R_n - R_n^a\Big] \delta M_n + \frac{1}{S_o}\Big[R_s - R_s^a\Big] \delta M_{ns} \Big\} ds \tag{20}$$

A detailed proof of the variational equation may be obtained by following the procedure given in Reference 6.

The variational statement of the plate bending problem is used to generate an approximate solution by selecting from a family of trial functions, the one that satisfied Equation 15 (the Ritz technique)(Reference 7).

FINITE ELEMENT SOLUTION

The selection of a family of trial functions for use in the Ritz technique is facilitated if the plate is represented by a series of finite elements (References 3 and 6)(Figure 2). These functions are formed by assuming that W, M_x, M_y and M_{xy} are linear within each triangle (expressed as a function of their values at the nodes) and continuous across the element interfaces. A particular member of the family is uniquely determined by assigning values to all the node values of the functions. These node values will be selected so as to satisfy Equation 15.

The plate is represented by Q elements; associated with these Q elements are N nodal points (a typical node point is denoted by n). The value of an unknown U at the n^{th} node point is denoted as U^n and the coordinates of the point by (x^n, y^n). A particular node n is common to M elements; in the consideration of the m^{th} element of this group, it will be represented as shown in Figure 3b. The coordinates of the vertices of this triangle are $x_i = x^n$, $x_j = x^r$, etc. The values of an unknown function at its vertices are denoted by $U_m^i = U^n$, $U_m^j = U^r$, etc.

Within an element m, the unknowns are expressed as

$$\left[W\ M_x\ M_y\ M_{xy}\right] = \left[1\ x\ y\right] \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \end{bmatrix}_m \qquad (21)$$

where continuity of the dependent variables is assured if the $\left[C\right]_m$ matrix is expressed in terms of the node values of the dependent variables

$$\left[C\right]_m = \left[T\right]_m \begin{bmatrix} W^i & M_x^i & M_y^i & M_{xy}^i \\ W^j & M_x^j & M_y^j & M_{xy}^j \\ W^k & M_x^k & M_y^k & M_{xy}^k \end{bmatrix}_m \qquad (22)$$

where

$$\left[T\right]_m = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_m = \frac{1}{2A_m} \begin{bmatrix} x_j y_k - x_k y_j & x_k y_i - x_i y_k & x_i y_j - x_j y_i \\ y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{bmatrix} \qquad (23)$$

and

$$A_m = \frac{1}{2}\left[x_j(y_k - y_i) + x_i(y_j - y_k) + x_k(y_i - y_j)\right] \qquad (24)$$

The function V, Equation 16, for the plate may be found by evaluating the integral over each element and summing the results (the area of the q^{th} element is denoted by A_q), i.e.,

$$V = \sum_{q=1}^{Q} \left\langle \iint_{A_q} \left\{ \cdots\cdots\cdots \right\} dx\, dy - \int_{S_{1q}} \left[\cdots\cdots\cdots \right] ds \right\rangle \qquad (25)$$

The bracketed expressions are identical to those given in Equation 16. The symbol S_{1q} denotes that portion of the S_1 boundary that lies in the q^{th} element.

The variation of Equation 16 with respect to the parameters M_x^n, M_y^n, M_{xy}^n and W^n yields the following equations:

$$\begin{bmatrix} \frac{\partial V}{\partial M_x^n} \\ \frac{\partial V}{\partial M_y^n} \\ \frac{\partial V}{\partial M_{xy}^n} \\ \frac{\partial V}{\partial W^n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

Upon substituting Equation 21 into Equation 25 and the results into Equation 26 the following expressions are obtained for each node point (note that only the M triangles surrounding node n contribute to these equations):

$$\sum_{m=1}^{M} \left[P\right]_m^T \left[U\right]_m^T = \sum_{m=1}^{M} \left[L\right]_m^T \tag{27}$$

where $\left[P\right]_m$, $\left[U\right]_m$ and $\left[L\right]_m$ are given by (the transpose of $\left[P\right]_m$ is written $\left[P\right]_m^T$)

$$\left[P\right]_m = \begin{bmatrix} F_1 - GB_{11} & -\nu F_1 & -GB_{31} & B_{11} \\ -\nu F_1 & F_1 - GB_{41} & -GB_{21} & B_{41} \\ -GB_{21} & -GB_{31} & 2(1+\nu)F_1 - G(B_{11}+B_{41}) & (B_{21}+B_{31}) \\ B_{11} & B_{41} & B_{21}+B_{31} & 0 \\ F_2 - GB_{12} & -\nu F_2 & -GB_{32} & B_{12} \\ -\nu F_2 & F_2 - GB_{42} & -GB_{22} & B_{42} \\ -GB_{22} & -GB_{32} & 2(1+\nu)F_2 - G(B_{12}+B_{42}) & (B_{22}+B_{32}) \\ B_{12} & B_{42} & B_{22}+B_{32} & 0 \\ F_3 - GB_{13} & -\nu F_3 & -GB_{33} & B_{13} \\ -\nu F_3 & F_3 - GB_{43} & -GB_{23} & B_{43} \\ -GB_{23} & -GB_{33} & 2(1+\nu)F_3 - G(B_{13}+B_{43}) & (B_{23}+B_{33}) \\ B_{13} & B_{43} & B_{23}+B_{33} & 0 \end{bmatrix} \quad (28)$$

$$\left[U\right]_m = \left[M_x^i \; M_y^i \; M_{xy}^i \; W^i \; M_x^j \; M_y^j \; M_{xy}^j \; W^j \; M_x^k \; M_y^k \; M_{xy}^k \; W^k\right]_m \quad (29)$$

$$\left[L\right]_m = \left[R \quad R \quad 0 \quad \frac{qA}{3}\right] \quad (30)$$

The summation sign ($\sum_{m=1}^{M}$) indicates that the equations obtained by considering the M triangles that have the common node n, must be combined. When node n lies on the boundary and Q_n^a and/or R_n^a and/or R_s^a are specified as non-trivial values, their contribution to the line integral in Equation 16 will appear as additional terms on the right hand side of Equation 27. The constants are given by the following expressions (ℓ = 1, 2, 3):

$$B_{1\ell} = A\, T_{21}\, T_{2\ell} \tag{31}$$

$$B_{2\ell} = A\, T_{21}\, T_{3\ell} \tag{32}$$

$$B_{3\ell} = A\, T_{31}\, T_{2\ell} \tag{33}$$

$$B_{4\ell} = A\, T_{31}\, T_{3\ell} \tag{34}$$

$$F_1 = -\frac{AS}{6S_o} \tag{35}$$

$$F_2 = -\frac{AS}{12S_o} \tag{36}$$

$$F_3 = -\frac{AS}{12S_o} \tag{37}$$

$$G = \frac{2L}{S_o} \tag{38}$$

$$R = -\frac{\nu LqA}{3S_o(1+\nu)} + \frac{SAm_T}{3S_o} \tag{39}$$

where T_{ij} and A are defined by Equations 23 and 24. Taking the variation of Equation 25 with respect to the 4N parameters M_x^n, M_y^n, M_{xy}^n and W^n yields a set of 4N equations which determine the approximate node values of the unknowns. Making use of these values, approximate values for the shears and rotations within each element may be found. By substituting Equation 21 into Equations 9 through 11, the following expressions for shears and rotations are obtained (within the q^{th} element):

$$Q_x = T_{21} M_x^i + T_{22} M_x^j + T_{23} M_x^k + T_{31} M_{xy}^i + T_{32} M_{xy}^j + T_{33} M_{xy}^k \tag{40}$$

$$Q_y = T_{31} M_y^i + T_{32} M_y^j + T_{33} M_y^k + T_{21} M_{xy}^i + T_{22} M_{xy}^j + T_{23} M_{xy}^k \tag{41}$$

$$R_x = S_o\left[T_{21} W^i + T_{22} W^j + T_{23} W^k\right] - 2LQ_x \tag{42}$$

$$R_y = S_o\left[T_{31} W^i + T_{32} W^j + T_{33} W^k\right] - 2LQ_y \tag{43}$$

At the boundaries of the plate, if one or more of the primary variables W, M_x, M_y or M_{xy} are specified, no corresponding equation is generated. In general, when moments are prescribed, these will have to be in terms of normal and tangential components (i.e., M_n and M_{ns}). Consequently, the values of the normal and tengential moments (M_n, M_s and M_{ns}) are utilized as unknowns at such points instead of the Cartesian components (M_x, M_y and M_{xy}). The transformation is made by employing the following relationships between the unknowns:

$$M_x = M_n \cos^2 \alpha + M_s \sin^2 \alpha - 2M_{ns} \sin \alpha \cos \alpha \tag{44}$$

$$M_y = M_n \sin^2 \alpha + M_s \cos^2 \alpha + 2M_{ns} \sin \alpha \cos \alpha \tag{45}$$

$$M_{xy} = (M_n - M_s) \cos \alpha \sin \alpha + M_{ns} (\cos^2 \alpha - \sin^2 \alpha) \tag{46}$$

In addition, it is necessary to recast the governing algebraic equations in terms of variational equations obtained by considering the variations δM_n, δM_s and δM_{ns}. This transformation is accomplished by using (at boundary point i) the equations arising from the variations δM_n^i, δM_s^i, δM_{ns}^i and δW^i. For example, the equation corresponding to the variation δM_n^i is

$$\frac{\partial V}{\partial M_n^i} = 0 \tag{47}$$

or

$$\frac{\partial V}{\partial M_n^i} = \frac{\partial V}{\partial M_x^i} \frac{\partial M_x^i}{\partial M_n^i} + \frac{\partial V}{\partial M_y^i} \frac{\partial M_y^i}{\partial M_n^i} + \frac{\partial V}{\partial M_{xy}^i} \frac{\partial M_{xy}^i}{\partial M_n^i} + \frac{\partial V}{\partial W^i} \frac{\partial W^i}{\partial M_n^i} \tag{48}$$

Making use of Equations 44 through 46 the last expression becomes

$$\frac{\partial V}{\partial M_n^i} = \frac{\partial V}{\partial M_x^i} \cos^2 \alpha + \frac{\partial V}{\partial M_y^i} \sin^2 \alpha + \frac{\partial V}{\partial M_{xy}^i} \cos \alpha \sin \alpha \tag{49}$$

In a similar manner, the equations corresponding to the variations of M_s^i and M_{ns}^i are obtained. The dependence of these equations on Equation 26 is

$$\begin{bmatrix} \frac{\partial V}{\partial M_s^i} \\ \frac{\partial V}{\partial M_n^i} \\ \frac{\partial V}{\partial M_{ns}^i} \end{bmatrix} = \begin{bmatrix} \sin^2 \alpha & \cos^2 \alpha & -\cos \alpha \sin \alpha \\ \cos^2 \alpha & \sin^2 \alpha & \cos \alpha \sin \alpha \\ (-2 \sin \alpha \cos \alpha) & (2 \sin \alpha \cos \alpha) & (\cos^2\alpha - \sin^2\alpha) \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial M_x^i} \\ \frac{\partial V}{\partial M_y^i} \\ \frac{\partial V}{\partial M_{xy}^i} \end{bmatrix} \tag{50}$$

Thus, Equation 50 gives the desired transformation of the governing equations at boundary point i. It is now possible to compute the contribution to the right-hand side of Equation 27 due to the boundary integral of Equation 16. These contributions will occur whenever boundary values of rotations and/or shears are specified. A typical term in the line integral of Equation 16 may be written as

$$LI = \int D^a \, I \, ds \tag{51}$$

where D^a is the specified value of the derived variable (shear or rotation) and I is the primary variable (scaled transverse deflection or moment). Figure 4 indicates a portion of the plate boundary along which the dependent variable D^a is specified. The assumed variation of the dependent variable I (Equation 21) is linear along the boundary S_i and thus, may be expressed as (with the surface coordinate S measured from the point (x^o, y^o)):

$$I = I^o + (I^o - I^-) \frac{S}{\ell_B} \qquad 0 \geq S \geq -\ell_B \tag{52}$$

$$I = I^o - (I^o - I^+) \frac{S}{\ell_F} \qquad \ell_F \geq S \geq 0 \tag{53}$$

Substituting these expressions into Equation 51 yields

$$LI = \int_{-\ell_B}^{0} D^a \left[I^o + (I^o - I^-) \frac{S}{\ell_B} \right] ds + \int_{0}^{\ell_F} D^a \left[I^o - (I^o - I^+) \frac{S}{\ell_F} \right] ds \tag{54}$$

Consider now the variation with respect to δI^o, i.e.,

$$\frac{\partial LI}{\partial I^o} = \int_{-\ell_B}^{0} D^a \left[1 + \frac{S}{\ell_B} \right] ds + \int_{0}^{\ell_F} D^a \left[1 - \frac{S}{\ell_F} \right] ds \tag{55}$$

This then is the expression that must be evaluated and added into the equation arising from the variation of I^o whenever shears or rotations are specified.

It is interesting to note the form of Equation 55 if $D^a(S)$ is specified as a constant D_B along the boundary B and as a constant D_F along the boundary F, Equation 55 becomes:

$$\frac{\partial LI}{\partial I^o} = D_B \frac{\ell_B}{2} + D_F \frac{\ell_F}{2}$$

EXAMPLES

A FORTRAN IV computer program was written to evaluate the analysis. Utilizing this program several examples were solved; the results of three solutions are given below.

In order to demonstrate the versatility and accuracy of the approximate analysis when applied to thin plates, the solution for two sample problems are given. Problems, for which exact solutions of the classical plate equations are available, were selected so that a comparison between the exact and approximate solutions could be made. The thickness of the plates were made small so as to minimize the effects of transverse shear deformations.

The first problem is a uniformly loaded rectangular plate that is simply supported along two opposing edges, fixed along a third side and free along the remaining side. The configuration of this plate is shown in Figure 5. Two approximate analyses of this plate were obtained; their finite element representations are also given in Figure 5. In the first case, only a very minimal number of elements were employed. Figures 6, 7 and 8 give comparisons of the approximate finite element solutions to that given by five terms of the infinite series solution (Reference 1). The plots of Figure 6 show the values found along the $y = 0.4$ line. The values presented in Figures 7 and 8 are the variations along the $X = 0.4$ line. Agreement between the finite element solution and the series solution is excellent.

The second example consists of a circular plate with a central hole, which is clamped at the inner and outer edges. The plate is loaded over an annular region as indicated in Figure 9. This example serves to illustrate the applicability of the solution to a plate with curved boundaries. Figures 10 and 11 are comparisons of the finite element solution to the exact solution along line L_1, see Figure 9. The agreement is excellent.

The final example illustrates the applicability of the solution to thick plates. The example consists of a portion of a thick circular plate simply supported at its outer edge and subjected to a uniform shear load along the boundary of a central hole. The finite element representation of the plate is shown in Figure 12. The importance of the thickness of the plate is measured by the parameter Lambda "λ" (Reference 8).

$$\lambda = \left(\frac{h}{2a}\right)^2$$

The thickness of the plate and the outer radius of the plate are respectively denoted by h and a. The exact thick plate solution was obtained by the perturbation approach given in Reference 8. Figure 12 gives a comparison of the maximum deflection as predicted by thin plate theory, thick plate theory and the approximate solution developed herein. The agreement between the approximate solution and the thick plate analysis is excellent.

CONCLUSIONS

A general approximate solution method applicable to the bending analysis of structural plates is presented. The plates may be of variable thickness and arbitrary shapes. Additionally, the plates may possess variable material properties. Arbitrary normal, thermal, and boundary loading conditions have been included. The analysis takes into account the effect of shearing deformations and as a result is applicable to both thin and moderately thick plates.

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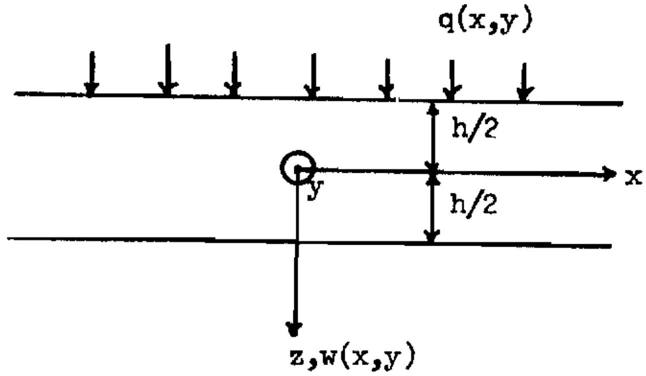
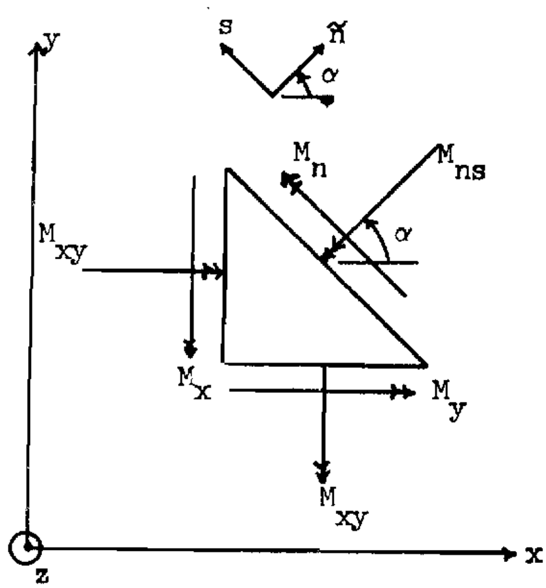
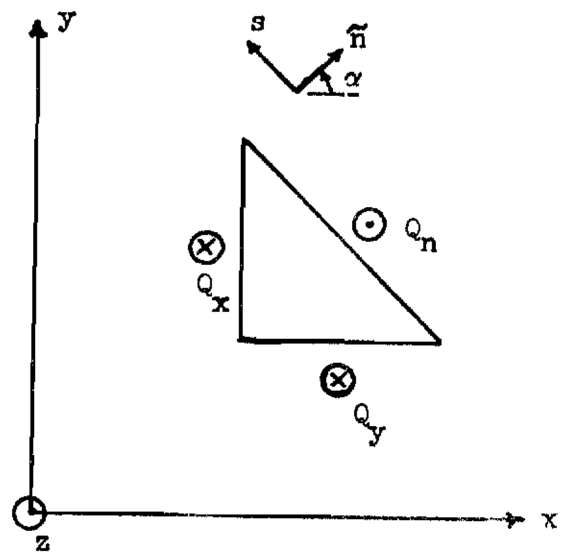


Load and Deflection Notation



Moment Notation



Shear Notation

Figure 1. Notation

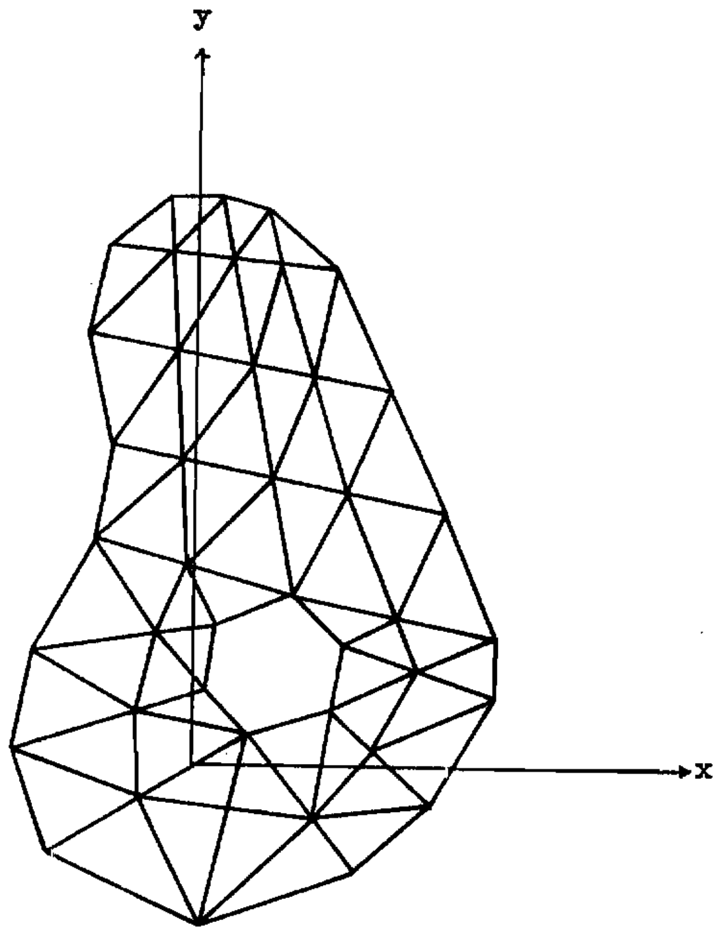
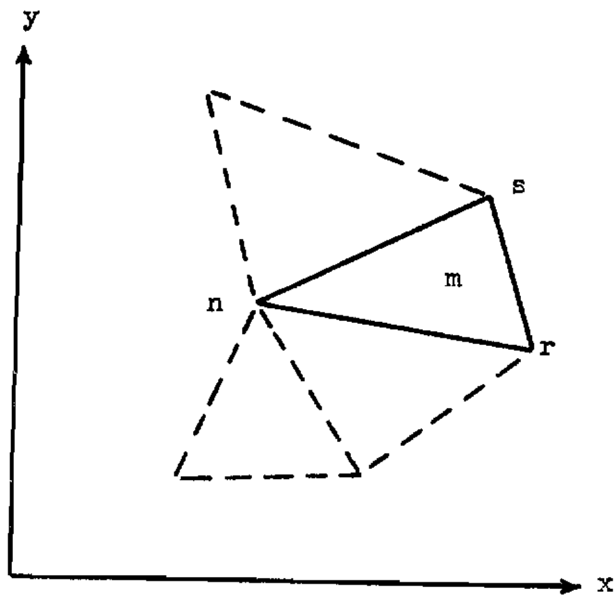
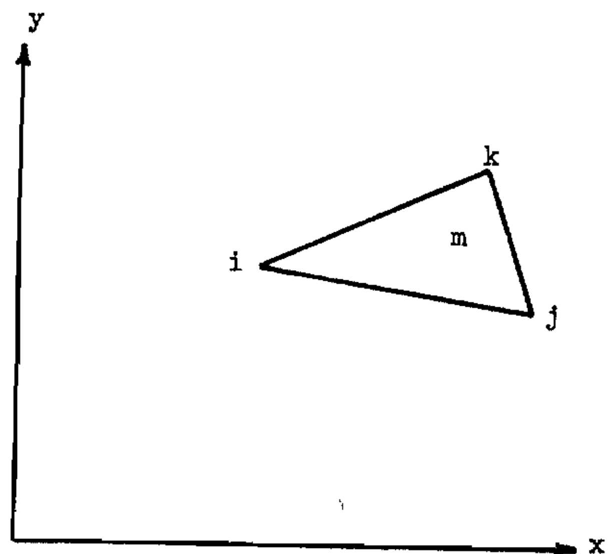


Figure 2. A Coarse Finite Element Representation of a Plate



a) Node Point n Surrounded by M Elements



b) The m^{th} Element

Figure 3. Detailed View of the Elements Surrounding Node Point n

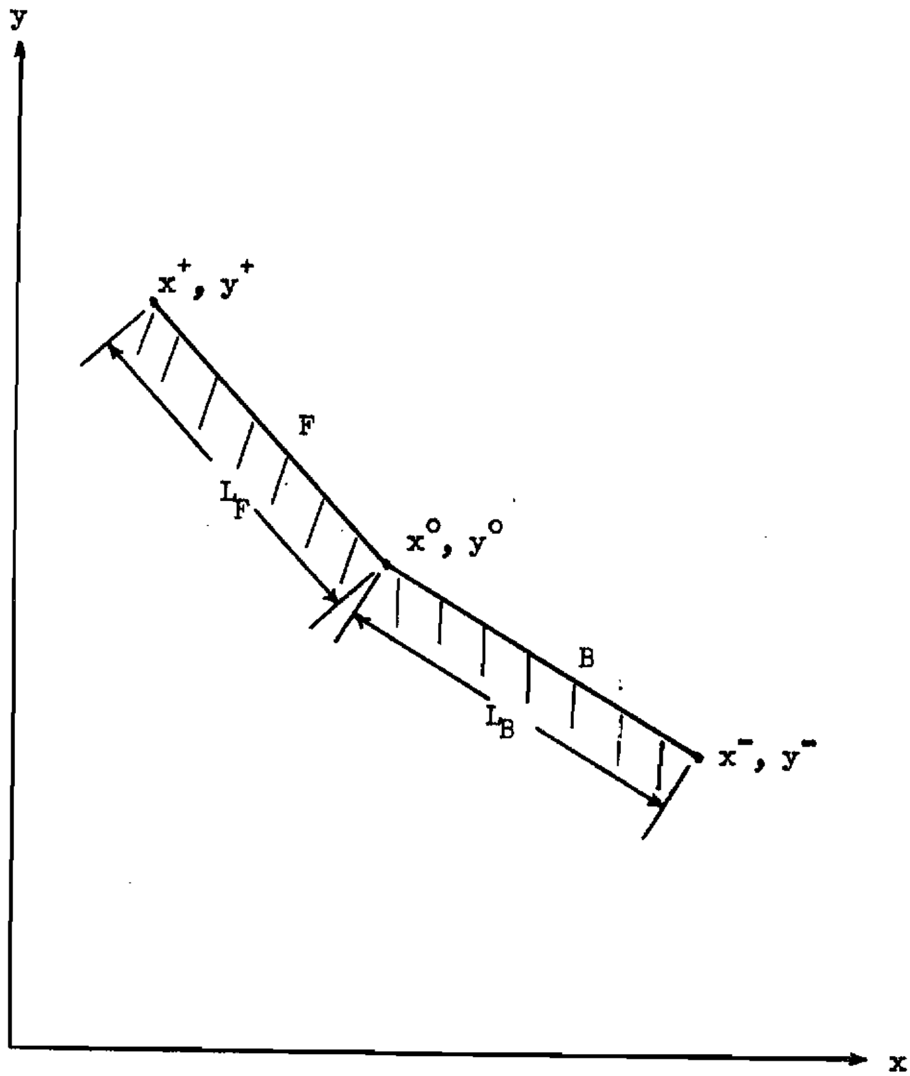


Figure 4. Notation Used in Calculation of Boundary Loads at Boundary Point x^o, y^o

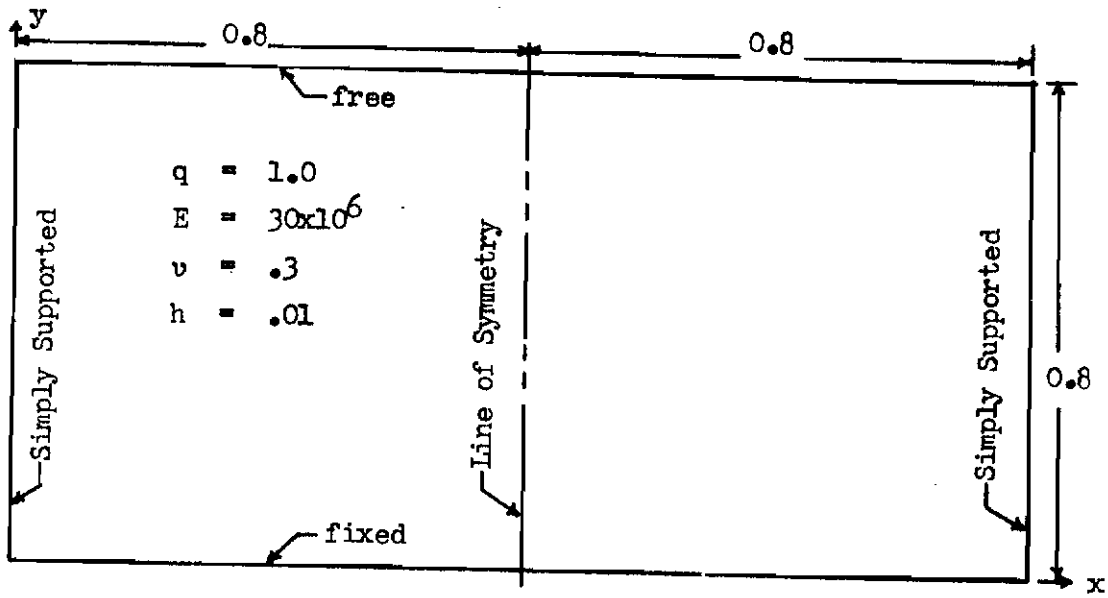
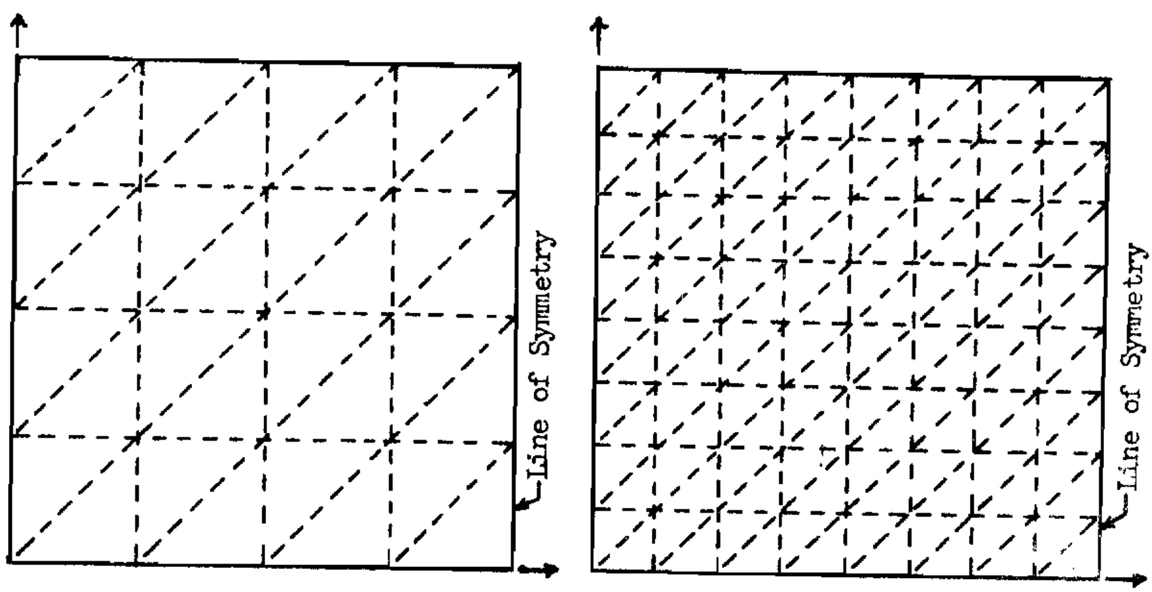


Plate Configuration



Coarse Grid

Fine Grid

Figure 5. Finite Element Representation of the Rectangular Plate

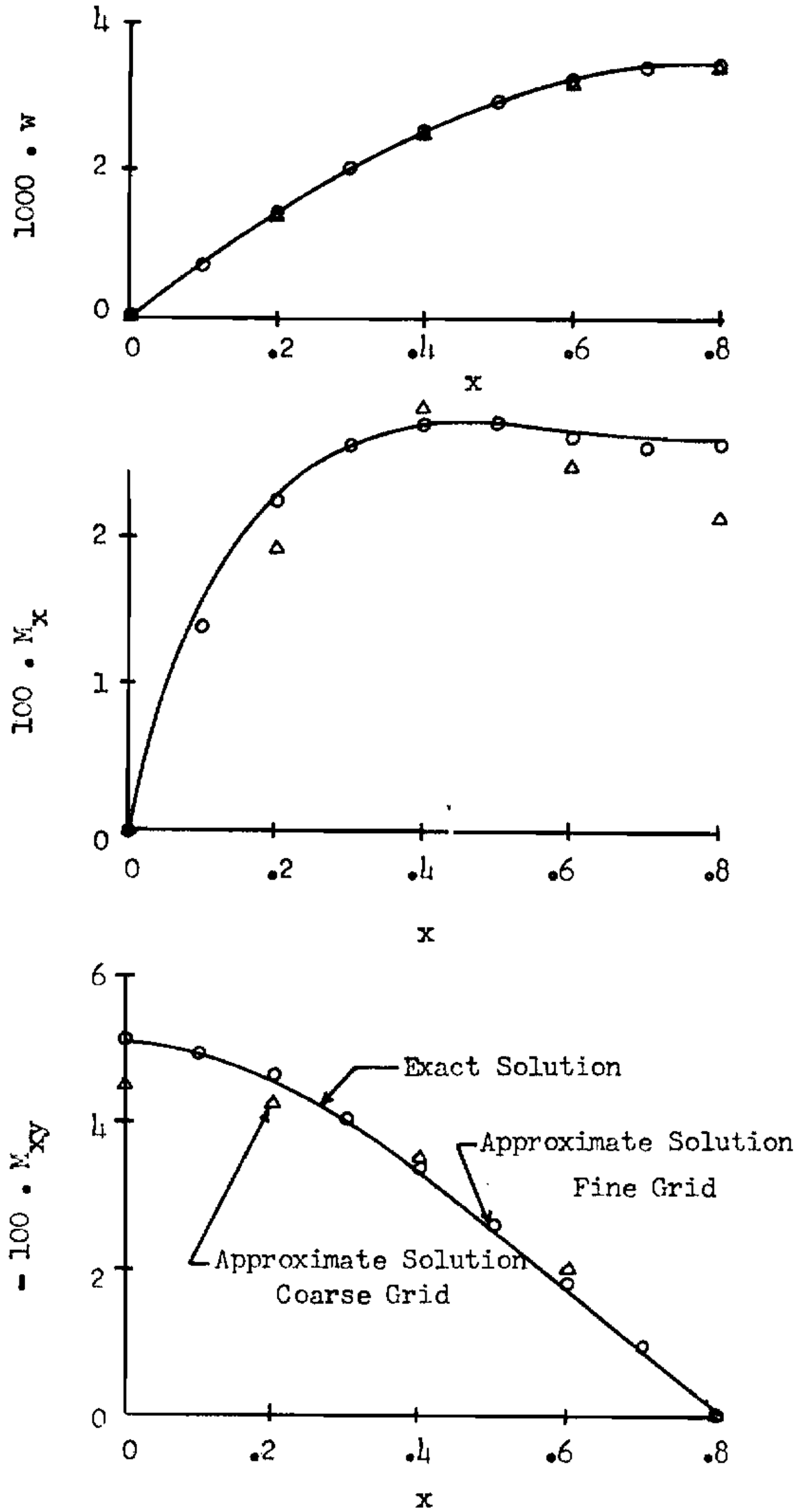


Figure 6. Approximate and Exact Values of w, M_x and M_{xy} Along Line y = 0.4 ~ Rectangular Plate

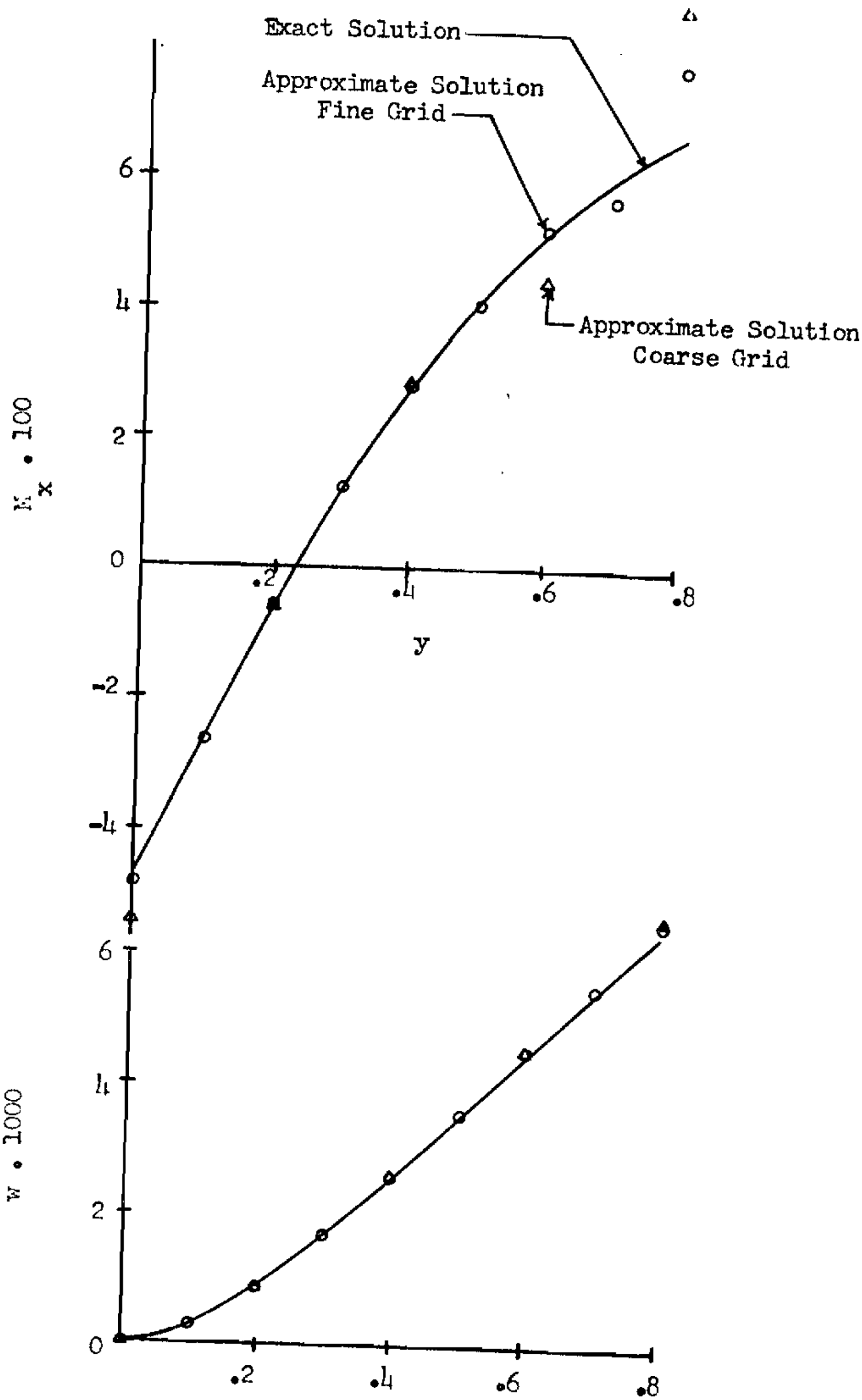


Figure 7. Approximate and Exact Values of w and M_x Along Line x = 0.4 ~ Rectangular Plate

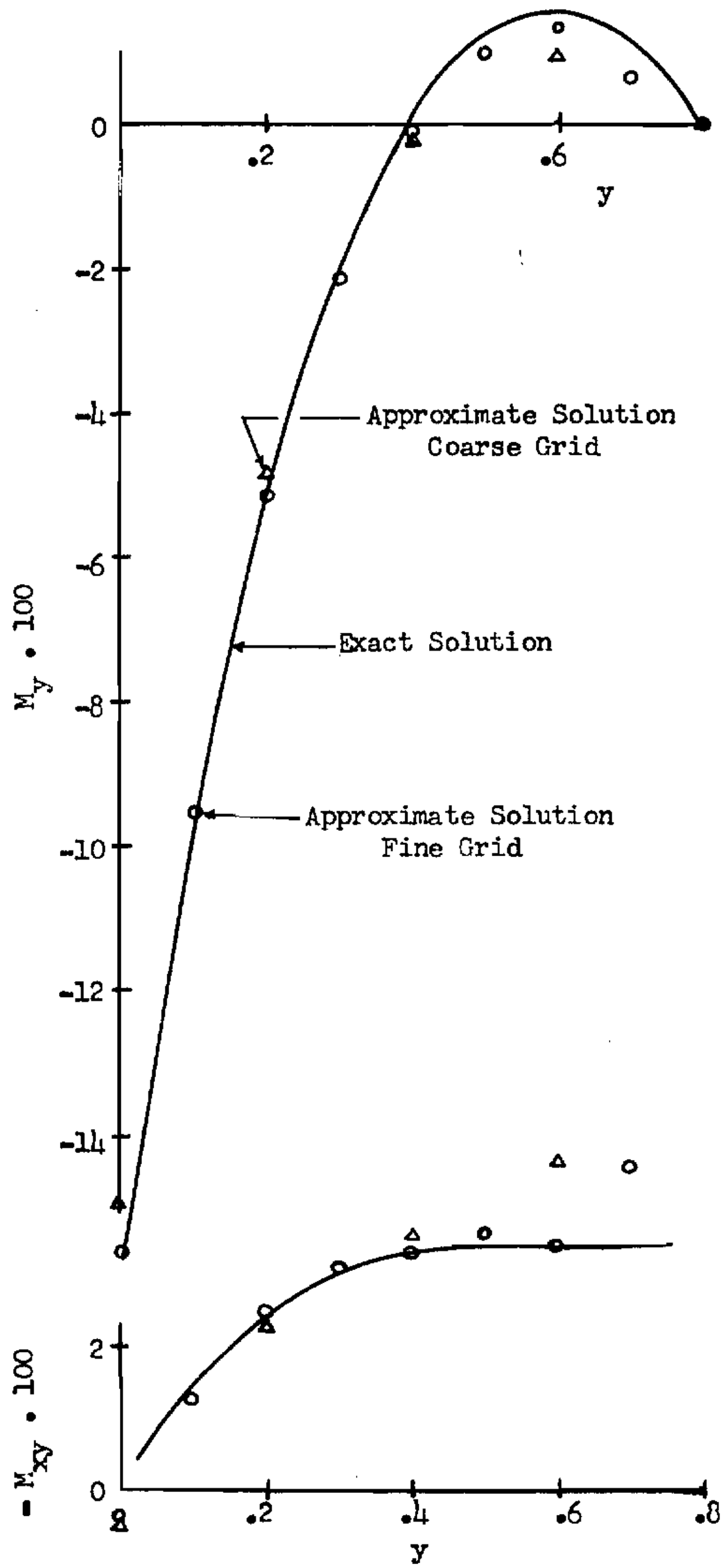


Figure 8. Approximate and Exact Values of M_y and M_{xy} Along Line x = 0.4 ∼ Rectangular Plate

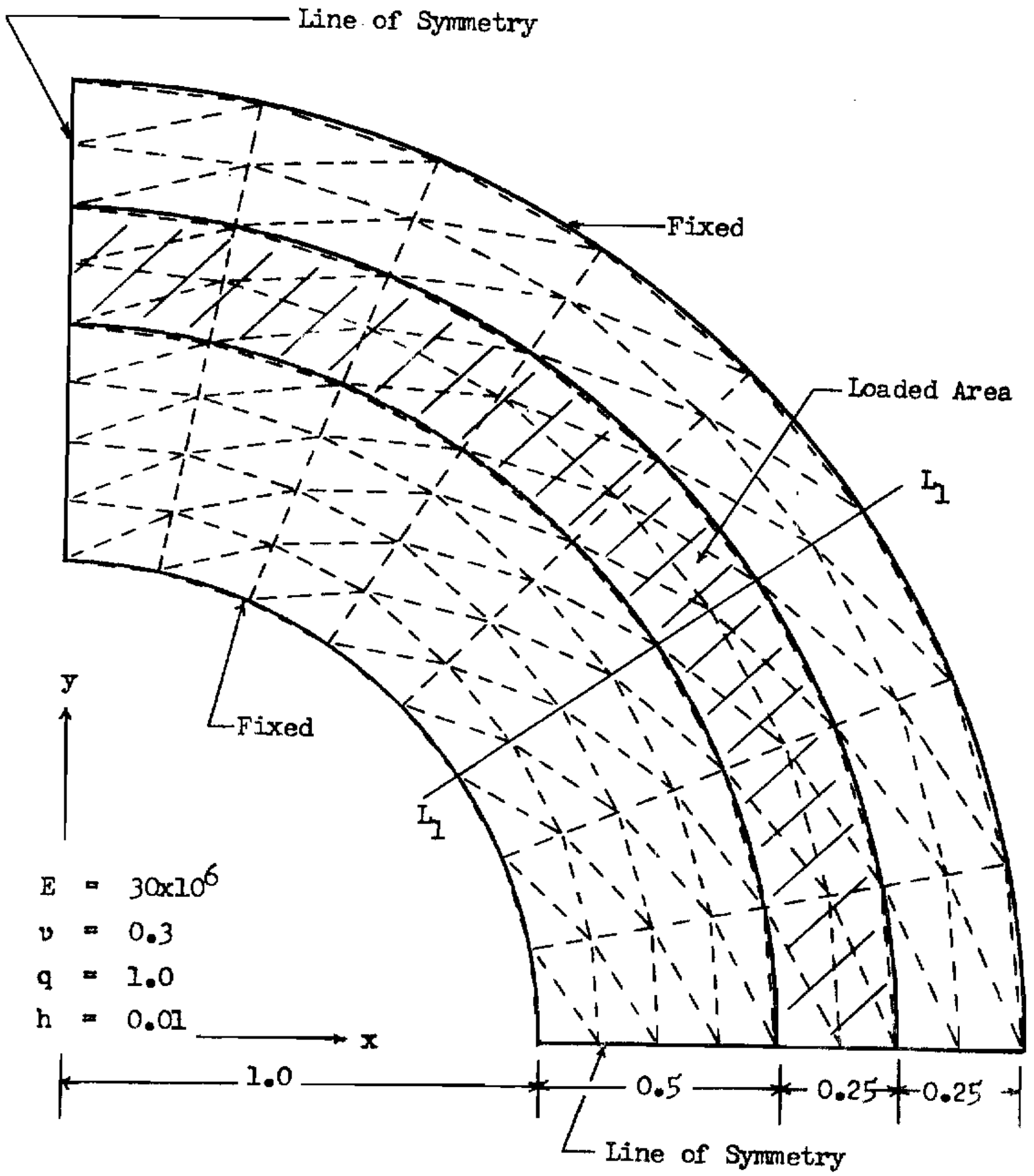


Figure 9. Finite Element Representation of a Quarter of the Circular Plate

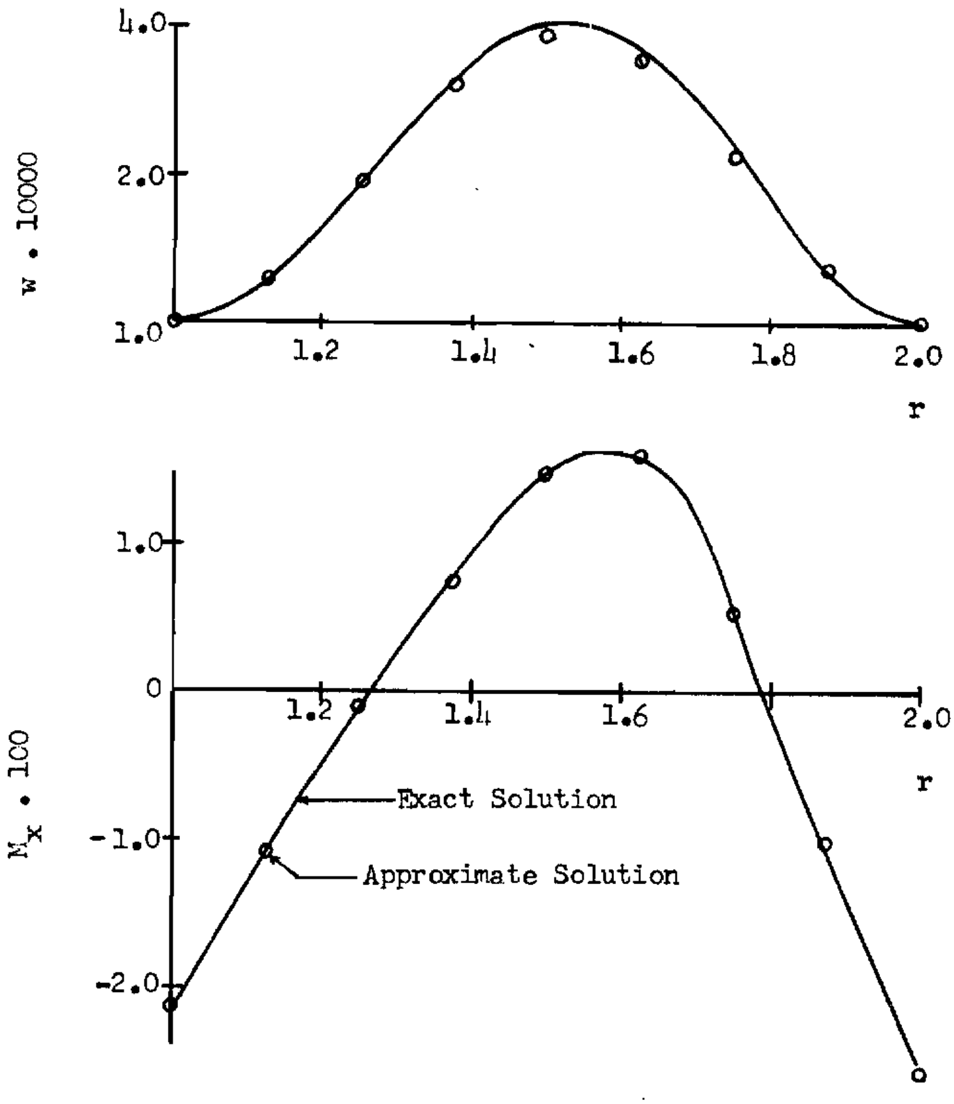


Figure 10. Approximate and Exact Values of w and M_x Along Line L_1 ~ Circular Plate

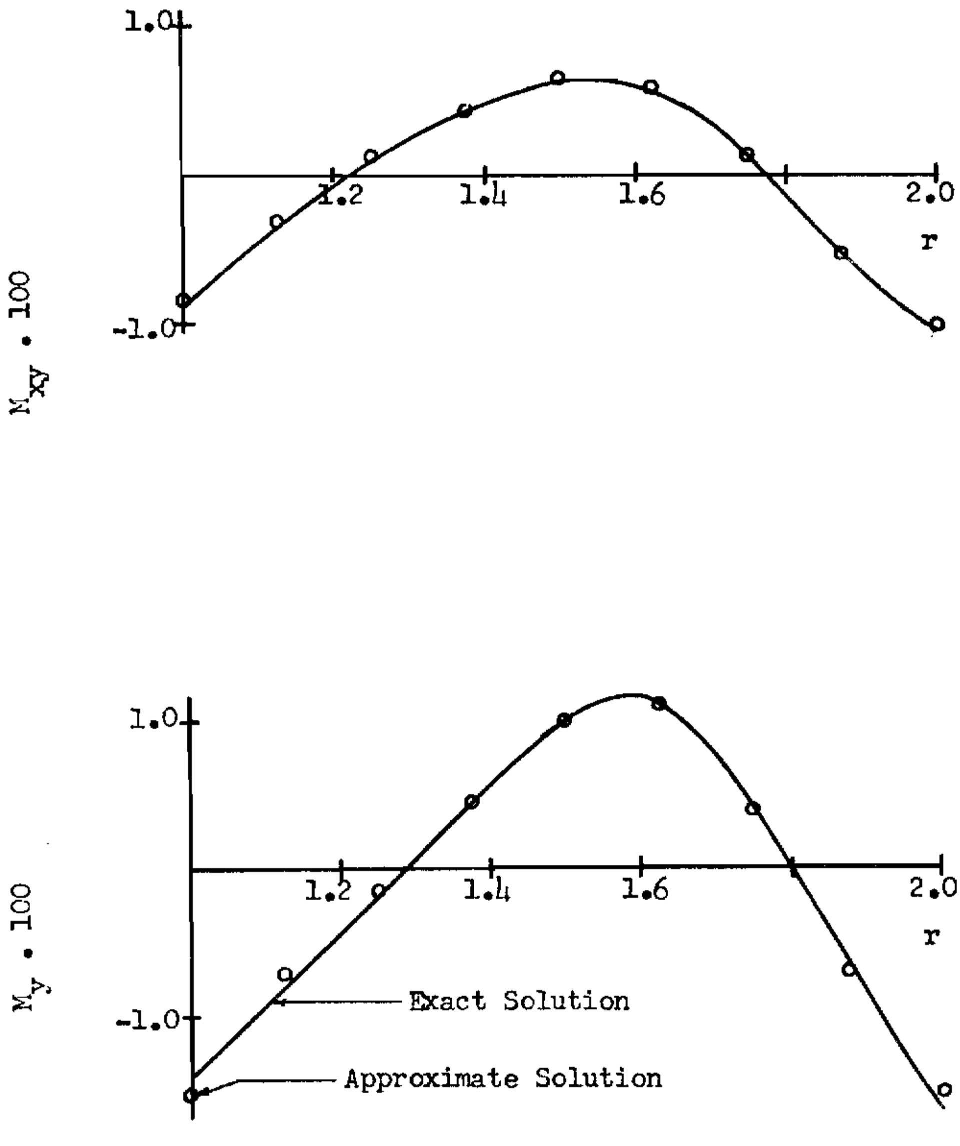
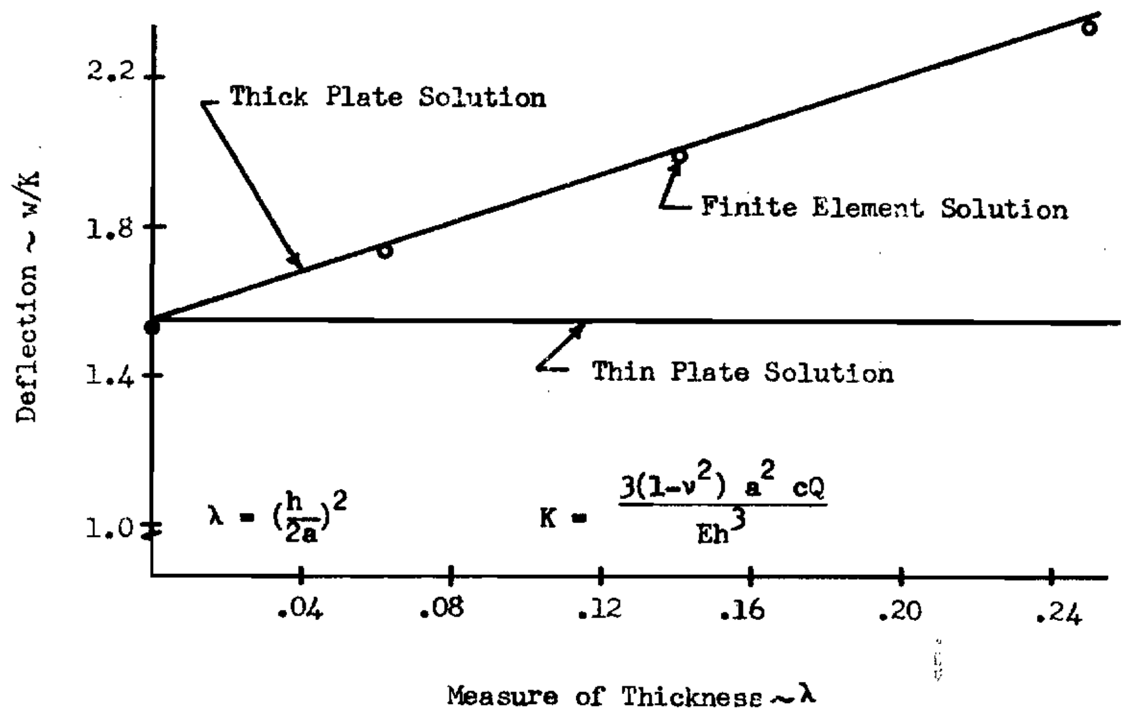
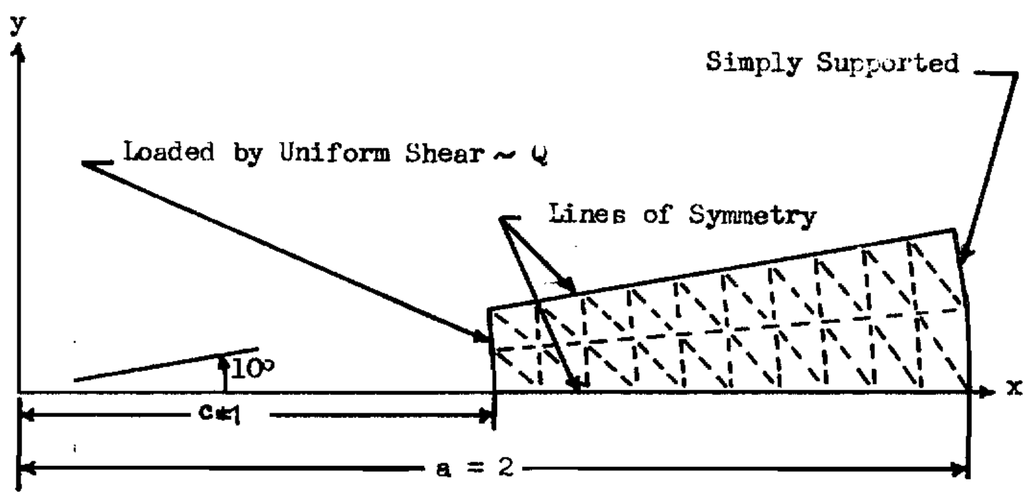


Figure 11. Approximate and Exact Values of M_{xy} and M_y Along Line L_1 ~ Circular Plate



Maximum Transverse Deflection



Finite Element Representation

Figure 12. Analysis of Thick Circular Plate