SESSION 5. STRUCTURAL APPLICATIONS

Session Chairman

D. S. Warren

Douglas Aircraft Co.
Long Beach, California

ELASTIC CRACK ANALYSIS BY A FINITE ELEMENT HYBRID METHOD*

T. H. H. Pian**, P. Tong*** and C. H. Luk****

Massachusetts Institute of Technology

The paper presents an efficient finite element method for evaluating the elastic stress intensity factors at the tip of a sharp crack. The method is based on a hybrid stress model for which special stress terms which represent the correct singularity behavior at the crack tip can be included. The magnitudes of such singular terms which are among the unknowns of the final matrix equations are, in fact, the stress intensity factors to be evaluated. Example solutions include plane stress cracks of both the opening type (Mode I) and in-plane shear type (Mode II) for isotropic materials and of the opening type for anisotropic materials.

---

*Work described in this paper was supported in whole by the Air Force Office of Scientific Research under Contract F44620-67-C-0019

**Professor of Aeronautics and Astronautics
***Associate Professor of Aeronautics and Astronautics
****Graduate Research Assistant

SECTION I

INTRODUCTION

The fundamental quantity to be considered in fracture mechanics is the "stress intensity factor" (Reference 1). For example, for an elastic plane stress or plane strain problem, in the immediate vicinity of the tip of a through crack the stresses varies as $1/\sqrt{r}$ where r is the radial coordinate of any point in the plane with the crack tip as the origin. The coefficient of this singular term is a measure of the strength of the singularity and is called the stress intensity factor k. It has a unit of ksi $\sqrt{\text{in}}$. The two types of crack governing the plane stress and plane strain problems are: (a) Mode I or the opening mode (b) Mode II or the in-plane shear mode (Figure 1). Another type of crack which involves out-of-plane is classified as Mode III crack.

For the two types of crack in the plane stress and plane strain problems, the singular terms of the stress distribution for isotropic materials are given respectively by:

Mode I:

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{k_I}{\sqrt{2r}} \begin{bmatrix} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \\ \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \end{bmatrix} \tag{1}$$

Mode II:

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{k_{II}}{\sqrt{2r}} \begin{bmatrix} -\sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \\ \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \end{bmatrix} \tag{2}$$

where θ is the angle measured from the x-axis; k_I and k_{II} are directly proportional to the magnitude of applied loading and is dependent on the geometry of the structure, the size and shape of the crack, and the nature of the applied loading. For an infinite sheet with a straight crack of length 2a under uniform tensile stress σ along the direction normal to the crack, the stress intensity factor k_I is equal to $\sigma\sqrt{a}$. The stress intensity factors for a single crack or for a crack array in an infinite sheet under different loading conditions can be determined directly using, for example, the complex function approach. However, for finite structures and for irregular cracks, approximate methods must be employed for evaluating the stress intensity factors. If the stress distribution near the tip of the crack can be estimated, then a fitting

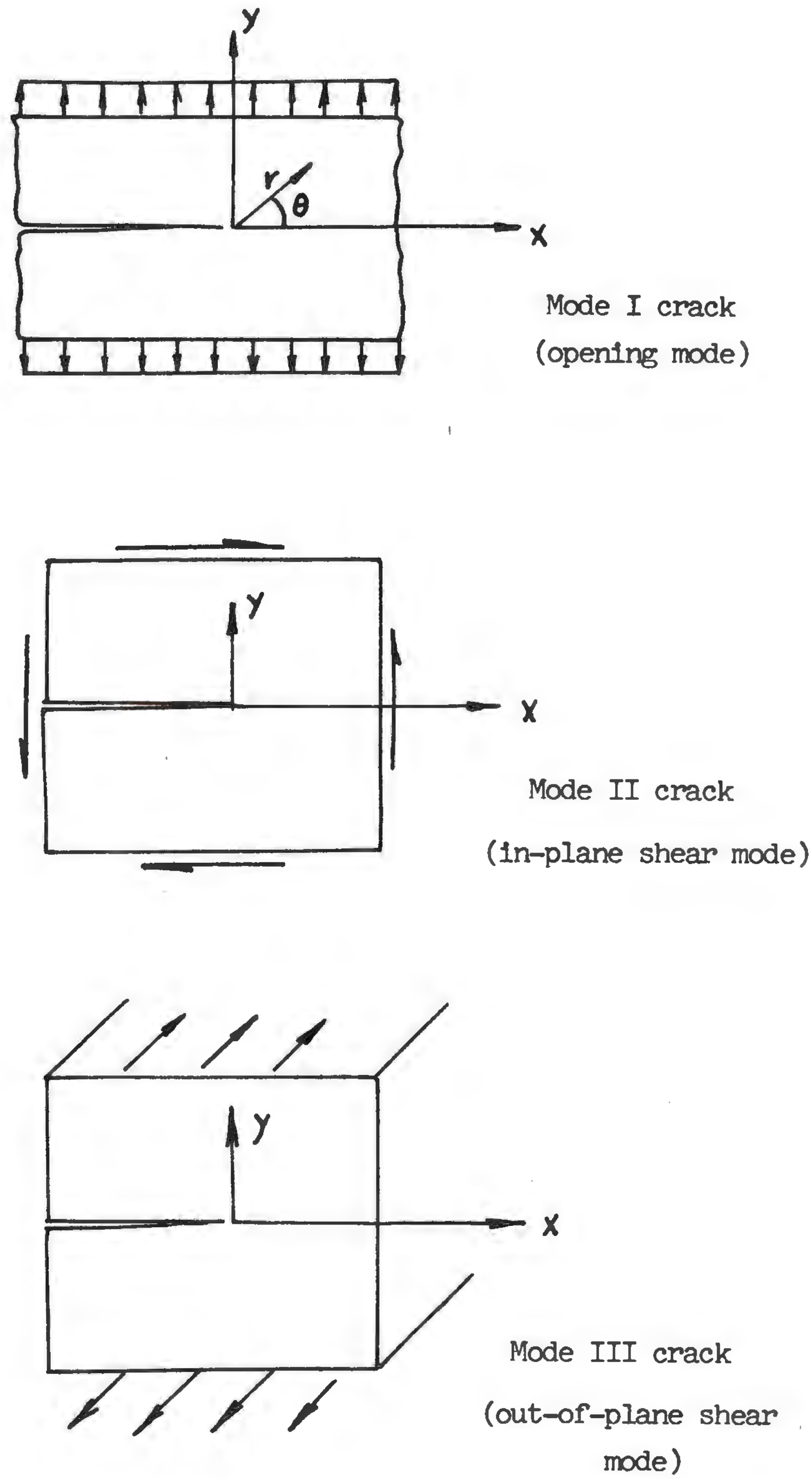


Figure 1. Three types of cracks

with a function of $1/\sqrt{r}$ shape will permit the determination of the quantity k.

The displacement field in the vicinity of the crack tip is also related to the stress intensity factors. For the two types of crack in plane stress and plane strain states, the displacement components u and v in the vicinity of the crack tip are given by,

Mode I:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{k_I (2r)^{1/2}}{8G} \begin{bmatrix} (2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \\ (2\kappa+1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2} \end{bmatrix} \quad (3)$$

Mode II:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{k_{II} (2r)^{1/2}}{8G} \begin{bmatrix} (2\kappa+3)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \\ -(2\kappa-3)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \end{bmatrix} \quad (4)$$

where G is the shear modulus and κ takes the value $(3-4\nu)$ for plane strain state; and, $(3-\nu)/(1+\nu)$, for plane stress state, ν being the Poisson's ratio. It is seen that the stress intensity factor k can also be evaluated by examining the displacements field near the crack tip. For example, for the Mode I crack a particularly convenient displacement component to be used to determine the quantity k_I is the opening of the crack.

Irwin (Reference 1) has shown that the stress intensity factor is related to the strain-energy release rate $\mathcal{G}$, i.e., the change in strain energy in the structure per unit distance of crack extension. For the two modes in the plane stress and plane strain problems, the relationship is given by

Mode I:

$$\mathcal{G}_I = \frac{\pi(\kappa+1)k_I^2}{8G} \quad (5)$$

Mode II:

$$\mathcal{G}_{II} = \frac{\pi(\kappa+1)k_{II}^2}{8G} \quad (6)$$

It is seen that a third approach for estimating the stress intensity factor is to determine the strain energy release rate. Another quantity which has been shown to be equal to $\mathcal{G}$ is the so-called J-integral, or a line integral involving the strain energy and the displacement gradient around any closed boundary surrounding the crack tip (Reference 2).

The critical value of k for which a crack becomes unstable is designated as k_c. The critical value of k_I is sometimes referred to as the fracture toughness k_{I_c}. The fracture toughness is a material property and is a function of plastic zone behavior at the tip of crack; hence it is related to the ductility of the material. For design purpose, the fracture toughness k_{I_c} is considered as an equivalent or fictitious stress intensity factor corresponding to the point of crack extension instability and is determined by assuming the material as elastic. When the fracture toughness k_{I_c} for a given material and plate thickness has been determined by tests, the estimation of the critical state of stress at the crack extension instability then is reduced simply of the determination of the elastic stress intensity factor k_I. The use of the concept of elastic stress intensity factor in structural design is valid for high strength materials used in aerospace applications such as aluminum alloy, steel and titanium, for which the plastic zone size at the tip of the crack is small in comparison with the crack length.

Elastic stress intensity factors for a number of crack configurations have been catalogued in various texts (Reference 3 and 4). For many configurations which involve cracks in an infinite plate closed form solutions for the stress intensity factor are available. Many of the cases given in Reference 3 were treated by approximate methods such as (1) the boundary collocation of stress functions (2) the conformal mapping technique and (3) continuous dislocation arrays and singular integral equations (Reference 2).

Practical fracture mechanics analysis often calls for the estimation of stress intensity factors for structural elements of irregular geometry and design. For such problems the use of the approximate methods listed above would require considerable algebraic manipulation for the set-up of each individual problem. The finite element method which has the advantage of easy adaptation to modern computer technology for obtaining a complete analysis of complex structures with very simple input instructions certainly becomes the most logic technique for the crack analysis.

The finite element method is also versatile in many other aspects. For example, the extension of the finite element analysis of isotropic materials to that of anisotropic materials usually involves the change of only a few cards in the complete computer program. The extension of stress analysis under mechanical loading to that under thermal loading again does not require a very extensive program revision. For the analysis of the stress distribution near a crack in a general 3-dimensional solid, the finite element method appears to be the only feasible scheme.

There have been a number of papers and reports dealing with the use of finite element methods for evaluating the elastic stress intensity factor of Mode I cracks (References 5 to 9). All of these are based on the assumed displacement approach. In the solutions of References 5, 6 and 7, the finite elements used at the tip of the crack contain no singularities in strains and stresses. As shown in Reference 10, in such a case, no matter how high the order of the polynomials is used for the assumed displacement functions, the rate of convergence of the finite element solution remains proportional to $\sqrt{\varepsilon}$ where ε is the element size. In order to obtain an accurate evaluation of the stress intensity factor, the size of the elements must be extremely small. Kobayashi and his co-authors (Reference 5), for example, indicated that in order to obtain sufficient accuracy for their assumed displacement finite element method the size of the element near the crack tip must be smaller than one-twentieth of the half-length of the crack. They, in fact, concluded that it is not reliable to obtain the stress intensity factor by fitting the approximate stress distribution obtained by the finite element method, and that, instead, the fitting of the crack surface displacement distribution is a more accurate scheme. This method of using crack surface displacement for determination of the stress intensity factor has also been followed by other authors (References 6 and 7). Anderson et. al. (Reference 6) also pointed out: the use of the energy release rate $\mathcal{G}$ for crack extension is a more efficient way of determining the elastic stress intensity factor. The energy method, however, is not applicable for the cases which involve the combination of Mode I and Mode II cracks (such as an inclined crack under uniform in-plane tensile stress or a crack in a general anisotropic material). The use of crack surface displacements is also not very convenient for problems which involve the singular behavior of both types of crack.

References 8 and 9 concern the use of special elements which contains singularities at the crack tip. Levy et. al. (Reference 9) used a special element which was derived mainly for elastic-perfectly-plastic material, hence it contains an $1/r$ type strain singularity. Thus, although they included in this report an elastic stress distribution solution obtained by using this special element, only a qualitative discussion of the accuracy for the stress intensity factor was given. Reference 8 involves the use of an assumed strain distribution which contains the correct singularity, but since the assumed displacement does not satisfy the interelement boundary compatibility, the convergence of the solution cannot be guaranteed.

The present paper consists of the formation of the finite element solution of the crack stress intensity problem using special finite elements derived by a procedure similar to the hybrid stress model (References 11 and 12). In the formulation of the hybrid stress model by Pian, a stress distribution in terms of a number of undetermined stress parameters is assumed within each element, and boundary displacements which are compatible with the neighboring elements are interpolated in terms of the nodal displacements. The stress parameters within each element are independent of those of the other elements and, hence, can be eliminated by the use of the Principle of Minimum Complementary Energy, and the resulting stiffness matrix of the element can be obtained. For the crack problem, special stress terms which represent the correct stress singularity behavior at the crack tip are included in addition to the non-singular stress terms for several elements in the neighborhood of the crack tip. The parameters for the non-singular stress terms which are independent for different elements can be eliminated first but those for the singular terms are kept, together with the nodal displacements, as the unknowns of the final matrix equations. The magnitudes of these singular terms are, in fact, the stress intensity factors k to be determined. Since in this method the singular part of the solution is extracted out in the correct analytical form, the nodal displacements of the finite element analysis correspond to a solution which has no singularity. Thus, the convergence behavior of this method remains the same as that of a common elasticity problem which contains no stress singularity.

Detailed discussion of the formulation of the assumed stress hybrid model for the crack analysis problem and numberical examples for the evaluation of this method are given in the following sections.

SECTION II

FORMULATION OF HYBRID STRESS MODEL

The variational principle which governs the assumed stress hybrid model is a modified principle of minimum complementary energy (References 12 and 13) for which the functional to be varied is

$$\Pi_{mc} = \sum_n \left(\int_{V_n} \frac{1}{2} S_{ijk\ell}\, \sigma_{ij}\, \sigma_{k\ell}\, dV - \int_{\partial V_n} T_i u_i \, dS + \int_{S_{\sigma_n}} \bar{T}_i u_i \, dS \right) \tag{7}$$

where

σ_{ij} = stress tensor

$S_{ijk\ell}$ = elastic compliance tensor

V = volume

S = surface

S_σ = portion of S over which the boundary tractions are prescribed

T_i = component of surface fraction

u_i = boundary displacement

n = nth finite element

∂V_n = entire boundary of nth element

The stress tensor σ_{ij} satisfies the equilibrium equations

$$\sigma_{ij,j} + \bar{F}_i = 0 \tag{8}$$

where $\bar{F}_i$ is the prescribed body force.

In the present finite element formulation for which the nature of the stress singularity near the crack tip is known it is a simple matter to include the correct singular behavior of the stress distribution in the assumed approximate functions for σ_{ij} for the elements in the vicinity of the crack tip. For simplicity in the present development we assume that the body forces are not present. The assumed functions for σ_{ij} are then divided into two parts, one of which contains no singularity while the other contains the proper singular behavior. In matrix form the assumed stresses are expressed as

$$\underset{\sim}{\sigma} = \underset{\sim}{P}\,\underset{\sim}{\beta} + \underset{\sim}{P_s}\,\underset{\sim}{\beta_s} \tag{9}$$

where $\underset{\sim}{P}\underset{\sim}{\beta}$ may be simply polynomials which satisfy the homogeneous stress equilibrium equations while $\underset{\sim}{P_s}\underset{\sim}{\beta_s}$ corresponds to the stress singularity terms. For example, $\underset{\sim}{\beta_s} = \{k_I \; k_{II}\}$ are the stress intensity factors for the Mode I and Mode II cracks and for isotropic materials the elements in $\underset{\sim}{P_s}$ matrix are the functions for these two types of crack given in Equations 1 and 2. For anisotropic materials the behavior of the stress singularity at the crack tip can also be obtained by the complex variable method (Reference 14). As a reference the singular stress terms for the opening type crack in an orthotropic panel is given in the Appendix. It should be noted that $\underset{\sim}{P_s}\underset{\sim}{\beta_s}$ represents a stress distribution which satisfies not only the equilibrium equations but also the compatibility equations.

The surface tractions for each element are related to the assumed stress distribution and can be expressed as

$$\utilde{T} = \utilde{R}\utilde{\beta} + \utilde{R_s}\utilde{\beta_s} \tag{10}$$

For elements away from the crack tip where the stress singularity effect is no longer important only the $\utilde{P}\utilde{\beta}$ parts appear in Equations 9 and 10. Similar to the procedure used in formulating the regular hybrid stress method the element boundary displacements $\utilde{u}_B$ are interpolated in terms of the nodal displacement q, i.e.

$$\utilde{u}_B = \utilde{L}\utilde{q} \tag{11}$$

Substituting of Equations 9, 10 and 11 into Equation 7 and realizing that the singular stress terms are included only in a certain number of elements, say p elements from n = 1 to n = p, in the vicinity of the crack tip, we obtain the following expression

$$\begin{aligned} \Pi_{mc} = \sum_{n=1}^{p} \Big(& \tfrac{1}{2}\utilde{\beta}^T \utilde{H}\utilde{\beta} + \utilde{\beta}^T \utilde{H_s}\utilde{\beta_s} + \tfrac{1}{2}\utilde{\beta_s}^T \utilde{H_{ss}}\utilde{\beta_s} \\ & - \utilde{\beta}^T \utilde{G}\utilde{q} - \utilde{\beta_s}^T \utilde{G_s}\utilde{q} + \utilde{\bar{Q}}^T \utilde{q} \Big) \\ + \sum_{n=p+1}^{m} \Big(& \tfrac{1}{2}\utilde{\beta}^T \utilde{H}\utilde{\beta} - \utilde{\beta}^T \utilde{G}\utilde{q} + \utilde{\bar{Q}}^T \utilde{q} \Big) \end{aligned} \tag{12}$$

where

$$\utilde{H} = \int_{V_n} \utilde{P}^T \utilde{S}\utilde{P}\, dV$$

$$\utilde{H_s} = \int_{V_n} \utilde{P}^T \utilde{S}\utilde{P_s}\, dV$$

$$\utilde{H_{ss}} = \int_{V_n} \utilde{P_s}^T \utilde{S}\utilde{P_s}\, dV$$

(continued on next page)

$$\underset{\sim}{G} = \int_{\partial V_n} \underset{\sim}{R}^T \underset{\sim}{L}\, ds$$

$$\underset{\sim}{G_s} = \int_{\partial V_n} \underset{\sim}{R_s}^T \underset{\sim}{L}\, ds \qquad (13)$$

$$\underset{\sim}{\bar{Q}}^T = \int_{S_{\sigma_n}} \underset{\sim}{\bar{T}}^T \underset{\sim}{L}\, ds$$

and m is the total number of elements.

The stationary conditions of the functional given by Equation 12 with respect to variations of $\underset{\sim}{\beta}$ which are independent from one element to the other then yields

$$\underset{\sim}{H}\,\underset{\sim}{\beta} + \underset{\sim}{H_s}\,\underset{\sim}{\beta_s} - \underset{\sim}{G}\,\underset{\sim}{q} = 0 \qquad \text{for} \quad n \leqq p$$

$$\underset{\sim}{H}\,\underset{\sim}{\beta} - \underset{\sim}{G}\,\underset{\sim}{q} = 0 \qquad \text{for} \quad n > p \qquad (14)$$

By solving for $\underset{\sim}{\beta}$ from Equation 14 and substituting back into Equation 12 we can express the functional π_{mc} in terms of the generalized displacement $\underset{\sim}{q}$ and the stress intensity factors $\underset{\sim}{\beta}_s$, i.e.

$$\begin{aligned}\pi_{mc} = \sum_{n=1}^{p} \Big(-\tfrac{1}{2}\underset{\sim}{q}^T \underset{\sim}{k}\,\underset{\sim}{q} - \underset{\sim}{\beta_s}^T \underset{\sim}{m}\,\underset{\sim}{q} \\ + \tfrac{1}{2}\underset{\sim}{\beta_s}^T \underset{\sim}{n}\,\underset{\sim}{\beta_s} + \underset{\sim}{\bar{Q}}^T \underset{\sim}{q} \Big) \\ + \sum_{n=p+1}^{m} \Big(-\tfrac{1}{2}\underset{\sim}{q}^T \underset{\sim}{k}\,\underset{\sim}{q} + \underset{\sim}{\bar{Q}}^T \underset{\sim}{q} \Big)\end{aligned} \qquad (15)$$

where

$$\underset{\sim}{k} = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G}$$

$$\underset{\sim}{m} = \underset{\sim}{G_s} - \underset{\sim}{H_s}^T \underset{\sim}{H}^{-1} \underset{\sim}{G} \qquad (16)$$

$$\underset{\sim}{n} = \underset{\sim}{H_{ss}} - \underset{\sim}{H_s}^T \underset{\sim}{H}^{-1} \underset{\sim}{H_s}$$

By expressing the element nodal displacements $\underset{\sim}{q}$ in terms of independent generalized global displacement $\underset{\sim}{q}^*$ and by realizing that $\underset{\sim}{\beta}_s$ is in common for the p element in the neighborhood of the crack tip we can write

$$\pi_{mc} = -\frac{1}{2}\underset{\sim}{q}^{*T}\underset{\sim}{K}\underset{\sim}{q}^* - \underset{\sim}{\beta}_s^T\underset{\sim}{M}\underset{\sim}{q}^* + \frac{1}{2}\underset{\sim}{\beta}_s^T\underset{\sim}{N}\underset{\sim}{\beta}_s + \underset{\sim}{\bar{Q}}^{*T}\underset{\sim}{q}^* \tag{17}$$

where $\underset{\sim}{K}$, $\underset{\sim}{M}$, $\underset{\sim}{N}$ and $\underset{\sim}{Q}^*$ are obtained by assembling the corresponding element matrices.

The stationary condition of the expression of π_{mc} with respect to $\underset{\sim}{q}^*$ and $\underset{\sim}{\beta}_s$ then yields

$$\underset{\sim}{K}\underset{\sim}{q} + \underset{\sim}{M}^T\underset{\sim}{\beta}_s = \underset{\sim}{\bar{Q}}^*$$

and

$$\underset{\sim}{M}\underset{\sim}{q}^* - \underset{\sim}{N}\underset{\sim}{\beta}_s = 0 \tag{18}$$

The solution of Equation 18 then yields the nodal displacements $\underset{\sim}{q}^*$ and the stress intensity factors $\underset{\sim}{\beta}_s$.

A few remarks should be made concerning this hybrid model formulation:

(1) The present formulation leads to a matrix mixed method with nodal displacements and some stress parameters as unknowns. Thus, although this formulation has the advantage of yielding the unknown stress intensity factors directly, it is not compatible with the most commonly used finite element formulation, i.e. the matrix displacement method. The formulation, however, can be easily modified by first grouping the p elements at the crack tip and then eliminating the stress parameters $\underset{\sim}{\beta}_s$ of the singular terms as explicit unknowns among these elements. An element stiffness matrix can thus be evaluated for the super finite element consisting of this group of elements. Thus except that this super element will have much larger number of degrees of freedom than that of the remaining conventional elements, the formulation now becomes a matrix displacement method.

(2) For most crack problems the crack surface is stress free. Thus, for those elements which contain the crack surface, the assumed functions $\underset{\sim}{P}\underset{\sim}{\beta}$ should be so chosen that the stress free boundary conditions are observed. Such step has been demonstrated to offer considerable improvement in the accuracy of the finite

element solutions by the hybrid stress model particularly when the number of finite elements is small (References 15 and 16). It will be shown in the following section that this step is even more crucial for the present formulation. Since the elastic stress intensity factor refers to the stress distribution at the immediate vicinity of the crack tip it is essential to provide an accurate description of the stresses for the elements near the crack tip.

(3) According to Equations 3 and 4 the displacements u and v corresponding to the stress singularity term should be proportional to the square root of r. This suggests that an appropriate interpolation function of the boundary displacements for the elements in the vicinity of the crack tip is one which includes a $\sqrt{r}$ term.

(4) In order to obtain a converging solution when the element size becomes smaller and smaller the singular stress terms must be included in a fixed region and not in only a certain fixed number of elements. For example, if the singular stress terms is included only in the elements which contain the crack tip, then if the element size approaches zero the stress singularity effect will completely disappear and the solution will certainly not converge to a correct stress intensity factor.

(5) Since $\underset{\sim}{P}_s \underset{\sim}{\beta}_s$ satisfies both compatibility and equilibrium equations, the volume integral for $\underset{\sim}{H}_{ss}$ (Equation 13) can be transformed into a line integral

$$\underset{\sim}{H}_{ss} = \int_{\partial V_n} \underset{\sim}{R}_s \underset{\sim}{L}_s \, d\ell \tag{19}$$

where L_s is a boundary displacement matrix corresponding to $\underset{\sim}{S}\underset{\sim}{P}_s$. In fact, in the case of isotropic materials L_s can be evaluated from Equations 3 and 4. Since $\underset{\sim}{R}_s$ and $\underset{\sim}{L}_s$ are proportional to $1/\sqrt{r}$ and $\sqrt{r}$ respectively the integrand in Equation 19 involves only sine and cosine functions in θ. Thus, although in the volume integral form the integrand in H_{ss} is proportional to $1/r$, in the transformed form the singularity disappears.

In the similar manner, since $\underset{\sim}{P}\underset{\sim}{\beta}$ satisfies the equilibrium equations the volume integral for $\underset{\sim}{H}_s$ can also be transformed into a surface integral,

$$\underset{\sim}{H}_s = \int_{\partial V_n} \underset{\sim}{R} \underset{\sim}{L}_s \, ds \tag{20}$$

It is seen that the integrands of $\underset{\sim}{H}_s$ and $\underset{\sim}{G}_s$ are proportional to $\sqrt{r}$ and $1/\sqrt{r}$ respectively. When the integrals in these two matrices are evaluated numerically special care must be taken to insure the accuracy of the integration when the range of r starts from zero. In the present formulation a transformation of variable is introduced prior to the application of the Gaussian quadrature. For example, for a line segment passing through r=0 the integration is carried out in terms of a variable z defined by $r=z^2$; thus, $dr=2zdz$, $\sqrt{r}dr=2z^2dz$ and $dr/\sqrt{r}=2dz$.

SECTION III

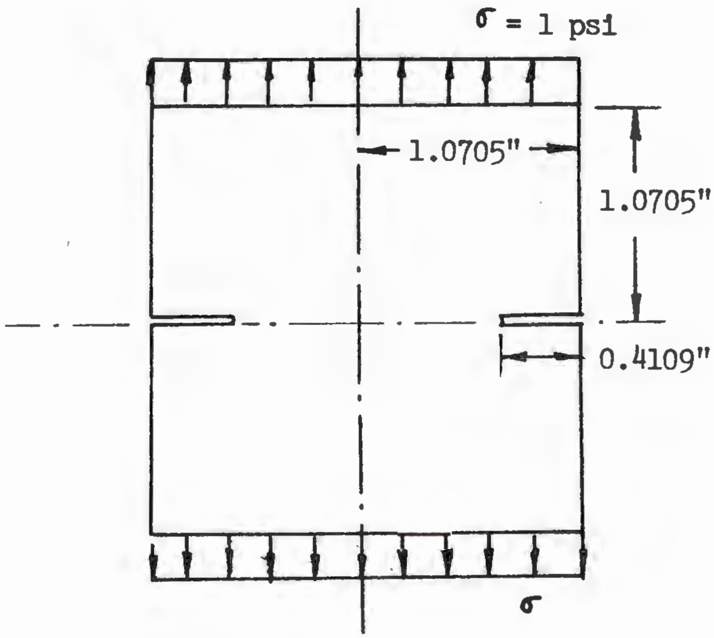
RESULTS OF DETERMINATION STRESS INTENSITY FACTORS

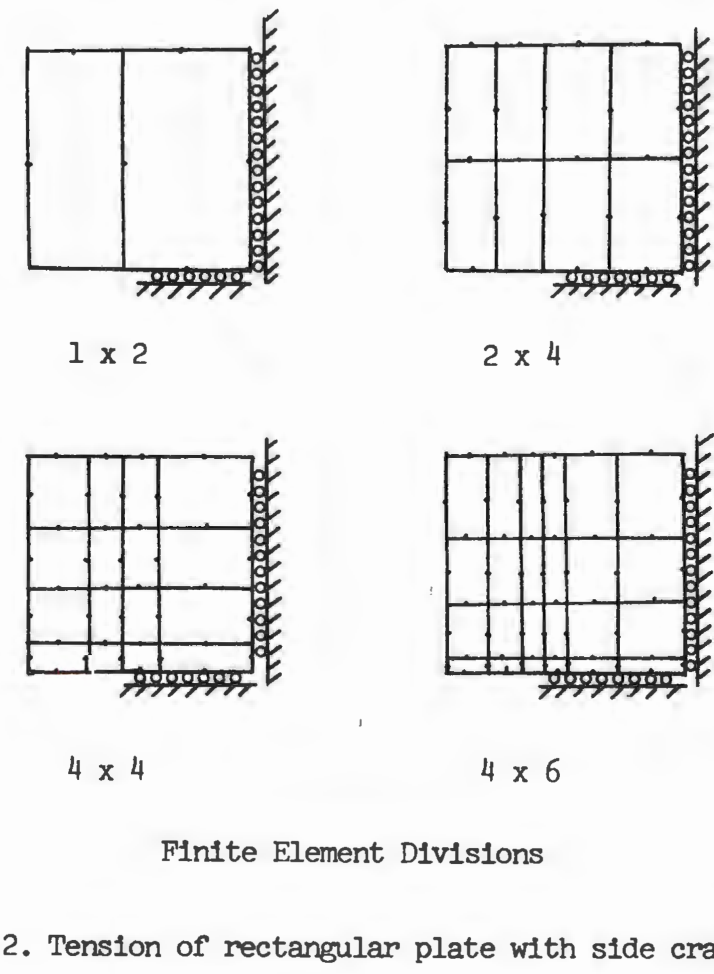
To illustrate the method outlined in this paper several numerical results are presented in this section.

1. The first problem concerns a rectangular panel with symmetric edge cracks under in-plane tension (Figure 2). This problem has been solved by Bowie (Reference 17) using the complex variable and conformal mapping technique. For the present finite element solutions the elements eight-node rectangular elements of various mesh sizes were used. Since the structure possesses double symmetry only one quarter of the panel is needed for the finite element analysis, and the only stress singularity term is the Mode I type. In all the solutions the stress singularity term is included only in the two elements at the crack tip.

In these solutions the non-singular stress terms include complete cubic functions in x and y for σ_x, σ_y and σ_{xy}. This means that 18 β's are used for those elements which do not have stress free boundary conditions or for which such conditions are not enforced. In the case that the stress free boundary conditions are enforced there remains only 10 independent β's. Since the elements used are rectangular elements with three nodes along each side, quadratic displacements are assumed for the boundaries of most of the elements. Only for the two elements at the crack tip the displacements are also assumed to be proportional to $a+b\sqrt{r}+cr$. As we have pointed out earlier, the $\sqrt{r}$ term is included here because it corresponds to the singular stress term.

Finite element solutions have been obtained using four different element sizes and arrangements as shown in Figure 2. The values of the stress intensity factor k_I were determined from the finite element solutions by three different methods: (1) by direct calculation of the stress coefficient β_s, (2) by using Equation 3 and considering the crack opening displacement v at the lower left corner of the element at the left of the crack tip, (3) by calculating the strain energy release rate $\mathcal{G}$ from the finite element solutions of two problems with the crack lengths differ by $\Delta\ell=0.05$. Values of k_I have been evaluated for the four element sizes by the first two methods. The third method was





Finite Element Divisions

Figure 2. Tension of rectangular plate with side cracks

applied only to the case of 2x4 mesh. These values are listed in Table I. By comparing with the analytical solution by Bowie we can conclude that all three methods provide about the degree of accuracy. The first method, of course, is the most direct one. The error of k_I obtained by the first method is less than 0.5% when the 4x6 mesh is used. It is only about 6% when the 1x2 mesh is used.

Table I also includes the result of an investigation of the effects of different assumed stresses and boundary displacements on the solution. The effect of not including the $\sqrt{r}$ term in the boundary displacement is apparently not very large although the solution for the case of 4x6 mesh is rather poor when compared with the result obtained when the $\sqrt{r}$ term is included. However, if the stress force boundary conditions for the element at crack tip are not enforced the resulting k_I would be completely unacceptable.

The second example consists of a rectangular isotropic panel (2bx2c) with a center crack of length 2L and four edge cracks of length a as shown in Figure 3. The stress intensity factors k_I and k_{II} for the edge cracks due to uniform tension loading are determined by the finite element method. The effect of the length of the center crack on k_I and k_{II} is investigated. In each solution a quarter of the panel is subdivided into forty-two 8-node elements which include special elements around the tips of the crack. A plot of k_I and k_{II} versus L/c is given in Figure 3.

The third example consists of a rectangular orthotropic panel (2bx2c) with a center crack of length 2L as shown in Figure 4. The crack stress intensity--k_I due to uniform tension loading are determined by the finite element method using twenty-four 8-node rectangular elements for a quarter of the panel. In formulating the special elements at the tip of the crack the stress singularity terms are given by Equation A-3 of the Appendix. Values of k_I are obtained for different ratios of the principal Young's moduli, Ex/Ey. The results are compared with the numerical results obtained by Bowie and Freese (Reference 19) by complex variable approach and extension of the modified mapping-collocation techniques. In determining k_I for the orthotropic panels in Reference 19, the parameter η_1 in Equation A.2 is kept as unity while η_2^2 $(=E_x/E_y)$ is left as a variable. For the present finite element analysis the individual orthotropic elastic constants are needed. A reference value of E_x is first chosen, E_y can then be determined by Equation A.2. The Poisson ratio ν_{xy} is then taken as 0.38 and the shear modulus G_{xy} can be calculated. It is seen that the comparison between the present finite element solutions and the complex variable solutions is, indeed, excellent.

TABLE I

STRESS INTENSITY FACTOR k_I FOR A FINITE PLATE WITH SYMMETRIC EDGE CRACKS BY USING SPECIAL EIGHT-NODE HYBRID STRESS ELEMENTS AT THE CRACK TIP

		Stress Intensity Factors k_I				
Assumed Element Stresses		Stress Free State Enforced				Stress free state not enforced
Assumed Boundary Displacements		$\sqrt{r}$ term included			$\sqrt{r}$ term not included	$\sqrt{r}$ term included
Methods for evaluating k_I*		(1)	(2)	(3)	(1)	(1)
Mesh Arrangement	Degrees of freedom					
1 x 2	24	0.840	0.904		0.823	0.508
2 x 4	72	0.822	0.797	0.780		
4 x 4	128	0.812	0.784			
4 x 6	184	0.791	0.790		0.838	0.634
Solution by complex variable method (Bowie)		0.793				

*The three methods used for evaluating k_I from the finite element solutions are:

(1) Direct calculation of stress coefficient

(2) From crack opening displacement v (at $\theta = \pi$)

(3) From strain energy release rate $\mathcal{G}$ (from two finite element solutions of two slightly different crack lengths).

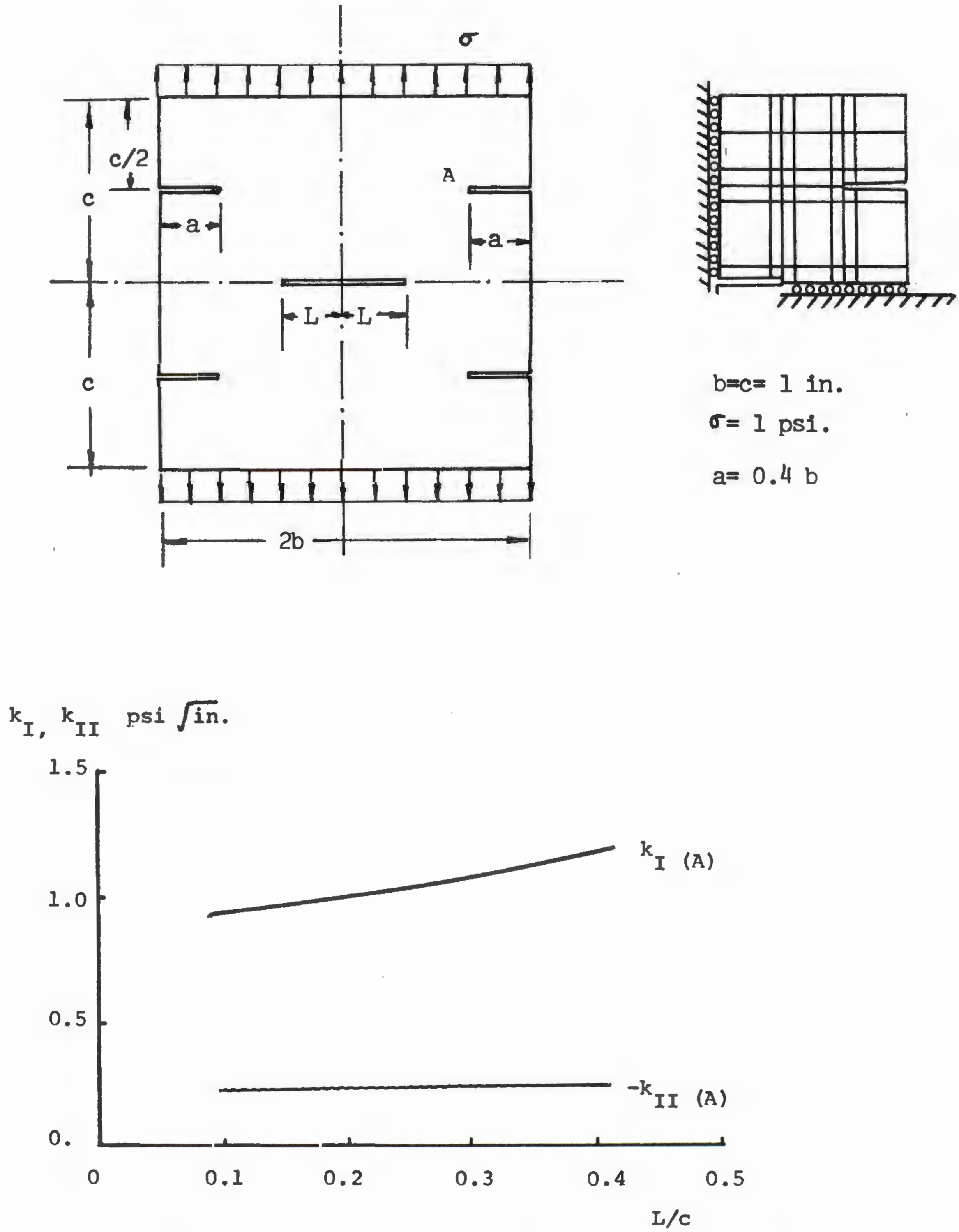


Figure 3. Effect of an adjacent crack on the stress intensity factors of a given crack

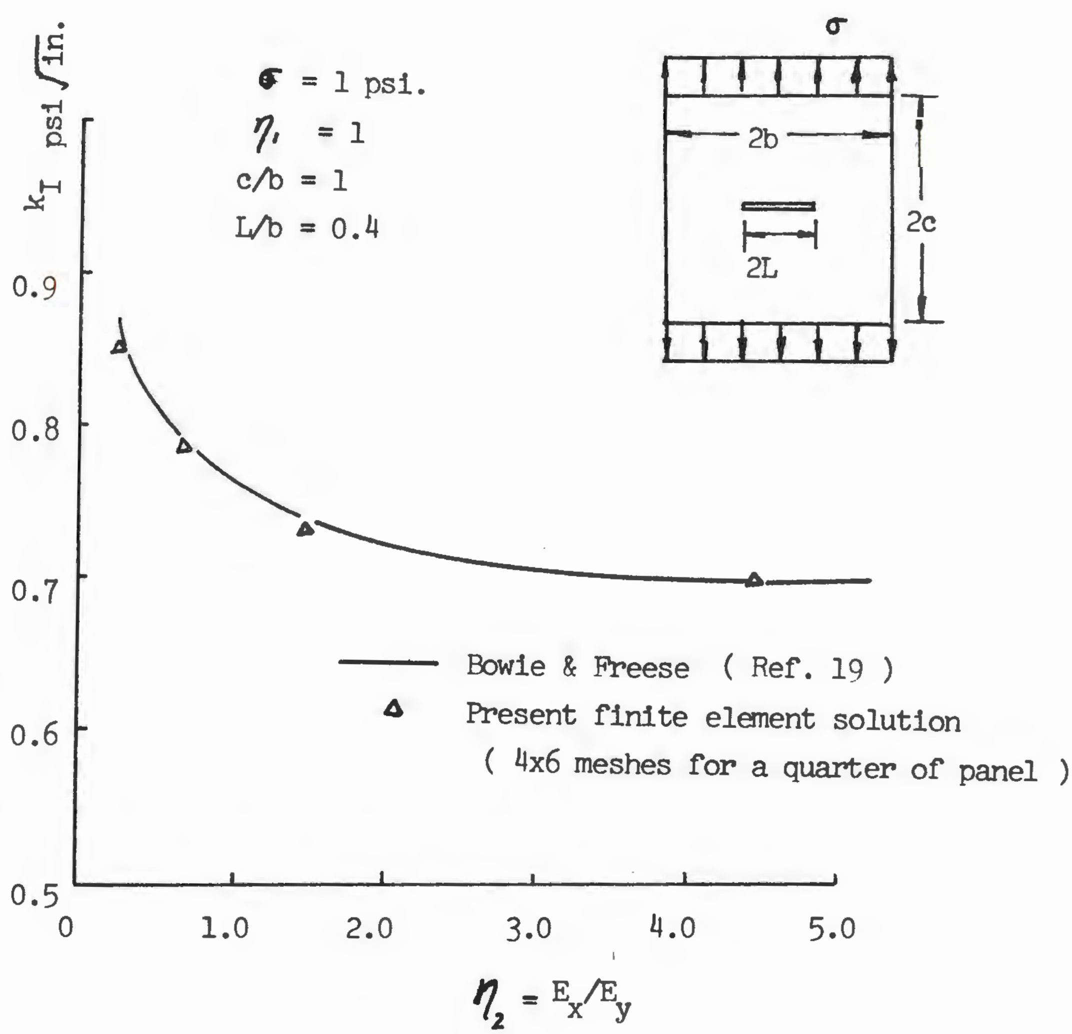


Figure 4

Stress Intensity Factor k_I of Orthotropic Plate with a Center Crack

SECTION IV

CONCLUSIONS AND REMARKS

The following conclusions and remarks can be made for the present method for evaluating the stress intensity factors of sharp cracks:

(1) The hybrid stress model is an ideal scheme for setting up a finite element method for evaluating the stress intensity factor. It can easily take into account the stress singularity in the elements near the crack tip. It is very convenient to take the correct behavior of boundary displacements and prescribed boundary traction into account.

(2) The numerical results indicate that fairly accurate estimation of the stress intensity factors can be obtained even when the size of the eight-node rectangular element is as large as the length of the crack.

(3) The present formulation leads to a matrix mixed method, but it can be easily modified to fit the conventional finite element programs based on the matrix displacement method.

(4) The method is suitable for isotropic as well as anisotropic materials and is adaptable to problems for which behaviors of Mode I and Mode II cracks exist simultaneously.

(5) For plates and shells with through cracks there also exist stress singularities at the crack tip and corresponding stress intensity factors can be defined. Hartranft and Sih (Reference 18) have concluded that for such problems one must employ the Reissner's refined plate theory which considers the transverse shear effect instead of the ordinary Kirchhoff plate theory. Since the hybrid stress model has been shown to be a convenient method for treating the transverse shear effect (Reference 11) it is also a logic scheme for the problem of through cracks in plates and shells.

SECTION V

REFERENCES

1. Irwin, G. R. "Fracture Mechanics," Structural Mechanics, Proc. First Symposium on Naval Structural Mechanics, Edited by J. N. Goodier and N. J. Hoff, Pergamon Press, 1960, pp. 557-594.

2. Rice, J. R. "Mathematical Analysis in the Mechanics of Fracture," Fracture - An Advanced Treatise, Vol. I, Edited by H. Liebowitz, Academic Press, 1968, pp. 191-311.

3. Wilhem, D. P. "Fracture Mechanics Guidelines for Aircraft Structural Applications," AFFDL-TR-69-111, Feb. 1970.

4. Paris, P. C. and Sih, G. C. "Stress Analysis of Cracks," Fracture Toughness Testing and Its Application, ASTM STP #381 (1965), pp. 30-83.

5. Kobayashi, A. S., Maiden, O. E., Simon, B. J., and Iida, S. "Application of the Method of Finite Element Analysis to Two-Dimensional Problems in Fracture Mechanics," University of Washington, Department of Mechanical Engineering, ONR Contract Nonr-477(39), NR 064 478, TR No. 5, Oct. 1968.

6. Anderson, G. P., Ruggles, V. L. and Stibor, G. S. "Use of Finite Element Computer Programs in Fracture Mechanics," Int. J. of Frac. Mech., Vol. 7, No. 1, March 1971, pp. 63-76.

7. Chan, S. K., Tuba, I. S. and Wilson, W. K. "On the Finite Element Method in Linear Fracture Mechanics," Engineering Fracture Mechanics, 1970, Vol. 2, pp. 1-17.

8. Byskov, E. "The Calculation of Stress Intensity Factors Using the Finite Element Method with Cracked Elements," Int. J. of Frac. Mech., Vol. 6, No. 2, June 1970, pp. 159-167.

9. Levy, N., Marcal, P. V., Ostergren, W. J. and Rice, J. R. "Small Scale Yielding near a Crack in Plane Strain--A Finite Element Analysis," Brown Univ., Div. of Engineering, TR NASA NIGL 40-002-080/1, Nov. 1969.

10. Tong, P. and Pian, T. H. H. "On the Convergence of the Finite Element Methods for Problems with Stress Singularities," to be published.

11. Pian, T. H. H. "Derivation of Element Stiffness Matrices by Assumed Stress Distribution," AIAA J., Vol. 2, No. 7, 1964, pp. 1333-1336.

12. Pian, T. H. H. and Tong, P. "Rationalization in Deriving Element Stiffness Matrix by Assumed Stress Approach," Proc. Second Conference on Matrix Methods in Structural Mechanics, AFFDL-TR-68-150, 1968, pp. 441-469.

13. Pian, T. H. H. and Tong, P. "Basis of Finite Element Methods for Solid Continua," Int. J. Numerical Methods in Engineering, Vol. 1, 1969, pp. 3-28.

14. Sih, G. C. and Liebowitz, H. "Mathematical Theories of Brittle Fracture," Fracture - An Advanced Treatise, Vol. II, Edited by H. Liebowitz, Academic Press, 1968, pp. 67-190.

15. Pian, T. H. H. "Element Stiffness Matrices for Boundary Compatibility and for Prescribed Boundary Stresses," Proc. First Conference of Matrix Methods in Structural Mechanics, AFFDL-TR-66-80, pp. 457-477.

16. Yamada, Y., Nakagiri, S. and Takatsuka, K., "Analysis of Saint-Venant Torsion Problem by a Hybrid Stress Model," Seisan-Kenkyu (Monthly Journal of Institute of Industrial Science, University of Tokyo), Vol. 21, No. 11, 1969, (paper presented at Japan-U.S. Seminar on Matrix Methods of Structural Analysis and Design, Aug. 25-30, 1969, Tokyo).

17. Bowie, O. L. "Rectangular Tensile Sheet with Symmetric Edge Cracks," J. of Appl. Mech. Trans. ASME, Vol. 31, Series E, No. 2, 1964, pp. 208-212.

18. Hartranft, R. J. and Sih, G. C. "Effect of Plate Thickness on the Bending Stress Distribution Around Through Cracks," J. Math and Physics, Vol. 47, No. 3, 1968, pp. 276-291.

19. Bowie, O. L. and Freese, C. E. "Central Crack in Rectangular Sheet with Rectilinear Anisotropy" Proc. Army Symposium on Solid Mechanics, 1970 - Lightweight Structures, AMMRC MS 70-5, Army Materials and Mechanics Research Center, Watertown, Massachusetts, Dec. 1970, pp. IV-1 - IV-14.

APPENDIX

The determination of the near crack tip stress distributions and the stress intensity factors for a general anisotropic material in plane stress or plane strain states has been presented, in details, in a survey paper by Sih and Liebowitz (Reference 14). The stress distributions are functions of two complex parameters s_1 and s_2 which are related to the anisotropic elastic constants. For the orthotropic panel with E_x and E_y as the principle Young's moduli, ν_{xy}, the Poisson ratio and G_{xy}, the shear modulus, the parameters s_1 and s_2 are taken as the roots of the characteristic equation

$$\frac{1}{E_x} s_j^4 + \left(\frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x} \right) s_j^2 + \frac{1}{E_y} = 0 \qquad \text{(A-1)}$$

The parameters s_1 and s_2 are purely imaginary, i.e. $s_1 = i\eta_1$ and $s_2 = i\eta_2$, and

$$\eta_1 \eta_2 = (E_x / E_y)^{1/2}$$

$$\eta_1 + \eta_2 = \sqrt{2} \left\{ (E_x/E_y)^{1/2} + E_x/(2G_{xy}) - \nu_{xy} \right\}^{1/2} \qquad \text{(A-2)}$$

The stress field at the tip of a opening crack which is parallel to the x-axis is given by

$$\sigma_x = \frac{k_I}{(2r)^{1/2}} Re\left[\frac{s_1 s_2}{s_1 - s_2} \left(\frac{s_2}{(\cos\theta + s_2 \sin\theta)^{1/2}} - \frac{s_1}{(\cos\theta + s_1 \sin\theta)^{1/2}} \right) \right]$$

$$\sigma_y = \frac{k_I}{(2r)^{1/2}} Re\left[\frac{1}{s_1 - s_2} \left(\frac{s_1}{(\cos\theta + s_2 \sin\theta)^{1/2}} - \frac{s_2}{(\cos\theta + s_1 \sin\theta)^{1/2}} \right) \right]$$

$$\sigma_{xy} = \frac{k_I}{(2r)^{1/2}} Re\left[\frac{s_1 s_2}{s_1 - s_2} \left(\frac{1}{(\cos\theta + s_1 \sin\theta)^{1/2}} - \frac{1}{(\cos\theta + s_2 \sin\theta)^{1/2}} \right) \right] \qquad \text{(A-3)}$$