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MODIFIED POTENTIAL ENERGY MASS REPRESENTATIONS FOR FREQUENCY PREDICTION†

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Mass representations are derived based on complementary energy requirements for computation of frequencies of systems composed of rods, torque tubes, and beams with nondiscrete masses. Modified mass representations are then derived which reduce the error for upper-bound frequency predictions for systems with uniformly distributed masses. Sample problems are solved illustrating that low-error estimates are obtained for a variety of boundary conditions with few elements.

1.0 INTRODUCTION

In predicting structural resonances analytically, mass and stiffness must be represented. If a set of assumptions is adapted in representing either the mass or stiffness, then these same assumptions can be imposed in representing the other quantity. In the case of finite differences, in which displacements are the unknowns and continuity of stresses is maintained, consistency of assumptions leads to lower-bound estimates of frequency. In the case of the potential energy approach, this consistency insures that upper-bound estimates of frequency are obtained.

Previous investigators have examined various representations of mass to predict resonant frequencies. Duncan (Reference 1) considered several finite difference representations for predicting beam resonances. He determined that the best lumping technique involved lumps midway between displacement stations. Leckie and Lindberg (Reference 2) demonstrated that the accuracy of finite difference predictions varied considerably depending on the boundary conditions being considered. They proposed a potential energy representation as one whose error could be defined and with which extrapolation could be employed. For their problems and the problems selected by Archer (Reference 3), frequency predictions using the potential energy approach had smaller error than those based on finite difference representations. Archer emphasized that the potential energy approach results in upper-bound estimates of frequency. It is noted that despite the fact that difference techniques resulted in larger error in predicting beam frequencies, they generally gave low frequency estimates.

None of these authors has considered the complementary energy representation of mass. This representation will provide lower estimates of frequency than those provided by the potential energy estimate. Moreover, the complementary energy mass representation can be used in conjunction with the potential energy estimate to obtain a modified potential energy mass matrix which yields lower upper-bound estimates of frequency than the potential energy estimate.

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In the next section, Section 2.0, of this paper, the complementary energy mass matrix representation is derived. Mass matrices for rods, torque tubes, and beams are presented. Section 3.0 defines a modified potential energy mass representation. Sections 4.0 and 5.0 include a number of examples illustrating the effectiveness of the modified mass matrix. Section 6.0, the final section, contains study conclusions.

2.0 DERIVATION OF THE MASS MATRICES

The basic requirements that must be satisfied in a complementary energy approach are that stresses must satisfy microscopic equilibrium within an element, and the stress resultants must be continuous across element interfaces. Satisfaction of the first requirement is contingent upon the nature of the stress field assumed within the element, and satisfaction of the second depends on the characteristics of the interelement connection. In meeting these requirements, the complementary energy mass matrix may be derived in three steps. The first step consists of developing the mass matrix for rigid body motions. In view of the fact that these involve no strain energy, they will trivially be consistent with the complementary energy requirements.

The second step consists of deriving the mass matrix for an equivalent set of mass forces imposed at the nodes of the structure. For both beams and rods, the ordinary potential energy stiffness matrix can be used to relate nodal forces and displacements. This relation is then operated upon by a scalar to obtain equivalent mass forces, which leads to the elastic portion of the complementary energy representation.

The third step consists of summing the rigid and the elastic contributions to the mass matrix. These matrices may be added because elastic motions are chosen which are orthogonal to the rigid motions.

In the remainder of this section, the mass representation for rods, tubes and beams will be obtained by executing these three steps. For completeness, the potential energy mass representations for these elements are also included.

Consider the straight rod element which provides resistance only to elongation, is stress-free along its sides, and obeys the laws of linear elasticity under deformation. The displacement function, when only end loads are permitted, is

$$u(x) = \frac{u_1 + u_2}{2} + \frac{u_2 - u_1}{a} x \tag{1}$$

where $u(x)$ is the displacement along the rod axis, u_1 and u_2 are displacements of the rod end points, "a" is the length of the rod, and x is measured along the rod with its origin at midspan. In Equation 1 the first term on the right represents the rigid body displacement of the rod, and the second term the elastic deformation.

Assuming time dependent generalized displacements, the kinetic energy corresponding to the rigid body mode is:

$$T = \frac{1}{2} \int_{-a/2}^{a/2} m(x) \left(\frac{\dot{u}_1 + \dot{u}_2}{2} \right)^2 dx \tag{2}$$

where $m(x)$ is the mass per unit length. Assuming the mass is distributed uniformly $[m(x) = m]$ and substituting Equation 2 into Lagrange's equation, the generalized forces are obtained as

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = ma \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = \mathbf{M}_R \, \ddot{\mathbf{u}} \tag{3}$$

where

$$\mathbf{M}_R = ma \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \tag{4}$$

Here $\mathbf{M}_R$ is the mass matrix due to rigid body motion, and the $\mathbf{F}$ are the generalized forces.

If the displacement function given by Equation 1 is used in a Rayleigh-Ritz approach, the following stiffness equation is obtained:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{a} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{5}$$

where A is the cross-sectional area, and E is Young's modulus of the rod. This relation defines end displacements for any end-loading, F_1 and F_2, based upon a stress distribution which satisfies microscopic equilibrium. Thus, by formulating a kinetic energy expression that reflects the characteristics of Equation 5, the generalized forces associated with this kinetic energy will imply stresses which satisfy the complementary energy requirements. Therefore, the form of the kinetic energy is taken as

$$T = \frac{\alpha}{2} \int_{-a/2}^{a/2} m(x)\, dx \; \lfloor \dot{u}_1 \; \dot{u}_2 \rfloor \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} \tag{6}$$

where α is a scalar. One remaining requirement is that this kinetic energy represent only the elastic energy in the rod, thus rigid body motion must be zero. This condition leads to $u_1 = -u_2$ or $\dot{u}_1 = -\dot{u}_2$, which when imposed in Equation 6 requires $\alpha = \frac{1}{4}$ in order to represent the total mass. Substituting Equation 6 into Lagrange's equation and combining the resultant mass matrix with Equation 4 yields the complementary energy mass matrix for the rod; namely,

$$\mathbf{M}_c = ma \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{7}$$

The potential energy mass representation is obtained by using the time derivative of the entire Equation 1 in Equation 2 rather than only the rigid body term, then using Lagrange's equation. This gives

$$\mathbf{M}_p = ma \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \tag{8}$$

Equations 7 and 8 define mass representations which will lead to predictions of resonant frequencies for rods with uniform cross-sections and uniform mass distribution. If $m\alpha$ is replaced by $\rho\alpha$ where ρ is the mass moment of inertia per unit length, these relations become mass representations for tubes with uniform torsional stiffness and uniform distribution of the mass moment. It is noted that the derivation procedure used can also provide mass representations for other than uniform mass distributions.

The procedure outlined for determining the complementary energy mass matrix for the rod element can be applied directly to the beam element. For a beam element having coordinates and displacement variables as shown in Figure 1, the rigid body displacement function is:

$$w_R = \frac{w_1 + w_2}{2} + \frac{w_2 - w_1}{a}\, x \tag{9}$$

Corresponding end point elastic deformations are

$$\begin{aligned} w_{1_E} &= 0, \quad \theta_{1_E} = \theta_1 - \frac{w_2 - w_1}{a} \\ w_{2_E} &= 0, \quad \theta_{2_E} = \theta_2 - \frac{w_2 - w_1}{a} \end{aligned} \tag{10}$$

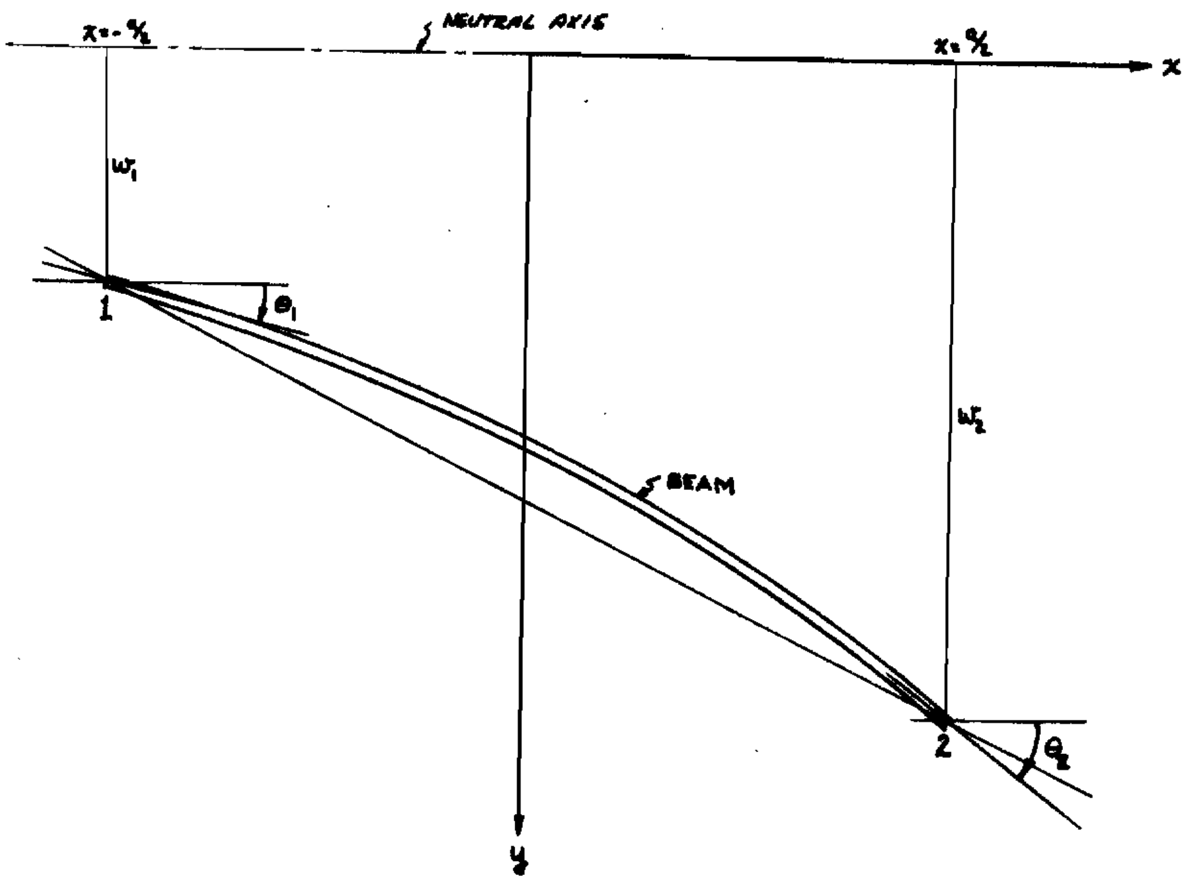


Figure 1. Beam Coordinates and Displacement Variables

Substitution of these functions into the appropriate kinetic energy expressions leads directly to the rigid body and elastic mass matrices. Note that only the end point moments need be considered in establishing the elastic mass matrix, since the transverse deflection at each node is accounted for entirely by the rigid body displacement function. In light of this condition the scalar a is determined by consideration of the total mass moment of inertia of the beam rather than the mass, as for the rods. Neglecting shear deformation for simplicity in later work, the complementary energy mass matrix is

$$\mathbf{M}_c = \frac{ma}{6}\begin{bmatrix} 2a^2 & & \text{SYM} & \\ 3a & 8 & & \\ a^2 & 3a & 2a^2 & \\ -3a & -5 & -3a & 8 \end{bmatrix} \tag{11}$$

The degrees of freedom are θ_1, ω_1, θ_2 and ω_2 with positive directions as shown in Figure 1.

The potential energy matrix has been given by Leckie and Lindberg (Reference 2), Archer (Reference 3), and others. It is

$$\mathbf{M}_p = \frac{ma}{420}\begin{bmatrix} 4a^2 & & & \text{SYM} \\ 22a & 156 & & \\ -3a^2 & -13a & 4a^2 & \\ 13a & 54 & -22a & 156 \end{bmatrix} \tag{12}$$

3.0 REDUCED RANGE MASS MATRICES

As already noted, a potential energy formulation of the dynamic equation leads to upper-bound eigenvalue estimates. However, it is possible to improve the mass representations given by Equations 8 and 12 so that the upper-bound estimate is lowered. This reduction can be achieved by a simple device. A modified potential energy mass matrix is defined as follows:

$$\mathbf{M}_{MP} = \mathbf{M}_P + \beta\,(\mathbf{M}_C - \mathbf{M}_P) \tag{13}$$

where β is a scalar factor. Note that for $\beta = 0$, $\mathbf{M}_{MP} = \mathbf{M}_P$, and for $\beta = 1$, $\mathbf{M}_{MP} = \mathbf{M}_C$. To evaluate β all possible end conditions are considered for a single element. Then, the value of β is selected which under no condition admits a frequency prediction less than the exact value for the lowest elastic frequency.

Consider the rod mass representations. Each segment of the rod is assumed to have a uniformly distributed mass. The effect of neighboring elements on a given segment can be represented by considering that the rod segment is connected by two springs to supports as shown in Figure 2. The scalar β is to be chosen so that no matter what value is given, the spring stiffness k_1, and k_2, the frequency estimates will be high.

For the rod configuration shown in Figure 2, exact eigenvalues are obtained by solving the wave equation subject to the spring constraint boundary conditions represented by k_1 and k_2. The transcendental frequency equation that is obtained is:

$$(4\lambda^2 - K_1 K_2)\tan 2\lambda - 2\lambda\,(K_1 + K_2) = 0 \tag{14}$$

where

$$\lambda^2 = \frac{\omega_n^2 a^2}{4C^2}, \quad C = \sqrt{\frac{AE}{m}}, \quad K_1 = \frac{k_1}{\frac{AE}{a}}, \quad K_2 = \frac{k_2}{\frac{AE}{a}},$$

and ω_n is the circular natural frequency.

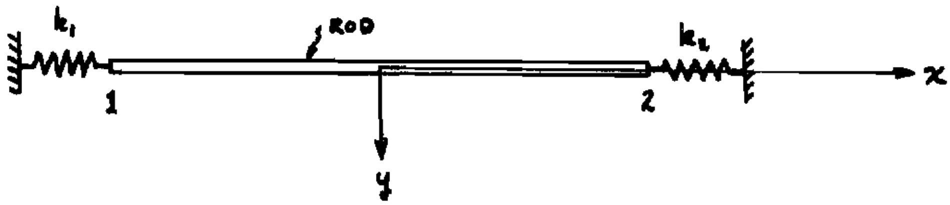


Figure 2. Rod Element with Node Constraints

Since high estimates are required, the value of the scalar can be established by examining the one element representation of the segment. Using the mass matrix defined by Equation 13, the single element dynamic matrix equation is:

$$\frac{2\lambda^2}{3}\begin{bmatrix} 2+\beta & 1-\beta \\ 1-\beta & 2+\beta \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1+K_1 & -1 \\ -1 & 1+K_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

which has the following frequency equation:

$$\lambda^2 = \frac{1}{2}\left\{\frac{6+(2+\beta)(K_1+K_2)}{2(1+2\beta)} \pm \sqrt{\left[\frac{6+(2+\beta)(K_1+K_2)}{2(1+2\beta)}\right]^2 - \frac{3(K_1+K_2+K_1K_2)}{1+2\beta}}\right\} \qquad (15)$$

Note that this equation can be used to obtain frequency estimates for the two mass representations. If $\beta = 0$, the potential energy estimate is obtained based on the displacement function (1), and if $\beta = 1$, the complementary energy estimate is obtained.

Solutions of Equations 14 and 15 for various values of the parameters k_1 and k_2 are given in Table I. This data indicates that the complementary estimate may be less than the exact solution, and the accuracy of the estimates is sensitive to end conditions.

In Table I the columns defining the scalar give the value of the scalar so that the frequency estimate equals the exact value. For the zero frequency case, the value of β for the first mode is immaterial since the rigid mode is simulated exactly. It can be seen that the critical value of $\beta = 0.108$ arises when the end springs are selected to simulate the free-free condition. Hence, this condition provides the largest value of β which guarantees that frequency estimates will be high. The mass matrix for $\beta = 0.108$ is:

$$M_{MP} = ma\begin{bmatrix} 0.3513 & 0.1487 \\ 0.1487 & 0.3513 \end{bmatrix}$$

Note that, as might have been anticipated, the critical value of β could have been defined from the idealized end-condition cases alone. Applying this consideration to the beam, it is

TABLE I

ROD SEGMENT EIGENVALUES

$$\left(\frac{\omega^2 a^2}{4c^2}\right)$$

Relative Stiffness		First Mode				Second Mode			
K_1	K_2	P.E.[1]	Exact	C.E.[2]	Scalar	P.E.[1]	Exact	C.E.[2]	Scalar
0	0	0.000	0.000	0.000	0.0	3.000	2.467	1.000	0.108
0	.1	0.024	0.024	0.024	14.6	3.076	2.517	1.026	0.111
0	1	0.197	0.185	0.191	1.84	3.803	2.934	1.309	0.151
0	10	0.605	0.510	0.451	0.554	12.39	4.635	5.550	1.58
0	∞	0.750	0.617	0.500	0.432	-	5.552	-	-
.1	.1	0.050	0.049	0.050	31.5	3.150	2.566	1.050	0.114
.1	1	0.233	0.216	0.227	2.41	3.867	2.980	1.323	0.151
.1	10	0.670	0.555	0.500	0.623	12.43	4.681	5.550	1.54
.1	∞	0.825	0.666	0.550	0.478	-	5.602	-	-
1	1	0.500	0.427	0.500	4.77	4.500	3.373	1.500	0.167
1	10	1.234	0.879	0.945	1.28	12.77	5.079	5.555	1.25
1	∞	1.500	1.029	1.000	0.915	-	6.035	-	-
10	10	5.000	1.726	5.000	4.71	18.00	7.042	6.000	0.778
10	∞	8.250	2.049	5.500	6.05	-	8.296	-	-

[1] Potential Energy Estimate

[2] Complementary Energy Estimate

found that the critical end conditions occur for the free-guided case. The modification to the potential energy matrix involves a scalar of 0.001709. The modified mass matrix corresponding to this value of β is

$$M_{MP} = \frac{ma}{420}\begin{bmatrix} 4.232 & & & \text{SYM} \\ 22.23 & 156.7 & & \\ -2.875 & -12.62 & 4.232 & \\ 12.62 & 53.31 & -22.23 & 156.7 \end{bmatrix}$$

4.0 ILLUSTRATIVE ROD (TUBE) PROBLEMS

Sections 2.0 and 3.0 provide three mass representations for predicting resonant frequencies for rods having uniformly distributed mass and stiffness (cross-sectional area times Young's modulus). For all distinct combinations of classical end conditions for the rod, the reciprocal square of the frequency parameter $\frac{c^2}{\omega^2 a^2}$ was computed for each of the three mass matrices. Results of these calculations are shown in Table II. For the free-free case, all matrices predict the zero frequency. The frequency parameter given in this case is for the first elastic mode.

Study of this table confirms the following conclusions:

1. All exact values are less than, or equal to, the modified potential energy predictions.
2. The modified energy predictions always are more accurate than either the potential or complementary energy estimates. The error in predictions for the modified case is reduced between 25 and 100 percent from the potential energy estimates.

It is of interest to examine the frequency parameter estimates as a function of the number of analysis elements. Table III summarizes such a study for the pinned-free end conditions. It can be seen that estimates continually involve less error as the number of elements is increased. Two segments are sufficient to obtain estimates with engineering accuracy. Potential and modified potential estimates are always low and complementary high.

TABLE II

ROD RECIPROCAL FREQUENCY PARAMETER

$$\left(\frac{c^2}{\omega^2 a^2}\right) \times 10$$

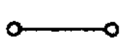
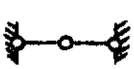
Geometry	Potential	Error (%)	Mod. Potential	Error (%)	Exact	Complementary	Error (%)
	0.8333	-17.8	1.013	0.0	1.013	2.500	48.0
	3.333	-17.8	3.513	-13.3	4.053	5.000	23.3
	0.8333	-17.8	0.8782	-13.3	1.013	1.250	18.9

TABLE III

PINNED-FREE ROD RECIPROCAL FREQUENCY PARAMETER*

$$\left(\frac{c^2}{\omega^2 a^2}\right) \times 10$$

No. Elements	Potential	Mod. Potential	Complementary
1	3.3333	3.5132	5.0000
2	3.8511	3.8961	4.2678
3	3.9615	3.9815	4.1467
4	4.0012	4.0124	4.1054
5	4.0016	4.0268	4.0861
6	4.0298	4.0348	4.0761
7	4.0359	4.0396	4.0699
8	4.0398	4.0426	4.0659
9	4.0432	4.0448	4.0631
10	4.0445	4.0463	4.0612
20	4.0508	4.0513	4.0550

*Exact Value = $\frac{4}{\pi^2}$ = .40528

5.0 ILLUSTRATIVE BEAM PROBLEMS

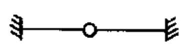
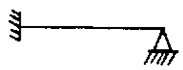
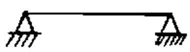
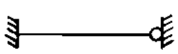
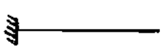
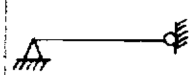
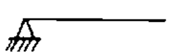
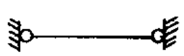
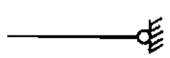
Table IV includes predictions of the reciprocal square of the frequency for beams for all distinct combinations of idealized end conditions. For the cases which include rigid modes (e.g., free-free), the zero frequencies are predicted exactly and therefore have been omitted in Table IV. The last column of Table IV indicates the precentage reduction in error when using the modified potential energy representation instead of the potential.

Conclusions 1 and 2 of Section 4.0 also apply to this data. The error in this case is reduced a minimum of 4.03 percent, however.

TABLE IV

BEAM RECIPROCAL FREQUENCY PARAMETER

$$\left(\frac{1}{\omega^2} \cdot \frac{EI}{ma^4} \right) \times 100$$

Geomentry	Potential	Mod. Potential	Exact*	Complementary	% Error Reduction	Scalar
	0.19345 0.01488	0.19431 0.01575	0.19978 0.02629	0.69444 0.52083	13.5 7.6	0.02255 0.01263
	0.23810	0.25193	0.42049	8.33333	7.6	0.02253
	0.83333 0.03968	0.84615 0.05386	1.02660 0.06417	8.33333 8.33333	6.6 58.0	0.02577 0.00295
	3.09524	3.10894	3.19640	11.1111	13.5	0.01262
	8.01190 0.08254	8.03186 0.09701	8.09086 0.20598	19.4444 8.33333	33.8 11.7	0.00697 0.01458
	16.2967 0.13190	16.3019 0.14604	16.4256 0.20278	19.4444 8.33333	4.03 20.0	0.04190 0.00857
	0.32488 0.02036	0.33938 0.03465	0.42049 0.04005	8.33333 8.33333	15.2 72.9	0.01127 0.00235
	1.01190	1.02560	1.02660	9.02777	93.2	0.00183
	3.18217 0.05196	3.19640 0.06650	3.19640 0.10946	11.1111 8.33333	100.0 25.3	0.00171 0.00676
	0.11949 0.01362	0.15408 0.02617	0.19978 0.02629	8.33333 8.33333	43.1 99.0	0.00685 0.00172

*Exact Values from Reference 4

6.0 SUMMARY AND CONCLUSIONS

This paper has presented the means for easily developing the complementary energy mass representation using the stiffness matrix and rigid mass matrix. It has shown that this matrix can be used to modify the potential energy mass matrix. Mass representations have been developed for uniformly distributed masses on uniform rods (tubes) and beams. Solutions for all possible distinct idealized end conditions have been presented using one or two element breakdowns. Based on theoretical considerations and this data, the following conclusions are drawn:

1. The modified potential energy matrix with the potential energy stiffness matrix defines an upper bound estimate of resonant frequencies.

2. Modified potential energy estimates always involve less error than potential energy estimates.

3. Complementary energy estimates are always less than both the potential and modified potential, but not necessarily less than the exact resonances.

4. Accuracy of estimates improves for all representations as the number of elements is increased.

5. The set of all possible distinct classical end conditions includes the case involving the severest test for accuracy for estimating frequencies by potential, modified potential, and complementary energy approaches.

6. The number of elements required to obtain satisfactory estimates of frequency is dependent on the mass representation selected, the end conditions, and the number of resonances required. Using the modified potential energy matrix, considering the worst end-conditions, and requiring the first resonance within five percent, two elements are required for uniform rods and at least two for beams.

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