THE EFFECT OF AXIAL LOADS ON THE STIFFNESS OF RIGID-JOINTED PLANE FRAMES

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The effect of axial loads on the members of a skeletal frame is to alter the stiffness of each from that usually taken in the orthodox "slope-deflection" analysis. By means of a Rayleigh-Ritz analysis it is shown that the use of tables of stiffness and carry-over factors may be avoided.

Ultimately it is shown that the stiffness matrix $\mathbf{K}$ for the frame may be expressed in the form

$$\mathbf{K} = \mathbf{K}_0 - \lambda \mathbf{L}_1 - \lambda^2 \mathbf{L}_2 - \lambda^3 \mathbf{L}_3 - - \infty$$

where $\mathbf{K}_0$ is the stiffness matrix obtained by ignoring the effect of axial loads

λ is a load factor

$\mathbf{L}_1, \mathbf{L}_2$ etc are matrices that can be calculated at "unit load", that is for $\lambda = 1$

INTRODUCTION

The most commonly used method of analysis of rigid-jointed skeletal frames, the slope-deflection stiffness method, ignores the effect of axial loads on the stiffness of members altogether and it is usually followed by a further set of calculations for the compression members, based on the behavior of isolated struts. This, obviously, is a most unsatisfactory system and some work has been done on methods of modifying the slope-deflection equations to take the axial loads into account from the beginning. Typical of such efforts is the method summarised by Livesley and Chandler (Reference 1) and by Merchant and Salem (Reference 2). Its major defect is that the stiffness and carryover factors have to be calculated afresh for each chosen value of the load factor, so that little can be carried on from one calculation to the next. Argyris (Reference 3) has produced a method, based on his idea of fictitious initial strains which reduces the amount of work after the first calculation, but which still leaves much to be done.

This paper suggests a method whereby most of the work can be done for a given set of loads — load factors $\lambda = 1$ — with comparatively small extra calculations for other values of λ.

ASSUMPTIONS

For the individual members the assumptions and simplifications implicit in the slope-deflection method are made. In addition, it is assumed that the axial load in each member can be calculated with reasonable accuracy at the beginning of the analysis and that it varies linearly with the load factor.

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STIFFNESS OF A SINGLE MEMBER

For the purpose of a Rayleigh-Ritz analysis it is assumed that the lateral displacement, y, of a single member, loaded and deformed as in Figure 1, may be expressed as the sum of a set of five modes, thus

$$y = a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4 + a_5 y_5 \tag{1}$$

In the absence of axial loads the three modes shown in Figure 2 are sufficient

$$\begin{aligned} y_1 &= \frac{x^3}{L^2} - \frac{2x^2}{L} + x \\ y_2 &= \frac{x^3}{L^2} - \frac{x^2}{L} \\ y_3 &= -\frac{2x^3}{L^2} + \frac{3x^2}{L} \end{aligned} \tag{2}$$

and their coefficients are

$$a_1 = \theta_A , \quad a_2 = \theta_B , \quad a_3 = \frac{\delta}{L} = \phi \tag{3}$$

These modes give the ordinary slope-deflection equations.

The effect of the axial load, P, is to alter the displacement, y, and to allow for this, Jennings's proposal (Reference 4) is taken up. This is to add two further modes as shown in Figure 3. Jennings's modes were the first and second collapse modes of the fixed-ended strut, but this is too restrictive and all that we need to specify is that there shall be no rotation at the ends of the member and that the modes shall have the same types of symmetry and the same critical loads as Jennings's modes. That is, the two modes, y_4 and y_5, may contain significant amounts of higher collapse modes of the fixed-ended strut, if necessary.

We now have

$$y = \theta_A \cdot y_1 + \theta_B \cdot y_2 + \phi \cdot y_3 + a_4 \cdot y_4 + a_5 \cdot y_5 \tag{4}$$

a_4 and a_5 may be regarded as typical displacements which specify the magnitude of y_4 and y_5 , but there is no need to identify them in more detail.

Since the structure obeys Hooke's Law, both the strain energy and the complementary energy are given by

$$U = \int_0^L \frac{EI}{2} \left(\frac{d^2 y}{dx^2} \right) dx \tag{5}$$

Using Equation 1 and performing the integration, we get

$$U = \frac{EI}{2} \sum k_{ij} \cdot a_i \cdot a_j \tag{6}$$

where

$$k_{ij} = \int_0^L \left(\frac{d^2 y_i}{dx^2} \right) \left(\frac{d^2 y_j}{dx^2} \right) dx \tag{7}$$

This expression for U is quadratic in the quantities a_i and so may be written in matrix form

$$U = \frac{EI}{2}\, \theta^T \cdot k \cdot \theta \tag{8}$$

where θ is the column matrix $\{a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5\}$

θ^T is the transpose of θ

and k is a square matrix whose elements are given by Equation 7

The change in potential energy of the external forces acting on the member is given by

$$V = \frac{P}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx + M_{AB} \cdot \theta_A + M_{BA} \cdot \theta_B + F \cdot L \cdot \phi \tag{9}$$

As in the case of the strain energy this may be written in matrix form

$$V = \frac{P}{2} \cdot \theta^T \cdot h \cdot \theta + \theta^T \cdot M \tag{10}$$

where h is a square matrix whose elements are given by

$$h_{ij} = \int_0^L \left(\frac{dy_i}{dx}\right)\left(\frac{dy_j}{dx}\right) dx \tag{11}$$

and M is the column of end loads corresponding to the modes y_i. If y is expressed as in Equation 4

$$M = \{ M_{AB} \quad M_{BA} \quad F \cdot L \quad 0 \quad 0 \} \tag{12}$$

The total change in energy from the initial state is

$$T = U - V = \frac{EI}{2}\, \theta^T\, k\, \theta - \frac{P}{2}\, \theta^T\, h\, \theta - \theta^T M \tag{13}$$

For equilibrium, the derivative of T with regard to each element of θ is zero and so

$$0 = EI\, k\, \theta - P\, h\, \theta - M \tag{14}$$

Rearranging Equation 14 gives

$$M = EI\; k - \frac{P}{EI}\; h\; \theta \tag{15}$$

Using $\quad P_E = \frac{\pi^2 EI}{L^2}$ and $\frac{P}{P_E} = \rho$

Equation 14 becomes

$$M = EI\left(k - \rho\, h\right)\theta \tag{16}$$

where $\boldsymbol{\ell}$ is a square matrix whose elements are given by

$$\ell_{ij} = \frac{\pi^2}{L^2} \cdot h_{ij} = \frac{\pi^2}{L^2} \int_0^L \left(\frac{dy_i}{dx}\right)\left(\frac{dy_j}{dx}\right) dx \tag{17}$$

If we write

$$\mathbf{S} = \left\{ M_{AB} \quad M_{BA} \quad F \cdot L \right\} \tag{18}$$

then $\mathbf{M}$ may be partitioned

$$\mathbf{M} = \left\{ \mathbf{S} \quad \mathbf{0} \right\} \tag{19}$$

Similarly, if we write

$$\begin{aligned} \boldsymbol{\theta} &= \left\{ \theta_A \quad \theta_B \quad \phi \right\} \\ \mathbf{z} &= \left\{ a_4 \quad a_5 \right\} \end{aligned} \tag{20}$$

then $\boldsymbol{\theta}$ may be partitioned

$$\bar{\boldsymbol{\theta}} = \left\{ \boldsymbol{\theta} \quad \mathbf{z} \right\} \tag{21}$$

With $\mathbf{k}$ and $\boldsymbol{\ell}$ partitioned to suit, Equation 16 becomes

$$\begin{pmatrix} \mathbf{S} \\ \mathbf{0} \end{pmatrix} = EI \left(\begin{bmatrix} k_1 & k_2 \\ k_2^T & k_3 \end{bmatrix} - \rho \begin{bmatrix} \ell_1 & \ell_2 \\ \ell_2^T & \ell_3 \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{z} \end{bmatrix} \tag{22}$$

The second of these two equations may be solved

$$0 = EI \left(\left(k_2^T - \rho \ell_2^T \right) \boldsymbol{\theta} + \left(k_3 - \rho \ell_3 \right) \mathbf{z} \right) \tag{23}$$

so that

$$\mathbf{z} = -\left(k_3 - \rho \ell_3 \right)^{-1} \left(k_2^T - \rho \ell_2^T \right) \boldsymbol{\theta}$$

Substituting for $\mathbf{z}$ in the first set gives

$$\mathbf{S} = EI \left(\left(k_1 - \rho \ell_1 \right) - \left(k_2 - \rho \ell_2 \right) \left(k_3 - \rho \ell_3 \right)^{-1} \left(k_2 - \rho \ell_2 \right)^T \right) \boldsymbol{\theta} \tag{24}$$

To determine the elements of these matrices, we first put $\rho = 0$, then Equation 24 becomes

$$\mathbf{s} = EI\left(\mathbf{k}_1 + \mathbf{k}_2 \; \mathbf{k}_3^{-1} \; \mathbf{k}_2^T \right) \boldsymbol{\theta} \tag{25}$$

and Equation 23 becomes

$$\mathbf{z} = - \mathbf{k}_3^{-1} \; \mathbf{k}_2^T \; \boldsymbol{\theta} \tag{26}$$

If $\rho = 0$, the modes y_4 and y_5 are not excited and so

$$\mathbf{z} = 0 \quad \text{for arbitrary } \boldsymbol{\theta}$$

Hence, since it is impossible for $\mathbf{k}_3$ to have infinite elements,

$$\mathbf{k}_2 = \underline{0}$$

Equation 24 is now

$$\mathbf{s} = EI\left(\mathbf{k}_1 - \rho \boldsymbol{\ell}_1 - \rho^2 \boldsymbol{\ell}_2 \left(\mathbf{k}_3 - \rho \boldsymbol{\ell}_3 \right)^{-1} \boldsymbol{\ell}_2^T \right) \boldsymbol{\theta} \tag{28}$$

Also for $\rho = 0$

$$\mathbf{s} = EI \; \mathbf{k}_1 \; \boldsymbol{\theta} \tag{29}$$

Since the slope-deflection equations are in fact

$$\begin{aligned} M_{AB} &= \frac{EI}{L}\left(4\theta_A + 2\theta_B - 6\phi\right) \\ M_{BA} &= \frac{EI}{L}\left(2\theta_A + 4\theta_B - 6\phi\right) \\ F\,L &= \frac{EI}{L}\left(-6\theta_A - 6\theta_B + 12\phi\right) \end{aligned} \tag{30}$$

$$\mathbf{k}_1 = \frac{1}{L}\begin{bmatrix} 4 & 2 & -6 \\ 2 & 4 & -6 \\ -6 & -6 & 12 \end{bmatrix} \tag{31}$$

From Equations 2 and 17

$$\boldsymbol{\ell}_1 = \frac{\pi^2}{30L}\begin{bmatrix} 4 & -1 & -3 \\ -1 & 4 & -3 \\ -3 & -3 & 36 \end{bmatrix} \tag{32}$$

By inspection of Figures 2 and 3, it is seen that owing to the different symmetries of the three modes y_3 , y_4 and y_5

$$\int_0^L \left(\frac{dy_3}{dx}\right)\left(\frac{dy_4}{dx}\right) dx = \int_0^L \left(\frac{dy_4}{dx}\right)\left(\frac{dy_5}{dx}\right) dx = 0$$

$$\text{i.e} \quad \ell_{34} = \ell_{43} = \ell_{45} = \ell_{54} = 0 \tag{33}$$

$$\int_0^L \left(\frac{d^2y_3}{dx^2}\right)\left(\frac{d^2y_4}{dx^2}\right) dx = \int_0^L \left(\frac{d^2y_4}{dx^2}\right)\left(\frac{d^2y_5}{dx^2}\right) dx = 0$$

$$\text{i.e} \quad k_{34} = k_{43} = k_{45} = k_{54} = 0 \tag{34}$$

From all this

$$k = \begin{bmatrix} \frac{4}{L} & \frac{2}{L} & \frac{-6}{L} & 0 & 0 \\ \frac{2}{L} & \frac{4}{L} & \frac{-6}{L} & 0 & 0 \\ \frac{-6}{L} & \frac{-6}{L} & \frac{12}{L} & 0 & 0 \\ 0 & 0 & 0 & k_{44} & 0 \\ 0 & 0 & 0 & 0 & k_{55} \end{bmatrix} \tag{35}$$

$$\ell = \begin{bmatrix} \frac{4\pi^2}{30L} & \frac{\pi^2}{30L} & \frac{-3\pi^2}{30L} & \ell_{14} & \ell_{15} \\ \frac{\pi^2}{30L} & \frac{4\pi^2}{30L} & \frac{-3\pi^2}{30L} & \ell_{24} & \ell_{25} \\ \frac{-3\pi^2}{30L} & \frac{-3\pi^2}{30L} & \frac{36\pi^2}{30L} & 0 & \ell_{35} \\ \ell_{41} & \ell_{42} & 0 & \ell_{44} & 0 \\ \ell_{51} & \ell_{52} & \ell_{53} & 0 & \ell_{55} \end{bmatrix} \tag{36}$$

$$z = -\rho\left(k_3 - \rho\,\ell_3\right)^{-1} \ell_2^T\, \theta$$

That is

$$\begin{bmatrix} a_4 \\ a_5 \end{bmatrix} = \rho \begin{bmatrix} k_{44} - \rho\ell_{44} & 0 \\ 0 & k_{55} - \rho\ell_{55} \end{bmatrix}^{-1} \begin{bmatrix} \ell_{14} & \ell_{24} & 0 \\ \ell_{15} & \ell_{25} & \ell_{35} \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \phi \end{bmatrix} \tag{37}$$

At $\rho = 4$ the member collapses in the first fixed-ended mode which is included in y_4. Hence a_4 becomes indefinite when $\theta = 0$

This means that

$$k_{44} - \rho \ell_{44} = 0$$

that is

$$k_{44} = 4 \ell_{44} \tag{38}$$

Similarly at $\rho = 8.183$ the member collapses in the second fixed-ended mode, giving

$$k_{55} = 8.183\, \ell_{55} \tag{39}$$

This leads to

$$z = +\rho \begin{bmatrix} \dfrac{1}{\ell_{44}(4-\rho)} & 0 \\ 0 & \dfrac{1}{\ell_{55}(8.183-\rho)} \end{bmatrix} \boldsymbol{\ell}_2^T \cdot \theta \tag{40}$$

and

$$S = EI \left(k_1 - \rho \boldsymbol{\ell}_1 - \rho^2 \boldsymbol{\ell}_2 \begin{bmatrix} \dfrac{1}{\ell_{44}(4-\rho)} & 0 \\ 0 & \dfrac{1}{\ell_{55}(8.183-\rho)} \end{bmatrix} \boldsymbol{\ell}_2^T \right) \theta \tag{41}$$

If we write

$$\boldsymbol{\ell}_2^T = \begin{bmatrix} \ell_A \\ \ell_B \end{bmatrix}$$

where

$$\boldsymbol{\ell}_A = \begin{bmatrix} \ell_{14} & \ell_{24} & 0 \end{bmatrix}$$

$$\boldsymbol{\ell}_B = \begin{bmatrix} \ell_{15} & \ell_{25} & \ell_{35} \end{bmatrix}$$

then

$$S = EI \left(k_1 - \rho \boldsymbol{\ell}_1 - \frac{\rho^2}{(4-\rho)} \frac{\boldsymbol{\ell}_A^T \boldsymbol{\ell}_A}{\ell_{44}} - \frac{\rho^2}{(8.183-\rho)} \frac{\boldsymbol{\ell}_B^T \boldsymbol{\ell}_B}{\ell_{55}} \right) \theta \tag{42}$$

The products $\dfrac{\boldsymbol{\ell}_A^T \boldsymbol{\ell}_A}{\ell_{44}}$ and $\dfrac{\boldsymbol{\ell}_B^T \boldsymbol{\ell}_B}{\ell_{55}}$ are obtained by fitting Equation 42 to the modified slope-deflection equations of References 1 and 2.

$$S = \frac{EI}{L} \begin{bmatrix} s & sc & -s(1+c) \\ sc & s & -s(1+c) \\ -s(1+c) & -s(1+c) & s^2(1-c^2) \end{bmatrix} \theta \tag{43}$$

By this

$$s = 4 - \rho\frac{4\pi^2}{30} - \frac{\rho^2 L}{(4-\rho)}\cdot\frac{\ell_{14}^2}{\ell_{44}} - \frac{\rho^2 L}{(8.183-\rho)}\cdot\frac{\ell_{15}^2}{\ell_{55}} \tag{44}$$

$$sc = 2 - \rho\frac{\pi^2}{30} - \frac{\rho^2 L}{(4-\rho)}\cdot\frac{\ell_{14}\ell_{24}}{\ell_{44}} - \frac{\rho^2 L}{(8.183-\rho)}\cdot\frac{\ell_{15}\ell_{25}}{\ell_{55}} \tag{45}$$

$$s^2(1-c^2) = 12 - \rho\frac{36\pi^2}{30} - \frac{\rho^2 L}{(4-\rho)}\cdot 0 - \frac{\rho^2 L}{(8.183-\rho)}\cdot\frac{\ell_{35}^2}{\ell_{55}} \tag{46}$$

These equations are already satisfied at $\rho = 0$, 4 and 8.183. If we now put $\rho = 2.0457$, when $s = 0$, $sc = 3.6033$ and $s^2(1-c^2) = -12.9838$, we get

$$\frac{\ell_A^T \ell_A}{\ell_{44}} = \frac{0.523}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{47}$$

$$\frac{\ell_B^T \ell_B}{\ell_{55}} = \frac{0.277}{L}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix} \tag{48}$$

It is not necessary to proceed further with the evaluation of the elements of $\boldsymbol{\ell}$, since the stiffness matrix, as can be seen in Equation 42, is now complete.

Since we are dealing with structures for which ρ is always less than 4, when the member concerned would collapse in the second Euler mode, we can write

$$\frac{\rho^2}{(4-\rho)} = \frac{\rho^2}{4}\left(1 + \frac{\rho}{4} + \left(\frac{\rho}{4}\right)^2 + \left(\frac{\rho}{4}\right)^3 + + \infty\right) \tag{49}$$

and

$$\frac{\rho^2}{(8.183-\rho)} = \frac{\rho^2}{8.183}\left(1 + \frac{\rho}{8.183} + \left(\frac{\rho}{8.183}\right)^2 + \left(\frac{\rho}{8.183}\right)^2 + + \infty\right) \tag{50}$$

If we put $\rho = \lambda\rho_1$ where ρ_1 is calculated at some "unit" or "working" load and λ is a load factor, and also write

$$\frac{\ell_A^T \ell_A}{\ell_{44}} = \alpha \quad \text{and} \quad \frac{\ell_B^T \ell_B}{\ell_{55}} = \beta \tag{51}$$

then the stiffness matrix of a single member is

$$k = EI \begin{bmatrix} k_1 - \lambda \rho_1 \ell_1 - \lambda^2 \left\{ \frac{\rho_1}{4} \rho_1 \alpha + \frac{\rho_1}{8.183} \rho_1 \beta \right\} - - \\ - - \lambda^n \left\{ \left(\frac{\rho_1}{4} \right)^{n-1} \rho_1 \alpha + \left(\frac{\rho_1}{8.183} \right)^{n-1} \rho_1 \beta \right\} - - \infty \end{bmatrix} \quad (52)$$

FRAME STIFFNESS

The method of forming the stiffness matrix for the complete structure from the member matrices is normal

$$K = A^T K_u A \quad (53)$$

where K_u is the stiffness matrix for the unassembled structure, being a diagonal list of member matrices, and **A** relates the member displacements to the frame displacements. Since **k**, in Equation 52, is in the form of the sum of a series of matrices, each multiplied by the appropriate power of λ, we may write Equation 53 in the form

$$K = A^T \left(K_{u1} - \lambda L_{u1} - \lambda^2 L_{u2} \ \text{etc} \right) A \quad (54)$$

where K_{u1}, L_{u1}, L_{u2} etc are assembled in the same way as K_u from the appropriate parts of the member matrices.

On multiplying out we get

$$K = K_0 - \lambda L_1 - \lambda^2 L_2 - \lambda^3 L_3 - \text{etc} \quad (55)$$

where

$$L_n = A^* \cdot L_{un} \cdot A \quad \text{with} \quad n = 1, 2, 3 \ldots .$$

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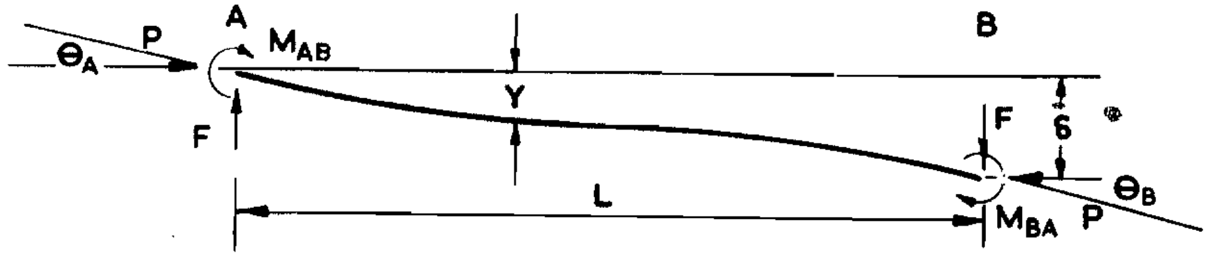

Figure 1. General Deformed Shape and Loading of Single Member

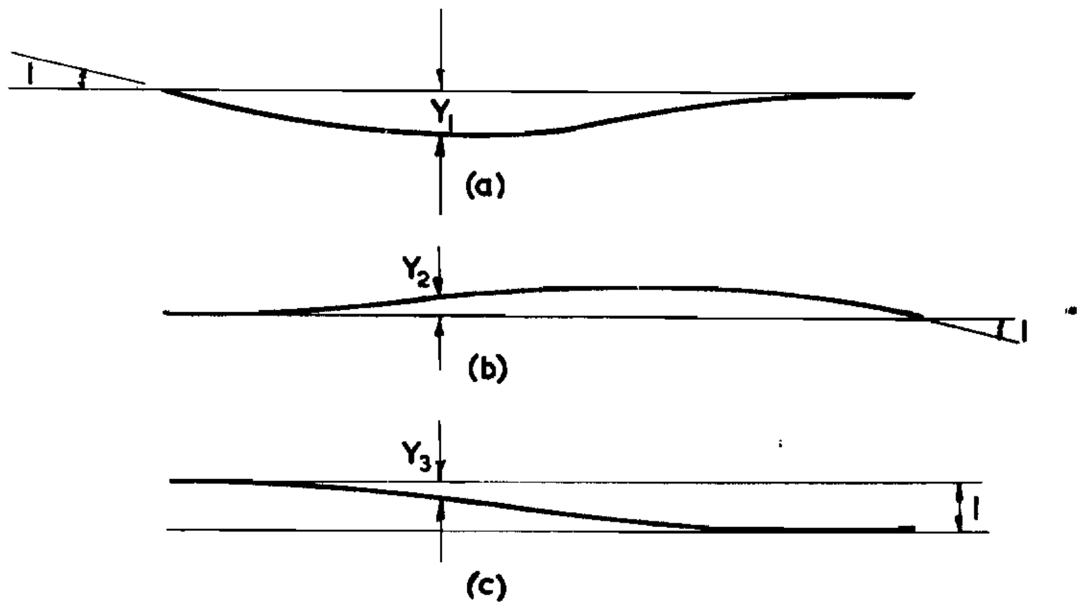

Figure 2. Basic "Slope Deflection" Shapes

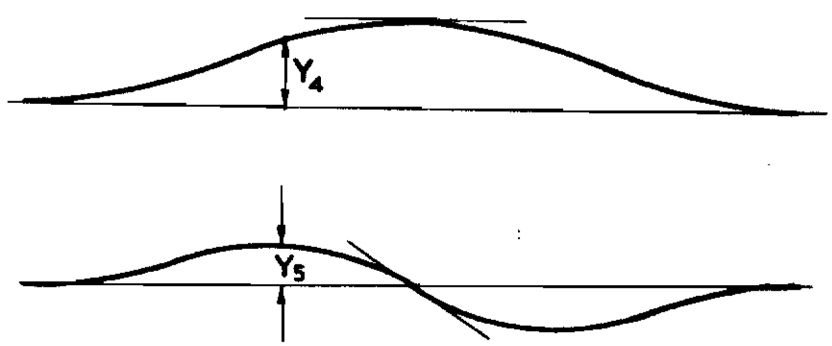

Figure 3. Additional "Stability" Shapes