A DOUBLY CURVED TRIANGULAR SHELL ELEMENT

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The stiffness matrix for a doubly-curved triangular shell element, suitable for the analysis of general nonsymmetric shells, is derived. The geometry of the element is considered and used to derive the coordinate transformations subsequently needed. The strains, strain-displacement relations, and stresses are then discussed on the basis of the shallow shell theory expounded by Novozhilov. Suitable assumed displacement fields are developed and are then used to obtain the desired stiffness matrix. Finally, sample calculations using the curved element, and comparing it for accuracy with a corresponding flat triangular element, are briefly described.

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SECTION I

INTRODUCTION

In the finite element analysis of thin elastic shells of general configuration, it is convenient to employ triangular elements. These make possible the approximation of highly general midsurface and edge geometries, and are readily adaptable to refinement of the mesh in regions where the solution is expected to vary rapidly. In recent applications (References 1 and 2) of this technique, flat triangular elements have been used. Such an element is formed by superposing plane stress and bending effects for a flat triangular plate element; a polyhedral approximation to the surface of the shell is then formed with an assemblage of such elements. Clough and his students, in particular, have demonstrated the feasibility of this approach, and have applied it, in various versions, to a wide variety of practical problems. However, even for relatively simple problems a large number of degrees of freedom, and the solution of a large system of equations, are usually required if good accuracy is to be obtained. It is therefore natural to seek possible improvements that would permit the attainment of good accuracy with, hopefully, a considerably coarser and more easily managed discretization of the shell.

In the analysis of shells using flat triangular elements, the state of stress in each shell element is assumed to be a combination of membrane stresses, constant through the thickness of the element, and bending stresses, varying linearly through the thickness. Hence the behavior of the element subject to such a combined stress distribution can be treated by a suitable combination of plane stress and bending finite element analyses. In the plane-stress analysis, the two in-plane displacement components at each of the three vertices of the triangle are taken as unknowns and related to the in-plane nodal force components by means of a membrane stiffness matrix. This stiffness matrix is usually derived by assuming that the in-plane displacements vary linearly from node to node of the element, and consequently that the membrane stress field within the element is constant. It is thus clear that for accuracy, the subdivision of the shell must be quite fine in regions where the stress is varying rapidly. In the bending analysis, the normal displacement component and the two in-plane components of rotation are taken as unknowns and are related to the lateral forces and the in-plane couples applied at the nodes by means of a bending stiffness matrix. The bending stiffness matrix for the element is derived from an assumed displacement function involving cubic variation of the normal displacement within the element.

A result of this approach is that there is no coupling of bending and membrane effects within an element: membrane displacements produce only membrane forces and, correspondingly, bending displacements, i.e, normal displacements and in-plane rotations, produce only bending forces, i.e., lateral forces and moments. The coupling between in-plane and bending effects, so characteristic of shell behavior, occurs only at nodes where successive elements not in the same plane adjoin. At such a node, in-plane forces in one element may be transmitted to another element lying in a somewhat different plane. For this second element these forces will have out-of-plane components giving rise to bending effects in the second element. Similarly, bending forces from the first element may give rise to membrane effects in the second. Thus, instead of a continuous coupling between bending and membrane effects, there is a coupling only at nodal points. Nevertheless, if the subdivision is sufficiently fine, that is, if the nodal points are sufficiently close together, such a representation of a shell should be adequate.

It is well known from shell analysis that relatively small slope discontinuities in the mid-surface of a shell can lead to large perturbations in the local stress distribution. It is quite possible that a similar effect tends to occur at the juncture of two elements not in the same plane. Indeed, in the finite element analysis of axisymmetric shells using conical frusta, where similar slope discontinuities do occur, they appear to cause perturbations of the stress fields under some conditions unless the subdivision is very fine so as nearly to eliminate the slope discontinuity (Reference 3).

The slope discontinuity between successive elements also has the effect of introducing a degree of incompatibility in the displacement fields of adjoining elements. For instance, in the first of two contiguous elements lying in somewhat different planes, suppose that the normal displacement has cubic variation along the common boundary. This means that for the second element there would have to be a cubic distribution of in-plane displacement for the maintenance of compatibility, but this is contrary to the usual assumption of linear variation of in-plane displacements. Such incompatibilities will tend to diminish as the mesh is refined and, therefore, as adjacent elements are brought more nearly into the same plane, since the displacement functions are chosen so as to give exact satisfaction of compatibility between two contiguous elements lying in the same plane.

One possibility for improving the accuracy of such analyses lies in the introduction of a doubly-curved triangular element. It has been demonstrated in the analysis of axisymmetric shells (Reference 4) that elements with meridional curvature often yield much greater accuracy than the conical frustum elements that are more often used for such analysis. It would appear that similar advantages would also result from the use of doubly-curved triangular elements, which would permit the modeling of a nonsymmetric shell without the introduction of slope discontinuities. One advantage would be the avoidance of any tendency for perturbations to arise from discontinuities in the idealized shell surface. In conferring this advantage, the curved triangular element may be directly compared to the axisymmetric element with meridional curvature. However, it also offers an additional advantage, for which there is no correspondence in axisymmetric finite element shell analysis, in that compatibility of displacements across boundaries between successive elements is relatively easy to achieve. This is contrary, as has been noted, to the situation for flat triangular elements.

A further potential advantage of the curved element, not directly associated with the two already alluded to, is that certain types of variation of membrane stresses within the element can readily be represented in a curved element, whereas in the usual formulation of a flat triangular element the membrane stresses are constant throughout the element. The variation of membrane stress within a curved element arises from the fact that in a curved element, as will be seen, the membrane stresses depend not only upon the tangential displacements, which vary linearly, but also upon the normal displacement, which varies cubically. Since, as is well known, it is primarily the rapid variation of the normal displacement in the vicinity of the edge of a shell that is responsible for the rapid variation of membrane stresses near the edge, it may be anticipated that the curved element will provide a more accurate treatment than the flat element of shell problems involving rapid variation of edge stresses.

In the present paper, the formulation of an arbitrary, doubly-curved, linear, elastic triangular shell element is presented. The anslysis of the element is based on shallow shell theory in which the elevation of the element above a base plane varies quadratically over the surface of the element. The displacement components tangential to the surface of the element are assumed to vary linearly; the normal displacement is assumed to vary cubically in the manner expounded by Bazeley, Cheung, Irons, and Zienkiewicz (Reference 5) for plate bending. In order to represent the various displacement functions efficiently, the "area" or "natural" coordinates for the triangle are utilized.

After the assumed displacement functions have been formulated, the strain energy of the element is obtained for an arbitrary nodal displacement vector, whereupon the stiffness matrix for the element is readily obtained for a particular coordinate system. A set of coordinate transformations is then derived which make possible the combination of the various element stiffness matrices into the total stiffness matrix for the complete shell.

Finally, to demonstrate the use of the curved element, it is used to solve a problem in shell theory and the results are compared with an analytical solution and also with results obtained from a similar treatment of the problem using flat triangular elements.

SECTION II

NATURAL COORDINATES

The subsequent calculations involving the specification of position within a triangular area are much simplified (Reference 5) by the introduction of "natural" or "area" coordinates for the triangle. We here define natural coordinates and present certain useful relations involving them.

To define natural coordinates for a particular triangle, suppose that the triangle is referred to cartesian coordinates, with vertices at the points (x_1, y_1), (x_2, y_2), and (x_3, y_3), as shown in Figure 1. An arbitrary point (x,y) within the triangle divides it into three smaller triangles, which can be numbered according to the vertex each is opposite. If the areas of these triangles are A_1, A_2, and A_3, and if the area of the original triangle is A, the natural coordinates of the point are

$$L_i = A_i/A \quad , \quad i = 1, 2, 3 \tag{1}$$

The three natural coordinates of the point are not completely independent; because of the relation among the areas

$$A_1 + A_2 + A_3 = A$$

the natural coordinates are connected by the relation

$$L_1 + L_2 + L_3 = 1 \tag{2}$$

If the expressions giving the A_i are substituted into Equation 1, more convenient expressions for the natural coordinates follow. For instance,

$$L_1 = (1/2A)(a_1 + b_1 x + c_1 y)$$

where

$$\begin{aligned} a_1 &= x_2 y_3 - x_3 y_2 \\ b_1 &= y_2 - y_3 = y_{23} \\ c_1 &= x_3 - x_2 = x_{32} \end{aligned} \tag{3}$$

and where similar expressions for L_2 and L_3 can be obtained by a cyclic interchange of the suffices 1, 2, 3.

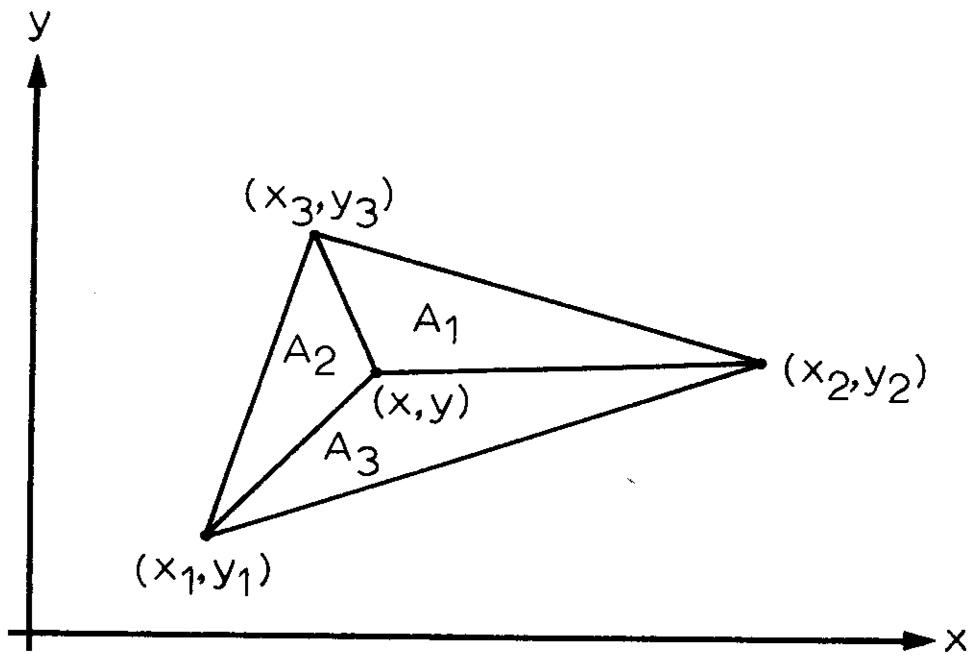


Figure 1. Area Coordinates

It is noteworthy that the natural coordinates have been defined so that L_i has the value 1 at the vertex (x_i, y_i) and zero at each of the other two vertices. Since the natural coordinates are linearly related to the cartesian coordinates, it then follows that the latter are given in terms of the former by the relations

$$x = x_1 L_1 + x_2 L_2 + x_3 L_3$$
$$y = y_1 L_1 + y_2 L_2 + y_3 L_3 \tag{4}$$

More generally, any quantity X with nodal values X_1, X_2, and X_3 which varies linearly with x and y may be written in the form

$$X = X_1 L_1 + X_2 L_2 + X_3 L_3$$

Expressions for the derivatives of the natural coordinates with respect to the cartesian coordinates follow immediately from Equation 2 and its analogs:

$$L_{i,x} = b_i / 2A \quad ; \quad L_{i,y} = c_i / 2A \tag{5}$$

Alternative expressions are obtained if the b_i and c_i are expressed in terms of the y_{jk} and x_{kj} using Equations 3 and similar relations.

In deriving stiffness and other matrices, we have occasion to integrate products of the natural coordinates over the area of the triangle. It is possible to derive the following formula:

$$\iint_A L_1^m L_2^n L_3^p \, dx \, dy = \frac{2A \, m! \, n! \, p!}{(m + n + p + 2)!} \tag{6}$$

SECTION III

MIDSURFACE OF ELEMENT

Suppose that we are given a curved triangular element of a shell. For the time being we may refer the element to a system of cartesian coordinates (x', y', z') associated with the element and oriented so that the x' - y' plane is parallel to the plane of the three corners of the element, as indicated in Figure 2. To determine the curvature of the element we will also consider the midpoint of each of its edges. We note that each point on the surface of the triangular element determines - and is determined by - its projection in the x' - y' plane. In particular, a point on the surface of the element can be determined by specifying the natural coordinates of the corresponding point in the base triangle in the x' - y' plane.

Assuming now that the local cartesian coordinates of each of the six nodal points of the element are given, we wish to obtain an analytical description of the surface of the element suitable for the application of shallow shell theory. For the latter, it is usually assumed that the elevation z' of the shell surface may be expressed as a polynomial of second degree in the coordinates of the base plane. Such a polynomial may be expressed in terms of natural coordinates as follows:

$$z' = C_1 L_1 + C_2 L_2 + C_3 L_3 + C_4 L_3 L_2 + C_5 L_3 L_1 + C_6 L_1 L_2$$

By substituting the coordinates of each nodal point successively into this equation, we obtain six equations to determine the six constants C_i. If $\mathbf{C}$ is the column vector of these constants and $\mathbf{Z}$ is the column vector of the z'-coordinates of the nodal points, these equations can be solved to give

$$\mathbf{C} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & -2 & -2 & 4 & \cdot & \cdot \\ -2 & \cdot & -2 & \cdot & 4 & \cdot \\ -2 & -2 & \cdot & \cdot & \cdot & 4 \end{bmatrix} \mathbf{Z}$$

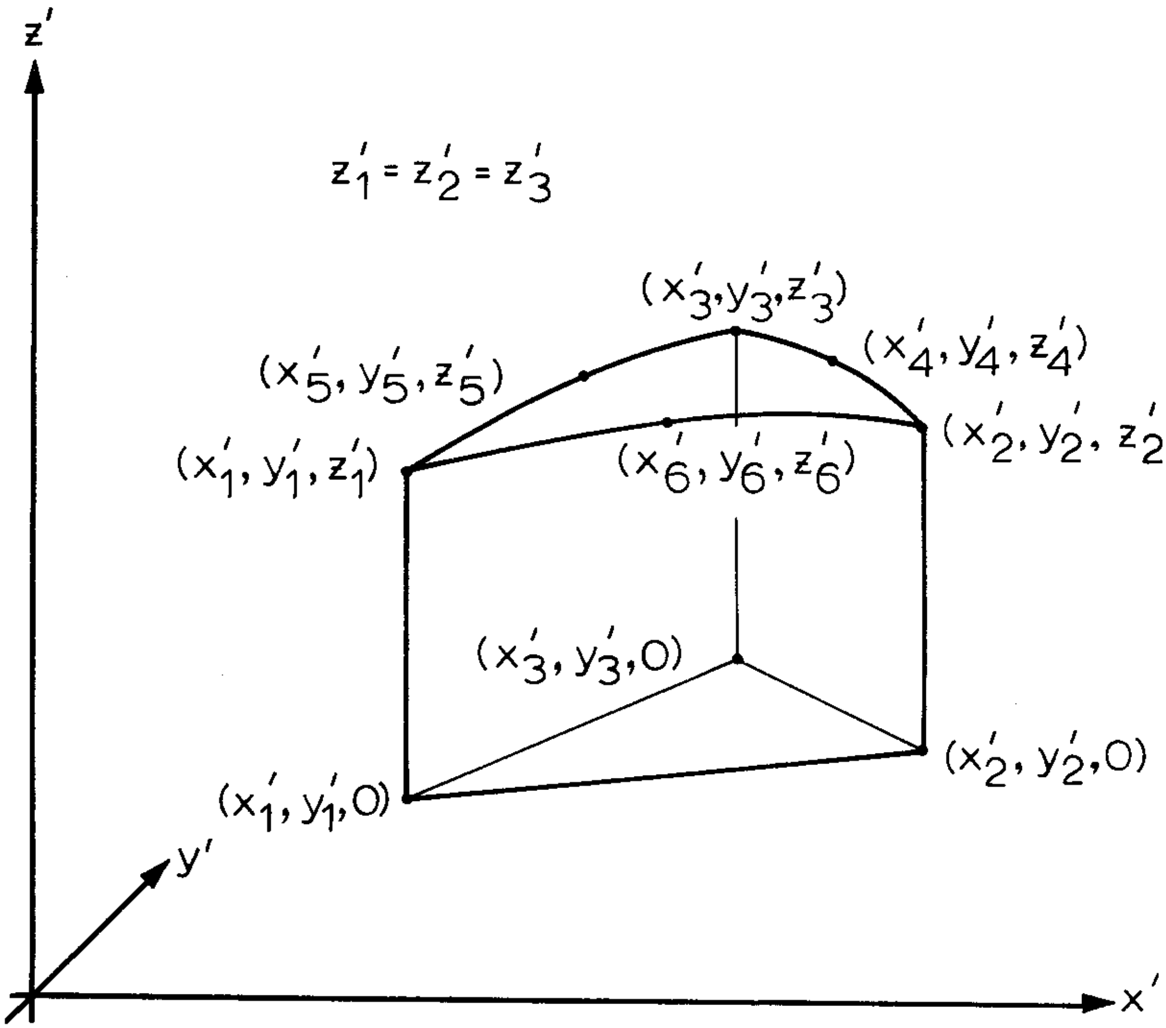


Figure 2. Curved Triangular Element

Hence the equation for the surface can be written in the form

$$z' = \mathbf{Z}^T \begin{bmatrix} 1 & \cdot & \cdot & \cdot & -2 & -2 \\ \cdot & 1 & \cdot & -2 & \cdot & -2 \\ \cdot & \cdot & 1 & -2 & -2 & \cdot \\ \cdot & \cdot & \cdot & 4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 4 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_2 L_3 \\ L_3 L_1 \\ L_1 L_2 \end{bmatrix} \qquad (7)$$

In deriving stiffness and other matrices governing the elastic behavior of a shell element, it is necessary to integrate various functions over the mid-surface of the element. That is, we are concerned with integrals of the form

$$\iint_S \mathcal{J}(\xi'_1, \xi'_2)\, ds$$

where the integration is taken over the area of the element. However, we will limit our attention to shallow shell elements where we can apply shallow shell theory and where in particular the square of the slope of the element relative to the base plane (the x' - y' plane) may be neglected. Under these conditions (Reference 6) we may equate the element of surface area to an element of area in the base plane, so that the above integral becomes

$$\iint_A \mathcal{J}(x'_1, x'_2)\, dx'_1\, dx'_2$$

where now the integration is carried out over the area of the base triangle.

SECTION IV

COORDINATE TRANSFORMATIONS

In the course of the analysis to be presented, it will be necessary to consider several coordinate transformations. We therefore describe here the various coordinate systems that arise, and derive the expressions giving the transformations from one coordinate system to another. The transformation matrices are needed partly to transform the coordinates themselves, but even more for transforming vectors and matrices from one coordinate system to another.

Suppose we have two coordinate systems, x_a $(=x_a, y_a, z_a)$ and x_b $(=x_b, y_b, z_b)$, and a fixed vector expressed in each of the two coordinate systems by $\mathbf{Q}_a = \{ Q_{a1}\ Q_{a2}\ Q_{a3} \}$ and $\mathbf{Q}_b = \{ Q_{b1}\ Q_{b2}\ Q_{b3} \}$, respectively. There will be a 3 x 3 matrix transformation giving $\mathbf{Q}_b$ in terms of $\mathbf{Q}_a$. Call this matrix $\mathbf{T}_{ba}$ so that we can write

$$\mathbf{Q}_b = \mathbf{T}_{ba}\ \mathbf{Q}_a \tag{8}$$

By considering the special case of a radius vector so that $\mathbf{Q}_b = \{ x_b\ y_b\ z_b \}$ and $\mathbf{Q}_a = \{ x_a\ y_a\ z_a \}$, we have

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \mathbf{T}_{ba} \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix}$$

showing that $\mathbf{T}_{ba}$ is the matrix of direction cosines arising in the coordinate transformation.

Note also that if we introduce a third coordinate system x_c in which the vector has the representation $\mathbf{Q}_c$, a second transformation matrix $\mathbf{T}_{cb}$ is implied such that

$$\mathbf{Q}_c = \mathbf{T}_{cb}\ \mathbf{Q}_b \tag{9}$$

We could also consider a transformation directly from x_a to x_c. The corresponding vector transformation would be

$$\mathbf{Q}_c = \mathbf{T}_{ca}\ \mathbf{Q}_a \tag{10}$$

The question of what is the relation between the various transformation matrices arises. Substitution of Equation 8 into Equation 9 and comparison of the resulting equations with Equation 10 yields the familiar chain rule

$$\mathbf{T}_{ac} = \mathbf{T}_{cb}\,\mathbf{T}_{ba} \tag{11}$$

We are also interested in how a matrix transforms under a coordinate transformation. Suppose we have a relation between two vectors $\mathbf{Q}_a$ and $\mathbf{R}_a$ in the x_a coordinate system

$$\mathbf{Q}_a = \mathbf{K}_a\,\mathbf{R}_a$$

It then follows that the corresponding relation in the x_b system is given by the matrix $\mathbf{K}_b$, where

$$\mathbf{K}_b = \mathbf{T}_{ba}\,\mathbf{K}_a\,\mathbf{T}_{ba}^T \tag{12}$$

The first specific coordinate transformation which we consider is that giving the local cartesian coordinates for an element, as defined in the previous section, in terms of a fixed, global coordinate system. Actually, as indicated in Figure 3, we specialize the local coordinates by assuming that the x' axis is parallel to the 1-2 side of the base triangle. The y' axis is also in the plane of the base triangle, and the x' axis is perpendicular. With these specifications, the derivation of the matrix $\mathbf{T}_{x'x}$ for the transformation of a vector from the global (x) to the local (x') coordinate system is a routine manner.

We next consider the transformation from local cartesian coordinates to what we call local surface coordinates (ξ'_1, ξ'_2, ξ'_3). The ξ'_1 and ξ'_2 coordinate lines are obtained by projecting the x' and y' axes, respectively, onto the surface of the element; the ξ'_3 coordinate lines are normal to the surface of the element. This coordinate system is needed only for points on the surface of the element; we need not define it elsewhere. It may be noted that the ξ'_1 and ξ'_3 coordinate lines will in general not be orthogonal. However, if the shell element is everywhere shallow, the departure from orthogonality will be a small effect of second order similar to others ignored in shallow shell theory (Reference 6).

In Figure 4 we indicate the local surface coordinates at one corner of the curved element, and also the local cartesian coordinates superposed to show how far unit lengths of the local surface coordinate lines deviate from the local cartesian coordinate lines. ϕ_y and ϕ_x are the rotations of the ξ'_1 and ξ'_2 axis out of a plane parallel to the x' - y' plane, and are here taken positive when in the direction shown. Hence

$$\phi_y = -\frac{\partial z'}{\partial x'} \quad ; \quad \phi_x = -\frac{\partial z'}{\partial y'} \tag{13}$$

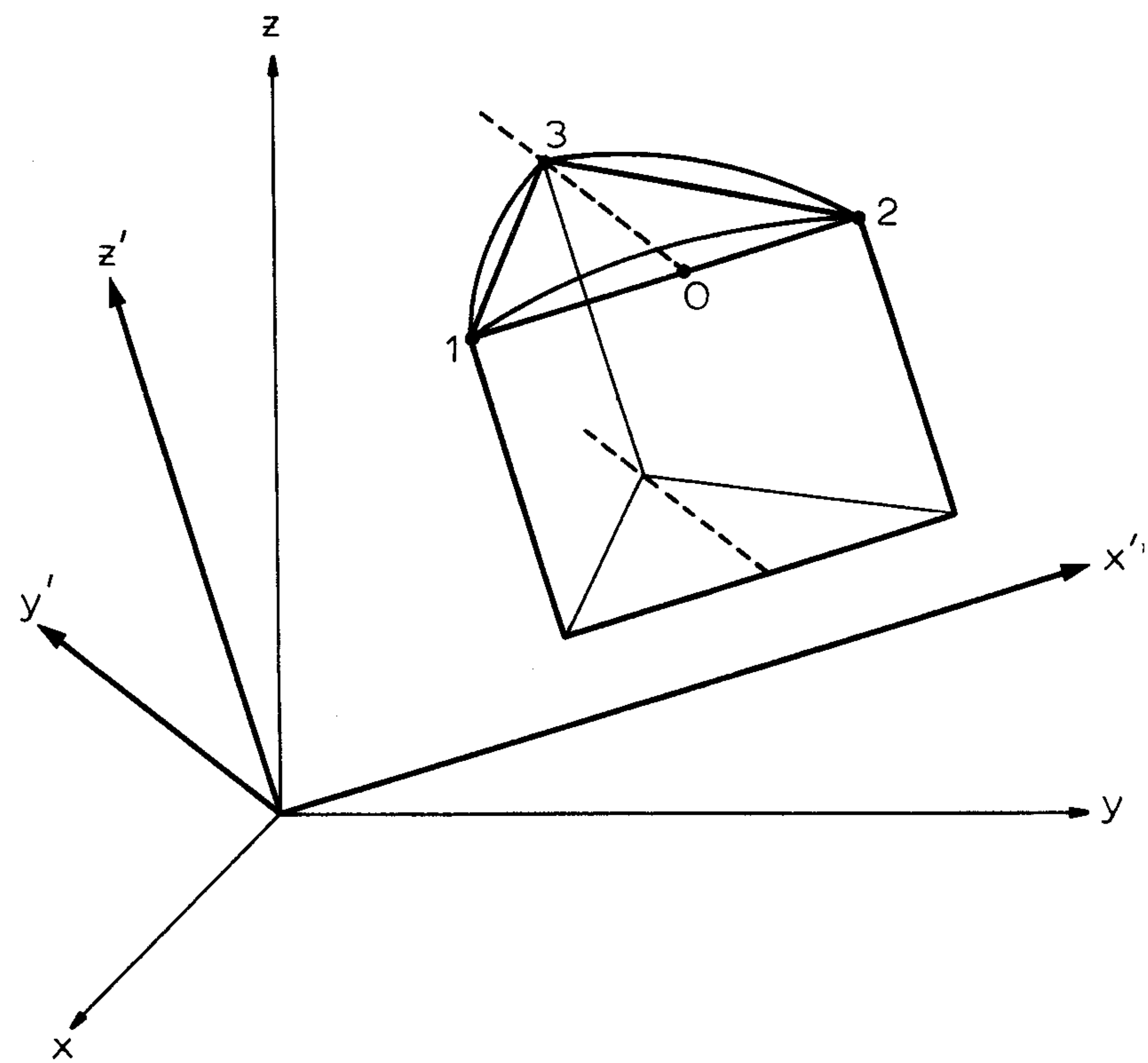
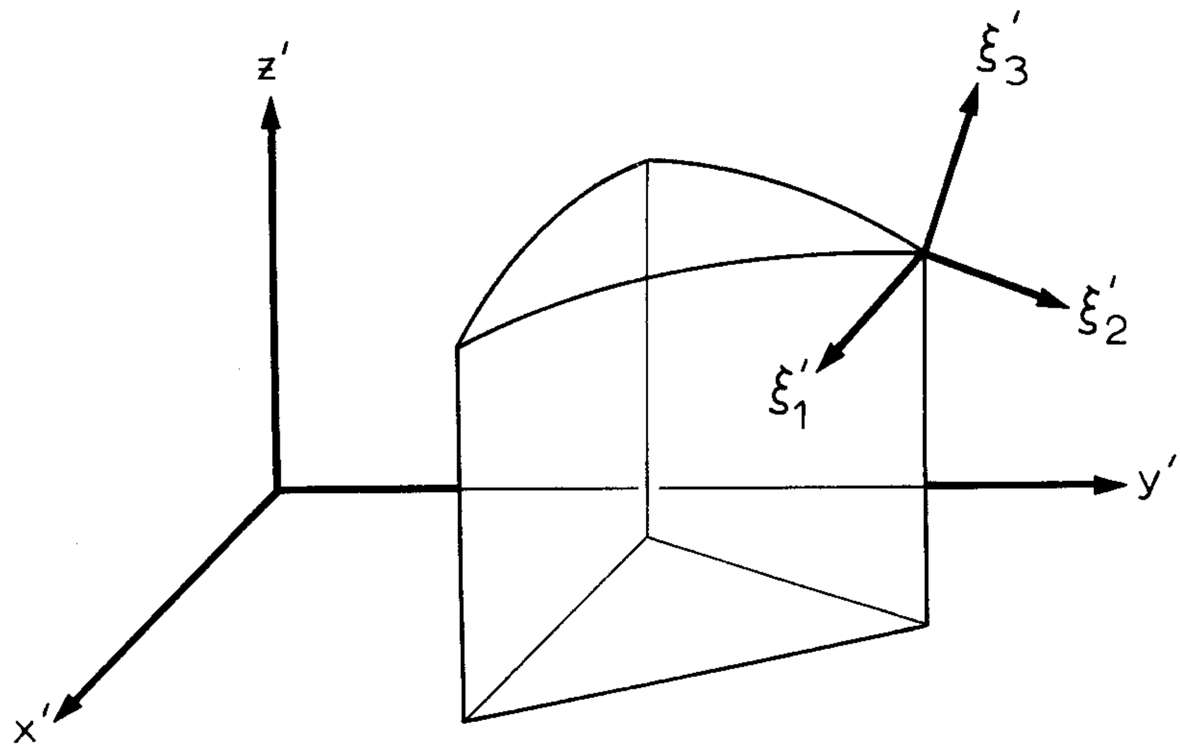
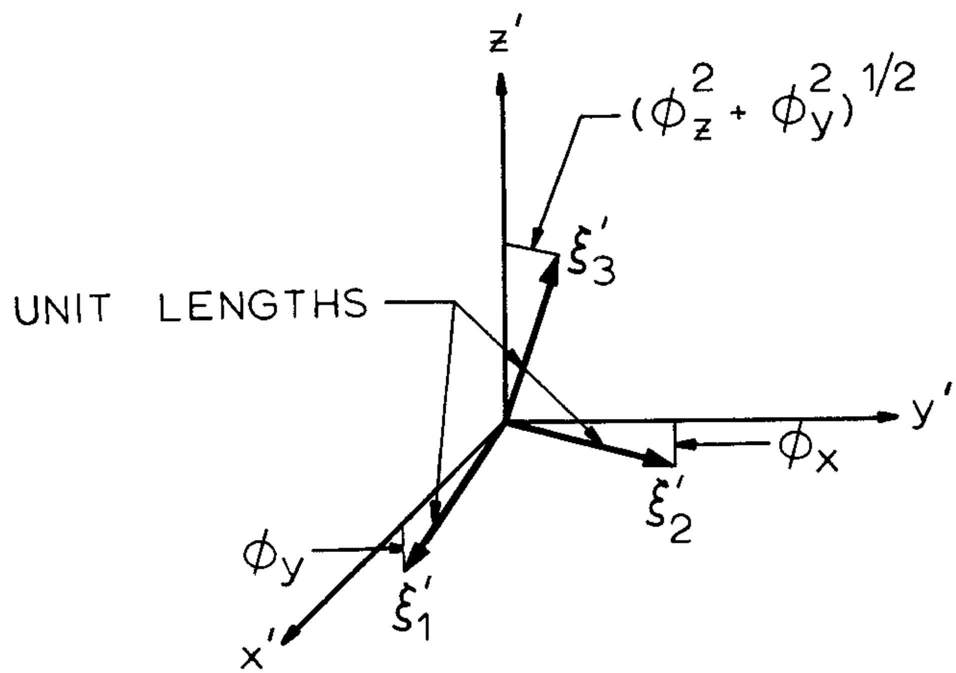


Figure 3. Global and Local Coordinates



(a) Relation of Local Surface and Local Cartesian Coordinates



(b) Superposition of Coordinate Systems

Figure 4. Local Surface and Cartesian Coordinates

As indicated in the figure, the angular deviation of ξ'_3 from the z' axis is $(\phi_x^2 + \phi_y^2)^{1/2}$ to within the accuracy of shallow shell theory. Evaluating the direction cosines of the local surface coordinates with respect to the local cartesian coordinates, we obtain the following transformation matrix:

$$\mathbf{T}_{\xi' x'} = \begin{bmatrix} (1-\phi_y^2)^{1/2} & 0 & -\phi_y \\ 0 & (1-\phi_x^2)^{1/2} & -\phi_x \\ \phi_y & \phi_x & (1-\phi_x^2-\phi_y^2)^{1/2} \end{bmatrix} \tag{14}$$

This matrix, and hence ϕ_x and ϕ_y, needs to be evaluated at each of the three corner nodes of the element, using Equations 5, 7, and 13.

The final transformation that will be needed is one connecting the global cartesian coordinate system with a global surface coordinate system independent of the orientation of the element boundaries. In most cases of practical interest a convenient global surface coordinate system is provided by the same coordinate system used to define the surface. For instance, fixing the radius of a spherical coordinate system defines a particular spherical surface; the remaining two spherical coordinates then define position on that surface. Given analytical relations between the global cartesian (x) and global surface (ξ) coordinate systems, the derivation of the corresponding $\mathbf{T}_{x\xi}$ transformation matrix is routine (Reference 7).

SECTION V

STRAINS AND STRESSES

In the following we use the shallow shell theory of Novozhilov (Reference 6). This is more convenient for our purposes since it refers quantities to the surface of the shell element rather than to the base plane, as in the theory of Marguerre. For instance, u and v are displacements in the ξ'_1 and ξ'_2 directions, not in the x' and y' directions.

In this theory the membrane strains have the form

$$\begin{aligned} \epsilon_1 &= \frac{\partial u}{\partial x'} - \frac{\partial^2 z'}{\partial x'^2} w \\ \epsilon_2 &= \frac{\partial v}{\partial y'} - \frac{\partial^2 z'}{\partial y'^2} w \\ \gamma_{12} &= \frac{\partial u}{\partial y'} + \frac{\partial v}{\partial x'} - 2 \frac{\partial^2 z'}{\partial x' \partial y'} w \end{aligned} \tag{15}$$

Here ϵ_1, ϵ_2, and γ_{12} are the strains in the surface of the element referred to the ξ'_1 and ξ'_2 axes, and w is the displacement normal to the surface.

The bending strains are given to within shallow shell approximation by

$$\begin{aligned} \kappa_1 &= \partial^2 w / \partial x'^2 \\ \kappa_2 &= \partial^2 w / \partial y'^2 \\ \kappa_{12} &= -2\partial^2 w / \partial x' \partial y' \end{aligned} \tag{16}$$

We divide the strain energy, u , of the element, as is usual in shell theory, into membrane and bending portions, u_m and u_b, respectively.

The membrane strain energy is the integral

$$u_m = \frac{1}{2} \iint_A (N_1 \epsilon_1 + N_2 \epsilon_2 + N_{12} \epsilon_{12}) \, dx' \, dy'$$

where the N's are the usual membrane stress resultants. If we utilize the matrix notation

$$\mathbf{N} = \{ N_1 \ N_2 \ N_{12} \}$$

$$\boldsymbol{\epsilon} = \{ \epsilon_1 \ \epsilon_2 \ \epsilon_{12} \}$$

the membrane strain energy can be written in the form

$$u_m = \frac{1}{2} \iint_A \mathbf{N}^T \boldsymbol{\epsilon} \, dx' dy' \tag{17}$$

In similar fashion we can write for the bending strain energy the integral

$$u_b = \frac{1}{2} \iint_A (M_1 \kappa_1 + M_2 \kappa_2 + M_{12} \kappa_{12}) \, dx' dy'$$

where the M's are the usual stress couples. Utilizing the additional matrix notation

$$\mathbf{M} = \{ M_1 \; M_2 \; M_{12} \}$$

$$\boldsymbol{\kappa} = \{ \kappa_1 \; \kappa_2 \; \kappa_{12} \}$$

we can rewrite the bending strain energy in the form

$$u_b = \frac{1}{2} \iint_A \mathbf{M}^T \boldsymbol{\kappa} \, dx' dy' \tag{18}$$

The stress-strain relations are those usually derived in works on shell theory (Reference 8). For our present purpose they are conveniently written in the form

$$\mathbf{N} = \mathbf{D}_m \boldsymbol{\epsilon} \tag{19}$$

$$\mathbf{M} = \mathbf{D}_b \boldsymbol{\kappa} \tag{20}$$

where $\mathbf{D}_m$ and $\mathbf{D}_b$ are matrices of elastic coefficients.

Substitution of Equations 9 and 20 into Equations 17 and 18, respectively, yields for the strain energy expressions

$$u_m = \frac{1}{2} \iint_A \boldsymbol{\epsilon}^T \mathbf{D}_m \boldsymbol{\epsilon} \, dx' dy' \tag{21}$$

$$u_b = \frac{1}{2} \iint_A \boldsymbol{\kappa}^T \mathbf{D}_b \boldsymbol{\kappa} \, dx' dy' \tag{22}$$

SECTION VI

ASSUMED DISPLACEMENT FUNCTIONS

To define the assumed displacement field we begin by taking a tangential displacement field which varies linearly over the surface of the base triangle of the element. Such a field can be expressed in terms of the natural coordinates of the base triangle by the expressions

$$u = u_1 L_1 + u_2 L_2 + u_3 L_3$$

$$v = v_1 L_1 + v_2 L_2 + v_3 L_3$$

where u and v are the tangential displacements in the ξ_1' and ξ_2' directions, and where the u_i and v_i are the nodal values of these displacements.

In matrix notation this becomes

$$\begin{aligned} u &= \mathbf{L}^T \mathbf{u} \\ v &= \mathbf{L}^T \mathbf{v} \end{aligned} \qquad (23)$$

where

$$\mathbf{u} = \{u_1 \; u_2 \; u_3\}$$

$$\mathbf{v} = \{v_1 \; v_2 \; v_3\}$$

$$\mathbf{L} = \{L_1 \; L_2 \; L_3\}$$

For the normal displacement of the element, we follow Reference 5 in many respects. The normal displacement is taken as the sum of two contributions:

$$w = w^R + w^*$$

The contribution w^R varies linearly over the element and has values at the nodes equal to the total normal displacement:

$$w^R = w_1 L_1 + w_2 L_2 + w_3 L_3$$

The contribution w* has zero nodal values; within the element it varies cubically in such a way that at the nodes it has slopes equal to the nodal values of the slopes of $w - w^R$. It is shown in Reference 5 that under these conditions w* can be written in the form

$$w^* = F_{x1} W_{11}^* + F_{y1} W_{21}^* + F_{x2} W_{12}^* + \cdots + F_{y3} W_{23}^*$$

Here W_{1i}^* and W_{2i}^* are the nodal values of $-w^*, \xi_2$ and w^*, ξ_1, respectively, and the F_{xi} and F_{yi} are functions of the natural coordinates derived in Reference 5, where there is also derived a transformation matrix **T** giving the reduced normal displacement vector

$$\boldsymbol{\delta}^* = \left\{ W_{11}^* \; W_{21}^* \; W_{12}^* \; W_{22}^* \; W_{13}^* \; W_{23}^* \right\}$$

in terms of the full normal displacement vector

$$\boldsymbol{\delta} = \left\{ w_1 \; W_{11} \; W_{21} \; w_2 \; W_{12} \; W_{22} \; w_3 \; W_{13} \; W_{23} \right\}$$

by means of the equation

$$\boldsymbol{\delta}^* = \mathbf{T} \boldsymbol{\delta}$$

In the above, W_{1i} and W_{2i} are the nodal values of $-w, \xi_2$ and w, ξ_1, respectively.

If we now introduce the vectors

$$\boldsymbol{\Lambda}_1 = \left\{ L_1 \; 0 \; 0 \; L_2 \; 0 \; 0 \; L_3 \right\}$$

$$\mathbf{F} = \left\{ F_{x1} \; F_{y1} \; F_{x2} \; F_{y2} \; F_{x3} \; F_{y3} \right\}$$

we can express the portions of the normal displacement as

$$w^* = \mathbf{F}^T \; \mathbf{T} \boldsymbol{\delta}$$

$$w^R = \boldsymbol{\Lambda}_1^T \; \boldsymbol{\delta}$$

Hence the total normal displacement is given by

$$w = (\boldsymbol{\Lambda}_1^T + \mathbf{F}^T \; \mathbf{T}) \boldsymbol{\delta} \qquad (24)$$

We are now in a position to express the strains and the strain energy in terms of the nodal displacement vectors. If we insert Equations 23 and 24 into the strain-displacement Relations 15, the result can be written as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \frac{\partial}{\partial x'} & 0 & k_1 \\ 0 & \frac{\partial}{\partial y'} & k_2 \\ \frac{\partial}{\partial y'} & \frac{\partial}{\partial x'} & k_3 \end{bmatrix} \begin{bmatrix} \mathbf{L}^T \ \mathbf{u} \\ \mathbf{L}^T \ \mathbf{v} \\ (\boldsymbol{\Lambda}_1^T + \mathbf{F}^T \ \mathbf{T}) \ \boldsymbol{\delta} \end{bmatrix}$$

where

$$k_1 = -\frac{\partial^2 z'}{\partial x'^2} \quad , \quad k_2 = -\frac{\partial^2 z'}{\partial y'^2} \quad , \quad k_3 = -2\,\frac{\partial^2 z'}{\partial x' \partial y'}$$

This can be rewritten in a form which separates the displacement vector:

$$\boldsymbol{\epsilon} = \begin{bmatrix} \frac{\partial}{\partial x'} & 0 & k_1 \\ 0 & \frac{\partial}{\partial y'} & k_2 \\ \frac{\partial}{\partial y'} & \frac{\partial}{\partial x'} & k_3 \end{bmatrix} \begin{bmatrix} \mathbf{L}^T & 0 & 0 \\ 0 & \mathbf{L}^T & 0 \\ 0 & 0 & (\boldsymbol{\Lambda}_1^T + \mathbf{F}^T \mathbf{T}) \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \boldsymbol{\delta} \end{bmatrix}$$

Combining the first two matrices given, we have

$$\boldsymbol{\epsilon} = \begin{bmatrix} \mathbf{L}^T_{,x'} & 0 & k_1(\boldsymbol{\Lambda}_1^T + \mathbf{F}^T \ \mathbf{T}) \\ 0 & \mathbf{L}^T_{,y'} & k_2(\boldsymbol{\Lambda}_1^T + \mathbf{F}^T \ \mathbf{T}) \\ \mathbf{L}^T_{,y'} & \mathbf{L}^T_{,x'} & k_3(\boldsymbol{\Lambda}_1^T + \mathbf{F}^T \ \mathbf{T}) \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \boldsymbol{\delta} \end{bmatrix}$$

If we introduce the notations

$$\boldsymbol{\mu} = \{ \mathbf{u} \quad \mathbf{v} \}$$

$$\mathbf{k} = \{ k_1 \ k_2 \ k_3 \}$$

$$\mathcal{L} = \begin{bmatrix} \mathbf{L}^T_{,x'} & 0 \\ 0 & \mathbf{L}^T_{,y'} \\ \mathbf{L}^T_{,y'} & \mathbf{L}^T_{,x'} \end{bmatrix}$$

the strain can finally be written in the form

$$\epsilon = \left[\mathcal{L} \quad k(\Lambda_1^T + F^T T \right] \left[\begin{matrix} \mu \\ \delta \end{matrix} \right] \tag{25}$$

SECTION VII

STIFFNESS MATRIX

To obtain the membrane portion of the stiffness matrix, we insert Equation 25 into the Expression 21 for the membrane strain energy. When this is done and the indicated matrix multiplications are partly carried out, we obtain:

$$U_m = \frac{1}{2}\left[\mu^T \ \delta^T\right] \iint_A \begin{bmatrix} \mathcal{L}^T D_m \mathcal{L}^T & \mathcal{L}^T D_m k(\Lambda_1^T + F^T T) \\ (\Lambda_1 + T^T F)\, k^T D_m \mathcal{L} & (\Lambda_1 + T^T F)\, k^T D_m k(\Lambda_1^T + F^T T) \end{bmatrix} dx'dy' \begin{bmatrix} \mu \\ \delta \end{bmatrix} \tag{26}$$

When the strain energy is written in this form the stiffness matrix is the one between the displacement vector and its transpose. We therefore see from Equation 26 that the membrane portion of the stiffness matrix can be written in the partitioned form

$$K_m = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix}$$

where

$$K_{11} = \iint_A \mathcal{L}^T D_m \mathcal{L} \ dx'dy'$$

$$K_{22} = \iint_A (\Lambda_1 + T^T F)\, k^T D_m k (\Lambda_1^T + F^T T)\, dx'dy'$$

$$K_{12} = \iint_A \mathcal{L}^T D_m k(\Lambda_1^T + F^T T)\, dx'dy'$$

Evaluation of the second and third integrals is somewhat tedious, but not too bad with proper organization and with the use of Equation 6.

To obtain the bending portion of the stiffness matrix we would insert the Expression 24 for the lateral displacement into the bending strain Expressions 16 and then insert the latter into Equation 22 for the bending strain energy. To do so, however, would be to retrace essentially the same steps, aside from notational differences, as were followed in Reference 5 in deriving the stiffness matrix for the flat plate. We have the same displacement field, the same strain expressions, and the same strain energy expression; therefore, the same stiffness matrix will arise.

Instead of this duplication, we may simply take the bending stiffness matrix of Reference 5. The only difficulty will be that this 9 x 9 matrix -- call it $\mathbf{K}_b'$ -- relates only to the bending displacements; before it can be combined with the membrane stiffness matrix by direct addition, it must be shifted into the lower right-hand corner of an otherwise null 15 x 15 matrix. Once this has been done, the resulting matrix, $\mathbf{K}_b$, say, can be added to the membrane stiffness matrix $\mathbf{K}_m$ to form the total stiffness matrix:

$$\mathbf{K} = \mathbf{K}_m + \mathbf{K}_b \tag{27}$$

SECTION VIII

FINAL TRANSFORMATIONS

The stiffness matrix given by Equation 27 gives the force vector corresponding to the displacement vector

$$\left\{ u_1 \;\; u_2 \;\; u_3 \;\; v_1 \;\; v_2 \;\; v_3 \;\; w_1 \;\; W_{11} \;\; W_{21} \;\; w_2 \;\; W_{12} \;\; W_{22} \;\; w_3 \;\; W_{13} \;\; W_{23} \right\}$$

However, we require a stiffness matrix for a displacement vector in which the rotation components appear rather than the W_{ij} terms above. The rotation components at each node are given by relations of the form

$$\theta_{1i} = W_{1i} + k_2 v_i \quad ; \quad \theta_{2i} = W_{2i} - k_1 u_i$$

It therefore follows that the above displacement vector can be obtained from the desired one by an equation of the form

$$U = T_2 U'$$

where T_2 is given by

$$T_2 = \begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -k_2 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
k_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -k_2 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & k_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -k_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & k_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix} .$$

By virtue of Equation 12, the stiffness matrix related to the desired displacement vector can be obtained from $\mathbf{K}$ by the equation

$$\mathbf{K}' = \mathbf{T}_2^T \, \mathbf{K} \, \mathbf{T}_2$$

Next, it is convenient to transform to a displacement vector in which the components are grouped node by node, for instance, in the order

$$\left\{ u_1 \ v_1 \ w_1 \ \theta_{11} \ \theta_{21} \ u_2 \ v_2 \ w_2 \ \theta_{12} \ \theta_{22} \ u_3 \ v_3 \ w_3 \ \theta_{13} \ \theta_{23} \right\}$$

The corresponding transformation of the stiffness matrix is accomplished by a straightforward interchange of rows and of columns of $\mathbf{K}'$ to give $\mathbf{K}''$, say.

Finally, we need to transform from the local surface coordinates to which the stiffness matrix presently refers to the global surface coordinates. For this we need the transformation matrix $\mathbf{T}_{\xi\xi'}$, which, by virtue of Equation 11, can be taken as

$$\mathbf{T}_{\xi\xi'} = \mathbf{T}_{\xi x}^T \, \mathbf{T}_{xx'} \, \mathbf{T}_{x'\xi'} \tag{28}$$

The three factors on the right of Equation 28 have all been previously developed; they will yield a 3 by 3 matrix $\mathbf{T}_{\xi\xi'}$ of the form

$$\mathbf{T}_{\xi\xi'_i} = \begin{bmatrix} \mathbf{T}_{P_i} & 0 \\ 0 & 1 \end{bmatrix} \quad , \qquad i = 1, 2, 3$$

which will vary from node to node. If we form the 5 by 5 matrices

$$\mathcal{T}_i = \begin{bmatrix} \mathbf{T}_{P_i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbf{T}_{P_i} \end{bmatrix}$$

and from them the 15 by 15 matrix

$$\mathbf{T}_3 = \begin{bmatrix} \mathcal{T}_1 & 0 & 0 \\ 0 & \mathcal{T}_2 & 0 \\ 0 & 0 & \mathcal{T}_3 \end{bmatrix}$$

we can obtain finally, again using Equation 12,

$$\mathbf{K}''' = \mathbf{T}_3^T \, \mathbf{K}'' \, \mathbf{T}_3 \tag{29}$$

as the stiffness matrix referred to the global surface coordinate system and ready for application of the direct stiffness method to form the stiffness matrix of the assembled structure.

SECTION IX

SAMPLE CALCULATIONS

The stiffness matrix given by Equation 29 has been programmed for the UNIVAC 1108 digital computer and incorporated into a general program for shell analysis. In the course of debugging, the curved element has been applied to a number of problems, which have also been solved with a program due to Clough and Johnson (Reference 2) which employs a flat triangular shell element. The results from both programs have been compared with analytical solutions, thereby allowing some idea to be formed of the relative value of the curved element.

A typical problem involved the analysis of a cylindrical roof panel with free sides and diaphragm end conditions, as illustrated in Figure 5. The curved element was used to obtain solutions for three different discretizations of a quadrant of the panel, namely, a 4 x 5, and 8 x 12, and a 12 x 18 mesh. Corresponding solutions were also obtained using the flat triangular element of Reference 2, and the convergence of each of the two approaches with increasing subdivision is indicated in Figure 5, where we plot the vertical displacement of the free edge at the plane of symmetry. The advantage provided by the curved element in this case is evident.

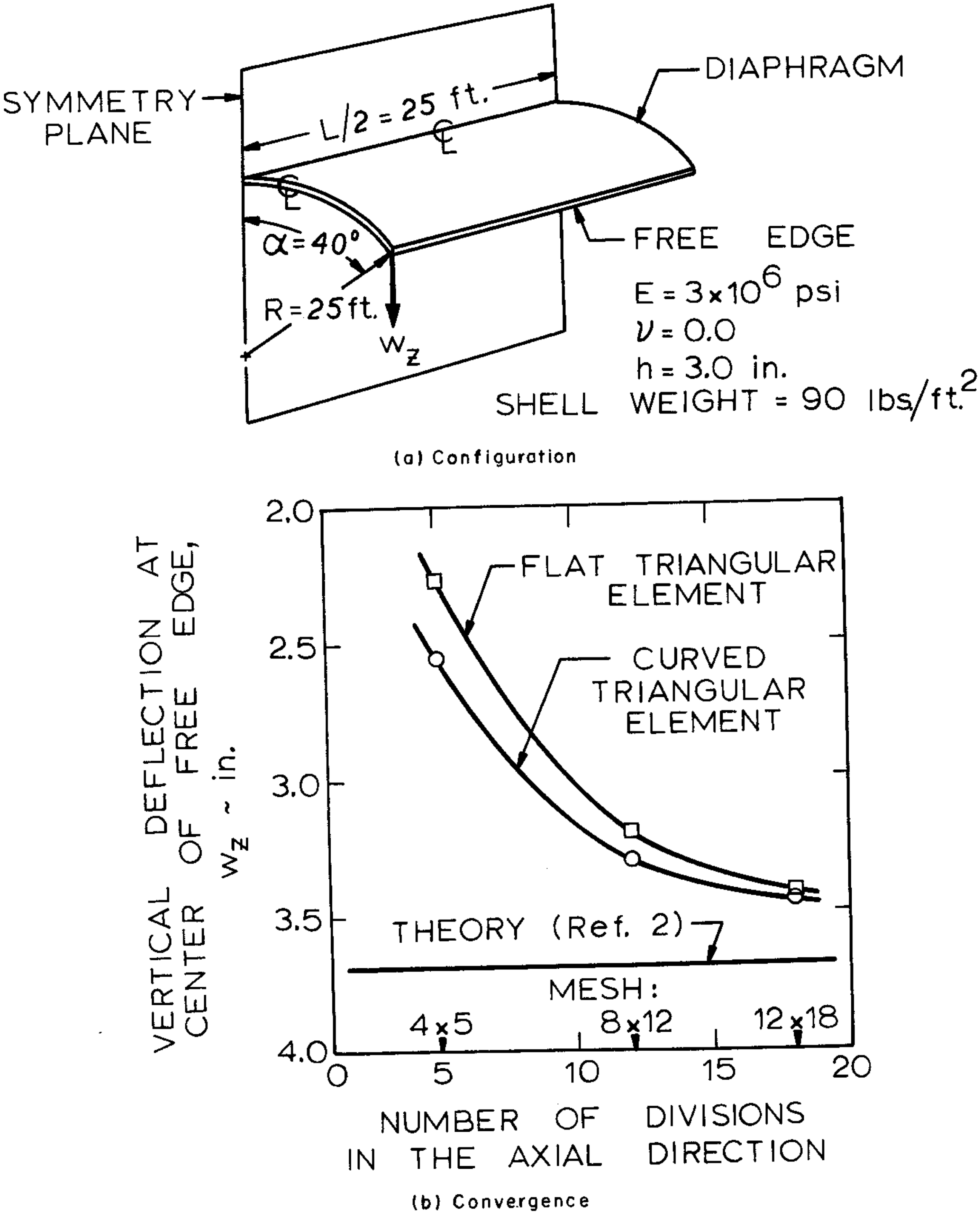


Figure 5. Sample Problem

SECTION X

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