AFFDL-TR-68-168

PARAMETRIC STUDY OF TSAI'S STRENGTH CRITERIA FOR FILAMENTARY COMPOSITES

R. S. SANDHU

FOREWORD

This report is the result of an in-house effort under Project No. 1473, "Exploratory Development in Structural Mechanics," Task No. 147306, "Stress and Stability Analysis of Heterogeneous Anisotropic Plates, Shells, and Arches." The work was carried out in the Advanced Theory and Analysis Groups of the Theoretical Mechanics Branch, Structures Division, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio. Mr. R. S. Sandhu (FDTR) was the task engineer.

The report was submitted by the author in December 1968 for publication.

The author gratefully acknowledges the assistance of Dr. W. A. Kapp in writing the computer program. He also extends his appreciation to Dr. N. S. Khot for suggestions during the preparation of the manuscript.

This technical report has been reviewed and is approved.

FRANCIS J. JANIK JR.
Chief, Theoretical Mechanics Branch
Structures Division
Air Force Flight Dynamics Laboratory

ABSTRACT

A parametric study of Tsai's strength criteria for unidirectional filamentary composites is presented. The purpose of this study is to determine the fiber orientation for the maximum strength under a given biaxial state of stress. The investigation indicates that the best fiber orientation depends upon the shear strength of the material in relation to its transverse strength. When the shear strength is less than the transverse strength, the optimum fiber orientation coincides with the principal stress direction. However, when the shear strength is greater than the transverse strength, the best fiber orientation does not always coincide with the principal stress direction. The results of this study are presented in the form of plots. The plots together with the equations presented in the text aid in determining the optimum orientation.

Distribution of this abstract is unlimited.

TABLE OF CONTENTS

ILLUSTRATIONS

TABLES

SECTION I
INTRODUCTION

Composite materials have been used for decades in structures. Reinforced concrete, asphaltic felts, reinforced rubber, and metal alloys are some familiar composite materials. Recently they have received more attention due to the advent of lightweight high-strength fibers. Composites formed by embedding fibers in matrix materials are nonhomogeneous and anisotropic. Methods of analysis and design developed for homogeneous and isotropic materials are not adequate for composites.

Often in structural design, the characteristic of interest is strength. For homogeneous and isotropic materials, well-established criteria are available; there is such a need for composites. Some work in this area has been done. Marin (Reference 1) developed a generalized theory of strength, in which he assumed the failure of an element subjected to triaxial stresses to be a function of the second stress invariant. It is identical to Hill's Theory (Reference 2) expressed in terms of principal stresses except for yield stress in shear. Marin determined the yield stress in shear by using an element with axes oriented at 45 degrees to the axes of material symmetry, whereas in Hill's work the orientation is the material axes. Tsai (Reference 3), using Hill's generalized Von Mises criteria, formulated a strength theory of composites. Briefly stated, the theory provides that for given strength characteristics and orientation of axes of material symmetry, stresses at failure can be determined. It is assumed the material layer is subjected to an inplane stress field determined by the ratios of the stresses.

The purpose of this report is to determine the optimum orientation of material axes for a given set of strength parameters and stress field using Tsai's criteria.

SECTION II

REVIEW OF YIELD CRITERIA

1. Yield Criteria for Isotropic Materials

A yield criteria function f (Reference 4) for isotropic materials can be defined as

$$f\left(I_1, I_2, I_3, \sigma_0\right) = 0 \tag{1}$$

where I_1, I_2, I_3, are the stress invariants and σ_0 is some significant material property. In the case of ductile isotropic materials, it has been observed that moderate mean normal stresses do not initiate yielding. Substracting the mean normal stresses from the stresses acting upon the material yields stress deviation. The function f expressed in terms of the stress invariants J_1, J_2, and J_3 of the stress deviation becomes

$$f\left(J_2, J_3, \sigma_0\right) = 0 \tag{2}$$

where J_1 is identically zero.

The frequently used form of Equation 2 is that of Von Mises. In this form it is assumed that J_2 is a constant. This formulation yields

$$2J_2 = \frac{1}{3}\left[\left(\sigma_{yy} - \sigma_{zz}\right)^2 + \left(\sigma_{zz} - \sigma_{xx}\right)^2 + \left(\sigma_{xx} - \sigma_{yy}\right)^2\right] + 2\left[\tau_{yz}^2 + \tau_{xz}^2 + \tau_{xy}^2\right] \tag{3}$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are the normal stresses and τ_{xy}, τ_{yz}, and τ_{xz} are the shear stresses referred to Cartesian coordinates x, y, and z.

In the case of a material subjected to uniaxial tensile yield stress σ_0, Equation 3 gives

$$J_2 = \frac{\sigma_0^2}{3} \tag{4}$$

Substitution of Equation 4 in Equation 3 yields

$$\frac{1}{2\sigma_0^2}\left[(\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + (\sigma_{xx} - \sigma_{yy})^2\right] + \frac{3}{\sigma_0^2}\left[\tau_{yz}^2 + \tau_{xz}^2 + \tau_{xy}^2\right] = 1 \tag{5}$$

Equation 5 is known as Von Mises' yield criteria.

2. Yield Criteria for Anisotropic Materials

To allow for the anisotropic properties of the materials, Hill (Reference 2) generalized Equation 5 as

$$F\,(\sigma_{yy} - \sigma_{zz})^2 + G\,(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 + 2\left[L\;\tau_{yz} + M\,\tau_{xz}^2 + N\,\tau_{xy}^2\right] = 1 \tag{6}$$

where F, G, H, L, M, and N are parameters characteristic of anisotropy of the material.

If X, Y, Z are the normal yield stresses and R, T, S are the yield stresses in shear with respect to the material axes x, y, and z of anisotropy, relations between parameters of anisotropy and the strength characteristics can be expressed as

$$\left.\begin{aligned} \frac{1}{X^2} &= H + G \\ \frac{1}{Y^2} &= H + F \\ \frac{1}{Z^2} &= G + F \\ \frac{1}{R^2} &= 2L \\ \frac{1}{T^2} &= 2M \\ \frac{1}{S^2} &= 2N \end{aligned}\right\} \tag{7}$$

Substitution of Equations 7 in Equation 6 gives

$$\frac{1}{2}\left[\left(\frac{1}{Y^2}+\frac{1}{Z^2}-\frac{1}{X^2}\right)(\sigma_{yy}-\sigma_{zz})^2+\left(\frac{1}{Z^2}+\frac{1}{X^2}-\frac{1}{Y^2}\right)(\sigma_{zz}-\sigma_{xx})^2\right.$$

$$\left.+\left(\frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}\right)(\sigma_{xx}-\sigma_{yy})^2\right]+\left[\left(\frac{\tau_{yz}}{R}\right)^2+\left(\frac{\tau_{xz}}{T}\right)^2+\left(\frac{\tau_{xy}}{S}\right)^2\right]=1 \tag{8}$$

Equation 8 is used as a strength criteria for an anisotropic material subjected to a three-dimensional stress field.

SECTION III

TSAI'S STRENGTH CRITERIA

In usual applications, most of the composites are thin and, therefore, a state of plane stress can be assumed to exist in them. Selecting z as the axis normal to the plane of plate-like material, the conditions for the plane stress field to exist are

$$\sigma_{zz} = \tau_{yz} = \tau_{xz} = 0 \tag{9}$$

Tsai (Reference 3) argued that fiber-reinforced composites can be treated as transversely isotropic, which means that

$$Z = Y \tag{10}$$

Substitution of results from Equations 9 and 10 in Equation 8 reduces it to the form

$$\sigma_{xx}^2 - \sigma_{xx}\,\sigma_{yy} + \frac{X^2}{Y^2}\,\sigma_{yy}^2 + \frac{X^2}{S^2}\,\tau_{xy}^2 = X^2 \tag{11}$$

In Equation 11 it is assumed that the stress field is defined with respect to the x and y axes. If it is defined relative to any other set of axes, σ_{xx}, σ_{yy}, and τ_{xy} can be computed by a simple transformation of the axes given in Equation 13.

In Figure 1, σ_1, σ_2, and τ_{12} are defined with respect to an orthogonal set of axes "1" and "2"; dividing by σ_1^2, Equation 11 can be rewritten in nondimensional form as

$$\left(\frac{\sigma_{xx}}{\sigma_1}\right)^2 - \left(\frac{\sigma_{xx}\,\sigma_{yy}}{\sigma_1^2}\right) + \left(\frac{X}{Y}\right)^2\left(\frac{\sigma_{yy}}{\sigma_1}\right)^2 + \left(\frac{X}{S}\right)^2\left(\frac{\tau_{xy}}{\sigma_1}\right)^2 = \left(\frac{X}{\sigma_1}\right)^2 \tag{12}$$

If x and y are the axes of material symmetry and α is the angle which x-axis makes with 1-axis, transformation of stresses from 1 and 2 axes to x and y axes yields

$$\begin{bmatrix} \frac{\sigma_{xx}}{\sigma_1} \\ \frac{\sigma_{yy}}{\sigma_1} \\ \frac{\tau_{xy}}{\sigma_1} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\,\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\,\cos\alpha \\ -\sin\alpha\,\cos\alpha & \sin\alpha\,\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \begin{bmatrix} 1 \\ n \\ k \end{bmatrix} \tag{13}$$

where

$$n = \frac{\sigma_2}{\sigma_1}$$

and

$$k = \frac{\tau_{12}}{\sigma_1}$$

Substitution of Equation 13 in Equation 12 (Reference 3) yields

$$\begin{aligned}
&\left[n^2 - n + a^2 + k^2\beta^2 \right] \sin^4 \alpha \; + \\
&2k\left[3n - 1 - 2a^2 - (n-1)\,\beta^2 \right] \sin^3\alpha \cos\alpha \; + \\
&\left[8k^2 - n^2 + 2n - 1 + (2n + 4k^2)\, a^2 + (n^2 - 2n + 1 - 2k^2)\,\beta^2 \right] \sin^2\alpha \cos^2\alpha \; + \\
&2k\left[3 - n - 2n\, a^2 + (n-1)\,\beta^2 \right] \sin\alpha \cos^3\alpha \; + \\
&\left[1 - n + a^2 n^2 + k^2\beta^2 \right] \cos^4\alpha = \left(\frac{X}{\sigma_1}\right)^2 = R_x^2
\end{aligned} \qquad (14)$$

or

$$R_x = f\,(n, k, a, \beta, \alpha) \qquad (15)$$

where

$$a = \frac{X}{Y}$$

$$\beta = \frac{X}{S}$$

$$R_x = \frac{X}{\sigma_1}$$

The strength factor, R_x, of the unidirectional filamentary layer, is a function of the strength parameters, a, β , n, k, the stress field, and the orientation α . For a given set of a, β , n, and k, the strength factor assumes an extremum value when

$$\frac{\partial}{\partial \alpha} f\,(n, k, a, \beta, \alpha) = 0 \qquad (16)$$

Equation 16 re-stated explicitly becomes

$$
\begin{aligned}
& 2k\left[3n-1-2a^2-(n-1)\beta^2\right]\sin^4\alpha \\
& +\left[8k^2(2+a^2-\beta^2)-2(n-1)(3n-1-2a^2-(n-1)\beta^2)\right]\sin^3\alpha\cos\alpha \\
& -\left[12k(n-1)(2+a^2-\beta^2)\right]\sin^2\alpha\cos^2\alpha \\
& +\left[-8k^2(2+a^2-\beta^2)+2(n-1)(n-3+2a^2n-(n-1)\beta^2)\right]\sin\alpha\cos^3\alpha \\
& +2k\left[n-3+2a^2n-(n-1)\beta^2\right]\cos^4\alpha = 0
\end{aligned}
\tag{17}
$$

or

$$
\begin{aligned}
& B_1\sin^4\alpha + B_2\sin^3\alpha\cos\alpha + B_3\sin^2\alpha\cos^2\alpha \\
& + B_4\sin\alpha\cos^3\alpha + B_5\cos^4\alpha = 0
\end{aligned}
\tag{18}
$$

where

$$
\left.
\begin{aligned}
B_1 &= 2k(3n-1-2a^2-(n-1)\beta^2) \\
B_2 &= 8k^2(2+a^2-\beta^2)-2(n-1)(3n-1-2a^2-(n-1)\beta^2) \\
B_3 &= -12k(n-1)(2+a^2-\beta^2) \\
B_4 &= -8k^2(2+a^2-\beta^2)+2(n-1)(n-3+2a^2n-(n-1)\beta^2) \\
B_5 &= 2k(n-3+2a^2n-(n-1)\beta^2)
\end{aligned}
\right\}
\tag{19}
$$

If $\cos\alpha \neq 0$, Equation 18 can be rewritten as

$$B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5 = 0 \tag{20}$$

where

$$\lambda = \tan\alpha \tag{21}$$

In case $\sin\alpha \neq 0$, Equation 18 can be expressed as

$$B_1 + B_2\bar{\lambda} + B_3\bar{\lambda}^2 + B_4\bar{\lambda}^3 + B_5\bar{\lambda}^4 = 0 \qquad (22)$$

where

$$\bar{\lambda} = \cot\alpha \qquad (23)$$

In three cases of simple states of stress, Equations 14 and 20 lead to the following results:

Case 1: $k = 0$ and $n = 1$

Case 1 corresponds to the "hydrostatic" state of stress. For this state Equation 14 is independent of α and the strength factor becomes

$$R_x = a = \frac{X}{Y}$$

or

$$\frac{\sigma_1}{X} = \frac{Y}{X} \qquad (24)$$

It indicates that for the conditions of stress stipulated in this case, the applied stress is governed by the transverse strength of the material.

Case 2: $k \neq 0$ and $n = 1$

For this state of stress, Equation 17 becomes

$$(\sin^2\alpha - \cos^2\alpha)\left[(1-a^2) + 2k(2+a^2-\beta^2)\sin\alpha\cos\alpha\right] = 0 \qquad (25)$$

Equation 25 yields that either

$$\alpha = \frac{\pi}{4} \qquad (26)$$

or

$$\sin 2\alpha = \frac{a^2-1}{k(2+a^2-\beta^2)} \qquad (27)$$

Substituting Equation 26 in Equation 14 yields Equation 28.

$$R_x^2 = a^2 (1-k^2) + 2k(1+k) \tag{28}$$

Equation 27 requires parametric study for its evaluation.

Case 3: k = 0 and n ≠ 1

For k = 0 and n ≠ 1, Equation 20 reduces to a form which yields

$$\lambda = 0 \tag{29}$$

and

$$\lambda = \sqrt{\frac{n-3+2na^2-\beta^2(n-1)}{3n-1-2a^2-\beta^2(n-1)}} \tag{30}$$

When $\lambda = 0$, R_x evaluated from Equation 14 becomes

$$R_x = \sqrt{1-n+a^2 n^2} \tag{31}$$

Equation 30, like Equation 27, needs to be studied parametrically in order to interpret the results in a meaningful way.

SECTION IV
PARAMETRIC STUDY OF STRENGTH CRITERIA

In Equations 14 and 17, parameters n and k define the state of stress, and a and β are parameters of strength of the anisotropic composite. All the possible stress fields are shown in Figure 2. It is evident from the figure that the value of n varies between -1.0 and 1.0, i.e., $(-1 \leq n \leq 1)$. For this study values of k are also assumed to vary between -1.0 and 1.0, i.e., $(-1 \leq k \leq 1)$. However, it is possible for k to take values beyond this range. Parameters a and β are assigned values between 1.0 and 100.

Coefficients B_i (i=1, 2, 3...5) in Equation 20 are the functions of n, k, a, and β. The quartic Equation 20 was solved for α and $\frac{1}{R_x}$ by varying one parameter at anytime. On substitution of real roots from Equation 20, corresponding values of $\frac{1}{R_x}$ were obtained from Equation 14. A maximum of the $\frac{1}{R_x}$ values and the corresponding angle of orientation were obtained.

A perusal of the computed values of α pertaining to the maximum of $\frac{1}{R_x}$ indicates that for certain values of n, k, a, and β, it is identical with α_1 computed from Equation 32.

$$\tan \alpha_1 = \frac{2k}{1-n} \tag{32}$$

where α_1 is the direction of the major principal stress.

It is also observed that changes in signs of k change the signs of α without affecting its magnitude. For that reason, only positive values of k have been used in Figures 3 to 8. Each curve in the plots shows the variation of $\frac{\sigma_1}{X}$ with n for a prescribed set of values of a and β indicated on each curve. Solid lines represent values of $\frac{\sigma_1}{X}$ for which α and α_1 are identical. Broken lines correspond to the cases when $\alpha \neq \alpha_1$. In all plots($\frac{\sigma_1}{X}$ vs n) curves tend to meet near some value n_c of n. For $n > n_c$, α and α_1 are identical, and $\frac{\sigma_1}{X}$ depends solely on a and is independent of β. $\frac{\sigma_1}{X}$ is independent of β also when $n < n_c$ provided $a < \beta$. For a greater than β, however, the strength improves with β.

This observation leads to the conclusion that if a is less than or equal to β, then $\frac{\sigma_1}{X}$ depends entirely upon a. The values of α computed from Equation 20 are the same as the values of α_1 obtained from Equation 32. This means that for maximum strength, fibers should be oriented in the principal stress direction. This simplifies the computation of maximum $\frac{\sigma_1}{X}$ considerably.

SECTION V

AXES OF MATERIAL SYMMETRY COINCIDENT WITH PRINCIPAL STRESS DIRECTIONS

It was seen in Section IV that for $a \leq \beta$, maximum strength is obtained when axes of material symmetry coincide with those of the principal stresses. This condition determines the orientation of the fibers and the criteria assumes a simpler form.

When α coincides with the principal stress direction, Equation 13 yields

$$\tau_{xy} = 0 \tag{33}$$

$$\tan 2\alpha_1 = \tan 2\alpha = \frac{2k}{1-n} \tag{34}$$

$$\frac{\sigma_{xx}}{\sigma_1} = \frac{1}{2}\left[(1+n) + \sqrt{(1-n)^2 + 4k^2}\right] \tag{35}$$

$$\frac{\sigma_{xy}}{\sigma_1} = \frac{1}{2}\left[(1+n) - \sqrt{(1-n)^2 + 4k^2}\right] \tag{36}$$

Substitution of Equations 33, 35, and 36 in Equation 12 result in Equation 37.

$$R_x^2 = \left(\frac{X}{\sigma_1}\right)^2 = \frac{a^2}{4}(1+n)^2 + \frac{1}{4}(a^2+2)\left[(1-n)^2 + 4k^2\right] + \frac{1}{2}(1-a^2)(1+n)\sqrt{(1-n)^2 + 4k^2} \tag{37}$$

For $k = o$, Equation 37 reduces to

$$R_x = \frac{X}{\sigma_1} = \sqrt{a^2 n^2 - n + 1} \tag{38}$$

which is the same as Equation 31.

The peaks observed in Figures 3 to 8 can be obtained from Equation 37 and the results are given in Table I. For small values of a, a peak position n_c depends upon a and k; however, for large values of a it can be defined by an approximate relationship

$$n_c \approx k^2 \tag{39}$$

The corresponding value of $\frac{1}{R_x}$ becomes

$$\frac{1}{R_x} = \frac{\sigma_1}{X} = \frac{1}{(1+n_c)} \tag{40}$$

In the derivation of the results discussed so far, it was assumed that X and Y were consistent with the nature of stresses σ_{xx} and σ_{yy} in Equation 12. It means that if σ_{xx} is tensile or compressive stress, the corresponding strength X has to be tensile or compressive. It holds equally for σ_{yy} and Y. In Equation 37, R_x is expressed in terms of a, n, and k, and it is, therefore, desirable to devise some method for the proper choice of a for given n and k without computing stresses σ_{xx} and σ_{yy}.

To devise a suitable technique, Equations 35 and 36 are rewritten as

$$\sigma_{xx} = \frac{1}{2}\left[(\sigma_1 + \sigma_2) + \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau_{12}^2}\right] \tag{41}$$

$$\sigma_{yy} = \frac{1}{2}\left[(\sigma_1 + \sigma_2) - \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau_{12}^2}\right] \tag{42}$$

and examined. It can be shown that the following conditions determine the choice of a.

$$\left.\begin{array}{llll}
(1.) & k > 0 \ (\sigma_1 > 0) \text{ and } n > k^2 & X = X_t \\
 & & X = Y_t \\
(2.) & k > 0 \ (\sigma_1 > 0) \text{ and } n < k^2 & X = X_t \\
 & & Y = Y_c \\
(3.) & k < 0 \ (\sigma_1 < 0) \text{ and } n > k^2 & X = X_c \\
 & & Y = Y_c \\
(4.) & k < 0 \ (\sigma_1 < 0) \text{ and } n < k^2 & X = X_c \\
 & & Y = Y_t
\end{array}\right\} \tag{43}$$

where X_t, X_c, Y_t, Y_c are the tensile and compressive strengths in x and y directions. Figure 9 represents a plot of Equation 43 indicating the zones of influence of X_t, X_c, Y_t, and Y_c.

SECTION VI

EFFECT OF ORIENTATION OF AXES OF MATERIAL SYMMETRY ON STRAINS

When a thin element of transversely isotropic material was subjected to a plane stress field, it was observed that the material did have preferred directions relative to the stress field so as to produce a maximum value of $\frac{\sigma_1}{X}$. It may, therefore, be reasonably postulated that there exist preferred directions which reduce the strains to the minimum. From Equation 13, stresses in the directions of the axes of material symmetry can be obtained as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \tag{44}$$

The strains in x and y directions are

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{xx}} & -\frac{\nu_{xy}}{E_{xx}} & 0 \\ -\frac{\nu_{yx}}{E_{yy}} & \frac{1}{E_{yy}} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \tag{45}$$

where

ϵ_{xx} = strain in x-direction

ϵ_{yy} = strain in y-direction

γ_{xy} = shear strain referred to the x and y axes

E_{xx} = modulus of elasticity in x-direction

E_{yy} = modulus of elasticity in y-direction

ν_{yx}, ν_{xy} = Poisson's ratios

G = modulus of rigidity

By using the strains computed from Equation 45, strains referred to any arbitrary set of axes x' and y' making an angle ϕ with the x and y axes are given by Equation 46

$$\begin{bmatrix} \epsilon_{x'x'} \\ \epsilon_{y'y'} \\ \frac{\gamma_{x'y'}}{2} \end{bmatrix} = \begin{bmatrix} \cos^2\phi & \sin^2\phi & 2\sin\phi\cos\phi \\ \sin^2\phi & \cos^2\phi & -2\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \sin\phi\cos\phi & \cos^2\phi - \sin^2\phi \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \frac{\gamma_{xy}}{2} \end{bmatrix} \tag{46}$$

$\epsilon_{x'x'}$ assumes an extremum value when

$$\gamma_{x'y'} = 0 \tag{47}$$

which means that

$$\tan 2\phi = \frac{\gamma_{xy}}{\epsilon_{xx} - \epsilon_{yy}} \tag{48}$$

Equation 48 is substituted in Equation 46 to yield

$$\epsilon_{x'x'} = \frac{1}{2}\left[(\epsilon_{xx} + \epsilon_{yy}) + \sqrt{(\epsilon_{xx} - \epsilon_{yy})^2 + \gamma_{xy}^2}\right] \tag{49}$$

An examination of Equation 49 indicates that $\epsilon_{x'x'}$ attains its minimum value when $\gamma_{xy} = 0$, i.e.,

$$\tan 2\alpha = \frac{2\tau_{12}}{\sigma_1 - \sigma_2} \tag{50}$$

Equation 50 implies that the minimum strain occurs along the direction of fibers.

SECTION VII

RESULTS AND CONCLUSIONS

A parametric study of Tsai's strength criteria for unidirectional composites was undertaken to determine the orientation which yielded the maximum strength for a given set of parameters of stress and strength. The results obtained from the study indicate that for materials whose shear strength S is less than the transverse strength Y, the orientation of the fibers for the maximum value of $\frac{\sigma_1}{X}$ corresponds to the principal stress direction and the maximum $\frac{\sigma_1}{X}$ does not depend upon the shear strength S. No similar conclusion can be drawn when shear strength S exceeds transverse strength Y; however, for $n > n_c$, $\frac{\sigma_1}{X}$ and α are independent of S.

The effect of the orientation of the material axis upon the strains was also examined. It was found that the minimum strains occur in the direction of the material axis.

From the above observation it is seen that for $S < Y$, a maximum strength $\frac{\sigma_1}{X}$ is obtained when the material axes are oriented along the principal stress directions and minimum strains occur in the directions of the material axes.

REFERENCES

1. J. Marin, "Theories of Strength for Combined Stresses and Non-Isotropic Materials," Journal Aeronautical Sciences, 24(4), April 1957.

2. R. Hill, The Mathematical Theory of Plasticity, Oxford University Press, London, 1950.

3. V. D. Azzi and S. W. Tsai, Anisotropic Strength of Composites, Society for Experimental Stress Analysis, Spring Meeting, May 1965.

4. J. C. Jaeger, Elasticity, Fracture, and Flow, Methuen and Co. Ltd, London, 1962.

5. C. R. Seliger, "Techniques for Finding Roots of Polynomial Equations of Degrees Seven or Lower," Instruments and Control Systems, Vol. 40, No. 2, February 1967.

6. C. R. Seliger "General Quartic Formula," Instruments and Control Systems, Vol. 40, August 1967.

TABLE I
PEAK POSITIONS AND STRENGTH FACTORS

k		RATIO OF STRENGTHS (α=X/Y) 10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0	100.0
0.	n_c	0.00500	0.00125	0.00056	0.00031	0.00020	0.00014	0.00010	0.00008	0.00006	0.00005
0.	$1/R_x$	1.00125	1.00031	1.00014	1.00008	1.00005	1.00003	1.00003	1.00002	1.00002	1.00001
0.1	n_c	0.01500	0.01125	0.01056	0.01031	0.01020	0.01014	0.01010	0.01008	0.01006	0.01005
0.1	$1/R_x$	0.99129	0.99040	0.99023	0.99017	0.99015	0.99012	0.99013	0.99010	0.99012	0.99011
0.2	n_c	0.04497	0.04124	0.04055	0.04031	0.04020	0.04014	0.04010	0.04008	0.04006	0.04005
0.2	$1/R_x$	0.96256	0.96179	0.96165	0.96160	0.96158	0.96157	0.96155	0.96154	0.96154	0.96154
0.3	n_c	0.09487	0.09122	0.09054	0.09030	0.09019	0.09014	0.09010	0.09008	0.09006	0.09005
0.3	$1/R_x$	0.91820	0.91762	0.91752	0.91748	0.91746	0.91744	0.91745	0.91744	0.91743	0.91745
0.4	n_c	0.16457	0.16114	0.16051	0.16029	0.16018	0.16013	0.16009	0.16007	0.16006	0.16005
0.4	$1/R_x$	0.86257	0.86219	0.86212	0.86210	0.86209	0.86208	0.86208	0.86207	0.86206	0.86207
0.5	n_c	0.25389	0.25098	0.25043	0.25024	0.25016	0.25011	0.25008	0.25006	0.25005	0.25004
0.5	$1/R_x$	0.80025	0.80006	0.80003	0.80001	0.80001	0.80000	0.80000	0.80000	0.80000	0.80000
0.6	n_c	0.36257	0.36065	0.36029	0.36016	0.36010	0.36007	0.36005	0.36004	0.36003	0.36003
0.6	$1/R_x$	0.73537	0.73531	0.73530	0.73530	0.73530	0.73529	0.73529	0.73529	0.73529	0.73529
0.7	n_c	0.49022	0.49006	0.49002	0.49001	0.49001	0.49001	0.49000	0.49000	0.49000	0.49000
0.7	$1/R_x$	0.67114	0.67114	0.67114	0.67114	0.67114	0.67114	0.67114	0.67114	0.67114	0.67114
0.8	n_c	0.63631	0.63906	0.63958	0.63976	0.63985	0.63990	0.63992	0.63994	0.63995	0.63996
0.8	$1/R_x$	0.60981	0.60977	0.60976	0.60976	0.60976	0.60976	0.60975	0.60975	0.60976	0.60976
0.9	n_c	0.80014	0.80748	0.80888	0.80937	0.80959	0.80972	0.80979	0.80984	0.80987	0.80990
0.9	$1/R_x$	0.55274	0.55255	0.55252	0.55250	0.55250	0.55249	0.55249	0.55249	0.55249	0.55249
1.0	n_c	0.98085	0.99506	0.99779	0.99875	0.99920	0.99945	0.99959	0.99969	0.99975	0.99980
1.0	$1/R_x$	0.50060	0.50015	0.50007	0.50004	0.50002	0.50002	0.50001	0.50001	0.50000	0.50000

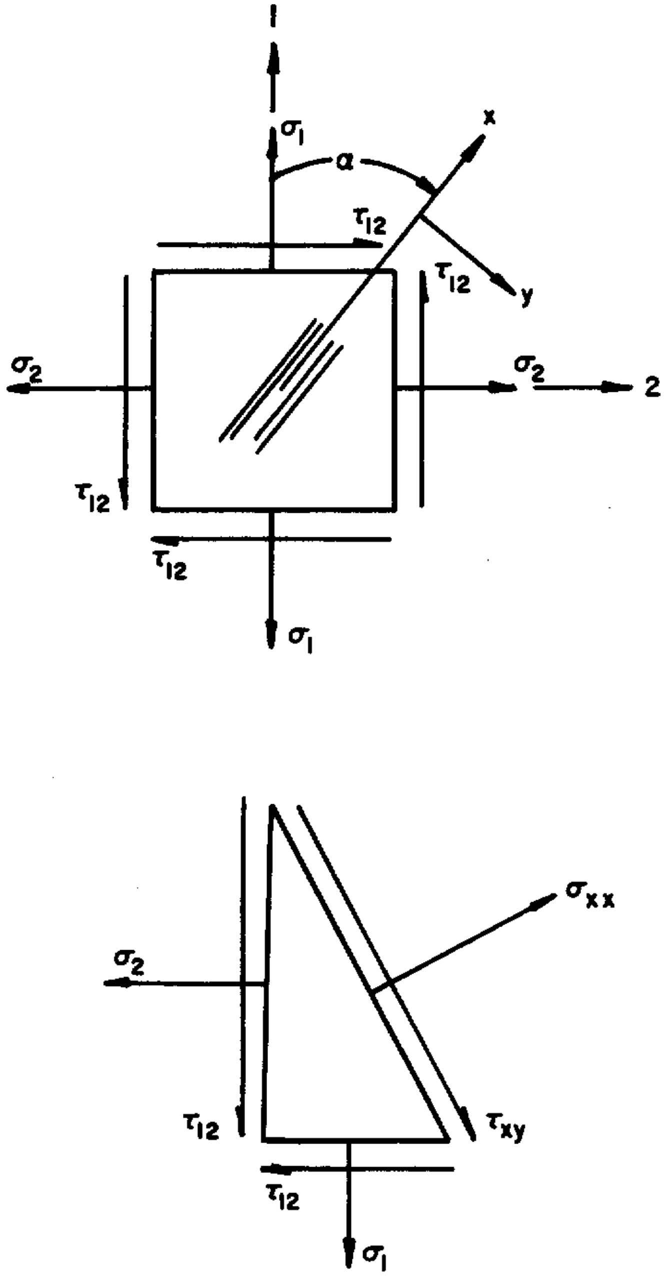

Figure 1. State of Plane Stress

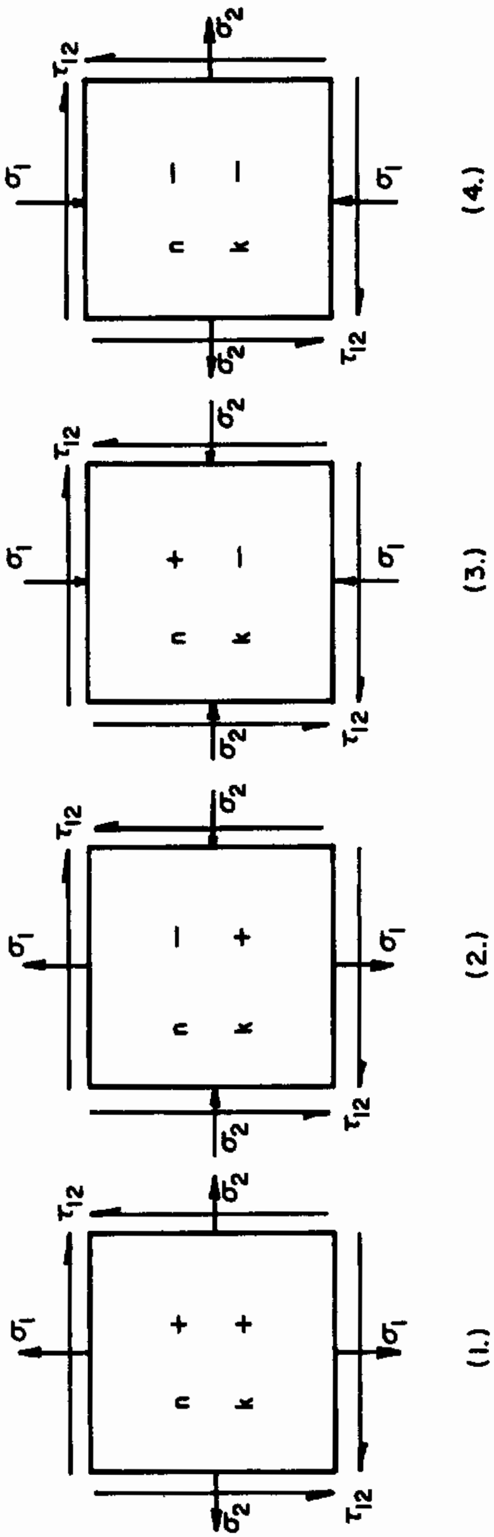

Figure 2. Possible States of Stress

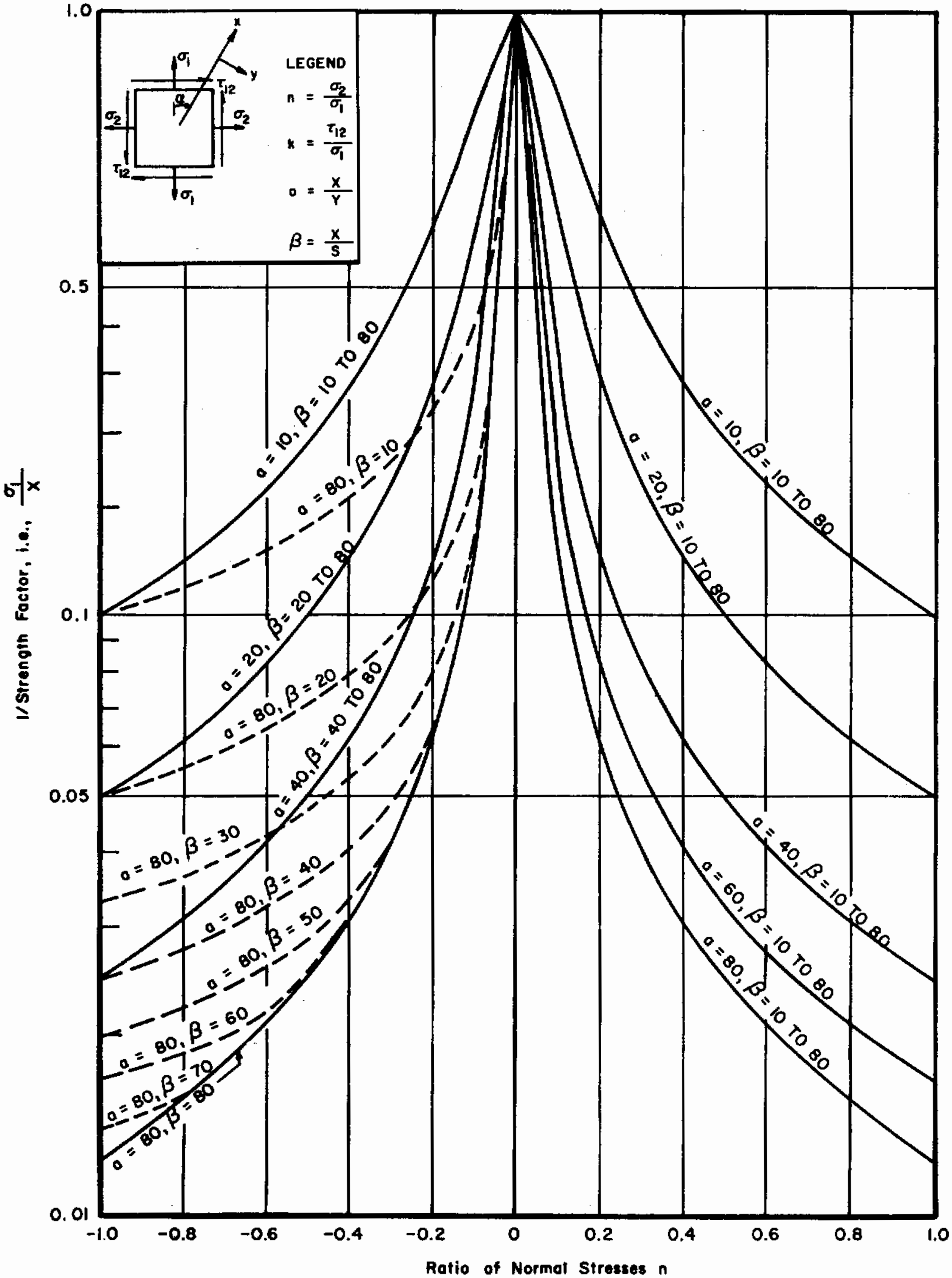

Figure 3. Variation of the Strength Factor as a Function of the Ratio of Normal Stresses for k Equal to Zero

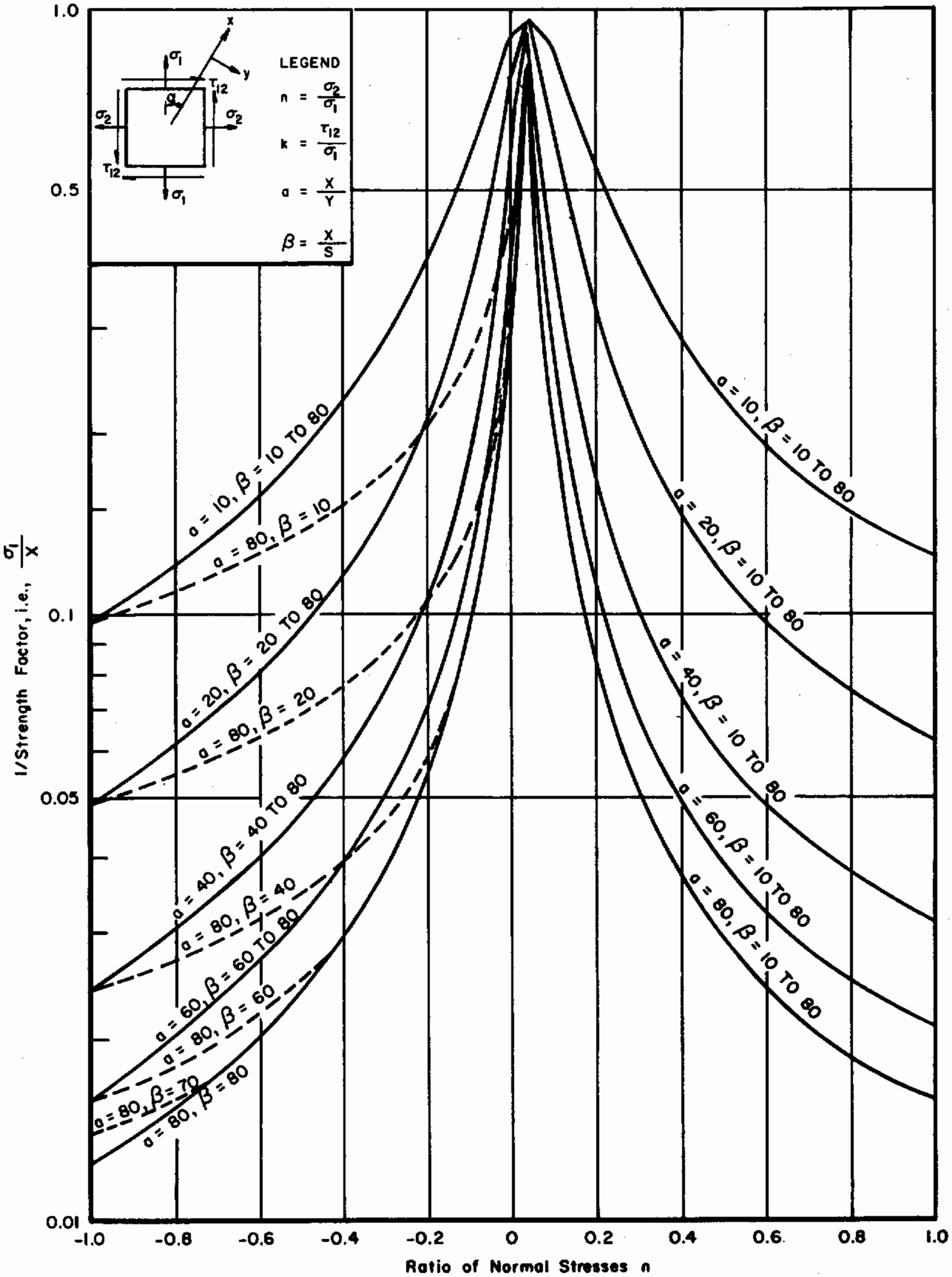

Figure 4. Variation of the Strength Factor as a Function of the Ratio of Normal Stresses for k Equal to 0.2

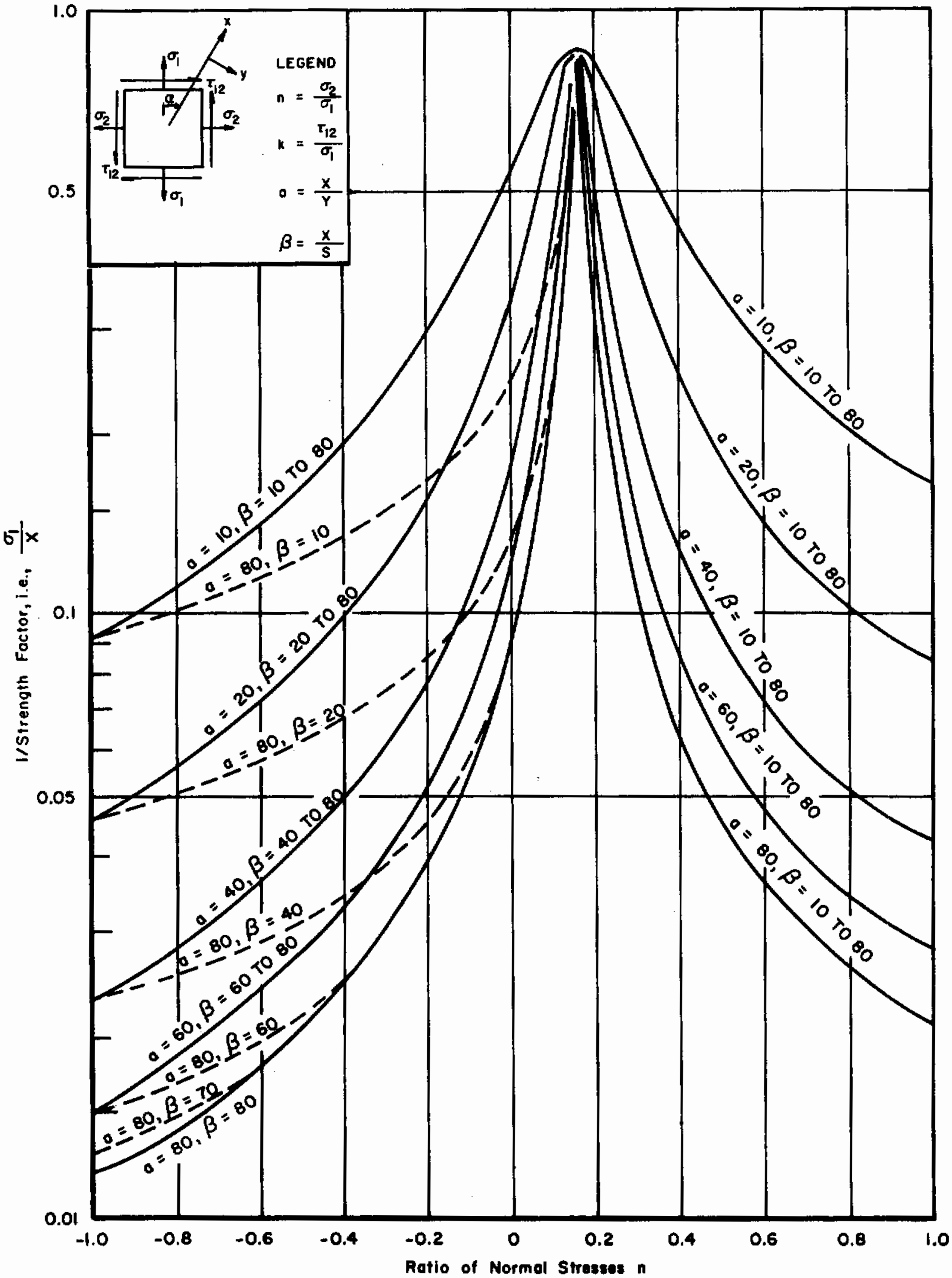

Figure 5. Variation of the Strength Factor as a Function of the Ratio of Normal Stresses for k Equal to 0.4

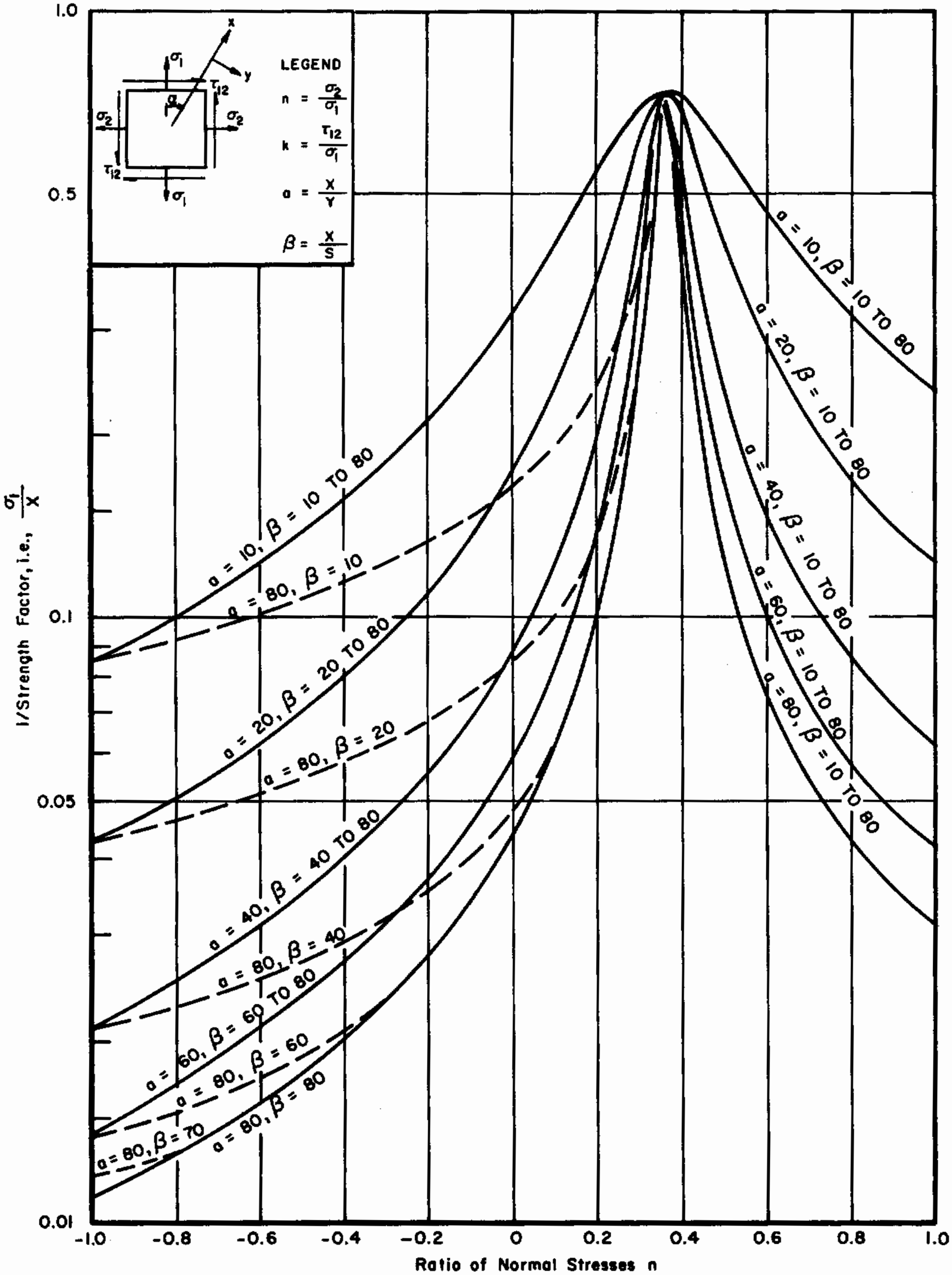

Figure 6. Variation of the Strength Factor as a Function of the Ratio of Normal Stresses for k Equal to 0.6

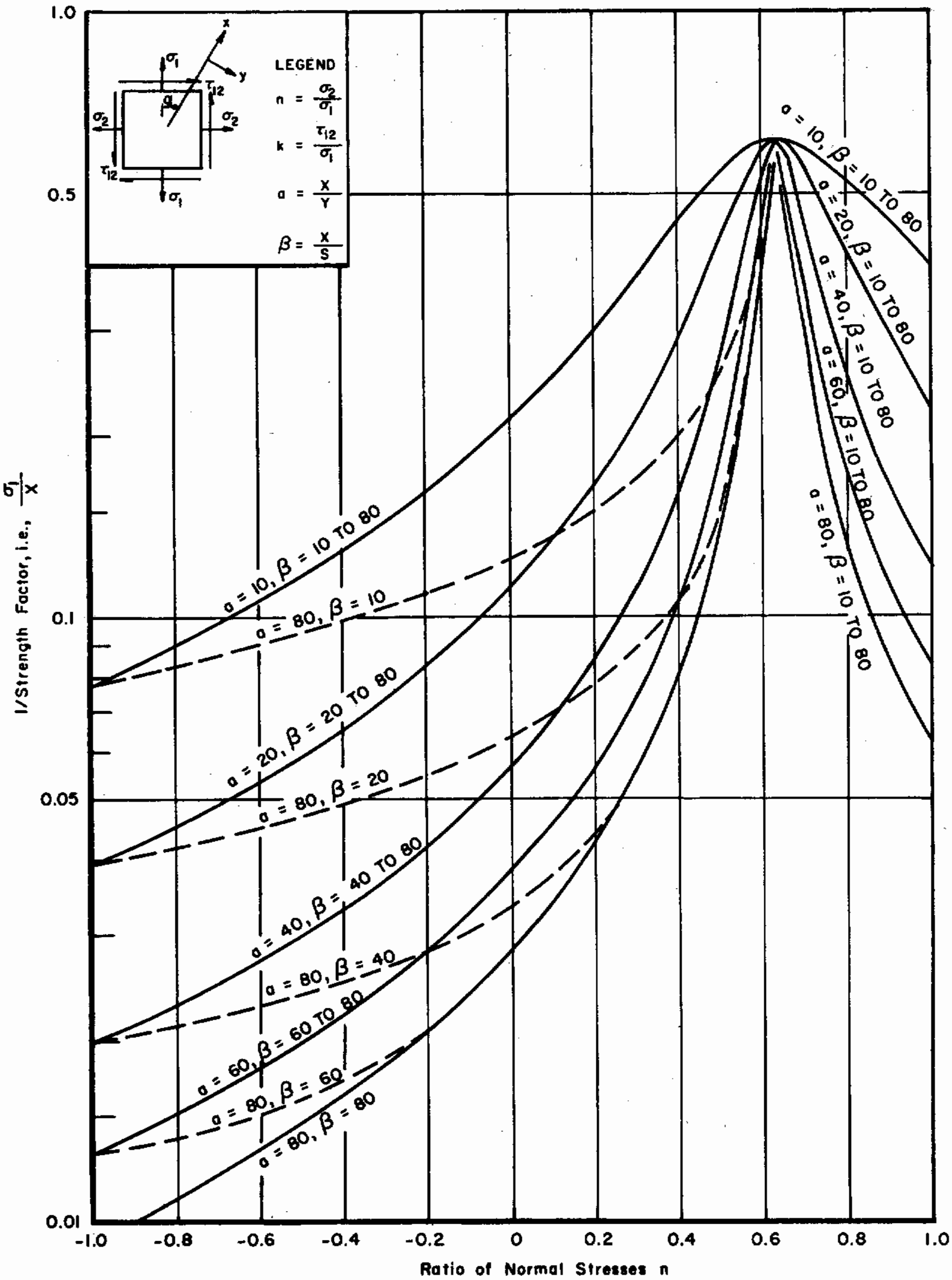

Figure 7. Variation of the Strength Factor as a Function of the Ratio of Normal Stresses for k Equal to 0.8

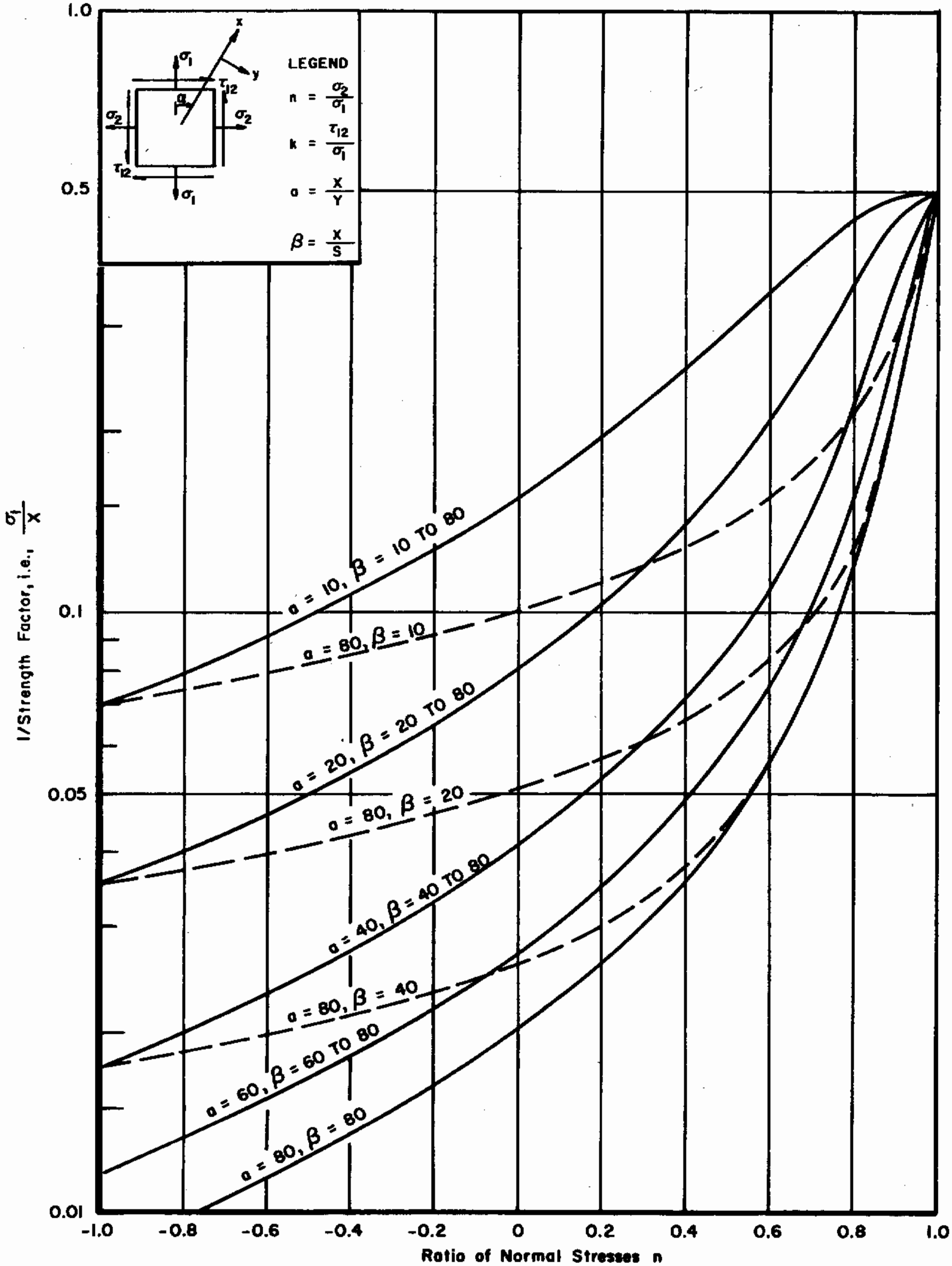

Figure 8. Variation of the Strength Factor as a Function of the Ratio of Normal Stresses for k Equal to 1.0

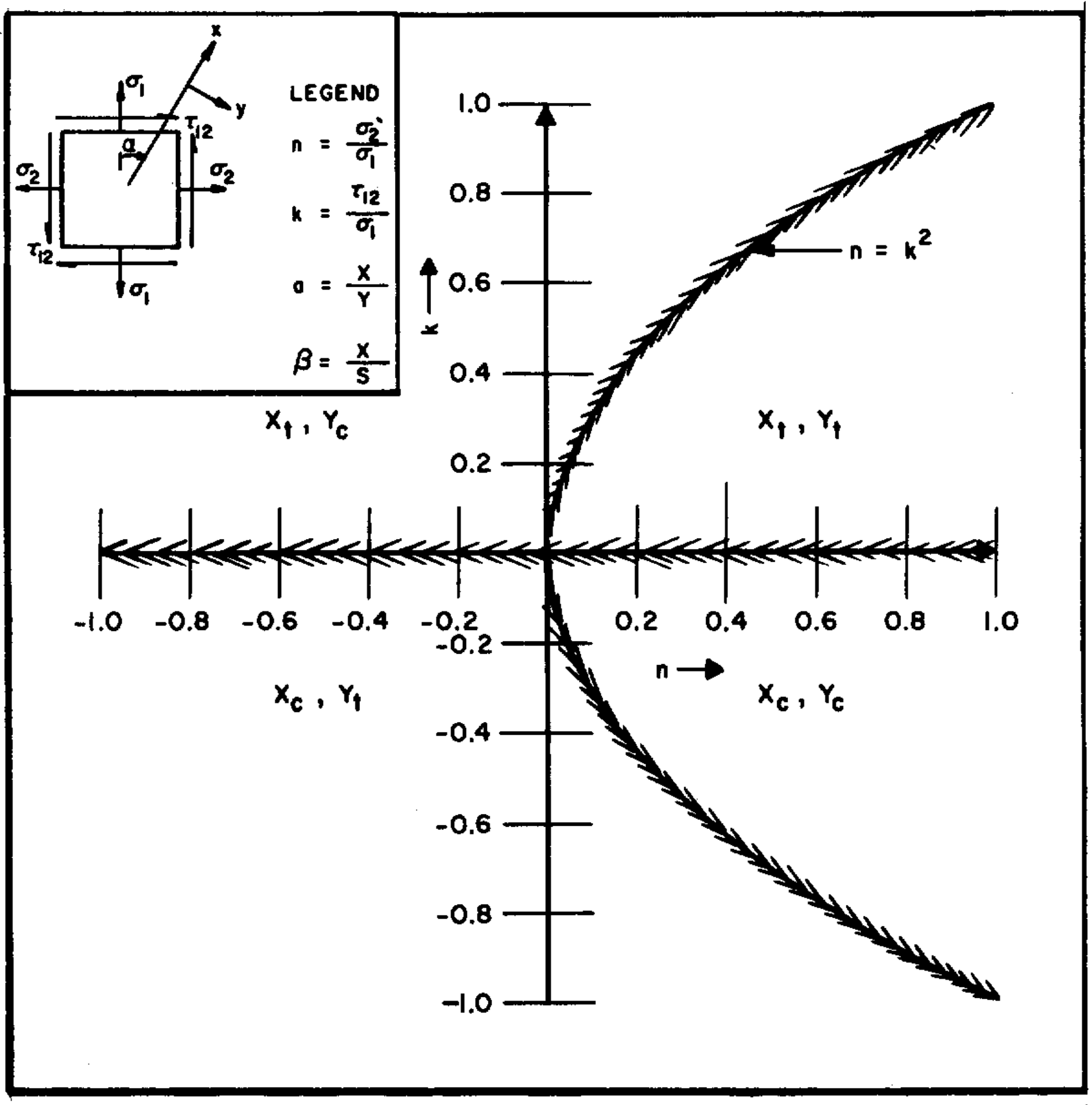

Figure 9. Zones of Tension and Compression for Plane Stress Field

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Air Force Flight Dynamics Laboratory (FDTR) Wright-Patterson Air Force Base, Ohio 45433	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP N/A

3. REPORT TITLE
"Parametric Study of Tsai's Strength Criteria for Filamentary Composites"

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)
Sandhu, R. S.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1969	26	6

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) AFFDL-TR-68-168
b. PROJECT NO. 1773	
c. Task 147306	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT
This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Air Force Flight Dynamics Laboratory Wright-Patterson Air Force Base, Ohio

13. ABSTRACT

A parametric study of Tsai's strength criteria for unidirectional filamentary composites is presented. The purpose of this study is to determine the fiber orientation for the maximum strength under a given biaxial state of stress. The investigation indicates that the best fiber orientation depends upon the shear strength of the material in relation to its transverse strength. When the shear strength is less than the transverse strength, the optimum fiber orientation coincides with the principal stress direction. However, when the shear strength is greater than the transverse strength, the best fiber orientation does not always coincide with the principal stress direction. The results of this study are presented in the form of plots. The plots together with the equations presented in the text aid in determining the optimum orientation.

DD FORM 1 NOV 65 1473

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Stress Analysis						
Strength of Composite Material						
Filamentary Composites						
Plates						