AFFDL-TR-68-150

A PRACTICAL CONPUTATIONAL METHOD FOR REDUCING A DYNAMICAL SYSTEM WITH CONSTRAINTS TO AN EQUIVALENT SYSTEM WITH INDEPENDENT COORDINATES

William C. Walton, Jr., and Earl C. Steeves*
NASA Langley Research Center
Langley Station, Hampton, Va.

A new method is presented by which equations of motion of a linear mechanical system can be derived in terms of independent coordinates when the system is described in terms of coordinates which are not independent but instead are governed by linear homogeneous equations of constraint. There is a discussion of the origin in practical vibrations analysis of dynamical systems involving equations of constraint. Methods previously used for handling such systems are discussed and the new method is demonstrated to have the following advantages: (1) For the most general constraint equations, solution of the equations is reduced in substance to computing the eigenvalues and eigenvectors of a symmetric matrix; and (2) the method is applicable when there are redundancies in the equations of constraint.

*Aerospace Technologists

SECTION I

INTRODUCTION

The purpose of this paper is to present a method by which equations of motion of a linear mechanical system can be derived in terms of independent coordinates when basic information about the system is available in terms of coordinates which are not independent but instead are governed by linear homogeneous equations of constraint. Necessity for this derivation arises frequently in practical vibration analysis. The method is believed to be new, and experience in analyzing the vibrations of shells has convinced the authors that it very often offers decided advantages over methods previously used.

There is a discussion of the reason dynamical systems involving constraint equations arise in practical analysis of oscillations of mechanical systems. There follows a description of methods previously used in dealing with such systems. Then a theorem designated the "zero eigenvalues theorem" which is basic to the method of this paper is proved, and the method is presented. Next, the result of the method of this paper is shown to include the result of the main older method as a special case. Two examples of application of the new method are presented and the paper closes with a discussion of numerical considerations involved in practical computing with the method.

SECTION II

BACKGROUND

In conventional analyses of small forced oscillations of mechanical systems, the physical system is idealized so that its configuration at any instant is determined by specification of a finite number of independent coordinates $q_1, q_2, \ldots, q_n, \ldots, q_N$. Then, with approximations allowable because of the assumed smallness of the oscillations, the Lagrangian of the system may be expressed in the form

$$L = \frac{1}{2}\,\dot{\mathbf{q}}'\mathbf{M}\dot{\mathbf{q}} - \frac{1}{2}\,\mathbf{q}'\mathbf{K}\mathbf{q} \tag{1}$$

where

(1) $\mathbf{q}$ is a column matrix the elements of which are the coordinates q_n.

(2) A prime denotes the transpose of a matrix.

(3) $\mathbf{M}$ and $\mathbf{K}$ are constant symmetric matrices of order N with $\mathbf{M}$ positive definite.

(4) A dot denotes differentiation with respect to time.

When the Lagrangian has the form shown by Equation 1, application of Hamilton's principle gives equations of motion of the system which have the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q} \tag{2}$$

In Equation 2 $\mathbf{Q}$ is a column matrix with N elements. The elements of $\mathbf{Q}$ are usually called generalized forces; and for forced oscillations, the problem under consideration, they are functions of time alone. The generalized forces are determined by the following requirement: Let δq be an arbitrary infinitesimal variation of the coordinates composing the Matrix $\mathbf{q}$. Then the work W done by the instantaneous forces applied to the system when these forces are moved through the displacements produced by the variation shall be given by the equation

$$W = \mathbf{Q}'\,\delta\mathbf{q} \tag{3}$$

Once the equations of motion have been brought to the form indicated by Equation 2 there is a well-established and quite effective body of mathematical theory and computational technique for determining the behavior of the system.

Often, however, it is much easier to express L and W if the system is described by coordinates which are not independent but which are governed by linear homogeneous equations of constraint. Letting $z_1, z_2, \ldots, z_p, \ldots, z_P$ represent such a set of coordinates, the constraint equations take the form

$$\mathbf{C}\,\mathbf{z} = 0 \tag{4}$$

where $\mathbf{z}$ is a column matrix the elements of which are the coordinates z_p, and where $\mathbf{C}$ is a constant matrix which has P columns and is, in general, rectangular.

In terms of the coordinates z_p the Lagrangian will generally take the form

$$L = \frac{1}{2}\,\dot{\mathbf{z}}'\overline{\mathbf{M}}\dot{\mathbf{z}} - \frac{1}{2}\,\mathbf{z}'\overline{\mathbf{K}}\mathbf{z} \tag{5}$$

where $\overline{\mathbf{M}}$ and $\overline{\mathbf{K}}$ are symmetric matrices of order P. The work W can be found in the form

$$W = \mathbf{Z}'\,\delta\mathbf{z} \tag{6}$$

where $\delta\mathbf{z}$ is an arbitrary variation of $\mathbf{z}$ compatible with the Equations of Constraint 4, and $\mathbf{Z}$ is a column matrix with P elements which are functions of time alone. It is sometimes possible to choose the coordinates z_p so that the Matrix $\overline{\mathbf{M}}$ is positive semidefinite rather than positive definite; but in this paper such choices are excluded and $\overline{\mathbf{M}}$ is assumed to be positive definite.

From what has been said it can be seen that it is useful to know a systematic procedure by which equations of motion in terms of independent coordinates, as in Equation 2, can be derived starting with the Lagrangian L and the work W in terms of coordinates governed by homogeneous equations of constraint as in Equations 5 and 6. It is the object of this paper to set forth such a procedure, but before doing so it is appropriate to discuss briefly how the problem has been dealt with in the past.

SECTION III

OLDER METHODS

In the past, the equations in terms of independent coordinates have been arrived at in two ways:

(1) Through consideration of particular physical or geometrical aspects of a problem the dependent coordinates z_p are chosen so as to impart a very simple form to the equations of constraint, rendering easy and obvious the determination of independent coordinates.

(2) Certain of the coordinates are selected to be independent coordinates, and the equations of constraint are then solved as simultaneous equations to express the remaining dependent coordinates in terms of those which have been selected to be independent.

Under the first approach come, for example, those finite element methods of structural analysis in which the coordinates of a free-body element are displacements and rotations at nodes. In such analyses the equations of constraint come down to equalities among appropriate displacements and rotations at moving nodes and equations in which appropriate displacements and rotations are set equal to zero at nodes where there are supposed to be rigid constraints. A set of independent coordinates is arrived at by the simple expedient of using a single symbol for all displacements and rotations which are equated at a node. This is the basis of the now widely used procedure of superimposing stiffness matrices or mass matrices of structural elements to arrive at a stiffness matrix or a mass matrix of an entire structure composed of the elements connected together.

As a basis for describing the advantages of the method to be presented, the second type of approach will be discussed formally. In this approach it is assumed (usually tacitly) that the rank R of the Matrix **C** is equal to the number of rows in **C** and that therefore Equation 4 may be written as

$$\mathbf{A}\mathbf{z}^{(a)} + \mathbf{B}\mathbf{z}^{(b)} = 0 \qquad (7)$$

where:

(1) **A** is an R by R nonsingular matrix the columns of which are R distinct colums of **C**.

(2) **B** is an R by (P-R) matrix the columns of which are those columns of **C** not included in **A**.

(3) $\mathbf{z}^{(a)}$ and $\mathbf{z}^{(b)}$ are column matrices the elements of which are elements of $\mathbf{z}$ corresponding to the columns in $\mathbf{A}$ and $\mathbf{B}$, respectively, and appropriately ordered.

By renumbering the coordinates z_p and making a corresponding rearrangement of the columns of $\mathbf{C}$ it can be contrived that the first R columns of $\mathbf{C}$ constitute the Matrix $\mathbf{A}$ and the last P-R columns of $\mathbf{C}$ constitute the Matrix $\mathbf{B}$. Correspondingly, the elements of $\mathbf{z}^{(a)}$ would be the first R elements of $\mathbf{z}$ and the elements of $\mathbf{z}^{(b)}$ the last P-R elements of $\mathbf{z}$. For convenience in the ensuing discussion it is assumed that such a rearrangement has been made. However, as a practical matter, it is very important to note that actually to carry out a suitable rearrangement one must be able to recognize R linearly independent columns of $\mathbf{C}$. This may not be easy.

Since $\mathbf{A}$ is nonsingular, an inverse of $\mathbf{A}$ exists and is unique. Equation 7 is satisfied therefore if — and only if

$$\mathbf{z}^{(a)} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{z}^{(b)} \tag{8}$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. It follows that the Equations of Constraint 4 are satisfied if — and only if

$$\mathbf{z} = \boldsymbol{\beta}\mathbf{z}^{(b)} \tag{9}$$

where

$$\boldsymbol{\beta} = \left[\begin{array}{c} -\mathbf{A}^{-1}\mathbf{B} \\ \hline \mathbf{I} \end{array}\right] \tag{10}$$

In Equation 10 $\mathbf{I}$ is an identity matrix of order P-R.

Substitution of Equation 9 into Equations 5 and 6 gives expressions for the Lagrangian L and the work W in terms of independent coordinates and in the forms shown by Equations 1 and 3, respectively. The ingredients of the resulting equations are

$$\mathbf{q} = \mathbf{z}^{(b)} \tag{11}$$

$$\mathbf{M} = \boldsymbol{\beta}'\overline{\mathbf{M}}\boldsymbol{\beta} \tag{12}$$

$$\mathbf{K} = \boldsymbol{\beta}'\overline{\mathbf{K}}\boldsymbol{\beta} \tag{13}$$

$$\mathbf{Q} = \boldsymbol{\beta}'\mathbf{Z} \tag{14}$$

The matrices $\mathbf{M}$ and $\mathbf{K}$ thus derived are symmetric and, since the columns of $\boldsymbol{\beta}$ are clearly linearly independent, the Matrix $\mathbf{M}$ is positive definite.

The preceding is a fair description of the textbook method for handling equations of constraint. For emphasis, it is noted once more that the method requires that the Rank R of the Matrix **C** be equal to the number of rows of **C** and that one be able to recognize R linearly independent columns of the Matrix **C**.

SECTION IV

THE ZERO EIGENVALUES THEOREM

The object here is to prove a theorem which is the foundation of the method of this paper.

Consider the equation

$$\mathbf{C z} = \mathbf{O} \tag{15}$$

where **C** is a matrix with any number of columns and any number of rows. Let P be the number of columns.

Let a Matrix **E** be defined by the equation

$$\mathbf{E} = \mathbf{C}'\mathbf{C} \tag{16}$$

and note that **E** is symmetric of order P.

Since **E** is symmetric one can always find a square Matrix **U** of order P such that

$$\mathbf{U}'\mathbf{U} = \mathbf{I} \tag{17}$$

and

$$\mathbf{U}'\mathbf{E}\mathbf{U} = \boldsymbol{\lambda} \tag{18}$$

where **I** is an identity matrix, and $\boldsymbol{\lambda}$ is a real diagonal matrix, each of order P. Any matrix **U** having these properties is called a modal matrix of **E**. The numbers occupying the main diagonal of $\boldsymbol{\lambda}$ are called the eigenvalues of **E**. Let λp represent the eigenvalue at the intersection of the p^{th} row and p^{th} column of $\boldsymbol{\lambda}$. Then we say that the p^{th} column of the modal matrix $\mathbf{U}$ is an eigenvector of the Matrix $\mathbf{E}$ corresponding to the eigenvalue λp. From Equation 17 then the P columns of a modal matrix constitute an orthonormal set of eigenvectors of $\mathbf{E}$. For convenience it is assumed that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_P \tag{19}$$

since the positions of the eigenvalues can be reordered simply by reordering the columns of $\mathbf{U}$.

Now, let a Matrix $\mathbf{D}$ be defined by the equation

$$\mathbf{D} = \mathbf{CU} \tag{20}$$

Then Equation 18 may be written

$$\mathbf{D}'\mathbf{D} = \boldsymbol{\lambda} \tag{21}$$

It is clear from the form of $\mathbf{E}$ shown in the defining Equation 16 that $\mathbf{E}$ is positive semidefinite. It follows by well-known theorems that the eigenvalues are positive or zero. Let S represent the number of them which are positive. Then the last P-S are zero. It follows from Equation 21 that the last P-S columns of $\mathbf{D}$ are null and the first S columns are linearly independent (in fact, orthogonal one to another).

Making use of Equation 17, Equation 15 may be written as

$$\mathbf{CUU}'\mathbf{z} = \mathbf{O} \tag{22}$$

or

$$\mathbf{D}\bar{\mathbf{z}} = \mathbf{O} \tag{23}$$

where

$$\bar{\mathbf{z}} = \mathbf{U}'\mathbf{z} \tag{24}$$

Further, substitution of Equations 20 and 24 into Equation 23 and subsequent substitution of Equation 17 returns Equation 15 uniquely.

Therefore, for a column $\mathbf{z}$ to satisfy Equation 15 it is both necessary and sufficient that the column $\bar{\mathbf{z}}$ defined by Equation 24 should satisfy Equation 23.

But $\bar{\mathbf{z}}$ satisfies Equation 23 if — and only if — the first S elements of $\bar{\mathbf{z}}$ are zero.

From Equations 17 and 24

$$\mathbf{z} = \mathbf{U}\bar{\mathbf{z}} \tag{25}$$

Thus we have the basic theorem that $\mathbf{z}$ is a solution of Equation 15 if — and only if — $\mathbf{z}$ may be expressed in the form

$$\mathbf{z} = \mathbf{T}\bar{\mathbf{q}} \tag{26}$$

where $\bar{\mathbf{q}}$ is a column matrix with P-S arbitrary elements and where $\mathbf{T}$ is a matrix the columns of which are those eigenvectors in $\mathbf{U}$ corresponding to eigenvalues with the value zero.

SECTION V

PROPOSED METHOD

With the basic theorem from the preceding section in hand, the following procedure may be proposed.

GIVEN:

(1) $\bar{\mathbf{K}}$ and $\bar{\mathbf{M}}$ each constant symmetric matrices of order P with $\bar{\mathbf{M}}$ postive definite.

(2) $\mathbf{C}$, a constant matrix with P columns and any number of rows.

(3) $\mathbf{Z}$, a column matrix with P elements each of which may be a function of time.

OBJECT:

(1) To compute a Matrix $\mathbf{T}$ such that:

(a) The transformation $\mathbf{z} = \mathbf{T}\mathbf{q}$ relates the dependent coordinates $\mathbf{z}$ appearing in Equations 5 and 6 to a set of independent coordinates $\mathbf{q}$ suitable for use in Equation 2.

(b) The transformation $\mathbf{Q} = \mathbf{T}'\mathbf{Z}$ produces a Matrix $\mathbf{Q}$ suitable for use in Equation 2.

(2) To compute Matrices $\mathbf{K}$ and $\mathbf{M}$ suitable for use in Equation 2.

PROCEDURE:

(1) Compute $\mathbf{E}$ where $\mathbf{E} = \mathbf{C}'\mathbf{C}$. $\mathbf{E}$ will be square-symmetric of order P and positive semidefinite.

(2) Compute a modal Matrix $\mathbf{U}$ and the eigenvalues λ_p of the Matrix $\mathbf{E}$. ($p = 1,2,3,\ldots,P$.) This is a standard operation at modern computing installations and, in fact, is one of the most successful applications of digital computers.

(3) Identify the columns of $\mathbf{U}$ which correspond to zero eigenvalues. This step requires attention because in principle one can fairly question the possibility of a rigorous distinction between finite eigenvalues and eigenvalues having the value zero when, as is normal, there is roundoff error in the process by which the eigenvalues are computed. The point is discussed in the section on numerics.

(4) Assemble a matrix the columns of which are the columns of $\mathbf{U}$ corresponding to the eigenvalues having the value zero. This matrix is the required transformation matrix $\mathbf{T}$. Its dimensions are P by (P-S) where S is the number of finite eigenvalues of $\mathbf{E}$.

(5) Compute $\mathbf{M}$ and $\mathbf{K}$ by the formulas $\mathbf{M} = \mathbf{T}'\bar{\mathbf{M}}\mathbf{T}$ and $\mathbf{K} = \mathbf{T}'\bar{\mathbf{K}}\mathbf{T}$. $\mathbf{M}$ and $\mathbf{K}$ will each be symmetric and $\mathbf{M}$ will be positive definite, provided sufficient numerical significance has been carried in all computations.

SECTION VI

GENERALIZATION OF THE TRANSFORMATION

Equation 26 gives the most general solution to the Equations of Constraint 4. Since the Matrix $\bar{q}$ appearing in Equation 26 is completely arbitrary, the solution can just as well be stated in the form

$$z = TH\bar{q} \tag{27}$$

where H is any nonsingular square matrix of order (P–S).

Therefore letting

$$\bar{T} = TH \tag{28}$$

the Matrix $\bar{T}$ may be used as the transformation matrix in place of T.

In order that a Matrix $\bar{T}$ may be written as in Equation 28, for some Matrix H, it is both necessary and sufficient that the columns of $\bar{T}$ constitute a set of linearly independent eigenvectors of E corresponding to the eigenvalues of E which have the value zero. The eigenvectors in $\bar{T}$ will not, in general, be orthonormal nor even orthogonal. The columns of $\bar{T}$ are orthonormal if — and only if — H is an orthogonal matrix and orthogonal if H is a diagonal matrix. Proof of these statements will not be made as they amount merely to a formal statement of the basic results of that portion of the theory of matrices which deals with repeated eigenvalues of a real symmetric matrix.

A connection may now be made between the method of this paper and the textbook method.

Assuming that the column and coordinate rearrangements leading to Equation 7 have been carried out, one may write

$$E = C'C = \begin{bmatrix} A' \\ \hline B' \end{bmatrix} \begin{bmatrix} A & \vdots & B \end{bmatrix} = \begin{bmatrix} A'A & \vdots & A'B \\ \hline B'A & \vdots & B'B \end{bmatrix} \tag{29}$$

It follows that

$$E\beta = \begin{bmatrix} A'A & \vdots & A'B \\ \hline B'A & \vdots & B'B \end{bmatrix} \begin{bmatrix} -A^{-1}B \\ \hline I \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{30}$$

where in Equation 30 the matrix on the right is a P by (P–R) null matrix. It has been previously noted that the columns of β are linearly independent. It is quite clear from Equation 30 that the columns of $\boldsymbol{\beta}$ are eigenvectors of **E** corresponding to P-R eigenvalues having the value zero. Here R represents the number of rows of **C**, and by the hypothesis made in order to apply the textbook method R represents also the rank of The Matrix **E** is clearly of rank R and therefore possesses no more than P-R eigenvalues with value zero.

Thus, the textbook solution is seen to be a special case of the general solution which would result if the method of this paper were applied after the rearrangements in **C** leading to Equation 7 were made.

SECTION VII
FIRST EXAMPLE

Here the method is applied to derive the equations of motion of a simple chain of spring-mass elements. The main intent is to illustrate application of the method. However, some points of general interest will arise.

The system consists of five point masses connected by linear massless springs as shown in Figure 1.

Figure 1. Spring-mass System

Each of the masses and each of the spring constants is assumed to have unit magnitude. The masses may displace only in the horizontal direction and the displacement of the n^{th} mass is denoted by x_n. A positive value of x_n is taken to mean displacement to the right and a negative value displacement to the left. A horizontal force F_n, positive to the right and a function of time only, acts upon the n^{th} mass.

The five displacements constitute a set of independent coordinates which determine the configuration of the system at any instant; and in terms of these coordinates, it is easy to write down directly equations of motion of the system in the form of Equation 1 with

$$q_n = x_n \qquad n = 1, 2, 3, 4, 5 \tag{31}$$

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{32}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

$$Q = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} \tag{34}$$

Thus, from a practical point of view, the method of this paper is not needed for an analysis of the system since the end result of the method, equations of motion in terms of independent coordinates, is readily obtained by inspection. However, as an object here is to illustrate the method, let the system be viewed in a different way as illustrated in Figure 2. There the system of Figure 1 is shown figuratively divided into four parts by cuts at the three inner masses, producing an eight-mass system.

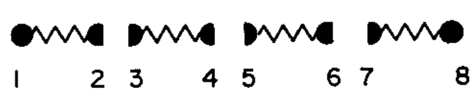

Figure 2. Cut System

The half circles represent masses of one-half-unit magnitude. The displacement of the p^{th} mass of this cut system is denoted by y_p.

It is assumed that three equations of constraint are imposed on the coordinates, namely

$$y_2 = y_3 \tag{35}$$

$$y_4 = y_5 \tag{36}$$

$$y_6 = y_7 \tag{37}$$

Thus the coordinates y_p are not independent, and from the simple geometric considerations involved it is clear that under these equations of constraint the systems of Figure 1 and Figure 2 are the same. In terms of the coordinates y_p a Lagrangian of the system may be expressed by an equation similar to Equation 5 with

$$z_p = y_p \qquad p = 1, 2, 3, 4, 5, 6, 7, 8 \tag{38}$$

$$\bar{K} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{39}$$

$$\bar{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{40}$$

The work W for the system may be expressed by an equation following the form of Equation 6 with

$$\mathbf{Z}' = \lfloor F_1, 0, F_2, 0, F_3, 0, F_4, F_5 \rfloor \tag{41}$$

It is noted that the form indicated by Equation 41 for the matrix $\mathbf{Z}$ is not unique. The following, form, for example, will serve equally well

$$\mathbf{Z}' = \lfloor F_1 \; (1/2)F_2, (1/2)F_2, (1/3)F_3, (2/3)F_3, (1/5)F_4, (4/5)F_4, F_5 \rfloor \tag{42}$$

All that is required is that $\mathbf{Z}'$ when introduced in Equation 6 should yield the work done during any displacement consistent with the equations of constraint.

Equations 35, 36, and 37, the equations of constraint, may be put in the form of Equation 4 with

C = (43)

0	1	-1	0	0	0	0	0
0	0	0	1	-1	0	0	0
0	0	0	0	0	1	-1	0

The first step in the application of the method is to compute the matrix **E** defined by Equation 16. This computation yields

E = (44)

0	0	0	0	0	0	0	0
0	1	-1	0	0	0	0	0
0	-1	1	0	0	0	0	0
0	0	0	1	-1	0	0	0
0	0	0	-1	1	0	0	0
0	0	0	0	0	1	-1	0
0	0	0	0	0	-1	1	0
0	0	0	0	0	0	0	0

The matrix **U** which follows is a modal matrix of the Matrix **E**, as may be easily verified by substitution into Equations 17 and 18.

U = (45)

0	0	0	1	0	0	0	0
$1/\sqrt{2}$	0	0	0	$1/\sqrt{2}$	0	0	0
$-1/\sqrt{2}$	0	0	0	$1/\sqrt{2}$	0	0	0
0	$1/\sqrt{2}$	0	0	0	$1/\sqrt{2}$	0	0
0	$-1/\sqrt{2}$	0	0	0	$1/\sqrt{2}$	0	0
0	0	$1/\sqrt{2}$	0	0	0	$1/\sqrt{2}$	0
0	0	$-1/\sqrt{2}$	0	0	0	$1/\sqrt{2}$	0
0	0	0	0	0	0	0	1

The Matrix λ containing the eigenvalues associated with the modal matrix is given by

$$\lambda = \begin{bmatrix} 2&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix} \quad (46)$$

The first three eigenvalues are finite and the last five have the value zero. Therefore the last five columns of **U** constitute a suitable transformation Matrix **T** . It follows that acceptable independent coordinates for describing any configuration of the system consistent with the equations of constraint are five coordinates q_n related to the displacements y_p by the equation

$$\begin{Bmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\\ y_8 \end{Bmatrix} = \begin{bmatrix} 1&0&0&0&0\\ 0&1/\sqrt{2}&0&0&0\\ 0&1/\sqrt{2}&0&0&0\\ 0&0&1/\sqrt{2}&0&0\\ 0&0&1/\sqrt{2}&0&0\\ 0&0&0&1/\sqrt{2}&0\\ 0&0&0&1/\sqrt{2}&0\\ 0&0&0&0&1 \end{bmatrix} \begin{Bmatrix} q_1\\ q_2\\ q_3\\ q_4\\ q_5 \end{Bmatrix} \quad (47)$$

From the Equation $\mathbf{Q} = \mathbf{T}'\mathbf{Z}$ it follows also, using either Equation 41 or Equation 42, that generalized forces suitable for use with the coordinates q_n, are given by

$$\begin{Bmatrix} Q_1\\ Q_2\\ Q_3\\ Q_4\\ Q_5 \end{Bmatrix} = \begin{Bmatrix} F_1\\ 1/\sqrt{2}\,F_2\\ 1/\sqrt{2}\,F_3\\ 1/\sqrt{2}\,F_4\\ F_5 \end{Bmatrix} \quad (48)$$

Completing the steps in the method gives

$$\mathbf{M} = \mathbf{T}'\overline{\mathbf{M}}\mathbf{T} = \begin{bmatrix} 1&0&0&0&0\\ 0&1/2&0&0&0\\ 0&0&1/2&0&0\\ 0&0&0&1/2&0\\ 0&0&0&0&1 \end{bmatrix} \quad (49)$$

$$
\mathbf{K} = \mathbf{T}'\overline{\mathbf{K}}\mathbf{T} = \begin{bmatrix} 1 & -1/\sqrt{2} & 0 & 0 & 0 \\ -1/\sqrt{2} & 1 & -1/2 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/\sqrt{2} \\ 0 & 0 & 0 & -1/\sqrt{2} & 1 \end{bmatrix} \tag{50}
$$

Equations 48, 49, and 50 give all the ingredients necessary for writing equations of motion for the spring-mass system in the form of Equation 2. By use of Equation 47 solutions of the equations giving time histories of the coordinates q_n can be transformed into time histories of the original coordinates y_p. If initial conditions consistent with the equations of constraint are given in terms of the coordinates y_p the equations

$$
\mathbf{q} = \mathbf{T}'\mathbf{y} \tag{51}
$$

and

$$
\dot{\mathbf{q}} = \mathbf{T}'\dot{\mathbf{y}} \tag{52}
$$

may be used to convert them into initial conditions on the coordinates q_n.

It may be noted that the matrices **K**, **M**, and **Q** given in the three preceding equations are not identical to the corresponding matrices which were written down directly from simple physical considerations in Equations 32, 33, and 34, respectively. Either of the sets of matrices forms a valid basis for equations of motion of the system of Figure 1. The difference between the matrices arises from the fact that the coordinates q_n arrived at by the method of this paper are not related to the coordinates y_p in the same way as are the coordinates x_n. Equation 47 shows the relationship between the coordinates q_n and y_p whereas coordinates x_n and y_p are related by the equation

$$
\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} \tag{53}
$$

Equation 53 may be written

$$\mathbf{y} = \bar{\mathbf{T}}\mathbf{x} \tag{54}$$

where

$$\bar{\mathbf{T}} = \mathbf{TH} \tag{55}$$

in which $\mathbf{T}$ is the transformation matrix in Equation 47, derived by the method of this paper, and

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{56}$$

Thus the coordinates x_n, which represent displacements of masses, are in the category of coordinates discussed in connection with Equation 27. The foregoing discussion illustrates a feature of the method of this paper which should be recognized by anyone using the method. That is: The coordinates q_n produced by the method are generally abstract in character and do not lend themselves to simple physical interpretations.

It is instructive to reexamine the matrix $\mathbf{C}$ in Equation 43 and think about the decisions involved in applying the textbook method. Consider the three pairs of displacements (y_2, y_3), (y_4, y_5), and (y_6, y_7) straddling the cuts in Figure 2. Let triplets of displacements be formed by taking one and only one displacement from each pair, for example (y_2, y_5, y_7) and (y_3, y_4, y_6). If the displacements in any such triplet are taken to make up the elements of the column $\mathbf{z}^{(a)}$ in Equation 7 the Matrix $\mathbf{A}$ formed from the corresponding columns will be nonsingular and the textbook method will succeed. If the elements of $\mathbf{z}^{(a)}$ are chosen from among the eight coordinates y_p in any other way, the Matrix $\mathbf{A}$ will be singular. In applying the textbook method to this simple problem, recognition of the combinations of coordinates suitable to form $\mathbf{z}^{(a)}$ must come about either from physical insight or from understanding of linear dependence among the columns of $\mathbf{C}$. In applying the method of this paper it is not necessary to think directly about the physics or about linear dependence. Instead, the problem becomes one of finding a modal matrix of $\mathbf{E}$ and identifying the columns associated with eigenvalues having the value zero. Due to the block-diagonal form of $\mathbf{E}$ in this case, it was possible by inspection to put down exactly a modal matrix and the eigenvalues of $\mathbf{E}$. Therefore, all decisions in application of the method of this paper could be made easily on a purely mathematical basis.

SECTION VIII

SECOND EXAMPLE

The purpose here is to discuss an example in which redundancies in the equations of constraint arise in a natural way. The mechanical system is shown in Figure 3. A cylindrical elastic shell is fixed at one end

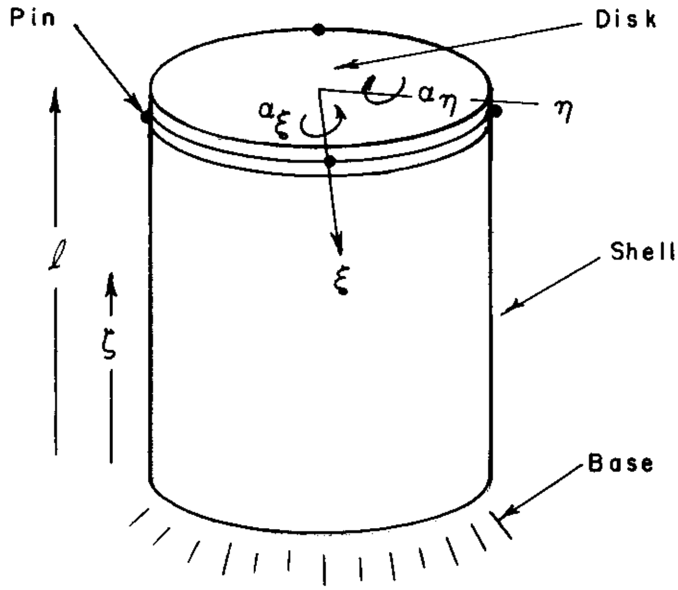

Figure 3. Shell With Attached Disk

to an immovable base. At the other end a thin massive rigid disk is attached to the wall of the shell by four symmetrically placed pins. Points in the shell wall are assumed to displace only longitudinally.

Adopting an approximation common in practical vibration analysis, the longitudinal displacement u of a general point in the shell wall is expressed as a linear combination of a finite number of displacement functions. The expansion assumed is

$$u = \sum (\delta_{m,o} + \delta_{m,4} \cos 4\theta) \sin m \frac{\pi\zeta}{2\ell} \tag{57}$$

where m takes on integral values and the summation sign indicates summation of the terms corresponding to some finite number of selected values of m. The coefficients $\delta_{m,o}$ and $\delta_{m,4}$ are functions of time alone and serve as coordinates which describe the instantaneous configuration of the shell.

Assuming small displacements, the instantaneous position of the disk is determined by specification of three coordinates δ_c, α_ξ, and α_η defined as follows:

(1) δ_c is the displacement of the center of the disk parallel to the longitudianal axis of the shell.

(2) α_ξ and α_η are small rotations about axis ξ and η, respectively, as shown in the sketch.

Equating the displacements of the disk to the displacements of the shell at each of the four pins gives

$$\delta_c - R\alpha_\xi = \sum (\delta_{m,o} + \delta_{m,4}) \sin \frac{m\pi}{2}$$

$$\delta_c + R\alpha_\eta = \sum (\delta_{m,o} + \delta_{m,4}) \sin \frac{m\pi}{2}$$

$$\delta_c + R\alpha_\xi = \sum (\delta_{m,o} + \delta_{m,4}) \sin \frac{m\pi}{2}$$

$$\delta_c - R\alpha_\eta = \sum (\delta_{m,o} + \delta_{m,4}) \sin \frac{m\pi}{2} \tag{58}$$

where R is the radius of the cylinder. If, in the summation on the right, only the terms corresponding to m = 1 are retained, the equations may be put in the form

$$\mathbf{C\,z} = 0 \tag{59}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & -1 & -1 \end{bmatrix} \tag{60}$$

and

$$\mathbf{z} = \begin{Bmatrix} \delta_c \\ R\alpha_\eta \\ R\alpha_\xi \\ \delta_{1,o} \\ \delta_{1,4} \end{Bmatrix} \tag{61}$$

It will be clear on inspection that an attempt to arrive at independent coordinates for this system by a straightforward application of the textbook method must fail because any choice of the Matrix **A** will lead to a matrix which has at least two identical columns and which is therefore singular. This difficulty stems from the fact that the system of equations is redundant which may be demonstrated by adding Rows one and three of Matrix **C** and subtracting from the result Row two producing Row four.

One way to arrive at independent coordinates would be to discard the fourth equation from the system and apply the textbook method to the first three equations. However, this approach requires, in general, the following:

(1) Recognition in the first place that the system is redundant.

(2) Recognition of dependent equations.

(3) Recognition of a nonsingular submatrix **A** after redundant equations are discarded.

For the example problem under consideration, the required understanding of the structure of the equations may be gained by inspection. In practical work, however, there may be many equations of constraint involving many unknowns, and the coefficients making up the Matrix **C** will usually not be small integers. Generally, in such situations, little of use can be deduced about the system merely by inspection of the matrix of coefficients. Also, one cannot always rely on physical insight to detect and understand redundancies. In fact, in the example being considered, there is nothing on the face of it in the physics to warn of a redundancy. Further, there are considerable theoretical and practical difficulties in making computational tests for singularity and redundancy when there is error, such as roundoff error, in the process by which the coefficients of the equations of constraint are generated.

Proceeding now to apply the method of this paper, the Matrix **E** is given by

$\mathbf{E} = \mathbf{C}'\mathbf{C} =$

4	0	0	-4	-4
0	2	0	0	0
0	0	2	0	0
-4	0	0	4	4
-4	0	0	4	4

The eigenvalues of **E** are

$$\lambda_1 = 12, \lambda_2 = 2, \lambda_3 = 2, \lambda_4 = 0, \lambda_5 = 0 \qquad (62)$$

It may be easily verified that the two columns of the Matrix **T** which follows are orthonormal eigenvectors of **E** corresponding to the two eigenvalues λ_4 and λ_5 which have the value zero.

$$\mathbf{T} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 0 & 0 \\ 0 & 0 \\ -1/\sqrt{6} & +1/\sqrt{2} \\ +2/\sqrt{6} & 0 \end{bmatrix} \qquad (63)$$

Therefore the system may be described by two independent coordinates q_1 and q_2 related to the coordinates in **z** by the equation

$$\mathbf{z} = \mathbf{T}\,\mathbf{q} \qquad (64)$$

As can be seen, direct concern with the number and nature of redundancies in the equations of constraint is unnecessary when the method of this paper is used. The problem reduces in substance to that of determining an orthonormal set of eigenvectors of **E** corresponding to the eigenvalues of **E** which have the value of zero.

SECTION IX

COMMENTS ON NUMERICS

In the examples it was possible to put down exactly the Matrix **C**, to carry out exactly the the multiplication **C′C** producing the Matrix **E**, and to determine exactly the eigenvalues of **E** and orthonormal eigenvectors corresponding to the eigenvalues with value zero. In practical work, however, numerical error due to roundoff and/or truncation may be introduced at any of these three stages of calculation. The extreme effect of such errors would be complete loss of numerical significance in the digits representing the eigenvalues of **E** and the elements of the eigenvectors of **E**. In the event the computation is subject to serious loss of significance, the Matrix **C** is said to be ill-conditioned with respect to the computing process used. The best indication of ill-conditioning is sensitivity of final results to small changes in the elements of **C**. The authors have applied the method of this paper a number of times in practical vibration analysis and have not as yet encountered a situation in which the Matrix **C** is ill-conditioned. Speaking from general experience, however, the possibility of ill-conditioning must be anticipated whenever simultaneous equations are solved numerically, and the method of this paper presents no exception to this statement. When an ill-conditioned system arises, the recourse most often open is to increase the number of digits carried in the computation. If this is attempted in connection with the method of this paper, it should be recognized that it may be necessary to increase the carried significance in the stage of the calculation in which the elements of **C** are generated as well as in the implementation of the multiplication **C′ C** and in calculating the eigenvalues and eigenvectors of **E**.

Another consequence of numerical error is that finite numbers may be generated for eigenvalues of **E** which would be precisely zero if there were no error in the computing process. This raises the question, in principle at least, of the possibility of rigorous distinction between finite numbers representing finite eigenvalues of **E** and finite numbers representing eigenvalues of **E** which are, in fact, zero. In the authors' experience this has proved to be more of a problem in principle than in practice. The authors use the threshold Jacobi Method to compute the eigenvalues and a modal matrix of **E**. Approximately 15 significant figures are carried throughout the calculation. With this procedure, inspection of the eigenvalues computed for **E** has always revealed two clearly distinguishable sets of numbers, the numbers in one set being many orders of magnitude smaller than the numbers in the other. The set of numbers with relatively large magnitudes are regarded as finite eigenvalues, and the remaining numbers are considered to be eigenvalues with value zero.

It is helpful to recognize that the number of finite eigenvalues of **E** can be anticipated if the rank R of **C** is known. For the rank of **E** is equal to R, and it is not difficult to show that the number of finite eigenvalues of **E** is therefore equal to R. Frequently, it is known from physical or geometric considerations that the equations of constraint are linearly independent in which case the rank R of **C** is equal to the number of rows of **C**.

As has been indicated, in the authors' experience with the method under discussion, it has always been possible to distinguish with confidence, on the basis of magnitude, between numbers representing finite eigenvalues and finite numbers representing eigenvalues which are, in fact, zero. If such a distinction could not be made, that is, if the eigenvalues of **E** were to decrease gradually from maximum to minimum, the authors would suspect ill-conditioning of the Matrix **C**.

SECTION X

CONCLUDING REMARKS

A computational method has been devised by which equations of motion of a linear mechanical system in terms of independent coordinates can be generated when basic information about the system is available in terms of coordinates which are not independent but, instead, are governed by linear homogeneous equations of constraint. Necessity for this derivation arises frequently in practical vibration analysis. The method is believed to be new, and experience in analyzing the vibrations of shells indicates that it will very often offer decided advantages over methods previously used. In the method, a real symmetric matrix is constructed by an operation which involves only the coefficients in the equations of constraint. The eigenvectors and corresponding eigenvalues of the symmetric matrix are computed. Then, a transformation matrix leading to independent coordinates is assembled from the eigenvectors corresponding to eigenvalues having the value zero. The main advantages of the method are: (1) For the most general constraint equations, the problem is reduced to calculating the eigenvectors and eigenvalues of a symmetric matrix. This calculation is one of the most successful applications of modern digital computers; (2) The method is applicable when there are redundancies in the equations of constraint.