A FINITE ELEMENT-METHOD FOR THE DETERMINATION OF NON-STATIONARY TEMPERATURE DISTRIBUTION AND THERMAL DEFORMATIONS

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In the finite element-method based on the principle of minimum potential energy, the so-called displacement-method, a given temperature distribution can easily be taken into account. In this paper a variational principle for linear heat flow problems has been formulated analogous to the principle of minimum potential energy. On the basis of this variational principle the temperature distribution can be derived by means of a finite element-method. The determination of the thermal deformations then follows as a natural sequel to these computations. For both successive problems the same division in elements is required. In this paper the necessary matrices have been derived. Two examples illustrate the application of the method.

1. INTRODUCTION

The determination of the thermal deformations and the accompanying stresses in a structure can be divided into two basic problems: 1) the determination of the temperature field; 2) the determination of the displacements and deformations due to this temperature field. Here it is assumed that the very weak interaction between the temperature- and the deformation problem may be neglected.

In practical applications a solution of the continuous field equations for the heat flow in a structure cannot be found but numerically with the exception of some simple one- and two-dimensional cases. For the solution of a deformation problem the finite element-method based on the principle of minimum potential energy, the so-called displacement method, has proved to be highly efficient. Here a known temperature distribution can easily be taken into account, provided that it can conveniently be translated into thermal components of generalized strains. A finite element-method for the solution of the temperature problem was devised that meets this requirement.

Analogous to the principle of minimum potential energy a variational principle for the heat flow problem can be formulated. An integral formula, quadratic in the temperatures, is shown to remain stationary with respect to temperature variations at the actual temperature distribution that satisfies the heat balance equation and boundary conditions. In the principle presented here, that of temperature dependence of thermal conductivity, specific heat of the material and heat transfer coefficients have to be neglected.

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On the basis of the variational principles governing the temperature and the displacement field, a finite element-method has been developed for the solution of both problems in one computational scheme. A necessary condition for the solution of these subsequent problems is that the division in elements for both problems is identical. The finite element-method for temperature problems leads to the solution of a set of simultaneous linear differential equations. The solution of two simple problems will illustrate this.

This variational principle for the heat flow problem resembles a principle derived previously by Biot (Reference 1). Biot's principle has the advantage that nonlinear problems can also be solved; it would also be possible to derive a finite-element method on his principle. However, for linear temperature problems the finite element-method derived in this paper, is considered preferable. Gurtin (Reference 2) presents a variational principle for the solution of some linear heat flow problems. The integral formula derived by him is also quadratic in the temperatures but since convolution integrals are used it is not quite clear in which way the finite element-method could be introduced to solve heat flow problems.

2. A VARIATIONAL PRINCIPLE FOR THE SOLUTION OF LINEAR TEMPERATURE PROBLEMS

Let a structure be subjected to a heat flux varying with time. The equation expressing balance of heat in an infinitisimal element of the structure is given by:

$$-\frac{\partial q_i}{\partial x_i} + Q^o - c_v \cdot \dot{\vartheta} = 0 \tag{1}$$

where, according to Fourier's law of heat conduction, for the heat flux vector holds:

$$q_i = -\lambda \cdot \frac{\partial \vartheta}{\partial x_i} \tag{2}$$

Here ϑ represents the temperature, λ the heat conductivity, Q° the heat production per unit volume and c_v the specific heat capacity. Moreover the summation-convention is applied here.

On the surface of the body three different types of boundary conditions will be considered:

1) prescribed temperature on the surface A_1

$$\vartheta = \vartheta^o \tag{3}$$

2) prescribed heat input through the surface A_2

$$-q_i^o \cdot n_i = -\lambda \cdot \frac{\partial \vartheta}{\partial x_i} \cdot n_i \tag{4}$$

3) heat flow through the surface A_3 that is proportional to the difference in temperature between the surface of the structure and the surrounding medium:

$$-q_i \cdot n_i = \lambda \frac{\partial \vartheta}{\partial x_i} \cdot n_i = -\alpha \, (\vartheta - \vartheta_m) \tag{5}$$

where α is the heat-transfer coefficient.

The surfaces A_1, A_2 and A_3 will form the total surface A of the structure to be investigated. The outward unit normals n_i on the surface A are taken positive. The differential Equations 1 and 2 together with their boundary conditions Equation 3 may be replaced by the requirement, that the following integral expression vanishes for all continuous, twice differentiable functions u, that are defined inside the volume, occupied by the structure and on its surface:

$$\begin{aligned} &\iint_{A_1} (\vartheta - \vartheta^{o}) \cdot u \cdot dA + \iint_{A_2} \left\{ - q_i^{o} \cdot n_i + \lambda \cdot \frac{\partial \vartheta}{\partial x_i} \cdot n_i \right\} \cdot u \cdot dA + \\ &+ \iint_{A_3} \left\{ + \alpha(\vartheta - \vartheta_m) + \lambda \cdot \frac{\partial \vartheta}{\partial x_i} \cdot n_i \right\} \cdot u \cdot dA + \\ &- \iiint_V \left\{ \frac{\partial}{\partial x_i} \left(\lambda \frac{\partial \vartheta}{\partial x_i} \right) + Q^{o} - c_v \dot{\vartheta} \right\} \cdot u \cdot dV = 0 \end{aligned} \tag{6}$$

In the above Equation 6 we shall restrict the functions u to those, that on the surface A_1, where the temperature is prescribed, are equal to zero. Then, according to the fundamental lemma of the calculus of variations the differential Equations 1 and 2 with their boundary conditions shown in Equations 3, 4 and 5 are equivalent to the variational condition:

$$\begin{aligned} \delta \bar{P} = &\iint_{A_2} \left\{ - q_i^{o} n_i + \lambda \cdot \frac{\partial \vartheta}{\partial x_i} \cdot n_i \right\} \delta \vartheta \cdot dA + \\ &+ \iint_{A_3} \left\{ + \alpha(\vartheta - \vartheta_m) + \lambda \cdot \frac{\partial \vartheta}{\partial x_i} \cdot n_i \right\} \cdot \delta \vartheta \cdot dA + \\ &- \iiint_V \left\{ \frac{\partial}{\partial x_i} \left(\lambda \cdot \frac{\partial \vartheta}{\partial x_i} \right) + Q^{o} - c_v \cdot \dot{\vartheta} \right\} \cdot \delta \vartheta \cdot dV = 0 \end{aligned} \tag{7}$$

where $\delta\vartheta$ are mathematically admissible variations of the temperature. This includes the conditions that $\vartheta = \vartheta^{o}$ on the surface A_1, where the temperature is prescribed and that ϑ is twice differentiable.

By application of the divergence theorem this expression is transformed into:

$$\begin{aligned} \delta \bar{P} = &\iiint_V \left\{ \lambda \frac{\partial \vartheta}{\partial x_i} \delta \left(\frac{\partial \vartheta}{\partial x_i} \right) - (Q^{o} - c_v \dot{\vartheta}) \delta \vartheta \right\} dV - \iint_{A_2} q_i^{o} \cdot n_i \cdot \delta \vartheta \cdot dA + \\ &+ \iint_{A_3} \alpha(\vartheta - \vartheta_m) \delta \vartheta \, dA = 0 \end{aligned} \tag{8}$$

Thus far no restriction has been made on the physical quantities λ, c_v and α. Hereafter these quantities will be considered as constants, independent of the temperature ϑ. Then the variational condition (8) can be interpreted as a stationary value principle for the following expression with respect to mathematically admissible variations of ϑ. In this variational process the time-derivative of ϑ is to be treated as a fixed quantity.

$$\bar{P} = \iiint\limits_V \left[\frac{1}{2} \lambda \cdot \frac{\partial \vartheta}{\partial x_i} \cdot \frac{\partial \vartheta}{\partial x_i} - Q^o \cdot \vartheta + c_v \cdot \vartheta \cdot \dot{\vartheta} \right] dV - \iint\limits_{A_2} q_i^o n_i \cdot \vartheta \cdot dA + \iint\limits_{A_3} \frac{1}{2} \alpha (\vartheta - \vartheta_m)^2 \, dA = \text{stationary} \tag{9}$$

Here Q°, q_i° and ϑ_m are given quantities that may vary with time. By means of the stationary principle, derived above, a new way of approximating heat-flow problems has been obtained. Particularly in the finite-element method this approximation is given as a set of simultaneous ordinary differential equations.

3. SOME GENERAL REMARKS ON THE FINITE ELEMENT METHOD

We shall restrict ourselves here to some remarks on the finite element method for the determination of the temperature distribution and for the determination of the deformations by the so-called displacement method.

The potential energy of a loaded structure is given by the expression:

$$P = \iiint\limits_V E(e_{ij}) \, dV - \iiint\limits_V f_i^o u_i \, dV - \iint\limits_{A_p} p_i^o u_i \, dA \tag{10}$$

Here $E(e_{ij})$ represents the specific elastic strain energy, f_i° the forces per unit volume, p_i° the forces per unit area and A_p the surface, where the forces have been prescribed.

According to the principle of minimum potential energy equilibrium of the structure is characterized by a stationary value of P with respect to kinematically admissible variations of the displacements. For linear deformation problems $E(e_{ij})$ is a quadratic expression in the first derivatives of the displacements and in the temperature. This integral (Equation 10) for the potential energy has a strong resemblance with the integral (Equation 9) that has been derived for linear temperature problems, e.g.:

$$\begin{aligned} \iiint\limits_V \frac{1}{2} \lambda \cdot \frac{\partial \vartheta}{\partial x_i} \cdot \frac{\partial \vartheta}{\partial x_i} \cdot dV &\rightleftharpoons \iiint\limits_V E(e_{ij}) \, dV \\ \iiint\limits_V Q \cdot \vartheta \cdot dV &\rightleftharpoons \iiint\limits_V f_i^o u_i \, dV \\ \iint\limits_{A2} q_i^o n_i \, dA &\rightleftharpoons \iint\limits_{A_p} p_i^o u_i \, dA \end{aligned} \tag{11}$$

When also inertial terms would be considered, we could extend this analogy to one more term:

$$\iiint_V c_v \cdot \vartheta \cdot \dot{\vartheta} \, dV \rightleftarrows -\iiint_V \rho \cdot u_i \, \ddot{u}_i \, dV$$

where ρ is the mass density.

So $\overline{P}$ may appropriately be called the temperature potential. As is well known the dimensions of the potential energy is work; on the other hand the dimension of the temperature potential is work x temperature.

It is seen from solutions of temperature problems in the book by Carslaw and Jaeger (Reference 6) that simple non-stationary temperature problems lead to complicated analytical analysis. Therefore the temperature potential could be a quite suitable tool for the solution of practical linear temperature problems by a finite element method, analogous to the element method based on the principle of minimum potential energy. In addition it will be possible to solve first the temperature problem and subsequently, using the same subdivision into elements, solve the thermal deformation problem.

In the discription of the element method some concepts will be used that will be described here briefly. In the element method we shall use the column vectors $\boldsymbol{\theta}$ and $\mathbf{u}$ to collect the values of the temperature or displacements respectively at distinct points of the structure. To indicate the difference between these vectors and the value of the temperature or displacement at an airbitrary point in the interior of an element the latter will be specified by a circumflex ($\wedge$).

The attack on a problem, that will be solved with one of the element methods described here, starts as follows:

a) the structure is divided by lines or surfaces into suitable elements.

b) at distinct points at the boundary of the elements a finite number of temperatures or displacements of unknown magnitude are assumed.

c) in the interior of the elements simple expressions are chosen, representing the temperature or the displacement field. These expressions contain just as many linear parameters as the number of temperatures or displacements mentioned under b. These forms are restricted by the requirement that the linear parameters shall be uniquely related to the temperature or displacements at all the distinct points of the structure.

In Sections 4 and 5 first the matrices will be derived for one individual element; this will be indicated by the superscript k. The prescribed quantities are indicated by a superscript o. For the structure as a whole, consisting of N elements, it is necessary to define new matrices. Generally for these matrices the following rule is applied: if $\mathbf{A}^k$ is the matrix for the element k then $\mathbf{A}$ is the corresponding matrix for the whole structure, where:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^1 & & \\ & \mathbf{A}^2 & \\ & & \ddots \\ & & & \mathbf{A}^N \end{bmatrix} \qquad (12)$$

The only exception to this rule is the definition of the location matrix for the whole structure. This matrix is defined as:

$$[L\ L^{o}] = \begin{bmatrix} L^{1} & L^{1o} \\ L^{2} & L^{2o} \\ | & | \\ L^{N} & L^{No} \end{bmatrix} \tag{13}$$

The non-prescribed and prescribed temperatures or displacements at distinct points of the structure are collected respectively in the vectors ϑ, ϑ^{o}, u and u^{o}. All other column vectors of the structure are defined as:

$$a = \begin{bmatrix} a^{1} \\ a^{2} \\ \vdots \\ a^{N} \end{bmatrix} \tag{14}$$

4. THE ELEMENT-METHOD FOR TEMPERATURE PROBLEMS

The value of the temperature distribution in the interior of an element can be approximated by:

$$\hat{\vartheta}^{k} = x'^{k}_{\vartheta} \cdot \beta^{k} \tag{15}$$

In this formula x'^{k}_{ϑ} is a row vector, containing functions of the coordinates;the linear parameters of the temperature distribution have been collected in the column vector β^{k}. They furnish an expression for $\hat{\vartheta}$ that fulfils the conditions b and c in Section 3.

From Equation 15 the three components of the column vector $\left[\frac{\partial \hat{\vartheta}}{\partial x_i}\right]$ can be evaluated:

$$\left[\frac{\partial \hat{\vartheta}_i}{\partial x_i}\right] = C^{k}_{\vartheta}\, \beta^{k} \tag{16}$$

Here the three rows of the matrix C^{k}_{ϑ} can be derived directly from the assumed row vector $\left(x'^{k}_{\vartheta}\right)^{T}$

The specific temperature potential due to heat conduction in the interior of an element, as it has been formulated in the Equation 9 for $\overline{P}$, becomes:

$$I^{k}_{\vartheta} = \frac{1}{2}\left[\frac{\partial \hat{\vartheta}}{\partial x_i}\right] G^{k}_{\vartheta} \left[\frac{\partial \hat{\vartheta}}{\partial x_i}\right] \tag{17}$$

where the matrix G^k_ϑ contains the heat conductivity λ. Substitution of Equation 16 into Equation 17 and integration of the matrix $(C^k_\vartheta)' G^k_\vartheta C^k_\vartheta$ over the volume of the element leads to a quadratic form in β^k representing the temperature potential due to conductivity of the element as a whole:

$$E^k_\vartheta = \iiint_V \frac{1}{2} \beta'^k C'^k_\vartheta G^k_\vartheta C^k_\vartheta \beta^k \, dV = \frac{1}{2} \beta'^k T^k \beta^k \tag{18}$$

By our choice of the approximation of the temperature distribution Equation 15 it is easy to find the relationship between the parameters of this temperature distribution β^k and the values of the temperature at distinct points of the element collected in the vector ϑ^k. The general expression for this linear transformation is represented by:

$$\beta^k = D^k_\vartheta \cdot \vartheta^k \tag{19}$$

Substitution of this transformation into Equation 18 leads to:

$$E^k_\vartheta = \frac{1}{2} \vartheta'^k D'^k_\vartheta T^k D^k_\vartheta \vartheta^k \tag{20}$$

In the integral expression Equation 9 part of the temperature potential is due to a given heat input. There are three types of heat input, that will be treated separately. We shall give here only formally the necessary transformation from the integrals to the matrix expressions. We can consider:

$$a: \quad -\iiint_V Q^o \vartheta \, dV \Longrightarrow -\vartheta'^k q_1^{ko}$$

$$b: \quad -\iint_{A_2^k} q_i^o n_i \cdot \vartheta \cdot dA \Longrightarrow -\vartheta'^k q_2^{ko}$$

and by adding (a) and (b):

$$-\vartheta'^k q_1^{ko} - \vartheta'^k q_2^{ko} = -\vartheta'^k q^{ko} \tag{21}$$

$$c: \quad -\iint_{A_3^k} \frac{1}{2} \alpha (\vartheta - \vartheta_m)^2 dA \Longrightarrow -\frac{1}{2} \left(\vartheta'^k - \vartheta'^k_m\right) Q^k \left(\vartheta^k - \vartheta^k_m\right) \tag{22}$$

As before, the matrix expression for that part of the temperature potential, which is due to the heating of an element, can be found by the following subsequent steps:

$$\begin{aligned} \iiint_{V^k} c_v \hat{\vartheta} \cdot \dot{\hat{\vartheta}} \, dV &= \iiint_{V^k} c_v \beta'^k x^k_\vartheta x'^k_\vartheta \dot{\beta}^k \, dV = \\ &= \beta'^k H^k \dot{\beta}^k = \vartheta'^k D'^k_\vartheta H^k D^k_\vartheta \cdot \dot{\vartheta}^k \end{aligned} \tag{23}$$

Thus far only one individual element has been treated. The following step is to consider an element as a part of the whole structure. This can be done by expressing the components of the vector ϑ^k in the prescribed and non-prescribed temperatures at distinct points of the structure. For that purpose the location matrix is used:

$$\vartheta^k = \begin{bmatrix} L_\vartheta^k & L_\vartheta^{ko} \end{bmatrix} \begin{bmatrix} \vartheta \\ \vartheta^o \end{bmatrix} \tag{24}$$

By means of the notation Equations 12, 13 and 14 the approximated temperature potential of the structure, divided into elements, will become:

$$\begin{aligned} \bar{P} = {} & \frac{1}{2} \begin{bmatrix} \vartheta^T & (\vartheta^o)^T \end{bmatrix} \begin{bmatrix} L_\vartheta^T \\ (L_\vartheta^o)^T \end{bmatrix} D_\vartheta^T \, T \, D_\vartheta \begin{bmatrix} L_\vartheta & L_\vartheta^o \end{bmatrix} \begin{bmatrix} \vartheta \\ \vartheta^o \end{bmatrix} - \begin{bmatrix} \vartheta^T & (\vartheta^o)^T \end{bmatrix} \begin{bmatrix} L_\vartheta^T \\ (L_\vartheta^o)^T \end{bmatrix} q^o + \\ & + \frac{1}{2} \begin{bmatrix} (\vartheta^T - \vartheta_m^T) \end{bmatrix} L_\vartheta^T \, Q \, L_\vartheta \begin{bmatrix} (\vartheta - \vartheta_m) \end{bmatrix} + \\ & + \begin{bmatrix} \vartheta^T & (\vartheta^o)^T \end{bmatrix} \begin{bmatrix} L_\vartheta^T \\ (L_\vartheta^o)^T \end{bmatrix} D_\vartheta^T \, H \, D_\vartheta \begin{bmatrix} L_\vartheta & L_\vartheta^o \end{bmatrix} \begin{bmatrix} \dot{\vartheta} \\ \dot{\vartheta}^o \end{bmatrix} = \text{stationary}, \end{aligned} \tag{25}$$

When a structure has a certain temperature distribution we have seen in Section 2 that the temperature potential $\bar{P}$ is stationary for the mathematical admissible variations of the temperature. The same will apply to Equation 25, but here the temperatures to be varied are the non-prescribed temperatures at a finite number of distinct points in the structure. This variational process leads to the following set of equations:

$$\begin{aligned} L_\vartheta^T D_\vartheta^T H D_\vartheta L_\vartheta \dot{\vartheta} = {} & -(L_\vartheta^T D_\vartheta^T T D_\vartheta L_\vartheta + L_\vartheta^T Q L_\vartheta) \cdot \vartheta + \\ & + L_\vartheta^T q^o - L_\vartheta^T D_\vartheta^T T D_\vartheta L_\vartheta^o \vartheta^o + L_\vartheta^T Q L_\vartheta \vartheta_m - L_\vartheta^T D_\vartheta H D_\vartheta L_\vartheta^o \dot{\vartheta}^o \end{aligned} \tag{26}$$

Since the expression $\iiint_V c_V \hat{\vartheta} \hat{\vartheta} \, dV$ is positive definite the matrix $L_\vartheta^T D_\vartheta^T H D_\vartheta L_\vartheta$ is non-singular. So by multiplication of Equation 26 by the inverse of this matrix the following set of simultaneous differential equations of the first order is obtained:

$$\dot{\vartheta} = -Y \cdot \vartheta + z(t) \tag{27}$$

where:

$$Y = (L_\vartheta^T D_\vartheta^T H D_\vartheta L_\vartheta)^{-1} (L_\vartheta^T D_\vartheta^T T D_\vartheta L_\vartheta + L_\vartheta^T Q L_\vartheta) \tag{28}$$

$$z(t) = (L_\vartheta^T D_\vartheta^T H D_\vartheta L_\vartheta)^{-1} (L_\vartheta^T q^o - L_\vartheta^T D_\vartheta^T T D_\vartheta L_\vartheta^o \vartheta^o + L_\vartheta^T Q L_\vartheta \vartheta_m - L_\vartheta^T D_\vartheta^T H D_\vartheta L_\vartheta \dot{\vartheta}^o) \quad (29)$$

The vector $z(t)$ is a known vector with respect to the time t. This can be specified by observing in Equation 29 that both the heat input (q^o), the prescribed temperatures (ϑ^o and $\dot{\vartheta}^o$) and the temperature of the medium, surrounding the structure (ϑ_m) can vary in a given way as functions of t.

As is known from the theory of matrix functions the solution of Equation 29 is:

$$\vartheta(t) = e^{-Yt} \left\{ \vartheta(o) + \int_0^t e^{Y\tau} \cdot z(\tau) \, d\tau \right\} \quad (30)$$

where the matrix e^{-Yt} is defined by the series:

$$e^{-Yt} = I - \frac{Yt}{1!} + \frac{Y \cdot Y \cdot t^2}{2!} - \frac{Y \cdot Y \cdot Y \cdot t^3}{3!} + \ldots \quad (31)$$

From physical considerations (see Section 6) and from a mathematical point of view it can be concluded that the matrix Y is singular if both the temperature is non-prescribed anywhere in the structure and the matrix Q is a zero matrix. However, the equations 29 and 30 are valid for all cases to be considered.

When Y is a non-singular matrix the most essential thermal problems can be solved in closed form with the aid of the eigenvalues and eigenvectors of the matrix Y. Two examples of these kinds of problems are given in Section 6.

When Y is singular a solution can only be obtained by means of a step by step numerical integration of Equation 27.

5. THE ELEMENT METHOD FOR THERMAL DEFORMATION PROBLEMS

In the preceding chapter we have found an approximation for the temperatures in a structure using an element method. In this chapter it is assumed that for the approximation of the thermal deformations the structure has been divided into elements in the same way as has been done for the determination of the temperatures.

The value of the displacements in the interior of an element can be approximated by

$$\hat{u}^k = X_u^k \cdot a^k \quad (32)$$

Generally this vector $\hat{u}^k$ consists of three components giving the displacements in the three coordinate directions. The matrix X_u^k contains functions of the coordinates, the vector a^k the linear parameters in the displacement field. The choice of X_u^k and the number of parameters is influenced by the conditions b and c of Section 3. From these displacements

Equation 32, the total deformation in the interior of an element can be evaluated; this is reflected in the following equation:

$$\hat{e}^k = C_u^k \quad \alpha^k \tag{33}$$

The local deformation due to a temperature field is given by the tensor formula:

$$e_{ij}^o = \frac{c}{3} \hat{\vartheta} \delta_{ij} \tag{34}$$

Here e_{ij}^o represents the strain tensor due to the temperature field $\hat{\vartheta}$, c is the coefficient of cubic expansion and δ_{ij} is the Kronecker delta. For the temperature distribution the approximation Equation 15 is used.

This has been written as:

$$\hat{\vartheta}^k = (x_\vartheta^k)^T \beta^k \tag{35}$$

This matrix formula for the deformation modes due to the temperature field in an element is given by:

$$\hat{e}^{ko} = B_\vartheta^k \beta^k \tag{36}$$

The elastic deformations (e^k) are the difference between the total deformations e^k and the thermal deformations e^{ko}:

$$\hat{s}^k = \hat{e}^k - \hat{e}^{ko} = C_u^k \alpha^k - B_\vartheta^k \beta^k \tag{37}$$

The specific elastic strain energy in the kth element may then be given by:

$$I_u^k = \frac{1}{2} (\hat{s}^k)^T G_u^k \hat{s}^k \tag{38}$$

where G_u^k is the matrix containing the elastic constants.

After substitution of Equation 37 and after integration over the volume of the element the total elastic strain energy of the element becomes:

$$E_u^k = \frac{1}{2} \left[(\alpha^k)^T (\beta^k)^T \right] \begin{bmatrix} S_{uu}^k & -S_{u\vartheta}^k \\ -S_{\vartheta u}^k & S_{\vartheta\vartheta}^k \end{bmatrix} \begin{bmatrix} \alpha^k \\ \beta^k \end{bmatrix} \tag{39}$$

where:

$$S_{uu}^k = \iiint_V (C_u^k)^T G_u^k \; C_u^k \; dV \; ;$$

$$S^k_{u\vartheta} = \left(S^k_{\vartheta u}\right)^T = \iiint_V \left(C^k_u\right)^T G^k_u \, B^k_\vartheta \, dV ;$$

$$S^k_{\vartheta\vartheta} = \iiint_V \left(B^k_\vartheta\right)^T G^k_u \, B^k_\vartheta \, dV .$$

By our choice of the Approximation 32 it is easy to find the relationship between the parameters of the displacements α^k and the values of the displacements at certain points of the element collected in the vector u^k. The general expression for this linear transformation is given by:

$$\alpha^k = D^k_u \cdot u^k \tag{40}$$

Substitution of Equations 40 and 19 into Equation 39 leads to:

$$E^k_u = \frac{1}{2} \left[\left(u^k\right)^T \left(\vartheta^k\right)^T\right] \begin{bmatrix} \left(D^k_u\right)^T & 0 \\ 0 & -\left(D^k_\vartheta\right)^T \end{bmatrix} \begin{bmatrix} S^k_{uu} & S^k_{u\vartheta} \\ S^k_{\vartheta u} & S^k_{\vartheta\vartheta} \end{bmatrix} \begin{bmatrix} D^k_u & 0 \\ 0 & -D^k_\vartheta \end{bmatrix} \begin{bmatrix} u^k \\ \vartheta^k \end{bmatrix} \tag{41}$$

The integral in the potential energy, which is due to loads on one element of the structure, can always be replaced by an approximation in matrix form. Here we shall not concern ourselves with the contribution due to forces of inertia. We restrict ourselves to quasi-static problems. We shall give here only formally the necessary transformation from the integrals to the matrix expression:

$$-\iiint_V f^o_i \, u_i \, dV - \iint_{A_P} P^o_i \, u_i \, dA \longrightarrow -\left(u^k\right)^T f^{ok} \tag{42}$$

In the same way as has been done in Section 4 we shall consider this element as a part of the whole structure. Again a location matrix is used to express the components of the vector u^k in the prescribed and non-prescribed displacements at distinct points of the structure. This transformation is:

$$u^k = \left[L^k \;\; L^{ko}\right] \begin{bmatrix} u \\ u^o \end{bmatrix} \tag{43}$$

By means of the Equations 12, 13 and 14 the matrix expression for the potential energy can now be formulated for the structure as a whole:

$$P = \frac{1}{2}\left[u^T (u^o)^T \vartheta^T (\vartheta^o)^T\right] \begin{bmatrix} L_u^T & 0 \\ (L_u^o)^T & 0 \\ 0 & L_\vartheta^T \\ 0 & (L_\vartheta^o)^T \end{bmatrix} \begin{bmatrix} D_u^T & 0 \\ 0 & -D_\vartheta^T \end{bmatrix} \begin{bmatrix} S_{uu} & S_{u\vartheta} \\ S_{\vartheta u} & S_{\vartheta\vartheta} \end{bmatrix} \begin{bmatrix} D_u & 0 \\ 0 & -D_\vartheta \end{bmatrix} \begin{bmatrix} L_u & L_u^o & 0 & 0 \\ 0 & 0 & L_\vartheta & L_\vartheta^o \end{bmatrix} \begin{bmatrix} u \\ u^o \\ \vartheta \\ \vartheta^o \end{bmatrix} + \tag{44}$$

$$- \left[u^T (u^o)^T\right] \begin{bmatrix} L_u^T \\ (L_u^o)^T \end{bmatrix} f^o$$

When a structure is in equilibrium the potential energy is stationary for kinematically admissible variations of the displacements. The same will apply to the matrix Expression 44 for P where the kinematical degrees of freedom have been collected in the vector u. This variational process leads to the following set of equations:

$$L_u^T D_u^T S_{uu} D_u L_u \cdot u = L_u^T f^o - L_u^T D_u S_{uu} D_u L_u^o u^o +$$

$$+ L_u^T D_u^T S_{u\vartheta} D_\vartheta \left[L \quad L_\vartheta^o\right] \begin{bmatrix} \vartheta \\ \vartheta^o \end{bmatrix} \tag{45}$$

We shall write this as:

$$K_u \cdot u = b + \left[K_\vartheta \; K_\vartheta^o\right] \begin{bmatrix} \vartheta \\ \vartheta^o \end{bmatrix} \tag{46}$$

where:

$$K_u = L_u^T D_u^T S_{uu} D_u L_u$$

$$b = L_u^T f^o - L_u^T D_u^T S_{uu} D_u L_u^o u^o \tag{47}$$

$$\left[K_\vartheta K_\vartheta^o\right] = L_u^T D_u^T S_{u\vartheta} D_\vartheta \left[L_\vartheta, L_\vartheta^o\right]$$

From Equation 46 we see how we can use the temperature vector ϑ, determined as a solution of Equations 27 or 28, to find the displacements in a thermal deformation problem.

It should be noted here that the notation, used here, differs slightly from the notation in the paper by Besseling (Reference 5). The matrices D_u^k are equivalent, not to the finite-difference matrices D^k, but to the transformation matrices T^k of Reference 5. As a consequence the stiffness matrix S_{uu}^k is singular and equivalent to the matrix $(C^k)^T S^k C^k$ in the notation of Reference 5.

6. SOLUTION OF TWO TEMPERATURE PROBLEMS BY MEANS OF THE FINITE ELEMENT-METHOD

6.1. The solution for two special cases.

We shall solve here two special linear heatflow problems using the element-method. These problems are specified by the prescribed conditions of the problem, or from the mathematical point of view: how does the vector $\mathbf{z}(t)$ in Equation 29 vary with the time t? We shall restrict ourselves to those vectors $\mathbf{z}(t)$ that are constant or vary linearly with time. The solution of these problems covers a great many practical applications.

Equation 30 governs the general solution of temperature problems by means of the element-method:

$$\boldsymbol{\theta}(t) = e^{-\mathbf{Y}t} \left\{ \boldsymbol{\theta}(0) + \int_0^t e^{\mathbf{Y}\tau} \mathbf{z}(\tau)\, d\tau \right\}. \tag{48}$$

In the following the matrix $\mathbf{Y}$ is assumed to be non-singular, so that Equation 48 can be integrated when $e^{\mathbf{Y}\tau}.\mathbf{z}(\tau)$ is an integrable function. If the structure is not completely isolated $\boldsymbol{\theta}(0)$ will not be of any influence for $t \Rightarrow \infty$ and we find the following equation:

$$\lim_{t \to \infty} e^{-\mathbf{Y}t} = 0 \tag{49}$$

6.1.1. The first problem to be considered will be:

$$\mathbf{z}(t) = \mathbf{z}_0 = \text{constant for } t > 0.$$

Integration of Equation 48 leads to:

$$(t) = e^{-\mathbf{Y}t} \cdot \boldsymbol{\theta}(0) + \mathbf{Y}^{-1} \left\{ 1 - e^{-\mathbf{Y}t} \right\} \mathbf{z}_0 \tag{50}$$

For $t \rightarrow \infty$ Equation 50 becomes:

$$\boldsymbol{\theta}(\infty) = \mathbf{Y}^{-1} \mathbf{z}_0 \tag{51}$$

Since non-singular matrices $\mathbf{Y}$ are only considered we know that $\boldsymbol{\theta}(\infty)$ exists. Substitution of Equation 51 into Equation 50 leads to:

$$\boldsymbol{\theta}(t) - \boldsymbol{\theta}(\infty) = e^{-\mathbf{Y}t} \left\{ \boldsymbol{\theta}(0) - \boldsymbol{\theta}(\infty) \right\} \tag{52}$$

With the notation:

$$\mathbf{b}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}(\infty) \tag{53}$$

Equation 52 becomes:

$$\mathbf{b}(t) = e^{-\mathbf{Y}t} \cdot \mathbf{b}(0) \tag{54}$$

6.1.2. In the second problem the vector $z(t)$ fulfils the expression:

$$z(t) = z_0 + z_1 \cdot t \tag{55}$$

Integration of Equation 48 leads to:

$$\theta(t) = e^{-Yt}\,\theta(0) + Y^{-1}\left\{1 - e^{-Yt}\right\} z_0 - Y^{-2}\left\{1 - e^{-Yt}\right\} z_1 + Y^{-1} z_1 t \tag{56}$$

$$c(t) = \theta(t) - Y^{-1} z_0 + Y^{-2} z_1 - Y^{-1} z_1 t \tag{57}$$

into Equation 56 leads to:

$$c(t) = e^{-Yt} \cdot c(0) \tag{58}$$

which expression is completely equivalent to Equation 54. Besides a constant increase in temperature: $Y^{-1} z_1$ t the temperature distribution for $t \Rightarrow \infty$ is given by: $Y^{-1} z_0 - Y^{-2} z_1$.

6.2. Further solution of the problems.

We have seen in paragraph 6.1. that the two problems treated there, can be reduced to one equation:

$$b(t) = e^{-Yt} \cdot b(0) \tag{59}$$

The solution of this set of equations can be found by the application of matrix functions (Reference 6). For a non-singular matrix Y the following matrix equation holds:

$$Y \cdot \Psi_y = \Psi_y \cdot \Lambda \tag{60}$$

where Ψ_y represents the matrix with eigenvectors and Λ the diagonal matrix containing the eigenvalues λ belonging to the eigenvectors of Y in Ψ_y.

Now we solve the following set of linear equations:

$$\Psi_y \cdot \omega_b = b(0) \tag{61}$$

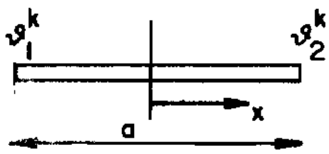
When the vector ω_b is known we can derive subsequently:

$$\begin{aligned} b(t) &= e^{-Yt}\, b(0) = e^{-Yt}\, \Psi_y \cdot \omega_b = \\ &= \Psi_y \cdot \bar{\Lambda}\, \omega_b = \Psi_y \cdot \Omega_b\, \bar{\lambda} \end{aligned} \tag{62}$$

where $\bar{\Lambda}$ represents the diagonal matrix with terms $e^{-\lambda t}$, $\bar{\lambda}$ the column vector with terms $e^{-\lambda t}$ and Ω_b the diagonal matrix with the terms of the vector ω_b. In this Expression 62 the product matrix $\Psi_y \cdot \Omega_b$ is known; so, if we have found a solution to Equation 59 in the form 62 the temperature distribution for all the requested values of t can easily be evaluated.

6.3. The cylindrical slab

The examples for the illustration of the finite element-method will deal with two different temperature problems of a cylindrical slab with zero initial temperature and with the surfaces at the end kept at the same, prescribed temperature. The slab has been completely isolated along its cylinder surface.



It is assumed that the temperature ϑ^k in a slab-element of length a can be approximated by:

$$\vartheta^k = x_\vartheta^{k\,T} \cdot \beta^k = \begin{bmatrix} 1 & \frac{x}{a} \end{bmatrix} \begin{bmatrix} \beta_1^k \\ \beta_2^k \end{bmatrix} \tag{63}$$

In agreement with Section 4 we need the matrices C_ϑ^k, G_ϑ^k, T^k, D_ϑ^k and H^k.

Since the temperature distribution in the slab is known these matrices can be derived. They have the following value:

$$C^k = \frac{1}{a}\begin{bmatrix} a & 1 \end{bmatrix}; \quad G_\vartheta^k = \lambda;$$

$$T^k = \frac{\lambda F}{a}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_\vartheta^k = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}; \quad H^k = \frac{c_v a F}{12}\begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix}$$

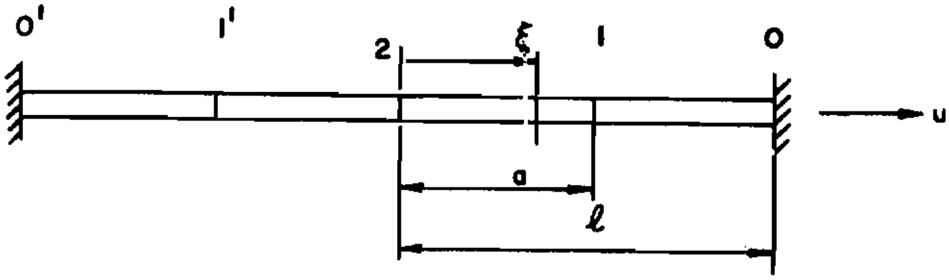
where F represents the cross-sectional area, a, the length, λ, the heat conductivity and c_v, the heat capacity of the slab. For the problems to be considered we need only the following two matrices:

$$\left(D_\vartheta^k\right)^T T^k D_\vartheta^k = \frac{\lambda F}{a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left(D_\vartheta^k\right)^T H^k D_\vartheta^k = \frac{c_v a F}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{64}$$

6.3.1. Use of the finite element-method

The slab to be considered has been divided into four identical elements. We shall use the symmetry of the problem to reduce the number of dependent variables to two.



In the following sections we shall find the general solution for two simple problems by means of the finite element-method; these solutions will be compared with theoretical results. If the centre of the slab is kept fixed in space ($u_2 = 0$) then the theoretical solution of the average temperature ϑ_a of the slab is a measure for the elongation of the slab:

$$u = \alpha \ell \; \vartheta_a$$

where α is the coefficient of linear expansion.

For the determination of the thermal deformations with the element-method the Formulas of Section 5 must be used.

Therefore the displacement $\hat{u}$ in a slab element will be approximated by:

$$\hat{u} = \begin{bmatrix} 1 & \frac{x}{a} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{65}$$

Elaboration of the equations in Section 5, with $u_2 = 0$, leads to the following set of equations for u_o and u_1:

$$\frac{EF}{a} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_o \\ u_1 \end{bmatrix} = \frac{1}{2} EF\alpha \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vartheta_o \\ \vartheta_1 \\ \vartheta_2 \end{bmatrix}$$

or: (66)

$$\frac{1}{\alpha \ell} \begin{bmatrix} u_o \\ u_1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} \vartheta_o \\ \vartheta_1 \\ \vartheta_2 \end{bmatrix}$$

From this equation we see that again $\frac{u_0}{q\ell}$ is equal to the average temperature ϑ_a in the slab using the approximate solution. This is only true in this case where all the slab elements are identical.

6.3.2. Example One

In the first example the surfaces at the end of the slab are kept at a constant temperature ϑ_0 for t > O. The matrix equation for the approximation is:

$$\frac{c_v aF}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\dot{\vartheta}_1\\ \dot{\vartheta}_2\end{bmatrix} = -\frac{\lambda F}{a}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} - \frac{\lambda F}{a}\begin{bmatrix}-1\\ 0\end{bmatrix}\cdot\vartheta_0 \quad (67)$$

With: ℓ = 2a. The matrix **Y** and the vector $\mathbf{z}_o$ are:

$$\mathbf{Y} = \frac{24\lambda}{7c_v\ell^2}\begin{bmatrix}5 & -3\\ -6 & 5\end{bmatrix} ; \quad \mathbf{z}_o = -\frac{24\lambda}{7c_v\ell^2}\begin{bmatrix}-2\\ 1\end{bmatrix}\vartheta_0 . \quad (68)$$

By the introduction of a dimensionless parameter $\bar{t}$ for the time:

$$\bar{t} = \frac{\lambda t}{c_v\ell^2} \quad (69)$$

the general solution of Equation 57 becomes:

$$\boldsymbol{\vartheta}(\bar{t}) = \begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}\vartheta_0 - \begin{bmatrix}0.859 & 0.141\\ 1.207 & -0.207\end{bmatrix}\begin{bmatrix}e^{-2.60\,\bar{t}}\\ e^{-31.7\,\bar{t}}\end{bmatrix}\vartheta_0 \quad (70)$$

The approximation of the average temperature is:

$$\vartheta_a(\bar{t}) = \vartheta_0 - [0.731 \quad 0.019]\begin{bmatrix}e^{-2.60\,\bar{t}}\\ e^{-31.7\,\bar{t}}\end{bmatrix}\vartheta_0 \quad (71)$$

The theoretical solution of this problem is given on page p. 100 in Reference 6 as a series expansion. If we take only the first two terms of this expansion and determine the values of the temperatures at the points 1 and 2 of the slab we get:

$$\boldsymbol{\vartheta}(\bar{t}) = \begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}\vartheta_0 - \begin{bmatrix}0.900 & 0.300\\ 1.273 & -0.424\end{bmatrix}\begin{bmatrix}e^{-2.47\,\bar{t}}\\ e^{-22.2\,\bar{t}}\end{bmatrix}\vartheta_0 \quad (72)$$

and the average temperature is:

$$\vartheta_a = \vartheta_0 - [0.811 \quad 0.090] \begin{bmatrix} e^{-2.47\bar{t}} \\ e^{-22.2\bar{t}} \end{bmatrix} \vartheta_0 \tag{73}$$

In the solution of the approximation of ϑ_a Equation 71 we see that for $\bar{t} = 0$ the average temperature becomes: $\vartheta_a = 0.25\ \vartheta_0$. This is due to the linear temperature distribution in the slab elements at the ends of the slab.

6.3.3. Example Two

In the second example the surfaces at the end of the slab are kept at a temperature $\dot{\vartheta}_0 \cdot t$ for $t > 0$.

The matrix equation for the approximation is:

$$\frac{c_v aF}{6} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix} = -\frac{\lambda F}{a} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} - \frac{\lambda F}{a} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \dot{\vartheta}_0 t - \frac{c_v aF}{6} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{\vartheta}_0 \tag{74}$$

With: $\ell = 2a$ the matrix $\mathbf{Y}$ and the vectors $\mathbf{z}_0$ and $\mathbf{z}_1$ are:

$$\mathbf{Y} = \frac{24\lambda}{7 c_v \ell^2} \begin{bmatrix} 5 & -3 \\ -6 & 5 \end{bmatrix}; \quad \mathbf{z}_0 = -\frac{1}{7} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \dot{\vartheta}_0; \quad \mathbf{z}_1 = -\frac{24\lambda}{7 c_v \ell^2} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \dot{\vartheta}_0 \tag{75}$$

By the introduction of a temperature parameter θ_0 defined by:

$$\theta_0 = \frac{c_v \ell^2 \dot{\vartheta}_0}{\lambda} \tag{76}$$

and by the use of the dimensionless $\bar{t}$ in Equation 69 the general solution of Equation 74 becomes:

$$\boldsymbol{\vartheta}(\bar{t}) = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{\vartheta}_0 t - \begin{bmatrix} 0.375 \\ 0.500 \end{bmatrix} \theta_0 + \begin{bmatrix} 0.364 & 0.011 \\ 0.515 & -0.015 \end{bmatrix} \begin{bmatrix} e^{-2.60\bar{t}} \\ e^{-31.7\bar{t}} \end{bmatrix} \theta_0 \tag{77}$$

The approximation of the average temperature is:

$$\vartheta_a(\bar{t}) = \dot{\vartheta}_0 t - 0.313\ \theta_0 + [0.311 \quad 0.002] \begin{bmatrix} e^{-2.60\bar{t}} \\ e^{-31.7\bar{t}} \end{bmatrix} \cdot \theta_0 \tag{78}$$

Again comparison with the theoretical solution of this problem (Reference 6 page 104) is possible. By restricting ourselves to the first two terms of the expansion in the theoretical solution, we get:

$$\vartheta(\bar{t}) = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{\vartheta}_0 t - \begin{bmatrix} 0.375 \\ 0.500 \end{bmatrix} \theta_0 + \begin{bmatrix} 0.365 & 0.014 \\ 0.515 & -0.019 \end{bmatrix} \begin{bmatrix} e^{-2.47\bar{t}} \\ e^{-22.2\bar{t}} \end{bmatrix} \qquad (79)$$

and

$$\vartheta_0(\bar{t}) = \dot{\vartheta}_0 t - 0.333\,\theta_0 + \begin{bmatrix} 0.328 & 0.004 \end{bmatrix} \begin{bmatrix} e^{-2.47\bar{t}} \\ e^{-22.2\bar{t}} \end{bmatrix} . \theta_0 \qquad (80)$$

These two simple examples show us the use of the finite element-method for linear heat flow problem. The corresponding treatment of more complicated structures is in preparation.

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