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AN ANALOGUE SERIES COMPUTER

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FOREWORD

This report covers studies conducted by Dr. Max Scherberg, who acted as project engineer for the Aeronautical Research Laboratory, WADC, Ohio, into the expansion of the use of the analogue series computer beyond that for which it was originally intended. The work was conducted under RDO 475-401 "Investigate Analogue Machine Computation of Certain Classes of Partial Differential Equations."

The assistance of Wolfgang G. Braun, who set up the electric circuits and built the prototype model, is gratefully acknowledged.

ABSTRACT

The analogue series computer was designed primarily to evaluate and find the roots of polynomials. A study was conducted to determine the possible expansion in the scope of this original use of the computer to other types of expansions. This report contains a brief description of the computer with equations that it currently solves included as background material. An additional use for the computer is discussed with a function solved to serve as an example. Graphs indicate the accuracy of the computer by showing the difference between computer values and hand-calculated values. Other possible functions that may be applied to the computer for solution are also suggested.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

Leslie B. Williams

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Colonel, USAF
Chief Aeronautical Research Laboratory
Directorate of Laboratories

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INTRODUCTION

The analogue series computer, invented by Dr. Scherberg for the Office of Air Research, was designed primarily for the purpose of evaluating and finding the roots of polynomials. In addition, it was considered to be capable of solving other types of expansions. The purpose of this report is to discuss the use of the computer with some basic types of expansions and to suggest that additional capabilities of the computer be investigated.

DESCRIPTION OF EQUIPMENT

Figure 1 is a block diagram of the analogue series computer. The cosine and sine generating signals originate from the function signal generator, pass through the respective left-upper and lower turning arms and through the coefficient potentiometer to isolating amplifiers. The multiplied signals from the amplifier pass into relay arms (R_2) and then into the precision condenser. This occurs when the timing voltage reaches zero, at which time, relays R_2 and R_3 trip their switch arms to the opposite poles. The sampled voltages in the upper condensers are now impressed through isolating amplifiers or integrators, which integrate this fixed voltage with respect to time. While this integration is going on, the lower condensers are initially being charged by the same signal voltages sampled in the upper condensers. This, however, is momentary, since relay R_1 is tried by the timing circuit soon after relays R_2 and R_3 are tripped, and the signal multiplied by the next coefficient is then being sampled in the lower condensers. Sampling and integrating is continued until all the terms of an expansion have been generated and added.

ILLUSTRATED USE OF THE ANALOGUE COMPUTER

To demonstrate the use of the analogue computer, a problem of finding the roots of the polynomial in a single variable Z, such as occurs in stability investigation, is discussed. The problem can be generalized to that of plotting the function:

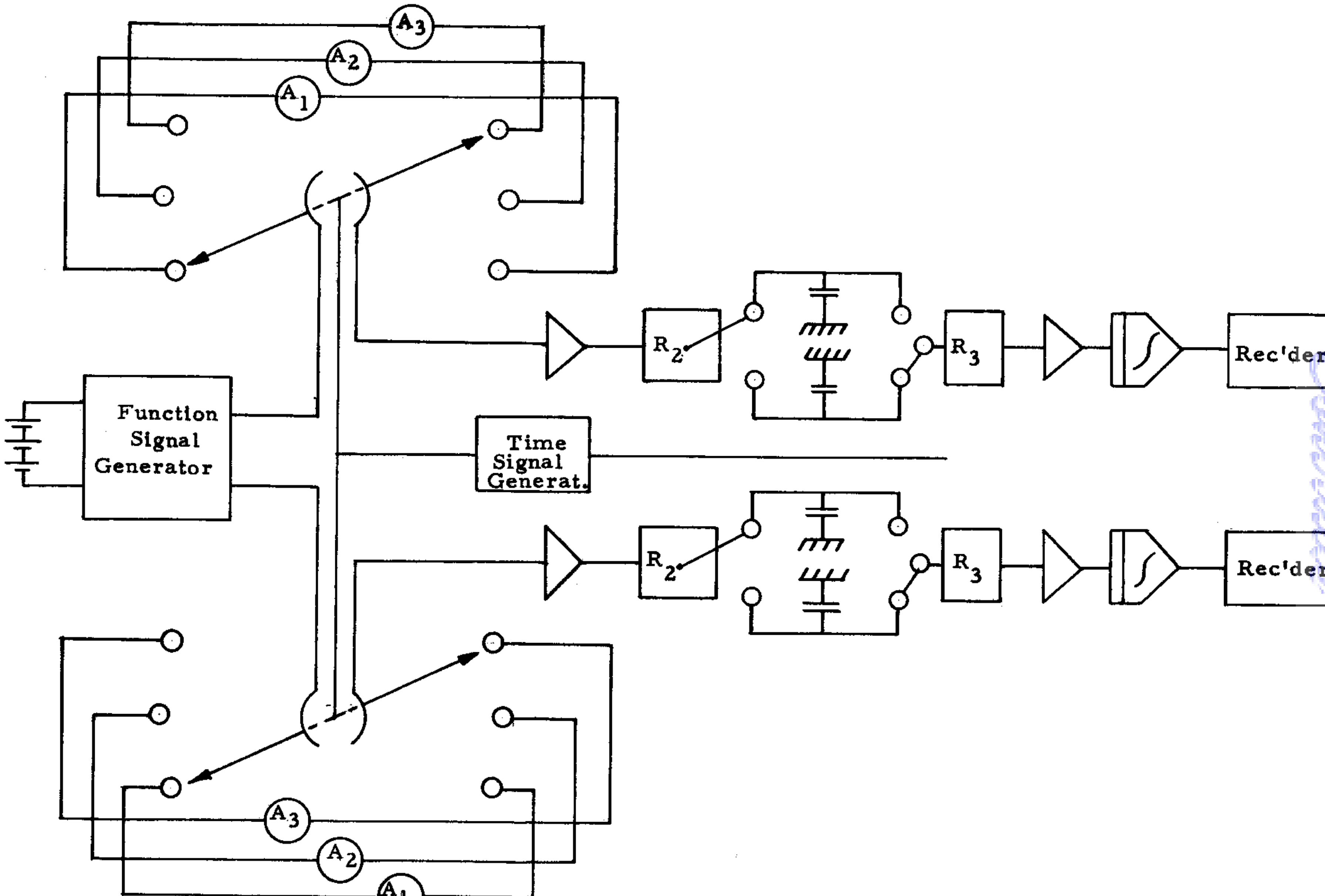


Fig. 1. Block Diagram of the Computer

(1) $f(Z) = a_o + a_1 Z + a_2 Z^2 + a_3 Z^3 \ldots\ldots + a_n Z^n$

for given contours in the Z plane. Assume, for the present, that the coefficients, a, are real.

For use in the electronic analogue computer, it is best to transform the function to trigonometric form by use of De Moivres transformation theorem $Z = r(\cos \theta + i \sin \theta)$ in which r, θ are the modulus and argument respectively of the complex number Z. The function f(Z) takes the form:

(2) $f(z) = (a_o + a_1 r \cos \theta + a_2 r^2 \cos 2\theta + \ldots + a_n r^n \cos n\theta)$

$+ j(a_1 r \sin \theta + a_2 r^2 \sin 2\theta + \ldots\ldots + a_n r^n \sin n\theta)$

In this form, the information is ready to be fed into the computer in the form of relative voltages representing the separate terms of the series. Appropriate addition of these voltages will result in a function of type (1) for some complex Z or coordinate pair (r, θ). Variation of (r, θ) through trial and error, until each of the sum voltages is zero, gives one root of f(Z).

Electronic circuits are devised to generate output voltages which are the solutions of:

(3) $\ddot{x} + 2a\dot{x} + (a^2 + b^2)x = 0$

(4) $\ddot{x} + (\pi/\tau)^2 x = 0$

Figures 2 and 3 illustrate the respective circuits.

As seen in Figure 2, the output voltages obtained from equation (3) are:

(5) $e^{-at} \sin(bt)$; $e^{-at} \cos(bt)$

Using the substitutions:

(6) $t = n\tau$, $\theta = b\tau$, $r = e^{-a\tau}$

the equations are transformed to:

(7) $r^n \sin n\theta$; $r^n \cos(n\theta)$

With suitable initial conditions, this signal may have the form:

(8) $x = e^{-at}$ when $b = 0$

or $x = \sin(bt)$ when $a = 0$

then using the same transforms given in equation (6):

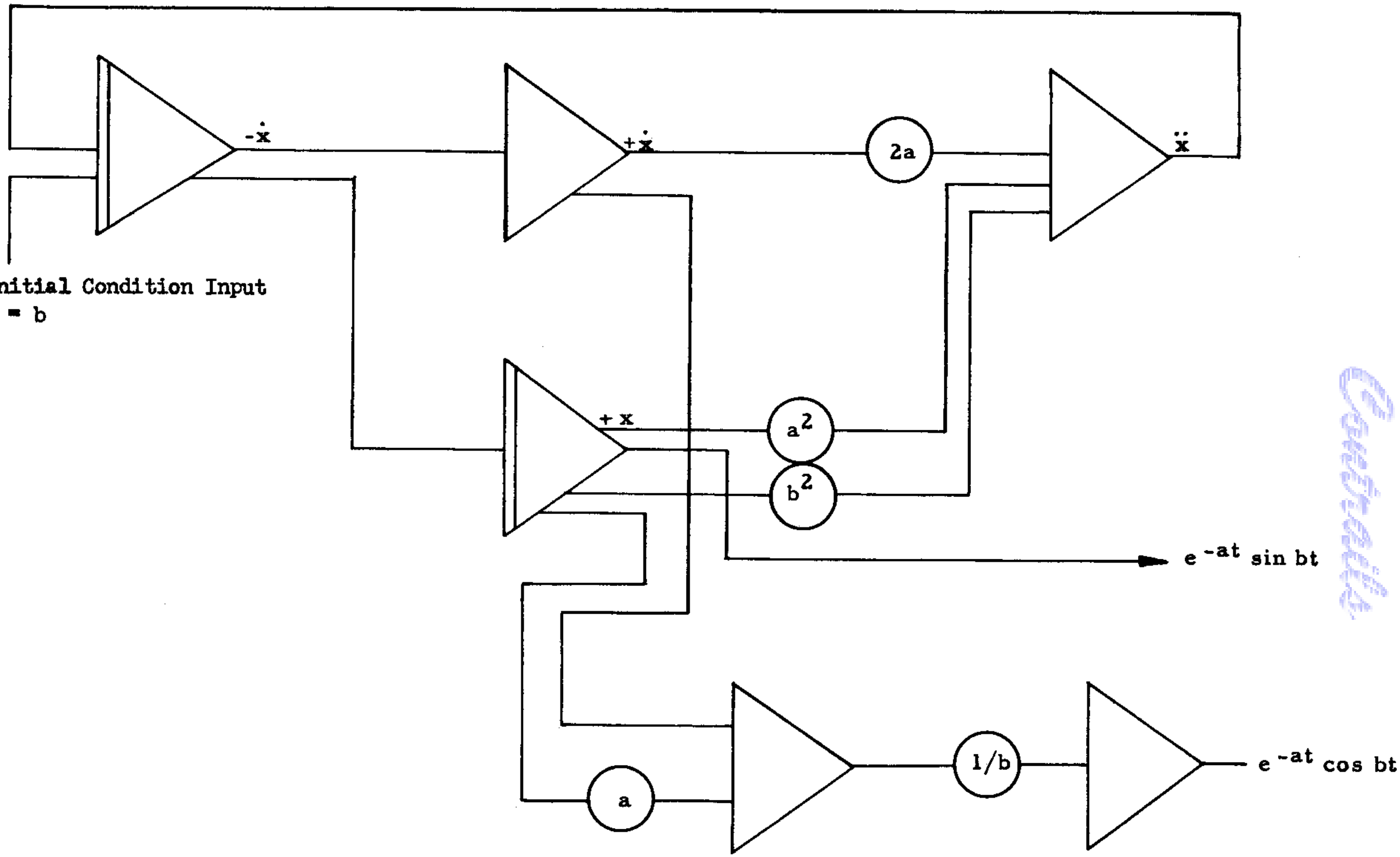
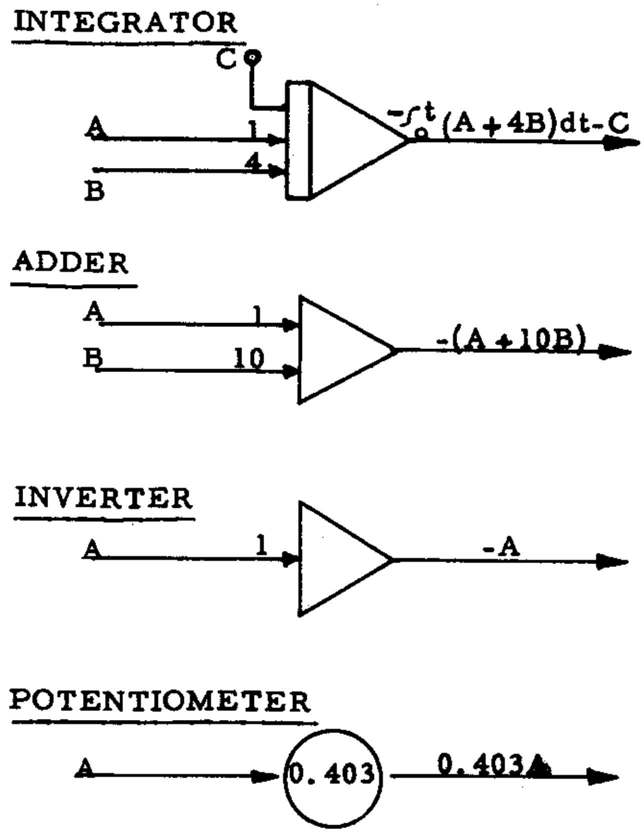


Figure 2. Wave Generator $(\ddot{x} + 2a\dot{x} + (a^2 + b^2)\,x = 0)$



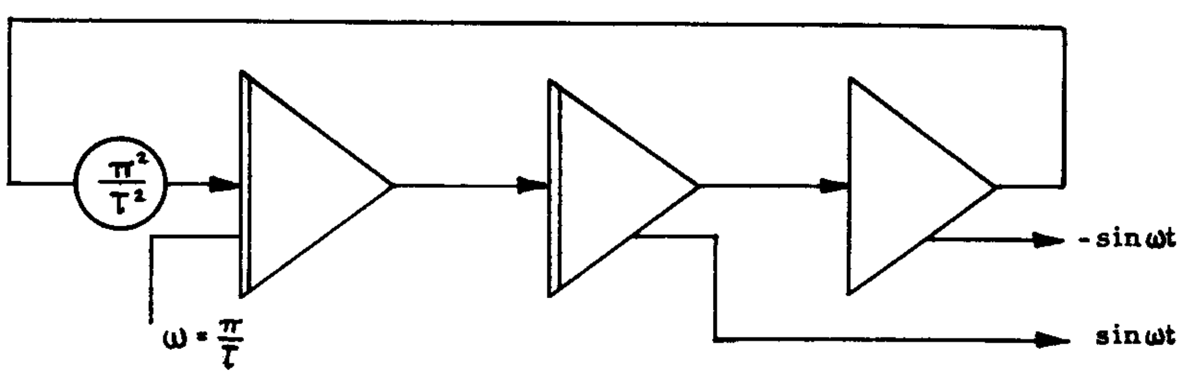


Figure 3. Timing Wave Generator $\left[\ddot{x} + \omega^2 x = 0 \begin{pmatrix} t = 0 \\ x = \omega \end{pmatrix}\right]$

(9) $x = r^n$ when $b = 0$

and $x = \sin(n\theta)$ when $a = 0$

Depending on initial conditions, periodic sampling of signals (7) and (9), when appropriately amplified by the coefficient multiplier (a_n) give the successive terms in expansions of the form:

(10) $\Sigma a_n r^n$ and $\Sigma a_n \sin n\theta$

The output of circuit solving equation (4) merely serves as a timing device and to operate sampling and switching relays. Addition of successive terms of the equation given by the circuit of Figure 1 will result in the value of the polynomial. Each of the terms may be successively fed into a summing integrator for a unit time so that the output of the integrator represents the successive sum of terms. Although some applications require the sampling period to be uniform, there are others in which this is not the case.

Figure 4 shows the component operation records made by a Brush Recorder for a sixth degree polynomial. In Figure 4, (a) is the generated signal and (b) is the signal multiplied successively by each of the coefficients. Part (c) shows the sampling in the upper condenser of Figure 1 followed by (d), a record of the sampling on condenser (2) of Figure 1. Line (e) is the recording of a succession of voltages impressed on an integrator and (f) shows the summed integrated values.

Figure 5 shows the actual computer connected to a REAC which was used to supply the function signals, the timing signals, and the coefficient potentiometer. Figure 6 is a close-up of the computer on which, group A is the condensers, and group B the sampling relays. The tubes are used for triggering the relays. Figure 7 is a rear view of the computer showing a stepper switch for the coefficient potentiometer.

For this particular application, there are three general parameters available, a, b, and τ, for the purpose of assigning the independent variable Z or the coordinate pair (r, θ). If the time period τ is fixed, a and b will determine r and θ, so that only one parameter need be varied to calculate the polynomial value of either central radii vectors or concentric circles in the complex plane. If τ is varied, while a and b remain fixed, the corresponding value of the polynomial will correspond to points on a spiral in the Z plane. On the other hand, if $a = 0$, then the contour in the Z plane corresponding to variations in τ will be parts of a fixed circle. In this case, the trigonometric form of the polynomial can be considered as representing the two parts of a conventional Fourier expansion if the respective coefficients in the two parts are made independent, thus making this method of analogue computation useful for summing trigonometric expansions.

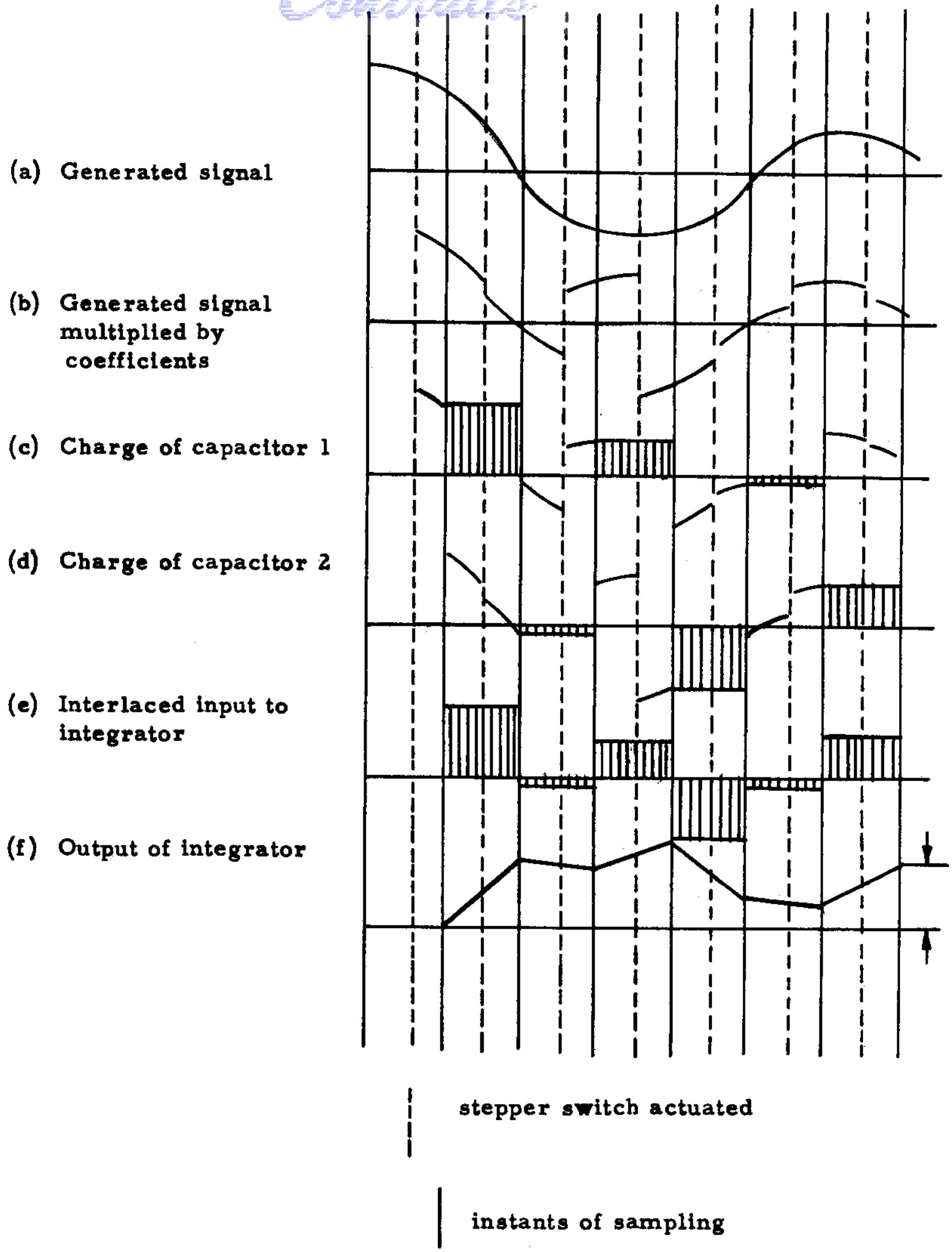


Fig. 4. Component Operation Record

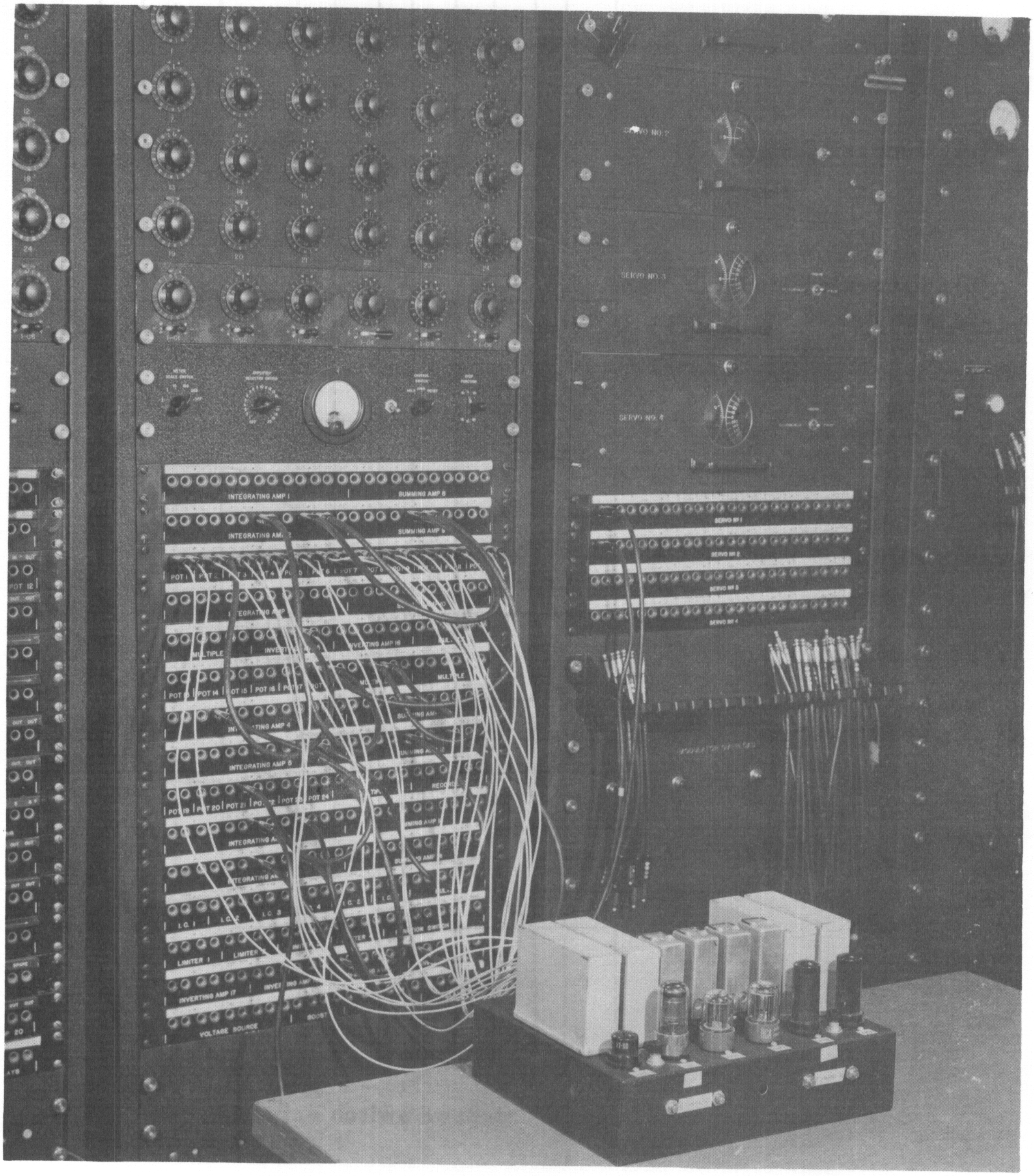


Figure 5. Computer Connected to a REAC

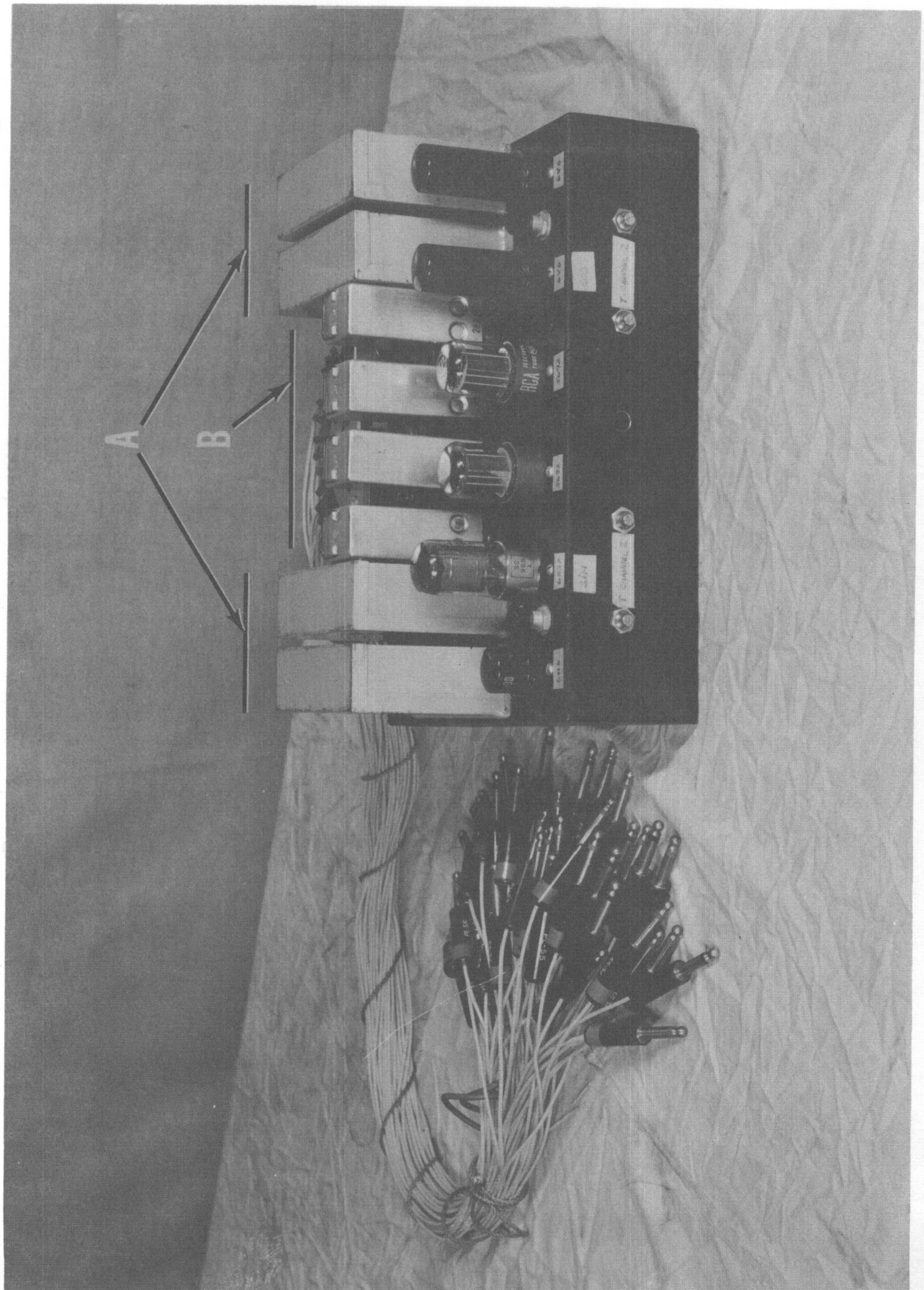


Figure 6. Close-Up of Computer

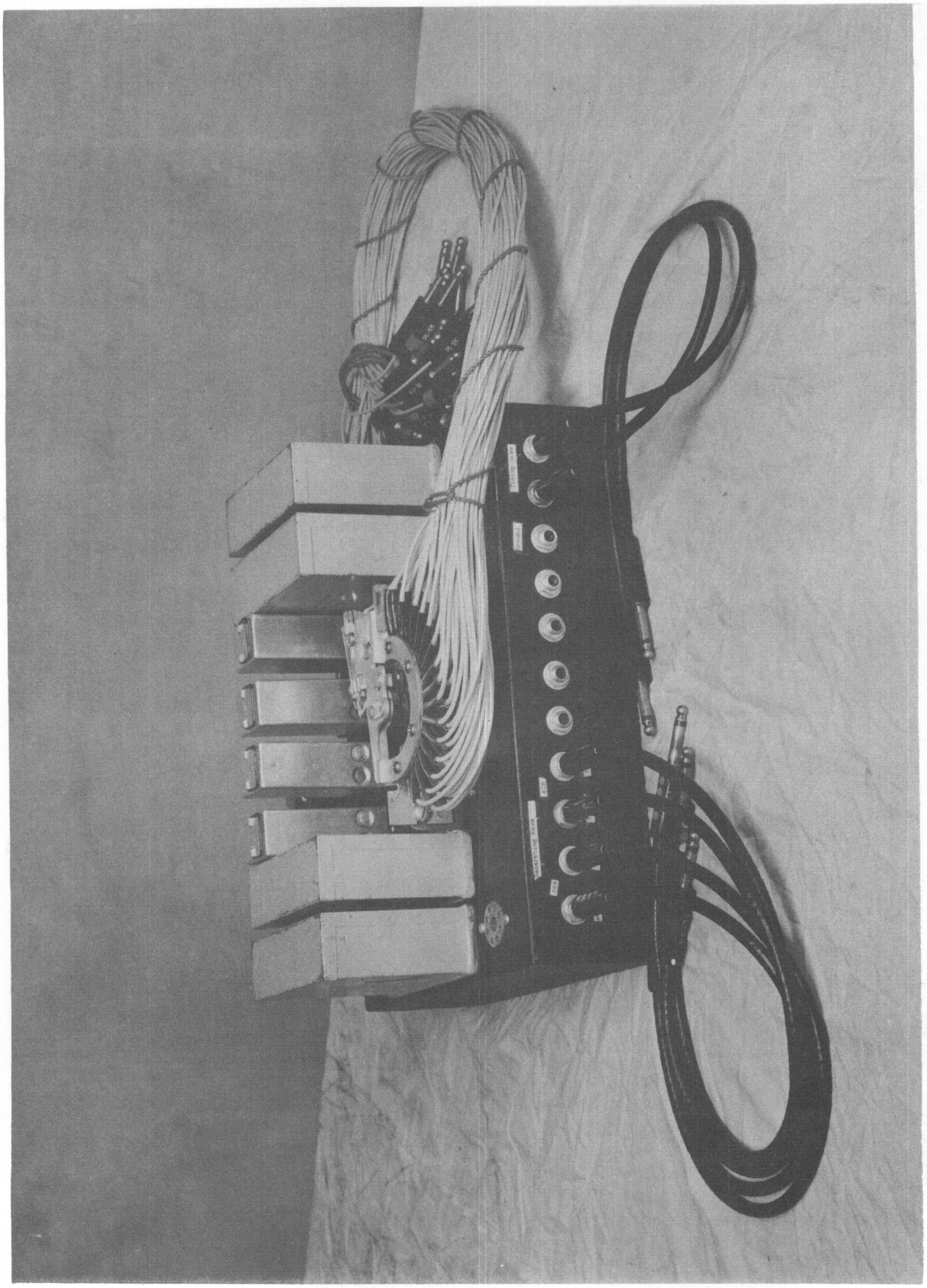


Figure 7. Rear View of Computer

When the coefficients in the polynomial are complex, the general terms of equation (2) become:

$$(11) \quad r^s \left[A_s \cos(s\theta) - B_s \sin(s\theta)\right] \quad - jr^s \left[A_s \sin(s\theta) + B_s \cos(s\theta)\right]$$

in which the general coefficient is:

$$(12) \quad a_s = A_s + jB_s$$

Since this form is still the basic form of the equation, the generalization to complex coefficients would represent only minor changes in the computer design.

To demonstrate the accuracy of the analogue series computer in solving this type problem, values of "a" were substituted in equation (1) for a sixth order polynomial which becomes:

$$(13) \quad 0 = -0.152 - 0.152Z - 0.3772Z^2 - 0.169Z^3 + Z^4 + 0.542Z^5 + 0.271Z^6$$

Figures 8 through 10 are graphical records of machine vs. hand calculated results obtained for equation (13). Figure 8 shows a graph in the negative-real axis for Z between 0.1 and 0.8. Some of the computer attained values are not on the real axis as they should be, but the error is actually quite small. If the point nearest the origin is taken to represent the real root, the machine would show a root at Z = 0.78 while the actual hand computed root was Z = 0.7778. A difference of only 0.0022. The second curve on the graph shows the curve corresponding to the Z vector (r, 170°) for $0.1 \leqq r \leqq 0.8$. This tends to verify the accuracy of the first computer calculated root. Figures 9 and 10 show a comparison of machine and hand computed curves for the valves corresponding to $\theta = 120°$ where $0.2 \leqq r \leqq 0.8$, and $\theta = 120°$ where $0.1 \leqq r \leqq 1.0$ respectively.

It is evident, that supplementary circuits must be devised for most problems in order for the calculator to solve the problem. Once the equations are broken down, into the timing and sampling functions, the problem becomes relatively easy. The following are several of the known types of expansions that may be summed by the series computer. The sub-equations are a sampling circuit and timing circuit signals.

(14) (a) $\Sigma a_n x^n$ where a_n, x are real

sampled circuit signal equation; $\dot{y} + ay = 0$

timing circuit signal; $\ddot{T} + (\pi/\tau)^2 T = 0$

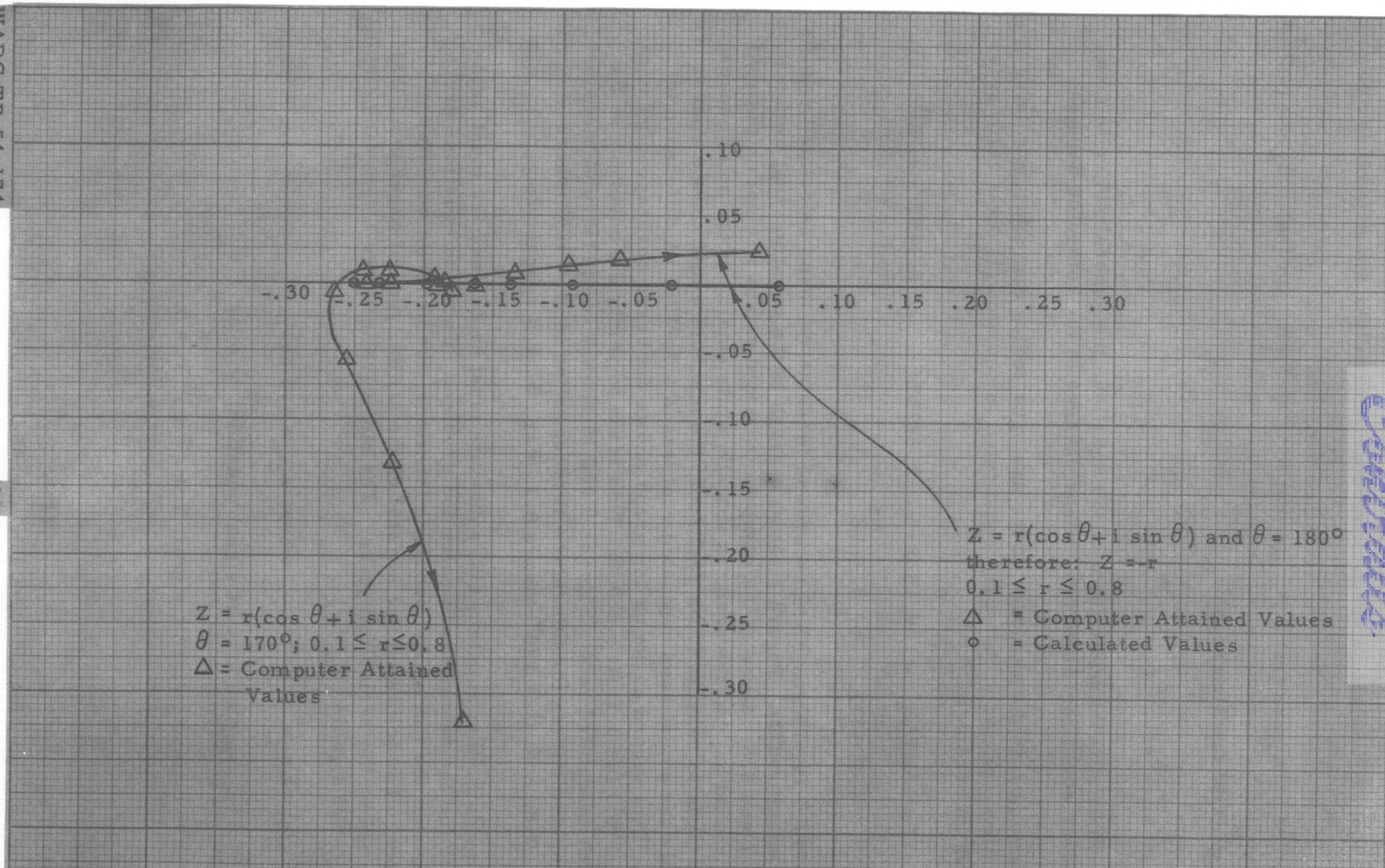


Figure 8. Machine vs. Hand Calculated Contours in "W" Plane (Z = -r)

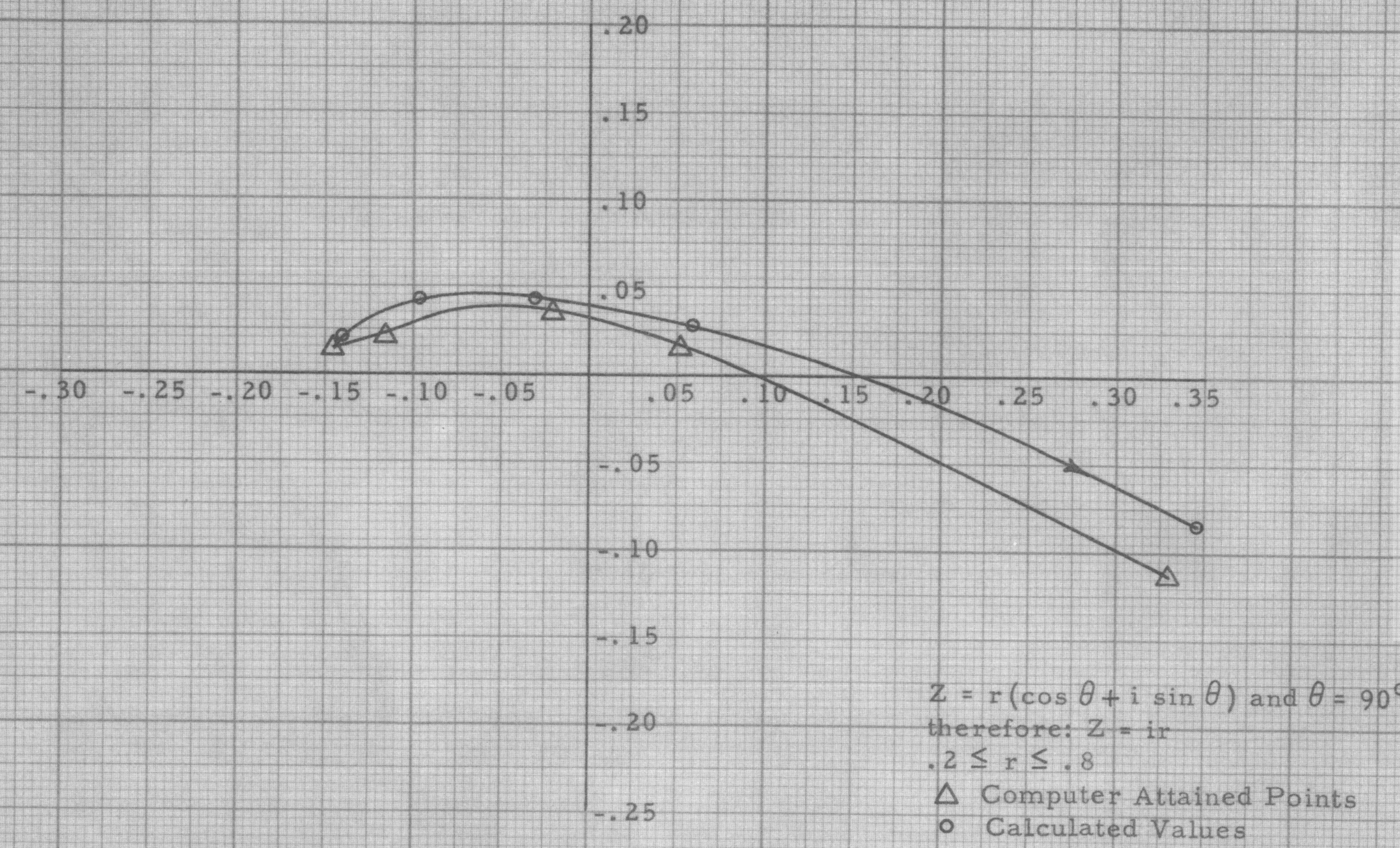


Figure 9. Machine vs. Hand Calculated Contour in "W" Plane (Z = ir)

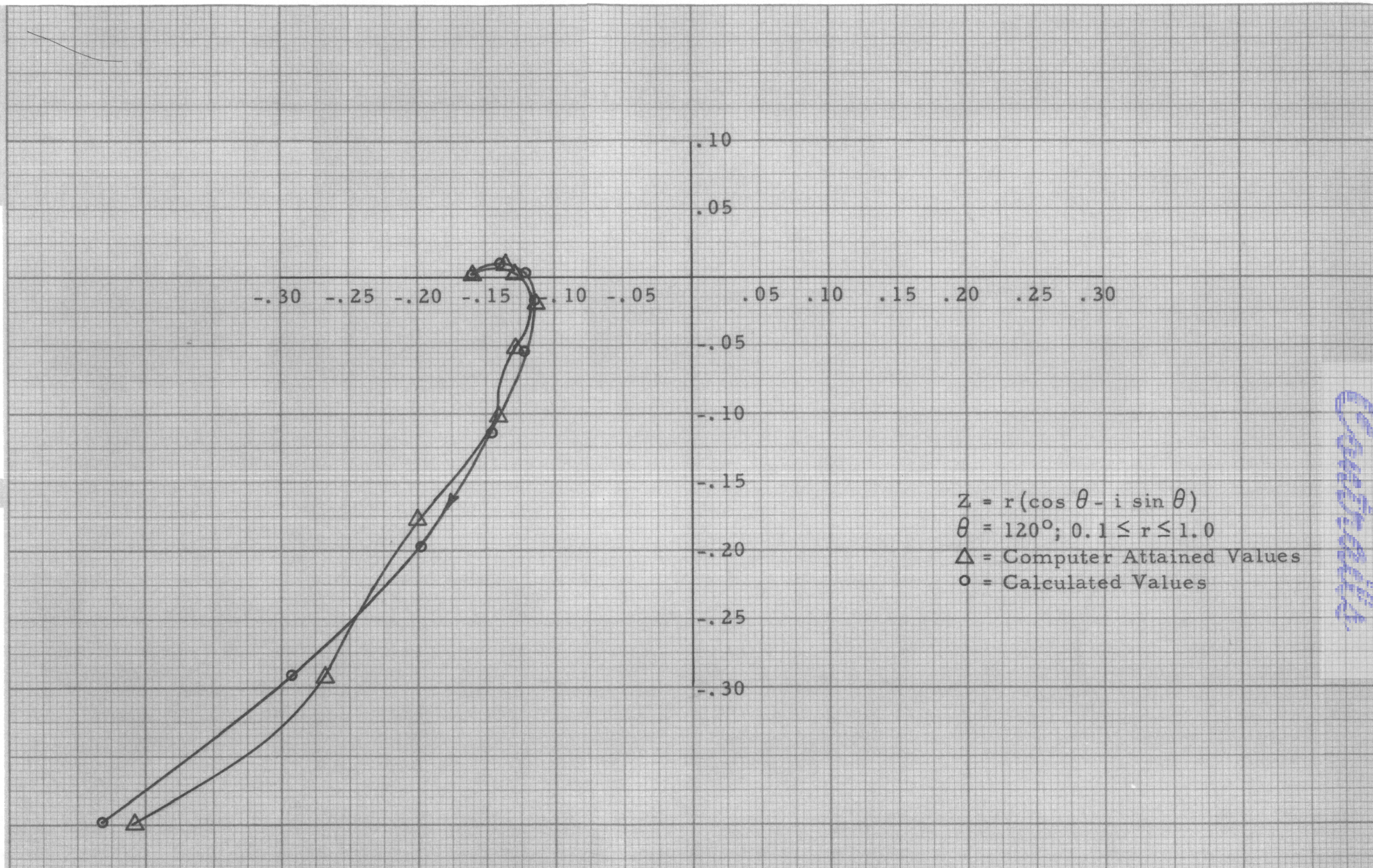


Figure 10. Machine vs. Hand Calculated Contour in "W" Plane (Z complex)

(b) $\Sigma (a_n \sin n\theta + b_n \cos n\theta)$ where a_n and θ are real:

sampled circuit signal equation; $\ddot{y} + b^2 y = 0$

timing circuit signal; $\ddot{T} + (\pi/\tau)^2 T = 0$

(c) $\Sigma a_n z^n$ where a_n and z_n are complex:

sampled circuit signal equation; $\ddot{y} + 2a\dot{y} + (a^2 + b^2) y = 0$

timing circuit signal; $\ddot{T} + (\pi/\tau)^2 T = 0$

(d) $\Sigma a_n n^x$ where a_n and x are real:

sampled circuit signal equation; $xy - ty = 0$

timing circuit signal; $\ddot{T} + (\pi/\tau)^2 T = 0$

(e) $\Sigma a_n J(\lambda_n x)$ where a_n and x are real and λ_n such that $J(\lambda) = 0$

sampled circuit signal equation; $a\ddot{y} + b\dot{y} + cy = 0$

and a, b, and c are functions of x and t and x is removable

by transformation of the independent variable;

timing circuit signal; $a_1\ddot{T} + b_1\dot{T} + c_1 T = 0$

and again a_1, b_1 and c_1, are functions of t.

As a special case of (e), the Schlomilch-Bessel expansions

are solved:

(f) $F(x) = \Sigma a_m J_n(mx)$:

sampled circuit signal equation; $\ddot{y} + \dot{y}/t + (1 - n^2/t^2)y = 0$

timing circuit equation; $\ddot{y} + (\pi/\tau)y = 0$

The Dini-Bessel expansion may also be treated as a special case of (e).

CONCLUSIONS

Generalization of application of this computer is quite evident. If the function generating circuit and the timing circuit were both set up to generate say the Bessel Function, $J_n(x)$ then this machine could be used to sum or to find the roots of an expansion of the form:

$$(15) \quad a_o + a_I J_n(\lambda_1 x) + a_2 J_n(\lambda 2x) \quad . . . . . \, a_m J_n(\lambda_m x)$$

in which λ are the roots of $J_n(x) = 0$. With additional investigation into the necessary timing and signal circuits it seems that the series analogue computer is capable of solving more complex and higher order equations.

BIBLIOGRAPHY

McCool, W. A. An AM-FM Electronic Analog Multiplier. Proceedings of the IRE Vol. 41, October 1953. No. 10.

Meneley, C. A. and Morrill, C. D. Application of Electronic Differential Analyzers to Engineering Problems. Proceedings of the IRE Vol. 41, October 1953. No. 10.

Scherberg, M. C. and Riordan, J.F. Other Aids to Computation; Analogue Calculation of Polynomial and Trigonometric Expansions. Mathematical tables and Other Aids to Computation VII, No. 41 January, 1953.