# DESCRIPTION OF STRUCTURAL DAMPING

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## ABSTRACT

Structural damping is frequently approximated in frequency domain by the constant hysteretic damping model. Transient vibrations of a member with constant hysteretic damping lead to a noncausal precursor response [1, 2]. Noncausal response can be avoided by introducing actual measured frequency dependent stiffness and damping behaviour of material, or by introducing constitutive equations of differential operator type with classical derivatives (integer order) or generalized type (fractional order).

This paper recalls and generalizes constitutive equations of viscoelastic behaviour of materials and members in time and frequency domain.

Weak frequency dependence of actual viscoelastic material can be fitted with only few parameters by adopting the fractional derivative concept.
The impulse response function of an oscillator with fractional derivative damping model is integrated in the present paper by a new efficient technique using inverse Fourier transform. This requires a unique definition of the constitutive equation in frequency domain. The response in shown to fulfill causality requirement. Amplitude decay of the considered damping models are compared after selection of equivalent damping properties.

The elastic-viscoelastic correspondence principle replaces Hooke's law describing a linear elastic material by the corresponding equations of a viscoelastic material in time domain of differential operator type or hereditary integral type [3]. For a one dimensional state of stress Hooke's law $\sigma(t) = E\varepsilon(t)$ is replaced by the constitutive equation of differential operator type

$$\sum_{k=0}^{N} p_k \frac{d^k}{dt^k}\sigma(t) = \sum_{k=0}^{M} q_k \frac{d^k}{dt^k}\varepsilon(t) \tag{1}$$

or hereditary integral type

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{dt}\,d\tau \quad . \tag{2}$$

Young's modulus E corresponds to the relaxation function $E(t)$ in equation (2). Thermorheological simple materials allow introducing the influence of space and time dependent temperature T on material behaviour by replacing the actual time t by a reduced time

$$\zeta(x,t) = \int_{0}^{t} \phi\,[T(x,\eta)]\,d\eta \quad . \tag{3}$$

This transformation is based on the shift function $\phi$ determined from experimental data [3]. As a generalization of the constitutive equation (1) with integer order derivatives in operators, fractional calculus can be used to respresent viscoelastic behaviour [4].
The derivative of fractional order $\alpha$

$$\frac{d^\alpha \varepsilon(t)}{dt^\alpha} = D^\alpha\{\varepsilon(t)\} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\varepsilon(t-\tau)}{\tau^\alpha}\,d\tau \quad 0<\alpha<1 \ , \tag{4}$$

defined with the gamma function $\Gamma(1-\alpha) = \int_0^\infty e^{-x}x^{-\alpha}dx$ is the inverse operation of fractional integration attributed to Riemann and Liouville [5].
Equation (1) with each integer order derivative being replaced by one of fractional order leads to

$$\sum_{k=0}^{N} p_k D^{\beta_k}\{\sigma(t)\} = \sum_{k=0}^{M} q_k D^{\alpha_k}\{\varepsilon(t)\} \tag{5}$$

Although the defining relationship (4) appears complicated with respect to computations both Laplace and Fourier transforms reveal the useful results

$$\begin{aligned} L\,[D^{\alpha}\{x(t)\}] &= s^{\alpha} L\,[x(t)] \\ F\,[D^{\alpha}\{x(t)\}] &= (i\omega)^{\alpha} F\,[x(t)] \end{aligned} \tag{6}$$

Harmonic functions of time such as $\varepsilon = \varepsilon^* exp\,(i\omega t)$ in steady state or the frequency domain of Fourier transform convert equations (2,5) with (6) to

$$\sigma^*(\omega) = E^*(\omega)\,\varepsilon^*(\omega) \tag{7}$$

Complex modulus $E^*(\omega)$ with storage modulus $E'(\omega)$ loss modulus $E''(\omega)$ and loss factor $\eta(\omega) = E''(\omega)/E'(\omega)$

$$E^*(\omega) = \frac{\sum_{k=0}^{M} q_k\,(i\omega)^{\alpha k}}{\sum_{k=0}^{N} p_k\,(i\omega)^{\beta k}} = E'(\omega) + i\,E''(\omega) = E'(\omega)\,[1 + i\eta\,(\omega)] \tag{8}$$

is related to the relaxation function $E\,(t)$ in equation (2) by the inverse Fourier transform $E\,(t) = F^{-1}\,[E^*(\omega)/(i\omega)]$.

Complex moduli of several viscoelastic models and associated rheological models are compared in Fig. 1 with the corresponding storage and loss moduli.

A frequency parameter $a_0 = \omega/\omega_n$ with a scaling frequency $\omega_n$ has been introduced which leads to a nondimensional time $\tau = \omega_n t$ according to $a_0\,\tau = \omega t$.

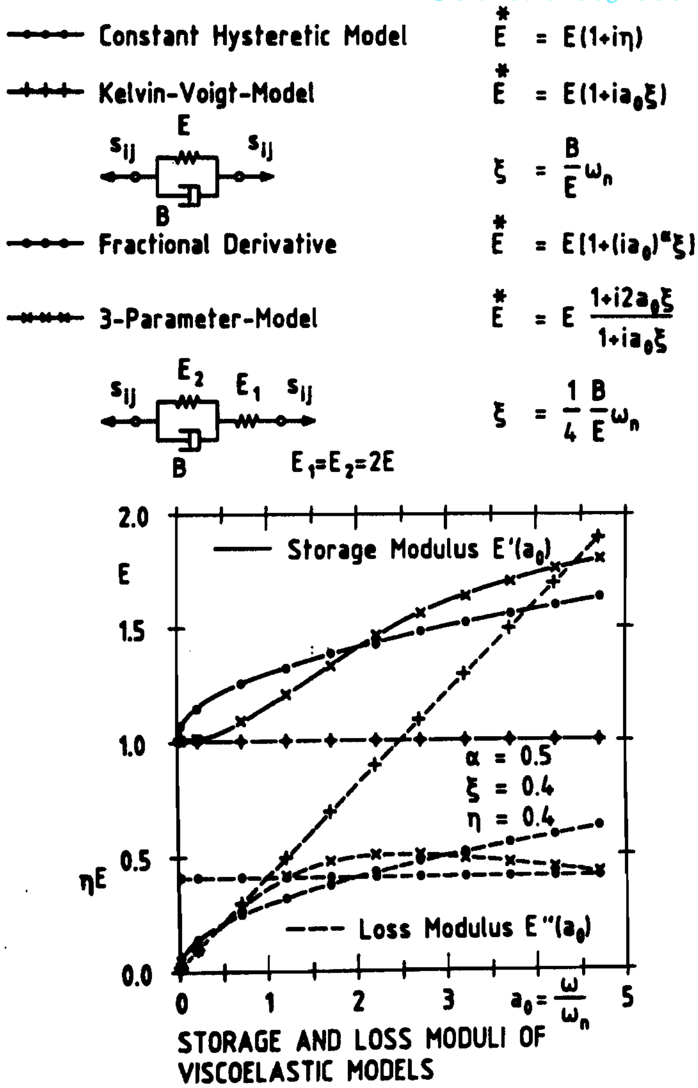


Fig. 1 Complex moduli, storage and loss moduli of viscoelastic models

In a limited frequency band, the frequency independent, so called constant hysteretic damping model approximates experimental results. This damping model is frequently extended over the entire range of positive frequencies to yield the complex modulus

$$E^* = E(1 + i\eta) \ . \tag{9}$$

Noncausal response occurs when results obtained with this model are transformed into the time domain [1, 2, 6, 7]. An analytical solution of the corresponding impulse response function of a SDOF oscillator is given in [1]. Noncausal response can be avoided by introducing actual measured frequency dependent stiffness and damping behaviour of material [2]. It is stated in the present paper, that noncausal response can be avoided by constitutive equations with fractional derivatives.

<u>Damping models of a member</u>

Transfer behaviour of a massless member in Fig. 2b is formulated by integrating equation (5) for a Kelvin-Voigt model (Fig. 1) with fractional derivative with respect to the volume of a member. This relates the force N to the tip displacement u in time domain

$$N(t) = k\,u(t) + cD^{\alpha}\{u(t)\} \quad 0 < \alpha < 1 \tag{10}$$

and in frequency domain of Fourier transform

$$N^*(a_o) = k\,|1 + \xi(ia_o)^\alpha|\; u^*(a_o) \tag{11}$$

with stiffness $k = EA/\ell$ and viscosity $\xi = c/k$, where $c = BA/\ell$. Constant hysteretic model (9) leads to

$$N^*(a_o) = k\,(1 + i\eta)\; u^*(a_o) \quad . \tag{12}$$

Both complex moduli of the member

$$k^*(a_o) = N^*(a_o)/u^*(a_o) = k'(a_o) + i\,k''(a_o) \tag{13}$$

(11, 12) do not as yet meet the requirements of a unique definition for the entire range of frequencies $-\infty < a_o < \infty$ of Fourier transform.

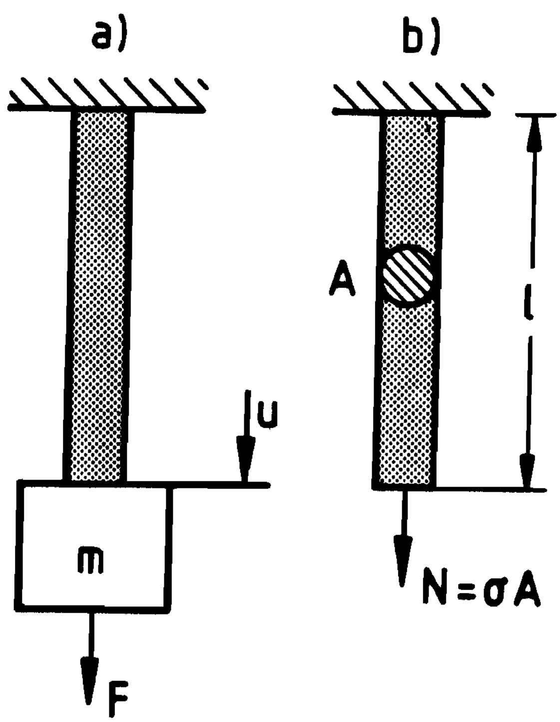


Fig. 2 (a) SDOF oscillator (b) Massless member

__Unique definition of constitutive equations in frequency domain__

Consider the term $(ia_o)^\alpha$ $a_o$, $\alpha \in \mathbb{R}$ in equation (11) as a special case of the complex expression $\Omega^z$ $\Omega$, $z \in \mathbb{C}$. A unique definition of $\Omega^z = \exp\,(zLn\Omega)$ is gained by restricting the arguments of $Ln\Omega$ to the principal values $-\pi < Im(Ln\Omega) < \pi$, leading to a branch cut along the negative real axis [8]. This definiton restricts the arguments $(ia_o)^\alpha = R\exp\,(i\phi)$ to

$$-\alpha\pi \leq \phi < \alpha\pi \qquad 0 < \alpha < 1 \quad . \tag{14}$$

Storage and loss moduli of the member in equation (11)

$$k'(a_o) = k\,|1 + \xi Re(ia_o)^\alpha|\,, \quad k''(a_o) = k\,\xi\, Im\,(ia_o)^\alpha \tag{15}$$

have a unique definition with equation (14). A physical interpretation supports the choice of proper roots. The force $N^* \exp(ia_o\tau)$, interpreted as a rotating vector in complex plane, causes the displacement $u^* \exp(ia_o\tau)$ and has to be ahead with minimal phaseshift counterclockwise for positive frequencies $a_o > 0$, $\exp(ia_o\tau)$, and clockwise for negative frequencies $a_o < 0$, $\exp(-i|a_o|\tau)$. Negative frequencies occur in two sided Fourier transform. This condition is fullfilled by equation (14) and requires the well known extension of complex modulus (12) according to

$$k^* = k(1 + i\eta \operatorname{sgn} a_o) \quad . \tag{16}$$

The fractional derivative of order $\alpha = 1/2$ can be derived to be relevant for polymeric materials from molecular physics [4].

According to (14) we have to choose

$$(ia_o)^{1/2} = \sqrt{|a_o|/2} \begin{Bmatrix} 1+i, & a_o \geq 0 \\ 1-i, & a_o < 0 \end{Bmatrix} \tag{17}$$

and obtain the complex modulus

$$k^*(a_o) = \begin{cases} k(1 + \xi\sqrt{a_o/2} + i\xi\sqrt{a_o/2}), & a_o \geq 0 \\ k(1 + \xi\sqrt{|a_o|/2|} + i\xi\sqrt{|a_o|/2}), & a_o < 0 \end{cases} \quad . \tag{18}$$

Complex moduli (16, 18) contain storage moduli, which are even functions of frequency and loss moduli, which are odd functions of frequency

$$k'(a_o) = k'(-a_o), \quad k''(a_o) = -k''(-a_o) \quad . \tag{19}$$

<u>Impulse response function of a damped oscillator</u>

Vibration response of the damped SDOF oscillator in Fig. 2a is calculated by Fourier transform, i.e. $u^*(a_o) = (1/\omega_n) \int_{-\infty}^{\infty} u(\tau) \exp(-ia_o\tau)\, d\tau$ , of Newtons equation of motion for mass m

$$m\omega_n^2 u''(\tau) = F(\tau) - N(\tau) \quad , \tag{20}$$

where $(\,)' = d/d\tau$ , $\omega_n = \sqrt{k/m}$. With Fourier transformed constitutive equation (11) of member, the frequency response function is given by

$$F(a_o) := \frac{k\,u^*(a_o)}{F^*(a_o)} = \frac{1}{1-a_o^2+\xi(ia_o)^\alpha} \quad . \tag{21}$$

Transient response can be obtained from inverse Fourier transform.
Dirac unit impulse excitation $F(\tau) = \delta(\tau)$ o—● $F^*(a_o) = 1$ leads to the impulse response function $u(\tau) = h(\tau)$

$$f(\iota) := m\omega_n h(\iota) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(a_o)\,e^{ia_o\iota}\,da_o \quad , \tag{22}$$

which governs the response for an arbitrary force $F(\tau)$ by the Duhamel convolution integral

$$u(\iota) = \frac{1}{\omega_n}\int_{-\infty}^{\iota} h(t-\xi)\,F(\xi)\,d\xi \quad . \tag{23}$$

By contour integration of equations (21, 22) and theory of residue the authors calculated the familiar impulse response function of the undamped oscillator as

$$f(\iota) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ia_o\iota}}{1-a_o^2}\,da_o = \begin{cases}\sin\iota, & \iota \geq 0\\ 0, & \iota < 0\end{cases} \quad . \tag{24}$$

A new efficient solution of the damped impulse response (22) is obtained by cutting the infinite integral in two intervals $-\infty > a_o \geq 0$ and $0 < a_o < \infty$ with associated unique definition of the integrand according to (17). Complex conjugate contributions lead to a real semiinfinite integral after reassembling

$$f(\iota) = \frac{1}{\pi}\int_0^{\infty}\frac{A(a_o)\cos(a_o\tau) + B(a_o)\sin(a_o\tau)}{A^2(a_o) + B^2(a_o)}\,da_o \quad , \tag{25}$$

where

$$A(a_o) = 1-a_o^2 + \xi\sqrt{a_o/2}, \quad B(a_o) = \xi\sqrt{a_o/2} \quad . \tag{26}$$

The corresponding solution for the member with constant hysteretic damping in equation (16) requires substitution of (26) by

$$A(a_o) = 1-a_o^2\,, \quad B(a_o) = \eta\ \ [2] \quad . \tag{27}$$

## Causality of impulse response function

Causal response of the damped oscillator due to the impulse excitation $F(\tau) = \delta(\tau)$ requires the system to be at rest $f(\tau) = 0$ for $\tau < 0$.
Causality of the transformation pair in equations (21, 23) $f(\tau)$ o–● $F(a_o)$ requires [9]

$$f(\tau) = \frac{2}{\pi}\int_0^\infty Re\,F(a_o)\cos(a_o\tau)\,da_o = -\frac{2}{\pi}\int_0^\infty Im\,F(a_o)\sin(a_o)\,da_o \quad (28)$$

for $\tau > 0$. The abbreviations in equation (25) lead to

$$f(\tau) = \frac{2}{\pi}\int_0^\infty \frac{A(a_o)\cos(a_o\tau)}{A^2(a_o) + B^2(a_o)}\,da_o = \frac{2}{\pi}\int_0^\infty \frac{B(a_o)\sin(a_o\tau)}{A^2(a_o) + B^2(a_o)}\,da_o \quad (29)$$

for $\tau > 0$.
The sum of the two expressions in (29) for $\tau > 0$ lead to equation (25), whereas for $\tau < 0$ equations (29) assure causality $f(\tau) = 0$ when they are inserted in equation (25). A numerical proof of the causality requirement (29) is given in the present paper for the damping model with fractional derivative and abbreviations (26). This is why the simple equations (29, 26) govern the impuls response $h(\tau) = f(\tau)/(m\omega_n)$ after numerical integration.
A published solution of the problem at hand is based on Laplace transform and contour integration [10]. It requires numerical root finding of the denominator of equation (21). Besides the residues at these poles, there is a contribution to the solution from integration along the earlier mentioned branch cut segments that requires numerical integration of a seminfinite integral as well.

Causality condition (28) is violated with equations (27) corresponding to the constant hysteretic damping model. Precursor response $f(\tau)$ for $\tau < 0$ is discussed in [1, 2].

## Numercial results of impulse response

Fig. 3 depicts the impulse response calculated from equations (25, 26) for positive and negative time $\tau = \omega_n t$. The response proves to be causal. Numerical integration of infinite integral converges rapidly.
No difference of impulse response was found when both simpler integrals (29) were integrated for $\tau > 0$. Negative time $\tau < 0$ characterizes the first integral (29)

as an even function of time, the second as an odd function of time. Parameters $\xi$ weighting the fractional derivative of order $\alpha = 1/2$ are chosen to be $\xi = \sqrt{2}/5$ and $\sqrt{2}/20$. Regarding equation (18), an increase of $\xi$ does not only increase the damping but also stiffens the member according to $k(1+\xi\sqrt{a_o/2})$. That is why stronger amplitude decay is associated with decreasing periods of zero crossing.

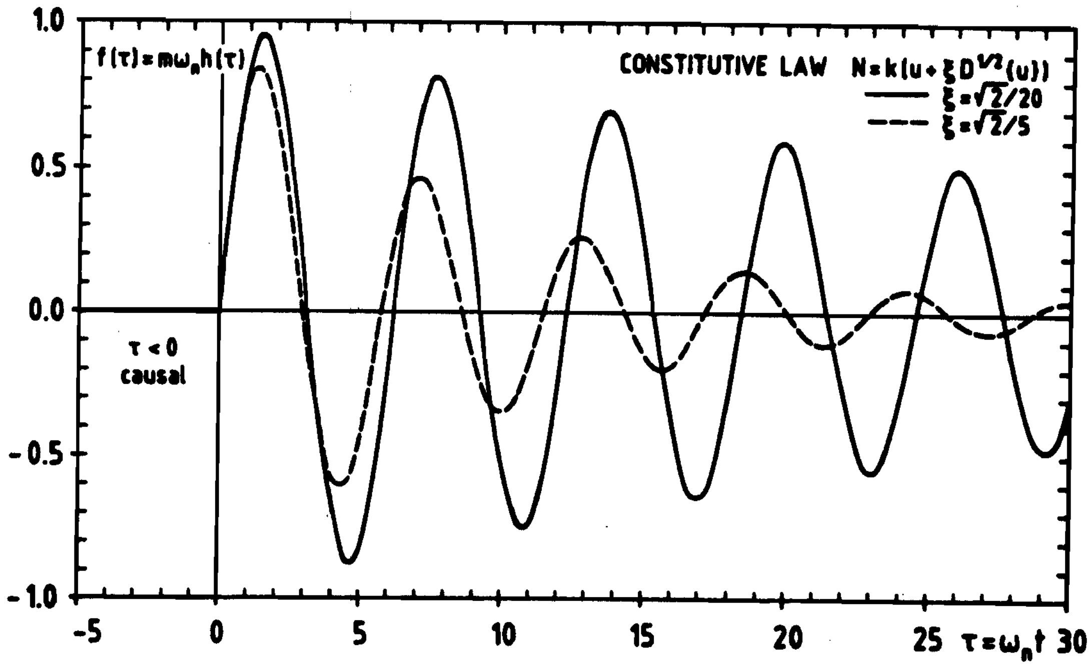


Fig. 3 Impulse response of oscillator with fractional derivative damping model

Impulse response for different damping models

Fig. 4 compares the impulse response functions corresponding to three damping models of Fig. 1:

- Constant hysteretic model [1]
- Kelvin-Voigt model with first oder derivative [1] and
- with fractional derivative of order 1/2

Equivalent loss factors have been chosen at $a_o = \omega/\omega_n = 1$ such that $\eta = a_o\xi\ (Kelvin\text{-}Voigt) = \sqrt{a_o/2}\ \xi\ (Frac.Der.) = 1/20$. This leads to nearly equal amplitude decay of all models, whereas the stiffening of the fractional derivative model decreases the periods of zero crossing.

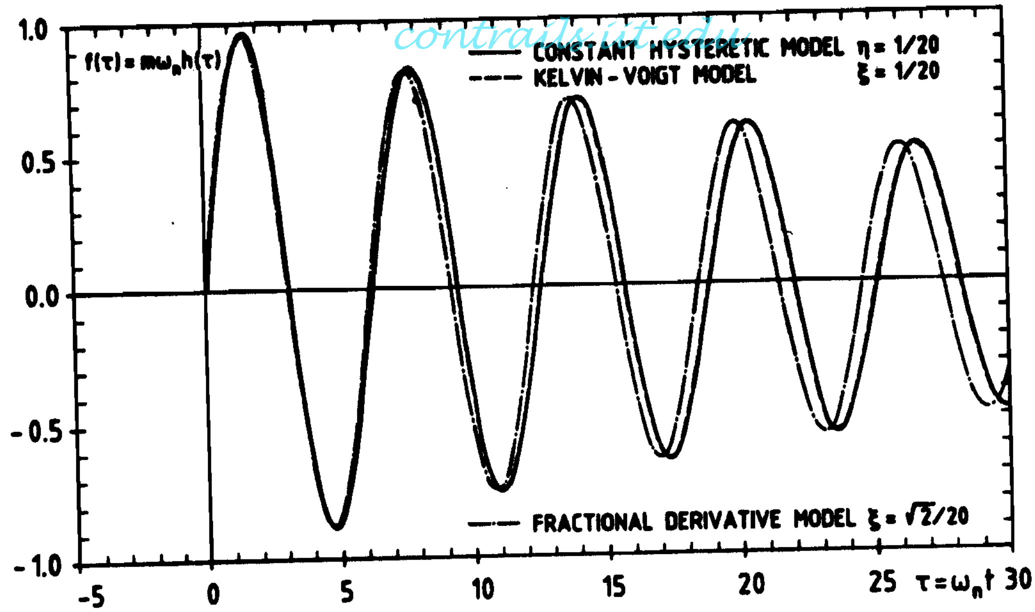


Fig. 4 Impulse response for different damping models

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