The Fractional Order State Equations for the Control of Viscoelastically Damped Structures

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Abstract

The fractional order state equations are developed to predict the effects of feedback intended to reduce motion in damped structures. The mechanical properties of damping materials are modeled using fractional order time derivatives of stress and strain. These models accurately describe the broad-band effects of material damping in the structure's equations of motion. The resulting structural equations of motion are used to derive the fractional order state equations. Substantial differences between the structural and state equations are seen to exist. The mathematical form of the state equations suggests the feedback of fractional order time derivatives of structural displacements to improve control system performance. Several other advantages of the fractional order state formulation are discussed.

Nomenclature

$\underline{\underline{A}}$: state Matrix

$-a^{\beta}$, : system eigenvalue

$-\underline{\underline{a}}^{\beta}$: diagonal matrix of eigenvalues

$\underline{\underline{B}}$: state control matrix

b : viscoelastic model parameter

D^{β} : the beta order fractional derivative

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$\hat{D}^{\beta}$: modified beta order fractional derivative

$E_o ; E_1$: viscoelastic model parameters

$E_{\beta}(x)$: the beta order Mittag-Leffler function

$\tilde{\underset{-}{F}}(t)$: applied loads prior to initial time, t = 0

$\tilde{\tilde{\underset{-}{F}}}(t)$: applied loads after initial time, t = 0

$\tilde{\underset{-}{f}}(t)$: modal loads prior to initial time, t = 0

$\tilde{\tilde{\underset{-}{f}}}(t)$: modal loads after initial time, t = 0

$\underset{-}{f}^{*}(t)$: stress operator acting on loads

$\tilde{\underset{-}{G}}(t)$: structural pseudo loads

g(t) : modal load

$\tilde{g}(t)$: modal psendo load

$-\underset{=}{G}$: feedback gain matrix

$I^{1-\alpha}$: the one minus α order fractional integral

$\underset{=}{k}_o$: structural stiffness matrix

$\underset{=}{k}_1$: structural visco-stiffness matrices

N : number of physical degrees of freedom

$\underset{=}{M}$: structural mass matrix

$\tilde{t}$: time starting at the onset of motion

t : time starting at the initial time

t_o : time interval between $\tilde{t}$ = 0 and t = 0

$\underset{-}{w}(t)$: structural displacements

x_i : spatial coordinates

$\underset{-}{x}_r$: the reduced state vector

$\underset{-}{x}(t)$: state vector

$\underset{-}{x}_o$: intial state vector

y(t) : modal response

$\tilde{y}(t)$: modal response for loading prior to t = 0

$\tilde{\tilde{y}}(t)$: modal response for loading after t = 0

$\underline{z}$: impulsive load coefficients vector

β : basis fraction (1/n) for the system

Γ : the gamma function

$\epsilon(t)$: strain history

ϕ : system orthonormal transformation

$\sigma(t)$: stress history

$(E_o+E_1D^\alpha)$: Strain operator

$(1+bD^\alpha)$: stress operator

Introduction

In the modeling of the linear elastic behavior of large space structures, damping has typically either been ignored or modeled as being linearly dependent on velocity. This damping model is adequate for very lightly damped structures and also allows a linear state space model to be defined for the structure's motion. This formulation is well suited to the design of active control systems using state space techniques.

However, for heavily damped structures ignoring the damping is imprudent and modeling it as being linearly dependent on velocity is inadequate. Velocity dependent damping models, while mathematically straightforward, fail to describe the broad band mechanical behavior of damping materials. Historically, the need for more refined models has pushed the development of viscoelasticity as a discipline within engineering mechanics. Applicable viscoelastic models relate time dependent stress and strain fields with series of ordinary time derivatives. These models yield acceptable broad band Bode plots of material properties, but they have drawbacks. Typically these models contain many terms, making them mathematically cumbersome and increasing the order of the differential equations describing the system.

As an alternative we will present accurate broad band viscoelastic damping models having only four parameters[2] and posed in terms of non-integer order time derivatives. The real strength of this approach is that these non-integer or fractional order derivatives describe inertial effects, damping effects, elastic effects and control effects with equal precision. Substantial accompanying benefits are that the order of differentiation in the system equations does not exceed three and that a potentially infinite number of additional feedback states arise to improve system performance.

To reap the benefits of this approach; however, one must become comfortable with the concept of fractional order differentiation. While the convolution operator that produces these time derivatives at first appears alien, frational differentiation in the Laplace transform domian is exceedingly simple. Multiplying a transform by s^{α}, in effect, produces the transform of the α order derivative.

The development and applications of fractional order derivatives in viscoelasticity and structural dynamics are well documented.[1-9,12-14,16,18] The models are consistent with themodynamic constraints[6] and have their foundation in classical molecular theories predicting the macro mechanical properties of viscoelastic materials.[4]

The resulting structural equations of motion serve as the foundation for the state equations, but they are substantially different. The hereditary nature of the structural equations suppresses the existence of homogeneous solutions found in the state equations. In addition, the two sets of equations employ different operators that lead to different requirements for initial conditions. It should come as no suprise that the generalized or fractional order state equations comprise a generalization of the initial value problem. The generalization begins with the structural equations of motion.

The Structural Equations of Motion

The structural equations of motion differ from classical formulations in that fractional order derivatives are used to model the viscoelastic damping phenomenon. The extended Riemann Liouville fractional derivative is a linear operator

$$D^{\alpha}_{(t)}[w(t)] = \frac{d}{dt}\int_0^t \frac{w(t)}{\Gamma(1-\alpha)(t-\tau)^{\alpha}}\,d\tau \qquad 0 \le \alpha \le 1 \qquad (1)$$

and serves as the basis of the generalized model of the viscoelastic phenomenon. The most general form of the models is

$$\sigma(t,x_i) + \sum_{p=1}^{N} b_p D^{\alpha_p}_{(t)}[\sigma(t,x_i)] = E_o\epsilon(t,x_i) + \sum_{p=1}^{N} E_p D^{\alpha_p}_{(t)}[\epsilon(t,x_i)] \qquad (2)$$

where the derivatives acting on the stress and strain fields are of real, rational fractional order. Note that this model becomes the classical viscoelastic model[10] when the orders of differentiation are taken to be integers.

The Fourier transform of the fractional derivative of a function has a special property when the function is zero for negative time.

$$F\,[D^{\alpha}_{(t)}[x(t)]] = (i\omega)^{\alpha}\,F[x(t)] \tag{3}$$

where

$$F\,[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt \tag{4}$$

This property, eqn 3, is that the transform of the α order derivatives is the transform parameter, $i\omega$, raised to the α power times the transform of the function. Note the similarity of this transform with the Fourier transform of an ordinary derivative.

The attractive feature of the fractional derivative operator is the ability to vary the degree of its frequency dependence through the choice of α. As a direct result, fractional derivative models are capable of modeling linear, frequency-dependent phenomena not easily captured by the transforms of ordinary derivatives. This leads to models accurate over several decades of frequency needing very few, typically four, parameters[6].

In the time domain the four parameter model for uniaxial deformation takes the form

$$(1 + bD^{\alpha})\sigma(t) = (E_o + E_1 D^{\alpha})\epsilon(t) \tag{5}$$

where b, E_o, E_1 and α are the parameters. This model has been used to construct the general three-dimensional constitutive equations for linear, homogeneous, isotropic viscoelastic materials[1]. When these general constitutive equations are employed, it can be shown that the general form of the finite element equations of motion take the form[3]

$$b\underline{\underline{M}}D^{2+\alpha}\underline{w}(t) + \underline{\underline{M}}D^{2}\underline{w}(t) + \underline{\underline{k}}_1 D^{\alpha}\underline{w}(t) + \underline{\underline{k}}_o\underline{w}(t) = bD^{\alpha}\underline{F}(t) + \underline{F}(t). \tag{6}$$

where $\underline{\underline{M}}$ is the mass matrix, $\underline{\underline{k}}_o$ is the stiffness matrix, $\underline{\underline{k}}_1$ is the visco-stiffness matrix, $\underline{F}(t)$ are the applied forces and $\underline{w}(t)$ are the structure's deflections. Note that the equations of motion are posed in terms of three real, square symmetric matrices. In general the visco-stiffness matrix $\underline{\underline{k}}_1$ will not be a linear combination of $\underline{\underline{M}}$ and $\underline{\underline{k}}_o$ and usually the equations of motion cannot be decoupled in their present form.

To overcome this obstacle to spectral analysis and begin the derivation of the state equations, we will pose the structural equations of motion in terms of two real, square symmetric matrices, for which an orthonormal transformation exists. To begin this process one takes advantage of the composition property of the fractional order derivative,

$$D^{\alpha}\left[D^{\gamma}\left[w(t)\right]\right] = D^{\alpha+\gamma}\left[w(t)\right], \tag{7}$$

and poses the structural equations of motion as

$$\left(b\underline{\underline{M}}(D^{\beta})^{m} + \underline{\underline{M}}(D^{\beta})^{r} + \underline{\underline{k}}_1(D^{\beta})^{q} + \underline{\underline{k}}_o\right)\underline{w}(t) = \left(1 + b(D^{\beta})^{q}\right)\underline{F}(t) \tag{8}$$

Here m, r and q are integers and $(D^{\beta})^{m}$ is the β order derivative taken m times.

$\beta m = 2 + \alpha$

$\beta r = 2$

$\beta q = \alpha$

$\beta = 1/n$

β is chosen to be the largest fraction of the form 1/n, where n is an integer, common to all the rational orders of differentiation in the structural equations of motion,. As we will see later, this form for β is necessary to insure that initial velocities appear in the fractional order state equations. The most general form of these equations of motion is

$$\sum_{p=0}^{m} \underline{\underline{c}}_p (D^{\beta})^{p}\,\underline{w}(t) = (1 + b(D^{\beta})^{q})\underline{F}(t) = \underline{f}^{*}(t). \tag{9}$$

Here the $\underline{\underline{c}}_p$ are real and constant, although many may be zero, and $\underline{f}^{*}(t)$ is the result of the viscoelastic stress operator acting on the applied forces, $\underline{F}(t)$, as shown in eqn 8.

Eqn 9 describes the structures with N degrees of freedom producing N equations of order βm that can be alternatively posed as m·N equations of order β. In matrix form the m·N equations of β order are

$$D^{\beta}\left[\begin{array}{c|c} \mathbf{0} & \\ \vdots & \\ \mathbf{0} & H \\ \mathbf{0} & \\ \hline \mathbf{c}_m & \cdots \mathbf{c}_3\,\mathbf{c}_2\,\mathbf{c}_1 \end{array}\right]\left\{\begin{array}{c} (D^{\beta})^{m-1}\underline{w}(t) \\ \vdots \\ (D^{\beta})^{2}\underline{w}(t) \\ (D^{\beta})^{1}\underline{w}(t) \\ \underline{w}(t) \end{array}\right\} \tag{10}$$

$$+\left[\begin{array}{c|c} & \mathbf{0} \\ & \vdots \\ -H & \mathbf{0} \\ & \mathbf{0} \\ \hline \mathbf{0}\cdots\mathbf{0}\;\mathbf{0} & \mathbf{c}_0 \end{array}\right]\left\{\begin{array}{c} (D^{\beta})^{m-1}\underline{w}(t) \\ \vdots \\ (D^{\beta})^{2}\underline{w}(t) \\ (D^{\beta})^{1}\underline{w}(t) \\ \underline{w}(t) \end{array}\right\} = \left\{\begin{array}{c} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \\ \underline{f}^{*}(t) \end{array}\right\}$$

where the lowest set of partitioned equations is seen to be eqn 9. The matrix [H] is chosen such that both square matrices of order m·N become symmetric and the top (m-1)·N equations are satisfied identically. This is accomplished by constructing H such that all matrices, $\mathbf{c}_p$ lying on any given diagonal running from lower left to upper right in the first matrix of eqn 10 are equal. We will refer to this form of the equations of motion as the expanded equations of motion.

For example, if α is one half in eqn 6, then β is one half making m=5 in eqn 8 and the expanded equations of motion become

$$D^{1/2}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & b\mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & b\mathbf{M} & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & b\mathbf{M} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & b\mathbf{M} & \mathbf{M} & \mathbf{0} & \mathbf{0} \\ b\mathbf{M} & \mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{k}_1 \end{bmatrix}\left\{\begin{array}{c} (D^{1/2})^{4}\underline{w}(t) \\ (D^{1/2})^{3}\underline{w}(t) \\ (D^{1/2})^{2}\underline{w}(t) \\ (D^{1/2})^{1}\underline{w}(t) \\ \underline{w}(t) \end{array}\right\} \tag{11}$$

$$+\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -b\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -b\mathbf{M} & -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & -b\mathbf{M} & -\mathbf{M} & \mathbf{0} & \mathbf{0} \\ -b\mathbf{M} & -\mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_0 \end{bmatrix}\left\{\begin{array}{c} (D^{1/2})^{4}\underline{w}(t) \\ (D^{1/2})^{3}\underline{w}(t) \\ (D^{1/2})^{2}\underline{w}(t) \\ (D^{1/2})^{1}\underline{w}(t) \\ \underline{w}(t) \end{array}\right\} = \left\{\begin{array}{c} \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ (1+bD^{1/2})\underline{F}(t) \end{array}\right\}$$

Both the general form (eqn 10) and the example in eqn 11 are now posed in terms of two real, square, symmetric matrices for which an orthonormal transformation exists,

$$\begin{Bmatrix} (D^{\beta})^{m-1}\underline{w}(t) \\ \vdots \\ (D^{\beta})^{2}\underline{w}(t) \\ (D^{\beta})^{1}\underline{w}(t) \\ \underline{w}(\bar{t}) \end{Bmatrix} = \begin{bmatrix} \\ \Phi \\ \\ \end{bmatrix} \begin{Bmatrix} y_{m\cdot N}(t) \\ \vdots \\ y_3(t) \\ y_2(t) \\ y_1(t) \end{Bmatrix} \qquad (12)$$

which leads to a system of m·N uncoupled differential equations of order β.

$$D^{\beta}\left[\diagdown I \diagdown\right]\left\{y(t)\right\} + \left[\diagdown a^{\beta} \diagdown\right]\left\{y(t)\right\} = \left[\, \phi \,\right]^{T}\left\{ f^{*}(t)\right\} = \left\{f(t)\right\} . \qquad (13)$$

<u>Constructing the Modal State Equations</u>

The decoupled structural equations of motion or basis equations (eqn 13) individually take the form

$$(D^{\beta} + a^{\beta})y(t) = f(t) \qquad \beta = 1/n \qquad (14)$$

where the subscripts have been dropped to simplify notation. Green's function solutions for these equations are relatively straightforward and the resulting expressions for the forced response of the structure can be shown to be real, continuous and causal (1:73). These solutions to eqn 14 may be viewed as particular solutions of the structural equations of motion.

It is important to note that the only homogeneous solution to eqn 14 is the trivial solution. This is consistent with a strict interpretation of eqn 2, the generalized visocoelastic constitutive model. Inherent in the model is the implication that at time zero the viscoelastic material should be in its virgin, undeformed state and the structure is commencing motion from a quiescent state. Attempting to impose non-trivial initial conditions implies the existence of previous motion that is inconsistent with the hereditary viscoelastic model and hence, homogeneous solutions are not needed.

To construct the modal state equations, one needs to shift the time scale such that the initial time occurs at some time, t_o, after the onset of structural motion. This shifted time scale is shown in figure 1. Posing the basis equations, eqn 14, in terms of this shifted time scale yields[19:48].

$$\frac{1}{\Gamma(1-\beta)} \frac{d}{d\tilde{t}} \int_{o}^{\tilde{t}} \frac{y(\tau - t_o)}{(\tilde{t}-\tau)^{\beta}} d\tau + a^{\beta} y(\tilde{t}-t_o) = f(\tilde{t}-t_o) \qquad (15)$$

The applied loads prior to $t_o (0 \le \tilde{t} \le t_o)$ are $\tilde{f}(\tilde{t}-t_o)$ and the corresponding response is $\tilde{y}(\tilde{t}-t_o)$. The equation predicting this response is

$$D^{\beta}\tilde{y}(\tilde{t}-t_o)+a^{\beta}\tilde{y}(\tilde{t}-t_o)=\tilde{f}(\tilde{t}-t_o) \tag{16}$$

The loads for the episode of interest $(\tilde{t} \ge t_o)$ are $\tilde{\tilde{f}}(\tilde{t}-t_o)$ and the equation for the corresponding response $\tilde{\tilde{y}}(\tilde{t}-t_o)$ is

$$D^{\beta}\tilde{\tilde{y}}(\tilde{t}-t_o) + a^{\beta}\tilde{\tilde{y}}(\tilde{t}-t_o) = \tilde{\tilde{f}}(\tilde{t}-t_o) \tag{17}$$

The total response for $t \ge t_o$ is $\tilde{\tilde{y}} + \tilde{y}$ and the general expression for the response is

$$\frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{\tilde{\tilde{y}}}(r-u) + \dot{\tilde{y}}(r-u)}{u^{\beta}} du + a^{\beta}\left[\tilde{\tilde{y}}(r) + \tilde{y}(r)\right] = \tilde{\tilde{f}}(r) + \tilde{g}(r) \tag{18}$$

where $r = \tilde{t}-t_o$, $u = \tilde{t}-\tau$. Here $g(r)$ is a pseudo forcing function that produces the residual response of the structure due to the prior application of $\tilde{f}(t-t_o)$.

$$\tilde{g}(r) = -\frac{1}{\Gamma(1-\beta)}\left\{ \int_r^{r+t_o} \frac{\dot{\tilde{y}}(r-u)}{u^{\beta}} du + \frac{\tilde{y}(-t_o)}{(r+t_o)^{\beta}} \right\} \tag{19}$$

Expressing eqn 18 in terms of the time t scale in Figure 1, where zero time is now t_o after the onset of structural motion, yields

$$\frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{y}(t-\tau)}{\tau^{\beta}} d\tau + a^{\beta}y(t) = \tilde{\tilde{f}}(t) + \tilde{g}(t) = g(t). \tag{20}$$

Note that here the order of differentiation and integration in the fractional derivative operator is the opposite of eqn 1. This reversal of operations occurred when Leibnitz's rule was used to differentiate in eqn 15, producing eqns 18 and 19. This change will prove crucial to solving the initial value problem, because in contrast with eqn 14, eqn 20 possesses both a particular solution, uniquely dependent on the forcing function, and a homogeneous solution, uniquely dependent on the initial value, y(0).

Before presenting these solutions it is important to address the relationship between the operator appearing in eqn 20 and the original definition shown in eqn 1. Using Leibnitz's rule to differentiate the integral in eqn 1 yields

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{w(t-\tau)}{\tau^\alpha} d\tau = \frac{1}{\Gamma(1-\alpha)} \left\{ \frac{w(0)}{t^\alpha} + \int_0^t \frac{\dot{w}\,(t-\tau)}{\tau^\alpha} d\tau \right\} \qquad (21)$$

or in operator form

$$D^\alpha\Big[w(t)\Big] = \frac{w(0)\,t^{-\alpha}}{\Gamma(1-\alpha)} + \hat{D}^\alpha\Big[w(t)\Big] = \frac{w(0)\,t^{-\alpha}}{\Gamma(1-\alpha)} + I^{1-\alpha}\Big[\dot{w}(t)\Big] \qquad (22)$$

$$\hat{D}^\alpha\Big[w(t)\Big] = I^{1-\alpha}\Big[\dot{w}(t)\Big]$$

where D^α is the definition and $\hat{D}^\alpha$ is the modified derivative operator appearing in eqn 21. In fact D^α is the Riemann-Liouville indefinite, fractional order $(1-\alpha)$ integral of the first derivative of the function or effectively an order $-\alpha$ integral of a function.[15,17] The key observation here is that the indefinite fractional order integral operator in effect produces a constant of integration in each modal response. These constants will be used to statisfy the initial conditions in the fractional order state equations.

Posing equation 20 in terms of the modified fractional derivative operator, $\hat{D}^\beta$,

$$(\hat{D}^\beta + a^\beta)\ y(t) = \tilde{\tilde{f}}(t) + \tilde{g}(t) = g(t) \qquad (23)$$

produces the modal state equations. Note the similar appearance of eqns 14 and 23. Recall that eqn 14 is based on the $\tilde{t}$ time scale and has a trivial homogeneous solution. On the other hand, eqn 25 is based on the t time scale, posses a non-trivial homogeneous solution and accounts for the effects of previous motion through the initial value, y (0), and pseudo forcing function, $\tilde{g}(t)$.

Constructing the Fractional Order State Equations

The overall goal is to determine the nature of the fractional order state equations from the modal state equations. The immediate goal is to use the modal state equations, eqn 23 to predict structural response, where the relaxation effects induced by previous motion are accounted for by the pseudo forcing functions, $\tilde{g}(t)$. The transient structural response will be a superposition of the homogeneous solutions of the modal state equations and will be shown to satisfy the initial conditions. The forced structural response will be constructed from the particular solutions to the modal state equations derived using Green's functions. Superimposing the transient and forced response produces the total structural response.

The transient structural response is constructed by first determining the general form of the homogeneous solution for the modal state equations, eqn 23. These solutions take the form

$$y_h(t) = y_h(0) \sum_{p=o}^{\infty} \frac{(-(at)^{\beta})^p}{\Gamma(1+p\beta)} \tag{24}$$

which is a special case of the beta order Mittag-Leffler function defined as (14:102)

$$E_{\beta}(x) = \sum_{p=o}^{\infty} \frac{(x)^p}{\Gamma(1+p\beta)} \; . \tag{25}$$

In Mittag-Leffler notation the homogeneous solution is

$$y_h(t) = y_h(0) \; E_{\beta}\left[-(at)^{\beta}\right], \tag{26}$$

where this special Mittag-Leffler function has the property

$$\hat{D}^{\beta} E_{\beta}\left[-(at)^{\beta}\right] = -a^{\beta} E_{\beta}\left[-(at)^{\beta}\right]. \tag{27}$$

The property should come as no surprise because the Mittag-Leffler function has long been viewed as a generalized exponential function[11]. In related work[12-14] Koeller has shown that the quasi-static fractional calculus viscoelastic formulation leads to Mittag-Leffler functions.

Including the particular solution, the total solution to each of the modified basis equation is

$$y(t) = y_h(0) \; E_{\beta}\left[-(at)^{\beta}\right] + \int_0^t D^{1-\beta}\left[E_{\beta}\left[-(a\tau)^{\beta}\right]\right] g(t-\tau)d\tau \tag{28}$$

which can be determined using Laplace transforms or other traditional solution techniques for integral-differential equations. The kernel in the convolution integral of eqn 28 is the unit impulse solution (Green's function) for the modified basis equations, and is singular. Note that $E_\beta(0)$ is not zero and that the singular behavior of the kernel can be determined through a straightforward application of eqn 1.

It is the singular nature of fractional order derivatives of $E_\beta(-(at)^\beta)$ that is useful in resolving an apparent paradox in the overall initial value problem. Recall that there are m·N (eqn 23) modal state equations needed to characterize the structure, where the solution for each equation has a homogeneous solution containing a different initial value. This paradox becomes apparent when eqn 12 is used to solve for the m·N initial values of the homogeneous basis functions in terms of the structure's initial displacements $\underline{w}_h(t)$ and their derivatives evaluated at time zero.

$$\left\{\begin{matrix} (D^\beta)^{m-1}\underline{w}_h(t) \\ \vdots \\ (D^\beta)^2\underline{w}_h(t) \\ (D^\beta)^1\underline{w}_h(t) \\ \underline{w}_h(t) \end{matrix}\right\}_{t=0} = \left[\ \Phi\ \right] \left\{\begin{matrix} y_{m\cdot N}(t) \\ \vdots \\ y_3(t) \\ y_2(t) \\ y_1(t) \end{matrix}\right\}_{t=0} \qquad (29)$$

The paradox is that at this point only $\underline{w}_h(0)$ and $D^1\underline{w}_h(0)$ can be specified, while the remaining elements in the state vector on the left of eqn 29 are undetermined. Note that the order of the differential equations of motion (eqn 6) is order $2 + \alpha$ or equivalently βm and that the state vector in eqn 31 calls for the initial values of derivatives up through $2 + \alpha - \beta$ or equivalently $\beta(m-1)$. In other words, when posing N, βm order differential equations as a system of m·N differential equations of order β, the corresponding initial value problem calls for all the initial values of the $p\beta$ order derivatives of the displacement vector, $\underline{w}_h(t)$: $p = 0,1,2,\cdots, m-1$. These requirements appear to be analogous to the traditional initial value problem, but also leave one with the requirement for yet more initial conditions.

It is proven in reference[18:54] that all of the non-integer derivatives of $\underline{w}_h(t)$ of order less than two appearing in the state vector have zero initial value. The initial values for acceleration and the accompanying higher order derivations appearing in the state vector can be determined by returning to the original equation of motion, eqn 6, and using successive applications, of eqn 22 to determine the singular terms in the equation of motion. The resulting equation of motion for the response to turning off the previous forcing function is

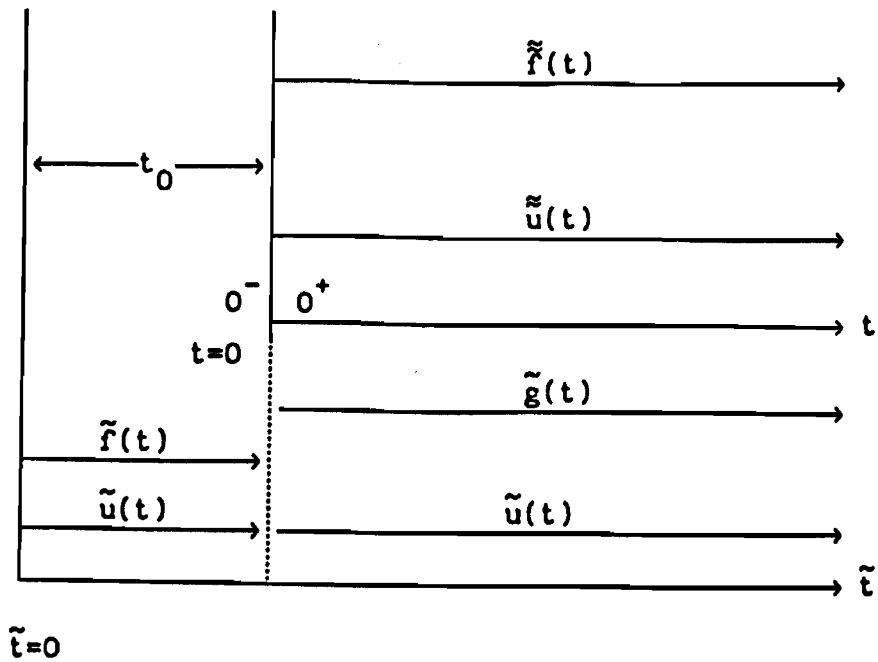

Figure 1 - Time Scales for the Loads and Responses of the Initial Value Problem.

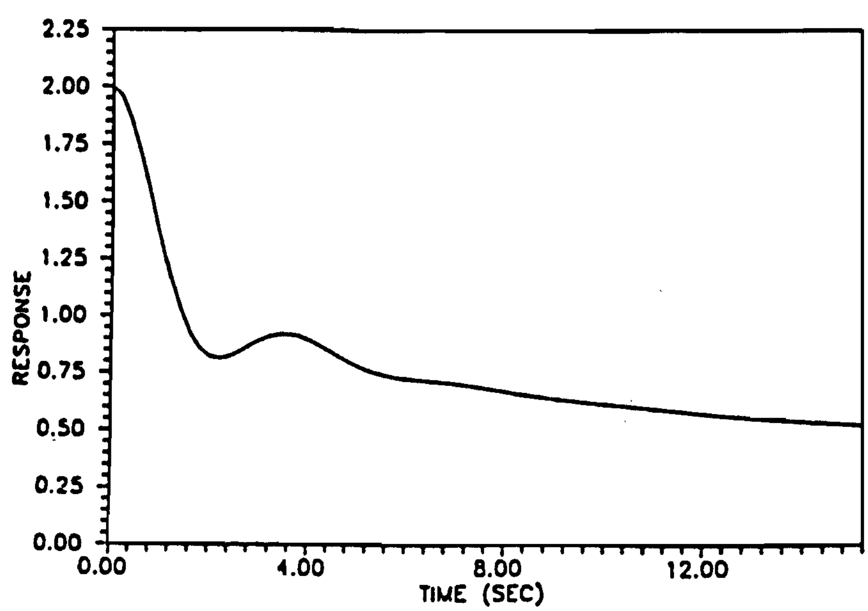

Figure 2 - The Response of the Damped Oscillator

$$-b\underline{\underline{M}}\frac{\ddot{\underline{\tilde{w}}}(0^-)t^{-\alpha}}{\Gamma(1-\alpha)} - b\underline{\underline{M}}\sum_{\ell=1}^{m-2n-1}\frac{t^{-\ell\beta}}{\Gamma(1-\ell\beta)}\hat{D}^{(\underline{m}-2n-\ell)\beta}\underline{\tilde{w}}(0^-)$$

$$+(1 + b\hat{D}^{\alpha})\,\underline{\underline{M}}\,\ddot{\underline{\tilde{w}}}(t) - \underline{\underline{k}}_1\frac{\underline{\tilde{w}}(0^-)t^{-\alpha}}{\Gamma(1-\alpha)} + (\underline{\underline{k}}_o + \underline{\underline{k}}_1\hat{D}^{\alpha})\underline{\tilde{w}}(t) \tag{30}$$

$$- b\,\frac{\underline{\tilde{F}}(0^-)t^{-\alpha}}{\Gamma(1-\alpha)} + \underline{\tilde{G}}(t)$$

The fractional derivatives in this equation of motion are evaluated for $t = 0^-$ or equivalently $\tilde{t} = t_o^-$. $\underline{\tilde{G}}(t)$ are the pseudo forces needed to produce the residual motion associated with the previous loading history, already accounted for in the modified basis equations. The singular forcing function is the result of the α order derivative of the step function turning of $\underline{\tilde{F}}(t)$. The remaining singular behavior is the result of repeatedly applying eqn 22 to separate out the singular behavior of the fractional derivatives of acceleration.

The corresponding equation of motion for the response to the new loads is

$$b\underline{\underline{M}}\frac{\ddot{\underline{\tilde{\tilde{w}}}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + b\underline{\underline{M}}\sum_{\ell=1}^{m-2n-1}\frac{t^{-\ell\beta}}{\Gamma(1-\ell\beta)}\hat{D}^{(\underline{m}-2n-\ell)\beta}\underline{\tilde{\tilde{w}}}(0^+)$$

$$+ (1 + b\hat{D}^{\alpha})\underline{\underline{M}}\,\ddot{\underline{\tilde{\tilde{w}}}}(t) + \underline{\underline{k}}_1\frac{\underline{\tilde{\tilde{w}}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + (\underline{\underline{k}}_o + \underline{\underline{k}}_1\hat{D}^{\alpha})\underline{\tilde{\tilde{w}}}(t) \tag{31}$$

$$= \frac{b\underline{\tilde{\tilde{F}}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + (1 + bD^{\alpha})\underline{\tilde{\tilde{F}}}(t)$$

where the singular forcing function results from again using eqn 22 to express the effects of the step function turning on $\underline{\tilde{\tilde{F}}}(t)$. The remaining singular behavior is also the result of using eqn 22 to separate out the singular behavior of the fractional derivatives of acceleration. Again the tilde and double tilde notation identify motion due to previous forces, $\underline{\tilde{F}}(t)$, and present forces, $\underline{\tilde{\tilde{F}}}(t)$, respectfully, as in eqns 16 and 17.

Equating the coefficients of the strongest singularities (order α) in eqn 30 and then in eqn 31 yields two equations needed to establish the initial conditions acceleration.

$$-b\underline{M}\,\ddot{\tilde{\underline{w}}}(0^-) - \underline{k}_1\tilde{\underline{w}}(0^-) = -b\tilde{\underline{F}}(0^-) \qquad (32)$$

$$b\underline{M}\,\ddot{\tilde{\tilde{\underline{w}}}}(0^+) + \underline{k}_1\,\tilde{\tilde{\underline{w}}}(0^+) = b\tilde{\tilde{\underline{F}}}(0^+) \qquad (33)$$

Adding these two equations produces the relationship needed to establish changes in the initial conditions due to stopping and starting of the load histories.

$$\underline{M}\left(\ddot{\tilde{\tilde{\underline{w}}}}(0^+) - \ddot{\tilde{\underline{w}}}(0^-)\right) + b^{-1}\underline{k}_1\left(\tilde{\tilde{\underline{w}}}(0^+) - \tilde{\underline{w}}(0^-)\right) = \tilde{\tilde{\underline{F}}}(0^+) - \tilde{\underline{F}}(0^-) \qquad (34)$$

Since this relationship is based on step loading, which is incapable of instantaneously changing the displacement or velocity time history between time 0^- and 0^+, one can conclude that

$$\tilde{\tilde{\underline{w}}}(0^+) = \tilde{\underline{w}}(0^-) \qquad (35)$$

$$\dot{\tilde{\tilde{\underline{w}}}}(0^+) = \dot{\tilde{\underline{w}}}(0^-) \qquad (36)$$

and eqn 36 can now be re-expressed as

$$\ddot{\tilde{\tilde{\underline{w}}}}(0^+) - \ddot{\tilde{\underline{w}}}(0^-) = \underline{M}^{-1}\left(\tilde{\tilde{\underline{F}}}(0^+) - \tilde{\underline{F}}(0^-)\right) \qquad (37)$$

Thus we see that the change in the initial accelerations is proportional to any instantaneous changes (steps) in the magnitudes of the applied loads at $t = 0$. It is reassuring to note that eqn 37 is strongly reminiscent of Newton's second Law. To determine the initial accelerations at time 0^+ one needs to determine the accelerations at time 0^- and then add to them the additional component of acceleration from the change in load histories. Should there be a continuous transition from one load history to the other, then

$$\ddot{\tilde{\tilde{\underline{w}}}}(0^+) = \ddot{\tilde{\tilde{\underline{w}}}}(0^-) \qquad (38)$$

and the accelerations at time 0^- are the accelerations used in the initial value problem. Satisfying the initial conditions on acceleration in this manner effectively removes the α order singular terms on both sides of eqns 30 and 31.

The remaining singular terms in these equations do not have corresponding terms on the respective force sides of the equations. To preserve the equality one must conclude that the coefficients of these singular terms are zero. Note that setting these coefficients to zero in effect generates the remaining initial conditions needed in eqn 29. From eqn 30

$$\hat{D}^{(m-2n-\ell)\beta}\left[\underline{\tilde{\ddot{w}}}(0^-)\right] = \underline{0} \qquad \ell = 1,2,3,\cdots,m-2n-1 \qquad (39)$$

and from eqn 31

$$\hat{D}^{(m-2n-\ell)}\left[\underline{\tilde{\tilde{w}}}(0^+)\right] = \underline{0} \qquad \ell = 1,2,3,\cdots,m-2n-1. \qquad (40)$$

Proof is given in reference[10:54]. Hence, one can see that the initial values of the fractional derivatives of displacement greater than second order and less than order βm must be zero to preserve the equation of motion. Adding the two equations of motion and recalling that

$$\underline{\tilde{\tilde{w}}}(t) + \underline{\tilde{w}}(t) = \underline{w}(t) \qquad t \geq 0 \qquad (41)$$

yields

$$\underline{\underline{M}}(1 + b\hat{D}^{\alpha})\underline{\ddot{w}}(t) + (\underline{\underline{k}}_o+\underline{\underline{k}}_1\hat{D}^{\alpha})\underline{w}(t) = (1+b\hat{D}^{\alpha})\ \underline{\tilde{\tilde{F}}}(t) + \underline{\tilde{G}}(t) \qquad (42)$$

which is identical to eqn 8 except for one very important detail. The fractional derivative operator has changed from the original definition, eqn 1, to the modified definition eqn 22. Recall that the modified basis functions use this modified definition as well.

In fact, the entire initial value problem (constituted by eqns 42, 10, 23, and 29) and its solutions (eqn 28) can be cast in terms of the modified definition of fractional differentiation. The composition property for the modified operator

$$\hat{D}^{\alpha}[\hat{D}^{\gamma}[w(t)]] = \hat{D}^{\alpha+\gamma}[w(t)], \qquad (43)$$

holds when the initial values of the fractional derivatives are zero as stipulated in the initial value problem. One can now straightforwardly demonstrate that eqn 42 leads to a corresponding form of eqn 10 where the D^{β} operator is replaced by $\hat{D}^{\beta}$. The D^{β} operators in eqn 29 can now be replaced by $\hat{D}^{\beta}$ as well. Noting that the particular solution in eqn 28 is independent of the initial value and may be viewed as an excitation from a quiescent state, one can show that the solution of the modal state equation takes the form

$$y_j(t) = y_j(0)\, E_\beta\left[-(a_j t)^\beta\right]$$

$$+ \int_0^t (-a_j^\beta)^{m-1}\, \hat{D}^{-1-\alpha}\left[E_\beta\left[-(a_j t)^\beta\right]\right] g_j(t-\tau)\,d\tau \qquad (44)$$

Proof is given in reference[19:62] and note that the kernel is now non-singular. One can now conclude that eqn 44 is the solution of a well-posed problem. The uniqueness of the solution follows immediately from Laplace transforms. Multiplying the initial value and the modal forcing function, $g_j(t)$, by $(1+\epsilon)$ and taking ϵ small demonstrates continuous dependence on the data, so long as the convolution integral is bounded.

To test the robustness of the modal state equations, one needs to ascertain its ability to generate the structural response to impulsive loading. The method entails solving the initial value problem for a step response (using initial accelerations, eqn 37) from a quiescent state and noting that the impulse response is the first derivative of the step response. The structural response for a unit impulse load at the z^{th} degree of freedom of the structure is

$$\underline{w}_{\delta z}(t) = b \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j} (-a_j^\beta)^{2n+2q-1} \hat{D}^{-1-\alpha}\left[E_\beta\left[-(a_j t)^\beta\right]\right] \underline{\phi}_{1j}^T \underline{z}$$

$$+ b \sum_{j=1}^{m\cdot N} \underline{\phi}_{1j} (-a_j)^{2n+q-1} \underline{\phi}_{1j}^T \underline{z} \cdot t \qquad (45)$$

where $\underline{z}$ is a N order column vector of zeroes except the zth element, which is one. Here $\underline{\phi}_{1j}$ are the structure's mode shapes which constitute the lowest N terms of the j^{th} eigenvector of the expanded equations of motion, eqn 10. Again the solution is seen to be continuous and is expressed in terms of the modified operator and the Mittag-Leffler function. Derivation of this expression is given elsewhere[19:67].

At this point one might be tempted to assert that the original definition of fractional order differentiation, eqn 1, is somehow inappropriate for the initial value problem. Not true. Recall that the initial value problem has insufficient numbers of physically motivated initial values to determine uniquely the overall homogeneous solution as a superposition of solutions to the modified basis equations. The additional auxiliary initial conditions, developed by suppressing singular behavior at time zero, provided precisely the number of needed initial conditions for a unique solution. Recall that the original definition, eqn 1, produced this singular behavior without which the initial value problem would flounder for lack of initial information[19].

Moreover, having derived the structural equations of motion in terms the modified fractional derivative operator, eqn 42, and having established the robustness of the formulation through the existence of its impulse response, one can now proceed to construct the structure's fractional order state equations. Casting equation 42 in terms of two real, square and symmetric matrices, as shown in eqn 10, produces the fractional order state equations.

$$\hat{D}^{\beta}\left[\begin{array}{c|c} \underline{\underline{0}} & \\ \vdots & H \\ \underline{\underline{0}} & \\ \underline{\underline{0}} & \\ \hline \underline{\underline{c}}_m & \cdots\,\underline{\underline{c}}_3\,\underline{\underline{c}}_2\,\underline{\underline{c}}_1 \end{array}\right]\left\{\begin{array}{c} (\hat{D}^{\beta})^{m-1}\underline{w}(t) \\ \vdots \\ (\hat{D}^{\beta})^{2}\underline{w}(t) \\ (\hat{D}^{\beta})^{1}\underline{w}(t) \\ \underline{w}(t) \end{array}\right\} \qquad (46)$$

$$+\left[\begin{array}{c|c} & \underline{\underline{0}} \\ & \vdots \\ -H & \underline{\underline{0}} \\ & \underline{\underline{0}} \\ \hline \underline{\underline{0}}\,\cdots\,\underline{\underline{0}}\;\underline{\underline{0}} & \underline{\underline{c}}_0 \end{array}\right]\left\{\begin{array}{c} (\hat{D}^{\beta})^{m-1}\underline{w}(t) \\ \vdots \\ (\hat{D}^{\beta})^{2}\underline{w}(t) \\ (\hat{D}^{\beta})^{1}\underline{w}(t) \\ \underline{w}(t) \end{array}\right\} = \left\{\begin{array}{c} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \\ \overset{\approx}{\underline{f}}^{*}(t)+\tilde{\underline{G}}(t) \end{array}\right\}$$

with straightforward matrix manipulation these state equations take the form

$$\hat{D}^{\beta}\underline{x}(t) = \underline{\underline{A}}\;\underline{x}(t) + \underline{\underline{B}}\;\underline{u}(t) + \underline{\underline{B}}\;\tilde{\underline{G}}(t)$$

Note that applying the orthogonal transformation given in eqn 12 to the state equations yields the modal state equations, eqn 23. In effect one has come full circle and derived equations of motion capable of describing the hereditary viscoelastic damping effects as well as characterize the system in terms of its initial states.

It is reassuring that the fractional order state equations predict a response that is, strictly speaking, a function of all its previous states, as it should be for a system that exhibits hereditary behavior. The pseudo forces $\tilde{\underline{G}}(t)$ describe the effects of previous internal viscoelastic deformation, and the initial states (taken at some time t_o after the onset of motion) describe the effects of immediately previous motion. To predict accurate short term system response, records of previous motion must be kept to construct the pseudo forces. This is necessary for heavily damped structures. In lightly damped structures the hereditary effects are much smaller and the pseudo forces may be ignored.

With or without the pseudo forces included the fractional order state equations one can predict open or closed loop system response. The closed loop feedback relationship between the state vector and the applied forces is

$$\tilde{\underline{F}}(t) = -\underline{\underline{\overset{*}{G}}}\,\underline{x}_r$$

$$\tilde{\underline{F}}^*(t) = -(1+b\hat{D}^\alpha)\underline{\underline{\overset{*}{G}}}\,\underline{x}_r = -\underline{\underline{G}}\,\underline{x} \tag{47}$$

Here $\underline{x}_r$ is the reduced state vector containing the displacements $\underline{w}(t)$ and all derivatives (including fractional order) of $\underline{w}(t)$ up to, but not including, the second derivative. When the stress operator takes the α order derivative of $\underline{x}_r$, this generates the higher order derivatives of $\underline{w}(t)$ in the full state vector. Here the $-\underline{\underline{\overset{*}{G}}}$ is the matrix of actual gain coefficients, $-\underline{\underline{G}}$ is the matrix of effective gain coefficients and $\underline{x}$ is the full state vector appearing in eqn 46.

Note that when b is zero in the stress operator the reduced state vector is the full state vector and no distinction is necessary between actual gains and effective gains. Substituting eqn 47 into eqn 46 produces the equations for the closed loop response.

$$\hat{D}^\beta \underline{x} = (\underline{\underline{A}} - \underline{\underline{B}}\,\underline{\underline{G}})\,\underline{x} \tag{48}$$

This equation includes the feedback of fractional order derivatives of the structure's response. Recall that the fractional derivatives actually being fed back are those of order less than two, namely those in the reduced state vector $\underline{x}_r$. The fact that the full state vector appears in eqn 48 is a consequence of the mathematics in eqn 47. However, eqn 48 is in fact the closed loop state equations. There is no a priori reason to exclude the fractional derivatives from feedback.

In fact Oldham[20] has developed RLC circuits that generate the fractional order derivatives and integrals of input signals over limited frequency ranges. It is possible to take signals proportional to structural deflections and accelerations and produce signals proportional to their fractional derivatives and feed them back.

<u>The Fractional Order Matrix Exponential Function</u>

Although the modal equations are an effective tool in deriving the fractional order state equations, solution formats for these state equations are not limited to modal analysis. When modal analysis is unwarranted, the fractional order analogue of the matrix exponential function can serve as an alternate solution format.

The development begins with the open loop state equations without the pseudo force.

$$\hat{D}^{\beta}\underline{x} = \underline{\underline{A}}\,\underline{x} \tag{49}$$

One can use the following approach to determine the closed loop response by substituting $\underline{\underline{A}} - \underline{\underline{BG}}$, into eqn 48 for $\underline{\underline{A}}$ here and replacing the orthogonal transformation that follows with a similarity transformation for the asymmetric matrix $\underline{\underline{A}} - \underline{\underline{B}}\,\underline{\underline{G}}$. For simplicity of notation the open loop case is considered.

One assumes a time series solution of the form.

$$\underline{x}(t) = \underline{x}_0 + \underline{x}_1\, t^{\beta} + \underline{x}_2\, t^{2\beta} + \cdots + \underline{x}_p\, t^{p\beta} + \cdots \tag{50}$$

Substituting this solution into eqn 49, evaluating the fractional derivatives term by term using the modified operator defined in eqn 22 and equating terms of like power in time yields the following solution.

$$\underline{x}(t) = \left[\, \underline{\underline{I}} + \frac{\underline{\underline{A}}\, t^{\beta}}{\Gamma(1+\beta)} + \frac{\underline{\underline{A}}^{2}\, t^{2\beta}}{\Gamma(1+2\beta)} + \cdots + \frac{\underline{\underline{A}}^{P} t^{P\beta}}{\Gamma(1+P\beta)} + \cdots \right] \underline{x}_o \tag{51}$$

or

$$\underline{x}(t) = E_{\beta}(\underline{\underline{A}}\, t^{\beta})\, \underline{x}_o \tag{52}$$

Here $E_{\beta}(\underline{\underline{A}}t^{\beta})$ is the fractional order matrix exponential function. It may be viewed as the generalized matrix form of the scalar Mittag-Leffler function given in eqn 24. Similar to its scalar counterpart, the fractional order matrix exponential function has the property

$$\hat{D}^{\beta}\left[E_{\beta}\,(\underline{\underline{A}}\, t^{\beta})\right] = \underline{\underline{A}}\, E_{\beta}(\underline{\underline{A}}\, t^{\beta}). \tag{53}$$

One can relate this form of the solution back to the modal solutions, eqn 22, by using the orthogonal transformation given in eqn 12

$$\underline{x} = \underline{\underline{\phi}}\, \underline{y} \tag{54}$$

to decouple the homogeneous form of eqn 46. The result is the homogeneous modal state equations.

$$\hat{D}^{\beta}\, \underset{\sim}{y} = -\underline{\underline{a}}^{\beta}\, \underset{\sim}{y} \qquad (55)$$

where $-\underline{\underline{a}}^{\beta}$ is a diagonal matrix containing the system's eigenvalues. Solutions of this equation take the form

$$y_j(t) = E_{\beta}(-(at)^{\beta})y_j(0) \qquad (56)$$

which are identical to those in eqn 26. However, using the orthogonal transformation to construct the structure's response from eqn 56 produces

$$\underset{\sim}{x}(t) = \underline{\underline{\phi}}\, E_{\beta}(-\, \underline{\underline{a}}^{\beta} t^{\beta})\, \underline{\underline{\phi}}^{T}\, \underset{\sim}{x}_o \qquad (57)$$

This result is equivalent to that shown in eqn 52.

Example Problems

To demonstrate the solution techniques developed for the fractional order state equations, one will first apply them to two simple cases. The first case is a homogeneous first order differential equation with constant coefficients. The second example is a second order differential equation for a single degree of freedom viscoelastically damped oscillator.

If one is to view the fractional order state formulation as a generalization of the initial value problem, its solution techniques should apply to ordinary differential equations with constant coefficients. The first order differential equation is

$$\hat{D}^{1}w + a^2\, w = 0 \qquad\qquad w(0) = w_0$$

which using the composition property can also be expressed as

$$\hat{D}^{2/2}w + a^2\, w = 0$$

Posed in fractional order state form this equation becomes

$$\hat{D}^{1/2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} \hat{D}^{1/2}\, w \\ w \end{Bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & a^2 \end{bmatrix}\begin{Bmatrix} \hat{D}^{1/2}\, w \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The associated eigenvalue problem is

$$\lambda\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} \phi \end{Bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & a^2 \end{bmatrix}\begin{Bmatrix} \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which has eigenvalues

$\lambda = \pm ia$

and associated eigenvectors of the form

$$\left\{ \phi \right\} = \left\{ \begin{matrix} \pm ia \\ 1 \end{matrix} \right\}$$

The solution takes the form

$$\left\{ \begin{matrix} \hat{D}^{1/2} w(t) \\ w(t) \end{matrix} \right\} = \begin{bmatrix} ia & -ia \\ 1 & 1 \end{bmatrix} \left\{ \begin{matrix} y_1(0) E_{1/2}(-(iat^{1/2})) \\ y_2(0) E_{1/2}(-(-iat^{1/2})) \end{matrix} \right\}$$

To determine the initial values $y_1(0)$ and $y_2(0)$, one evaluates this expression at $t = 0$

$$\left\{ \begin{matrix} 0 \\ w_0 \end{matrix} \right\} = \begin{bmatrix} ia & -ia \\ 1 & 1 \end{bmatrix} \left\{ \begin{matrix} y_1(0) \\ y_2(0) \end{matrix} \right\}$$

and solves for $y_1(0)$ and $y_2(0)$

$$\left\{ \begin{matrix} y_1(0) \\ y_2(0) \end{matrix} \right\} = \frac{1}{2ia} \begin{bmatrix} 1 & ia \\ -1 & ia \end{bmatrix} \left\{ \begin{matrix} 0 \\ w_0 \end{matrix} \right\} = \left\{ \begin{matrix} w_0/2 \\ w_0/2 \end{matrix} \right\}$$

Substituting these values into the solution for w(t) given above yields

$$w(t) = \frac{w_0}{2} E_{1/2}(-(iat^{1/2})) + \frac{w_0}{2} E_{1/2}(-(-iat^{1/2}))$$

Using the series representation of the Mittag-Leffler function given in eqn 25 and summing the two series, the terms having fractional order powers of time add out and one is left with

$$w(t) = w_0 \sum_{p=0}^{\infty} \frac{(-a^2 t)^p}{\Gamma(1+p)}$$

or

$$w(t) = w_0 e^{-a^2 t}$$

as expected.

In the second example the fractional order time behavior does not add out, but instead describes the decaying motion of a damped oscillator.

$$\hat{D}^2 w(t) + 2\hat{D}^{1/2} w(t) + w(t) = 0$$

For simplicity the coefficient of the 1/2 order derivative in the stress operator is taken to be zero. The remaining half order derivative describes the low frequency viscoelastic damping and the mass and stiffness coefficients are taken be to one. Again using the composition property the equation may be posed as

$$\hat{D}^{4/2}w(t) + 2\hat{D}^{1/2}w(t) + w(t) = 0$$

In expanded form the equations become

$$\hat{D}^{1/2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \hat{D}^{3/2}w(t) \\ \hat{D}^{2/2}w(t) \\ \hat{D}^{1/2}w(t) \\ w(t) \end{Bmatrix} + \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \hat{D}^{3/2}w(t) \\ \hat{D}^{2/2}w(t) \\ \hat{D}^{1/2}w(t) \\ w(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The eigenvalues and eigenvectors for this system appear in Table 1. Applying the initial conditions

$\lambda_1 = -0.5437$ $\quad\lambda_2 = -1.0$ $\quad\lambda_3 = 0.7718+1.1151i$ $\quad\lambda_4 = 0.7718-1.1151i$

$$\phi_1 = \begin{Bmatrix} \lambda_1^3 \\ \lambda_1^2 \\ \lambda_1 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} \lambda_2^3 \\ \lambda_2^2 \\ \lambda_2 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} \lambda_3^3 \\ \lambda_3^2 \\ \lambda_3 \\ 1 \end{Bmatrix} \quad \phi_4 = \begin{Bmatrix} \lambda_4^3 \\ \lambda_4^2 \\ \lambda_4 \\ 1 \end{Bmatrix}$$

Table 1 - The Eigenvalues and Eigenvectors of the Fractional Order State Equation for the Damped Oscillator.

$x(0) = 2.0 \quad \hat{D}^{1/2}w(0) = 0 \quad \hat{D}^{2/2}w(0) = 0 \quad \hat{D}^{3/2}w(0) = 0$

and solving for the coefficients of the Mittag-Leffler functions as before yields the response of the heavily damped oscillator. A plot of the response is given in figure 2.

Conclusions

The fractional derivative model of viscoelastic damping appears to be a useful tool in constructing state equations that describe the motion of damped structures. The essential value of this viscoelastic model lies in its use of generalized derivative operators. When the model is incorporated into equations of motion, the accelerations describing inertial effects can be expressed in terms of the same operator that describes viscoelastic effects. Furthermore when the external loads are related to structural responses through constant gain feedback, the feedback forces can be described in terms of this operator as well. Given that these fractional order state equations contain fractional order time derivatives of structural motion in the state vector, this formulation suggests the feedback of rational order time derivatives of structural response.

These fractional order state equations appear to constitute a generalization of the classical initial value problem. Posing a system of integro-differential equations as higher order matrix equations with lower, fractional order differential operators produces additional homogeneous solutions with accompanying requirements for additional initial conditions. These additional or auxiliary initial conditions are developed by suppressing singular behavior in the equations of motion. Eliminating the singular behavior in the equations of motion also leads to the use of a modified fractional order derivative that accommodates initial conditions (initial states) in the state equations. Thus the state equations are seen to be related to the original structural equations of motion, but not identical as they would be in a classical formulation.

Moreover, this formulation apppears to be a strong candidate for the general description of linear systems exhibiting strong hereditary behavior with weak frequency dependence. The advantages for the controls engineer are numerous. First, one avoids the use of time dependent coefficients in the state equations. Also the fractional derivative models are compact, making least squares fits to data tractable and manipulation of the model practical. The resulting state equations have analytic solutions and the solution techniques are similar to classical approaches. Finally, the inclusion of the fractional order derivatives in the state vector provides additional forms of feedback to improve system performance. Given that a fractional derivative model accurately captures the hereditary effects, the fractional order state equations appear to be a useful tool in the design and analysis of a feedback control system.

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