TECHNIQUES IN DESIGN AND USING VE DAMPERS

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ABSTRACT

The efficiency of energy dissipation through dampers has been the major concern of damper design. This consideration is based on the assumption that the structure being worked on is proportionally damped. However, most real multi-degree-of-freedom structures are actually non-proportionally damped. For such structures, dampers not only dissipate energy but also transform energy. In this paper, some new dimensions of identifying and using dampers are investigated. A new method of design and re-design of visco-elastic dampers is suggested. Such a method is applicable to many non-proportionally damped systems.

INTRODUCTION

Minimizing excessive vibrations of structures induced by earthquake ground motions or other dynamic loadings has been a major research topic in recent years. To control such vibrations, systemic modifications of the structure's damping ratio by means of incorporating VE dampers into the system have achieved considerable success. Many recent studies as reported in the literatures contain one common restriction that the systems considered are classically damped. Since in engineering practice we often deal with non-classically damped systems, there is a need to develop approaches for non-classically damped systems. This paper presents an approach which can be used for both classically and non-classically damped systems.

Consider the fundamental equation of motion

$$\mathbf{M}\,\ddot{X}(t) + \mathbf{C}\,\dot{X}(t) + \mathbf{K}\,X(t) = F(t) \tag{1}$$

where **M**, **C** and **K** are mass, damping and stiffness matrices; They are symmetric in general. $X(t)$, $\dot{X}(t)$ and $\ddot{X}(t)$ are displacement, velocity and acceleration vectors. $F(t)$ is a forcing function. Equation (1) can be written in a monic form

$$\ddot{Y}(t) + \tilde{C}\,\dot{Y}(t) + \tilde{K}\,Y(t) = \tilde{F}(t) \tag{2}$$

by using the following notations

$$\begin{aligned} Y(t) &= \mathbf{M}^{-1/2}X(t) \\ \tilde{C} &= \mathbf{M}^{-1/2}C\,\mathbf{M}^{-1/2} \\ \tilde{K} &= \mathbf{M}^{-1/2}K\,\mathbf{M}^{-1/2} \end{aligned} \tag{3}$$

Next, we introduce a matrix product

$$\tilde{K}^{1/2} = (\mathbf{M}^{-1/2}\,K\,\mathbf{M}^{-1/2})^{1/2} = \mathbf{M}^{-1/4}\,K^{1/2}\,\mathbf{M}^{-1/4}$$

which will not be discussed in detail in this paper. However, it is worth while to note that $\tilde{C} = 2\,\xi\,\tilde{K}^{1/2}$ is a damping representation and this representation plays a special role in VE damper design.

Examine the following case. In traditional viscous-elastic (VE) damper design, damping ratio is given by Equation (4)

$$\xi = \frac{W_d}{4\pi\,W_p} \tag{4}$$

where W_d is energy dissipated during a cycle. W_p is the maximum potential energy (or maximum kinetic energy) before the cycle. An extension of this SDOF formula can give the i^{th} damping ratio of a proportionally damped MDOF system (see Zhang (1986))

$$\xi_i = \frac{W_{di}}{4\pi W_i} = \frac{\sum_j \gamma_{ij}^2 G_i V_j}{2 Q_i^T \tilde{K} Q_i} \qquad (5)$$

where the subscript i indicates the formula is for the i^{th} mode. j indicates the quantity is related to the j^{th} VE damper. Q_i is the corresponding system eigenvector and V_j is the volume of the damper. With the damping ratios given as in (5), damping coefficient matrix $\tilde{C}$ can be obtained by

$$\tilde{C} = Q \,\mathrm{diag}(\; 2\, \xi_i \omega_i)\; Q^T \qquad (6)$$

Suppose damping ratio ξ_i = constant for i = 1, ... ,n. Then

$$\xi_i = \frac{W_{di}}{4\pi W_i} = \frac{(G_i\; A \,/\, t\;)\pi\, Q^T \tilde{K} Q \,/\, a}{4\;\pi\;\; Q^T \tilde{K} Q \,/\, 2} = \frac{G_i\; A}{2\; t\; a} = \text{constant} \qquad (7)$$

where A is the area of the VE material, t is the corresponding thickness, V = At; and a is a proportional coefficient. Thus

$$\tilde{C} = Q^T \mathrm{diag}(\; 2\; \frac{G_i\; A}{2\; t\; a}\; \omega_i)Q = \frac{G_i A}{ta}\; Q^T \mathrm{diag}(\omega_i)Q = 2\xi\; \tilde{K}^{1/2} = \delta\; \tilde{K}^{\;1/2}$$

where the proportional coefficient $\delta = \dfrac{G_i A}{ta} = 2\;\xi$.

From the above explanation we can see that, if a system is *strictly-proportionally damped*, (i.e. ξ_i = constant for all i), then its damping matrix is proportional to $\tilde{K}^{1/2}$ with a coefficient of 2 ξ .

A GENERAL METHOD FOR EVALUATION OF THE DAMPING COEFFICIENT MATRIX

Among the three coefficient matrices, mass **M**, damping **C** and stiffness **K**, of the dynamic system (1), **K** and **M** are comparatively easy to determine. For example, we can use the finite element method to construct the mass and stiffness matrices respectively. In general we do not have explicit formula for damping matrices. This is why we consider matrix representation (6). In the following, we will show that, if mass and stiffness matrices of the structures and loss factors or loss modulus of damping materials are known, we can establish a method to determine the damping matrix within sufficient accuracy.

GENERAL PROPORTIONALLY DAMPED SYSTEMS

For a strictly-proportionally damped system, the damping matrix is

$$\tilde{C} = 2\ \xi\ \tilde{K}^{1/2}$$

Note that $\tilde{K}$ has the eigen-decomposition

$$\tilde{K} = Q\ \mathrm{diag}\ (\omega_i^2)\ Q^T$$

and $Q = [Q_1,\ Q_2,\ \dots\ Q_n]$. Thus

$$\tilde{C} = 2\ \xi\ Q\ \mathrm{diag}\ (\omega_i)\ Q^T = 2\ \xi \sum_{j=1}^{n} \omega_j\ Q_j\ Q_j^T \tag{8}$$

From Equation (8) we see that, for a strictly-proportionally damped system, the damping matrix is the sum of the outer product of mode shapes Q_i and modal damping coefficients $2\xi\omega_i$. This representation of damping matrix is very useful because it can be extended to both proportionally damped and non-proportionally damped systems. To understand this idea, we try to obtain Equation (8) by a force-method. This method is different from the energy-method used earlier to obtain $\tilde{C} = 2\ \xi\ \tilde{K}^{1/2}$. The energy-method is often employed to determine stiffness matrices (see Timosinko (1982)), but for a system with complicated energy interactions it can not produce the general damping matrix. Without loss of generality, we use the monic equation again. For simplicity in notations, we omit the overhead bar ~.

$$\ddot{X}(t) + C\ \dot{X}(t) + K\ X(t) = F(t)$$

If we can find a forcing function F(t) such that the inertia force $\ddot{X}(t)$ constantly cancels the spring force $KX(t)$, then we have

$$C\ \dot{X}(t) = F(t) \tag{9}$$

and from Equation (9), we may be able to determine the damping matrix C. For the easiest case of strictly-proportionally damped system, it can be seen that such a force corresponding to the i^{th} mode of the system can be given by

$$F_i(t) = 2\ \xi\ \omega_i^2\ Q_i\ \cos\ (\omega_i t)$$

Since

$$X(t) = Q_i \sin\ (\omega_i t),\ \text{and}\ \ddot{X}(t) = -\omega_i^2\ Q_i \sin\ (\omega_i t)$$

so

$$K\ X(t) = \omega_i^2\ Q_i \sin\ (\omega_i t) = -\ \ddot{X}(t)$$

and

$$C\ \dot{X}(t) = C\ \omega_i\ Q_i \cos(\omega_i t) = 2\ \xi\ \omega_i^2 Q_i\ \cos(\omega_i t) = F_i(t).$$

Now let the forcing function be

$$F(t) = \sum_{i=1}^{n} F_i(t) = \sum_{i=1}^{n} 2\ \xi\ \omega_i^2\ Q_i\ \cos\ (\omega_i t)$$

Then

$$X(t) = \sum_{i=1}^{n} Q_i\ \sin\ (\omega_i t)$$

$$C\ \dot{X}(t) = F(t) = \sum_{i=1}^{n} 2\ \xi\ \omega_i^2\ Q_i \cos\ (\omega_i t) \qquad (10)$$

Therefore, for the strictly-proportionally damped systems, the damping matrix **C** must be in the form of Equation (10). For a more general case of proportionally damped systems, the damping ratios are not necessarily identical. However, since the damping matrix must have the same eigenvector matrix as the stiffness matrix, so

$$\mathbf{C} = \mathbf{Q}\ \text{diag}\ (2\ \xi_i \omega_i)\ \mathbf{Q}^T = \sum_{j=1}^{n} 2\ \xi_j \omega_j\ Q_j\ Q_j^T \qquad (11)$$

Equations (8) and (11) appear to be trivial for proportionally damped systems. However, if all the modal damping ratios ξ_i are known, then this representation of damping matrix is unique.

NON-PROPORTIONALLY DAMPED SYSTEMS

In the case of non-proportionally damped systems, the damping matrix may still be expressed in the form

$$\mathbf{C} = \mathbf{P}\ \text{diag}\ (2\ \eta_i \omega_i)\ \mathbf{P}^T = \sum_{j=1}^{n} 2\ \eta_j \omega_j\ P_j\ P_j^T$$

where **P** is the system eigenvector matrix, which is now different from **Q**, the eigenvector matrix of **K**. The quantities η_i are also different from ξ_i, the i^{th} damping ratio of the system. Generally speaking, we can no longer find all P_i by the same approach described in the previous section. For the time being, we will not discuss how to overcome this difficulty which is left to the next section. Let us now examine the force-method again in a more general setting, by considering a simple damper shown in Fig. 1.

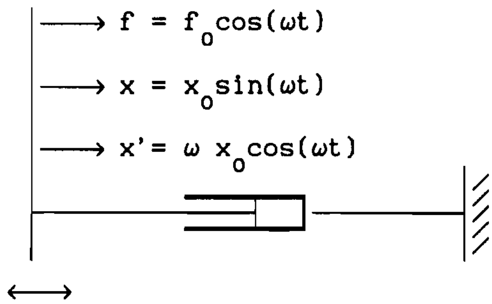

Fig. 1 A damper fixed at one end

If work is done on this damper by external force $f = f_0\cos(\omega t)$, then the forced displacement will have 90° phase difference from the given force. The velocity will be in phase with the force, that is $x = x_0\sin(\omega t)$ and $\dot{x} = \omega\, x_0\cos(\omega t)$. The subscript o denotes amplitude, ω denotes the driving frequency. Work W_f done by the external force in one cycle $0\text{-}2\pi/\omega$ is given by

$$W_f = \int_0^{2\pi/\omega} f\,dx = \int_0^{2\pi/\omega} f_0\cos(\omega t)\;\omega\, x_0\cos(\omega t)\,dt$$

$$= \int_0^{2\pi/\omega} f_0\, x_0\cos^2(\omega t)\,d\omega t = f_0\, x_0/2\,(\omega t + \sin(\omega t)\cos(\omega t))\Big|_0^{2\pi/\omega}$$

$$= \pi\, f_0\, x_0 \qquad (12)$$

Now consider a 3-DOF system shown in Figure 2, where k_i and $c_{(i)}$ are the stiffness and damping between the $(i-1)^{th}$ and the i^{th} floors respectively. Suppose the structure itself is damping free and the system damping si entirely provided by the added dampers, each has an identical value. Then $c_{(i)}$ is the number of dampers mounted in between the $(i-1)^{th}$ and the i^{th} floors. If both k_i and $c_{(i)}$ are constant, or the ratios $k_i/c_{(i)}$ are constants for all i, then the system is proportionally damped. Otherwise, it is non-proportionally damped.

By using the force-method, we can determine the damping matrix of the structure. The procedures are: (1) Create a sub damping matrix for each mode with natural frequency ω_i. Denote the matrix by $\mathbf{C}_i$. (2) By applying the properties of lightly damped systems, add all sub matrices $\mathbf{C}_i$ together. The result will be the damping matrix $\mathbf{C}$, i.e.

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3$$

Note that, for a general damped system, without determining the damping matrix, we can not obtain the exact values of natural frequencies. However, if the system is lightly damped (see Liang et al 1991), we can use ω_{ni}, the square root of the corresponding eigenvalue of generalized stiffness matrix $\tilde{K}$, to approximate the natural frequency ω_i. The error in such an approximation is negligibly small. Now, consider the first mode, with natural frequency ω_1.

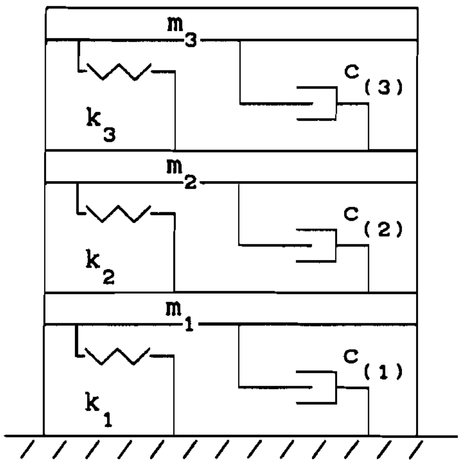

Fig. 2   3-DOF system with springs and dampers

Assume first that the second and third floors are fixed, and only the first

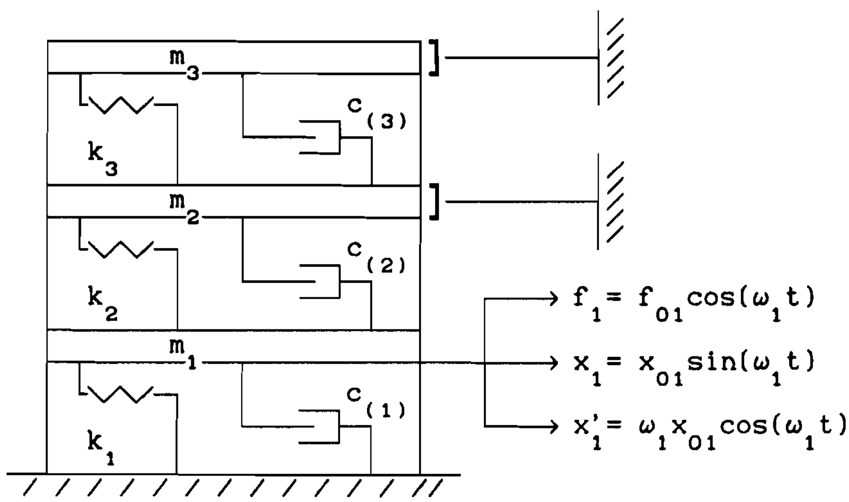

Fig. 3   3-DOF system with 2nd and 3rd floors fixed

floor may vibrate under loading $f_1 = f_{01}\cos(\omega_1 t)$. In a stabilized state, there is an instant when the inertia and the spring forces cancel each other. Say at t_{01}, the damping force exerted on mass No.1 equals to the external loading. Namely,

$$c_{11}^{(1)}\dot{x}_1(t_{01}) = f_1(t_{01}) \qquad (13)$$

Here the damping coefficient c_{ij} is quantitatively equal to the amount of force exerted on the i^{th} mass if the j^{th} mass is moving with a unit velocity and all other masses have zero velocity. The superscript (i) is used to denote the damping effect caused by the i^{th} natural frequency. Therefore, we can write

$$c_{11}^{(1)} = f_1(t_{01}) / x_1'(t_{01}) = f_{01} / (x_{01}\omega_1) = f_{01} \Big|_{x_{01}\omega_1 = 1}$$

Note that, within one cycle, the work done by the external force f_1 is equal to the energy dissipated in the damper. That is $W_f = W_d$. In a later section it will be shown that in certain cases such as VE damper design, (the loss modulus G" and total volume V of the damping material are given), the energy dissipated in a cycle can be calculated by

$$W_d = \pi \bar{\beta} \gamma^2$$

where γ is the strain caused by deformation of damping material, $\bar{\beta}$ is a proportional coefficient. If the damper is a shear-resisting damper, as in the above example, then

$$\gamma = x_0 / t$$

where t is the thickness of the damping material. Thus, we have

$$f_0 x_0 \pi = W_f = W_d = \pi \bar{\beta} \gamma^2 = \pi \bar{\beta}/t \; x_0^2 \qquad (14)$$

or,

$$f_0 = \beta x_0$$

where the proportional coefficient $\beta = \bar{\beta}/t$. Denote the coefficient β of the damping $c_{(i)}$ by β_i. We refer to β_i as the *damping factor*. We then have

$$f_{01} = \beta_1 x_{01} \qquad (15)$$

With the help of Equation (15), damping coefficient $c_{11}^{(1)}$ can be determined by

$$c_{11}^{(1)} = f_{01} / (x_{01}\omega_1) = f_{01} \Big|_{x_{01}\omega_1 = 1} = (\beta_1 + \beta_2)/\omega_1$$

Following the same procedure, we can have

$$c_{12} = - \beta_2/\omega_1$$

and

$$c_{13} = 0$$

since there is no relative motion between the second floor and the third floor.

Next let us assume the first floor is fixed and the second floor is free, as shown in Fig. 4

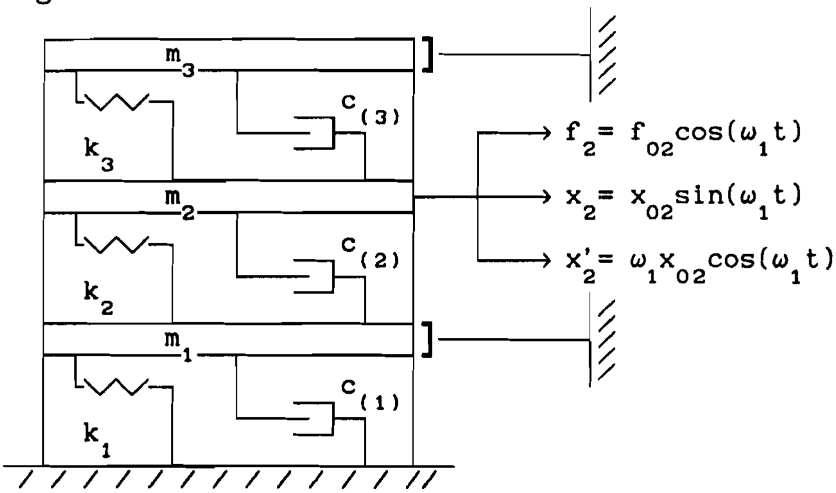

Fig 4 3-DOF system with 1st and 3rd floors fixed to grounds

Similar to the first case, we have

$$c_{21} = - \beta_2/\omega_1 \quad , \quad c_{22} = (\beta_2 + \beta_3)/\omega_1 \quad , \quad \text{and } c_{23} = - \beta_3/\omega_1$$

Finally, we free the third floor and fix the second floor, as shown in Fig. 5

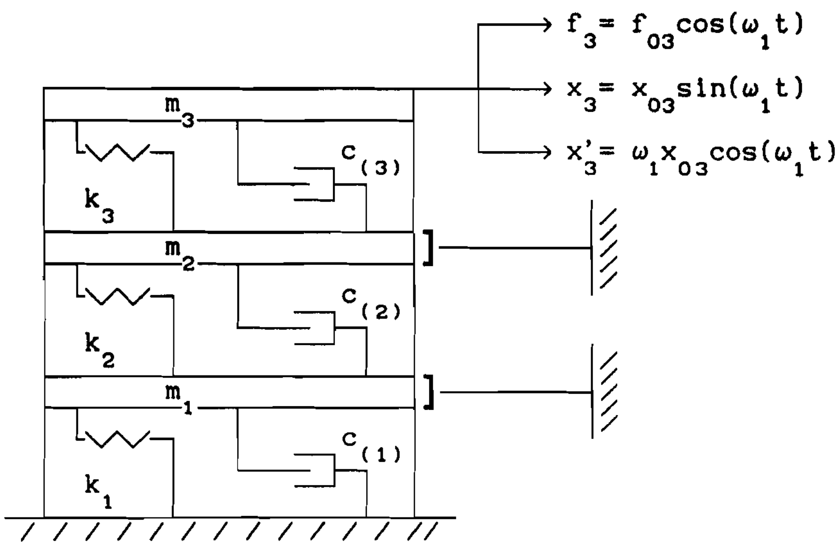

Fig. 5 3-DOF system with 1st and 2nd floors fixed to grounds

We know $c_{31} = 0$, $c_{32} = -\beta_3/\omega_1$ and $c_{33} = \beta_3/\omega_1$. Therefore, for harmonic driving frequency ω_1, we have

$$C_2 = \begin{bmatrix} \beta_1 + \beta_2 & -\beta_2 & 0 \\ -\beta_2 & \beta_2 + \beta_3 & -\beta_3 \\ 0 & -\beta_3 & \beta_3 \end{bmatrix} / \omega_2$$

Similarly, for harmonic driving frequency ω_2, we have

$$C_3 = \begin{bmatrix} \beta_1 + \beta_2 & -\beta_2 & 0 \\ -\beta_2 & \beta_2 + \beta_3 & -\beta_3 \\ 0 & -\beta_3 & \beta_3 \end{bmatrix} / \omega_3$$

By adding up all the components, The damping matrix is obtained

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 = \begin{bmatrix} \beta_1 + \beta_2 & -\beta_2 & 0 \\ -\beta_2 & \beta_2 + \beta_3 & -\beta_3 \\ 0 & -\beta_3 & \beta_3 \end{bmatrix} (1/\omega_1 + 1/\omega_2 + 1/\omega_3)$$

This idea can be extended to n DOF systems, without loss of generality, we write

$$\mathbf{C} = \sum_{i=1}^{n} \mathbf{C}_i = \mathbf{B} \sum_{i=1}^{n} 1/\omega_i \qquad (16)$$

where **B** is the damping factor matrix

$$\mathbf{B} = \begin{bmatrix} \beta_1 + \beta_2 & -\beta_2 & \cdots\cdots & 0 \\ -\beta_2 & \beta_2 + \beta_3 & -\beta_3 \cdots & 0 \\ & \cdots\cdots\cdots\cdots & & \\ 0 & & -\beta_n & \beta_n \end{bmatrix}$$

The above procedure is further explained by using the following example.

EXAMPLE

Consider again the structure shown in Fig. 1. Assume that the mass matrix is I, and the stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} 600 & -200 & 0 \\ -200 & 300 & -100 \\ 0 & -100 & 100 \end{bmatrix}$$

First, incorporate a number of dampers on the structure to make the system proportionally damped with the damping ratio 0.0625. Denote the corresponding damping matrix by C_0 . Then add two sets of same dampers between the first floor and the ground. Add another set of the same dampers of the original kind between the first and the second floors. Denote the damping matrix regarding to these added dampers only by C_1. Now, let us try to determine C matrix. According to the earlier discussion, if the system is lightly damped, we have

$$C = C_0 + C_1$$

The eigenvector matrix of **K** is

$$Q = \begin{bmatrix} .1706 & -.4317 & .8857 \\ .4732 & -.7526 & -.4579 \\ .8643 & .4973 & .0759 \end{bmatrix}$$

and $\omega_1 = 6.7268$, $\omega_2 = 15.8539$, $\omega_3 = 26.5218$. Therefore

$$(1/\omega_1 + 1/\omega_2 + 1/\omega_3) = .2494$$

$$C_0 = Q \operatorname{diag}(2 \times .0625 \times \omega_i)\, Q^T$$

$$= \begin{bmatrix} 2.9947 & -.6330 & -.0786 \\ & 2.0059 & -.5129 \\ & & 1.1372 \end{bmatrix}$$

From C_0, we have $B_0 = C_0 \,/\, (1/\omega_1 + 1/\omega_2 + 1/\omega_3)$

$$= \begin{bmatrix} 12.0056 & -2.5376 & -0.3150 \\ & 8.0417 & -2.0564 \\ & & 4.5590 \end{bmatrix}$$

So, $\beta_1 = 9.1530$ and $\beta_2 = 3.4477$.

Thus $$B_2 = \begin{bmatrix} 21.7536 & -3.4477 & 0 \\ & 3.4477 & 0 \\ & & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 5.4262 & -0.8600 & 0 \\ & 0.8600 & 0 \\ & & 0 \end{bmatrix}$$

Thus $$C = \begin{bmatrix} 8.4209 & -1.4930 & -0.0786 \\ & 2.8659 & -0.5129 \\ & & 1.1373 \end{bmatrix}$$

Table 1 lists complex damping ratios of each cases.

Table 1 Complex Damping Ratios

system with	1st mode	2nd mode	3rd mode
C_0 only	.0625 + .0000j	.0625 + .0000j	.0625 + .0000j
C_1 only	.0157 + .0005j	.0294 + .0016j	.0971 - .0022j
$C_0 + C_1$	.0782 + .0005j	.0920 + .0017j	.1597 - .0022j

From Table 1, we know that for i = 1, 2, 3, approximately

$$\xi_i(C) = \xi_i(C_0) + \xi_i(C_1)$$

and

$$\zeta_i(C) = \zeta_i(C_0) + \zeta_i(C_1)$$

With non-proportional damping C, we can calculate the natural frequencies $\omega_1 = 6.7303$, $\omega_2 = 15.8804$ and $\omega_3 = 26.4637$. Compare the results with that of the corresponding proportionally damped system, we know the errors are small.

FORMULA FOR DESIGN OF VISCOUS-ELASTIC DAMPERS

In the previous section, we showed the procedure of how to determine the sub damping matrix when the system is not proportionally damped. In this section, we discuss a remaining issue of how to determine the damping factor β_i. For a given material, it is usually difficult to determine the damping factors. However, for certain damping materials, such as visco-elastic (VE) material, since the loss modulus are known, the damping ratios can be calculated by equation

$$\xi_i = \frac{W_{di}}{4\pi W} = \frac{\sum_j \gamma_{ij}^2 G_i V_j}{2 Q^T \tilde{K} Q} \qquad (17)$$

For more detailed information of this formula, readers may refer to Lin and Liang (1989). Now, for the case of certain proportionally damped system, Equation (17) can be used together with Equation (7) to determine the damping matrix.

In general cases, Equation (17) may not be valid. However, The energy dissipated by a damper during one cycle, W_d can be calculated by

$$W_d = \pi G''(\omega) A/t \; x_0^2$$

Comparing with Equation (8), we know that

$$\beta_i^{(j)} = G''(\omega_j)\ A_i/t_i \tag{18}$$

where the subscript i denotes the i^{th} damper, and j denotes the j^{th} mode or driving frequency. Note that, for VE material, the loss modulus G" is a function of frequency, denoted by $G''(\omega_j)$. Therefore, the global damping matrix for VE damper design can be written as

$$\mathbf{C} = \sum_{i=1}^{n} \mathbf{C}_i \tag{19a}$$

$$\mathbf{C}_i = \begin{bmatrix} \beta_1^{(i)} + \beta_2^{(i)} & -\beta_2^{(i)} & \cdots & & 0 \\ -\beta_2^{(i)} & \beta_2^{(i)} + \beta_3^{(i)} & -\beta_3^{(i)} & \cdots & 0 \\ & \cdots\cdots & & & \\ 0 & & & -\beta_n^{(i)} & \beta_n^{(i)} \end{bmatrix} / \omega_i \tag{19b}$$

CONCLUSION

The procedure described in this paper is the first attempt to develop a way of design VE dampers for non-proportionally damped systems. Also the procedure provides a good approach to identify the system's damping matrix. It is hoped that actual applications of this method can bring out its practical meanings.

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