A STRONG CRITERION FOR TESTING PROPORTIONALLY DAMPED SYSTEMS

Z. Liang, M. Tong, and G. C. Lee

412 Bonner Hall
State University of New York at Buffalo
Amherst, NY 14260
Tel. (716) 636-2771

ABSTRACT

To check whether a structural system is proportionally damped in dynamic analysis, Caughey advanced criterion to examine if the generalized damping and stiffness matrices of the system commute. This criterion depends on the data of generalized damping which are not directly measurable. This is a sever restriction. In this paper, we present an alternative criterion to determine the proportionality of damping which is rigorous and dynamically flexible. In particular, it can be used in both forward and inverse problems.

INTRODUCTION

Modal analysis is a rapidly-growing field, both in theory and in engineering applications. In recent years, the theoretical basis of modal analysis has progressed extensively from the level of one-degree-of-freedom systems to the level of multi-degree-of-freedom systems by many individuals including: Frazer (1946), Foss (1956), Lancaster (1966), Clough (1975), Meirovitch (1980), Vold (1982), Ewins (1984), Juang (1985) and Mitchell (1990), et al. etc. In dealing with an MDOF system, the simplest model is one that can be decoupled into a group of SDOF systems. Such an MDOF system has great advantage over other MDOF systems in that it can be easily treated by using various techniques developed for SDOF systems.
For some time, it was not obvious on how to identify the decoupleable system until Caughey and O'Kelly published a necessary and sufficient criterion (1965) that states: A system of the form

$$\mathbf{I}_n\ddot{X} + \mathbf{C}_1\dot{X} + \mathbf{K}_1X = 0$$

possesses a complete decoupling, if and only if the matrix $\mathbf{C}_1$ and $\mathbf{K}_1$ commute, i.e.

$$\mathbf{C}_1\mathbf{K}_1 = \mathbf{K}_1\mathbf{C}_1.$$

Based on this criterion, the notion of *proportional/non-proportional damping* was introduced. A system is said to be proportionally damped if and only if the above expression holds.

Caughey and O'Kelly's criterion is a widely accepted approach in modal analysis. However, the criterion depends on the data of generalized damping which are not directly measurable. Such a limitation restricted the further applications of the criterion in many areas of modal analysis such as system identifications, forward and inverse problems. In this paper, we present an alternative criterion which in many aspects is better than the Caughey criterion in a dynamic context.

FUNDAMENTAL VIBRATION SYSTEMS AND THEIR MODES

Consider an n dimensional free vibration system described by the following matrix equation.

$$M\ddot{X} + C\dot{X} + KX = 0 \tag{1}$$

where **M**, **C**, **K** are the mass, damping and stiffness matrices of size n×n such that **M** and **K** are positive definite, and **C** may be positive semi-definite. X, $\dot{X}$ and $\ddot{X}$ are the displacement, velocity and acceleration vectors. In general, we also require ($C^2 - 4K$) to be negative definite, (see Inman, 1989). Since **M** is non-singular, it is easy to see that Equation (1) is equivalent to

$$I_n \ddot{X}_1 + M^{-1/2} C M^{-1/2} \dot{X}_1 + M^{-1/2} K M^{-1/2} X_1 = 0 \qquad (2)$$

where $M^{-1/2}$ is the square root of the inverse of **M**, and I_n is the identity matrix of size n; and $X_1 = M^{1/2} X$. Let C_1 and K_1 denote $M^{-1/2} C M^{-1/2}$ and $M^{-1/2} K M^{-1/2}$ respectively. Then we can write

$$I_n \ddot{X}_1 + C_1 \dot{X}_1 + K_1 X_1 = 0 \quad . \qquad (3)$$

Note that matrices C_1 and K_1 have the same definiteness as those of **C** and **K** respectively. And, both C_1 and K_1 are symmetric.

Let $X_1 = X_0 e^{\lambda t} = \begin{bmatrix} x1 \\ x2 \\ \vdots \\ xn \end{bmatrix} e^{\lambda t}$ and substitute it back in (3). We have

$$(I_n \lambda^2 + C_1 \lambda + K_1) X_0 = 0 \quad . \qquad (4)$$

In the case of proportional damping, it is known that C_1 and K_1 commute (i.e. $C_1 K_1 = K_1 C_1$). So there exists an orthogonal matrix **Q** such that $Q^T Q = I_n$, and it diagonalizes C_1 and K_1 simultaneously.

$$Q^T C_1 Q = \begin{bmatrix} c_{11} & & & \\ & c_{22} & & \\ & & \ddots & \\ & & & c_{nn} \end{bmatrix} = \Lambda_c, \qquad Q^T K_1 Q = \begin{bmatrix} k_{11} & & & \\ & k_{22} & & \\ & & \ddots & \\ & & & k_{nn} \end{bmatrix} = \Lambda_k .$$

It follows that Equation (4) can be further modified to have the form

$$Q\,(I_n \lambda^2 + \Lambda_c \lambda + \Lambda_k) Q^T X_0 = 0 \ . \qquad (5)$$

Let P denote $Q^T X_0$. Then multiplying Q^T to Equation (5) from the left, we get

$$(I_n \lambda^2 + \Lambda_c \lambda + \Lambda_k) P = 0 \ . \qquad (6)$$

We define this as the characteristic equation of a Λ_c-Λ_k system. Since the determinate $|I_n \lambda^2 + \Lambda_c \lambda + \Lambda_k|$ equals

$$\begin{vmatrix} \lambda^2 + c_{11}\lambda + k_{11} & & \\ & \ddots & \\ & & \lambda^2 + c_{nn}\lambda + k_{nn} \end{vmatrix} = \prod_{i=1}^{n}(\lambda^2 + c_{ii}\lambda + k_{ii}),$$

with $(C\,-^{2}4K)$ negative definite, we can solve $|I_n\lambda +^{2}\Lambda_c\lambda + \Lambda_k| = 0$ for n pairs of conjugate roots $\lambda_1, \bar{\lambda}_1; \lambda_2, \bar{\lambda}_2; \; \ldots \; \lambda_n, \bar{\lambda}_n$; where λ_i is defined as the i^{th} system eigenvalue (eigenfrequency). They can be further expressed as

$$\lambda_i = -\xi_i\omega_i + j\sqrt{1-\xi_i^2}\;\omega_i$$

where ξ_i and ω_i are the i^{th} damping ratio and undamped natural frequency of the system. In the following, we may omit the subscript i of λ_i , P_i and ω_i, etc., for simplicity. Note that in the normal mode case, ω_i^2 are also the eigenvalues of the matrix $\mathbf{K}_1$.

Corresponding to each λ_i there is a system eigenvector $\mathbf{e}_i$ (the unit vector at i^{th} direction in an n dimensional space) which satisfies

$$(I_n\lambda_i^2 + \Lambda_c\lambda_i + \Lambda_k)\mathbf{e}_i = 0$$

$$(I_n\bar{\lambda}_i^2 + \Lambda_c\bar{\lambda}_i + \Lambda_k)\mathbf{e}_i = 0 \; .$$

Write

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad P_I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

We call Λ and P_I the *eigenvalue* and *mode shape* matrices respectively. It is now clear that the system has been completely *self-decoupled.* (A more rigorous definition of self-decoupled system will be introduced later.)

In the case of non-proportional damping, we know that $C_1K_1 \neq K_1C_1$. So the above analysis is no longer valid. However, there still exists an orthogonal matrix Q that diagonalizes K_1. Applying the matrix Q to Equation (4), we have

$$(I_n\lambda^2 + Q^TC_1Q\,\lambda + \Lambda_k)P = 0. \tag{7}$$

Next denote a symmetric matrix by

$$D = Q^T C_1 Q = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{12} & d_{22} & \cdots & d_{2n} \\ & & \cdots & \\ d_{1n} & d_{2n} & \cdots & d_{nn} \end{bmatrix}.$$

Note that, in equation (7), the diagonal matrix Λ_k has the same form as equation (5). Therefore, we have

$$(I_n \lambda^2 + D\lambda + \Lambda_k)P = 0. \tag{8}$$

This equation is called the *characteristic equation* of the $D\text{-}\Lambda_k$ system. The $D\text{-}\Lambda_k$ system has similar eigenproblem as system (1), that is, they both have the same eigenvalues and their eigenvectors are related by the formula

$$P = Q^T X_0 = Q^T M^{-1/2} X$$

where P , X_0 and X denote the eigenvector matrices of Systems (8), (3) and (1) respectively.

Because of the above mentioned property of the $D\text{-}\Lambda_k$ system simplifies the eigenproblem, we refer this system as a *fundamental vibration system* (FVS) or *canonical vibration system*. Fig 1 shows a 3 degree-of-freedom FVS.

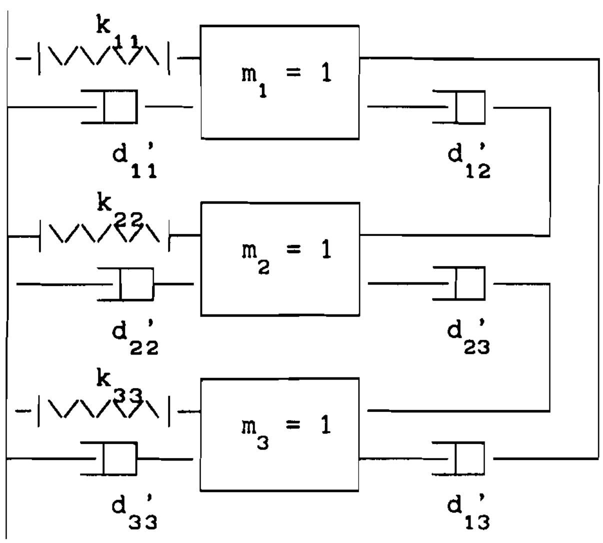

Fig. 1 A 3-DOF FVS

Comparing Fig. 1 with Equation (8), we see that

$$d_{11} = d'_{11} + d'_{12} + d'_{13}, \quad d_{12} = -d'_{12}, \quad d_{13} = -d'_{13}, \quad \text{etc.}$$

It is noted that the previously mentioned proportional damping model is a special case of the FVS. Following the procedure descried above in dealing with the Λ_c-Λ_k system, we can similarly obtain an eigenvalue matrix and a mode shape matrix. However, in order to understand the properties of these two matrices, some new concepts will first be introduced.

<u>DEFINITION</u>: A non-zero system eigenvector P is said to be *true-complex* or *strongly-complex* for the corresponding eigenvalue λ if

$$(I_n\lambda^2 + D\lambda + \Lambda_k)P = 0$$

holds and P can not be expressed as a complex linear combination of the real-valued eigenvectors of Λ_k. A system eigenvector P is *weakly-complex* if it is not true-complex.

It is clear that a proportionally damped system has only weakly-complex system eigenvectors. Hence the system has a real-valued mode shape matrix.

<u>LEMMA 1</u>: Given an FVS. If P is a system eigenvector such that

$$(I_n\lambda^2 + D\lambda + \Lambda_k)P = 0$$

for some λ, and P is weakly-complex, then the corresponding eigenvalue λ satisfies $\lambda\bar{\lambda} = k_{ii}$ for some $i = 1, \ldots, n$, (where k_{ii} is the i^{th} diagonal element of Λ_k).

<u>COROLLARY 1</u>: Suppose λ is a system eigenvalue of a given FVS such that $\lambda\bar{\lambda} \neq k_{ii}$ for all $i = 1, 2, \ldots, n$. Then its corresponding eigenvector P is strongly-complex.

<u>LEMMA 2</u>: Given an FVS, if P is a system eigenvector, then the following conditions are equivalent.

(1) P is an eigenvector of the damping matrix D.

(2) P is an eigenvector of the stiffness matrix Λ_k.

(3) P is weakly-complex.

<u>LEMMA 3</u>: Given an FVS. If there is a vector P such that P is an eigenvector of both D and Λ_k, then it is also a system eigenvector. Moreover, P is weakly-complex.

A CRITERION FOR DETERMINING PROPORTIONALLY DAMPED SYSTEMS

In this section we present a theorem for determining a proportionally damped systems. Before starting with the theorem itself we need to consider a lemma of Rayleigh quotient.

<u>LEMMA 4</u>: For any Hermitian matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, denoting its first (smallest) and last (largest) eigenvalues by a_1 and a_n respectively. Then the Rayleigh quotient equals a_1 or a_n, i.e.

$$\frac{X^H \mathbf{A} X}{X^H X} = a_1 \tag{9}$$

or

$$\frac{X^H \mathbf{A} X}{X^H X} = a_n \tag{10}$$

if and only if its corresponding non-zero vector $X \in \mathbb{C}^n$ is the first or last eigenvector of **A**. Here the superscript H denotes the Hermitian transpose, $\mathbb{C}^{n\times n}$ and $\mathbb{C}^n$ stand for the set of all nxn matrices and vectors, real or complex, respectively, (see Ortega, 1987).

<u>DEFINITION</u> : For an FVS, if the damping matrix D is of the following form, then we say that the system is *self-decoupled* into two subsystems.

$$D = \left[\begin{array}{ccc|ccc} d_{11} & \cdots & d_{1t} & & & \\ & \cdots & & & 0 & \\ d_{1t} & \cdots & d_{tt} & & & \\ \hline & & & d_{t+1} & \cdots & d_{t+1,n} \\ & 0 & & & \cdots & \\ & & & d_{t+1,n} & \cdots & d_{nn} \end{array}\right].$$

An FVS with diagonal D is self-decoupled into n subsystems. In this case, the system is said to be *completely self-decoupled.*

<u>DEFINITION</u> : Two FVS $D_1 - \Lambda_{k1}$ and $D_2 - \Lambda_{k2}$ are said to be *equivalent* if there exists an orthogonal matrix T such that

$$T^T D_1 T = D_2$$
$$T^T \Lambda_{k1} T = \Lambda_{k2} .$$

Theorem 1: For a given FVS, all the system eigenvectors are weakly-complex, if and only if, there exists an one-one correspondence between the system eigenvalues λ_i and the eigenvalues k_{ii} of Λ_k such that

$$\lambda_i \bar{\lambda}_i = k_{ii} \qquad i = 1, \ldots, n.$$

Proof: Sufficiency.

First, we discuss the case where all the eigenvalues of Λ_k are distinct. Consider the smallest k_{ii}, say k_{11}. Let P_1 be a system eigenvector whose eigenvalue satisfies $\lambda_1 \bar{\lambda}_1 = k_{11}$. By the characteristic equation

$$P_1^H (I_n \lambda^2 + D\lambda + \Lambda_k) P_1 = 0$$

we have

$$\lambda_1^2 + \frac{P_1^H D P_1}{P_1^H P_1} \lambda_1 + \frac{P_1^H \Lambda_k P_1}{P_1^H P_1} = 0 \,. \tag{11}$$

Since the Rayleigh quotients $\dfrac{P_1^H D P_1}{P_1^H P_1}$ and $\dfrac{P_1^H \Lambda_k P_1}{P_1^H P_1}$ are real, it follows that both λ_1 and its conjugate $\bar{\lambda}_1$ are roots of Equation (11). We know that a real quadratic equation has a unique irreducible expression in the real field. Thus $\dfrac{P_1^H \Lambda_k P_1}{P_1^H P_1} = \lambda_1 \bar{\lambda}_1 = k_{11}$. By Lemma 4, P_1 must be the first eigenvector of Λ_k. Namely, $P_1 = pe_1$, where p is a scalar. Then from

$$(I_n \lambda_1^2 + D\, \lambda_1 + \Lambda_k) pe_1 = 0$$

we have

$$\begin{aligned} \lambda_1^2 p + d_{11} p\, \lambda_1 + k_1\, p &= 0 \\ 0 + d_{12} p\, \lambda_1 + 0 &= 0 \\ \ldots\ldots \\ 0 + d_{1n}\, p\, \lambda_1 + 0 &= 0 \,. \end{aligned} \tag{12}$$

Equations (12) imply that

$$d_{1m} = 0 \,, \qquad m = 2, \ldots, n.$$

Note that **D** is symmetric. So it must be of the following form

$$D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & d_{2n} \\ & & \dots & \\ 0 & d_{2n} & \dots & d_{nn} \end{bmatrix} .$$

Therefore, the D-Λ_k system is self-decoupled into two independent subsystems each forms a D-Λ_k model with lower dimensions.

Now apply the same procedure to the subsystem with n-1 dimension we can show that the damping matrix **D** can be further divided. Hence it must be in the following form

$$D = \left[\begin{array}{cc|ccc} d_{11} & 0 & & & \\ 0 & d_{22} & & \mathbf{0} & \\ \hline & & d_{33} & \dots & d_{3n} \\ & \mathbf{0} & & \dots & \\ & & d_{3n} & \dots & d_{nn} \end{array}\right] .$$

Repeat the same procedure, **D** will finally be shown to be a diagonal matrix. Thus it is proportional damping, and the system eigenvectors are weakly-complex.

Secondly, we consider the case where Λ_k has a repeated eigenvalue k_{rr} with multiplicity r. Without loss of generality, assume k_{rr} is the smallest among all k_{ii}, and Λ_k is in the form

$$\Lambda = \begin{bmatrix} k_{rr} & & & & \\ & \ddots & & & \\ & & k_{rr} & & \\ & & & k_{r1} & \\ & & & & \ddots \\ & & & & & k_{nn} \end{bmatrix}$$

Choose an orthogonal matrix **T** such that

$$T = \left[\begin{array}{ccc|c} t_{11} & \dots & t_{1r} & \\ & \dots & & \mathbf{0} \\ t_{r1} & \dots & t_{rr} & \\ \hline & 0 & & I_{n-r} \end{array}\right]$$

$$T^T \Lambda_k T = \Lambda_k , \quad \text{and}$$

$$T^T D\, T = \hat{D} = \left[\begin{array}{ccc|ccc} \hat{d}_{11} & . & & \hat{d}_{1,r+1} & \cdots & \hat{d}_{1n} \\ & \ddots & & & \cdots & \\ & & \hat{d}_{rr} & \hat{d}_{r,r+1} & \cdots & \hat{d}_{rn} \\ \hline \hat{d}_{1,r+1} & \cdots & \hat{d}_{r,r+1} & \hat{d}_{r+1,r+1} & \cdots & \hat{d}_{r+1,n} \\ & \cdots & & & \cdots & \\ \hat{d}_{1n} & \cdots & \hat{d}_{rn} & \hat{d}_{r+1,n} & \cdots & \hat{d}_{nn} \end{array}\right] .$$

Now we apply the same procedure described in the previous case to this induced $\hat{D} - \Lambda_k$ system until Equation (11) is obtained. According to the assumption there are r system eigenvalues, say λ_1, λ_2, ..., λ_r , such that $\lambda_i \bar{\lambda}_i = k_{rr}$, $i = 1, 2, \ldots, r$. By Lemma 4, their corresponding eigenvectors P_i are eigenvectors of Λ_k with eigenvalue k_{rr}. Hence

$$P_i = \beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \ldots + \beta_r \mathbf{e}_r , \quad i = 1, 2, \ldots, r.$$

If in $\hat{\mathbf{D}}$, $\hat{d}_{11}$, ..., $\hat{d}_{rr}$ are distinct. Then we claim P_i is of the form

$$P_i = \beta_{ij} \mathbf{e}_j , \quad 1 \le j \le r.$$

Otherwise, if P_i is the linear combination of more than one $\mathbf{e}_j$, then we find there are at least two eigen-equations, say

$$\lambda_i^2 + d_{11}\lambda_i + k_{rr} = 0$$

$$\lambda_i^2 + d_{22}\lambda_i + k_{rr} = 0 .$$

Since $d_{11} \ne d_{22}$, this will contradict the uniqueness of a real coefficient irreducible quadratic expression.

Under this circumstance, we can use the same argument given in (12) and thus see that

$$\begin{bmatrix} \hat{d}_{1,r+1} & \cdots & \hat{d}_{1n} \\ & \cdots & \\ \hat{d}_{r,r+1} & \cdots & \hat{d}_{rn} \end{bmatrix}^T = \begin{bmatrix} \hat{d}_{1,r+1} & \cdots & \hat{d}_{1n} \\ & \cdots & \\ \hat{d}_{r,r+1} & \cdots & \hat{d}_{rn} \end{bmatrix} = \mathbf{0} .$$

Therefore, we have

$$\hat{D} = \left[\begin{array}{ccc|ccc} \hat{d}_{11} & & & & & \\ & \ddots & & & 0 & \\ & & \hat{d}_{rr} & & & \\ \hline & & & \hat{d}_{r+1,r+1} & \cdots & \hat{d}_{r+1,n} \\ & 0 & & & \cdots & \\ & & & \hat{d}_{r+1,n} & \cdots & \hat{d}_{nn} \end{array}\right] .$$

Since $\hat{D} = T^T D\, T$, by pre- and post-multiplying T and T^T to $\hat{D}$, we have

$$D = T\, \hat{D}\, T^T = \left[\begin{array}{ccc|ccc} d_{11} & \cdots & d_{1r} & & & \\ & \cdots & & & 0 & \\ d_{1r} & \cdots & d_{rr} & & & \\ \hline & & & d_{r+1,r+1} & \cdots & d_{r+1,n} \\ & 0 & & & \cdots & \\ & & & d_{r+1,n} & \cdots & d_{nn} \end{array}\right] .$$

It is clear that D is self-decoupled. Hence we can continue work on the subsystems and eventually show that $D\text{-}\Lambda_k$ is equivalent to a $D_1 - \Lambda_{k1}$ system where D_1 is diagonal.

As for the case where there are some repetitions among d_{11}, d_{22}, ..., d_{rr}, say $d = d_{11} = d_{22} = \ldots = d_{ss}$, we claim that the system has a repeated eigenvalue λ with multiplicity s. Because for each P_i, $1 \le i \le s$, the eigen-equation

$$\lambda_i^2 + \frac{P_i^H \hat{D}\, P_i}{P_i^H P_i} \lambda_i + \frac{P_i^H \Lambda_k P_i}{P_i^H P_i} = 0$$

has the same coefficients

$$\frac{P_i^H \hat{D}\, P_i}{P_i^H P_i} = d \qquad \text{and} \qquad \frac{P_i^H \Lambda_k P_i}{P_i^H P_i} = k_{rr} .$$

Therefore, $\lambda_1 = \lambda_2 = \ldots = \lambda_s = \lambda$ and the coefficient matrix $(\lambda^2 + \hat{D}\lambda + \Lambda_k)$ has rank n-s. It follows that we can simply choose e_1, e_2, ..., e_s as our P_1, P_2, ..., P_s. Then following the same argument we can see that $\hat{D}$ is self-decoupled and eventually reduce $D\text{-}\Lambda_k$ system to an equivalent $D_1\text{-}\Lambda_{k1}$ system with D_1 diagonal.

We now know that under the given condition a $D\text{-}\Lambda_k$ system must be equivalent to a $D_1 - \Lambda_{k1}$ system where the damping matrix D_1 is diagonal.

Consequently, $D\Lambda_k = \Lambda_k D$ and the system is proportionally damped. Therefore, all the system eigenvectors are weakly-complex. Thus the sufficiency of the Theorem is proven.

Necessity.

The necessary condition of theorem 1 is implied by Lemma 1.

COROLLARY 2: Given a System (1), either it has no strongly-complex eigenvector in which case the system is proportionally damped or there are at least two strongly-complex eigenvectors in which case it is non-proportionally damped.

EXTENSION OF THE CRITERION IN DISTRIBUTED MASS SYSTEMS

For linearly distributed mass systems, a similar theorem holds just as for the discrete lumped mass systems. Let U be a variable defined in a bounded domain $\mathcal{D}$ and satisfy the equation:

$$U_{tt} + L_1 U_t + L_2 U = 0 \qquad (13)$$

with homogeneous linear boundary conditions specified on the bounding surface Γ. L_1 and L_2 are compact self-adjoint spatial operators. Suppose that the boundary conditions for a higher-order operator are derivable from a compatible set of boundary conditions associated with a lower-order operator. We begin our discussion by considering the properties of eigenfunctions.

DEFINITION: An *eigenfunction* of System (13) is a non-zero function ϕ associated with a scalar λ such that the function is differentiable by both L_1 and L_2 and satisfies

$$(\lambda^2 + \lambda L_1 + L_2)\, \phi = 0 \qquad (14)$$

and the boundary conditions attached to (13). The scalar λ is called an *eigenvalue* of System (13).

DEFINITION: An eigenfunction ϕ associated with eigen value λ is said to be *true-complex* or *strongly-complex* if it can not be expressed as a linear combination of the real-valued eigenfunctions of λ. If ϕ is not strongly-complex, then it is called *pseudo-complex* or *weakly-complex*.

LEMMA 5: For a given System (13), if Φ is an eigenfunction, then the following conditions are equivalent.

(1) Φ is an eigenfunction of the viscous damping operator L_1.

(2) Φ is an eigenfunction of the stiffness operator L_2.

(3) Φ is weakly-complex.

PROOF: (1) is equivalent to (2).

Suppose $$L_1\phi = \alpha\ \phi\ . \tag{15}$$

Substituting Equation (15) into Equation (14), we get

$$\lambda^2\phi + \lambda(\alpha\ \phi) + L_2\phi = 0.$$

So $$L_2\phi = -(\lambda^2 + \lambda\alpha)\phi.$$

Since $-(\lambda^2 + \lambda\ \alpha\ )$ is a scalar, ϕ is an eigenfunction of L_2.

Now, suppose $$L_2\phi = \beta\phi. \tag{16}$$

we can similarly prove (1) if (2) holds.

(1) or (2) is equivalent to (3).

Suppose ϕ is complex valued, i.e. $\phi = \psi + j\eta$, where ψ and η are real-valued functions. Then $L_1\phi = L_1\psi + j\ L_1\eta$. It is easy to see that $L_1\psi$ is real and $j\ L_1\eta$ is imaginary.

On the other hand, $L_1\phi = \alpha\phi = \alpha\psi + j\alpha\eta$. Since L_1 is self-adjoint, all of its eigenvalues must be real. Therefore, $\alpha\psi$ is real and $j\alpha\eta$ is imaginary. Comparing the above two equations, we conclude

$$L_1\psi = \alpha\ \psi, \qquad L_1\ \eta = \alpha\ \eta\ .$$

That is, both ψ and η are real-valued eigenfunctions of L_1. Since ϕ is a linear combination of ψ and η, ϕ is weakly-complex.

LEMMA 6: Given a System (13), if there is a function Φ such that Φ is an eigenfunction of both L_1 and L_2, then it is also a system eigenfunction. Moreover, Φ is weakly-complex.

PROOF: Substituting Equations (15) and (16) into (14) and denoting the left hand side by

$$a = \lambda^2\ \phi + \lambda\ L_1\phi + L_2\phi\ ,$$

we have

$$a = (\ \lambda^2 + \alpha\ \lambda + \beta\ )\ \phi\ .$$

Choosing a λ such that $(\ \lambda^2 + \alpha\ \lambda + \beta\ ) = 0$, then we have the desired result.

<u>**Lemma 7**</u>: For any compact and self-adjoint operator L, let its smallest (greatest) eigenvalue be denoted by λ_1 (λ_n). If the Rayleigh quotient equals λ_1, i.e.

$$\frac{< \Psi , L\Psi >}{< \Psi , \Psi >} = \lambda_1 \ (\lambda_n), \tag{17}$$

where $< (.) , (.) >$ denotes an inner product, then the non-zero function Ψ is an eigenfunction of L associated with $\lambda_1(\lambda_n)$, and vice versa.

Proof:

Since operator L is compact and self-adjoint, there is a complete set of eigenfunctions for L, i.e. for any function f in a Hilbert space,

$$f = \sum_{i=1}^{\infty} <f, \ \phi_i> \phi_i$$

and

$$Lf = \sum_{i=1}^{\infty} L < f, \ \phi_i \ > \phi_i \ = \ \sum_{i=1}^{\infty} < f, \ \phi_i > \lambda_i \phi_i$$

where λ_i is associated eigenvalues. Suppose λ_1 is the smallest eigenvalue. Consider $< \Psi , L\Psi > = \lambda_1 < \Psi, \Psi >$. We also have

$$\Psi = \sum_{i=1}^{\infty} < \Psi, \ \phi_i \ > \phi_i$$

and

$$L\Psi = \sum_{i=1}^{\infty} < \Psi, \ \phi_i \ > \lambda_i \phi_i$$

Therefore

$$< \Psi, L\Psi > = \sum_{i=1}^{\infty} < \Psi, \ \phi_i >^* < \Psi, \ \phi_i > \lambda_i = \lambda_1 \sum_{i=1}^{\infty} < \Psi, \ \phi_i >^* < \Psi, \ \phi_i >$$

Or

$$\sum_{i=1}^{\infty} < \Psi, \ \phi_i >^* < \Psi, \ \phi_i > (\lambda_i - \lambda_1) = 0 \tag{18}$$

where * denotes the complex conjugate of functions.

Since λ_1 is the smallest eigenvalue, $(\lambda_i - \lambda_1) \neq 0$ when $i \neq 1$. This fact forces $<\Psi, \ \phi_i> = 0$, for all $i \neq 1$ and $< \Psi, \ \phi_1 > \neq 0$ due to non-zero Ψ. Here subscript 1 denotes the eigenfunction being associated to λ_1. we therefore conclude $\Psi = < \Psi, \ \phi_1 > \phi_1$. This proves that Ψ is an eigenfunction of L associated with the smallest eigenvalue λ_1. In a similar fashion, we can show the case for λ_n, the greatest eigenvalue.

THEOREM 2: For a given System (13), all the system eigenfunctions are weakly-complex, if and only if, for any system eigenvalues λ_i, we can find an eigenvalues β_i of operator L_2 such that

$$\lambda_i \lambda_i^* = \beta_i \tag{19}$$

PROOF. Necessity.

Consider an eigenfunction ϕ_i associated with eigenvalue λ_i. Suppose it is weakly-complex. From

$$\lambda_i^2 \phi_i + \lambda_i L_1 \phi_i + L_2 \phi_i = 0$$

and using the Equations (15) and (16) with the subscript i, we have

$$(\lambda_i^2 + \lambda_i \alpha_i + \beta_i) \phi_i = 0 .$$

Since $\phi_i \neq 0$, so

$$(\lambda_i^2 + \lambda_i \alpha + \beta) = 0 .$$

Note that both α_i and β_i are real, so Equation (19) holds.

Sufficiency.

We only prove the case that all eigenvalues are distinct. For the case with repeated eigenvalues, the proof is similar but more complicated. Consider the smallest eigenvalue β_1 of L_2.

$$\lambda_1 \lambda_1^* = \beta_1$$

with the corresponding eigenfunction ϕ_1 and its associated system eigenvalue λ_1, that

$$\lambda_1^2 \phi_1 + \lambda_1 L_1 \phi_1 + L_2 \phi_1 = 0.$$

Taking the inner product of the above equation with ϕ_1, we get

$$\langle \phi_1, \lambda_1^2 \phi_1 \rangle + \langle \phi_1, \lambda_1 L_1 \phi_1 \rangle + \langle \phi_1, L_2 \phi_1 \rangle = 0.$$

Then

$$\lambda_1^2 + \lambda_1 \left\{ \frac{\langle \phi_1, L_1 \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} \right\} + \left\{ \frac{\langle \phi_1, L_2 \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} \right\} = 0$$

Note that the terms in above { . } are real scalars since $\{ . \}^* = \{ . \}$ for both L_1 and L_2 are self-adjoint. Therefore,

$$\left\{ \frac{\langle \phi_1, L_2 \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} \right\} = \lambda_1 \lambda_1^* = \beta_1 .$$

β_1 is the smallest eigenvalue of L_2. Thus ϕ_1 is an eigenfunction of L_2. From Lemma 5, we know that ϕ_1 is weakly-complex. Now we show that ϕ_1 is an eigenfunction of L_1. Since L_1 is compact, it has a complete set of orthogonal base $\mathrm{Span}_{i=1}^{\infty}\{ \psi_i \}$. So ϕ_1 can be represented in $\mathrm{Span}_{i=1}^{\infty}\{ \psi_i \}$. Also L_1 is closed on $\mathrm{Span}_{i=2}^{\infty}\{\psi_i\}$ and L_2 is closed on $\mathrm{Span}_{i=2}^{\infty}\{ \psi_i \}$. It follows that for any scalar λ, $S(\lambda) = (\lambda^2 + \lambda L_1 + L_2)$ is a closed operator on $\mathrm{Span}_{i=2}^{\infty}\{ \psi_i \}$ and $\mathrm{Span}\{ \phi_1 \}$. Consider the second smallest

$$\lambda_2 \lambda_2^* = \beta_2.$$

Let ϕ_2 be the eigenfunction of $S(\lambda_2)$, i.e. $S(\lambda_2)\ \phi_2 = 0$. Then ϕ_2 can be express ϕ_2 as

$$\phi_2 = a_1\ \phi_1 + a_2\ \Psi$$

where $\Psi \in \mathrm{Span}_{i=2}^{\infty}\{ \psi_i \}$. Let $S(\lambda_2)$ acts on ϕ_2.

$$S(\lambda_2)\phi_2 = S(\lambda_2)(a_1\phi_1 + a_2\Psi) = a_1 S(\lambda_2)\phi_1 + a_2 S(\lambda_2)\Psi = 0$$

Since ϕ_1 is orthogonal to Ψ, we have $a_1 S(\lambda_2)\phi_1 = 0$ and $a_2\ S(\lambda_2)\Psi = 0$. But

$$S(\lambda_2)\phi_1 \neq 0$$

because ϕ_1 is not an eigenfunction associated with λ_2. Therefore we have $a_1 = 0$ and

$$\phi_2 = a_2\Psi \ . \tag{20}$$

From Equation (20), we know that $\phi_2 \in \mathrm{Span}_{i=2}^{\infty}\{ \psi_i \}$, so we can repeat the same procedure for the case

$$S(\lambda_2)\ \phi_2 = 0$$

and show that ϕ_2 is weakly-complex as was shown for ϕ_1. This eventually gives the proof that all ϕ_i are weakly-complex.

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