THE EVALUATION OF YOUNG'S COMPLEX MODULUS OF VISCOELASTIC MATERIALS

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The motion of a root-excited cantilever beam coated on both sides with a vibration-damping viscoelastic material is investigated.

Measurements of relative amplitudes and phase lag between the free and driven ends of the beam are used to characterize the variation of the viscoelastic material's complex Young's modulus with frequency. Effects of other parameters such as temperature or humidity on the complex modulus can be evaluated provided that tests are done in an environmental chamber.

The method is readily applicable to beams with coatings of viscoelastic material of equal thickness on both sides and also to any beam manufactured out of a single, self-supporting material. It is efficient, fast and accurate. It is a valuable alternative to the ASTM E756-83 "Standard Method for Measuring Vibration-Damping Properties of Materials."

Key words: vibration, viscoelastic, damping, complex, modulus, experimental, measurement.

INTRODUCTION

Over the years, numerous methods were developed for the purpose of evaluating the damping characteristics of non-self-supporting viscoelastic materials. A large share of those methods are used to evaluate extensional damping properties with the Young's complex modulus approach. This is explained by the fact that simpler analytical developments and experimental rigs are required.

Van Oort [1] and Oberst and Frankenfeld [2,3] first studied the behaviour of thin fixed-free beams coated with viscoelastic materials on one, or both sides (Van Oort only). The relationships derived by Van Oort, because of their assumptions, are not used for the investigation of vibration damping materials having a high loss factor. The work by Oberst and Frankenfeld was aimed at vibration damping materials having a low elastic modulus E and a high loss factor η. The so-called *Oberst beam method* has since been generally accepted and is now standardized by the DIN [4] and the ASTM [5].

Schwarzl analyzed, in a more rigourous manner, vibrations of beams made up of two viscoelastic materials [6]. He concluded that Van Oort and Oberst and Frankenfeld theories were simplified versions of his own approach because, in their assumptions, they had neglected the effects of coupling between flexural and extensional motions. He also pointed out that, for an asymmetric beam made of two materials having different loss factors, the neutral fibre was moving through the cross-section at a frequency twice that of lateral vibrations.

Ross, Kerwin and Ungar's [7] analysis was developed for a three-layer system and included extensional and shear type damping treatments for plates as well as for beams. In the special case of unconstrained damping treatment of a beam (zero thickness of third layer), RKU equations simplify to those reported by Oberst and Frankenfeld.

Nashif [8] developed a method using the Oberst apparatus [9] with a metal supporting beam coated on both sides with equals thicknesses of a viscoelastic material. Bending problems at high -or low- temperature caused by the large diference between thermal coefficient of expansion of viscoelastic materials and metals were eliminated. Surprisingly, no mention to Schwarzl's [6] neutral fibre movement conclusion was made by Nashif to further justify the use of symmetric specimens.

The ASTM has published a "Standard Method for Measuring Vibration-Damping Properties of Materials" [5] which is based on the equations proposed by Oberst and Frankenfeld, Nashif and Ross, Kerwin and Ungar. Unfortunately, these methods contain a number of assumptions that prevent them from being generally applicable:

- The damping effects of the supporting material are neglected;
- Eigenvalues equations are derived without considering the effects of damping (added stiffness, phase lag, etc.);
- The global loss factor is calculated with methods that were developed for lightly damped, single degree of freedom systems (half power bandwidth or logarithmic decrement).

To eliminate the restrictions of existing methods, an approach based on the study of lateral vibrations of root-excited cantilever beams is proposed. It is an extension of the work on damping properties of rigid polymers by Ostiguy and Evan-Iwanowski [10], Horio and Onogi [11], Bland and Lee [12] and Strella [13].

THEORETICAL ANALYSIS

The use of a root-excited beam enables one to use both amplitude and phase lag measurements for the characterization of damping. The viscoelastic material's elastic modulus and loss factor can be determined from experimental measurements, without using any approximations or assumptions. This proves valuable particularly for materials having high loss factors. Symmetric test sections are used so that the neutral fibre is remains in the geometric center of the cross-section and that no thermally induced bending occurs.

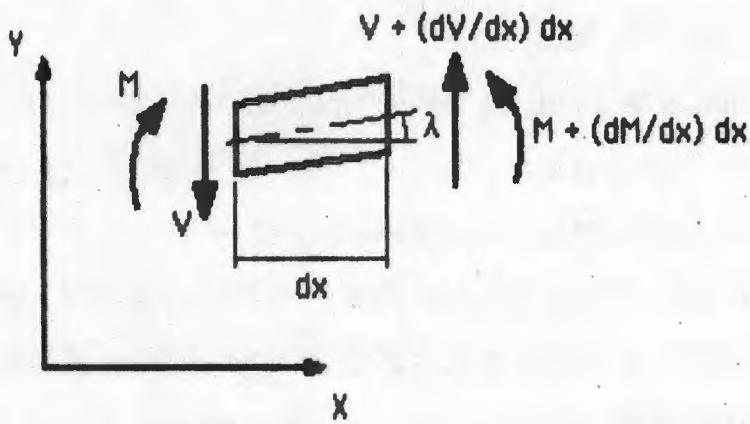

Figure 1: Free body diagram of differential element

A free body diagram of a differential element of length *dx* of a beam is shown in figure 1. By equating the forces in the **Y** direction to the corresponding inertia force and by summing the moments about the element's center of gravity, we obtain the standard equation of motion for Euler beams (shear and rotatory inertia effects neglected)

$$m\,\partial^2 y/\partial t^2 + \partial^2 M/\partial x^2 = 0 \qquad (1)$$

where m is the mass per unit length. From the classical theory of pure bending of beams, the bending moment is related to the lateral motion through the flexural rigidity term. This equation can be applied to viscoelastic materials by replacing the standard elastic Young's modulus E by the complex modulus E*. We then have

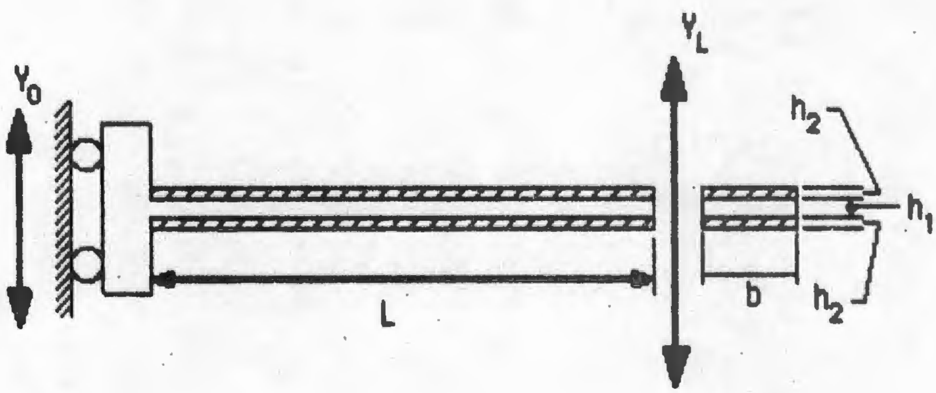

Figure 2: Test beam geometry and test layout

$$M = E^* I \, \partial^2 y/\partial x^2 \quad (2)$$

Combining equations (1) and (2) we get

$$m \, \partial^2 y/\partial t^2 + (E^* I) \, \partial^4 y/\partial x^4 = 0 \quad (3)$$

For the beam shown in the test section schematic view (figure 2), mass, stiffness and inertia properties are

$$m = \rho_1 S_1 + \rho_2 S_2 = \rho_1 b h_1 + 2\rho_2 b h_2 \quad (4)$$

$$E^* I = E_1(1+i\eta_1) I_1 + E_2(1+i\eta_2) I_2 \quad (5)$$

$$I_1 = b h_1^3/12 \quad (6)$$

$$I_2 = b h_1^2 h_2/2 + b h_1 h_2^2 + 2 b h_2^3/3 \quad (7)$$

With the use of the following parameters

$$H = h_2/h_1 \quad (8)$$

$$K_H = 6H + 12H^2 + 8H^3 \quad (9)$$

equations (3) to (7) give

$$12\,(\rho_1 + 2\rho_2 H)\,\partial^2 y/\partial t^2 +$$
$$h_1^2\,\{E_1(1+i\eta_1) + K_H\,E_2(1+i\eta_2)\}\,\partial^4 y/\partial x^4 = 0 \qquad (10)$$

By separating space and time solutions and by defining the **q** parameter as

$$q^4 = \frac{12\,(\rho_1 + 2\rho_2 H)\,\omega^2}{h_1^2\,[(E_1 + K_H\,E_2) + i(\eta_1 E_1 + K_H\,\eta_2 E_2)]} \qquad (11)$$

we have

$$d^4Y/dx^4 - q^4\,Y = 0 \qquad (12)$$

For the beam shown in figure 2, the boundary conditions are

$$Y = Y_0 \qquad dY/dx = 0 \qquad \text{at } x = 0;$$
$$d^2Y/dx^2 = 0 \qquad d^3Y/dx^3 = 0 \qquad \text{at } x = L. \qquad (13)$$

Only the motion at the free end is of interest. It is found by solving equation (12) with the above boundary conditions and then putting **x=L**. Dividing by the motion at the driven end gives the ratio of amplitude **AR** and the phase lag $\boldsymbol{\theta}$ between the free and driven ends

$$AR\,e^{-i\theta} = \frac{\cos\Psi + \cosh\Psi}{1 + \cos\Psi\,\cosh\Psi} \qquad (14)$$

where

$$\Psi = \left[\frac{12(\rho_1 + 2\rho_2 H)\,\omega^2 L^4}{h_1^2\,[(E_1 + K_H E_2) + i(\eta_1 E_1 + K_H \eta_2 E_2)]} \right]^{1/4} = qL = \alpha + i\,\beta \qquad (15)$$

Equation (14) can be transformed into two functions of unknown parameters α and β by equating the real and imaginary parts on both sides of the equation. Once simplified, these two functions are

$$\begin{aligned} AR\,\{&1 + \cos\alpha \cos\beta \cosh\alpha \cosh\beta + \sin\alpha \sin\beta \sinh\alpha \sinh\beta\} \\ &- \cos\theta\,\{\cos\alpha \cosh\beta + \cosh\alpha \cos\beta\} \\ &- \sin\theta\,\{\sin\alpha \sinh\beta - \sinh\alpha \sin\beta\} = 0 \end{aligned} \qquad (16)$$

and

$$\begin{aligned} AR\,\{&\cos\alpha \sin\beta \sinh\alpha \cosh\beta - \sin\alpha \cos\beta \cosh\alpha \sinh\beta\} \\ &+ \cos\theta\,\{\sin\alpha \sinh\beta - \sinh\alpha \sin\beta\} \\ &- \sin\theta\,\{\cos\alpha \cosh\beta + \cosh\alpha \cos\beta\} = 0 \end{aligned} \qquad (17)$$

These non-linear equations are solved numerically by a Newton-Raphson scheme [14]. Reasonably close starting values (α_0, β_0) are required. For that purpose, we define the following two parameters

$$A = \frac{h_1^2\,(E_1 + K_H E_2)}{12\,(\rho_1 + 2\rho_2 H)} \qquad (18)$$

and

$$B = \frac{h_1^2 (\eta_1 E_1 + K_H \eta_2 E_2)}{12 \omega (\rho_1 + 2\rho_2 H)} \tag{19}$$

Equation (12) then becomes

$$(A + i\omega B)\, d^4Y/dx^4 - \omega^2 Y = 0 \tag{20}$$

which is identical to equation (10) in Strella's paper [13]. With the current symbols, equations (30), (31) and (32) of Strella become

$$A = \left[\frac{(16\, \omega_r L^2)}{(16\, a_0^2 - F^2)}\right]^2 \tag{21}$$

$$B = \frac{F L^2}{a_0^3} \frac{(16\, \omega_r L^2)}{(16\, a_0^2 - F^2)} \tag{22}$$

$$F = \frac{-5.478 + 2\sqrt{7.502 + 6.15\, AR^2}}{1.689\, AR^2} \tag{23}$$

where ω_r is a resonant frequency, a_0 is the eigenvalue of the equivalent mode number for a fixed-free beam (a_0=1.875, 4.694, 7.855, etc.) and AR is as previously defined. Approximate values for E_2 and η_2 are found with equations (18) and (19). These approximations are given by

$$E_{02} = \frac{12(\rho_1 + 2\rho_2 H)A}{K_H h_1^2} - \frac{E_1}{K_H} \qquad (24)$$

and

$$\eta_{02} = \frac{12(\rho_1 + 2\rho_2 H)\omega_r B}{K_H h_1^2 E_{02}} - \frac{\eta_1 E_1}{K_H E_{02}} \qquad (25)$$

We then obtain from equation (15)

$$\alpha_0 = \sqrt[4]{R}\ \cos(\phi/4)$$

$$\beta_0 = \sqrt[4]{R}\ \sin(\phi/4) \qquad (26)$$

where

$$R = \frac{12(\rho_1 + 2\rho_2 H)\omega_r^2 L^4}{h_1^2 \sqrt{[(E_1 + K_H E_{02})^2 + (\eta_1 E_1 + K_H \eta_{02} E_{02})^2]}} \qquad (27)$$

and

$$\phi = tg^{-1} - \frac{(\eta_1 E_1 + K_H \eta_{02} E_{02})}{(E_1 + K_H E_{02})} \qquad (28)$$

These starting values α_0 and β_0 are now used to iterate to the final solution for α and β with the Newton-Raphson method. The numerical value of the complex angle $\Psi = (\alpha + i\beta)$ is now known. Again rearranging equation (15), we obtain

$$(E_1 + K_H E_2) + i(\eta_1 E_1 + K_H \eta_2 E_2) = \frac{12(\rho_1 + 2\rho_2 H)\omega^2 L^4}{h_1^2(\alpha + i\beta)^4} \qquad (29)$$

The numerical values of elastic modulus E_2 and loss factor η_2 of the viscoelastic coating are found by equating the real and imaginary parts on both sides of equation (29). After simplifications, we have

$$E_2 = \frac{12(\rho_1 + 2\rho_2 H)\omega^2 L^4}{K_H h_1^2}\left[\frac{\alpha^4 + \beta^4 - 6\alpha^2\beta^2}{\alpha^8 + \beta^8 - 6\alpha^4\beta^4 - 4\alpha^2\beta^2(\alpha^4 + \beta^4)}\right] - \frac{E_1}{K_H} \qquad (30)$$

and

$$\eta_2 = \frac{12(\rho_1 + 2\rho_2 H)\omega^2 L^4}{K_H h_1^2}\left[\frac{4\alpha\beta(\beta^2 - \alpha^2)}{\alpha^8 + \beta^8 - 6\alpha^4\beta^4 - 4\alpha^2\beta^2(\alpha^4 + \beta^4)}\right] - \frac{\eta_1 E_1}{K_H E_2} \qquad (31)$$

To evaluate the complex modulus $E^*_2(i\omega)$ of a non-self-supporting viscoelastic material, the procedures outlined below must be followed.

1- record the following parameters, with appropriate units: L, ρ_1, h_1, E_1, η_1, ρ_2 and h_2;

2- evaluate H and K_H with equations (8) and (9);

3- record amplification **AR**, phase lag θ, resonant frequency ω_r and mode number so that Strella's approximate method can be used as a first approximation;

4- evaluate **A**, **B** and **F** as per equations (21), (22) and (23) with appropriate resonant frequency ω_r and eigenvalue a_0;

5- evaluate E_{o2} and η_{o2} with equations (24) and (25);

6- find starting values α_0 and β_0 with equations (26), (27) and (28);

7- iterate toward final values α and β;

8- evaluate elastic modulus E_2 and loss factor η_2 with equations (30) and (31).

CONCLUSIONS AND RECOMMENDATIONS

When testing a non-self-supporting material, the support beam can be manufactured out of a viscoelastic material because its own damping characteristics were carried throughout the derivation of the equations. For self supporting materials that can be shaped as a beam, the equations defined in this paper are simplified by eliminating all terms containing 2 as a subscript. The equations then become identical as those derived by Ostiguy and Evan-Iwanowski [10].

An experimental setup similar to those used by Ostiguy and Evan-Iwanowski [10] or Strella [13] is recommended. Strella's setup is particularly useful because it allows quick free length changes to be made. The length/thickness ratio should remain greater than 50 so that shear and inertia effects can be neglected. Non-contacting electro-optical or laser instrumentation should be used for amplitude and phase lag measurements. Tests should be done inside an environmental chamber to evaluate the effects of temperature, humidity, vacuum, etc. Frequency and temperature effects can be combined, with the use of a reduced frequency nomogram

[15, 16], to provide a complete description of damping properties of a material on a single chart.

The approach proposed in this paper allows one to evaluate quickly and precisely the Young's complex modulus of viscoelastic materials. Additional work is being done to adapt this method for complex shear modulus evaluation. The method can be used for any material, without any restriction. It is fast, accurate and its repeatability has been demonstrated [10]. It brings significant improvements over existing test methods.

NOMENCLATURE

a_0	eigenvalues for a clamped-free beam
A, B	parameters defined by equations (18) and (19)
AR	amplitude ratio of free vs driven end
b	beam width (m)
E	elastic modulus, real part of E* (N/m^2)
E_0	approximate value of E (N/m^2)
E*	Young's complex modulus (N/m^2)
F	parameter defined in reference [13]
G*	complex shear modulus (N/m^2)
h	thickness (m)
H	thickness ratio
i	unit imaginary number ($i^2 = -1$)
I	area moment of inertia (m^4)
K_H	$= 6H + 12H^2 + 8H^3$
L	free length of beam (m)
m	mass per unit length (kg/m)
M	bending moment (N.m)
qL	complex frequency parameter
R	parameter defined by equation (27)
S	cross-section (m^2)
t	time (s)
V	shear force (N)
x	station along beam (m)

$y(x,t)$	transverse displacement of beam (m)
$Y(x)$	vibration amplitude (m)
Y_0	vibration amplitude at driven end (m)
Y_L	vibration amplitude at free end (m)
α, β	real and imaginary parts of Ψ
α_0, β_0	approximate values of α and β
η	loss factor
η_0	approximate value of η
λ	angular deformation (rad)
θ	phase lag between free and driven ends (rad)
ρ	density (kg/m^3)
ϕ	angle defined by equation (28) (rad)
Ψ	complex angle (rad)
ω	circular frequency of vibration (rad/s)
ω_r	resonant frequency (rad/s)
1,2	subscript for beam materials

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