Modal Analysis of Kelvin Viscoelastic Solids Under Arbitrary Excitation: Circular Plates under Moving Loads

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ABSTRACT

The response of a finite, inhomogeneous, Kelvin viscoelastic solid under arbitrary excitation is determined by modal analysis. Through the reciprocal theorem of viscoelasticity, vibration modes of the Kelvin viscoelastic solid satisfy orthogonality conditions and the system response under any excitation is represented in a modal series. This formulation technique is illustrated on an asymmetric, classical, circular plate containing Kelvin inclusions excited by a constant transverse force rotating at constant speed. The viscosity of the inclusions suppresses the instability excited at supercritical speed in the elastic plate, but it may or may not suppress instability excited at subcritical speed depending on the geometry and location of the Kelvin inclusions.

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1. Introduction

Viscoelastic components are often introduced to elastic structures to suppress excessive vibration and to reduce noise level produced by the structures [1-3]. Instead of adding additional damping material to elastic structures, which is common in damping design such as tuned dampers [4,5] and surface treatments [6,7], viscoelastic components can be introduced as inclusions in the structure [8]. In addition, the location and geometry of the viscoelastic inclusions can be specified to significantly strain the inclusions during particular vibration modes.

In an earlier study [8] eigenfunctions and Green's function have been determined for a three dimensional, finite, elastic solid with Kelvin viscoelastic inclusions through an integral equation and a perturbation iteration method. The purposes of this paper are to develop the orthogonality of the eigenfunctions of the viscoelastic solid and to present the response of the solid to arbitrary excitation.

Following the viscoelastic reciprocal theorem [9], eigenfunctions of the Kelvin viscoelastic solid satisfy orthogonality conditions in a state space representation. Eigenfunction expansion of the response in a modal series then discretizes an action integral whose stationarity governs the response of the viscoelastic solid under arbitrary excitation. Stationarity of the action integral and the state space orthogonality conditions give a set of decoupled equations governing the generalized coordinates of the modal series.

This technique is illustrated on an asymmetric, classical, circular plate containing viscoelastic inclusions excited by a constant transverse force rotating at constant speed. The steady state response of the plate is obtained through the modal analysis.

2. Orthogonality

Consider an inhomogeneous, isotropic, Kelvin viscoelastic solid with Lamé distributions $\lambda(\mathbf{r})$, $\mu(\mathbf{r})$, density distribution $\rho(\mathbf{r})$, and damping distributions $\lambda^*(\mathbf{r})$, $\mu^*(\mathbf{r})$. The solid occupies a three dimensional domain τ with zero displacements on the boundary σ_1 and vanishing traction on boundary σ_2. The complex-valued eigenfunction $\boldsymbol{\psi}(\mathbf{r}) \equiv [\psi^{(1)}(\mathbf{r}), \psi^{(2)}(\mathbf{r}), \psi^{(3)}(\mathbf{r})]^T$ and eigenvalue ν satisfy

$$\frac{d}{dx_j}[\sigma_{ij}(\boldsymbol{\psi}(\mathbf{r}),\nu;\lambda,\mu,\lambda^*,\mu^*)] = \nu^2\rho(\mathbf{r})\psi^{(i)}(\mathbf{r}), \quad i=1,2,3 \tag{1}$$

with boundary conditions

$$\psi(\mathbf{r}) = 0, \quad \text{on } \sigma_1 \tag{2a}$$

$$\sigma_{ij}(\boldsymbol{\psi}(\mathbf{r}),\nu;\lambda,\mu,\lambda^*,\mu^*)n_j = 0, \quad \text{on } \sigma_2, \quad i=1,2,3 \tag{2b}$$

where

$$\sigma_{ij}(\mathbf{u},\nu;\lambda,\mu,\lambda^*,\mu^*) = \lambda\delta_{ij}\varepsilon_{kk}(\mathbf{u})+2\mu\varepsilon_{ij}(\mathbf{u}) + \nu\,[\lambda^*\delta_{ij}\varepsilon_{kk}(\mathbf{u})+2\mu^*\varepsilon_{ij}(\mathbf{u})] \tag{3}$$

In addition, the divergence theorem [9] gives

$$\int_{\sigma_2}\sigma_{ij}(\mathbf{u},\nu;\lambda,\mu,\lambda^*,\mu^*)n_j u'_i\, d^2\mathbf{r} - \int_\tau \frac{d}{dx_j}[\sigma_{ij}(\mathbf{u},\nu;\lambda,\mu,\lambda^*,\mu^*)]u'_i\, d^3\mathbf{r}$$

$$= \int_\tau I(\mathbf{u},\mathbf{u}';\lambda,\mu)\, d^3\mathbf{r} + \nu \int_\tau I(\mathbf{u},\mathbf{u}';\lambda^*,\mu^*)\, d^3\mathbf{r} \tag{4}$$

with

$$I(\mathbf{u},\mathbf{u}';\lambda,\mu) = \int_\tau [\lambda\varepsilon_{kk}(\mathbf{u})\varepsilon_{ll}(\mathbf{u}')+2\mu\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{u}')]\, d^3\mathbf{r} \tag{5}$$

where $\mathbf{u}(\mathbf{r})e^{\nu t}$ and $\mathbf{u}'(\mathbf{r})e^{\nu t}$ are harmonic displacement fields vanishing on σ_1, and $\varepsilon_{ij}(\cdot)$ is infinitesimal strain.

Replace the unprimed and primed $\mathbf{u}$ in (4) by $\psi_m(\mathbf{r})$ and $\psi_n(\mathbf{r})$. Since eigenfunctions satisfy (1) and (2a,b), (4) implies

$$-\nu_m^2 \int_\tau \rho(\mathbf{r})\psi_m\cdot\psi_n\, d^3\mathbf{r} = \int_\tau I(\psi_m,\psi_n;\lambda(\mathbf{r}),\mu(\mathbf{r}))\, d^3\mathbf{r} + \nu_m \int_\tau I(\psi_m,\psi_n;\lambda^*(\mathbf{r}),\mu^*(\mathbf{r}))\, d^3\mathbf{r} \tag{6a}$$

and

$$-\nu_n^2 \int_\tau \rho(\mathbf{r})\psi_n\cdot\psi_m\, d^3\mathbf{r} = \int_\tau I(\psi_n,\psi_m;\lambda(\mathbf{r}),\mu(\mathbf{r}))\, d^3\mathbf{r} + \nu_n \int_\tau I(\psi_n,\psi_m;\lambda^*(\mathbf{r}),\mu^*(\mathbf{r}))\, d^3\mathbf{r} \tag{6b}$$

Subtract (6a) from (6b), apply the symmetry of I, and normalize

$$(\nu_m+\nu_n)\int_\tau \rho(\mathbf{r})\psi_m\cdot\psi_n\, d^3\mathbf{r} + \int_\tau I(\psi_m,\psi_n;\lambda^*(\mathbf{r}),\mu^*(\mathbf{r}))\, d^3\mathbf{r} = \delta_{mn} \tag{7a}$$

Multiply (7a) by ν_n and add to (6b)

$$-\nu_m\nu_n \int_\tau \rho(\mathbf{r})\psi_m\cdot\psi_n\, d^3\mathbf{r} + I(\psi_m,\psi_n;\lambda(\mathbf{r}),\mu(\mathbf{r}))\, d^3\mathbf{r} = -\nu_n\,\delta_{mn}, \quad (n \text{ no sum}) \tag{7b}$$

Orthonormality (7a,b) can also be written in a compact form

$$<\Phi_m,\Phi_n>_{\mathbf{A}} = \int_\tau \Phi_m^T \mathbf{A}\Phi_n\, d^3\mathbf{r} = \delta_{mn} \tag{8a}$$

$$<\Phi_m,\Phi_n>_{\mathbf{B}} = \int_\tau \Phi_m^T \mathbf{B}\Phi_n\, d^3\mathbf{r} = -\nu_n\,\delta_{mn} \tag{8b}$$

where Φ_m, $\mathbf{A}$, and $\mathbf{B}$ are

$$\Phi_m = \begin{bmatrix} \nu_m\psi_m \\ \psi_m \end{bmatrix} \tag{9a}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \rho(\mathbf{r}) \\ \rho(\mathbf{r}) & I(\cdot,\cdot;\lambda^*(\mathbf{r}),\mu^*(\mathbf{r})) \end{bmatrix} \tag{9b}$$

$$\mathbf{B} = \begin{bmatrix} -\rho(\mathbf{r}) & 0 \\ 0 & I(\cdot,\cdot;\lambda(\mathbf{r}),\mu(\mathbf{r})) \end{bmatrix} \tag{9c}$$

The first and second entries of I in (9b,c) operate on the premultiplied and postmultiplied functions, respectively.

3. Response Under Arbitrary Excitations

The response $\mathbf{w}(\mathbf{r},t)$ of the solid under arbitrary excitation $\mathbf{f}(\mathbf{r},t)$ satisfies stationarity of the following action

$$\delta J = \int_{t_1}^{t_2} \left[\delta T - \delta V + \delta W_D + \delta W_F \right] dt \tag{10}$$

where

$$\delta T = \int_\tau \rho(\mathbf{r})\dot{\mathbf{w}}(\mathbf{r},t)\delta\dot{\mathbf{w}}(\mathbf{r},t)\,d^3\mathbf{r} \tag{11a}$$

$$\delta V = \int_\tau I(\mathbf{w},\delta\mathbf{w};\lambda(\mathbf{r}),\mu(\mathbf{r}))\,d^3\mathbf{r} \tag{11b}$$

$$\delta W_D = -\int_\tau I(\dot{\mathbf{w}},\delta\mathbf{w};\lambda^*(\mathbf{r}),\mu^*(\mathbf{r}))\,d^3\mathbf{r} \tag{11c}$$

$$\delta W_F = \int_\tau \mathbf{f}(\mathbf{r},t)\cdot\delta\mathbf{w}(\mathbf{r},t)\,d^3\mathbf{r} \tag{11d}$$

are the variations in the kinetic and strain energies plus the virtual work done in the viscoelastic material and by the external load. By (11a-d), (10) can be rewritten as

$$\delta J = \int_{t_1}^{t_2} \left\{ \int_\tau \rho(\mathbf{r})\frac{d}{dt}\left[\dot{\mathbf{w}}\delta\mathbf{w}\right]d\tau - \left[<\dot{\Psi},\delta\Psi>_{\mathbf{A}} + <\Psi,\delta\Psi>_{\mathbf{B}}\right] + \int_\tau \mathbf{F}\cdot\delta\Psi\,d\tau \right\} dt \tag{12}$$

where

$$\Psi(\mathbf{r},t) = \begin{bmatrix} \dot{\mathbf{w}}(\mathbf{r},t) \\ \mathbf{w}(\mathbf{r},t) \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 0 \\ \mathbf{f}(\mathbf{r},t) \end{bmatrix} \tag{13}$$

If the eigenfunctions $\Phi_m(\mathbf{r})$ in (9a) are complete, then $\Psi(\mathbf{r},t)$ allows a representation as an eigenfunction expansion

$$\Psi(\mathbf{r},t) = \sum_{m=1}^{\infty} q_m(t)\Phi_m(\mathbf{r}) \tag{14}$$

Substitute (14) into (12) and recall the orthogonality (8a,b) to obtain

$$\delta J = \int_\tau \int_{t_1}^{t_2} \rho(\mathbf{r})\frac{d}{dt}\left[\dot{\mathbf{w}}\delta\mathbf{w}\right]dt\,d\tau - \sum_{n=1}^{\infty}\int_{t_1}^{t_2}\left[(\dot{q}_n - \nu_n q_n - Q_n)\delta q_n\right]dt \tag{15}$$

where

$$Q_n(t) = \int_\tau \Phi_n^T(\mathbf{r})\cdot\mathbf{F}\,d^3\mathbf{r} = \int_\tau \psi_n^T(\mathbf{r})\cdot\mathbf{f}(\mathbf{r},t)\,d^3\mathbf{r} \tag{16}$$

The stationarity of J, $\delta J = 0$, and $\delta q_n(t_1) = \delta q_n(t_2) = 0$, $n = 1, 2, \ldots$, imply that

$$\dot{q}_n(t) - \nu_n q_n(t) = Q_n(t), \quad n = 1, 2, 3, \cdots \tag{17a}$$

with the initial condition

$$q_n(0) = <\Phi_n(\mathbf{r}),\Psi(\mathbf{r},0)>_{\mathbf{A}}, \quad n = 1, 2, 3, \cdots \tag{17b}$$

The complete response is then

$$\mathbf{w}(\mathbf{r},t) = \sum_{n=1}^{\infty} q_n(t)\psi_n(\mathbf{r}) = \sum_{n=1}^{\infty}\left[<\Phi_n(\mathbf{r}),\Psi(\mathbf{r},0)>_{\mathbf{A}} e^{\nu_n t} + \int_0^t e^{\nu_n(t-\tau)}Q_n(\tau)d\tau\right]\psi_n(\mathbf{r}) \tag{18}$$

4. Applications to Asymmetric Circular Plates

The steady state response of a stationary, classical, asymmetric, viscoelastic circular plate under a rotating force is determined by modal analysis. Classical plate theory requires

$$\mathbf{w}(\mathbf{r},t) = w(\mathbf{r},t)\hat{k}, \quad \mathbf{f}(\mathbf{r},t) = f(\mathbf{r},t)\hat{k}$$

where $\hat{k}$ is the unit vector normal to the middle surface of the plate. The eigenfunctions are

$$\psi_{mn}(\mathbf{r})\hat{k} \quad \text{and} \quad \overline{\psi}_{mn}(\mathbf{r})\hat{k}\ , \quad m=0,1,\ldots, \quad n=0,\pm1,\cdots$$

where $\psi_{mn}(\mathbf{r})$ and $\psi_{m,-n}(\mathbf{r})$ and their complex conjugates (denoted by the overbar) are four orthonormal complex eigenfunctions of the plate with m nodal circles and n nodal diameters. Therefore, the plate response is

$$w(\mathbf{r},t) = \sum_{\alpha=1}^{2}\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} \psi_{mn}^{(\alpha)}(\mathbf{r})\, q_{mn}^{(\alpha)}(t) \tag{19}$$

where $\psi_{mn}^{(1)}(\mathbf{r}) \equiv \psi_{mn}(\mathbf{r})$ and $\psi_{mn}^{(2)}(\mathbf{r}) \equiv \overline{\psi}_{mn}(\mathbf{r})$.

For a unit concentrated force rotating along a circle $r=r_0$ at constant speed Ω;

$$f(\mathbf{r},t) = \frac{1}{r_0}\,\delta(r-r_0)\,\delta(\theta-\Omega t) \tag{20}$$

the modal response is

$$\dot{q}_{mn}^{(\alpha)}(t) - \nu_{mn}^{(\alpha)}\, q_{mn}^{(\alpha)}(t) = \psi_{mn}^{(\alpha)}(r_0, \Omega t) \tag{21}$$

Because $\psi_{mn}^{(\alpha)}(r,\theta)$ is periodic in θ

$$\psi_{mn}^{(\alpha)}(r,\theta) = \sum_{p=-\infty}^{\infty} a_{mn}^{(\alpha)}(r;p)\, e^{ip\theta}\ , \tag{22}$$

and the steady state $q_{mn}^{(\alpha)}(t)$ is

$$q_{mn}^{(\alpha)}(t) = \sum_{p=-\infty}^{\infty} \frac{a_{mn}^{(\alpha)}(r_0;p)}{ip\,\Omega-\nu_{mn}^{(\alpha)}}\, e^{ip\Omega t} \tag{23}$$

Resonance occurs when $\nu_{mn} = ip\,\Omega$ and $a_{mn}^{(\alpha)}(r_0;p)$ is nonzero.

The average strain energy of vibration is

$$<E_s> = \frac{1}{T}\int_0^T\!\!\int_A I(w,\overline{w};\lambda,\mu)\, dA\, dt\ , \quad T = \frac{2\pi}{\Omega} \tag{24}$$

Substitute (19) and (23) into (24) to obtain

$$<E_s> = -\sum_{\alpha\beta mnkl} \int_A I(\psi_{mn}^{(\alpha)}, \overline{\psi}_{kl}^{(\beta)};\lambda,\mu)\, dA \left[\sum_{p=-\infty}^{\infty} \frac{a_{mn}^{(\alpha)}(r_0;p)\,\overline{a}_{kl}^{(\beta)}(r_0;p)}{(ip\,\Omega-\nu_{mn}^{(\alpha)})\,(ip\,\Omega+\overline{\nu}_{kl}^{(\beta)})}\right] \tag{25}$$

where $\sum_{\alpha\beta mnkl} \equiv \sum_{\alpha=1}^{2}\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{\beta=1}^{2}\sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty}$. Similarly, total dissipation per cycle is

$$<E_d> = \int_0^T\!\!\int_A I(\dot{w},\dot{\overline{w}};\lambda^*,\mu^*)\, dA\, dt \tag{26}$$

Substitute (19) and (22) into (26) to obtain

$$<E_d> = -\sum_{\alpha\beta mnkl} \int_A I(\psi_{mn}^{(\alpha)}, \overline{\psi}_{kl}^{(\beta)};\lambda^*,\mu^*)\, dA \left[\sum_{p=-\infty}^{\infty} \frac{2\pi\Omega p^2\, a_{mn}^{(\alpha)}(r_0;p)\,\overline{a}_{kl}^{(\beta)}(r_0;p)}{(ip\,\Omega-\nu_{mn}^{(\alpha)})\,(ip\,\Omega+\overline{\nu}_{kl}^{(\beta)})}\right] \tag{27}$$

An averaged loss factor $<\eta>$ is [7]

$$<\eta> = \frac{<E_d>}{<E_s>} \tag{28}$$

As a numerical illustration of the moving load instability, consider an uniform, elastic, circular plate with three evenly spaced radial inclusions. Each inclusion spans an angle 0.035 rad ($\approx 2°$) and extends from $r = 0.75b$ and $r = b$, where b is the outer radius of the plate. The inclusions considered are elastic or viscoelastic. Material properties of the plate and the inclusions, and plate eigenvalues are described in Table 1. Eigenfunctions $\psi_{mn}(\mathbf{r})$ of the plate are calculated by the method of perturbation iteration [8, 10].

Figure 1 shows the average strain energy of the asymmetric plate. The thin lines are for elastic, and the thick lines are for viscoelastic inclusions. When the inclusions are elastic, resonances occur at $\frac{\Omega}{\omega_{cr}} = \frac{\beta_{mn}}{n}$, where ω_{cr} is the critical speed of the axisymmetric plate without the inclusions and β_{mn} is a natural frequency of the asymmetric plate in Table 1. Increasing modal damping (Table 1) results in greater amplitude reduction in Fig. 1. Figure 2 shows the strain energy of the plate at subcritical speed. Subcritical speed resonances, which do not exist in axisymmetric plates [11], do occur here at rotation speeds

$$\frac{\Omega}{\omega_{cr}} = \frac{\beta_{mn}}{|3j \pm n|}, \quad j = \pm 1, \pm 2, \cdots \tag{29}$$

because [10]

$$a_{mn}^{(\alpha)}(r_0; p) \begin{cases} \neq 0, & p = 3j \pm n, \quad j = \pm 1, \pm 2, \cdots \\ = 0, & \text{Otherwise} \end{cases} \tag{30}$$

The resonance around $\Omega \approx 0.704\omega_{cr}$ is caused by $\psi_{00}(\mathbf{r})$, and the ones near $\Omega \approx 0.592\,\omega_{cr}$ and $\Omega \approx 0.829\,\omega_{cr}$ are caused by the repeated modes $\psi_{04}(\mathbf{r})$ and $\psi_{0,-4}(\mathbf{r})$. The subcritical speed instability may or may not be suppressed by damping in the inclusions depending on the modal damping ζ_{mn}. The resonance by $\psi_{00}(\mathbf{r})$ is slightly suppressed because of the minimal modal damping in $\psi_{00}(\mathbf{r})$ (cf. Table 1).

5. Conclusions

1. For an inhomogeneous, isotropic, Kelvin viscoelastic solid, eigenfunctions $\psi_m(\mathbf{r})$ and $\psi_n(\mathbf{r})$ satisfy orthonormality conditions (8a,b).

2. A lateral force rotating at constant speed will excite asymmetric plates containing elastic or viscoelastic inclusions to subcritical speed resonances that do not exist in axisymmetric plates. The occurrence of the subcritical resonances depends on the plate asymmetry and can be predicted analytically.

3. The viscosity of the inclusions may or may not suppress the subcritical resonances depending on the geometry and location of the inclusions. The suppression of resonances can be predicted by the modal damping of each vibration mode.

Table 1 - Normalized Eigenvalues of a Circular Plate with Three Inclusions

Normalized natural frequencies:

$$\omega_{mn} = \frac{\overline{\omega}_{mn}}{\omega_{cr}} \text{ (axisymmetric plates)}, \ \beta_{mn} = \frac{\overline{\beta}_{mn}}{\omega_{cr}} \text{ (asymmetric plates)}$$

Modal damping ratio: ζ_{mn}

Plate: fixed at inner rim at $0.5b$, free at outer rim at b; $\frac{W_0}{D_0} = 0.7$.†

Inclusions extend from $r = 0.75b$ to $r = b$; $\varepsilon = 0.035$.

For elastic inclusions: $\frac{\rho'_0}{\rho_0} = \frac{D'_0}{D_0} = \frac{W'_0}{W_0} = 0.5$.†

For viscoelastic inclusions: $D_0^* = 0.00489$, $\frac{W_0^*}{D_0^*} = 0.7$; $\frac{\rho'_0}{\rho_0} = \frac{D'_0}{D_0} = \frac{W'_0}{W_0} = 0.5$.‡

(m,n)	Mode	No Inclusions	With Three Inclusions		
			Elastic †	Viscoelastic ‡	
		ω_{mn} †	β_{mn}	ζ_{mn}	β_{mn}
(0,0)	axisym.	2.1050	2.1121	4.5481×10^{-4}	2.1122
(0,1)	cos	2.1479	2.1549	4.9882×10^{-4}	2.1549
	sin	2.1479	2.1549	4.9882×10^{-4}	2.1549
(0,2)	cos	2.3764	2.3842	8.1266×10^{-4}	2.3843
	sin	2.3764	2.3842	8.1266×10^{-4}	2.3843
(0,3)	cos	3.0000	3.0212	1.2677×10^{-3}	3.0215
	sin	3.0000	2.9927	2.8305×10^{-3}	2.9931
(0,4)	cos	4.1368	4.1429	4.1766×10^{-3}	4.1443
	sin	4.1368	4.1429	4.1766×10^{-3}	4.1443

† Converted from Table 1 of [10]
‡ Converted from Table 1 of [8]

References

1. Nashif, Ahid D., Jones, David I. G., and Henderson, John P., *Vibration Damping*, John Wiley & Sons, Inc., New York, NY., 1985.
2. Rogers, Lynn, "Vibration Damping as a Design Parameter," *Mechanical Engineering*, vol. 108, pp. 66-69, Jan. 1986.
3. Drake, Michael L., "Passive Damping: Has Its Time Finally Come," *Mechanical Engineering*, vol. 107, pp. 20-24, Dec. 1985.
4. Yamaguchi, H. and Saito, H., "Vibrations of Beams with an Absorber Consisting of a Viscoelastic Solid and a Beam," *Earthquake Engineering and Structural Dynamics*, vol. 12, pp. 467-479, 1984.
5. Snowdon, J. C., "Platelike Dynamic Vibration Absorbers," *Journal of Engineering for Industry*, vol. 97(1), pp. 88-93, Feb. 1975.
6. El-Raheb, M. and Wagner, P., "Damped Response of Shells by a Constrained Viscoelastic Layer," *Journal of Applied Mechanics*, vol. 53, pp. 902-908, Dec. 1986.
7. Torvik, P. J., "The analysis and Design of Constrained Layer Damping Treatments," in *Damping Applications for Vibration Control*, ed. Peter J. Torvik, pp. 85-112, ASME, New York, NY., 1980.
8. Shen, I. Y. and Mote, C. D. Jr., "Dynamic Analysis of Finite, Three Dimensional, Linear Elastic Solids with Kelvin Viscoelastic Inclusions: Theory with Application to Asymmetrically Damped Circular Plates," *The Journal of the the Acoustical Society of America*. (Submitted)
9. Christensen, R. M., *Theory of Viscoelasticity*, Academic Press, Inc., New York, N.Y., 1982.
10. Shen, I. Y. and Mote, C. D. Jr., "Dynamic Analysis of Three Dimensional, Finite, Linear Elastic Solids Containing Small Elastic Imperfections: Theory with Application to Asymmetric Circular Plates," *Journal of Sound and Vibration*. (Submitted)
11. Mote, C. D. Jr., "Stability of Circular Plates Subjected to Moving Loads," *Journal of the Franklin Institute*, vol. 290(4), pp. 329-344, October 1970.

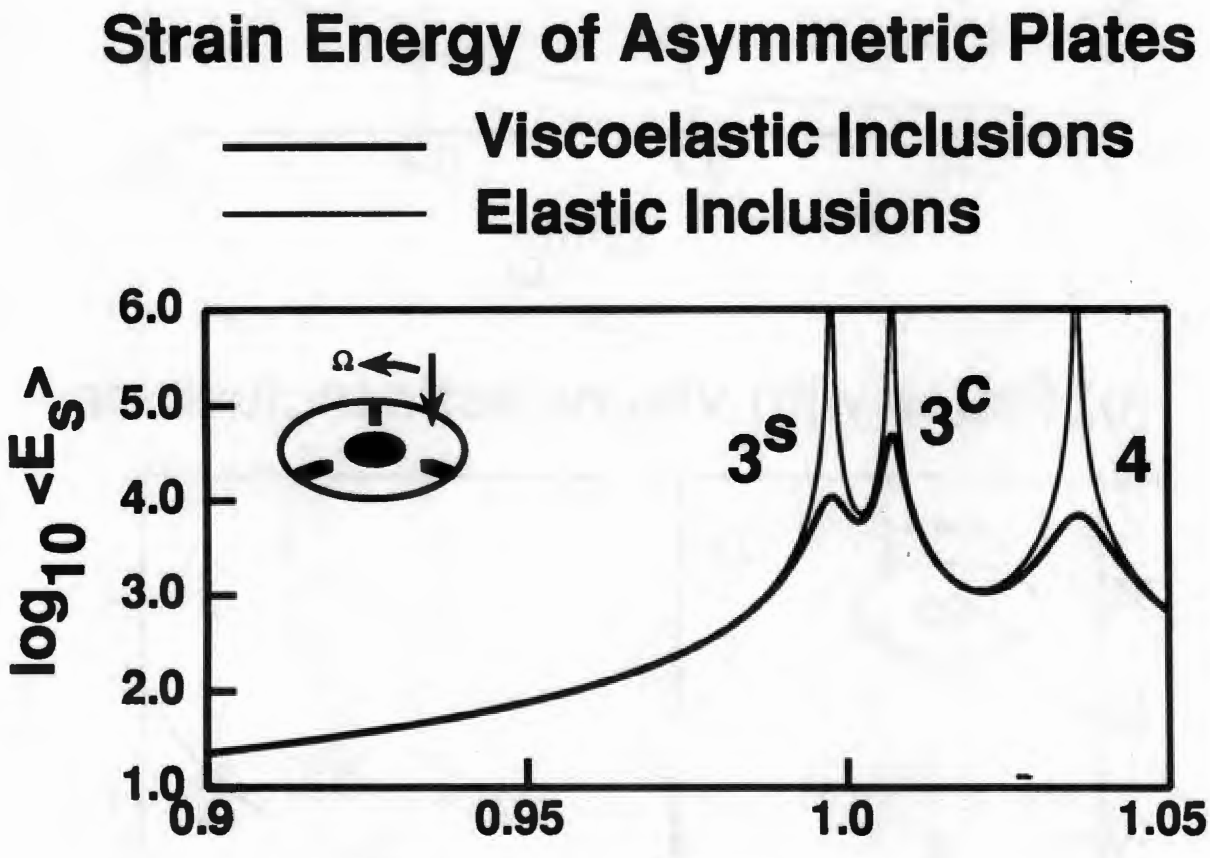
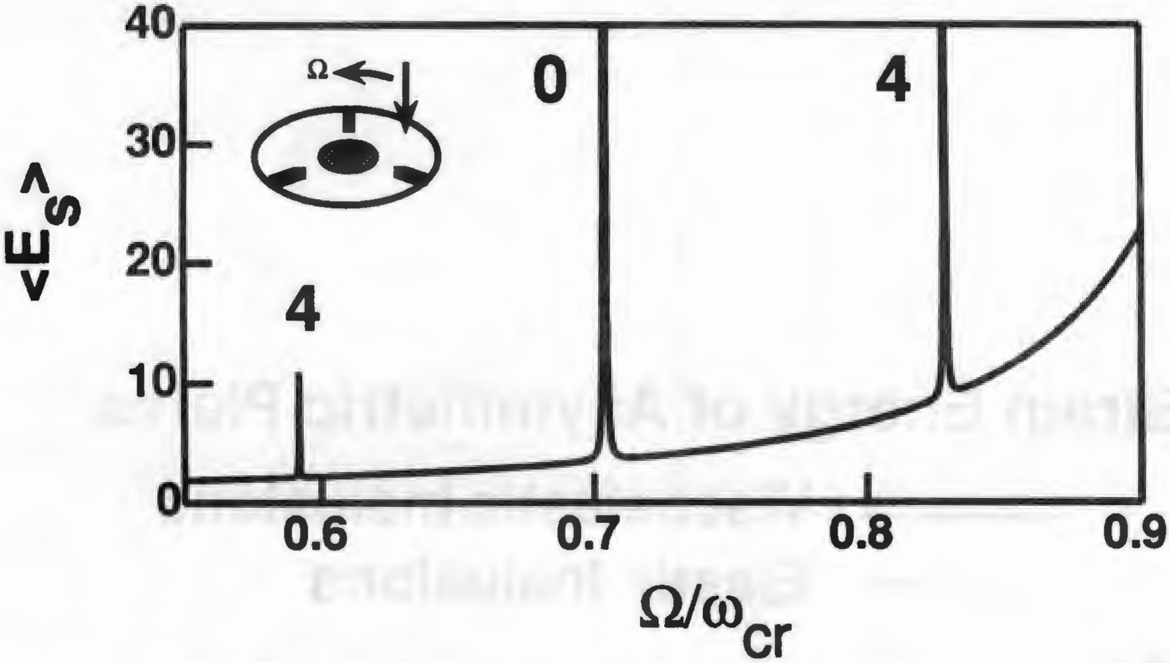


Fig. 1 Average strain energy of asymmetric plates with three elastic or viscoelastic inclusions excited by a rotating force at supercritical speed

Strain Energy of Asymmetric Plates

(a) Plates with Elastic Inclusions



(b) Plates with Viscoelastic Inclusions

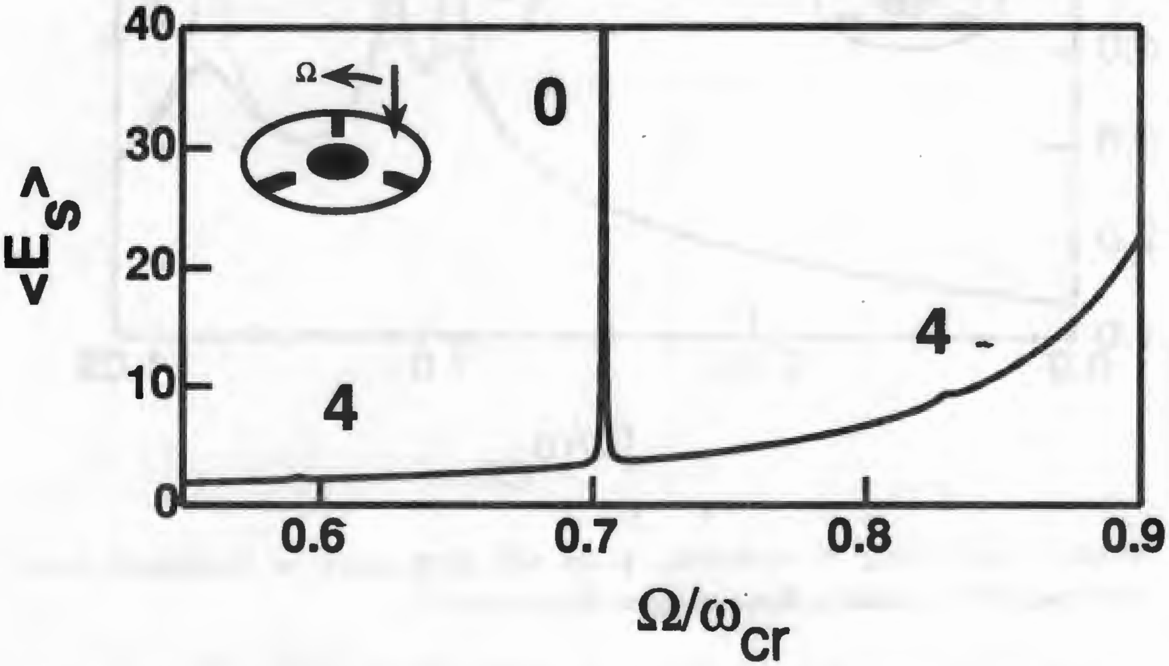


Fig. 2 Average strain energy of asymmetric plates excited by a rotating force at subcritical speed; (a) plate with elastic inclusions, and (b) plate with viscoelastic inclusions