SECTION 7

AN IMPROVED ESTIMATE FOR THE ERROR OF TRUNCATION FOR AN INFINITE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT: Many of the partial differential equations arising from physical problems may be reduced to an infinite system of ordinary differential equations. An approximate solution of the original partial differential equation can be obtained from the infinite system by truncation. In this section some explicit estimates for the error of this type of approximation are obtained. An extension of Gronwall's Lemma is also proved. This section is an extension of a previous paper by the author and P. K. C. Wang, "Estimates for Truncation Errors of Infinite Dimensional Systems of Linear Ordinary Differential Equations," IBM Research Report, 1964.

7.1 INTRODUCTION

Many of the partial differential equations arising from physical problems may be separated into space-dependent and time-dependent parts, where the latter assumes the form of an infinite system of ordinary differential equations. An approximate solution to the original partial differential equation may be obtained from the infinite system by truncation.

Some estimates of the error of this type of approximation were obtained for certain classes of infinite systems of linear ordinary differential equations, in a previous paper by the author and P. K. C. Wang (Section 6). In the present paper these estimates are refined and extended.

Use of the familiar Gronwall's Lemma in the previous paper suggested the more general theorem, i.e.,

Theorem A : Let $A(t)$, $B(t)$, $C(t)$, $D(t)$, $P_1(t)$, and $P_2(t)$ be non-negative and twice continuously differentiable functions in the interval $[0, T]$. Let $U_1(t)$ and $U_2(t)$ be continuously differentiable functions in $[0, T]$, and, in this interval, also satisfy

$$U_1(t) \le P_1(t) + \int_0^t [A(\tau)\, U_1(\tau) + B(\tau)\, U_2(\tau)]\, d\tau, \qquad \text{and}$$

$$U_2(t) \le P_2(t) + \int_0^t [C(\tau)\, U_1(\tau) + D(\tau)\, U_2(\tau)]\, d\tau.$$

Then,

$$U_1(t) \le P_1(t) + \int_0^t [f_1(\tau) + g_1(\tau)]\, d\tau, \qquad \text{and}$$

$$U_2(t) \le P_2(t) + \int_0^t [f_2(\tau) + g_2(\tau)]\, d\tau,$$

where f_1 and f_2 depend on P_1, A, and B, while g_1 and g_2 depend only on P_2, C, and D.

This theorem appears to be new, but its proof will be deferred to a separate paper, since a special case of this theorem is sufficient in this paper. (See Lemma I.)

Section 7.1 contains the derivation of bounds for the error of truncation. Knowledge about the solution of the truncated system can be used to sharpen bounds.

Section 7.2 shows that our results, applied to the system discussed in 7.1 yield very precise error estimates.

7.2 ERROR ESTIMATES

Let there be given a denumerably infinite system of first order ordinary differential equations of the form,

$$\frac{dx_n(t)}{dt} + a_n x_n(t) = \sum_{m=1}^{\infty} a_{nm} x_m(t), \qquad n = 1, 2, \ldots, \tag{7.2-1}$$

$$x_n(0) = X_n^o ,$$

where the a_n and a_{nm} are constants and $a_n \geq -p^2 > -\infty$. Clearly we may take $a_{nn} = 0$. Let a solution to (7.2-1) exist and be denoted by

$$X(t) = (x_1(t), x_2(t), \ldots) .$$

Further, for $0 < t < T \leq \infty$, assume that $X(t) \in \ell^1$ (or ℓ^2). Also let the Nth order truncated system, corresponding to (7.2-1), be

$$\frac{dY_{Nn}(t)}{dt} + a_n Y_{Nn}(t) = \sum_{m=1}^{N} a_{nm} Y_{Nm}(t), \quad n = 1, 2, \ldots, N,$$

$$Y_{Nn}(0) = X_n^o , \tag{7.2-2}$$

where the a_n, a_{nm}, and X_n^o are the same as in (7.2-1), and let the solution to (7.2-2) be denoted by

$$Y_N(t) = (Y_{N1}(t), Y_{N2}(t), \ldots, Y_{NN}(t)) .$$

We define the error of truncation to be the difference between the solutions of (7.2-1) and (7.2-2), i.e.,

$$E(t) = (e_1(t), e_2(t), \ldots) ,$$

$$\text{where } e_n(t) = \begin{cases} x_n(t) - Y_{Nn}(t), & n = 1, 2, \ldots, N , \\ x_n(t), & n > N . \end{cases} \tag{7.2-3}$$

Our purpose here is to derive estimates for the magnitude of $E(t)$, which depend on the order N of truncation. The systems of differential equations, which we are considering, are derived from certain kinds of partial differential equations. Thus we must choose a measure of the magnitude of $E(t)$ which can be related to the original partial differential equation. There are two obvious choices which arise quite naturally. The first is an ℓ^2 type,

$$\|E\|_2 = \left[\sum_{n=1}^{\infty} |e_n|^2 \right]^{1/2} ,$$

which directly yields an estimate of the error of the solution of the original partial differential equation. The second is an ℓ^1 type,

$$\|E\|_1 = \sum_{n=1}^{\infty} |e_n| ,$$

which can be compared with the ℓ^2 measure, i.e., if $\|E\|_1 \leq 1$, then $\|E\|_1 \geq \|E\|_2$.

The components of the error $E(t)$ are of two kinds. The first kind, $e_n(t)$ for $n > N$, represents, so to speak, the tail of the solution, which the truncation process simply cuts off. The second kind, $e_n(t)$ for $1 \leq n \leq N$, represents the effect of the coupling of the first N components of the solution with the aforementioned tail. We will, for this reason, derive separate error estimates for each of these kinds of error.

From equations (7.2-1), (7.2-2), and (7.2-3), we obtain a system of differential equations satisfied by $E(t)$, so that

$$\left[\begin{array}{ll} \dfrac{dE_n(t)}{dt} + a_n E_n(t) = \begin{cases} \displaystyle\sum_{m=1}^{\infty} a_{nm} E_m(t), & n = 1,2,\ldots,N, \\ \displaystyle\sum_{m=1}^{\infty} a_{nm} E_m(t) + \sum_{m=1}^{N} a_{nm} Y_{Nm}(t), & n > N, \end{cases} \\ E_n(0) = \begin{cases} 0, & n = 1,2,\ldots,N, \\ X_n^o, & n > N. \end{cases} \end{array} \right. \tag{7.2-4}$$

We rewrite equation (7.2-4) as an equivalent system of integral equations,

$$E_n(t) = \begin{cases} \displaystyle\int_o^t \left[\sum_{m=1}^{N} a_{nm} E_m(\tau) + \sum_{m>N} a_{nm} E_m(\tau) \right] \exp\left[-a_n(t-\tau)\right] d\tau, \\ \qquad\qquad n = 1,2,\ldots,N, \quad (7.2\text{-}5) \\ X_n^o \exp(-a_n t) + \displaystyle\int_o^t \left[\sum_{m=1} a_{nm} Y_{Nm}(\tau) \right] \exp\left[-a_n(t-\tau)\right] d\tau_o + \end{cases}$$

$$+\int_0^t \Big[\sum_{m=1}^{N} a_{nm}E_m(\tau) + \sum_{m>N} a_{nm}E_m(\tau)\Big] \exp[-a_n(t-\tau)]\, d\tau,$$

$$n > N\,.$$

We define, for each of the two kinds of error, a pair of norms as follows:

$$\|E\|_{i,N} = \Big[\sum_{n=1}^{N} |E_n|^i\Big]^{1/i}, \quad \|E\|_{i,-N} = \Big[\sum_{n>N}^{\infty} |E_n|^i\Big]^{1/i}, \quad i = 1,2.$$

In order to continue we must impose upon system (7.2-1), the requirement: either

$$\sum_{n=1}^{\infty} \sup_m |a_{nm}| < \infty, \qquad \text{for } i = 1 \tag{7.2-6a}$$

or

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm}^2 < \infty\,, \qquad \text{for } i = 2\,. \tag{7.2-6b}$$

With conditions (7.2-6a,b), we obtain, for i = 1, 2,

$$\|E\|_{i,N} \exp(k_N t) \le \int_0^t [A_1^i \|E\|_{i,N} + A_2^i \|E\|_{i,-N}] \exp(k_N\tau)\, d\tau, \tag{7.2-7a}$$

$$\|E\|_{i,-N} \exp(k_{-N}t) \le \|X^o\|_{i,-N} + \int_0^t A_3^i \|Y_N\|_{i,N} \exp(k_{-N}\tau)\, d\tau +$$

$$+ \int_0^t [A_3^i \|E\|_{i,N} + A_4^i \|E\|_{i,-N}] \exp(k_{-N}\tau)\, d\tau, \tag{7.2-7b}$$

where $k_N = \min\limits_{n\le m} a_n$, $\qquad k_{-N} = \inf\limits_{n>N} a_n$,

$$A_1^1 = \sum_{n=1}^{N} \max_{m\le N} |a_{nm}|, \qquad A_1^2 = \Big[\sum_{n=1}^{N} \sum_{m=1}^{N} |a_{nm}|^2\Big]^{1/2},$$

$$A_3^1 = \sum_{n>N}^{\infty} \max_{m\le N} |a_{nm}|, \qquad A_3^2 = \Big[\sum_{n>N}^{\infty} \sum_{m=1}^{N} |a_{nm}|^2\Big]^{1/2},$$

$$A_2^1 = \sum_{n=1}^{N} \sup_{m>N} |a_{nm}|, \qquad A_2^2 = \Big[\sum_{n=1}^{N} \sum_{m>N}^{\infty} |a_{nm}|^2\Big]^{1/2},$$

$$A_4^1 = \sum_{n>N}^{\infty} \sup_{m>N} |a_{nm}|, \qquad A_4^2 = \Big[\sum_{n>N}^{\infty} \sum_{m>N}^{\infty} |a_{nm}|^2\Big]^{1/2}.$$

(Conditions (7.2-6a,b) insure uniform convergence of the summations which arise in the derivation of (7.2-7a,b) from (7.2-5), allowing interchange of summations and integrations.)

At this point, we require:

Lemma I. Let A, B, C, and D be non-negative constants, and $P_1(t)$, and $P_2(t)$ be non-negative and continuously differentiable functions in $0 \le t \le T \le \infty$. Let $U_1(t)$ and $U_2(t)$ be continuous functions in $[0, T]$ and in this interval satisfy

$$U_1(t) \le P_1(t) + \int_0^t [A\, U_1(\tau) + B\, U_2(\tau)\, e^{\alpha\tau}]\, d\tau ,$$

$$U_2(t) \le P_2(t) + \int_0^t [C\, U_1(\tau)\, e^{-\alpha\tau} + D\, U_2(\tau)]\, d\tau .$$

Then,

$$U_1(t) \le P_1(t) + \int_0^t [k_1 P_1(\tau) + k_2 P_2(\tau)\, e^{\alpha\tau}] \exp[(1/2)(A+D+\alpha+2\lambda)(t-\tau)] d\tau ,$$

$$U_2(t) \le P_1(t) + \int_0^t [k_3 P_1(\tau)\, e^{-\alpha\tau} + k_4 P_2(\tau)] \exp[(1/2)(A+D-\alpha+2\lambda)(t-\tau)]\, d\tau,$$

where

$$\lambda = [1/2][(\alpha + D - A)^2 + 4BC]^{1/2} ,$$

$$k_1 = \max [A,\ (2BC - A(\alpha + D - A))/2\lambda] ,$$

$$k_2 = \max [B,\ (2BD - B(\alpha + D - A))/2\lambda] ,$$

$$k_3 = \max [C,\ (2AC + C(\alpha + D - A))/2\lambda] ,$$

$$k_4 = \max [D,\ (2BC + D(\alpha + D - A))/2\lambda] ,$$

The proof of Lemma I will be found in the Appendix.

Since $\|E\|_{i,N} + \|E\|_{i,-N} < \infty$ and is continuous, we may apply Lemma I to inequalities (7.2-7a,b) to obtain

$$\|E\|_{i,N} \le k_{i,2} R_i(t) \exp(-k_{-N} t) , \tag{7.2-8a}$$

$$\|E\|_{i,-N} \le [P_{i,2}(t) + k_{i,4} R_i(t)] \exp(-k_{-N} t) , \tag{7.2-8b}$$

where

$$R_i(t) = \int_o^t P_{i,2}(\tau) \exp(\beta_i(t-\tau))\, d\tau ,$$

$$P_{i,2}(t) = \|X^o\|_{i,-N} + \int_o^t A_3^i \ \|Y_N\|_{i,N} \exp(k_{-N}\tau)\, d\tau ,$$

$$k_{i,2} = A^i{}_2 \max(1, [2A_4^i - (\alpha_N + A_4^i - A_1^2)]/2\lambda_i) ,$$

$$k_{i,4} = \max(A_4^i, [2A_2^i A_3^i + A_4^i (\alpha_N + A_4^i - A_1^i)]/2\lambda_i) ,$$

$$\beta_i = [1/2]\, [A_1^i + A_4^i - \alpha_N + 2\lambda_i] ,$$

$$\alpha_N = k_N - k_{-N} ,$$

$$\lambda_i = [(1/4)(\alpha_N + A_4^i - A_1^i)^2 + A_2^i A_3^i]^{1/2} .$$

Using the definitions above, inequality (7.2-8b) can take the form,

$$\|E\|_{i,-N} \le [\ \|X^o\|_{i,-N} + A_3^i \int_o^t \|Y_N\|_{i,N} \exp(k_{-N}\tau)\, d\tau] \{1 + $$

$$+ [k_{i,4} (1-\exp(\beta_i t))]/\beta_i \} [\exp(-k_{-N} t)] . \tag{7.2-8b'}$$

In those instances where the solution, $Y_N(t)$, of the truncated system is available, or where an estimate of $\|Y_N\|_{i,N}$ is available, inequalities (7.2-8a) and (7.2-8b') provide bounds for the errors of truncation. If neither is known, then we may use an estimate for $\|Y_N\|_{i,N}$, which we obtain using the method of Bandy and Wang. (Section 6.3, theorem 1), i.e.,

$$\|Y_N\|_{i,N} \quad \|X^o\|_{i,N} \exp(t(A_1^i - k_N)) . \tag{7.2-9}$$

This yields an estimate for $P_{i,2}$:

$$P_{i,2}(t) \le \|X^o\|_{i,-N} + \|X^o\|_{i,N} A_3^i (A_1^i - \alpha_N)^{-1} [\exp(t(A_1^i - \alpha_N))-1]. \tag{7.2-10}$$

Using (7.2-9) and (7.2-10) in (7.2-8a) and (7.2-8b'), we obtain estimates which involve only the coefficients and initial conditions of (7.2-1) :

$$\left[\begin{array}{l} \|E\|_{i,N} \le \{ \|X^o\|_{i,-N} R_1^i(t) + \\ \qquad\qquad + \|X^o\|_{i,N} A_3^i (R_2^i - R_1^i)/(A_1^i - \alpha_N - \beta_i)\} k_{i,2} \exp(-k_{-N}t) , \\ \|E\|_{i,-N} \le [\|X^o\|_{i,-N} + \|X^o\|_{i,N} A_3^i R_2^i(t)] [1 + k_{i,4} R_1^i(t)] \exp(-k_{-N}t) , \end{array}\right.$$

where

$$R_1^i(t) = [\exp(\beta_i t) - 1]/\beta_i ,$$

$$R_2^i(t) = [\exp(t(A_1^i - \alpha_N)) - 1]/[A_1^i - \alpha_N] .$$

Conditions 7.2-6a, b) and the condition that the a_n are bounded from below imply first, that the $R_j^i(t) \exp(-k_{-N}t)$ for $i,j = 1,2$ may be bounded by expressions independent of N and finite for all finite t; second, that the A_1^i for $i = 1,2$ are bounded for all N; and third, that A_2^i, A_3^i, A_4^i, $k_{i,2}$, and $k_{i,4}$ for $i = 1,2$ go to zero as $N \to \infty$. Thus for all fixed and finite t, if the initial condition is bounded in ℓ^1, both $\|E\|_{i,N}$ and $\|E\|_{i,-N}$ go to zero as $N \to \infty$.

7.3 EXAMPLE

In Section 6.1, the example,

$$\left[\begin{array}{lll} \dfrac{da_1}{dt} + a_1 \pi^2/4 = 0 , & & \\ \dfrac{da_n}{dt} + a_n \pi^2 n^2 = -(-1)^n 3a_1 \pi^2/4n , & & n = 2,3, \ldots, \\ a_n(0) = [(-1)^{n+1}]/[n\pi(4n^2-1)] , & & n = 1,2, \ldots, \end{array}\right.$$

was considered. We will now calculate the error obtained by truncating this system after the Nth equation. Condition (7.2-6a) is not valid here, so we use condition (7.2-6b), i.e.,

$$\sum_{n=2}^{\infty} [(-1)^n 3\pi^2 (1/4n)]^2 = 9\pi^4 (1/16) \sum_{n=2}^{\infty} (1/n)^2$$

$$= 3\pi^4 (1/32)(\pi^2 - 6) < \infty .$$

We can easily compute the various constants required for estimates (7.2-8a) and (72-8b'):

$$k_{2,2} = 0\,, \qquad k_{2,4} = 0\,, \qquad k_{-N} = \pi^2 (N+1)^2\,,$$

$$P_{2,2}(t) = (1/\pi)\,[\sum_{n=N+1}^{\infty} [n\,(4n^2-1)]^{-2}\,]^{1/2} +$$

$$+ (3\pi^2/4)\,[\sum_{n=N+1}^{\infty} n^{-2}\,]^{1/2} \int_0^t \|\tilde{a}_N\|_{2,N} \exp(\pi^2 (N+1)^2\,\tau)\,d\tau\,,$$

where $\tilde{a}_N$ denotes the solution to the truncated system. The error estimate than is

$$\|E\|_{2N} = 0\,, \tag{7.3-1a}$$

$$\|E\|_{2,-N} = P_{2,2}(t) \exp(-\pi^2 (N+1)^2\, t)\,. \tag{7.3-1b}$$

We can solve the truncated system and compute an exact value for

$\|\tilde{a}_N\|_{2,N}$:

$$\|\tilde{a}_N\|_{2,N} = (1/\pi)\,[\sum_{n=1}^{N} [n\,(4n^2-1)]^{-2}]^{1/2} \exp(\pi^2 t(-1/4))\,. \tag{7.3-2}$$

From (7.3-1a), (7.3-1b), and (7.3-2) we obtain the estimate for error,

$$\|E\|_{2,N} = 0\,, \tag{7.3-3a}$$

$$\begin{aligned}\|E\|_{2,-N} \le {} & [(1/\pi)\exp(-\pi^2 (N+1)^2\, t)]\{[\sum_{n=N+1}^{\infty} [n\,(4n^2-1)]^{-2}]^{1/2} \\ & -3\,[4\,(N+1)^2 - 1]^{-1}[\sum_{n=N+1}^{\infty} n^{-2}]^{1/2}\,[\sum_{n=1}^{N} [n(4n^2-1)]^{-2}]^{1/2}\} \\ & +3\,[4\pi(N+1)^2 - \pi^{-1}\{[\sum_{n=N+1}^{\infty} n^{-2}]\,[\sum_{n=1}^{N} (4n^3-n)^{-2}]\}^{1/2} \exp(-\pi^2 t/4)\,.\end{aligned} \tag{7.3-3b}$$

The first term (i.e. within the first brace) of equation (7.3-3b) is in fact negative and therefore may be discarded, so we have as our estimate of the error of truncation:

$$\|E\|_{2,N} = 0 , \tag{7.3-4}$$

$$\|E\|_{2,-N} \leq 3[4\pi(N+1)^2-\pi]^{-1}\{ \sum_{n=N+1}^{\infty} n^{-2}] [\sum_{n=1}^{N} (4n^3-n)^{-2}]\}^{1/2} \exp(-\pi^2 t/4).$$

Comparing (7.3-4) with the exact result, i.e.,

$$\|E\|_{2,N} = 0 , \tag{7.3-5}$$

$$\|E\|_{2,-N} = [1/\pi] [\sum_{n=N+1}^{\infty} (4n^3-n)^{-2}]^{1/2} \exp(-\pi^2 t/4) ,$$

we note that the exponential parts of both (7.3-4) and (7.3-5) are the same, and the constant factors are nearly equal. (For $N = 1$, this difference, which is just the first term of (7.3-3b), is approximately 0 .00590.)

7.4 APPENDIX A. The Proof of Lemma I.

We are given,

$$U_1(t) \le P_1(t) + \int_0^t [A\, U_1(\tau) + B\, U_2(\tau)\, e^{\alpha\tau}]\, d\tau\,,$$

$$U_2(t) \le P_2(t) + \int_0^t [C\, U_1(\tau)\, e^{-\alpha\tau} + D\, U_2(\tau)]\, d\tau\,,$$

for $0 \le t \le T \le \infty$, with the conditions on U_1, U_2, P_1, P_2, A, B, C, and D as given in Section 7.1. Further let

$$\begin{aligned}
\bar{U}_1(t) &= U_1(t) - P_1(t),\\
\bar{U}_2(t) &= U_2(t) - P_2(t),\\
\bar{P}_1(t) &= A\, P_1(t) + B\, P_2(t)\, e^{\alpha t},\\
\bar{P}_2(t) &= C\, P_1(t)\, e^{-\alpha t} + D\, P_2(t)\,, \qquad (7.4\text{-}1)\\
W_1(t)e^{At} &= \int_0^t [A\, U_1(\tau) + B\, U_2(\tau)\, e^{\alpha\tau} + \bar{P}_1(\tau)]\, d\tau\,,\\
W_2(t)e^{Dt} &= \int_0^t [C\, U_1(\tau)\, e^{-\alpha\tau} + D\, U_2(\tau) + \bar{P}_2(\tau)]\, d\tau\,,\\
a &= \alpha + D - A, \qquad \lambda^2 = a^2/4 + BC.
\end{aligned}$$

It is easy to show that

$$W_1(t)e^{At} \ge \bar{U}_1(t)\,, \qquad W_1(0) = 0\,,$$

$$W_2(t)e^{Dt} \ge \bar{U}_2(t), \qquad W_2(0) = 0\,,$$

and

$$\begin{aligned}
\frac{dW_1(t)}{dt} &\le \bar{P}_1(t)e^{-At} + B\, W_2(t)\, e^{at}\,,\\
\frac{dW_2(t)}{dt} &\le \bar{P}_2(t)e^{-Dt} + C\, W_1(t)e^{-at}\,.
\end{aligned} \qquad (7.4\text{-}2)$$

We may decouple inequalities (7.4-2) to obtain

$$\frac{dW_1(t)}{dt} \leq \bar{P}_1(t)e^{-At} + B\,e^{at}\int_0^t \bar{P}_2\,(\tau)\,e^{-D\tau}\,d\tau + BCe^{at}\int_0^t W_1(\tau)e^{-a\tau}d\tau\,,$$

(7.4-3)

$$\frac{dW_2(t)}{dt} \leq \bar{P}_2(t)e^{-Dt} + Ce^{-at}\int_0^t \bar{P}_1(\tau)e^{-A\tau}\,d\tau + BCe^{-at}\int_0^t W_2(\tau)e^{+a\tau}d\tau\,,$$

Let $W_3(t)$ and $W_4(t)$ be solutions to

$$\frac{d[W_3(t)e^{at/2}]}{dt} = \bar{P}_1(t)\,e^{-At} + B\,e^{at}\int_0^t \bar{P}_2(\tau)\,e^{-D\tau}\,d\tau$$

$$+ BCe^{at}\int_0^t W_3(\tau)\,e^{-a\tau/2}d\tau\,,\quad W_3(0) = 0\,,$$

$$\frac{d[W_4(T)e^{-at/2}]}{dt} = \bar{P}_2(t)\,e^{-Dt} + Ce^{-at}\int_0^t \bar{P}_1(\tau)\,e^{-A\tau}\,d\tau \qquad (7.4\text{-}4)$$

$$+ BCe^{-at}\int_0^t W_4(\tau)e^{a\tau/2}d\tau,\quad W_4(0) = 0\,.$$

Equations (7.4-4) become, after differentiation,

$$\frac{d^2W_3(t)}{dt^2} - \lambda^2\,W_3(t) = e^{at/2}\;\frac{d[\bar{P}_1(t)e^{-(a+A)\,t}]}{dt} + B\bar{P}_2(t)e^{-t(D-a/2)}\,,$$

$$W_3(0) = 0\,,\quad \frac{dW_3(0)}{dt} = \bar{P}_1\,(0)\,,$$

$$\frac{d^2W_4(t)}{dt^2} - \lambda^2W_4(t) = e^{-at/2}\;\frac{d[\bar{P}_2(t)e^{(a-D)t}]}{dt} + C\bar{P}_1(t)e^{-t(A+a/2)}\,,$$

$$W_4(0) = 0,\quad \frac{dW_4(0)}{dt} = \bar{P}_2(0)\,.$$

Solving,

$$W_3(t) = (1/\lambda)\{\bar{P}_1\,(0)\,\sinh\;\lambda t\,+$$

$$+\int_0^t e^{a\tau/2}\,[\sinh\,\lambda(t-\tau)]\,[(\bar{P}_1(\tau)e^{-(a+A)\tau}) + B\,\bar{P}_2(\tau)\,e^{-D\tau}]\,d\tau\}\,,$$

(7.4-5)

$$W_3(t) = (1/\lambda)\{\bar{P}_2(O)\sinh\lambda t +$$

$$+\int_o^t e^{-a\tau/2}[\sinh\lambda(t-\tau)][(\bar{P}_2(\tau)e^{(a-D)\tau}) + C\bar{P}_1(\tau)e^{-A\tau}]\,d\tau\}.$$

Equations (7.4-5), after elimination of $\bar{P}_1$ and $\bar{P}_2$ through use of equations (7.4-1) and simplification, take the form,

$$W_3(t) \leq e^{\lambda t}\int_o^t [k_1 P_1(\tau)e^{-\alpha\tau/2} + k_2 P_2(\tau)e^{\alpha\tau/2}]e^{-(D+A+2\lambda)\tau/2}d\tau,$$

$$W_4(t) \leq e^{\lambda t}\int_o^t [k_3 P_1(\tau)e^{-\alpha\tau/2} + k_4 P_2(\tau)e^{\alpha\tau/2}]e^{-(D+A+2\lambda)\tau/2}d\tau,$$

where k_1, k_2, k_3, and k_4 are as given in the statement of Lemma I, and equations (7.4-6) are valid for all t, for $0 \leq t < T$.

Comparing $W_1(t)$ and $W_2(t)$ with $W_3(t)$ and $W_4(t)$ respectively, we see from equations (7.4-3) and (7.4-4),

$$\frac{d[W_1(t) - W_3(t)e^{at/2}]}{dt} \leq BCe^{at}\int_o^t [W_1(\tau) - W_3(\tau)e^{a\tau/2}]e^{-a\tau}d\tau,$$

$$W_1(0) - W_3(0) = 0,$$

$$\frac{d[W_2(t) - W_4(t)e^{-at/2}]}{dt} \leq BCe^{-at}\int_o^t [W_2(\tau) - W_4(\tau)e^{-a\tau/2}]e^{a\tau}d\tau,$$

$$W_2(0) - W_4(0) = 0.$$

By Lemma II (See Appendix B),

$$W_1(t) - W_3(t)e^{at/2} \leq 0,$$

$$W_2(t) - W_4(t)E^{-at/2} \leq 0, \qquad \text{for } 0 \leq t < T.$$

Thus,

$$U_1(t) \leq P_1(t) + W_3(t)\, e^{(A + D + \alpha) t/2} \quad ,$$

$$U_2(t) \leq P_2(t) + W_4(t)\, e^{(A + D - \alpha) t/2} \qquad (7.4\text{-}7)$$

and finally, from equations ((7.4-7) and (7.4-6), we obtain

$$U_1(t) \leq P_1(t) + \int_0^t [k_1 P_1(\tau) + k_2 P_2(\tau)\, e^{\alpha\tau}]\, e^{(D + A + \alpha + 2\lambda)(t-\tau)/2} d\tau,$$

$$U_2(t) \leq P_2(t) + \int_0^t [k_3 P_1(\tau)\, e^{-\alpha\tau} + k_4 P_2(\tau)]\, e^{(D + A + \alpha + 2\lambda)(t-\tau)/2} d\tau.$$

7.5 APPENDIX B. Lemma II.

Let U(t) and $\frac{d}{dt}$ U(t) be continuous functions for $0 \leq t < T \leq \infty$ and satisfy there

$$\frac{d}{dt} U(t) \leq K e^{\alpha t} \int_0^t U(\tau)\, e^{-\alpha\tau}\, d\tau \text{ , where } \alpha \text{ and } K \text{ are constants.}$$

If $U(0) \leq 0$ and $K > 0$, then $U(t) \leq 0$ for $0 \leq t < T$.

Proof. Set

$$V(t)\, e^{\alpha t/2} = K e^{\alpha t} \int_0^t U(\tau)\, e^{-\ \tau} d\tau .$$

It is easy to show that

$$\frac{d^2}{dt^2} V(t) - \lambda^2 V(t) = -f^2(t) \leq 0 ,$$

$$V(0) = 0 , \qquad \frac{d}{dt} V(0) \leq 0 , \qquad \lambda^2 = \alpha^2/4 + K ,$$

for some function f(t) . Solving for V(t),

$$V(t) = (1/\lambda) \sinh \lambda t \frac{d}{dt} V(0) - \int_0^t [\sinh \lambda(t-\tau)]\, f^2(\tau)\, d\tau \leq 0 .$$

Thus,

$$\frac{d}{dt} U(t) \leq V(t)\, e^{\alpha t/2} \leq 0 ,$$

$$U(t) \leq U(0) \leq 0 .$$