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STRESS WAVES IN A LAMINATED MEDIUM GENERATED BY TRANSVERSE FORCES

L. E. VOELKER and J. D. ACHENBACH
The Technological Institute
Northwestern University
Evanston, Illinois 60201

FOREWORD

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W. J. Trapp

W. J. TRAPP
Chief, Strength and Dynamics Branch
Metals and Ceramics Division
Air Force Materials Laboratory

Abstract

A laminated medium composed of alternating layers of two homogeneous isotropic elastic solids is suddenly subjected to a spatially uniform distribution of transverse forces, which are applied in a plane normal to the layering. The resulting two-dimensional transient wave propagation problem is analyzed by means of modal analysis. The normal and shear stresses at the interfaces are expressed as infinite integrals which are integrated numerically for not too large values of time. For larger values of time, the integrals are approximated by the method of stationary phase. The predominant contribution to the interface shear stress comes from the head-of-the-pulse approximation. The normal stress at the interface, which is composed of several contributions, is oscillatory, and the interface bonds may thus be subjected to tensile stresses.

This abstract is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of the Metals and Ceramics Division (MAMD), Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio 45433.

ILLUSTRATIONS

Introduction

If a lamellar composite consisting of alternating layers of two different homogeneous solids is suddenly disturbed, the resulting transient wave motion is of a complicated nature. In the particular problem that is considered here, a laminated body is subjected to a spatially uniform distribution of body forces normal to the layering and acting in a plane normal to the direction of the layering. The body forces vary in time as Heaviside step functions. The attention is focused on the normal stresses and the shear stresses at the interfaces. These stresses are of interest because, for tensile stresses or shear stresses exceeding the bond strengths, the layers separate and delamination occurs, which generally reduces the load-bearing capacity of the layered body.

To compute the interface stresses it is necessary to go into considerable detail in unraveling the complexity of the wave pattern. The transient waves propagating in the direction of the layering are analyzed by using the equations of the theory of linear elasticity for the layers. By means of the technique of modal analysis the interface stresses are computed as a summation over integrals, each representing the contribution of a particular mode. Under the assumption that the contribution of the lowest mode is the most significant one, the stresses are obtained as single infinite integrals. For small values of time the integrals are evaluated numerically; for large values of time the method of stationary phase is employed to evaluate the integrals approximately.

In analyzing wave motions in a layered medium the approach to be followed depends, of course, on the information that is desired. If, as in this paper, one wishes to determine the magnitudes of certain field quantities at specific locations, it is necessary to consider the layers as separate continua with appropriate continuity conditions at the interfaces. If, on the other hand, all that is desired are average values of field quantities over several layers, it conceivably is sufficient to apply the "effective modulus theory." An application of this theory implies that the layered medium is replaced by a homogeneous but transversely isotropic continuum whose elastic moduli and average mass density are expressed in terms of the thicknesses and the material constants of the constituent layers. The computation of the effective material constants was discussed by several authors,[1-4] both on the basis of static and dynamic considerations. The effective modulus theory suffers from the obvious drawback that the effect of dispersion cannot be described, which limits its applicability not only to averaged values, but also to long wavelength phenomena. In some recent papers,[5,6] a theory was proposed which is based on certain kinematical assumptions regarding the deformations of the individual layers, and which improves on the effective modulus theory in that the effect of the structuring on such gross dynamic quantities as frequencies and phase velocities of free time-harmonic motion can be accounted for. For plane harmonic waves propagating in the direction of the layering, and normal to the layering, the approximate phase velocities computed in Ref. 5 agree with exact phase velocities, which were computed earlier,[7,8] over a substantial range of wavenumbers. The approximate theory of Refs. 5 and 6 remains, as yet, unassessed for transient wave propagation problems.

Statement of the Problem

We consider an unbounded stratified medium consisting of alternating plane parallel layers of two homogeneous isotropic materials. The Lamé elastic constants, the mass densities, and the thicknesses of the reinforcing layers and the matrix layers are denoted by $\lambda_f, \mu_f, \rho_f, d_f$ and $\lambda_m, \mu_m, \rho_m, d_m$, respectively. The medium is at rest, when at time $t = 0$ body forces are suddenly applied in the plane $x = 0$, see Fig. 1. The body forces are uniformly distributed in the plane $x = 0$,

$$\underset{\sim}{P}(x,y,z,t) = - \underset{\sim}{j} P_0 \delta(x) H(t), \tag{1}$$

where $\delta(x)$ and $H(t)$ are the Dirac delta function and the Heaviside step function, respectively. The externally applied forces generate transient waves propagating in the positive and negative x-directions. The motion is two-dimensional (plane strain), because both the applied loads and the geometry are independent of the z-coordinate.

The uniformity of the externally applied load distribution, in conjunction with the periodicity of the structuring, allows several preliminary observations with regard to the deformation of the layered medium. For the present system of loads the deformation in the unbounded medium is antisymmetric with respect to any midplane, which implies that the horizontal displacement $u(x,y,t)$ and the normal stress $\sigma_y(x,y,t)$ vanish in all midplanes. It is also apparent that all reinforcing layers are undergoing identical deformations, and the same conclusion can be drawn with regard to the deformations of the matrix layers. In view of these observations it suffices to consider the composite layer $0 \leq y^f \leq \frac{1}{2} d_f$, $-\frac{1}{2} d_m \leq y^m \leq 0$ shown in Fig. 2, with the following set of boundary conditions relative to the local coordinates (x,y^f) and (x,y^m):

at $y^f = 0$:

$$u^f(x,0,t) = 0; \qquad \sigma_y^f(x,0,t) = 0 \tag{2a,b}$$

at $y^m = 0$:

$$u^m(x,0,t) = 0; \qquad \sigma_y^m(x,0,t) = 0 \tag{3a,b}$$

At the interface $y^f = \frac{1}{2} d_f$, $y^m = -\frac{1}{2} d_m$ we have

$$u^f(x,\tfrac{1}{2} d_f,t) = u^m(x,-\tfrac{1}{2} d_m,t); \qquad v^f(x,\tfrac{1}{2} d_f,t) = v^m(x,-\tfrac{1}{2} d_m,t) \tag{4a,b}$$

$$\sigma_y^f(x,\tfrac{1}{2} d_f,t) = \sigma_y^m(x,-\tfrac{1}{2} d_m,t); \qquad \sigma_{xy}^f(x,\tfrac{1}{2} d_f,t) = \sigma_{xy}^m(x,-\tfrac{1}{2} d_m,t) \tag{5a,b}$$

Since the medium is initially at rest, the initial conditions are

$$u^f(x,y,0) = \dot{u}^f(x,y,0) = u^m(x,y,0) = \dot{u}^m(x,y,0) \equiv 0 \tag{6a,b,c,d}$$

with similar equations for the displacements in the y-direction.

It is of interest to note that the solution of the formulated problem can also be regarded as the response of a layered half-space $x \geq 0$ to the following mixed boundary conditions at $x = 0$:

$$\sigma_{xy}^f(0,y,t) = \sigma_{xy}^m(0,y,t) = \frac{1}{2} P_o H(t) \tag{7}$$

$$u^f(0,y,t) = u^m(0,y,t) = 0 \tag{8}$$

For a half-space it is somewhat more realistic to require that the normal stress rather than the normal displacement vanish. That problem, which is much more difficult, remains as yet unsolved. For single rods and plates it has, however, been observed experimentally that the exact distribution of the stresses on the boundary does not greatly affect the major contribution of the response at sufficiently large distances from the boundary.

In this paper the technique of modal analysis is employed, i.e., the displacements and the stresses are expressed in terms of modes of time-harmonic motion. This approach is suggested by the representation of the delta function in Eq. 1, in terms of a Fourier integral

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dk = \frac{1}{\pi} \int_{0}^{\infty} \cos kx \, dk \tag{9}$$

Thus, we first determine the desired field quantities for a body-force distribution of the form

$$\underset{\sim}{P} = \underset{\sim}{j} F_y \cos kx \, H(t), \tag{10}$$

and then use a superposition integral as in Eq. 9 to obtain a formal solution of the original problem. The method of solution is similar to the one used by Jones[9] to study transverse impact waves in a bar under conditions of plane-strain elasticity.

Antisymmetric Modes

For plane strain the equations of motion of a homogeneous isotropic elastic solid are

$$L_1[u,v] + P_x = \rho \ddot{u} \tag{11}$$

$$L_2[u,v] + P_y = \rho \ddot{v}, \tag{12}$$

where the operators $L_1[u,v]$ and $L_2[u,v]$ are defined as

$$L_1[u,v] = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 u}{\partial y^2} \tag{13}$$

$$L_2[u,v] = (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u}{\partial x \partial y} + \mu \frac{\partial^2 v}{\partial x^2} \tag{14}$$

In Eqs. 11-14, λ and μ are Lamé's elastic constants, ρ is the mass density, u and v are the displacements in the x- and y-directions, respectively, and P_x and P_y are body forces. The relevant stress components are

$$\sigma_y = (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda \frac{\partial u}{\partial x} \tag{15}$$

$$\sigma_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{16}$$

As a first step toward a solution by modal analysis, we consider Eqs. 11 and 12 for $P_x \equiv 0$ and $P_y \equiv 0$, and we investigate solutions of the form

$$u(x,y,t) = - U(y) \sin kx \cos \omega t \tag{17}$$

$$v(x,y,t) = V(y) \cos kx \cos \omega t \tag{18}$$

General solutions of the displacement equations of motion of the linear theory of elasticity have been studied extensively.[10] Appropriate expressions for U(y) and V(y) satisfying the conditions at y = 0 of vanishing horizontal displacement and vanishing normal stress are

$$U(y) = kA \sin qy + sD \sin sy \tag{19}$$

$$V(y) = qA \cos qy - kD \cos sy \tag{20}$$

The corresponding stresses are

$$\sigma_y = \Sigma_y \cos kx \cos \omega t \tag{21}$$

$$\sigma_{xy} = \Sigma_{xy} \sin kx \cos \omega t, \tag{22}$$

wherein

$$\Sigma_y = \mu[A(k^2 - s^2)\sin qy + 2Dks \sin sy] \tag{23}$$

$$\Sigma_{xy} = \mu[- 2Akq \cos qy + D(k^2 - s^2)\cos sy] \tag{24}$$

In Eqs. 23 and 24,

$$q^2 = (\omega/c_1)^2 - k^2, \qquad s^2 = (\omega/c_2)^2 - k^2 \tag{25a,b}$$

$$c_1^2 = (\lambda + 2\mu)/\rho, \qquad c_2^2 = \mu/\rho \tag{26a,b}$$

With appropriate subscripts and superscripts attached to the constants and the stresses and displacements, Eqs. 11-26 apply to both the reinforcing layer and the matrix layer.

By employing Eqs. 17-22 to write out expressions for the displacements and the stresses for the matrix and the reinforcing layers, and by subsequently using these expressions in Eqs. 4-5, a system of four homogeneous equations for A^f, D^f, A^m, and D^m is obtained,

$$\xi A_f \sin\left(\tfrac{1}{2}\, q_f\right) + s_f D_f \sin\left(\tfrac{1}{2}\, s_f\right) = -\,\xi A_m \sin\left(\tfrac{1}{2}\, q_m/\zeta\right) - s_m D_m \sin\left(\tfrac{1}{2}\, s_m/\zeta\right) \qquad (27)$$

$$q_f A_f \cos\left(\tfrac{1}{2}\, q_f\right) - \xi D_f \cos\left(\tfrac{1}{2}\, s_f\right) = q_m A_m \cos\left(\tfrac{1}{2}\, q_m/\zeta\right) - \xi D_m \cos\left(\tfrac{1}{2}\, s_m/\zeta\right) \qquad (28)$$

$$\gamma\left(\xi^2 - s_f^2\right) A_f \sin\left(\tfrac{1}{2}\, q_f\right) + 2\gamma\xi s_f D_f \sin\left(\tfrac{1}{2}\, s_f\right) =$$

$$-\left(\xi^2 - s_m^2\right) A_m \sin\left(\tfrac{1}{2}\, q_m/\zeta\right) - 2\xi s_m D_m \sin\left(\tfrac{1}{2}\, s_m/\zeta\right) \qquad (29)$$

$$2\gamma\xi q_f A_f \cos\left(\tfrac{1}{2}\, q_f\right) - \gamma\left(\xi^2 - s_f^2\right) D_f \cos\left(\tfrac{1}{2}\, s_f\right) =$$

$$2\xi q_m A_m \cos\left(\tfrac{1}{2}\, q_m/\zeta\right) - \left(\xi^2 - s_m^2\right) D_m \cos\left(\tfrac{1}{2}\, s_m/\zeta\right) \qquad (30)$$

The requirement that the determinant of the coefficients must vanish yields the frequency equation. The frequency determinant was derived earlier in Ref. 7, and it can also be found in worked-out form in Ref. 8. Here we reproduce the result of Ref. 8 in slightly modified notation as

$$q_m \Omega^2 (PU-QT) + \theta q_f \Omega^2 (RU-ST) + (2\gamma\xi^2 - 2\xi^2 + \Omega^2 - \theta\Omega^2)(QR-PS) = 0, \qquad (31)$$

wherein

$$P = \xi^2 \tan\left(\tfrac{1}{2}\, q_f\right) + q_f s_f \tan\left(\tfrac{1}{2}\, s_f\right) \qquad (32)$$

$$Q = \gamma\left(\xi^2 - s_f^2\right) \tan\left(\tfrac{1}{2}\, q_f\right) + 2\gamma q_f s_f \tan\left(\tfrac{1}{2}\, s_f\right) \qquad (33)$$

$$R = \xi^2 \tan\left(\frac{1}{2} q_m/\zeta\right) + q_m s_m \tan\left(\frac{1}{2} s_m/\zeta\right) \tag{34}$$

$$S = \left(\xi^2 - s_m^2\right)\tan\left(\frac{1}{2} q_m/\zeta\right) + 2q_m s_m \tan\left(\frac{1}{2} s_m/\zeta\right) \tag{35}$$

$$T = s_f \tan\left(\frac{1}{2} s_f\right) + s_m \tan\left(\frac{1}{2} s_m/\zeta\right) \tag{36}$$

$$U = 2\gamma s_f \tan\left(\frac{1}{2} s_f\right) + 2s_m \tan\left(\frac{1}{2} s_m/\zeta\right) \tag{37}$$

In Eqs. 27-37, we have introduced the following dimensionless quantities:

$$\xi = kd_f, \qquad \Omega = \omega d_f/(\mu_m/\rho_m)^{\frac{1}{2}} \tag{38a,b}$$

$$\zeta = d_f/d_m, \qquad \gamma = \mu_f/\mu_m, \qquad \theta = \rho_f/\rho_m \tag{39a,b,c}$$

$$q_f = \left[(\theta/\delta_f\gamma)\Omega^2 - \xi^2\right]^{\frac{1}{2}} \tag{40}$$

$$s_f = [(\theta/\gamma)\Omega^2 - \xi^2]^{\frac{1}{2}} \tag{41}$$

$$q_m = \left[(1/\delta_m)\Omega^2 - \xi^2\right]^{\frac{1}{2}} \tag{42}$$

$$s_m = [\Omega^2 - \xi^2]^{\frac{1}{2}} \tag{43}$$

In Eqs. 40 and 42, the constants δ_f and δ_m are defined as

$$\delta_f = \frac{2(1-\nu_f)}{1-2\nu_f}, \qquad \delta_m = \frac{2(1-\nu_m)}{1-2\nu_m}, \tag{44}$$

where ν denotes Poisson's ratio.

For prescribed dimensionless wavenumber ξ, the transcendental frequency equation 31 yields an infinite number of roots Ω_n for the dimensionless frequency, whose numerical values must be determined on a digital computer. Close inspection shows that $\Omega \equiv 0$ satisfies Eq. 31, which is particularly obvious for the special case $\gamma = 1$. The case $\Omega \equiv 0$ yields, however, not an actual mode of motion. In the limit of vanishing ξ, i.e., for infinitely long waves, Eq. 31 splits up into the product of two frequency equations governing the frequencies of uncoupled antisymmetric thickness stretch and antisymmetric thickness shear motions. These frequency equations, which are given in Ref. 11, provide the cut-off frequencies of the higher modes. The frequency of the lowest mode vanishes in the limit of vanishing ξ, and the slope of the frequency curve, which is the dimensionless phase velocity of the lowest mode for infinitely long waves, is obtained as

$$c_s = \left\{ \frac{\gamma}{[(1-\eta) + \eta\theta][(1-\eta)\gamma + \eta]} \right\}^{\frac{1}{2}} , \tag{45}$$

where the phase velocity was rendered dimensionless by dividing by $(\mu_m/\rho_m)^{\frac{1}{2}}$, and where

$$\eta = d_f/(d_f + d_m) \tag{46}$$

The solutions of Eq. 31 plotted in the (Ω,ξ) plane yield an infinite number of branches, each corresponding to a particular mode of wave propagation. In the following analysis the dimensionless wavenumber ξ is treated as an independent variable, and the frequencies Ω_n have been solved on a digital computer. The frequencies of the first four modes are plotted in Fig. 3.

Once the frequency has been computed for a prescribed value of ξ, and for a particular mode, the result can be substituted back into Eqs. 27-29, and the three constants A^f, A^m, D^f can be expressed in terms of the remaining D^m. Subsequent substitution of the results into Eqs. 23 and 24 yields expressions for $\Sigma_y^f, \Sigma_{xy}^f$ and $\Sigma_y^m, \Sigma_{xy}^m$ in terms of the remaining constant D^m and y^f/d_f and y^m/d_m, respectively. After normalizing the normal and shear stresses by dividing by the value at the interface, the stresses have been plotted in Fig. 4 and Fig. 5. The results show the mode shapes for several values of the dimensionless wavenumber ξ. It is noted that for very long waves, the stress distribution is virtually linear. The above computations also provide a check on the computed frequencies, in that the stresses that are computed for the matrix and the reinforcing layer must have the same value at the interface.

Returning to Fig. 2, we assume that a vector-valued function $\underset{\sim}{F}^f(x,y^f,t)$ is defined for $0 \le y^f \le \frac{1}{2} d_f$,

$$\underset{\sim}{F}^f = \underset{\sim}{i} \sin kx \, F_x^f(y^f) + \underset{\sim}{j} \cos kx \, F_y^f(y^f) \tag{47}$$

A similar function $\underset{\sim}{F}^m$ is defined for $-\frac{1}{2} d_m \le y^m \le 0$. It is further assumed that $\underset{\sim}{F}^f$ and $\underset{\sim}{F}^m$ can be expanded in the modes $U_r^f(y^f), V_r^f(y^f)$ and $U_r^m(y^m), V_r^m(y^m)$, which follow from Eqs. 19 and 20

$$\underset{\sim}{F}^f = \sum_{r=1}^{\infty} \left(\underset{\sim}{i} \sin kx \, f_x^r \, U_r^f + \underset{\sim}{j} \cos kx \, f_y^r \, V_r^f \right) \tag{48}$$

An expansion of the type 48 is meaningful only if the characteristic functions $U_r^f(y^f)$, $V_r^f(y^f)$, $U_r^m(y^m)$, and $V_r^m(y^m)$ form an orthogonal set. The process of deriving the generalized relation of orthogonality is straightforward, but rather lengthy. Here we present only the result:

$$(\underset{\sim}{U}_r, \underset{\sim}{U}_s) = (\underset{\sim}{U}_r, \underset{\sim}{U}_r)\delta_{rs}, \tag{49}$$

where δ_{rs} is the Kronecker delta, and the inner product (U_r, U_s) is defined as

$$(\underset{\sim}{U}_r, \underset{\sim}{U}_s) = \rho_f \int_0^{\frac{1}{2}d_f} \underset{\sim}{U}_r^f \cdot \overline{\underset{\sim}{U}}_s^f \, dy^f + \rho_m \int_{-\frac{1}{2}d_m}^{0} \underset{\sim}{U}_r^m \cdot \overline{\underset{\sim}{U}}_s^m \, dy^m \tag{50}$$

In Eq. 50 a bar denotes a complex conjugate, and the dot indicates a scalar product of two vectors. The vector $\underset{\sim}{U}_r^f$ is defined as

$$\underset{\sim}{U}_r^f(y) = \underset{\sim}{i}\, U_r^f(y) + \underset{\sim}{j}\, V_r^f(y) \tag{51}$$

The vector $\underset{\sim}{U}_r^m(y)$ is defined similarly.

Motions Generated by a Transverse Force

In this section we return to Eqs. 11 and 12, and we consider the case that $P_x \equiv 0$, and P_y is defined by Eq. 10. For the reinforcing layer the displacement equations of motion then are

$$L_1^f[u^f, v^f] = \rho_f \ddot{u}^f \tag{52}$$

$$L_2^f[u^f, v^f] + F_y \cos kx\, H(t) = \rho_f \ddot{v}^f, \tag{53}$$

where the operators are defined by Eqs. 13 and 14. Similarly, we have for the matrix layers

$$L_1^m[u^m, v^m] = \rho_m \ddot{u}^m \tag{54}$$

$$L_2^m[u^m, v^m] + F_y \cos kx\, H(t) = \rho_m \ddot{v}^m \tag{55}$$

The solutions of Eqs. 52-55 are sought as expansions over the previously determined system of normal modes. Thus

$$u^f(x,y,t) = \sum_{r=1}^{\infty} U_r^f(y)\varphi_r(t) \sin kx \tag{56}$$

$$v^f(x,y,t) = \sum_{r=1}^{\infty} V_r^f(y)\varphi_r(t) \cos kx \tag{57}$$

$$u^m(x,y,t) = \sum_{r=1}^{\infty} U_r^m(y)\varphi_r(t) \sin kx \tag{58}$$

$$v^m(x,y,t) = \sum_{r=1}^{\infty} V_r^m(y)\varphi_r(t) \cos kx \tag{59}$$

The expansions are substituted in Eqs. 52-55, and the resulting equations are multiplied by $\overline{U}_s^f$, $\overline{V}_s^f$, $\overline{U}_s^m$, and $\overline{V}_s^m$, respectively. Of the four resulting equations the first two are integrated over half the thickness of the reinforcing layer, and the last two over half the thickness of the matrix layer. Upon adding the four integrated equations, the following ordinary differential equation for $\varphi_r(t)$ is deduced:

$$\ddot{\varphi}_r + \omega_r^2 \varphi_r = M_r F_y H(t), \tag{60}$$

where

$$M_r = \left\{ \int_0^{+\frac{1}{2}d_f} \overline{V}_r^f \, dy^f + \int_{-\frac{1}{2}d_m}^{0} \overline{V}_r^m \, dy^m \right\} \Big/ (\underset{\sim}{U}_r, \underset{\sim}{U}_r) \tag{61}$$

The inner product $(\underset{\sim}{U}_r, \underset{\sim}{U}_r)$ follows from Eq. 50 and is here written out in detail as

$$(\underset{\sim}{U}_r, \underset{\sim}{U}_r) = \rho_f \int_0^{\frac{1}{2}d_f} \left(|U_r^f|^2 + |V_r^f|^2 \right) dy^f + \rho_m \int_{-\frac{1}{2}d_m}^{0} \left(|U_r^m|^2 + |V_r^m|^2 \right) dy^m \tag{62}$$

It is seen that $(\underset{\sim}{U}_r, \underset{\sim}{U}_r)$ is proportional to the kinetic energy stored per unit length, and consequently the expression is always positive, except for $r = 1$, when it vanishes for $\xi = \omega_1 \equiv 0$.

The solution of Eq. 60 subject to the initial conditions (6a,b,c,d) is now obtained as

$$\varphi_r = F_y \Phi_r (1 - \cos \omega_r t) H(t), \tag{63}$$

wherein

$$\Phi_r = M_r / \omega_r^2 \tag{64}$$

The solutions for the displacements follow immediately upon substitution of Eq. 63 into Eqs. 56-59. The stresses in the reinforcing layer are obtained by substituting U_r^f and V_r^f in the stress-strain relations of the types 15 and 16. The interface stresses, which are subsequently obtained by setting $y^f = \frac{1}{2} d_f$, and which are denoted by the symbols $\tau_y(x,t)$ and $\tau_{xy}(x,t)$, emerge in the following forms:

$$\tau_y(x,t) = (F_y d_f) \sum_{r=1}^{\infty} S_y^r (1 - \cos \omega_r t) \cos kx \tag{65}$$

$$\tau_{xy}(x,t) = -(F_y d_f) \sum_{r=1}^{\infty} S_{xy}^r (1 - \cos \omega_r t) \sin kx, \tag{66}$$

where the summation is carried out over the roots of Eq. 31. The rather lengthy expressions for S_y^r and S_{xy}^r in terms of ω_r and k are not reproduced here, but for the lowest mode their numerical values are as functions of the dimensionless wavenumber ξ displayed in Fig. 6. It is noted that $S_{xy}^1(\xi)$ shows a singularity for $\xi = 0$. It can be shown that $S_{xy}^1(\xi)$ is odd in ξ, and $S_y^1(\xi)$ is even in ξ.

Equations 65 and 66 yield the interface stresses for the body-force distribution given by Eq. 10. The interface stresses for the distribution of body forces given by Eq. 9 then immediately follow as

$$\tau_y(X,T) = -\frac{1}{\pi} P_o \sum_{r=1}^{\infty} \int_0^{\infty} S_y^r(\xi)(1 - \cos \Omega_r T) \cos \xi X \, d\xi \tag{67}$$

$$\tau_{xy}(X,T) = \frac{1}{\pi} P_o \sum_{r=1}^{\infty} \int_0^{\infty} S_{xy}^r(\xi)(1 - \cos \Omega_r T) \sin \xi X \, d\xi \tag{68}$$

In Eqs. 67 and 68 we have introduced the dimensionless time T and the dimensionless coordinate X by

$$T = (\mu_m/\rho_m)^{\frac{1}{2}} t/d_f, \qquad X = x/d_f \tag{69a,b}$$

The formal solution of the problem at hand, given by Eqs. 67 and 68, is much too complicated to be of practical value. It is, however, well known that for problems of the type considered here the main part of the response is associated with the lowest mode,[9] i.e., with the first term of the summations 67 and 68. Thus, we approximate

$$\tau_y(X,T) = -\frac{1}{\pi} P_o \int_0^\infty S_y^1(\xi)(1 - \cos \Omega_1 T) \cos \xi X \, d\xi \tag{70}$$

$$\tau_{xy}(X,T) = \frac{1}{\pi} P_o \int_0^\infty S_{xy}^1(\xi)(1 - \cos \Omega_1 T) \sin \xi X \, d\xi \tag{71}$$

To evaluate the integrals in Eqs. 70 and 71 we can resort to numerical integration or to approximate evaluation by the method of stationary phase. Numerical integration is feasible for small values of T, and the method of stationary phase can be employed for large T. Both methods of integration are applied in the next section.

A more elementary approach to the problem discussed here consists in describing the mechanical behavior of the laminated medium by the stress-strain relations for a homogeneous, but transversely isotropic medium. The effective elastic constants of the transversely isotropic solid were computed by Postma[1] in terms of the elastic constants and the thicknesses of the layers. A body-force distribution of the type Eq. 1 applied in a transversely isotropic medium generates a plane shear wave propagating in the positive and negative x-directions. Thus, for positive x we find

$$(\sigma_{xy})_{ef} = \frac{1}{2} P_o \; H(t - x/c_{ef}), \tag{72}$$

where

$$c_{ef} = (\mu_{ef}/\rho_{ef})^{\frac{1}{2}}, \tag{73}$$

and, according to Ref. 1,

$$\mu_{ef} = \frac{(d_f + d_m)\mu_f \mu_m}{d_f \mu_m + d_m \mu_f} \tag{74}$$

$$\rho_{ef} = \frac{\rho_f d_f + \rho_m d_m}{d_f + d_m} \tag{75}$$

Introducing the dimensionless coordinates X and T, Eqs. 69a,b, the shear stress is rewritten as

$$(\sigma_{xy})_{ef} = \frac{1}{2} P_0 \; H(T - X/C_s), \tag{76}$$

where $C_s = c_{ef}/(\mu_m/\rho_m)^{\frac{1}{2}}$, which agrees with the expression for the limiting phase velocity for vanishing wavenumber of the lowest mode in the layered medium, see Eq. 45. Normal stresses do not enter at all, thus $(\sigma_y)_{ef} \equiv 0$.

The Interface Stresses

Numerical evaluation of the integrals in Eqs. 70 and 71 is feasible for not too large values of the dimensionless time T. Though the function $S^1_{xy}(\xi)$ has a pole at $\xi = 0$, the complete integrand remains finite at $\xi = 0$; the function $S^1_y(\xi)$ is well behaved over the whole range of ξ. As ξ increases, a point is reached where further increase of ξ does not add appreciably to the integrals. Up to that point Simpson's rule was used to compute $\tau_y(X,T)$ and $\tau_{xy}(X,T)$ for $X = 5$ and for $T \leq 5$. For higher values of T the integrands become highly oscillatory and the numerical approach becomes too time-consuming. It is then more efficient to employ approximate analytical methods that are valid for large values of T. The numerically computed interface stresses are shown in Figs. 7 and 8 for the following values of the material parameters: $\gamma = 50$, $\theta = 3$, $\nu_f = 0.3$, $\nu_m = 0.35$, and $\zeta = 4$ (i.e., $\eta = 0.8$).

To evaluate the integrals by approximate methods, Eqs. 70 and 71 are rewritten as

$$\tau_y(X,T) = -\frac{P_o}{2\pi}\int_{-\infty}^{\infty} S_y^1(\xi)\left[e^{i\theta_1} - \frac{1}{2}e^{i\theta_2} - \frac{1}{2}e^{i\theta_3}\right]d\xi \qquad (77)$$

$$\tau_{xy}(X,T) = \frac{P_o}{2\pi i}\int_{-\infty}^{\infty} S_{xy}^1(\xi)\left[e^{i\theta_1} - \frac{1}{2}e^{i\theta_2} - \frac{1}{2}e^{i\theta_3}\right]d\xi, \qquad (78)$$

where we have used that $S_y^1(\xi)$ is even, and $S_{xy}^1(\xi)$ is odd in ξ, and where

$$\theta_1 = \xi X, \qquad \theta_2 = \xi X + \Omega_1(\xi)T, \qquad \theta_3 = \xi X - \Omega_1(\xi)T \qquad (79a,b,c)$$

For large values of T, the integrals 77 and 78 can be approximated by the method of stationary phase. This method, which was discussed in general terms by Eckart,[12] is based on the observation that the major contributions to the integrals come from the vicinities of points $\xi = \xi^*$, where ξ^* satisfies the stationary phase condition

$$d\theta/d\xi\big|_{\xi=\xi^*} = 0 \qquad (80)$$

For such a value of ξ, the approximation to the integral is obtained by expanding θ as a Taylor series about $\xi = \xi^*$, $S_y^1(\xi^*)$ and $S_{xy}^1(\xi^*)$ being treated as constants. Assuming that $d^2\theta/d\xi^2$ does not vanish at $\xi = \xi^*$, and also that T is large enough so that the Taylor expansion for θ need not be taken beyond the second term, the simplified integral can then be evaluated. When $\theta = \xi X$, the stationary phase condition cannot be satisfied for $X \neq 0$, and the first integral in Eqs. 77 and 78 thus vanishes in the present approximation. For $\theta = \theta_2$ and $\theta = \theta_3$, the condition 80 yields

$$X = \pm (d\Omega_1/d\xi)T = \pm C_g T, \tag{81}$$

where C_g is the dimensionless group velocity. For the lowest modes, the dimensionless group velocities are shown in Fig. 9. For n = 1, the group velocity and the higher derivatives $d^2\Omega_1/d\xi^2$ and $d^3\Omega_1/d\xi^3$ are plotted in Fig. 10. For any value of X/T, the appropriate roots ξ^* can now be obtained from Fig. 9. Since the frequency equation 31 depends on Ω^2 and ξ^2, both $\xi = +\xi^*$ and $\xi = -\xi^*$ are points of stationary phase. For the normal stress $\tau_y(X,T)$ at positive X, the contribution from the vicinity of these points is then obtained as[12,9]

$$\tau_y(X,T) = \frac{P_o S_y^1(\xi)}{(2\pi T)^{\frac{1}{2}}} \left[\pm \frac{d^2\Omega_1}{d\xi^2}\right]^{-\frac{1}{2}} \cos\left(\xi X - \Omega_1 T \mp \frac{\pi}{4}\right)\Big|_{\xi=\xi^*}, \tag{82}$$

where the alternative sign applies if $d^2\Omega_1/d\xi^2$ is negative.

The approximation 82 is not valid if $d^2\Omega_1/d\xi^2$ also vanishes at $\xi = \xi^*$. The modification which is required for that case is, however, well known,[12,9] and we simply state the result as

$$\tau_y(X,T) = P_o S_y^1(\xi)\left(\frac{2}{T}\right)^{\frac{1}{3}}\left(\pm \frac{d^3\Omega_1}{d\xi^3}\right)^{-\frac{1}{3}} Ai(s)\cos(\xi X - \Omega_1 T)\Big|_{\xi=\xi^*}, \tag{83}$$

where

$$s = \mp \left(X - \frac{d\Omega_1}{d\xi}\right)\left(\frac{2}{T}\right)^{\frac{1}{3}}\left(\pm \frac{d^3\Omega_1}{d\xi^3}\right)^{-\frac{1}{3}}, \tag{84}$$

and Ai(s) is the Airy integral

$$Ai(s) = \frac{1}{\pi}\int_0^\infty \cos\left(sz + \frac{1}{3}z^3\right) dz \tag{85}$$

The alternate sign applies again when $d^3\Omega_1/d\xi^3 < 0$, i.e., when the group velocity is a minimum, and for $\tau_y(X,T)$ it applies in particular to the point of stationary phase at $\xi = 0$.

The group velocity asymptotically approaches another stationary value as ξ increases beyond bounds. The contributions for large values of ξ are associated with short wavelength components, and the approximation is obtained by replacing the integrand by its limiting form for large values of ξ.[9] As ξ increases, the phase velocity of the lowest mode either approaches the velocity of Stoneley waves or the slower of the two possible velocities of shear waves. For the present combination of material properties Stoneley waves do not exist, and the limiting phase velocity is thus the velocity of shear waves in the matrix layer. For large ξ, we can then approximate

$$\Omega_1(\xi) \simeq \xi + b\xi^{-1}, \tag{86}$$

where b is best computed numerically from the Ω-ξ data. Also

$$S_y^1(\xi) \simeq \bar{S}_y^1 \xi^{-2} \tag{87}$$

Upon substituting 86 and 87, the integral 77 can be evaluated in terms of a Bessel function of order two, to yield for $(T-X) > 0$

$$\tau_y(X,T) = \frac{1}{2}\,\bar{S}_y^1 (bT)^{-\frac{1}{2}} (T-X)^{\frac{1}{2}} J_1[2(bT)^{\frac{1}{2}}(T-X)^{\frac{1}{2}}] \tag{88}$$

The integral vanishes identically for $X < T$.

An important special case, for which both 82 and 83 fail, occurs if the function multiplying the exponential in the integrand has a pole on the real axis at the point of stationary phase, and if, moreover, $d^2\Omega_1/d\xi^2$ vanishes at that point. For the present problem such a pole is found in the integral for $\tau_{xy}(X,T)$, since, as shown in Fig. 6, $S^1_{xy}(\xi)$ becomes unbounded at $\xi = 0$. For $\tau_{xy}(X,T)$, the contribution from the vicinity of the pole is actually the predominant one because the contribution does not decay for increasing X or increasing T. To determine this contribution we follow, by and large, the discussion by Skalak,[13] and we write the essential contribution as

$$\tau_{xy} = \frac{1}{2\pi i} P_o \int_{-\epsilon}^{+\epsilon} \frac{1}{\xi}\left(e^{i\theta_1} - \frac{1}{2} e^{i\theta_2} - \frac{1}{2} e^{i\theta_3}\right) d\xi, \tag{89}$$

where θ_1, θ_2, and θ_3 are defined by Eqs. 79a,b,c, and where we have used that for small ξ

$$S^1_{xy}(\xi) \simeq \frac{1}{\xi} \tag{90}$$

By extending the limits of integration to $\pm\infty$, the first term in Eq. 89 is easily evaluated. For the remaining integrals, two terms of the expansion are used to approximate $\Omega_1(\xi)$ in the vicinity of $\xi = 0$,

$$\Omega_1(\xi) = C_s\xi + a\xi^3, \tag{91}$$

where C_s is defined by Eq. 41, and a is defined as

$$a = \frac{1}{6} \left.\frac{d^3\Omega_1}{d\xi^3}\right|_{\xi=0} \tag{92}$$

For certain specific values of the material constants, the constant a follows from Fig. 10. Introducing the approximation 91 in Eq. 89 yields

$$\tau_{xy} = \frac{1}{2} P_o\{[H(X) - H(-X)] - I_1 - I_2\}, \tag{93}$$

where H(X) is the Heaviside unit function, and

$$I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin\left[\xi X + C_s \xi T + a\xi^3 T\right]}{\xi} d\xi \tag{94}$$

$$I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin\left[\xi X - C_s \xi T - a\xi^3 T\right]}{\xi} d\xi \tag{95}$$

It is permissible to extend the limits from $\pm \epsilon$ to $\pm \infty$ because the integrals so added are of order $1/T^{\frac{1}{2}}$,[13] which is negligible in the present approximation. Apart from the sign of the constant a, the integrals 94 and 95 are of the same form as discussed by Skalak,[13] and they can be reduced to integrals of Airy's function. We rewrite

$$I_1 = \frac{1}{\pi} \int_0^{\infty} \frac{\sin[\xi z'' + a\xi^3 T]}{\xi} d\xi \tag{96}$$

$$I_2 = \frac{1}{\pi} \int_0^{\infty} \frac{\sin[\xi z' - a\xi^3 T]}{\xi} d\xi, \tag{97}$$

where

$$z' = X - C_s T, \qquad z'' = X + C_s T \tag{98a,b}$$

By a change of variables

$$I_1 = \frac{1}{\pi} \int_0^{\infty} \frac{\sin\left(\alpha'' \xi + \frac{1}{3}\xi^3\right)}{\xi} d\xi \tag{99}$$

$$I_2 = -\frac{1}{\pi} \int_0^{\infty} \frac{\sin\left(-\alpha' \xi + \frac{1}{3}\xi^3\right)}{\xi} d\xi, \tag{100}$$

where

$$\alpha' = z'/(3aT)^{1/3}, \qquad \alpha'' = z''/(3aT)^{1/3} \tag{101a,b}$$

The integrals 99 and 100 can be recognized as integrals of Airy's function, see Eq. 85, and we obtain

$$I_1 = \int_0^{\alpha''} Ai(s)ds + \frac{1}{6} \tag{102}$$

$$I_2 = -\int_0^{-\alpha'} Ai(s)ds - \frac{1}{6}\ , \tag{103}$$

where the term 1/6 enters as the values of I_1 and I_2 for $\alpha'' = 0$ and $\alpha' = 0$, respectively. For large values of T, the integrals in Eqs. 102 and 103 approach 1/3, and for large values the contributions of I_1 and I_2 in Eq. 93 thus cancel, reducing the interface shear stress to the value according to the effective modulus theory, which was presented in Eq. 76, and which is also the result for the exact problem as T increases beyond bounds.

The approximate normal interface stress is computed from Eqs. 82 and 83 and is also shown in Fig. 8. The solutions according to the two expressions are matched at T = 1.4 and T = 3.6. These matching points are somewhat arbitrary because the two approximations are nearly indistinguishable in the vicinities of these points. For X = 5 and T < 1.4 (i.e., X/T > 3.57), Eq. 83 is required since the maximum of the group velocity is located in this region, see Fig. 9. In the range 1.4 < T < 3.6 (i.e., 1.39 < X/T < 3.57), we can use Eq. 83 for the two roots. Finally, for 3.6 < T < 5 (i.e., 1.0 < X/T < 1.39), Eq. 82 is used for the root at the larger value of ξ, while Eq. 83 is employed to deal with the minimum of the group velocity at $\xi = 0$. For values of T > 5, which are not considered here, we would have to use Eq. 88. It is noted from Fig. 8 that the approximations agree quite well with the numerical computations with respect to the phase and the amplitude. The mean values tend to be somewhat at variance as T increases.

For the interface shear stress it is sufficient to use Eq. 93 over the whole range of T ($T < 10$), since this "head of the pulse" approximation predominates the other contributions. It is noted from Fig. 7 that the approximate result agrees well with the numerical computations in the region of overlap. The head-of-the-pulse approximation approaches the correct limit as T increases. In Fig. 7, we have also plotted the elementary result according to the effective modulus theory given by Eq. 76. The latter solution shows a sharp wavefront and an instantaneous jump to the large time value of the exact solution.

Concluding Remarks

It is shown in this paper that a laminated medium shows a marked dispersive behavior and that a careful analysis is required to determine the interface stresses for dynamic loading conditions. The results indicate that under transverse loading the interface shear stresses are not larger than the shear stresses predicted by the elementary effective modulus theory. The normal stresses at the interface may, however, be tensile and acquire appreciable magnitudes, considering the usually low tensile strength of interface bonds. Normal stresses are not accounted for by the elementary theory.

Preliminary investigations indicate that the inclusion of higher modes in the analysis does not appreciably add to the interface shear stress. The contributions of higher modes to the normal interface stress are relatively more significant and may further increase the magnitudes of the tensile stresses.

The method of solution used in this paper is applicable to distributions of body forces in the x-direction other than the delta function. Body forces acting in the direction of the layering can also be treated. Although the distribution in the y-direction must be the same for all reinforcing layers and for all matrix layers, respectively, an extension from the uniform distribution to an arbitrary antisymmetric distribution across the thicknesses is easily achieved. To investigate the response to body forces that are symmetrically distributed across the thicknesses of the layers, the frequency spectrum of symmetric modes, which was considered in Ref. 11, must be used.

An analogous analysis of a layer composed of a large number of laminations is considerably more complicated because the two free surfaces destroy the simplifying conditions of periodicity and antisymmetry of the deformation pattern which made the analysis feasible for the unbounded body. The solutions presented in this paper should be very useful in determining the validity for transient wave propagation of the improved continuum theories for a laminated medium, which were presented in Refs. 5 and 6. A subsequent application of these approximate continuum theories to bounded laminated bodies would not present great difficulties.

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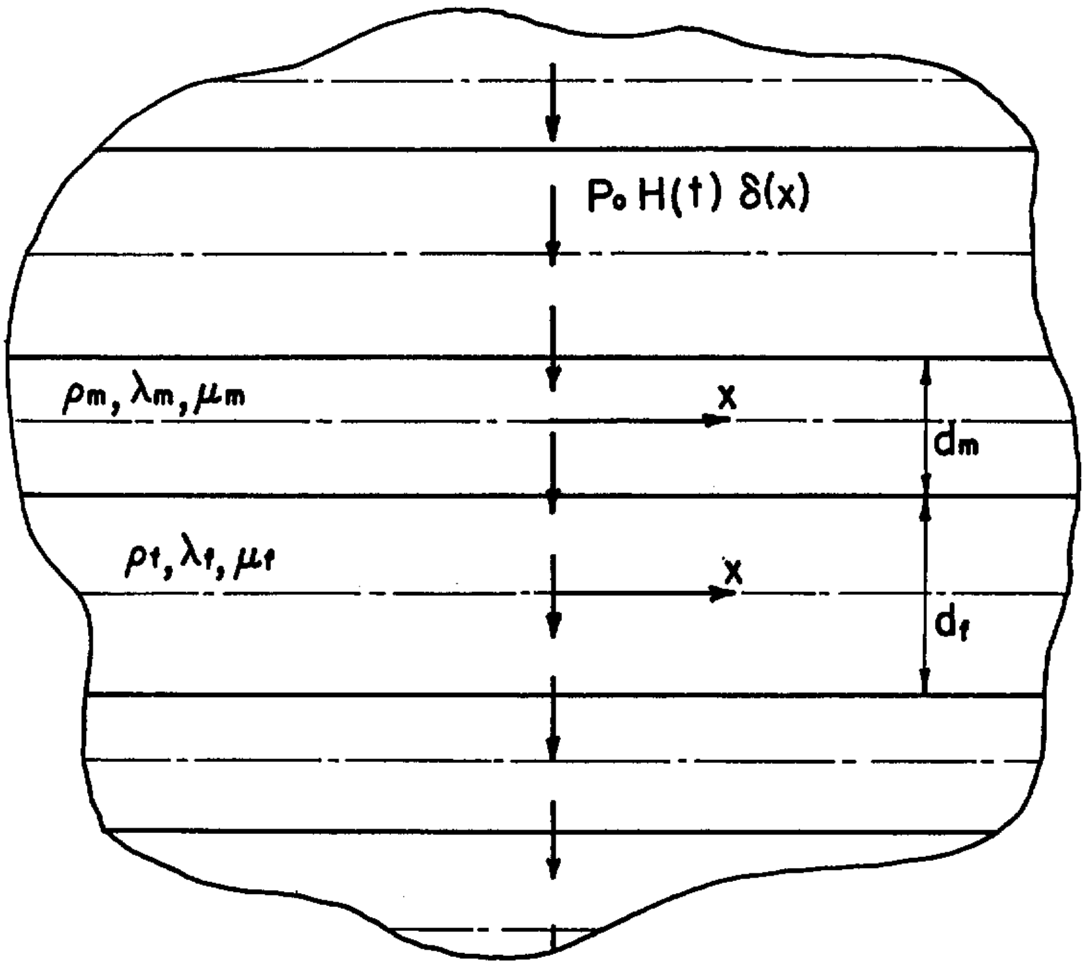

Fig. 1. Laminated medium

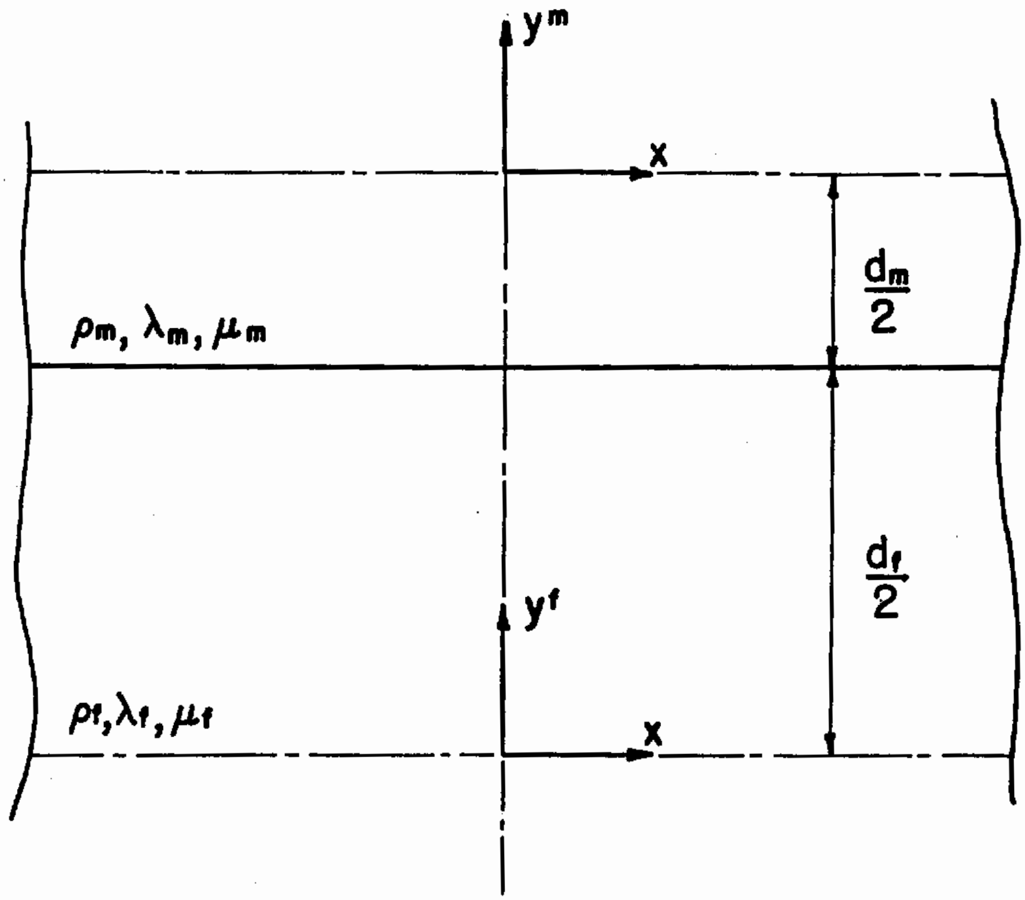

Fig. 2. Composite layer

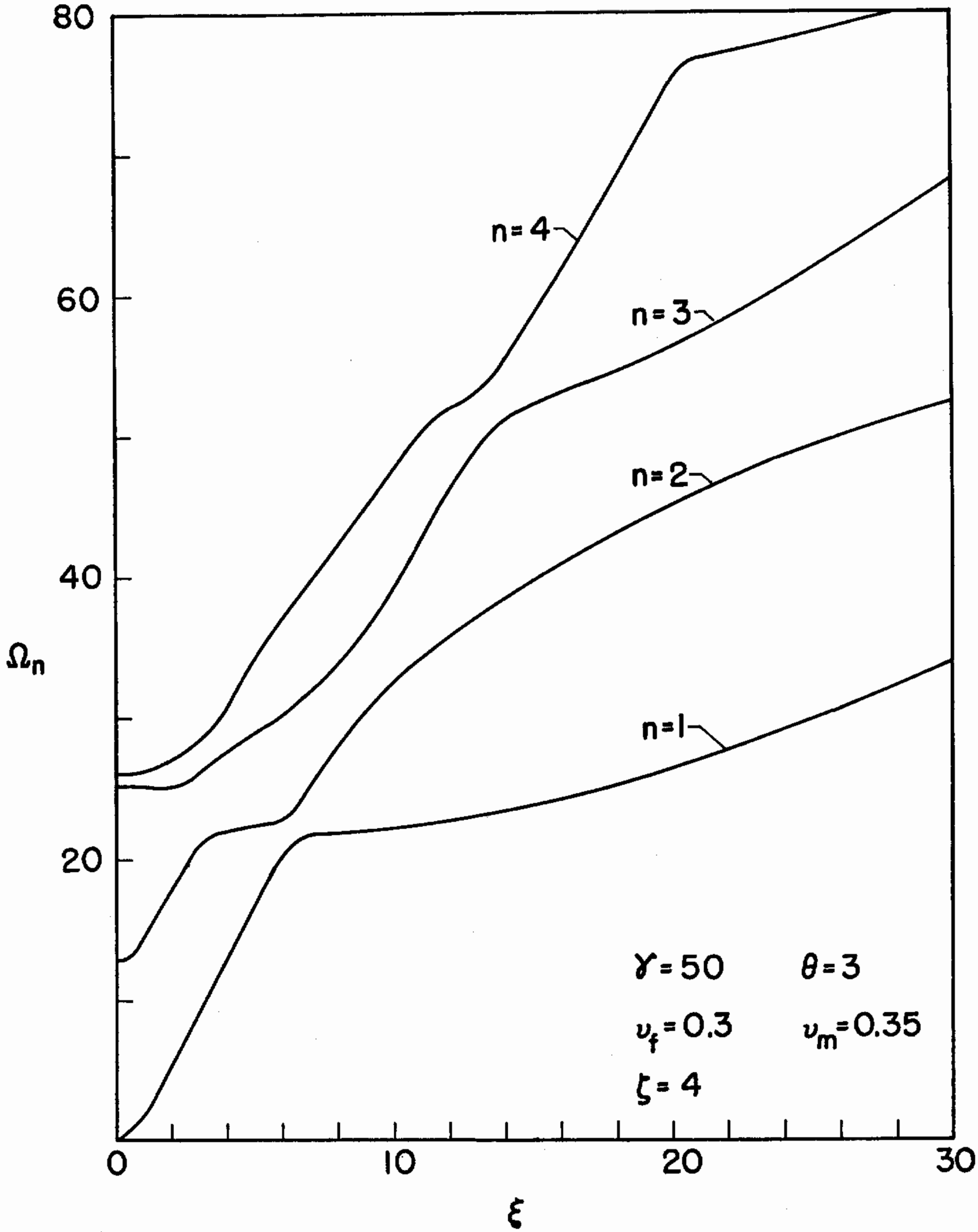

Fig. 3. Dimensionless frequencies vs dimensionless wavenumber for antisymmetric modes

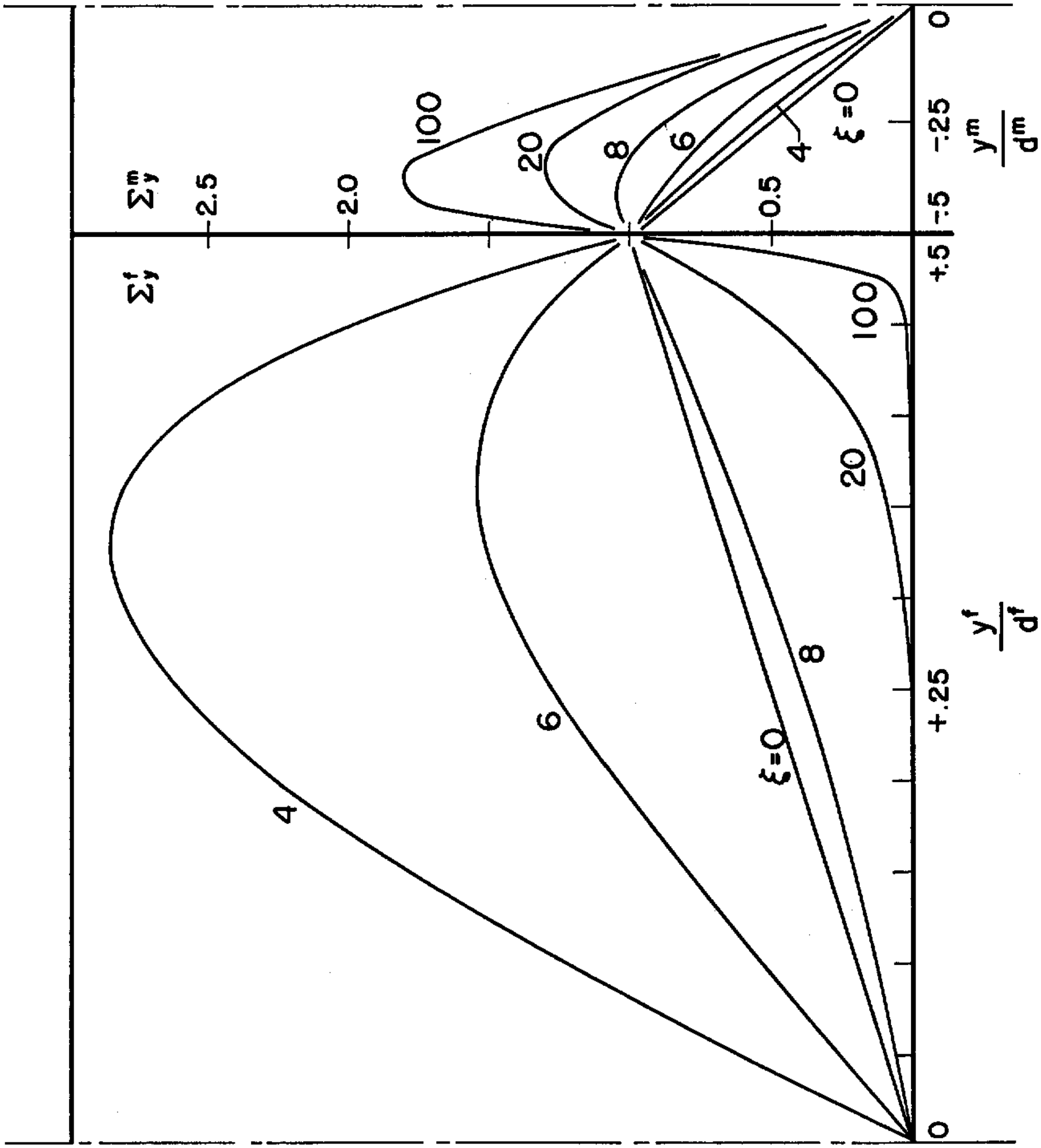

Fig. 4. Normal stresses for the lowest mode

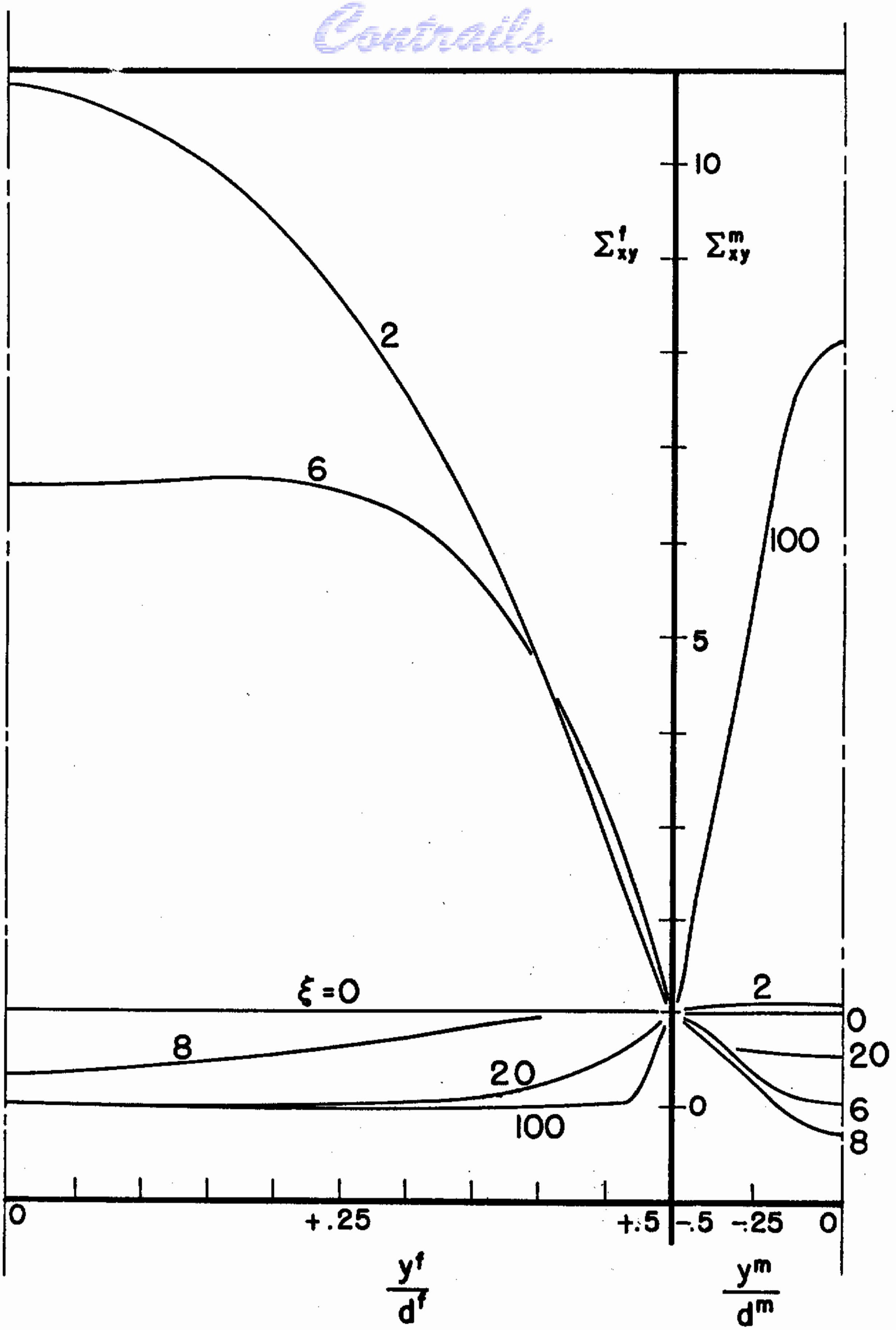

Fig. 5. Shear stresses for the lowest mode

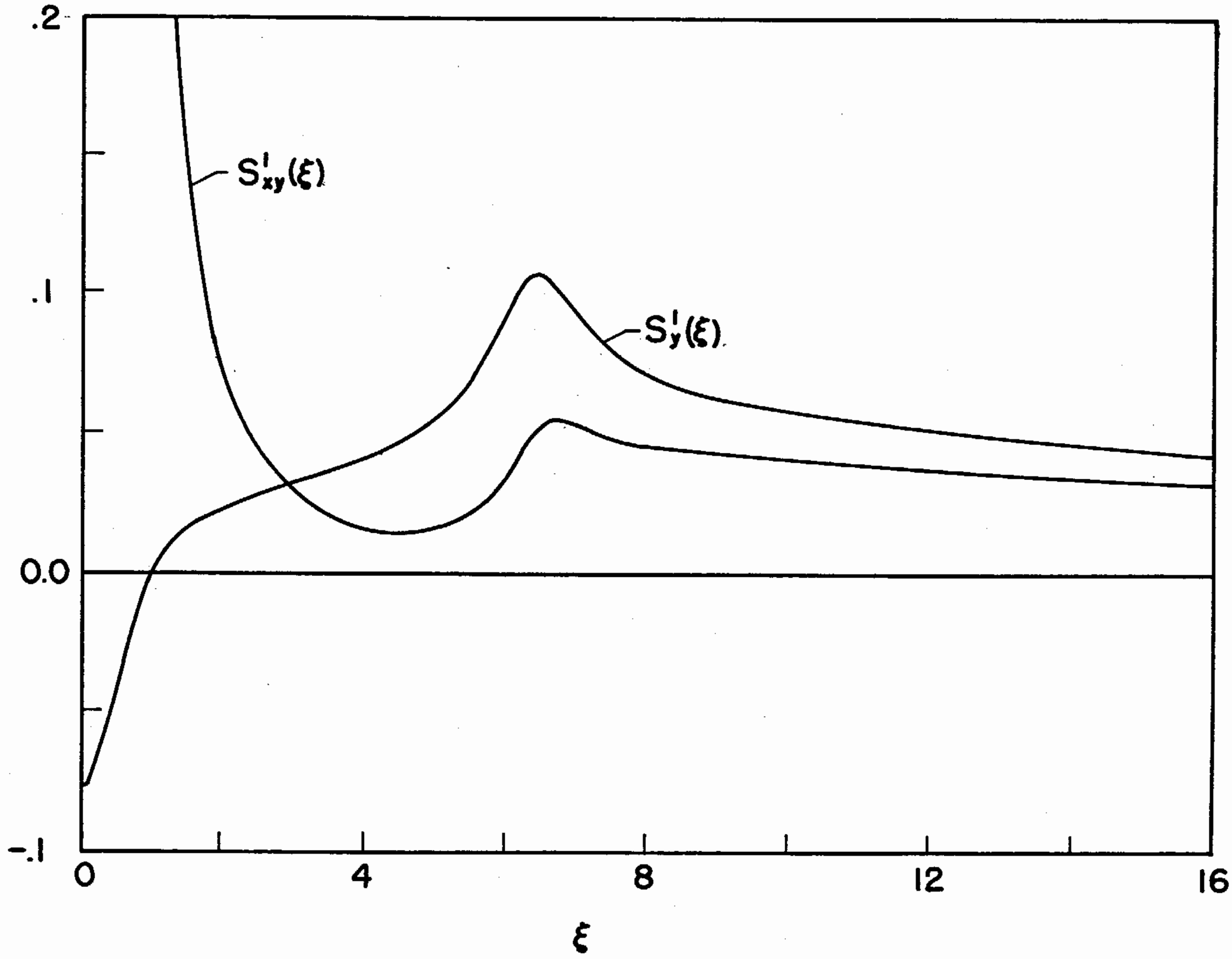

Fig. 6. Amplitude functions for the interface stresses

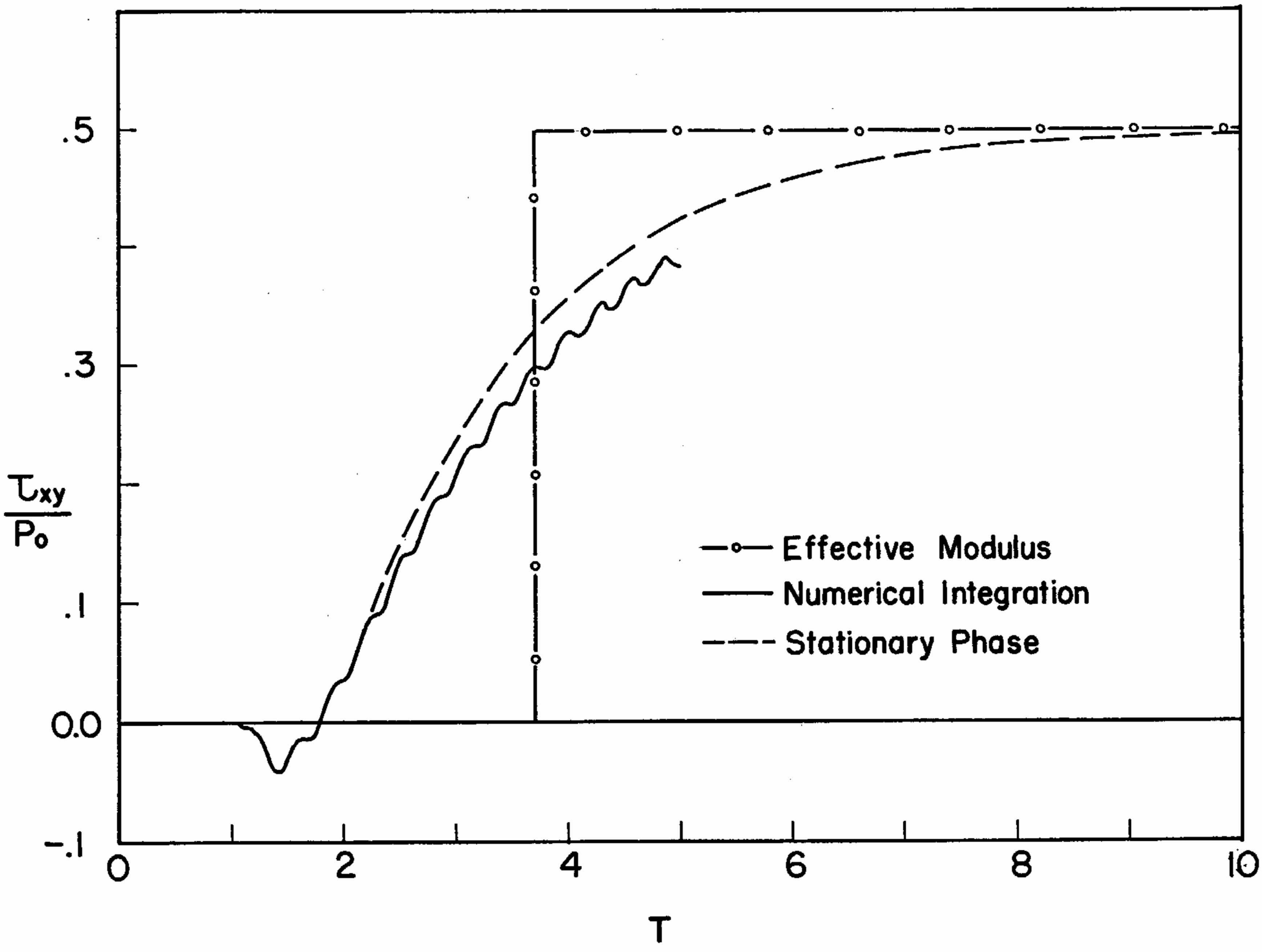

Fig. 7. Interface shear stress at X = 5

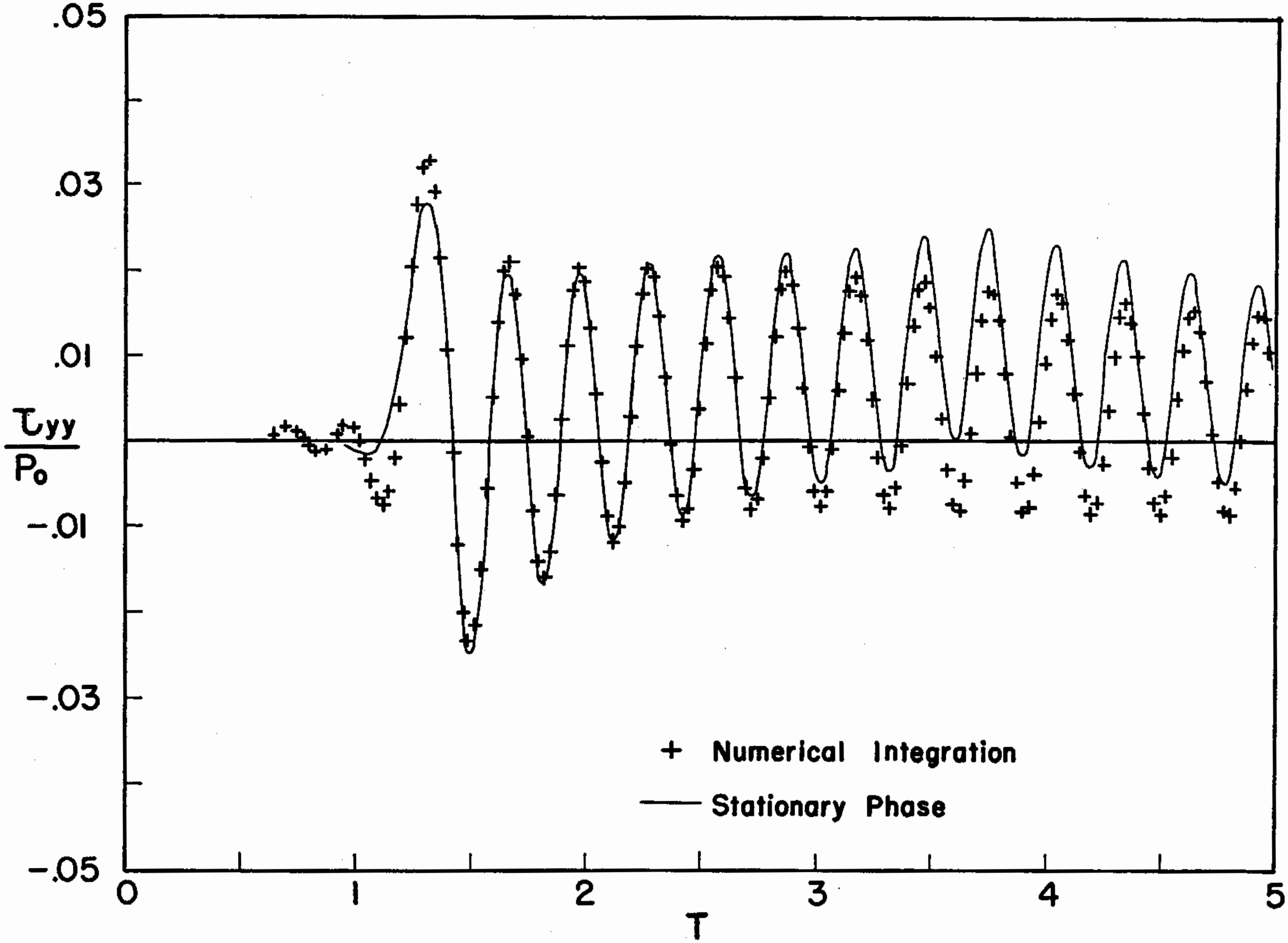

Fig. 8. Normal interface stress at X = 5

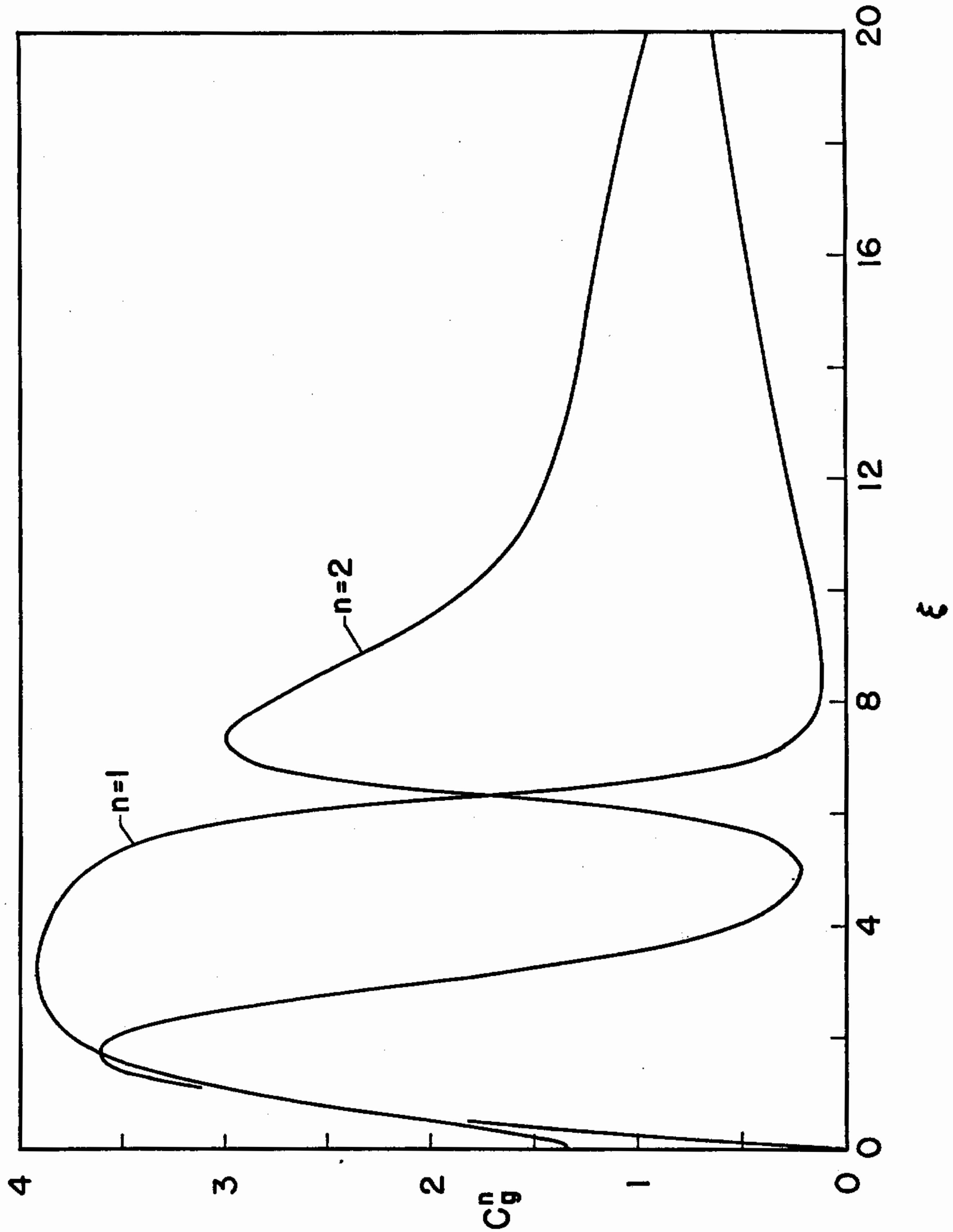

Fig. 9. Dimensionless group velocity

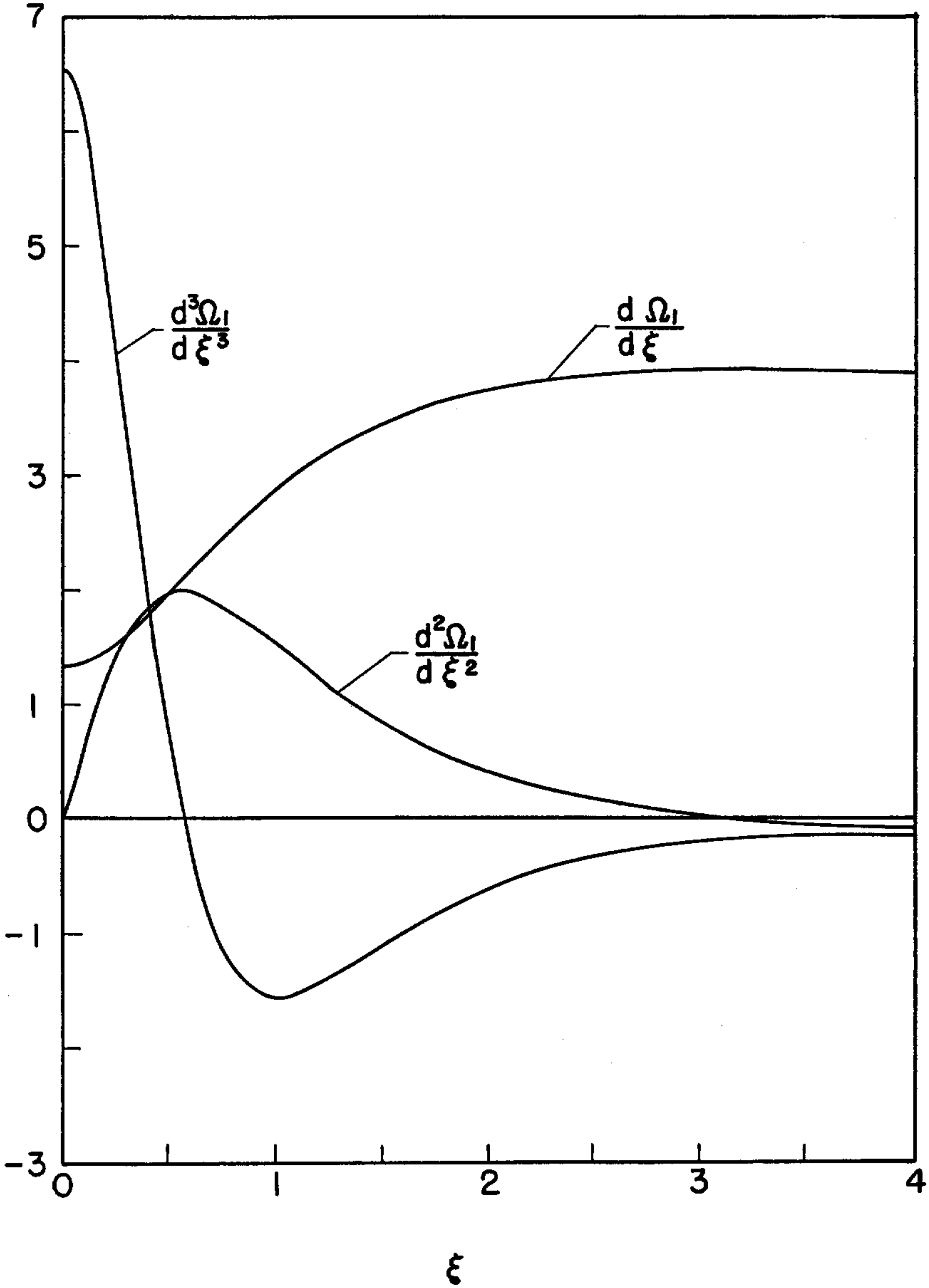

Fig. 10. Dimensionless group velocity and higher derivatives of $\Omega_1(\xi)$

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13. ABSTRACT

A laminated medium composed of alternating layers of two homogeneous isotropic elastic solids is suddenly subjected to transverse forces. The resulting two-dimensional transient wave propagation problem is analyzed by means of modal analysis. The normal and shear stresses at the interfaces are expressed as infinite integrals which are integrated numerically for not too large values of time. For larger values of time, the integrals are approximated by the method of stationary phase. The predominant contribution to the interface shear stress comes from the head-of-the-pulse approximation. The normal stress at the interface, which is composed of several contributions, is oscillatory, and the interface bonds may thus be subjected to tensile stresses.

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