SECTION 2

APPLICATION OF LYAPUNOV'S DIRECT METHOD TO STABILITY PROBLEMS IN ELASTIC AND AEROELASTIC SYSTEMS

by

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ABSTRACT: This section discusses the formulation of stability problems in both elastic and aeroelastic systems in the framework of Lyapunov stability theory. The applicability of Lyapunov's direct method to these problems is further explored along the direction set forth by Movchan and Slobodkin. A number of specific examples are worked out in detail so as to reveal some of the difficulties and to give some idea on the class of problems for which Lyapunov's direct method is readily applicable.

2.1 INTRODUCTION

The problem of determining the stability of equilibrium states of elastic and aeroelastic systems is of considerable practical importance. For example, in the recent development of space vehicles having light flexible structures, the aeroelastic stability problems are intensified by the presence of aerodynamic heating. Effective analytical tools for solving these problems are helpful in practical design.

Although there exists a large amount of literature devoted to this subject, many of the works are concerned with special systems under strong idealized conditions. Because of the stringent assumptions made in deriving systems' mathematical models, and the limitations of the techniques used in the stability analysis, the results are often erroneous or contradictory to the actual physical situations. For example, in applying the often-used modal approach to the stability study of aeroelastic systems, an analysis based on a finite number of modes may not reflect any information on the behavior of all the remaining modes. In fact, it is quite possible that the first few modes have stable behavior while the remaining modes exhibit instability. The energy contained in these high frequency modes may be sufficiently high to cause actual structural damage. In view of the fact that intricate phenomena can occur in many elastic and aeroelastic systems, the development of a unified approach to the solution of their stability problems is questionable. However, effective analytical techniques can be developed for various classes of systems. These techniques can have considerable practical value provided that precautions are taken regarding their range of validity.

One of the most useful mathematical tools for studying the stability of equilibrium of dynamical systems governed by ordinary differential equations has been the direct method of Lyapunov[1]. Although the original work of Lyapunov is devoted primarily to systems having finite degrees of freedom, many of his results have been extended to distributed systems or systems having infinite degrees of freedom[2-6]. Recently, Movchan[7] discussed elastic stability in the framework of Lyapunov's stability theory. Also, Slobodkin[8] considered the stability of equilibrium of conservative linear elastic systems on the basis of Lyapunov's ideas. However, their results are primarily of orienting nature. Therefore, a full assessment of the range of application of Lyapunov's theory to elastic systems is not possible. Similar to Lyapunov's theory for finite-dimensional dynamical systems, the application of his ideas to distributed systems involves the construction of certain functionals having prescribed properties. Unfortunately, this task is an art in itself. Recent attempts have been made in developing systematic methods for constructing these functionals for particular classes of distributed systems (e.g., the work of Brayton and Miranker [9]).

The objective of this section is to explore further the applicability of Lyapunov's direct method to the study of the stability of equilibrium of both elastic and aeroelastic systems. A number of specific examples are worked out in detail so as to reveal some of the difficulties and to give some idea on the class of problems for which Lyapunov's direct method is readily applicable.

2.2 PRELIMINARIES

In order to apply Lyapunov's stability theory to elastic and aeroelastic systems, the physical meaning of elastic stability in the sense of Lyapunov should be clarified.

In what follows, the mathematical terminologies and precise definitions for various degrees of Lyapunov stability for distributed parameter systems will be established first. Then, the physical significance of these stability definitions in the framework of elastic and aeroelastic systems will be discussed. Finally, the pertinent stability theorems will be stated.

2.2.1 Definitions

Consider a dynamical system whose state at any fixed time t is specified by S_t, an element of the metric space Γ with a metric ρ. The distance between two arbitrary states S_t and S_t^* in Γ is specified by $\rho(S_t, S_t^*)$, and two states are considered to be identical when $\rho(S_t, S_t^*) = 0$. For a distributed parameter dynamical system defined on a spatial domain Ω, S_t consists of a set of real-valued functions $\{ u_i(t,X),\ i = 1, \ldots, N\}$ defined on Ω, or an element of a function space $\Gamma(\Omega)$, where X is the spatial coordinate vector. Also, for a distributed parameter system which is coupled to a lumped-parameter sub-system (for example: an elastic beam with lumped masses attached to it at various spatial points), the state space Γ can be taken to be the Cartesian product of a function space and a finite-dimensional Euclidean space.

The system motion starting from an initial state S_{t_o} at time t_o is defined by a two-parameter family of transformations $\Phi_{t_o}^t$, $t \geq t_o \geq 0$, with domain Γ and having the following properties:

(i) For any initial state $S_{t_o} \in \Gamma$, the range of $\Phi_{t_o}^t(S_{t_o})$ is also in Γ. Furthermore,

$$\lim_{t \to t_o} \Phi_{t_o}^t(S_{t_o}) = S_{t_o} .$$

(ii) $\Phi_{t_1}^{t_2}(\Phi_{t_o}^{t_1}(S_{t_o})) = \Phi_{t_o}^{t_2}(S_{t_o})$, with $t_2 \geq t_1 \geq t_o \geq 0$ (the semi-group property).

(iii) $\Phi_{t_o}^{t}$ is continuous in S_{t_o} (in the sense of the metric ρ).

The semi-group property implies that the knowledge of the system state at any time completely determines its future behavior. In general, the problem of establishing precise conditions for which a given mathematical model possesses the above properties is by no means trivial. In the present work, it will be assumed that the given mathematical model describes a dynamical system under certain restrictive conditions.

Definition: An equilibrium state of a dynamical system is an element $S_{eq} \in \Gamma$ such that $\rho(\Phi_{t_o}^{t}(S_{eq}), S_{eq}) = 0$ for all $t \geq t_o$. The set of all equilibrium states of a dynamical system will be called the equilibrium set $\mathcal{E}$.

Definition: An invariant set $\mathcal{J}$ of a dynamical system is a subset of Γ such that for any initial state $S_{t_o} \in \mathcal{J}$, $\Phi_{t_o}^{t}(S_{t_o})$, for any fixed $t \in [t_o, \infty)$, will also remain in $\mathcal{J}$. Obviously, $\{S_{eq}\}$ and $\mathcal{E}$ are invariant sets of the system.

The distance of a particular state S from an invariant set $\mathcal{J}$ is defined by:

$$\rho(S, \mathcal{J}) = \inf_{S' \in \mathcal{J}} \rho(S, S')$$

Also, the distance of a particular motion $\Phi_{t_o}^{t}(S_{t_o})$ from an invariant set is defined by:

$$\rho(\Phi_{t_o}^{t}(S_{t_o}), \mathcal{J}) = \sup_{t,\ S \in \Phi_{t_o}^{t}(S_{t_o})} \rho(S, \mathcal{J})$$

Within the framework set by the foregoing discussions, precise definitions for various degrees of stability of the invariant set of a distributed dynamical system in the sense of Lyapunov can be stated as follows:

Definition: An invariant set $\mathcal{J}$ of a dynamical system is said to be stable with respect to a specific metric ρ, if for every real number $\epsilon > 0$, there exists a real number $\delta(\epsilon, t_o) > 0$ such that the inequality

$$\rho(S_{t_o}, \mathcal{J}) < \delta$$

implies

$$\rho(\Phi_{t_o}^{t}(S_{t_o}), \mathcal{J}) < \epsilon \quad \text{for all} \quad t \geq t_o.$$

If in addition, $\rho(\Phi_{t_o}^{t}(S_{t_o}), \mathcal{J}) \to 0$ as $t \to \infty$, then the invariant set $\mathcal{J}$ is said to be asymptotically stable.

Definition: If an invariant set $\mathcal{J}$ is stable (asymptotically stable) for all initial states $S_{t_o} \in \Gamma$, then $\mathcal{J}$ is said to be stable (asymptotically stable) in the large.

In the case where the stability of a particular equilibrium S_{eq} is of interest, it is convenient to formulate the system equations about S_{eq} so that S_{eq} corresponds to the trivial solution. Note also that stability in the sense Lyapunov is identically the uniform continuity of $\Phi_{t_o}^{t}(S_{t_o})$ in $[t_o, \infty)$ with respect to the initial state S_{t_o}.

In applying the Lyapunov stability definition to elastic and aeroelastic systems, on should define the state space Γ so that it consists of only physically admissible state functions. In most cases, the state space Γ is taken to be a set consisting of only functions possessing spatial derivatives up to sufficiently high order, and satisfying certain conditions at the boundary of the spatial domain. Also, the metric ρ should be chosen so that the stability definitions are compatible with the physical situations. The above points can be clarified by the following example:

Consider an elastic beam of unit length, simply supported at both ends, whose bending motion is governed by:

$$\frac{\partial^2 w(t,x)}{\partial t^2} = \frac{-\partial^4 w(t,x)}{\partial x^4} \tag{2.2-1}$$

with boundary conditions

$$w(t,0) = w(t,1) = \left.\frac{\partial^2 w(t,x)}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 w(t,x)}{\partial x^2}\right|_{x=1} = 0 \tag{2.2-2}$$

A possible choice for the state variables of this system is $\{w(t,x), \partial w(t,x)/\partial t\}$ since the future behavior of the system can be determined completely by specifying its initial displacement and velocity for all $x \in (0,1)$. With this choice of state variables, one may introduce a metric ρ defined by

$$\rho_1(S_t, 0) = \sup_{x \in (0,1)} |w(t,x)| + \sup_{x \in (0,1)} \left| \frac{\partial w(t,x)}{\partial t} \right| \qquad (2.2\text{-}3)$$

which is simply a measure of the "closeness" between the state S_t at time t and the equilibrium null state, in terms of the maximum displacement and velocity of the beam along its entire length.

In order that the equilibrium null state is to be Lyapunov stable with respect to to metric ρ_1, there must exist a δ for every ϵ such that when $\rho_1(S_o, 0) < \delta$, we have $\rho_1(\Phi_0^t(S_o), 0) < \epsilon$ for all $t \geq 0$.

Consider all solutions of (2.2-1) corresponding to the following particular set of initial states:

$$\Gamma_o = \left\{ S_o : w(0,x) = c_o \sin(n\pi x), \left. \frac{\partial w(t,x)}{\partial t} \right|_{t=0} = 0, \right.$$

$$\left. \text{for } c_o = \text{arbitrary real number, } n = 1, 2, \ldots \right\}$$

(2.2-4)

and satisfying boundary conditions (2.2-2):

$$w_n(t,x) = c_o \sin(n\pi x) \cos(n\pi t), \quad n = 1, 2 \ldots. \qquad (2.2\text{-}5)$$

In view of (2.2-3), for any initial state $S_o \in \Gamma_o$, $\rho_1(S_o, 0) = |c_o|$, $\rho_1(S_t, 0) = \rho_1(\{w_n(t,x), \partial w_n(t,x)/\partial t\}, 0) = |c_o \cos(n\pi t)| + |c_o n \pi \sin(n\pi t)|$. Consider the set of all initial states S_o for which $\rho_1(S_o, 0) < c'$, where c' is a specified positive number. Since the maximum value of $\rho_1(\Phi_0^t(S_o), 0)$ with respect to t is $|c_o| (1+n^2\pi^2)^{1/2}$, it can be made arbitrarily large by choosing a S_o with large n, although $\rho_1(S_o, 0) = |c_o| < c'$. Thus, there does not exist a δ for every ϵ such that whenever $\rho_1(S_o, 0) < \delta$, we have $\rho_1(\Phi_o^t(S_o), 0) < \epsilon$ for all $t \geq 0$. On the other hand, it can be easily verified that Lyapunov's stability condition with respect to metric ρ_1 can be satisfied if we take the set of initial states to be

$$\Gamma_1 = \left\{ W_o : w(0,x) = \frac{c_1}{n^2} \sin(n\pi x), \left. \frac{\partial w(t,x)}{\partial t} \right|_{t=0} = 0, \right.$$

$$\left. \text{for } c_1 = \text{arbitrary real number, } n = 1, 2, \ldots \right\}$$

(2.2-6)

which implies that the maximum absolute amplitude of the initial displacements tends to zero as $n \rightarrow \infty$. Alternatively, if we retain the set of initial states Γ_0, but redefine the metric to be:

$$\rho_2(S_t, 0) = \left[\int_0^1 \left\{ \left(\frac{\partial w(t,x)}{\partial t} \right)^2 + \left(\frac{\partial^2 w(t,x)}{\partial x^2} \right)^2 \right\} dx \right]^{1/2} \qquad (2.2\text{-}7)$$

then, for any initial state $S_0 \in \Gamma_0$,

$$\rho_2(S_t, 0) = \frac{1}{\sqrt{2}} \; n\pi \; |c_0| \; (1-(1-n^2\pi^2)\cos^2 n\pi t)^{1/2} \qquad (2.2\text{-}8)$$

and

$$\max_{t \geq 0} \; \rho_2(S_t, 0) = \rho_2(S_0, 0) = \frac{1}{\sqrt{2}} \; n^2\pi^2 \, |c_0| \, . \qquad (2.2\text{-}9)$$

Clearly, the Lyapunov stability condition with respect to ρ_2 is satisfied.

The foregoing conclusions have the following physical interpretation: In the case where the initial states $S_0 \in \Gamma_0$ and the metric is defined by ρ_1, although the maximum absolute initial displacement amplitude is small, the initial strain energy $\mathcal{E}_0$ defined by

$$\mathcal{E}_0 = \frac{1}{2} \int_0^1 \left(\frac{\partial^2 w(0,x)}{\partial x^2} \right)^2 dx$$

can be arbitrarily large. As a result, $\rho_1(S_t, 0)$ can be also arbitrarily large and hence we do not have Lyapunov stability. On the other hand, Lyapunov stability can be achieved if we restrict the initial states to members of a class of functions Γ_1 for which the foregoing possibility is avoided. Now, consider the case where a metric ρ_2 (defined by(2.2-7)) is used. Since $\rho_2(S_t, 0)$ is identically $\sqrt{2\,\mathcal{E}_t}$, where $\mathcal{E}_t$ is the total energy of the system (2.2-1) (kinetic plus strain energy) at any time t, Lyapunov stability with respect to this metric implies that if the total energy of the initial state is bounded, then the total energy of the system will remain bounded for all $t > 0$. This is certainly true for system (2.2-1), since it is conservative. In view of the foregoing discussions, it should be apparent that the choice of the initial state space Γ and the metric ρ is crucial in the formulation of the elastic stability problem in the framework of Lyapunov stability theory, which implies, in physical terms, that one must first define the class of admissible initial perturbations and a measure for the distance between states.

In many physical situations, it is of interest to establish boundedness of some of the state variables or some given function of the state variables. For example, one may be interested only in ensuring boundedness of the maximum deflections in an elastic system. In these cases, it is meaningful to define Lyapunov stability with respect to two metrics.[7,8,10] To fix ideas, consider again a dynamical system whose motion starting from an initial state $S_{t_o} \in \Gamma$ t_o is defined by a transformation $\Phi_{t_o}^{t}$. Let the range of $\Phi_{t_o}^{t}$ be denoted by Γ_s, and let there be defined a continuous mapping $\mathcal{M}$ from $\Gamma_s \rightarrow \widetilde{\Gamma}$ which will be referred to hereafter as an auxiliaty state space. For example, in the case of system (2.2-1) with its state defined by $S_t = \{w(t,x), \partial w(t,x)/\partial t\}$, a possible form for $\mathcal{M}$ may be:

$$\widetilde{S}_t = \mathcal{M} S_t = \begin{bmatrix} \int_0^1 g_1(x)\,(\cdot)dx & 0 \\ 0 & \int_0^1 g_2(x)\,(\cdot)dx \end{bmatrix} \begin{bmatrix} w(t,x) \\ \dfrac{\partial w(t,x)}{\partial t} \end{bmatrix} \qquad (2.2\text{-}10)$$

where $g_i(x)$ are specified spatial weighting functions. Thus, the components of vector $\widetilde{S}_t$ (referred to hereafter as "stability variables") correspond to spatially averaged displacement and velocity of the beam. Note here that $\widetilde{\Gamma}$ is a two-dimensional Euclidean space. The notion of "closeness" between two initial states S_t and S'_t in Γ, and two elements $\widetilde{S}_t$ and $\widetilde{S}'_t$ in $\widetilde{\Gamma}$ can be established by defining a metric $\rho(S_t, S'_t)$ in Γ, and another metric $\widetilde{\rho}(\widetilde{S}_t, \widetilde{S}'_t)$ in $\widetilde{\Gamma}$. Also, the notions of an invariant set and an equilibrium set in $\widetilde{\Gamma}$ can be defined in the same manner given previously. Here, we assume that $\mathcal{M}$ is continuous and has the property that it maps an invariant set in Γ onto a corresponding invariant set $\widetilde{\mathcal{J}}$ in $\widetilde{\Gamma}$.

With the above notions, one may define Lyapunov stability with respect to two metrics ρ and $\widetilde{\rho}$ as follows:

Definition: An invariant set $\mathcal{J}$ of a dynamical system is said to be stable with respect to two metrics ρ and $\widetilde{\rho}$, if for every real number $\epsilon > 0$, there exists a real number $\delta(\epsilon, t_o) > 0$ such that the inequality

$$\rho(S_{t_o}, \mathcal{J}) < \delta$$

implies

$$\widetilde{\rho}(\mathcal{M}\Phi_{t_o}^{t}(S_{t_o}), \mathcal{J}) < \epsilon \quad \text{for all } t \geq t_o .$$

If in addition, $\widetilde{\rho}(\mathcal{M}\Phi_{t_o}^{t}(S_{t_o}), \mathcal{J}) \rightarrow 0$ as $t \rightarrow \infty$, then the invariant set $\mathcal{J}$ is said to be asymptotically stable with respect to two metrics ρ and $\widetilde{\rho}$.

The physical meaning of the above definition can be clarified by considering again system (2.2-1) with mapping $\mathcal{M}$ given by:

$$\mathcal{S}_t = \mathcal{M}S_t = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w(t,x) \\ \frac{\partial w(t,x)}{\partial t} \end{bmatrix} \qquad (2.2\text{-}11)$$

Let the metric ρ be defined by ρ_2 given by (2.2-7), and $\tilde{\rho}$ be defined by:

$$\rho_3(\mathcal{M}S_t, 0) = \sup_{x \in (0,1)} |w(t,x)| \qquad (2.2\text{-}12)$$

The stability of the equilibrium null state of this system with respect to metrics ρ_2 and ρ_3 implies that if the total energy of the initial state is bounded, then the maximum deflection of the beam will remain bounded for all $t > 0$.

2.2.2 Stability Theorems

Fundamental to Lyapunov's direct method as applied to stability problems with two metrics is the selection of a family of functionals V which will give some estimate of $\tilde{\rho}(\mathcal{M}\Phi^t_{t_o}(S_{t_o}), \mathcal{J})$. If it is possible to show that the maximum value of $V(\mathcal{M}\Phi^t_{t_o}(S_{t_o}))$ (V evaluated along any perturbed motion $\mathcal{M}\Phi^t_{t_o}(S_{t_o})$) will be small whenever $\rho(S_{t_o}, \mathcal{J})$ is small, then we have stability. If, in addition, $V(\mathcal{M}\Phi^t_{t_o}(S_{t_o})) \to 0$ as $t \to \infty$, then we have asymptotic stability.

The following theorems are essentially those of Zubov[2] except that they are stated in terms of two metrics. The proofs for these theorems follow directly from those of Zubov with minor modifications. For clarification, a proof for Theorem 1 will be outlined in Appendix 1.

Theorem 1: In order for an invariant set $\mathcal{J}$ of a dynamical system to be stable with respect to two metrics ρ and $\tilde{\rho}$, it is necessary and sufficient that there exist a one-parameter family of functionals V_t defined on $\tilde{\Gamma}$ for all $\mathcal{S}$ satisfying $\tilde{\rho}(\mathcal{S}, \mathcal{J}) \le r$, and having the following properties:

(i) V_t is continuous with respect to metric ρ, i.e., for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that the inequality $|V_t(\mathcal{M}S_{t_o})| < \epsilon$ holds for all $S_{t_o} \in \Gamma$ satisfying the condition $\rho(S_{t_o}, \mathcal{J}) < \delta$.

(ii) V is positive definite with respect to metric $\tilde{\rho}$, i.e., for any sufficiently small number $\alpha_1 > 0$, there exists a number $\alpha_2 > 0$ such that when $\tilde{\rho}(\mathcal{M}S_t, \tilde{\mathcal{J}}) > \alpha_1$, we have $V_t(\mathcal{M}S_t) > \alpha_2$ for all $t > t_o$; furthermore, $V_t(\mathcal{M}S_t) \to 0$ uniformly with respect to t as $\tilde{\rho}(\mathcal{M}S_t, \tilde{\mathcal{J}}) \to 0$.

(iii) V_t evaluated along any perturbed motion $\mathcal{M}\Phi^t_{t_o}(S_{t_o})$ is nonincreasing for all $t > t_o$, provided that $\rho(S_{t_o}, \mathcal{J}) \leq \delta_o$ where δ_o is a sufficiently small positive number.

Theorem 2: In order for an invariant set $\mathcal{J}$ of a dynamical system to be asymptotically stable with respect to two metrics ρ and $\tilde{\rho}$, it is necessary and sufficient that there exist a one-parameter family of functionals V_t having the following properties:

(i) All conditions of Theorem 1 are satisfied.

(ii) V_t evaluated along any perturbed motion $\mathcal{M}\Phi^t_{t_o}(S_{t_o})$ tends to zero as $t \to \infty$, provided that $\rho(S_{t_o}, \mathcal{J}) \leq \delta_o$, where δ_o is a sufficiently small positive number.

Note that Theorems 1 and 2 reduce to those of Zubov if we take $\mathcal{M}$ to be an identity transformation and $\rho = \tilde{\rho}$.

2.3 GENERAL DISCUSSION

In the previous section, we have outlined the basic concepts and main results in Lyapunov's theory of stability for distributed parameter dynamical systems, and discussed briefly the physical meaning of the stability of equilibrium for elastic systems in the sense of Lyapunov. Now, we shall proceed to discuss the formulation of stability problems for general elastic and aeroelastic systems in the framework of Lyapunov under reasonable physical assumptions, and to derive stability conditions for certain special classes of systems.

2.3.1 Elastic Systems

Consider an arbitrary (perfect) elastic body (i.e., a body whose stress depends only on the deformation and not on its deformation history) which, at any time t, occupies certain spatial domain Ω_t. In general, the motion of the elastic body may be induced by the presence of a certain body force F (per unit mass) acting on the body, which may vary from point to point in Ω_t. Also, the motion may be caused by the presence of a certain surface force T (per unit area) acting on all or portions of the material boundary $\partial\Omega_t$ of the body.

In addition to external influences, the deforming motion of the elastic body depends on its internal energy which is functionally related to its configuration and the temperature distribution within the body. The motion also depends on the rate at which energy is extracted from or injected into the elastic body through the material boundary $\partial \Omega_t$, and the rate at which energy is transformed from one form to another within the body. An example of the latter phenomenon is the generation of heat due to mechanical friction between the material particles during the deforming motion. In many physical situations, certain simplifying assumptions can be made in deriving the mathematical model governing the motion of the elastic body by considering the special nature of the deforming process. For example, in the case where the deforming motion is rapid, it may be reasonable to assume that the process is adiabatic.

Finally, the motion of the elastic body is affected by the existence of certain physical constraints imposed on portions of the body. These constraints usually take the form of a set of prescribed conditions on the displacement, moments, forces, temperature, etc., at the boundary of Ω_t . A typical example of these constraints is the attachment of portions of the body to certain rigid reference frame as in the case of a cantilever beam.

With all the surface and body forces, constraints, and material properties of the elastic body well-defined, the elastic body usually possesses one or more equilibrium states at which the temperature distribution and the positions and velocities of all the material particles of the body measured with respect to certain reference frame remain invariant with time. The basic problem of elastic stability is to determine whether or not a particular mechanical equilibrium state (equilibrium positions and velocities of material particles) or a particular set of mechanical equilibrium states is stable. Here, the stability of mechanical equilibrium, from the physical standpoint, implies that the system's deforming motion (i.e., motions of all material particles of the elastic body) corresponding to any sufficiently small initial perturbation about the equilibrium state (including possibly perturbations of temperature about its equilibrium distribution) will remain bounded for all subsequent times. As discussed earlier in section 2.2, the above notion of stability can be expressed mathematically in the framework of Lyapunov's stability theory by introducing appropriate metrics in the initial state and auxiliary state spaces. In essence, these metrics establish precise measures of "smallness" of perturbations and "boundedness" of motions of the elastic system about the equilibrium.

In order to apply Lyapunov's direct method to the elastic stability problem, a mathematical description of the motion of the elastic body under the influences of certain prescribed body and surface forces, and constraints must be established. Here, an important and difficult problem is to express the stress-strain and strain-displacement relations in appropriate mathematical forms. Since these relations are generally nonlinear, the solutions to the resulting equations of motion may not be unique. However, this fact does not contribute any difficulties in applying Lyapunov's direct method since uniqueness of solutions is not assumed in proving the stability theorems. On the other hand, the main difficulty in applying the direct method lies in the construction of functionals V satisfying the properties prescribed in the stability theorems.

In the sequel, we shall only consider the class of elastic systems for which stability conditions can be derived by taking V to be an energy functional.

2.3.1.1 Equations of Motion

Let X_i be a fixed Cartesian coordinate system with respect to which the motion of an elastic body may be described. Let the coordinates of a particular material particle of the elastic body in the stress-free state be denoted by a_i.

Upon application of certain time-invariant initial surface force $T_i^{(o)}$, initial body force $F_i^{(o)}$, and possibly heat, the body deforms into one of its possible equilibrium states. A material particle whose coordinates are a_i in the stress-free state moves to $x_i^{(o)}$ in the new equilibrium state, and hence its displacement r_i is given by:

$$r_i = x_i^{(o)} - a_i \tag{2.3-1}$$

By the assumption that there is no body moment, the stress tensor is symmetric. Thus, the following conditions for mechanical equilibrium must be satisfied:

$$\rho_o F_i^{(o)} + \frac{\partial \sigma_{ij}^{(o)}}{\partial x_j^{(o)}} = 0 \tag{2.3-2}$$

and boundary conditions:

$$T_i^{(o)} = \sigma_{ij}^{(o)} n_j^{(o)} \tag{2.3-3}$$

where $\sigma_{ij}^{(o)}$ is the initial stress tensor, $n_j^{(o)}$ are the components of the outward unit normal vector at a point on the equilibrium material boundary surface, and ρ_o is the mass density at equilibrium.

The relationship between the initial stress and strain for a general anisotropic, inhomogeneous elastic body as given by Murnaghan[11] can be expressed in the form:

$$\sigma_{ij}^{(o)} = \rho_o \frac{\partial \mu_o}{\partial \epsilon_{k\ell}^{(o)}} \frac{\partial x_i^{(o)}}{\partial a_k} \frac{\partial x_j^{(o)}}{\partial a_\ell} \tag{2.3-4}$$

where μ_o is the internal energy per unit mass, and $\epsilon_{k\ell}$ is the Lagrangian strain tensor given by:

$$\epsilon^{(o)}_{k\ell} = \frac{1}{2}\left[\frac{\partial r_k}{\partial a_\ell} + \frac{\partial r_\ell}{\partial a_k} + \frac{\partial r_m}{\partial a_k}\,\frac{\partial r_m}{\partial a_\ell}\right] \qquad (2.3\text{-}5)$$

In order to study the stability of a particular equilibrium state via Lyapunov's direct method, it is necessary to derive the dynamical equations for the perturbed motion of the elastic body about the equilibrium state under consideration.

In physical situations, there exists, in addition to the initial static forces $T_i^{(o)}$ and $F_i^{(o)}$, external perturbing forces which are defined about the equilibrium state. Let the total surface and body forces be denoted by $(T_i^{(o)} + \hat{T}_i)$ and $(F_i^{(o)} + \hat{F}_i)$ respectively, where both $\hat{T}_i$ and $\hat{F}_i$ vanish when the body is at equilibrium. Also, we shall denote the position of a material particle by x_i and its displacement about the equilibrium $x_i^{(o)}$ by:

$$u_i = x_i - x_i^{(o)}\,, \qquad (2.3\text{-}6)$$

and the perturbed stress tensor by $(\sigma_{ij}^{(o)} + \hat{\sigma}_{ij})$.

The equations for the perturbed motion about the equilibrium can be written as:

$$(\rho_o + \hat{\rho})\,\frac{D^2 u_i}{Dt^2} = (\rho_o + \hat{\rho})\,(F_i^{(o)} + \hat{F}_i) + \frac{\partial(\sigma_{ij}^{(o)} + \hat{\sigma}_{ij})}{\partial x_j} \qquad (2.3\text{-}7)$$

with boundary conditions:

$$(T_i^{(o)} + \hat{T}_i) = (\sigma_{ij}^{(o)} + \hat{\sigma}_{ij})\,(n_j^{(o)} + \hat{n}_j) \qquad (2.3\text{-}8)$$

where $(n_j^{(o)} + \hat{n}_j)$ are the components of an outward normal vector at a point on the perturbed material boundary surface, $(\rho_o + \hat{\rho})$ is the perturbed mass density, and D/Dt denotes material differentiation.

Equations (2.3-7) and (2.3-8), in view of (2.3-2) and (2.3-3), reduce to:

$$(\rho_o + \hat{\rho})\,\frac{D^2 u_i}{Dt^2} = (\rho_o + \hat{\rho})\,\hat{F}_i + \hat{\rho}\,F_i^{(o)} + \frac{\partial\hat{\sigma}_{ij}}{\partial x_j} \qquad (2.3\text{-}7^*)$$

and

$$\hat{T}_i = (\sigma_{ij}^{(o)} + \hat{\sigma}_{ij})\,\hat{n}_j + \hat{\sigma}_{ij}\,n_j^{(o)} \qquad (2.3\text{-}8^*)$$

The perturbed stress increment $\hat{\sigma}_{ij}$ is related to the strain by:

$$\hat{\sigma}_{ij} = (\rho_o + \hat{\rho}) \frac{\partial(\mu_o + \hat{\mu})}{\partial(\epsilon^{(o)}_{k\ell} + \hat{\epsilon}_{k\ell})} \frac{\partial x_i}{\partial a_k} \frac{\partial x_j}{\partial a_\ell} - \partial^{(o)}_{ij} \qquad (2.3\text{-}9)$$

where

$$\hat{\epsilon}_{k\ell} = \frac{1}{2} \left[\frac{\partial u_m}{\partial a_k} \frac{\partial u_m}{\partial a_\ell} + \frac{\partial x^{(o)}_m}{\partial a_\ell} \frac{\partial u_m}{\partial a_k} + \frac{\partial x^{(o)}_m}{\partial a_k} \frac{\partial u_m}{\partial a_\ell} \right]. \qquad (2.3\text{-}10)$$

In the case where the elastic body is incompressible so that the deformation is isochoric, (2.3-7*) simplifies to

$$\rho_o \frac{D^2 u_i}{Dt^2} = \rho_o \hat{F}_i + \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \qquad (2.3\text{-}9^*)$$

The foregoing equations describe only the perturbed motion of an elastic body about it s mechanical equilibrium. In general, the temperature distribution within the body may vary with time as a result of straining and extraction (or injection) of heat energy through its boundary surface. The consideration of thermoelastic effects generally leads to an additional equation governing the variation in the temperature distribution or the internal energy density of the elastic body.[12] This equation is coupled with (2.3-7*). Thus, a complete specification of the state of the elastic body at any time includes both the elastic and thermodynamic state variables. However, in many practical situations, the effect of thermal stress may be included in (2.3-7*) by introducing a fictitious body force.

2.3.1.2 Stability of Equilibrium

In order to formulate the Lyapunov stability problem with respect to two metrics for an elastic body described in the previous section, it is necessary to specify:

(i) an initial state space Γ along with metric ρ, consisting of all the admissible initial perturbations of the elastic body about its particular equilibrium state under consideration, which generally includes both elastic and thermodynamic (e.g., temperature) perturbations:

(ii) an auxiliary state space $\tilde{\Gamma}$ along with metric $\tilde{\rho}$, defining the set of all functions which the stability variables can take on at any fixed time.

Here, we shall consider the stability problem in which ρ corresponds to a measure of the total energy of the elastic system. To apply Lyapunov's stability theorems to this problem, we take V to be the total energy:

$$V = \int_{\Omega_t} (\rho_o + \hat{\rho}) \left[\frac{1}{2} \frac{Du_i}{Dt} \frac{Du_i}{Dt} + \mu_o + \hat{\mu} \right] d\,\Omega_t \qquad (2.3\text{-}11)$$

where $\hat{\mu}$ is the increment in internal energy per unit mass.

The rate of change of V with respect to time, in view of (2.3-7*), is given by:

$$\frac{DV}{Dt} = \int_{\Omega_t} \left[(\rho_o + \hat{\rho})\, \hat{F}_i + \hat{\rho}\, F_i^{(o)} + \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \right] \frac{Du_i}{Dt}\, d\,\Omega_t + \int_{\Omega_t} (\rho_o + \hat{\rho}) \frac{D\hat{\mu}}{Dt}\, d\,\Omega_t$$

(2.3-12)

If we introduce a metric $\tilde{\rho}$ with respect to which V is positive definite, then from Theorem 1, the equilibrium state will be stable with respect to ρ, $\tilde{\rho}$, if DV/Dt given by (2.3-12) is non-positive for all $t \geq t_o$.

For the case where the deformation is sufficiently small and can be considered to be isothermal, we can approximate the internal energy density by a three-term Taylor series. Thus,

$$\hat{\mu} \approx \frac{\partial \mu_o}{\partial \epsilon_{k\ell}^{(o)}} \hat{\epsilon}_{k\ell} + \frac{1}{2} \frac{\partial^2 \mu_o}{\partial \epsilon_{k\ell}^{(o)} \partial \epsilon_{mn}^{(o)}} \hat{\epsilon}_{k\ell}\, \hat{\epsilon}_{mn} \qquad (2.3\text{-}13)$$

Substituting (2.3-13) into (2.3-12) leads to the following sufficient condition for stability:

$$\frac{DV}{Dt} = \int_{\Omega_t} \left[(\rho_o + \hat{\rho})\, \hat{F}_i + \hat{\rho}\, F_i^{(o)} + \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \right] \frac{Du_i}{Dt}\, d\,\Omega_t$$

$$+ \int_{\Omega_t} (\rho_o + \hat{\rho},) \left[\frac{\partial \mu_o}{\partial \epsilon_{k\ell}^{(o)}} \frac{D\hat{\epsilon}_{k\ell}}{Dt} + \frac{1}{2} \frac{\partial^2 \mu_o}{\partial \epsilon_{k\ell}^{(o)} \partial \epsilon_{mn}^{(o)}} \left(\frac{D\hat{\epsilon}_{k\ell}}{Dt} \hat{\epsilon}_{mn} + \hat{\epsilon}_{k\ell} \frac{D\hat{\epsilon}_{mn}}{Dt} \right) \right] d\Omega_t$$

$$\leq 0 \quad \text{for all } t \geq t_o. \qquad (2.3\text{-}14)$$

For adiabatic deformations, the quantity DV/Dt, in view of the First Law of Thermodynamics, is equal to the rate at which work is being done on the system by the external forces, i.e.,

$$\frac{DV}{Dt} = \frac{DW}{Dt} = \int_{\Omega_t} (\rho_o + \hat{\rho})\, (F_i^{(o)} + F_i) \frac{Du_i}{Dt}\, d\,\Omega_t + \int_{\partial\Omega_t} (T_i^{(o)} + T_i) \frac{Du_i}{Dt}\, d(\partial\,\Omega_t)$$

(2.3-15)

The second integral in (2.3-15) can be transformed into a volume integral by using boundary conditions (2.3-8) and Green's theorem:

$$\frac{DW}{Dt} = \int_{\Omega_t} \left\{ \left[(\rho_o + \hat{\rho}) \hat{F}_i + \hat{\rho} F_i^{(o)} + \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \right] \frac{Du_i}{Dt} + (\sigma_{ij}^{(o)} + \hat{\sigma}_{ij}) \frac{\partial}{\partial x_j} \frac{Du_i}{Dt} \right\} d\,\Omega_t \qquad (2.3\text{-}16)$$

which, in view of(2.3-9), reduces to:

$$\frac{DW}{Dt} = \int_{\Omega_t} \left\{ \left[(\rho_o + \hat{\rho}) \hat{F}_i + \hat{\rho} F_i^{(o)} + \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \right] \frac{Du_i}{Dt} + (\rho_o + \hat{\rho}) \frac{\partial(\mu_o + \hat{\mu})}{\partial(\epsilon_{kl}^{(o)} + \hat{\epsilon}_{kl})} \frac{\partial x_i}{\partial a_k} \frac{\partial x_j}{\partial a_l} \frac{\partial}{\partial x_j} \frac{Du_i}{Dt} \right\} d\,\Omega_t \qquad (2.3\text{-}17)$$

For stability, we require DW/DT to be ≤ 0 for all $t \geq t_o$.

In order to derive sufficient conditions for stability in terms of the parameters of the given system, it is necessary to obtain estimates of DV/Dt for a sufficiently wide class of functions which includes the solution space as a subset. This task, in general, is by no means trivial. For conservative systems, DV/Dt is identically zero for all t. It follows from Theorem 1 that the positive definiteness of V with respect to ρ is both necessary and sufficient for stability. For a general system, it is most likely that useful estimates of the stability regions in the parameter space cannot be derived by taking V directly to be the total energy of the system, but instead a functional which is continuous with respect to ρ is considered. In general, this functional may not have direct physical meaning.

2.3.2 Aeroelastic Systems

An aeroelastic system basically consists of an elastic solid system interacting with an aerodynamic system. Since very complex phenomena can occur in aeroelastic systems, only a brief qualitative discussion of the stability problems will be given here.

The formulation of stability problems for an aeroelastic system in the framework of Lyapunov generally necessitates a mathematical description of the perturbed motions of both the solid and fluid systems about their equilibrium states. The basic mathematical model is composed of two sets of equations in the form of (2.3-7), one for the fluid and the other for the solid. Also, additional equations may be required to describe the thermodynamic states of the solid and fluid systems. In many physical situations, the fluid can be assumed to obey Newton's hypothesis that the components of the stress tensor at a fixed point in space and time depend linearly on the components of the rate of deformation tensor at that point. This assumption leads to the well-known Navier-Stokes equation. The interaction between the solid and fluid systems is expressed in the form of a set of boundary conditions which may be given in terms of surface forces, velocities, temperatures, etc. For example, for a viscous fluid, it may be justifiable in certain cases to assume a no-slip condition implying that the local relative velocity between the fluid and the solid is zero at the boundary surface.

In many physical situations where the velocities of the perturbed motion of the elastic solid are small as compared to those of the fluid, crude but useful mathematical models may be derived by neglecting the fluid dynamics completely. The effect of the fluid on the elastic solid is essentially of a quasi-steady-state nature.

In aeroelastic stability problems, the behavior of the perturbed motion of the elastic solid about its mechanical equilibrium state is of primary interest. Therefore, it is meaningful to define the metric $\tilde{\rho}$ on the auxiliary state space corresponding to the state variables of the elastic solid only, and the metric ρ on the state space of the complete aeroelastic system. A physical interpretation of the stability of equilibrium of an aeroelastic system in the sense of Lyapunov with respect to two metrics ρ, $\tilde{\rho}$ is that if the initial perturbations of both the elastic solid and fluid systems are sufficiently small in the sense of metric ρ, then the subsequent motions of the elastic solid will remain bounded in the sense of metric $\tilde{\rho}$.

2.4 PARTICULAR SYSTEMS

In this section, explicit conditions for the stability of equilibrium of particular elastic and aeroelastic systems will be derived via Lyapunov's direct method. The objective here is to give specific examples of classes of systems for which Lyapunov's direct method are readily applicable.

2.4.1 Elastic Panel

An elastic panel with finite chord, infinite span, and uniform thickness as shown in Figure 2-1 will be considered here. It is assumed that two edges of the panel are pinned. Furthermore, the distance between two edges can be varied externally by a small amount Δs, and the panel can also undergo thermal expansion due to its temperature variation.

Assuming that the panel is thin and deflections are sufficiently small, and the unstressed panel is perfectly flat, the equation of motion for the dimensionless deflection $w(t,y)$, normalized with respect to the chord length ℓ, can be expressed in the form:

$$\frac{\partial^4 w(t,y)}{\partial y^4} - \frac{\ell^2 N_y}{D} \frac{\partial^2 w(t,y)}{\partial y^2} + \frac{\partial^2 w(t,y)}{\partial t^2} + \frac{\zeta(y)\,\ell^2}{\sqrt{D\rho_s h}} \frac{\partial w(t,y)}{\partial t} = 0 \tag{2.4-1}$$

where y is the dimensionless spatial coordinate normalized with respect to ℓ, t is the dimensionless time normalized with respect to the quantity $(\rho_s h \ell^4/D)^{1/2}$, ρ_s is the mass density, D is the bending rigidity of the panel, and $\zeta(y)$ is a distributed viscous damping coefficient.

The compressive load $-N_y$ due to displacement of the edges and temperature rise ΔT of the panel above that of the edges can be expressed in the form:[13]

$$-N_y = \frac{-E_1 h}{2} \int_0^1 \left(\frac{\partial w(t,y)}{\partial y}\right)^2 dy - \frac{E_1 h}{\ell} \Delta s + E_2 h \alpha \Delta T \tag{2.4-2}$$

where, if the spanwise stress vanishes,

$E_1 = E_2 = E$, the Young's modulus

and if the spanwise strain vanishes,

$E_1 = E/(1-\nu^2), \quad E_2 = E/(1-\nu)$

where ν is the Poisson's ratio and α is the coefficient of linear thermal expansion.

The boundary conditions for this panel with pinned edges are given by:

$w(t,0) = w(t,1) = 0$

$$\left.\frac{\partial^2 w(t,y)}{\partial y^2}\right|_{y=0} = \left.\frac{\partial^2 w(t,y)}{\partial y^2}\right|_{y=1} = 0 \tag{2.4-3}$$

It is of interest here to establish conditions for the asymptotic stability of the equilibrium state or the trivial solution of (2.4-1) with boundary conditions (2.4-3), with respect to a metric ρ_4 defined by:

$$\rho_4 = \left[\int_0^1 \left\{ \sum_{i=0}^{2} \left(\frac{\partial^i w(t,y)}{\partial y^i}\right)^2 + \left(\frac{\partial w(t,y)}{\partial t}\right)^2 \right\} dy \right]^{1/2} \tag{2.4-4}$$

To apply Lyapunov's direct method to this problem, we consider the following functional:

$$V_1 = \frac{1}{2} \int_0^1 \left\{ \left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{\ell^2}{D} \left[\frac{E_1 h}{4} \int_0^1 \left(\frac{\partial w}{\partial y}\right)^2 dy + \frac{E_1 h}{\ell} \Delta s - E_2 h \alpha \Delta T \right] \cdot \left(\frac{\partial w}{\partial y}\right)^2 \right\} dy \qquad (2.4\text{-}5)$$

From Theorem 2, it is required that V_1 is positive definite with respect to metric ρ_4 and V_1 is nonincreasing and $\to 0$ along any perturbed motion as $t \to \infty$.

To establish the positive-definiteness of V_1 with respect to ρ_4, it is required to show that there exists a positive function $\eta(\rho_4)$ such that

$$V_1 \geq \eta(\rho_4) \text{ for all } t \geq 0.$$

This can be accomplished with the aid of the following inequalities:

$$\int_0^1 \left(\frac{\partial^2 w}{\partial y^2}\right)^2 dy \geq \pi^2 \int_0^1 \left(\frac{\partial w}{\partial y}\right)^2 dy \geq \pi^4 \int_0^1 w^2(t,y)\, dy \qquad (2.4\text{-}6)^*$$

First, note that if the [. . .] term in (2.4-5) is ≥ 0, V_1 is obviously positive definite with respect to ρ_4. Consider the case where the [. . .] term is < 0. From 2.4-6), it follows that

$$V_1 \geq \frac{1}{2} \int_0^1 \left\{ \left(\frac{\partial w}{\partial t}\right)^2 + \beta_1 \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \left[\frac{\ell^2}{D} \left(\frac{E_1 h}{4} \int_0^1 \left(\frac{\partial w}{\partial y}\right)^2 dy + \frac{E_1 h}{\ell} \Delta s - E_2 h \alpha \Delta T \right) + \pi^2 \beta_2 \right] \left(\frac{\partial w}{\partial y}\right)^2 + (1 - \beta_1 - \beta_2)\, \pi^4 w^2 \right\} dy \qquad (2.4\text{-}7)$$

where β_1 and β_2 are positive constants satisfying $\beta_1 + \beta_2 < 1$. Clearly, if

$$\frac{\ell^2}{D} \left(\frac{E_1 h}{\ell} \Delta s - E_2 h \alpha \Delta T \right) + \pi^2 \beta_2 < \beta_1 \qquad (2.4\text{-}8)$$

then,

$$V_1 \geq \frac{1}{2} \left[\text{Min} \left\{ \beta_1, \pi^4 (1 - \beta_1 - \beta_2) \right\} \right] \rho_4^2 \qquad (2.4\text{-}9)$$

and V_1 is positive definite with respect to ρ_4.

* These inequalities are valid for functions defined on (0, 1) which are four times differentiable and satisfy boundary conditions (2.4-3).

To ensure that V_1 is nonincreasing and $\to 0$ along any perturbed motion as $t \to \infty$, we require $dV_1/dt < 0$. Differentiating V_1 with respect to t and integrating by parts lead to:

$$\frac{dV_1}{dt} = \int_0^1 \left[\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial y^4} \right] \frac{\partial w}{\partial t} \, dy + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial y \partial t} \Bigg|_0^1 - \frac{\partial^3 w}{\partial y^3} \frac{\partial w}{\partial t} \Bigg|_0^1$$
$$+ \frac{\ell^2}{D} \left[\frac{E_1 h}{2} \int_0^1 \left(\frac{\partial w}{\partial y} \right)^2 dy + \frac{E_1 h}{\ell} \Delta s - E_2 h \alpha \Delta T \right] \left[\frac{\partial w}{\partial y} \frac{\partial w}{\partial t} \Bigg|_0^1 - \int_0^1 \frac{\partial^2 w}{\partial y^2} \frac{\partial w}{\partial t} \right] dy \qquad (2.4\text{-}10)$$

which, in view of 2.4-1), (2.4-2) and (2.4-3), immediately reduces to:

$$\frac{dV_1}{dt} = - \int_0^1 \frac{\zeta(y) \ell^2}{\sqrt{D \rho_s h}} \left(\frac{\partial w}{\partial t} \right)^2 dy \qquad (2.4\text{-}11)$$

It is evident that condition (2.4-8) and $\zeta(y) > 0$ for almost all $y \in (0,1)$ are sufficient for ensuring asymptotic stability of the trivial solution.

2.4.2 Aeroelastic Panel

An elastic panel similar to that discussed in the previous section will be considered. Here, we assume that there are no displacement of the supporting edges and temperature rise in the panel, but there exists aerodynamic pressure due to the exposure to supersonic flow on one side of the panel and stagnant air on the other side. The free-stream velocity is assumed to be normal to the supporting edges.

For this panel, the equation of motion for the dimensionless deflection $w(t,y)$ is:

$$\frac{\partial^4 w(t,y)}{\partial y^4} + k_1 \frac{\partial w(t,y)}{\partial y} + k_2(y) \frac{\partial w(t,y)}{\partial t} + \frac{\partial^2 w(t,y)}{\partial t^2} = 0 \qquad (2.4\text{-}12)$$

with boundary conditions given by (2.4-3), where

$$k_1 = \frac{\rho_a v_s^2 \ell^3}{D \sqrt{M_a^2 - 1}}, \quad k_2(y) = \frac{\ell^2}{\sqrt{D \rho_s h}} \left[\zeta(y) + \frac{\rho_a v_s (M_a^2 - 2)}{(M_a^2 - 1)^{3/2}} \right] \qquad (2.4\text{-}13)$$

where ρ_a is the mass density of the undisturbed air, M_a is the Mach number, and v_s is the velocity of sound. The normalizations are taken with respect to the same quantities given in section 2.4.1. The above mathematical description is valid for $M_a >> \sqrt{2}$.

Again, we wish to establish conditions for asymptotic stability of the equilibrium null state with respect to ρ_4 defined by (2.4-4). This can be accomplished by considering a functional of the form:

$$V_2 = \int_0^1 \left\{ \left(\frac{\partial w}{\partial t} + cw \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right\} dy \qquad (2.4\text{-}14)$$

where c is a positive constant. It can be easily verified with the aid of inequalities (2.4-6) that V_2 is positive definite with respect to ρ_4.

The derivative of V_2 with respect to t, in view of system equation (2.4-12), is given by:

$$\frac{dV_2}{dt} = -\int_0^1 \left[(k_2(y) - c)\left(\frac{\partial w}{\partial t}\right)^2 + c(k_2(y) - c)\, w \frac{\partial w}{\partial t} + k_1 \frac{\partial w}{\partial t}\frac{\partial w}{\partial y} + k_1 c\, w \frac{\partial w}{\partial y} + c\left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right] dy + \left(\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial y \partial t} - \frac{\partial^3 w}{\partial y^3}\frac{\partial w}{\partial t} - cw\frac{\partial^3 w}{\partial y^3} + c\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} \right)\Bigg|_0^1 \qquad (2.4\text{-}15)$$

Using boundary conditions (2.4-3) and inequalities (2.4-6), (2.4-15) can be bounded by:

$$\frac{dV_2}{dt} \leq -\int_0^1 U^T(t,y)\, \mathcal{A}\, U(t,y)\, dy \qquad (2.4\text{-}16)$$

where $U = \text{Col}\,(w, \partial w/\partial y, \partial w/\partial t)$ and $\mathcal{A}$ is a matrix linear operator given by:

$$\mathcal{A} = \begin{bmatrix} c(1-\alpha)\pi^4 & k_1 c/2 & c(k_2(y) - c)/2 \\ k_1 c/2 & c\,\alpha\,\pi^2 & k_1/2 \\ c(k_2(y)-c)/2 & k_1/2 & (k_2(y)-c) \end{bmatrix} \qquad (2.4\text{-}17)$$

where $0 < \alpha < 1$.

For asymptotic stability, $\mathcal{A}$ must be a positive definite operator. Assuming that k_2 is a continuous function of y, $\mathcal{A}$ will be positive definite if the Sylvester's inequalities are satisfied for any fixed $y \in (0, 1)$:

$$c(1 - \alpha)\,\pi^4 > 0, \tag{2.4-18}$$

$$4\,\alpha(1 - \alpha)\,\pi^6 - k_1^2 > 0, \tag{2.4-19}$$

$$\pi^2(1 - \alpha)\,[4c\,\alpha\pi^2(k_2(y)-c) - k_1^2] - \alpha c^2(k_2(y) - c)^2 > 0 \text{ for all } y\in(0,1) \tag{2.4-20}$$

The first condition (2.4-18) is automatically satisfied. The remaining conditions, expressed explicitly in terms of the system parameters, become:

$$4\,\alpha(1 - \alpha)\,\pi^6 > \rho_a^2 v_s^4 \ell^6 / D(M_a^2 - 1), \tag{2.4-19*}$$

$$\left[\frac{\ell^2}{\sqrt{D\rho_s h}}\left(\zeta(y) + \frac{\rho_s v_s (M_a^2 - 2)}{(M_a^2 - 2)^{3/2}}\right) - c + 2\,\pi^4 c^{-1}(1 - \alpha)\right]^2 <$$

$$\frac{\pi^2(1-\alpha)}{c^2\,\alpha}\left[4\pi^6(1-\alpha)\,\alpha - \frac{\rho_a^2 v_s^4 \ell^6}{D^2(M_a^2 - 1)}\right] \text{ for all } y\in(0,1). \tag{2.4-20*}$$

The above equalities constitute a sufficient condition for asymptotic stability. For the case where D is a continuous function of y, similar inequalities can be obtained in a straightforward manner.

2.4.3 Cantilever Wing

Consider a cantilever chordwise-rigid wing with straight elastic axis as shown in Figure 2-2. The direction of the free-stream velocity v_o is parallel to the x-axis. Neglecting rotary inertia and shear deformation, the equations of motion for the dimensionless bending deflection $w(t,y)$ (normalized with respect to the wing length ℓ), and torsional deflection angle $\alpha(t,y)$ of the cantilever wing based on linear theory of elasticity are:

$$m(y)v_o^2\ell^2 \frac{\partial^2 w(t,y)}{\partial t^2} - S_y(y)v_o^2\ell \frac{\partial^2 \alpha(t,y)}{\partial t^2} + \frac{\partial^2}{\partial y^2} EI(y) \frac{\partial^2 w(t,y)}{\partial y^2}$$

$$= \ell^3 \left[L(w, \frac{\partial w}{\partial t}, \alpha, \frac{\partial \alpha}{\partial t}) + D_w(w, \frac{\partial w}{\partial t})\right] \tag{2.4-21}$$

$$-S_y(y)v_o^2\ell \frac{\partial^2 w(t,y)}{\partial t^2} + I_y(y)v_o^2 \frac{\partial^2 \alpha(t,y)}{\partial t^2} - \frac{\partial}{\partial y} GJ(y) \frac{\partial \alpha(t,y)}{\partial y}$$

$$= \ell^2 \left[M_y(w, \frac{\partial w}{\partial t}, \alpha, \frac{\partial \alpha}{\partial t}) + D_\alpha (\alpha , \frac{\partial \alpha}{\partial t}) \right] \quad (2.4\text{-}22)$$

where y is the dimensionless spatial coordinate (normalized with respect to the wing length ℓ) and t is the dimensionless time (normalized with respect to the quantity ℓ/v_o). S_y and I_y are the static moment per unit length and mass moment of inertia per unit length about the elastic axis respectively. GJ is the torsional rigidity. L is the aerodynamic force per unit span, and M_y is the aerodynamic moment per unit span. Both L and M_y depend on the parameters and motion of the wing, and the free-stream velocity v_o. D_w and D_α are damping force and moment distributions respectively.

The boundary conditions for this system are:

$$w(t,0) = \left.\frac{\partial w(t,y)}{\partial y}\right|_{y=0} = \alpha(t,0) = 0$$

$$EI_{(y)} \left.\frac{\partial^2 w(t,y)}{\partial y^2}\right|_{y=1} = \frac{\partial}{\partial y} EI(y) \left.\frac{\partial^2 w(t,y)}{\partial y^2}\right|_{y=1} = 0$$

$$GJ(y) \left.\frac{\partial \alpha(t,y)}{\partial y}\right|_{y=1} = 0 \quad (2.4\text{-}23)$$

Here, we assume that the wing root does not undergo vertical translational motion, and

$$L(w=0, \frac{\partial w}{\partial t} = 0, \alpha = 0, \frac{\partial \alpha}{\partial t} = 0) = 0; \; D_w(w=0, \frac{\partial w}{\partial t} = 0) = 0$$

$$M_y(w=0, \frac{\partial w}{\partial t} = 0, \quad = 0, \frac{\partial \alpha}{\partial t} = 0) = 0; \; D_\alpha(\alpha = 0, \frac{\partial \alpha}{\partial t} = 0) = 0$$

Thus, the trivial solution is an equilibrium state of the system. It is of interest here to derive explicit conditions for the asymptotic stability of the trivial solution of (2.4-21) and (2.4-22), in terms of the system parameters.

First, consider the following functional which is proportional to the total energy of the wing:

$$V_3 = \frac{1}{2} \int_0^1 \left[m(y)v_o^2\ell^2 \left(\frac{\partial w}{\partial t}\right)^2 + EI(y)\left(\frac{\partial^2 w}{\partial y^2}\right)^2 - 2S_y(y)v_o^2\ell \frac{\partial w}{\partial t}\frac{\partial \alpha}{\partial t} + GJ(y)\left(\frac{\partial \alpha}{\partial y}\right)^2 + I_y(y)v_o^2\left(\frac{\partial \alpha}{\partial t}\right)^2 \right] dy \quad (2.4\text{-}24)$$

A straightforward computation shows that dV_3/dt can be expressed in the form:

$$\frac{dV_3}{dt} = \int_0^1 \left[m(y)v_o^2 \ell^2 \frac{\partial^2 w}{\partial t^2} - S_y(y)v_o^2 \ell \frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial^2}{\partial y^2} EI(y) \frac{\partial^2 w}{\partial y^2} \right] \frac{\partial w}{\partial t} dy$$

$$+ \int_0^1 \left[I_y(y)v_o^2 \frac{\partial^2 \alpha}{\partial t^2} - S_y(y)v_o^2 \ell \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial y} GJ(y) \frac{\partial \alpha}{\partial y} \right] \frac{\partial \alpha}{\partial t} dy$$

$$+ EI(y) \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial y \partial t} \Bigg|_0^1 - \frac{\partial}{\partial y} EI(y) \frac{\partial^2 w}{\partial y^2} \frac{\partial w}{\partial t} \Bigg|_0^1 + GJ(y) \frac{\partial \alpha}{\partial y} \frac{\partial \alpha}{\partial t} \Bigg|_0^1$$

(2.4-25)

In view of system equations (2.4-21) and (2.4-22), and boundary conditions (2.4-23), (2.4-25) reduces to:

$$\frac{dV_3}{dt} = \int_0^1 \ell^3 \frac{\partial w}{\partial t} \left[L(w, \frac{\partial w}{\partial t}, \alpha, \frac{\partial \alpha}{\partial t}) + D_w (w, \frac{\partial w}{\partial t}) \right] dy$$

$$+ \int_0^1 \ell^2 \frac{\partial \alpha}{\partial t} \left[M_y(w, \frac{\partial w}{\partial t}, \alpha, \frac{\partial \alpha}{\partial t}) + D_\alpha (\alpha, \frac{\partial \alpha}{\partial t}) \right] dy$$

(2.4-26)

Equation (2.4-26) is proportional to the rate at which work is being done on the wing by the aerodynamic loading. If we define stability of equilibrium with respect to a metric which is a measure of the total energy of the wing, then clearly $dV_3/dt < 0$ for all $t \geq t_o$ is sufficient for ensuring asymptotic stability.

In the sequel, we shall consider the special case where both the aerodynamic load, damping force and moment distributions can be approximated by linear functions of their arguments:

$$L + D_w = k_{w1}(t,y)w(t,y) + k_{w_t 1}(t,y) \frac{\partial w(t,y)}{\partial t} + k_{\alpha 1}(t,y)\, \alpha(t,y)$$

$$+ k_{\alpha_t 1}(t,y) \frac{\partial \alpha(t,y)}{\partial t} \qquad (2.4\text{-}27)$$

$$M_y + D = k_{w2}(t,y)w(t,y) + k_{w_t 2}(t,y)\frac{\partial w(t,y)}{\partial t} + k_{\alpha 2}(t,y)\,\alpha(t,y) + k_{\alpha_t 2}(t,y)\frac{\partial \alpha(t,y)}{\partial t} \tag{2.4-28}$$

We wish to establish conditions for which the trivial solution is asymptotically stable with respect to two metrics ρ_5, ρ_6 defined by:

$$\rho_5 = \left\{\int_0^1 \left[\left(\frac{\partial w}{\partial t}\right)^2 + \sum_{i=0}^{2}\left(\frac{\partial^i w}{\partial y^i}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2 + \alpha^2 + \left(\frac{\partial \alpha}{\partial t}\right)^2\right]dy\right\}^{1/2} \tag{2.4-29}$$

and

$$\rho_6 = \left\{\int_0^1 \left[\left(\frac{\partial w}{\partial t}\right)^2 + w^2 + \left(\frac{\partial \alpha}{\partial t}\right)^2 + \alpha^2\right] dy\right\}^{1/2} \tag{2.4-30}$$

For this case, the total energy given by (2.4-24) is not directly useful in deriving the stability conditions. Instead, we consider the following functional:

$$V_4 = V_3 + \int_0^1 \Big\{c_1 w\left[m(y)v_o^2\ell^2 \frac{\partial w}{\partial t} - S_y(y)v_o^2\ell\frac{\partial \alpha}{\partial t}\right] + c_2\,\alpha\left[I_y(y)v_o^2\frac{\partial \alpha}{\partial t} - S_y(y)v_o^2\ell\frac{\partial w}{\partial t}\right]\Big\}dy \tag{2.4-31}$$

where c_1 and c_2 are positive constants. It can be readily verified that V_4 is continuous with respect to ρ_5. To satisfy condition (ii) in Theorem 1, V_4 must be positive definite with respect to ρ_6. This can be established by making use of the following inequalities:

$$\int_0^1 EI(y)\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dy \geq \left[\underset{y}{\text{Min}}\; EI(y)\right]\int_0^1 w^2\,dy \tag{2.4-32}$$

and

$$\int_0^1 GJ(y)\left(\frac{\partial \alpha}{\partial y}\right)^2 dy \geq \left[\underset{y}{\text{Min}}\; GJ(y)\right]\int_0^1 \alpha^2\,dy \tag{2.4-33}$$

It follows that

$$V_4 \geq \int_0^1 U^T(t,y)\,\mathcal{B}_1\,U(t,y)\,dy \tag{2.4-34}$$

where $U = \mathrm{Col}(w,\ \partial w/\partial t,\ \alpha,\ \partial\alpha/\partial t)$, and $\mathcal{B}_1$ is a matrix linear operator:

$$\mathcal{B}_1 = \frac{1}{2}\begin{bmatrix} \underset{y}{\mathrm{Min}}\ EI(y) & c_1 m(y) v_o^2 \ell^2 & 0 & -c_1 S_y(y) v_o^2 \ell \\ c_1 m(y) v_o^2 \ell^2 & m(y) v_o^2 \ell^2 & -c_2 S_y(y) v_o^2 \ell & -S_y(y) v_o^2 \ell \\ 0 & -c_2 S_y(y) v_o^2 \ell & \underset{y}{\mathrm{Min}}\ GJ(y) & c_2 I_y(y) v_o^2 \\ -c_1 S_y(y) v_o^2 \ell & -S_y(y) v_o^2 \ell & c_2 I_y(y) v_o^2 & I_y(y) v_o^2 \end{bmatrix}$$

(2.4-35)

Clearly, V_4 will be positive definite with respect to ρ_6, if $\mathcal{B}_1$ is positive definite for any fixed $y \in (0,1)$, or the following conditions are satisfied:

$$\underset{y}{\mathrm{Min}}\ EI(y) > c_1^2 m(y) v_o^2 \ell^2 > 0 \qquad (2.4\text{-}36)$$

$$c_2^2 v_o^2 I_{cg}(y) [c_1^2 v_o^2 \ell^2 m(y) - \underset{y}{\mathrm{Min}}\ EI(y)] > (\underset{y}{\mathrm{Min}}\ GJ(y))\ c_1^2 m(y) v_o^2 \ell^2$$
$$- (\underset{y}{\mathrm{Min}}\ EI(y))\ [c_2^2 v_o^2 m(y) d^2 - \underset{y}{\mathrm{Min}}\ (GJ(y))] < 0 \qquad (2.4\text{-}37)$$

The above inequalities are reduced from those of Sylvester. Also, the following explicit forms for S_y and I_y have been used in reduction

$$S_y(y) = m(y) d(y), \qquad I_y(y) = I_{cg}(y) + m(y) d^2(y) \qquad (2.4\text{-}38)$$

where I_{cg} is the moment of inertia per unit length about the wing's center of gravity curve, and d is the distance between the center of gravity curve and the elastic axis.

The derivative of V_4 with respect to t can be expressed in the form:

$$\frac{dV_4}{dt} = \int_0^1 \Big\{ c_1 m(y) v_o^2 \ell^2 \left(\frac{\partial w}{\partial t}\right)^2 - (c_1 + c_2) S_y(y) v_o^2 \ell \frac{\partial w}{\partial t}\frac{\partial \alpha}{\partial t} + c_2 I_y(y) v_o^2 \left(\frac{\partial \alpha}{\partial t}\right)^2$$
$$+ (c_1 w + \frac{\partial w}{\partial t})\ \ell^3\ (L + D_w) + (c_2 \alpha + \frac{\partial \alpha}{\partial t}) \ell^2\ (M_y + D_\alpha) \Big\}\ dy$$
$$- \int_0^1 \Big\{ c_2 GJ(y) \left(\frac{\partial \alpha}{\partial y}\right)^2 + c_1 EI(y) \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \Big\} dy \qquad (2.4\text{-}39)$$

where $(L + D_w)$ and $(M_y + D_\alpha)$ are given by (2.4-27) and (2.4-28) respectively.

In view of inequalities (2.4-32) and (2.4-33), dV_4/dt satisfies

$$\frac{dV_4}{dt} \leq \int_0^1 U^T(t,y)\ \mathcal{B}_2\ U(t,y)\ dy \tag{2.4-40}$$

where $U = \text{Col}\ (w,\ \partial w/\partial t,\ \alpha,\ \partial\alpha/\partial t)$ and $\mathcal{B}_2$ is a matrix linear operator:

$$\begin{bmatrix} 2c_1(\ell^3 k_{w1} - \underset{y}{\text{Min}}\ EI(y)) & \ell^3(c_1 k_{w_t 1} + k_{w1}) & \ell^2(c_1 \ell k_{\alpha 1} + c_2 k_{w2}) & \ell^2(c_1 \ell k_{\alpha_t 1} + k_{w2}) \\ \ell^3(c_1 k_{w_t 1} + k_{w1}) & 2\ell^2(c_1 m(y) v_0^2 + k_{w_t 1}\ell) & \ell^2(c_2 k_{w_t 2} + \ell k_{\alpha 1}) & \ell(\ell^2 k_{\alpha_t 1} + \ell k_{w_t 2} - (c_1 + c_2) v_o^2 S_y) \\ \ell^2(c_1 \ell k_{\alpha 1} + c_2 k_{w2}) & \ell^2(c_2 k_{w_t 2} + \ell k_{\alpha 1}) & 2c_2(\ell^2 k_{\alpha 2} - \underset{y}{\text{Min}}\ GJ(y)) & \ell^2(c_2 k_{\alpha_t 2} + k_{\alpha 2}) \\ \ell^2(c_1 \ell k_{\alpha_t 1} + k_{w2}) & \ell(\ell^2 k_{\alpha_t 1} + \ell k_{w_t 2} - (c_1 + c_2) S_y v_o^2) & \ell^2(c_2 k_{\alpha_t 2} + k_{\alpha 2}) & 2(c_2 I_y v_o^2 + \ell^2 k_{\alpha_t 2}) \end{bmatrix} \tag{2.4-41}$$

A sufficient condition for asymptotic stability of equilibrium is that the inequalities (2.4-36) and (2.4-37) are satisfied and $\mathcal{B}_2$ is a negative definite operator for all $t \geq t_o$ or the parameters of $-\mathcal{B}_2$ satisfy the Sylvester inequalities for all $y \epsilon (0,1)$ and all $t \geq t_o$.

For the case where a quasi-steady-state approximation[14] for the aerodynamic loading of a thin airfoil in an incompressible flow is valid, the aerodynamic loading coefficients have the following explicit forms:

$$k_{w1} = 0,\quad k_{w_t 1} = 2\pi\rho_o v_o^2 b + \zeta_1(y) v_o,$$

$$k_{\alpha 1} = 2\pi\rho_o v_o^2 b,\quad k_{\alpha_t 1} = 2\pi\rho_o v_o^2 b^2 \ell^{-1}(1-\eta),$$

$$k_{w2} = 0,\quad k_{w_t 2} = \pi\rho_o v_o^2 b^2(2\eta + 1), \tag{2.4-42 *}$$

$$k_{\alpha 2} = -\pi\rho_o v_o^2 b^2(2\eta + 1),\quad k_{\alpha_t 2} = -\pi\rho_o v_o^2 b^3 \ell^{-1}(\eta + 1)(1-2\eta) + \zeta_2(y) v_o \ell^{-1}$$

* Here, we have neglected the effect of accelerations $\partial^2\alpha/\partial t^2$ and $\partial^2 w/\partial t^2$ at the aeroelastic loading.

where ζ_1 and ζ_2 are distributed viscous damping coefficients (for other notations, see Figure 2-3). A sufficient condition for negative definiteness of $\mathcal{B}_2$ can be established by substituting (2.4-42) directly into the Sylvester inequalities associated with (2.4-41). However, the explicit forms of these inequalities are quite lengthy and will not be given here.

Note that the conditions derived in the foregoing manner are only sufficient for asymptotic stability, hence they may not give sharp estimates of the stability regions in the parameter space. Also, their usefulness is limited by the particular choice made for the form of system (structural) damping which is known to be a crucial factor in flutter analysis.

2.4.4 Cantilever Wing with Lumped Masses

Here, we consider a cantilever wing which is identical to the one described in the previous section except for the addition of two lumped masses which are coupled to each other by means of a nonlinear spring and damper; one of the masses is rigidly attached to the wing tip.

Let the dimensionless position of mass M measured with respect to the x-y plane be denoted by z_M as shown in Figure 2-4, where z_M is normalized with respect to the wing length ℓ. We assume that the motion of M is describable by the following nonlinear ordinary differential equation:

$$M v_o^2 \ell \frac{d^2 z_M}{dt^2} + v_o \ell^2 f_d (z_M + w(t,1)) \left[\frac{dz_M}{dt} + \frac{dw(t,1)}{dt} \right] + \ell^2 k(z_M + w(t,1)) = 0 \qquad (2.4\text{-}43)$$

where t is the dimensionless time (normalized with respect to ℓ / v_o), f_d and k are specified functions of their arguments corresponding to the damping coefficient and spring force respectively. Here, we assume that the aerodynamic forces acting on M' and M are negligible; also, the center of mass of M is directly below the spring.

The equations of motion for the elastic wing are given again by (2.4-21) and (2.4-22); but the boundary conditions at y = 1 are now given by:

$$EI(y) \left. \frac{\partial^2 w(t,y)}{\partial y^2} \right|_{y=1} = 0, \quad GJ(y) \left. \frac{\partial \alpha(t,y)}{\partial y} \right|_{y=1} = \frac{I_1 v_o^2}{\ell} \frac{d^2 \alpha(t,1)}{dt^2}$$

$$\frac{\partial}{\partial y} \left. EI(y) \frac{\partial^2 w(t,y)}{\partial y^2} \right|_{y=1} = M' v_o^2 \ell \frac{d^2 w(t,1)}{dt^2} + v_o \ell^2 f_d (z_M + w(t,1)) \left[\frac{dz_M}{dt} + \frac{dw(t,1)}{dt} \right] + \ell^2 k(z_M + w(t,1)) \qquad (2.4\text{-}44)$$

where I_1 is the moment of inertia of M and M' about the elastic axis.

It is of interest to establish conditions for asymptotic stability of equilibrium (the trivial solution) with respect to the following two metrics:

$$\rho_7 = \left[\rho_5^2 + (z_M + w(t,l))^2 + \left(\frac{dz_M}{dt}\right)^2 + \left(\frac{dw(t,l)}{dt}\right)^2 + \left(\frac{d\alpha(t,l)}{dt}\right)^2\right]^{1/2} \qquad (2.4\text{-}45)$$

and

$$\rho_8 = \left[\rho_6^2 + (z_M + w(t,l))^2 + \left(\frac{dz_M}{dt}\right)^2 + \left(\frac{dw(t,l)}{dt}\right)^2 + \left(\frac{d\alpha(t,l)}{dt}\right)^2\right]^{1/2} \qquad (2.4\text{-}46)$$

By considering a functional of the form

$$V_5 = V_4 + \frac{1}{2} v_o^2 \ell \left[M \left(\frac{dz_M}{dt}\right)^2 + M' \left(\frac{dw(t,l)}{dt}\right)^2 + I_1 \ell^{-2} \left(\frac{d\alpha(t,l)}{dt}\right)^2 \right. \\ \left. \int_0^{z_M + w(t,l)} \ell^2 k(\xi)\, d\xi \right] \qquad (2.4\text{-}47)$$

where V_4 is given by (2.4-31), it can be readily shown that in addition to the conditions derived in the previous section, the following conditions are sufficient for ensuring asymptotic stability of the trivial solution with respect to two metrics ρ_7, ρ_8 :

$$\left.\begin{array}{l} f_d(z_M + w(t,l)) > 0, \\ (z_M + w(t,l))\, k\,(z_M + w(t,l)) > 0 \text{ for } (z_M + w(t,l)) \neq 0 \end{array}\right\} \qquad (2.4\text{-}48)$$

The cantilever wing considered here is a simple example of a system which is a composite of both distributed and lumped parameter subsystems.

2.5 CONCLUSIONS

In formulating stability problems in elastic and aeroelastic systems in the framework of Lyapunov's stability theory, the initial state space and the metrics with respect to which stability is defined must be chosen with care so that the physical meanings of the problems are preserved. In applying, Lyapunov's direct method to these problems, the main difficulty lies in the fact that there are no systematic methods for constructing the functionals having the properties prescribed by the stability theorems. In this paper, it has been demonstrated that sufficient conditions for the stability of equilibrium of a few nontrivial elastic and aeroelastic systems can be derived via Lyapunov's direct method. These conditions may be very weak since they are derived without detailed knowledge of the solutions. However, they can serve as a valuable guide for practical design and for more detailed stability analysis. In this respect, Lyapunov's direct method can become a useful mathematical tool for the stability study of elastic and aeroelastic systems.

2.6 APPENDIX

Proof for Theorem 1

Necessity: Assume that the invariant set $\mathcal{J}$ is stable with respect to two metrics ρ, $\tilde{\rho}$. Then, by definition, for any $\epsilon > 0$ there exists a $\delta > 0$ such that when $\rho(S_{t_o}, \mathcal{J}) < \delta$ we have $\tilde{\rho}(\mathcal{M}\Phi^t_{t_o}(S_{t_o}), \tilde{\mathcal{J}}) < \epsilon$ for all $t \geq t_o$. We let:

$$V_t(\mathcal{M}\Phi^t_{t_o}(S_{t_o})) = \sup_{t \geq t_o} \tilde{\rho}(\mathcal{M}\Phi^t_{t_o}(S_{t_o}), \tilde{\mathcal{J}}) \quad \text{for} \quad \rho(S_{t_o}, \mathcal{J}) < \delta \tag{2.6-1}$$

The expression (2.6-1) defines a family of functionals V_t depending on parameter t, defined for any S_{t_o} satisfying $\rho(S_{t_o}, \mathcal{J}) < \delta$. It follows from (2.6-1) and the stability definition that $V_t(\mathcal{M}\Phi^t_{t_o}(S_{t_o})) \leq \delta_1$ for $\rho(S_{t_o}, \mathcal{J}) < \delta_1$, where $\delta_1 < \delta$, and δ_1 corresponds to a chosen $\epsilon_1 > 0$.

Hence, $V_t(\mathcal{M}\Phi^t_{t_o}(S_{t_o})) = V_t(\mathcal{M}S_{t_o}) \to 0$ as $\rho(S_{t_o}, \mathcal{J}) \to 0$. Also, from (2.6-1) we have $V_t(\mathcal{M}\Phi^t_{t_o}(S_{t_o})) \geq \tilde{\rho}(\mathcal{M}S_{t_o}, \tilde{\mathcal{J}})$, thus V_t is positive definite with respect to $\tilde{\rho}$ or V_t has property (ii).

To show that V_t has property (iii), we consider the quantity $\tilde{\rho}(\mathcal{M}\Phi^t_{t_1}(S_{t_1}), \tilde{\mathcal{J}})$ where $S_{t_1} \in \Phi^{t_1}_{t_o}(S_{t_o})$, $t_1 > t_o$. In view of the semi-group property of a dynamical system, we have

$$\Phi^t_{t_1}(S_{t_1}) \subset \Phi^t_{t_o}(S_{t_o}) \quad \text{for} \quad t \geq t_1 \quad \text{and} \quad S_{t_1} \in \Phi^{t_1}_{t_o}(S_{t_o}).$$

From the assumption that is continuous, we have

$$\tilde{\rho}(\mathcal{M}\Phi^t_{t_1}(S_{t_1}), \tilde{\mathcal{J}}) \leq \tilde{\rho}(\mathcal{M}S_{t_o}, \quad) \quad t \geq t_1 \text{ and for any } S_{t_1} \in \Phi^{t_1}_{t_o}(S_{t_o}).$$

It follows that

$$V_t(\mathcal{M}\Phi^t_{t_1}(S_{t_1})) = \sup_{t \geq t_1} \tilde{\rho}(\mathcal{M}\Phi^t_{t_1}(S_{t_1}), \tilde{\mathcal{J}}) \leq \sup \tilde{\rho}(\mathcal{M}S_{t_o}, \tilde{\mathcal{J}}) \leq V_t(\mathcal{M}\Phi^t_{t_o}(S_{t_o}))$$

for any $t_1 \geq t_o$, or V_t has property (iii).

Finally, the fact that V_t has property (i) follows directly from (2.6-1) and the stability definition, i.e., for any $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that when $\rho(S_{t_o}, \mathcal{J}) < \delta_1$, we have

$$0 < V_t(\mathcal{M}\Phi_{t_o}^{t}(S_{t_o})) = \sup_{t \geq t_o} \tilde{\rho}(\mathcal{M}\Phi_{t_o}^{t}(S_{t_o}), \tilde{\mathcal{J}}) \leq \epsilon_1.$$

This completes the proof for the necessity of conditions (i) - (iii) in Theorem 1. Sufficiency: Assume that there exists a family of functionals V_t depending on parameter t and having properties (i) - (iii). We wish to show that the invariant set $\mathcal{J}$ is stable.

Let us take an ϵ such that $0 < \epsilon < r$, and set

$$\lambda = \inf V_t(\mathcal{M}S_t) \text{ for } \epsilon \leq \tilde{\rho}(\mathcal{M}S_t, \tilde{\mathcal{J}}) \leq r.$$

Also, let us choose δ such that when $\rho(S_t, \mathcal{J}) < \delta$ we have

$$V_t(\mathcal{M}S_t) < \lambda \text{ for } t \geq 0.$$

We wish to show that the quantity δ corresponds to the chosen ϵ in accordance with the stability definition, i.e., when $\rho(S_t, \mathcal{J}) < \delta$ we have

$$\tilde{\rho}(\mathcal{M}\Phi_t^{t'}(S_t), \tilde{\mathcal{J}}) < \epsilon \quad \text{for } t' > t \geq 0.$$

Assume the opposite is true. Then, there exists a S_t satisfying $\rho(S_t, \mathcal{J}) < \delta$ such that at a certain finite $t_1 > 0$, the equality $\tilde{\rho}(\mathcal{M}\Phi_t^{t_1}(S_t), \mathcal{J}) = \epsilon$ holds. Thus,

$$V_t(\mathcal{M}\Phi_t^{t'}(S_t)) \geq \lambda$$

But, by virtue of property (iii) in the theorem,

$$V_t(\mathcal{M}\Phi_t^{t'}(S_t)) \leq V_t(\mathcal{M}S_t) < \lambda,$$

which implies a contradiction. This completes the proof.

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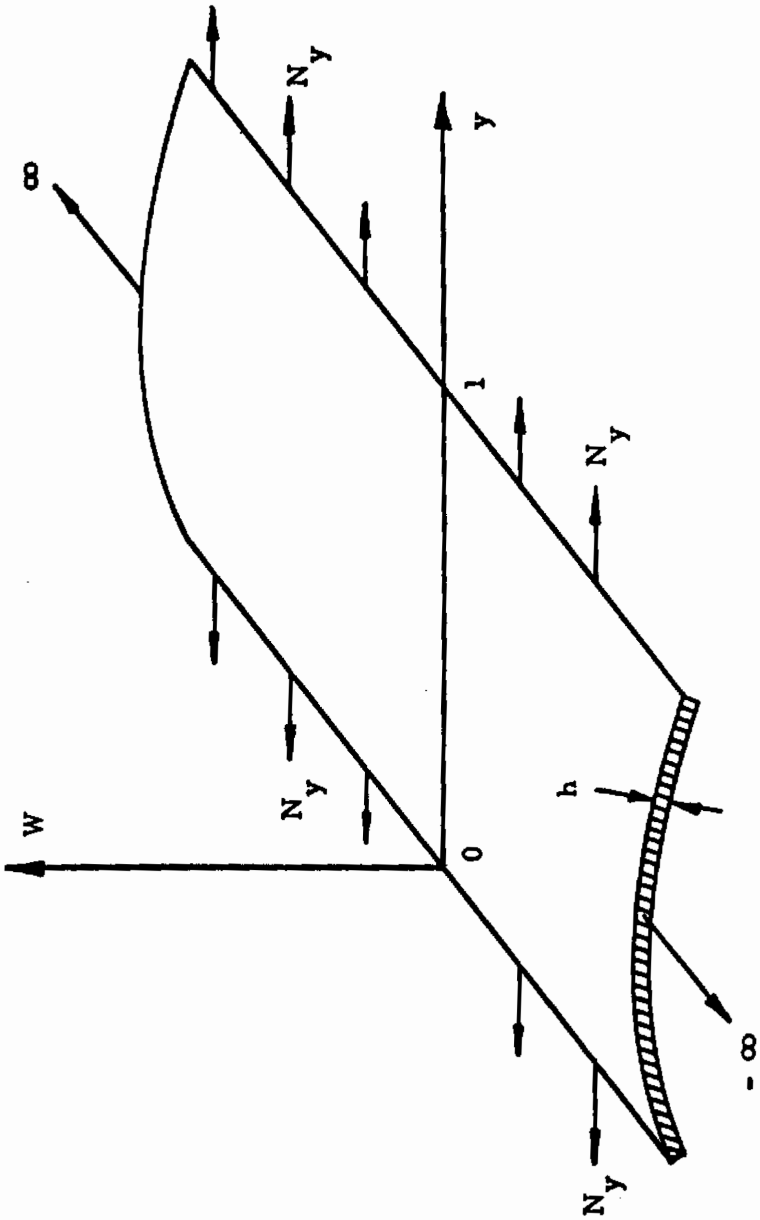

Figure 2.1

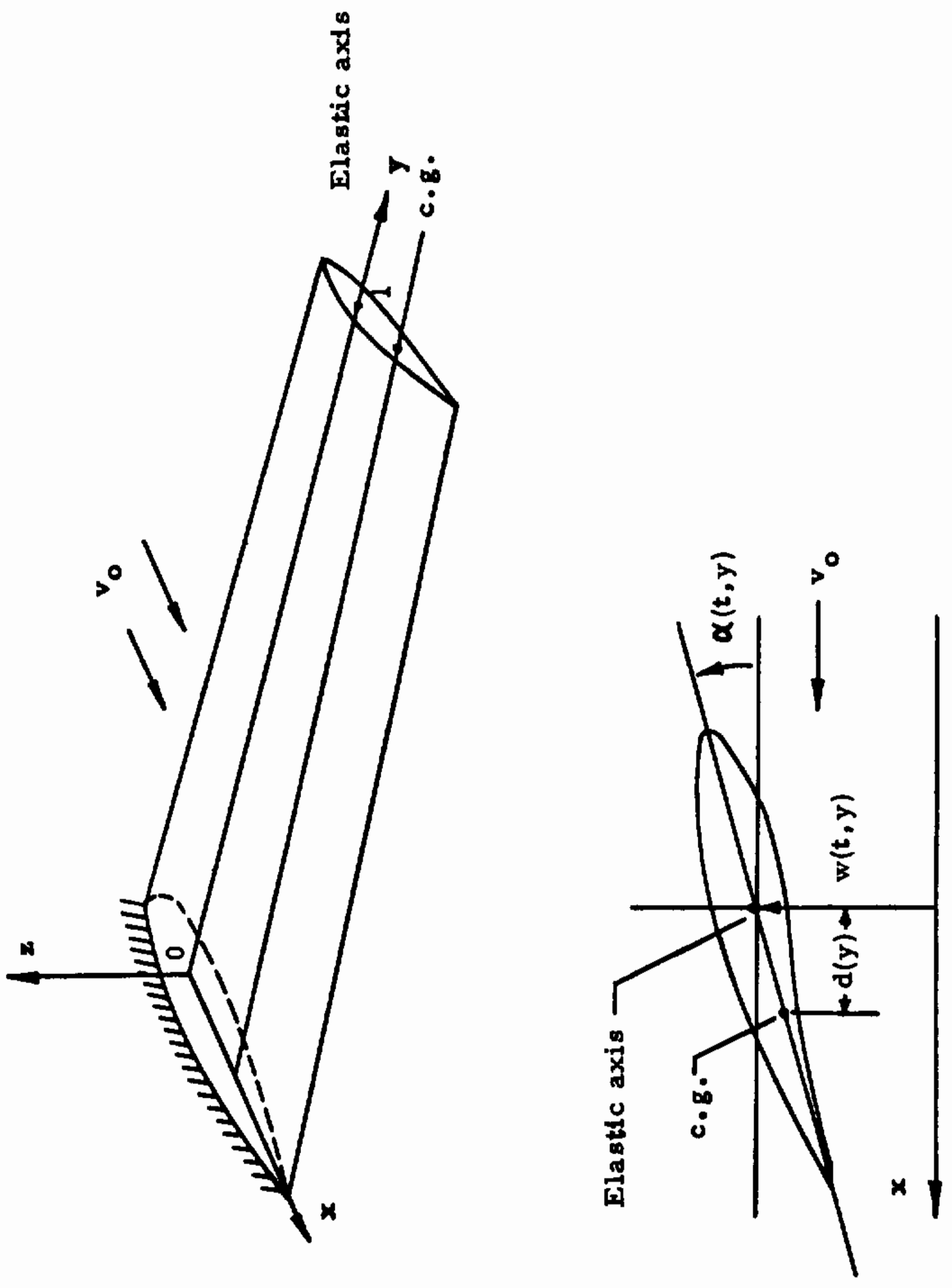

Figure 2.2

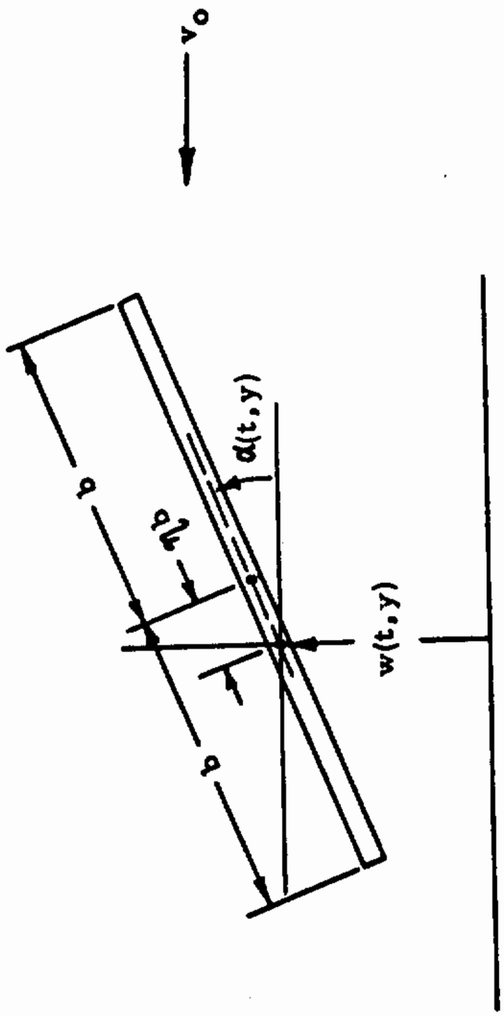

Figure 2.3

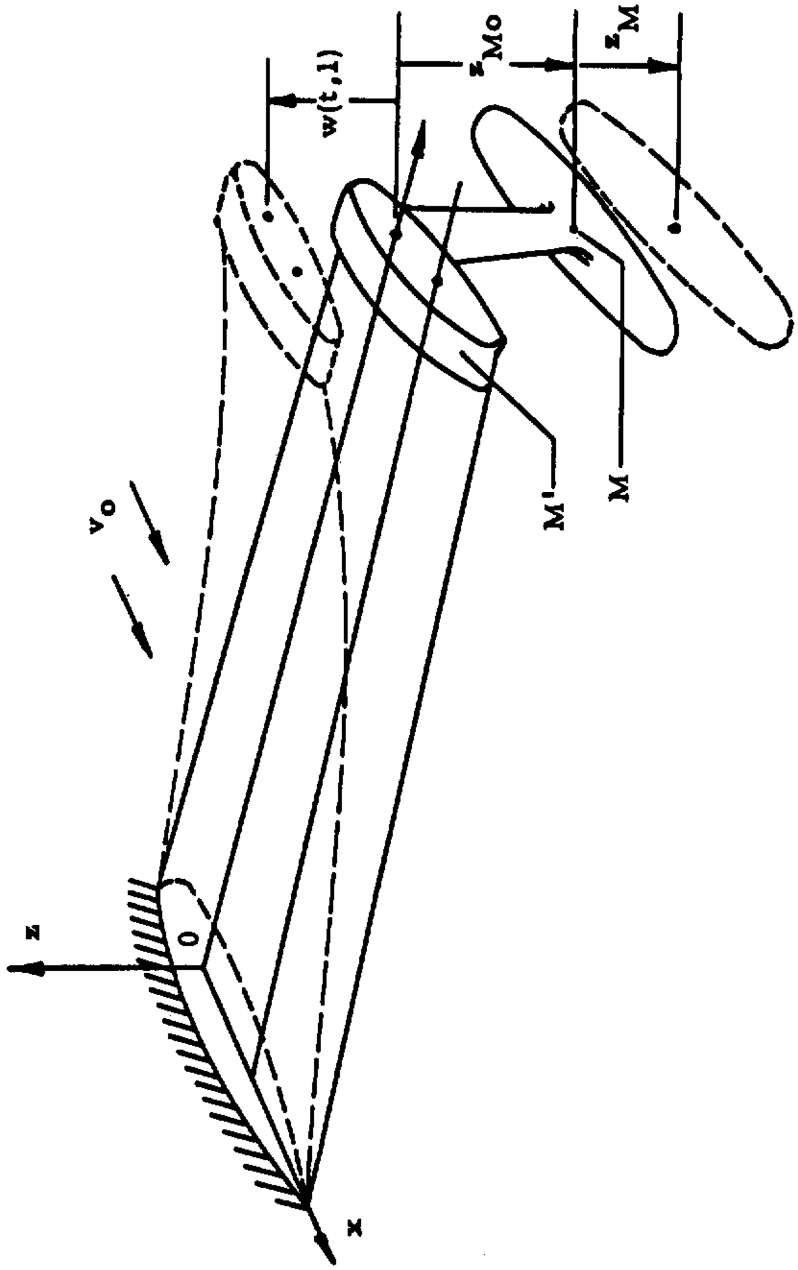

Figure 2.4