STRUCTURAL SURVIVABILITY ANALYSIS

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Structural survivability studies require analysis of many similar structures. This paper describes a modification of the multiple configuration analysis technique of Melosh and Luik (Reference 1) to perform fail-safe and invulnerability evaluations. The process incorporates the displacement approach and the finite-element concept for analysis of the initial structural system. Response predictions for the modified structures use the direct, the complementary energy and the potential energy methods. The approach yields approximate response predictions with accuracy decreasing directly with the number of damage steps. Applications illustrate the nature of the fail-safe analysis and the accuracy of invulnerability analyses. Described herein, a technique for approximate analysis, using the fail-safe results as a basis, shows its accuracy as a function of the number of damage steps.

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SECTION I

INTRODUCTION

A structure with survivability is defined as one which is fail-safe and invulnerable. A fail-safe structure continues to perform satisfactorily if any one structural component fails (Reference 2). No member will be over-stressed and the system does not undergo excessive deformation nor have unsatisfactory dynamic characteristics. Fail-safe evaluation requires analysis for the design loading conditions of as many structures as there are structural elements.

An invulnerable structure performs satisfactorily despite mechanical damage. Satisfactory performance can involve reduced requirements in the loading conditions or the performance specifications from those of the undamaged structure. Mechanical damage can be induced by any source, impact, fatigue, or corrosion. This initial damage can affect any part of the structural system. Invulnerability is distinct from fail safety since the damage progresses through a number of damage steps. The region of damage grows and may cascade until the structure becomes a mechanism. Invulnerability analysis requires evaluation of each of the imposed damage conditions for the invulnerability loading conditions. The performance of the structure must be compared with the appropriate performance specifications.

Survivability evaluation is warranted whenever human life or high cost structure is involved. Fail-safe analysis can be the basis for structural reliability and failure resistance evaluation. It can also be used to provide a low cost measure of structural invulnerability. Invulnerability evaluation may be used to assess damage, determine recoverability, or measure safety. As a basis for identifying structural failure modes, it can lead to safer and more economical designs.

This paper presents rapid analysis procedures for evaluating structural survivability. It describes a fail-safe analysis procedure which yields exact predictions of structural stresses and displacements. A modification of this process gives approximate invulnerability predictions with accuracy depending upon the number of damage steps until failure. Applications to the analysis of truss, frame, and compound systems illustrate the accuracy of the method and the nature of survivability characterizations.

The next section of this paper discusses the basic analysis approach for the survivability evaluation. The third section expounds the use of this approach for fail-safe analysis. The fourth section describes a modification of this approach for invulnerability studies. The fifth section discusses the use of fail-safe data for invulnerability studies. The final section contains the conclusions.

SECTION II

BASIC ANALYSIS APPROACH

The fundamental problem is to perform rapid analysis of many structures which are mutations of the initial configuration. For this paper, analysis consists of the prediction of stresses and deformations of the structural system. The types of mutations of the initial system include the omission of one member or the condition when several members are acting with reduced effectiveness and other members are missing.

Reanalysis of each mutation reusing the initial structure analysis process is extremely costly compared with analysis modification. Methods of exact analysis of modified structures, based on modifying an initial analysis, have been described by other authors. Pipes (Reference 3) describes the matrix manipulations to be employed if the force method is the basis for the initial configuration analysis. Sack, Carpenter, and Hatch (Reference 4) describe an equivalent process when the displacement method provides the initial analysis basis. Both of these procedures are unsatisfactory because they require the construction of transformation matrices based on geometry and topology and the operation with data dependent on the structural changes not normally saved in the solution process.

Since estimates of response may be satisfactory for invulnerability studies the economy of an approximate procedure recommends its use. The method of Melosh and Luik (Reference 1) is selected as the basis for analysis. This process provides exact estimates of behavior for fail-safe analysis and it guarantees monotonic convergence to the exact answers when applied to invulnerability studies. The rate of convergence is rapid and is an exponential function of the number of analysis cycles used (Reference 5). The process is also modular, thus well suited for implementation on a digital computer.

The four basic concepts of the Melosh and Luik multiple-configuration-analysis approach are:

1. Develop assumed behavior vectors based on the initial structure. The initial structure for survivability analyses is an "inclusive configuration" in that all members that will ever be included in a configuration are contained in the initial system. Thus, vectors of generalized forces which are based upon this design implicitly contain all the geometric and topological information for all the subset structures of interest.

2. Develop assumed behavior vectors associated with the members being changed. This approach can lead to exact predictions of internal forces and deflections for a fail-safe analysis because the process can then be regarded as superposition.

3. Use an energy approach for approximate analyses.

4. Base the modified structure response predictions upon considering a few vectors at a time and cycling through the complete relevant set of vectors until the response predictions are as exact as desired. The complete relevant set of vectors includes only those vectors or components associated with the members changed. Cycling through this set using an energy criterion to select the best behavior estimates is sufficient to insure monotonic convergence toward the exact behavior predictions for the modified structures. Since only a few vectors are treated at a time, very large structural systems can be handled despite a limited high-speed storage capability.

To understand the analytical process, let the stiffness method be considered as the basis of the initial configuration analysis. This analysis approach produces an element stiffness matrix and an element stress matrix for each element. When an element stiffness matrix is multiplied by the system displacements, it defines a set of generalized forces for the element. When an element stress matrix is multiplied by the joint displacements, it provides a measure of the mean stresses in the element.

These matrices are normally produced by computer codes based upon the stiffness method. In addition, these codes can easily yield a set of transformation matrices, one for each element, which will transform the element stiffness matrices into a set of independent self-equilibrating loadings for each member. These member self-equilibrating force systems can also be found by defining constraints on the element corresponding to imposing determinate support conditions. Alternately, with the known rigid body modes, the Gram-Schmit process can be used to develop self-equilibrating vectors from the element stiffness matrix. The analyst could also obtain the same self-equilibrating vectors by obtaining the eigenvectors of the elements stiffness matrices and discarding those associated with zero eigenvalues.

In the displacement approach the total set of load-deflection equations are formed and the equations are solved to obtain the deflections. It is common to solve these equations by decomposing the stiffness matrix. The recommended procedure, to maximize numerical accuracy (Reference 6) is to decompose this matrix into three matrices as follows:

$$K = LDL^T \qquad (1)$$

where K is the square symmetric stiffness matrix

L is a lower triangular matrix

D is with a diagonal matrix

L^T is the transpose of L.

It is easy to save the L matrix as a representation of the load-deflection equations for the equations for the initial structural configuration.

To describe the reanalysis process, suppose that all members through e-1 have been revised. Then in order to reflect the changes in structural response due to the change in effectiveness of member e, the following steps are performed:

a. Find deflections of the initial structural systems for each of the independent self-equilibrating loadings associated with the member being changed. Internal forces for the changed member are evaluated for these loadings. The deflections are developed directly by forward and backward substitution using the matrices L and D. A complete set of force vectors could be developed initially, if desired. However, with existing computer hardware, it is cheaper to develop these as needed using the L and D matrices, rather than reading them because they can be regenerated faster. Solve for the internal forces by multiplying the element stiffness matrix by the displacements. These forces will satisfy equilibrium if rigid body states are included in the element stiffness matrix.

b. Assume that the response of the modified structure can be obtained by superimposing the response of the initial configuration, the initial configuration response to the self-equilibrating loadings, and the cumulative change in response between the initial configuration and the response estimate for all the changes through member e-1. This assumption will lead to a set of simultaneous equations of maximum order J+2 for each loading condition, where J is the number of self-equilibrating states for the member. These equations are solved for the superposition scalars.

c. Superimpose the loadings to obtain an effective external loading for the initial configuration.

d. Evaluate deflections, as in step a, using the effective loading.

e. Determine member stresses, using the stress matrices.

The differences in the fail-safe and invulnerability analysis approaches arise in implementing step b. Steps c and d define one of a variety of ways of implementing the superposition. The procedure described herein minimizes the amount of new data to be stored in each calculation.

SECTION III

FAIL-SAFE ANALYSIS

In fail-safe analysis, step b of the general process involves direct application of the requirement that stresses in the cut member must be zero. Since each member failure is treated independently, the cumulative change vectors are zero, thus the self-equilibrating internal forces must be superimposed with the initial internal forces such that the internal forces are zero in the cut member. This requirement is expressed mathematically as

$$S_I^c + \sum_{j=1,2}^{J} \beta_j \, S_j^c = 0 \qquad (2)$$

β are undefined scalars

S_j^c are the internal forces in the cut member due to the j^{th} self-equilibrating state.

S_I^c are the internal forces in the cut member predicted for the initial configuration.

J is the number of self-equilibrating vectors.

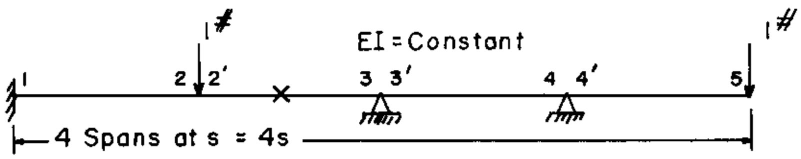
Table I provides an illustration of the calculations for fail-safe analysis of a member of a continuous beam. The beam consists of finite elements of equal length and uniform stiffness. The real loads on the structure consist of unit loads applied at the second and fifth joints and acting laterally to the beam axis. The system is clamped at the left and has two simple supports: one at station three and one at station four. The initial structure has two force redundants and six displacement unknowns.

The data given in Table I show the results when a fail-safe analysis is made for member 2'-3. Columns two, three, and four are results of calculations for step a of the analysis procedure. The second column defines external loads, joint displacements, and internal forces for the real loads on the initial configuration. The third and fourth columns display comparable information for the member self-equilibrating loadings.

The self-equilibrating loadings consist of the first two columns of the stiffness matrix for element 2' - 3. Associated displacements apply to the initial configuration. The internal forces must be modified to define an internal self-equilibrating state. Therefore, the cut member internal forces obtained by multiplying the element stiffness matrix times the

TABLE I*

BEAM FAIL-SAFE ANALYSIS



	Variable	Initial Structure	Member 2'-3 Self-Equil. Vector 1	Member 2'-3 Self-Equil. Vector 2	Member 2'-3 Cut
Effective Loading	P_2	1.0	2.0	-3.0	$.70000^{1}$
	M_2	0	-1.0	2.0	$-.36667^{1}$
	M_3	0	-1.0	1.0	$-.23333^{1}$
	M_4	0	0	0	0
	P_5	1.0	0	0	$.10000^{1}$
	M_5	0	0	0	0
Displacements** $\left(\times \frac{EI}{s^3}\right)$	W_2	$.79167^{-1}$	$.70833^{-1}$	$-.13750^{0}$	$.33333^{0}$
	θ_2	$-.37500^{-1}$	$-.11250^{0}$	$.26250^{0}$	$-.50000^{0}$
	θ_3	$.15000^{-1}$	$-.50000^{-1}$	$-.50000^{-1}$	$.16667^{0}$
	θ_4	$-.32500^{0}$	$.25000^{-1}$	$.25000^{-1}$	$-.33333^{0}$
	W_5	$.65833^{0}$	$-.25000^{-1}$	$-.25000^{-1}$	$.66667^{0}$
	θ_5	$-.82500^{0}$	$.25000^{-1}$	$.25000^{-1}$	$.83333^{0}$
Internal Forces	S_{p2}'	$.27500^{0}$	$-.17500^{0}$	$.75000^{-1}$	0
	S_{M2}'	$-.32500^{0}$	$.25000^{-1}$	$-.22500^{0}$	0
	S_{M3}	$.50000^{-1}$	$.15000^{0}$	$.15000^{0}$	0

$$\begin{bmatrix} -.17500^{0} & .75000^{-1} \\ .25000^{-1} & -.22500^{0} \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} = - \begin{Bmatrix} .27500^{0} \\ -.32500^{0} \end{Bmatrix}$$

$$\therefore \quad \beta_1 = .10000^{1} \qquad \beta_2 = -.13333^{1}$$

*Exponents indicate power of ten. i.e. $0.5^{-1} = 0.5 \times 10^{-1}$

**Displacements multiplied by EI/s^3

deflection vector are adjusted by adding the external imposed member loads with opposite sign. This step produces the data in the third group of rows of the table in columns three and four.

The realization of Equation 2 when member 2' - 3 is cut is given below the table entries. The total effective loads are given by

$$P_e = P_r + \sum_{j=1,2}^{J} \beta_j \quad P_j \tag{3}$$

where P_e is the total effective external load

P_I is the real initial configuration load

P_j is the j^{th} external loading for the self-equilibrating load

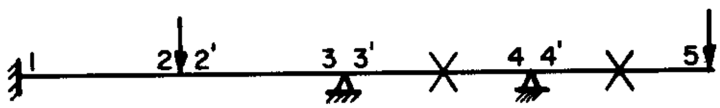
This yields data in the fifth column of Table I. The remainder of the data in this column follows from manipulations with L, D, and the cut member stiffness matrix. As expected, these data agree with exact response predictions for the structure remaining when member 2' - 3 is omitted.

The process must produce exact response predictions since the superposition guarantees satisfaction of equilibrium and compatibility everywhere, within the limitations of the finite-element representation. Thus, for a plane framework, regardless of the degree of force redundancy, solution of two simultaneous equations for each external load condition is the key to the analysis. If a triangular membrane is to be cut from a compound system, three equations are involved; for a rectangular prism, 18. In general, the number of equations to be solved is J.

Table II shows the results of similar fail-safe calculations for the cases when member 3' - 4 and separately, 4' - 5 are cut. In both of these cases, no set of β can be found to satisfy the equations. This means that the model reduces to a mechanism if either member is cut; or in general, if the matrix of β coefficients is singular, the reduced model cannot sustain an arbitrary external loading. The vanishing of all the coefficients when member 4' - 5 is cut indicates that the forces in this member can be established by equilibrium equations alone. Similarly, the vanishing of one of the two rows of coefficients when member 3' - 4 is cut indicates knowledge of a single force redundant (say the vertical shear at joint 4) is sufficient to define all the internal forces in member 3' - 4.

TABLE II*

BEAM FAIL-SAFE ANALYSIS BREAKDOWNS



	Variable	Member 3'-4 Self Vector 1	Member 3'-4 Self Vector 2	Member 4'-5 Self Vector 1	Member 4'-5 Self Vector 2
Effective Loading	P_2	0	0	0	0
	M_2	0	0	0	0
	M_3	-1.0	2.0	0	0
	M_4	-1.0	1.0	-1.0	2.0
	P_5	0	0	-2.0	3.0
	M_5	0	0	-1.0	1.0
Displacements $(\times \frac{EI}{s^3})$	W_2	$-.25000^{-1}$	$.75000^{-1}$	0	0
	θ_2	$.25000^{-1}$	$-.75000^{-1}$	0	0
	θ_3	$-.10000^{0}$	$.30000^{0}$	0	0
	θ_4	$-.20000^{0}$	$.10000^{0}$	0	0
	W_5	$.20000^{0}$	$-.10000^{0}$	.16666	$.50000^{0}$
	θ_5	$-.20000^{0}$	$.10000^{0}$	0	$-.50000^{0}$
Internal Forces	$S_{M3'}$	.20000	-.60000	0	0
	S_{M4}	0	0	0	0

Member 3-4 Cut

$$\begin{bmatrix} .20000^{0} & -.60000^{0} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} = - \begin{Bmatrix} -.50000^{1} \\ -.10000^{1} \end{Bmatrix}$$

Member 4'-5 Cut

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} = - \begin{Bmatrix} .10000^{1} \\ 0 \end{Bmatrix}$$

* Exponents indicate power of ten. i.e. $.5^{-1} = .5 \times 10^{-1}$

The fail-safe analysis procedure was applied to the structure illustrated in Figure 1. The structure is a four-level tower. The cross section of the tower is a square, four units on a side. The height of each bay is two units. The tower has a total height of eight units. The structure consists of 72 elements meeting at 20 joints. It is fixed at the lower four joints. All members have unit area and are aluminum. Examination of the structure reveals that there are 24 force redundants and 42 displacement redundants in the system.

The tower is subjected to two independent loading systems. All external loads are applied in the top plane. The first loading system consists of a side loading; a set of 250-pound loads applied to points 17, 18, 19, 20 in the positive y direction. The second loading condition is a torsion load: A 1000-pound load applied to joint 17 in the positive x direction, to joint 18 in the positive y direction, to joint 19 in the negative x direction and to joint 20 in the negative y direction.

The fail-safe analysis consists of removing each member in turn and studying the resulting stress distribution in the remaining structure. If the removal of a member causes no overstress, then that member is termed fail-safe. If the structure is fail-safe, then the removal of one member cannot cause overstress in any other member. (If a structure is not fail-safe, it could be made so in two ways: the existing members can be given larger areas or more members can be added. Which way results in a more economical fail-safe design can be decided with an optimum design procedure.)

The results of the fail-safe analysis for the tower are depicted in Table III in matrix form. The row codes of the matrix denote the members which were removed and the column codes indicate the resulting member of failures. If a matrix coefficient is blank or zero, no failure has occurred. A number in the appropriate box denotes the loading condition under which the member fails. For instance, the element in row 1-8, column 4-5 is a 2. This means that removal of element 1-8 causes element 4-5 to fail when subjected to loading condition 2.

Some interesting failure characteristics can be deduced by studying this array. Consider the first loading. Repeated failures of members 1-5, 2-6, 3-7, and 4-8 imply that these members are highly stressed in this loading in the initial structure. In fact, these members are fully stressed in this condition. It is observed that though failure of some members can make some of these members safe, no failure can make them all safe.

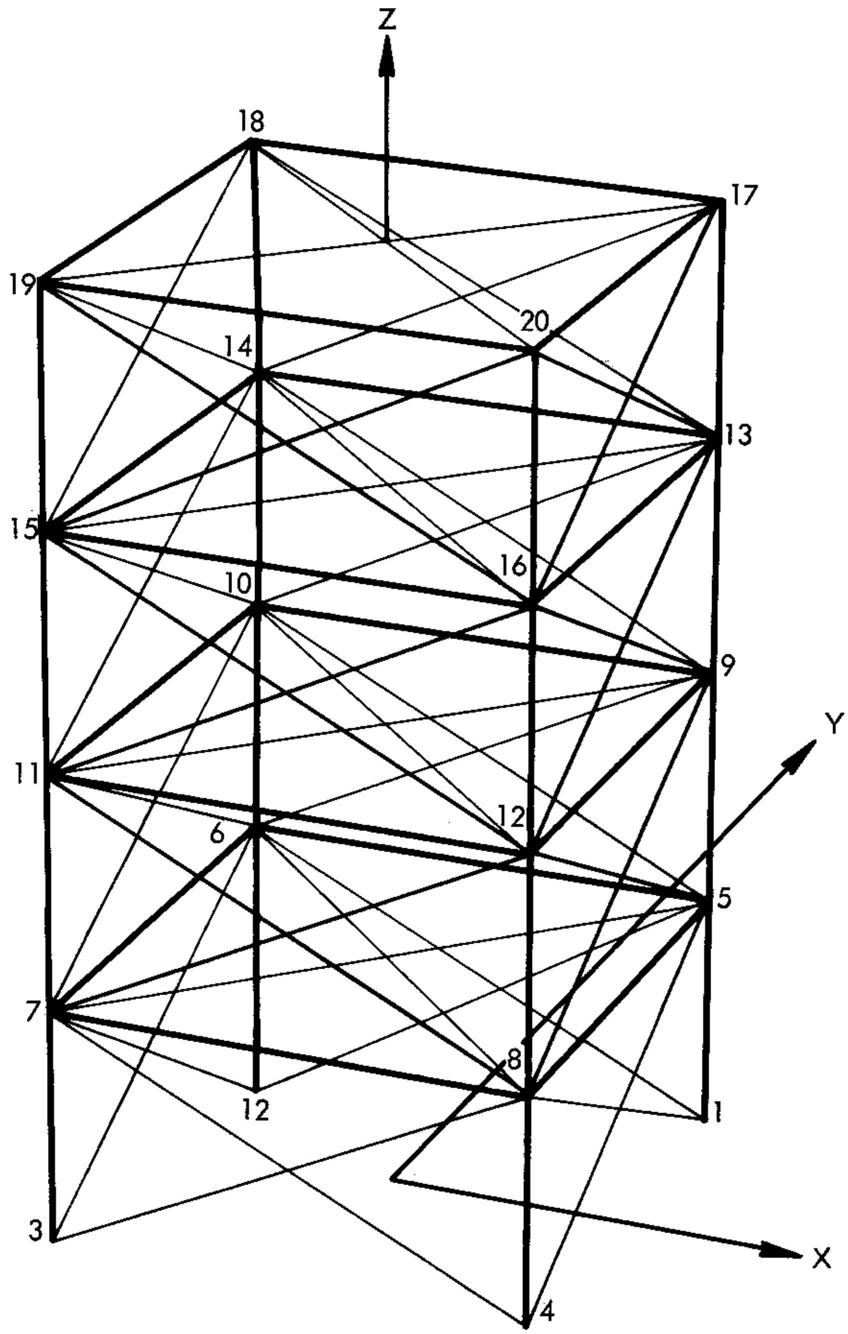


Figure 1. Four-Level Tower

TABLE III

TOWER FAIL-SAFE MATRIX*

	Ground Level												First Level												Second Level												Third Level							
No.	1-5	1-6	1-8	2-5	2-6	2-7	3-6	3-7	3-8	4-5	4-7	4-8	5-9	5-10	5-12	6-9	6-10	6-11	7-10	7-11	7-12	8-9	8-11	8-12	9-13	9-14	9-16	10-13	10-14	10-15	11-14	11-15	11-16	12-13	12-15	12-16	13-18	13-20	14-17	14-19	15-18	15-20	16-17	16-19
1-5	3				1			1									1			1																								
1-6	1	3		2	1			1				1																																
1-8	1		3		1			1		2																																		
2-5	1	2		3				1				1																																
2-6	1				3							1	1																															
2-7	1				1	3	2					1																																
3-6	1					2	3	1				1																																
3-7	1							3				1	1																															
3-8	1				1			1	3		2	1																																
4-5			2		1	1				3		1																																
4-7	1				1			1	2		3	1																																
4-8					1			1				3					1			1																								
5-9					1			1					3				1			1																								
5-10	1				1			1				1		3		2																												
5-12	1				1			1				1			3							2																						
6-9	1				1			1				1		2		3																												
6-10	1											1	1				3							1																				
6-11	1				1			1				1						3	2																									
7-10	1				1			1				1						2	3																									
7-11	1											1	1							3				1																				
7-12	1				1			1				1									3		2																					
8-9	1				1			1				1			2							3																						
8-11	1				1			1				1									2		3																					
8-12					1			1				1					1			1				3																				
9-13					1			1									1			1					3																			
9-14	1				1			1				1														3		2																
9-16	1				1			1				1															3							2										
10-13	1				1			1				1														2		3																
10-14	1											1	1											1					3															
10-15	1				1			1				1																		3	2													
11-14	1				1			1				1																		2	3													
11-15	1											1	1											1								3												
11-16	1				1			1				1																					3		2									
12-13	1				1			1				1															2							3										
12-15	1				1			1				1																					2		3									
12-16					1			1																												3								
13-18	1				1			1				1																									3		2					
13-20	1				1			1				1																										3				2		
14-17	1				1			1				1																									2		3					
14-19	1				1			1				1																												3	2			
15-18	1				1			1				1																												2	3			
15-20	1				1			1				1																														3		2
16-17	1				1			11				1																											2				3	
16-19	1				1			1				1																														2		3

*Elements not listed are fail-safe

Fail-safe inadequacies under loading 1 arise only in the first two tower levels. Except at the ground level, failure at any level induces failures only at a lower level. Failure in the upper level, however, results in no new element failures.

A failure in loading condition 2 results in overstress only for another member in the same face and joining the same levels. The symmetry of the fail-safe array in this loading reflects the symmetry of the structural geometry and internal forces.

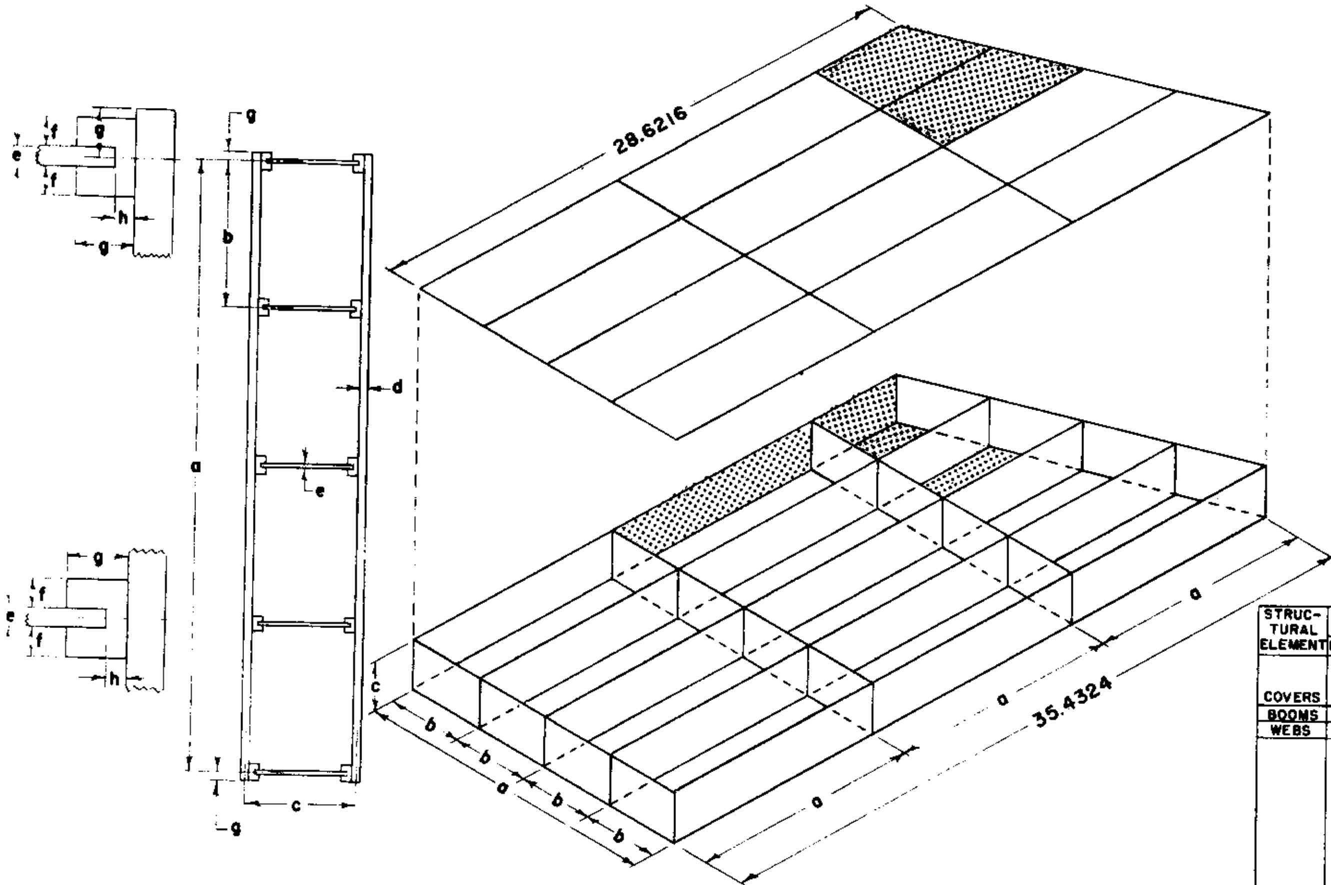
No single member failure can induce a kinematic instability. Thus no member stress is independent of stress in another member or can be determined without considering compatibility requirements. If kinematic failure had occurred, this would have been indicated by a "K" on the diagonal of the fail-safe array.

Even disregarding the failures in members 1-5, 2-6, 3-7, and 4-8, the structure is not fail-safe. Defining the percent of elements for which the system is fail-safe as a fail-safe measure, the tower is 45.8% fail-safe since 33 of the 72 members can fail without causing any new element failures.

The fail-safe analysis of the wing shown in Figure 2 provides another example. This wing has been investigated experimentally and analytically. (Reference 7, 8, 9, 10, and 11) The wing is of uniform depth. It has five parallel spars. Three ribs interlace the spars and lie perpendicular to them. The structure is cantilevered. Two independent tip loadings are considered: the first composed of equal force at the leading and trailing edges; the second of a single force at the trailing edge. The first loading will induce primarily bending; the second twisting. The total normal load in each condition is 2207.5 pounds (1000 kgm).

The finite element model of the structure is illustrated in Figure 3. It consists of 60 joints, 132 axial force members, 8 triangular cover skin elements near the root and 77 rectangular panels for the webs and remaining cover skins. A summary of the geometric and material properties of the elements appears in Table IV. The caps and cover skins are assumed to have no bending stiffness.

Table V shows a diagonal submatrix of the fail-safe matrix. Failure for this assessment is based on overstress (20,000 psi). The structural elements involved consist of a group near the root and the trailing edge of the wing. The region is shown shaded in Figure 2. Two's indicate an element failing under loading 2. Three's indicate elements failing with either conditions 1 or 2. No failures occur due to loading 1 only which do not also occur when loading 2 only is applied.



STRUCTURAL ELEMENT	E (γ = .3) Kg/mm²	E (γ = .3) PSI×10⁶
COVERS	7400	10.525
BOOMS	7450	10.596
WEBS	6900	9.814

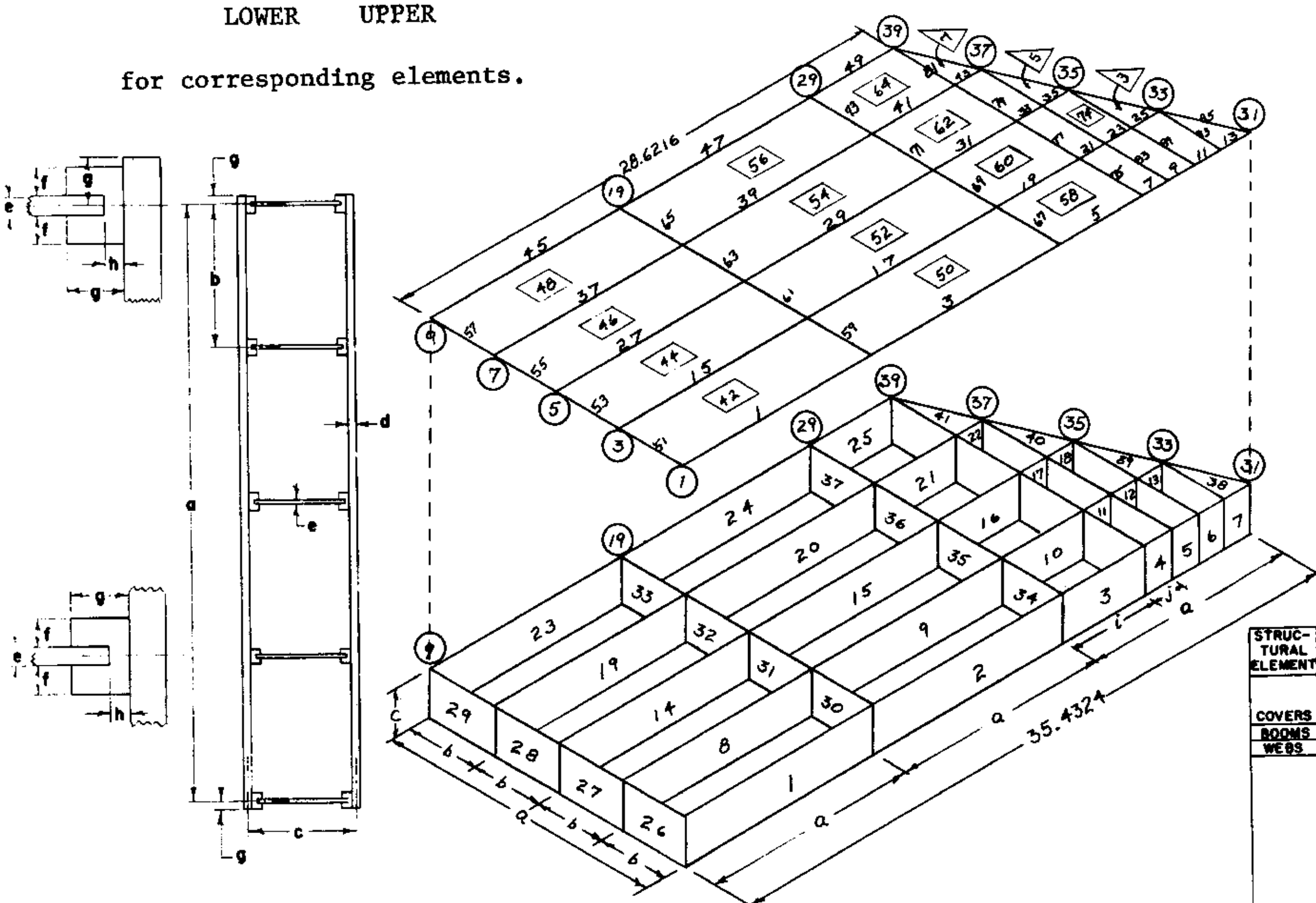
DIMENSIONS	MM	IN.
a	300	11.8108
b	75	2.9527
c	60	2.3622
d	3	0.1181
e	1.5	0.0590
f	4	0.1575
g	5	0.1968
h	1	0.0394
FRONT SPAR	900	35.43
REAR SPAR	727	28.62
CONVERSION	25.4	1
	.453Kg	1 LB.

Figure 2. Swept Wing Model

NOTE: Lower surface indexing is given by

$$n_{LOWER} = n_{UPPER} + 1$$

for corresponding elements.



STRUCTURAL ELEMENT	$\nu = .3$ E Kg/mm²	PSI·10⁶
COVERS	7400	10.525
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DIMENSIONS	MM	IN.
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b	75	2.9527
c	60	2.3622
d	3	0.1181
e	1.5	0.0590
f	4	0.1575
g	5	0.1968
h	1	0.0394
i	12.7	5.0000
j	43.25	1.7027
FRONT SPAR	900	35.43
REAR SPAR	727	28.62
CONVERSION	25.4	1
	.453Kg.	1 LB.

Figure 3. Idealized Swept Wing

TABLE IV

PROPERTIES OF STRUCTURAL ELEMENTS

Bar Element No.	Displacement Analysis			
	Bar Area		Young's Modulus, E	
	in^2	cm^2	lb/in^2 $\times 10^{-6}$	kgm/cm^2 $\times 10^{-3}$
1 - 14	0.0652	0.4206	10.596	745
15 - 44	0.0466	0.3006	10.596	745
45 - 50	0.0652	0.4206	10.596	745
51 - 58	0.0652	0.4206	10.596	745
59 - 66	0.0466	0.3006	10.596	745
67 - 74	0.0466	0.3006	10.596	745
75 - 82	0	0	10.596	745
83 - 94	0	0	10.596	745
95 - 102	0	0	10.596	745
103	1.0	6.452	9.814	690
104 - 106	0	0	9.814	690
107	1.0	6.452	9.814	690
108 – 132	0	0	9.814	690

Panel Element No.	Displacement Analyses							
	Panel Thickness		Young's Modulus, E		†Shear Modulus, G		*Poisson's Ratio, ν	
	in	cm	lb/in^2 $\times 10^{-6}$	kgm/cm^2 $\times 10^{-3}$	lb/in^2 $\times 10^{-6}$	kgm/cm^2 $\times 10^{-3}$		
1 - 8	0.118	0.030	10.525	740	4.048	284.6	0.3	
1 - 41	0.059	0.015	9.814	690	3.774	265.4	0.3	} Triangular
42 - 47	0.118	0.030	10.525	740	4.048	284.6	0.3	} Rectangular

† calculated from $G = E/2(1+\nu)$

* assumed

The fail-safe submatrix indicates that an increase in thickness in members 50 and 49 will make a big improvement in fail-safety. Examining the rest of the row partitions of the diagonal submatrix confirms this. Only failure of element 25 induces failures for elements not listed in Table V. The part of the structure involved in the submatrix has a fail-safety of only 6.3 percent without redesign of 50 and 49. Redesign will bring fail-safety to at least 58 percent.

These computer assessments of fail-safety take about the same amount of computer time as required for the stress and displacement analysis of the initial structure. Response predictions are exact. The economy is attributed to avoidance of repeated triangularizations of the matrix of coefficients of the load-deflection equations. The economy of the approach decreases as the number of elastic degrees of freedom of an element is increased.

TABLE V

WING FAIL-SAFE MATRIX

Failed Element \ Subsequent Element Failure	50	49	65R	64R	42	41	63R	62R	32	31	25R	24R
50	3	2										
49	2	3										
65R	3		3									
64R		3		3								
42	2	2			3							
41	2	2				3						
63R	3	2	3				3					
62R	2	3		3				3				
32	2	2							3			
31	2	2								3		
25R	3	3									3	
24R												3

R designates a rectangular panel. Unmodified numbers refer to bars

SECTION IV

INVULNERABILITY ANALYSES

For invulnerability analyses, step b of the general process uses energy approaches. In treating a particular member change, the complementary energy method is tried first. If this fails, the potential energy method is used.

The complementary energy method is used first because when approximate analyses are made in this context it yields more accurate response predictions than the potential approach (Reference 1). Because the complementary approach also involves somewhat less calculation effort, it is the clear choice over the potential. This process, however, will not reflect displacement changes when the structure becomes statically determinate. Then the potential energy method is evoked to define displacements.

Consider the development of the generalized load-deflection equations when the complementary energy method is used. These equations are given by

$$C\beta = b \tag{4}$$

C is a square symmetric array with coefficients

β is a vector of the β_j

b is a vector with j components

Then the complementary energy evaluates the coefficients as

$$C_{ij} = \sum_{m=1,2,\cdots}^{M} S_i \; A_m \; S_j \tag{5}$$

$$b_j = -\sum_{m=1,2,\cdots}^{M} S_I \; A_m \; S_j \tag{6}$$

where M is the number of members

A_m is the flexibility of element m

S_I is the vector of internal forces due to the real loads on the initial configuration

S_i and S_j defines the internal forces for the self-equilibrating load on the initial configuration

To simplify the calculation of the β, each self-equilibrating vector is written as the sum of prestress internal forces and the forces induced by deformation,

$$S_i = \bar{s}_i + \bar{S}_i \tag{7}$$

where $\bar{s}_i$ is vector of prestress internal forces involving only contributions to the member being changed

$\bar{S}_i$ is the vector of internal forces induced in the initial configuration by $\bar{s}_i$

substituting Equation 7 in Equation 5 gives

$$C_{ij} = \sum_{m=1,2,\cdots}^{M} (\bar{s}_i \; A_m \; \bar{s}_j + \bar{s}_i \; A_m \; \bar{S}_j + \bar{S}_i \; A_m \; \bar{s}_j + \bar{S}_i \; A_m \; \bar{S}_j) \tag{8}$$

But, the first right-hand term contains the external work in magnitude. As a consequence of the Maxwell reciprocity theorem, the second and third terms are equal. Furthermore, energy can be calculated by adding to the energy in the initial configuration the change in energy due to the change in flexibility. i.e. let

$$A_m = A_{mI} + a_m \tag{9}$$

where A_{mI} is the initial flexibility of member m

a_m is the change in flexibility of member m

Then since $\bar{s}$ is zero except for member e, Equation 8 can be rewritten as

$$C_{ij} = -\bar{s}_i \; \delta_j + \sum_{m=1,2,\cdots}^{E} (2\bar{s}_i \; A_{mI} \; \bar{S}_j + \bar{s}_i \; A_{mI} \; s_j + S_i \; a_m \; S_j) \tag{10}$$

where δ_j are the displacements associated with the j^{th} equilibrating loading; and the summation of m extends only those members whose flexibility has changed.

E is the number of elements changed.

Calculation of the coefficients of b_j can be facilitated by recognizing that the b must vanish when $A_m = A_{mI}$. Therefore, using Equation 9, Equation 6 can be written

$$b_j = -\sum_{m=1,2,\cdots}^{E} S_I \; a_m \; S_j \tag{11}$$

The calculation of C_{ij} and b_j by Equations 10 and 11 requires only displacements for the initial configuration and the stiffness matrix for the elements changed. The element stiffness matrices can be transformed to flexibility matrices by imposing determinant boundary conditions and inverting.

When the potential energy method is used, the displacements are given by

$$\delta = \sum_{j=1,2,\cdots}^{J} \beta_j^u \, \delta_j \tag{12}$$

where δ is a vector of joint displacements

The coefficients of the generalized load-deflection equations are given by

$$C_{ij} = \sum_{m=1,2,\cdots}^{M} \delta_i \, K_m \, \delta_j \tag{13}$$

$$b_j = P_I \, \delta_j \tag{14}$$

where the K_m are the element stiffness matrices

Again the β coefficients can be evaluated using only displacements of the initial configuration and the stiffnesses of the changed elements. Development of the equation in the appropriate form follows lines similar to development of Equation 10. The coefficients are recast as

$$C_{ij} = P_i \, \delta_j + \sum_{m=1,2,\cdots}^{E} \delta_i \, k_m \, \delta_j \tag{15}$$

where k_m is the change in stiffness for element m

Then, Equations 14 and 15 define the coefficients in the generalized load-deflection equations using the potential energy method.

It is seen that the complementary approach requires the solution of J+1 simultaneous equations while the potential requires J+2. If a group of members are changed, the energy estimates will approach the exact energy with the error reducing monotonically in successive steps for repeated cyclic treatment of the members. Since the set of self-equilibrating vectors associated with the members being changed is complete, the convergence will be toward the exact answers, barring truncation errors.

To illustrate the invulnerability calculations for a structure, consider again the continuous beam. It will be assumed that the flexibility of members 1 - 2 and 2 - 3 are doubled due to the initial damage condition. The analysis steps for treating initial damage are no different than those for treating progressive damage. The only distinction between initial damage and progressive damage is that the sequence of member treatment is known ab initio for the initial damage. For progressive damage, however, the e+1st member to treat is selected based on member stresses after representing damage to member 1 through e. Thus evaluation of the initial damage effects is sufficient to illustrate all the calculation steps.

Table VI shows the complementary energy calculation results, developed in evaluating response, when the flexibility of member 2' - 3 is doubled. In the case of the first member change, the cumulative change vector is zero. The basic data required for the analysis is contained in the first two groups of rows (shown as related to vector 1 and 2) of the table and the element 2 - 3 flexibility. This is given by

$$A_{2-3} = \frac{1}{6\,(EI)_{2-3}} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad (16)$$

This is referenced to the internal moment at 2' and 3.

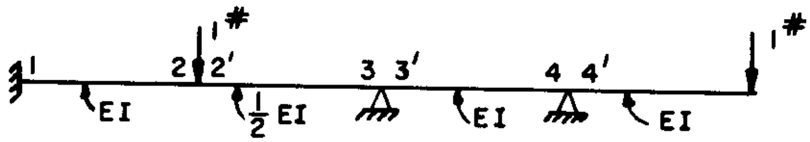
The third group of rows in Table VI shows the components formed to evaluate the coefficients in the generalized load-deflection equation given below this group. The effective load P_e is formed using the β ; and Equation 3. The predicted responses given are exact and would be regardless of the number of force redundants.

The next step in the calculations involves applying the complementary energy method to predict response predictions with the flexibility of member 1 - 2 doubled. In this case, the cumulative change vector is nonzero. The three simultaneous equations involved are singular because the structure involves only two force redundants. Despite this singularity, a unique solution exists for the β because the set is consistent. For this particular problem, predictions of response are also exact for the second member change with iteration because force redundacy is less than J+1.

Table VII shows the calculation data when representation of the second member damage is based on the potential energy method. In this case, four simultaneous equations are involved. The b coefficients are generated in the development of the C_{ij} coefficients. The change in the C_{ij} due to reductions in stiffness of members 1 - 2 and 2' - 3 are given in

TABLE VI*

FIRST MEMBER DAMAGE ANALYSIS



	Variable	Displ. × $\frac{EI}{s^3}$	S	= $\bar{s}$	+ $\bar{S}$
Vector 1	P_2'	$.70833^{-1}$	$-.17500^{0}$	-2.0	$.18250^{1}$
	M_2'	$-.11250^{0}$	$.25000^{-1}$	1.0	$-.97500^{0}$
	M_3	$-.50000^{-1}$	$.15000^{0}$	1.0	$-.85000^{0}$
Vector 2	P_2'	$-.13750^{0}$	$.75000^{-1}$	3.0	$-.29250^{1}$
	M_2	$.26250^{0}$	$-.22500^{0}$	-2.0	$.17750^{1}$
	M_3	$-.50000^{-1}$	$.15000^{0}$	-1.0	$.11500^{1}$

C_{ij}	$-\bar{s}_i\delta_j$	$\bar{s}_i A_{2-3}\bar{S}_j$	$\bar{S}_i A_{2-3}\bar{s}_j$	$\bar{s}_i A_{2-3}\bar{s}_j$	$S_i a_{2-3} S_j$	Σ
C_{11}	$.30416^{0}$	$-.30416^{0}$	$-.30416^{0}$	$.33333^{0}$	$.64583^{-2}$	$.35625^{-1}$
C_{12}	$-.48750^{0}$	$.48750^{0}$	$.48750^{0}$	$-.50000^{0}$	$.10625^{-1}$	$-.18750^{-2}$
C_{22}	$.88750^{0}$	$-.88750^{0}$	$-.88750^{0}$	$.10000^{1}$	$.35625^{-1}$	$.148125$
b_1	—	—	—	—	$+.77083^{-2}$	$+.77083^{-2}$
b_2	—	—	—	—	$+.36875^{-1}$	$+.36875^{-1}$

$$\begin{bmatrix} .35625^{-1} & -.18750^{-2} \\ -.18750^{-2} & .148125 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} = \begin{Bmatrix} -.77083^{-2} \\ -.36875^{-1} \end{Bmatrix}$$

$$\therefore \quad \beta_1 = -.2\overline{296} \qquad \beta_2 = -.2\overline{518}$$

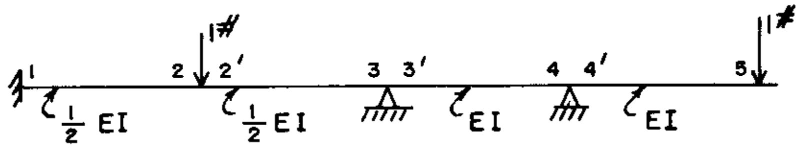
	P_2	M_2	M_3	M_4	P_5	M_5
P_e	$.12963^{1}$	$-.27\overline{407}^{0}$	$-.2222\overline{2}^{-1}$	0	1.0	0

	W_2	θ_2	θ_3	θ_4	W_5	θ_5
δ	$.97531^{-1}$	$-.7777\overline{7}^{-1}$	$.17\overline{407}$	$-.33\overline{703}$	$.67\overline{037}$	$-.83\overline{703}$

* Exponents indicate power of ten. i.e. $.5^{-1} = .5 \times 10^{-1}$

TABLE VII*

SECOND MEMBER DAMAGE ANALYSIS



	Variable	Initial Vector	Cum. Change Vector	Member (1-2) Vector 1	Self-equilibrium Vector 2
Effective Load	P_2	1.0	$.29633^{-1}$	−2.0	3.0
	M_2	0	$-.27407^{0}$	−1.0	1.0
	M_3	0	$.22222^{-1}$	0	0
	M_4	0	0	0	0
	P_5	1.0	0	0	0
	M_5	0	0	0	0
Displacements $\times \frac{EI}{S^3}$	W_2	$.79167^{-1}$	$.18364^{-1}$	$-.95833^{-1}$	$.15000^{0}$
	θ_2	$-.37500^{-1}$	$-.40277^{-1}$	$-.11250^{0}$	$.10000^{0}$
	θ_3	$.15000^{0}$	$.24074^{-1}$	$-.50000^{-1}$	$.10000^{0}$
	θ_4	$-.32500^{0}$	$-.12037^{-1}$	$.25000^{-1}$	$-.50000^{-1}$
	W_5	$.65833^{0}$	$.12037^{-1}$	$-.25000^{-1}$	$.50000^{-1}$
	θ_5	$-.82500^{0}$	$-.12037^{-1}$	$.25000^{-1}$	$-.50000^{-1}$

	C_{11}	C_{12}	C_{13}	C_{14}	C_{22}
$P^T \delta$	$.73750^{0}$	$.30401^{-1}$	−.12083	$.20000^{0}$	$.11048^{-1}$
$\delta^T K \delta$	$.04330^{0}$	$.97814^{-2}$	$.49167^{-1}$	$-.77500^{-1}$	$.22078^{-2}$
Total	$.69420^{0}$	$.20620^{-1}$	$-.71447^{-1}$	.12250	$.88403^{-2}$

	C_{23}	C_{24}	C_{33}	C_{34}	C_{44}
$P \delta$	$.35500^{0}$	$.14815^{0}$	$.30416^{0}$	$-.40000^{0}$	$.55000^{0}$
$\delta^T K \delta$	$-.35500^{0}$	$-.13714^{0}$	$-.14836^{0}$	$.19250^{0}$	$-.26000^{0}$
Total	0	$.11011^{-1}$	$.15580^{0}$	$-.20750^{0}$	$.29000^{0}$

Then, $\beta_1 = 1.0000$ $\beta_2 = 2.7793$ $\beta_3 = -2.1092$ $\beta_4 = -1.3467$

*Exponents indicate power of ten, ie. $.5^{-1} = .5 \times 10^{-1}$

the table. These are developed using the element stiffnesses and the assumed displacements. The displacements (and internal forces) associated with the β are exact because the force redundancy is less than J+1. The exactness is reflected by the fact that β is exactly one.

This problem illustrates some important characteristics of the approach. In particular, response predictions for the first member change are exact. Whenever one of the two energy methods will yield exact predictions, the other will. When the results are approximate, the complementary energy approach takes less calculation effort than the potential.

Figure 4 exhibits the accuracy of the analysis technique in applying it to the four-level tower. The figure shows the error in element force as a function of the number of iteration cycles for an initial damage condition. The assumed damage involves the mechanical destruction of elements 4-8, 5-9, 12-16, 14-18, and 2-6. In the analysis, elements were cut one-by-one in the given sequence. The set of self-equilibrating vectors was cycled through six times after all members changes were made.

The element curves show an exponential diminution of the error in element forces after the first iteration. The total strain energy must decrease monotonically in each cycle of iteration. This fact could be used to tailor the number of iterations to the desired accuracy.

Since the exact solution involves zero forces for each of the members, the application shows that the complementary energy process does not degenerate despite the theoretical singularity. Since the exact forces are zero, the relative error is defined as the error in element force divided by the maximum element force. With this definition, the maximum error diminishes from 19.3 percent in member 14-18 after all changes have been introduced to 10.5 percent in member 4-8 at the end of the first iteration cycle. After six cycles the maximum relative error is less than 0.1 percent.

These data pertain to internal loads under loading condition 1. The initial damage results in an increase in internal strain energy from 156 to 671 inch-pounds. Thus the average deflections increases by a factor of about five. When the invulnerability analysis involving cascading failure is executed, the analysis predicts immediate collapse as a mechanism due to first step failures in the first level support system. Since only a small increase in energy can be absorbed before collapse the system can be expected to fail rapidly. The failure mode will appear like the death throes of the leaning tower of Pisa. Most of the members will remain intact as the system rotates to its destruction.

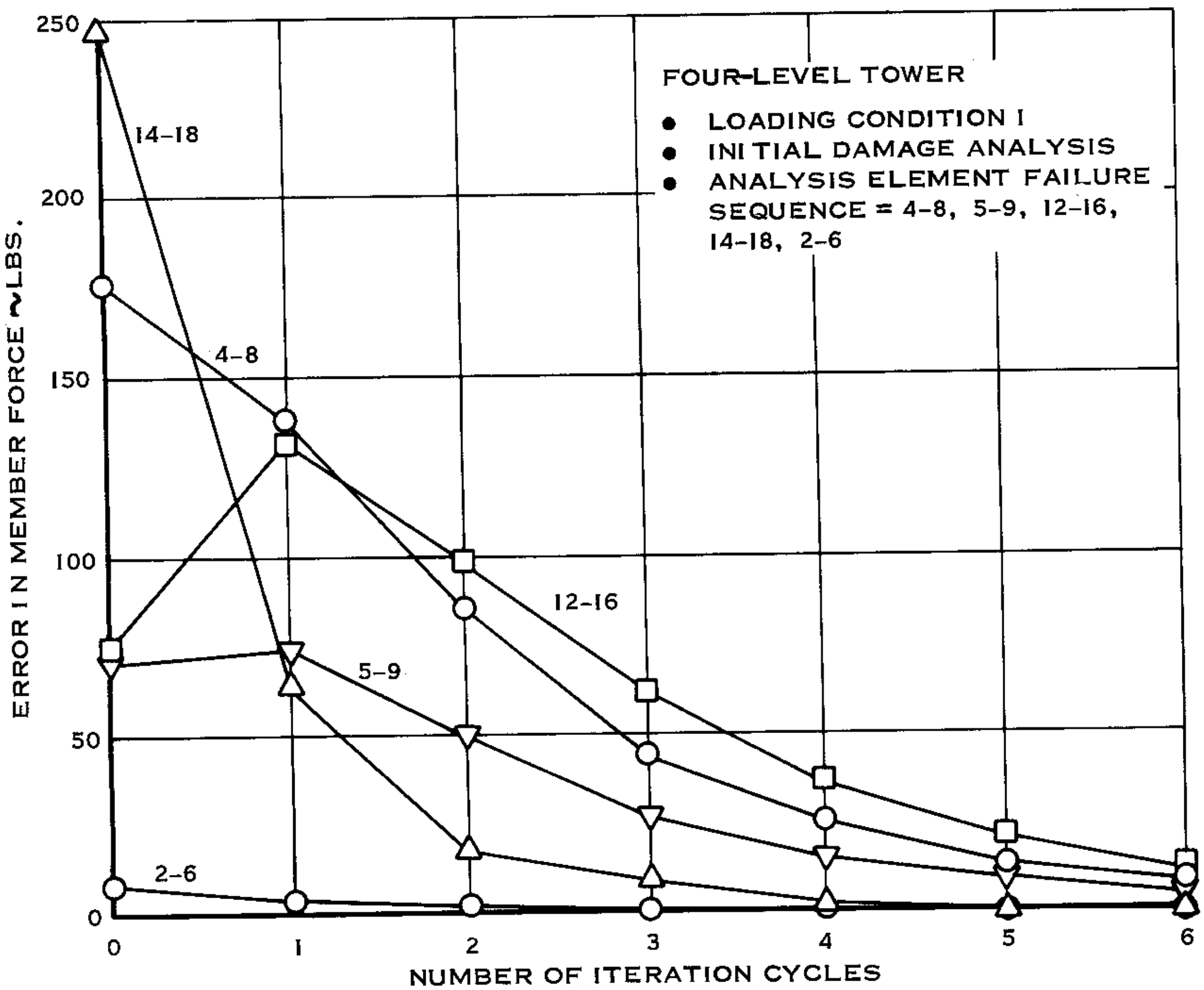


Figure 4. Member Force Error Variation

Table VIII summarizes results of an invulnerability analysis of the wing. Initial damage was assumed to occur in element 50. The cascading failure was explored through six steps with the indicated associated failures. At each step, the element which had the maximum overstress in the previous step was removed. This table shows that removal of more than one element in a step would lead to erroneous results.

Failures arise in loading condition 2 which is assumed to act with unreduced magnitude when the initial damage is incurred. The structure stabilizes after the sixth member failure. Thus the static analysis confirm that the wing will survive this particular damage condition.

Table IX summarizes results of an invulnerability analysis when initial damage destroys elements 50, 49, and 64R and loading condition 1 is acting. This analyses shows that the wing fails due to excessive deflections after six element failures. Examination of the failure sequence shows that it is progressing across the wing root. Even if excessive deflection were not experienced, collapse due to overstress would be expected with this bending loading.

Figure 5 shows the wing-tip deflections for the initial damage and the cascade step 5. The large increases in deflection for the twist loading occur when a cover skin element fails. The first large increase for the bending loading occurs when a cover skin element fails. The second occurs when the trailing edge web fails. The monotonic increase in deflection is a symptom of the monotonic strain energy increase that is associated with successive failure steps.

TABLE VIII

WING-TWISTING INVULNERABILITY ANALYSIS

Cascade Step	Element Destroyed	Subsequent Failures
1	50	49
2	50,49	65R
3	40,49,65R	42,44,63R,8T
4	50,49,65R,42,	63R
5	50,49,65R,42,63R	32,25R,61R
6	50,49,65R,42,63R,32	None

TABLE IX

WING-BENDING INVULNERABILITY ANALYSIS

Cascade Step	Element Destroyed	Subsequent Failures
1	50,49,64	41,43,7T,62R
2	50,49,64,41,	62,7T
3	50,49,64,41,62	31,33,35,5T,25R,60R
4	50,49,64,41,62,31	25,33,5T,60R
5	50,49,64,41,62,31,25	34,36,6T,21R,61R 69R,71R,75R
6	Deflections Excessive	

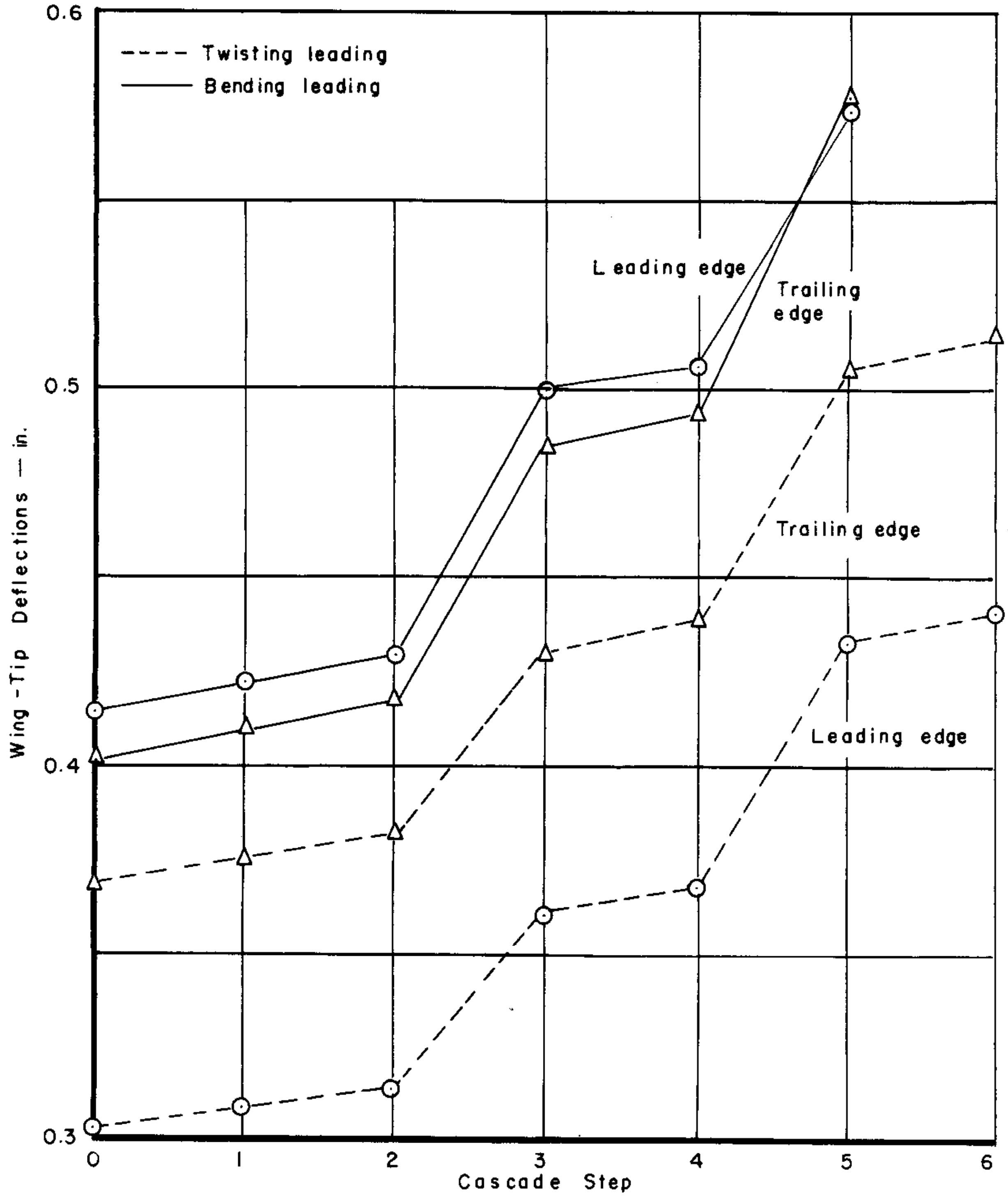


Figure 5. Wing-Invulnerability Analysis Deflections

SECTION V

COMBINED ANALYSES

Use of fail-safe analysis data for invulnerability analyses can result in improving calculation economy. One process of achieving this consists of super-imposing the internal force distributions of the fail-safe analysis to reflect progressive member damage.

The process is like the fail-safe analysis procedure described. Self-equilibrating force vectors are formed by subtracting the internal force vectors for a failed member analysis from the force vector for the initial configuration. Corresponding displacements are developed with a similar subtraction of displacement vectors. The β's are then found by equations of the form of Equation 2. With the β's known, internal forces and deflections are obtained by superposition.

The results of a combined analysis of the swept wing are described. Using this method for invulnerability analysis of the swept wing gives excellent predictions of deflections. The maximum error in any of the six steps is only three parts in the third significant decimal digit.

The combined method also yield good stress predictions for the wing. Table X cites the maximum relative error in stress estimates for the failure cascade steps. Most element stress predictions are accurate to two or more significant decimal digits. Indication of the next element to remove was always the same as for the exact analysis.

In these calculations, the maximum stress was selected as the representative stress for a panel. For cover skins, this stress was the spanwise normal stress. For the spar and rib webs, this stress was the shear stress.

This combined analysis method offers an economical way of identifying critical damage conditions. For example, in a membrane problem, analysis effort is reduced by at least a factor of three using the combined approach. If the failure does not cascade through many steps, excellent accuracy can be anticipated for deflection predictions, good predictions of the sequence of member overstressing, and fair to good predictions of element stresses. Since accuracy will deteriorate as the number of damage steps increases a check on compatibility should be included in the analysis to identify spurious results.

TABLE X

COMBINED ANALYSES STRESS ERROR*

Cascade Step	Elements Destroyed	Maximum Relative Error	Element With Error
1	50	0	All
2	50, 49	0	All
3	50, 49, 65R	2.15%	24R
4	50, 49, 65R, 42	1.70%	24R
5	50, 49, 65R, 42, 63R	3.18%	32
6	50, 49, 65R, 42, 63R, 32	0.736%	24R

*Twist Loading

SECTION VI

CONCLUSIONS

Study of survivability analysis reveals the following:

1. Fail-safe analysis can be performed at a cost comparable to the cost of analysis of the initial configuration by extending the approach of Melosh and Luik (Reference 1). This approach yields exact predictions of static response without reference to initial geometry. It uses instead element stiffness matrices in a modular process.

2. Exhibiting fail-safe data in a matrix array leads to rapid recognition of elements and loadings which are critical to obtaining a fail-safe design. This array can also be used to provide an indication of failure cascade when the initial damage consists of the destruction of structural elements.

3. Extension of the fail-safe analysis procedure to invulnerability studies can be done by providing for analysis iteration. With this process, good estimates of failure progression are expected with a single iterative cycle. By repeated iterations, responses can be evaluated as exactly as desired.

4. Use of fail-safe response predictions for approximate invulnerability analyses is satisfactory and economical for large order systems involving elements with multiple elastic degrees of freedom when few failure cascade steps are considered. For truss systems, where exact behavior predictions would result, the method is superior to the iterative approach.

SECTION VII

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