STATISTICAL ASPECTS OF ACOUSTICAL FATIGUE

by

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The theory of statistics deals with random variables, that is, quantities which have to be described by a distribution function $F(x)$ indicating the probability that the variable takes a value equal to or less than x. This function involves one parameter of location, one of scale, and in some cases one or more parameters of shape. The random variable may equally well be defined by the function $f(x) = dF(x)/dx$ which, to avoid confusion, will here be called the probability density function.

In the field of material fatigue failure there are three principal random quantities: the fatigue strength of the object, the load imposed on the object, and the resulting cycle life. The fatigue strength and the cycle life are random variables while the load in general is a random function of time. In conventional fatigue tests, a load pulsating with a constant stress or strain amplitude until fatigue failure occurs is substituted for the random load occurring in actual service. The fatigue damaging effect of one single stress cycle is then uniquely determined by two stress components, for example, the stress amplitude S_a and the mean stress S_m, the maximum stress S_{max} and the stress ratio $R = S_{min}/S_{max}$, or some other pair of components. A definite distribution of fatigue life is, consequently, associated with any pair of stress components, which in this simple case, constitutes a necessary and sufficient <u>fatigue damage representation</u> of the load.

Since the effect of the frequency, the speed effect, is, within wide limits, negligible compared to the scatter in fatigue life, there is no necessity of knowing the frequency actually used. An addition of the frequency to the representation would therefore provide unnecessary information.

A complete fatigue test is composed of several stress levels, each level defined by one of the stress components, while some other component remains the same for all levels. Each observed cycle life N_i within a group of observations belonging to a certain stress level S_j indicates a certain percentage point of the distribution function $F(N)$ (infected with a sampling error which can be reduced only by increasing the number of observations within the group). Furthermore, it can be proved - on the rather safe assumption that an increased load results in a decreased cycle life, and vice versa - that the stress level S_j indicates exactly the same percentage point as above of the distribution function $F(S)$ corresponding to the assigned cycle life N_i. Consequently, the fatigue strength distribution can be determined from a sufficiently large number of properly located data points. It should, however, be noted that the average (mean or median) S-N curve and the distribution functions $F(S)$

depend not only on that stress component which defines the stress level but also on that stress component which remains constant throughout the test. For example, the distribution function $F(S_{max})$ for a constant S_m differs from that for a constant R, and the deduction of one of them from the other is impossible without having a complete family of appropriate S-N curves.

In the general and more complicated case when the load is a random function of time, it has to be represented by a set of characteristics (stress components and distribution parameters). This set should be necessary and sufficient, that is, any given set should define a load which is uniquely associated with a specific distribution of fatigue life. Unnecessary information is provided by the representation, if variation of one of the characteristics does not influence the damaging effect of the specified load. Insufficient information is presented, if the specified load corresponds to more than one fatigue life distribution.

Various representations of a random time function will now be examined. Let $y(t)$ denote this function. The logical extension from the concept random variable to the random function is performed by regarding the values $y(t_k)$ as individual values of a random variable Y, where t_k are time points taken at random or at constant time intervals, sufficiently small compared to the period of the highest frequency involved in the random function.

The probability density function $f(y)$ of the random variable Y will be called <u>the probability representation</u> of the random time function $y(t)$. Two important statistics of this distribution are: the mean $\bar{y}$ and the variance σ^2 defined by

$$\bar{y} = \lim_{T \to \infty} \frac{1}{T-t_o} \int_{t_o}^{T} y(t)\, dt \qquad (1)$$

$$\sigma^2 = \lim_{T \to \infty} \frac{1}{T-t_o} \int_{t_o}^{T} (y - \bar{y})^2\, dt \qquad (2)$$

If $\bar{y}$ and σ^2 are independent of the lower limit t_o of the integral, the random function is termed <u>stationary</u>.

Since by experience and in contradiction to the Miner's law of cumulative damage, the fatigue life is influenced by the sequence of high and low stress cycles imposed on the test piece, it is obvious that, even if the random load is stationary, the fatigue life will depend on the starting point t_o of the given load $y(t)$.

If, however, identical test pieces are repeatedly subjected to this stationary random load $y(t)$, the average of observed cycle lives will with increasing number of tests tend to a fixed value which defines the average cycle life corresponding to average random load.

The scatter in cycle life is thus composed of one portion due to the test piece and another portion due to the randomness of the load. This fact may be mathematically expressed in the following way.

Suppose that the median S-N curve can be put in the form

$$\breve{S} - \breve{S}_e = b\,(\breve{N}/B + 1)^{-a} = b \,.\, \breve{U} \qquad (3)$$

where U is a function of the cycle life, determined by the parameters a and B.

The P-S-N diagram may in some cases (which have been called the P-cases A) be put in the more general form

$$S - S_e = b \,.\, U \qquad (4)$$

where S is the random load, S_e the random fatigue limit and U a function of the random life N.

The variances of these random variables are related by

$$\mathrm{var}(S) + \mathrm{var}(S_e) = b^2\,\mathrm{var}\,(U) \qquad (5)$$

since S and S_e are independent variables.

Another representation of random time function is obtained by considering the function to be composed of a finite or infinite number of sinusoidal components with circular frequencies ω between 0 and ∞. It should, however, be pointed out that this splitting up into components is a purely mathematical procedure, in most cases without any physical sense.

In the particular case that $y(t)$ is a periodic function it may be represented by a Fourier series but in general a Fourier integral is required being

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\,e^{i\omega t}\,d\omega \qquad (6)$$

where $G(\omega)$ is a <u>frequency representation</u> of $y(t)$ which may be computed from $y(t)$ by the reciprocal relation

$$G(\omega) = \int_{-\infty}^{\infty} y(\tau)\,e^{-i\omega\tau}\,d\tau \qquad (7)$$

Another frequency representation is obtained by means of the average power $\overline{y^2(t)}$ defined by

$$\overline{y^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T [y(t)]^2 dt \qquad (8)$$

Denoting by $\Phi(\omega)$ that portion of $\overline{y^2(t)}$ which arises from components having frequencies between ω and $\omega + d\omega$, we have

$$\overline{y^2(t)} = \int_0^\infty \Phi(\omega)\, d\omega \qquad (9)$$

The function $\Phi(\omega)$ is called the power spectrum of $y(t)$. It may be evaluated from observed data by means of the correlation function $R(\tau)$ defined by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T y(t) \cdot y(t+\tau)\, dt \qquad (10)$$

The functions $\Phi(\omega)$ and $R(\tau)$ are reciprocally related by the Fourier cosine transformation and are consequently equivalent representations.

The use of frequency representations is an adequate method for relating the response of a system subjected to a random load on two conditions only; (1) that the system is linear, because only then the response of two simultaneous components is equal to the sum of the responses of each separate component, (2) that the system responds differently to different frequencies, because otherwise there is no use of knowing how the "power" is distributed over the frequency band. If these conditions are fulfilled there is a simple relation between the power spectra of the output and of the input through the frequency-response function, which is a characteristic of the linear system.

An equally simple relation does not exist between the imposed random load and the fatigue damage which is, in fact, a non-linear, almost frequency-insensitive function of the load. For this reason it is to be suspected that neither the probability representation nor any frequency representation is an appropriate representation of the damaging effect of a random load.

Searching for a representation, providing necessary and sufficient information on the damaging effect of a load varying randomly with time, the following considerations may be useful:

Based on the fact that the stress cycles in Fig.1 will produce equal fatigue damage, it is apparent that there is no use of knowing the shape of these stress cycles and that the maximum and the minimum stresses (or the range and the mean stress) will in this particular case provide necessary and sufficient information on the damaging effect.

It seems possible to formulate a general law stating that two stress cycles which can be transformed into each other by a continuous deformation of the time scale will have equal damaging effect. As a corollary it follows that a necessary representation should be invariant with any reasonable deformation of the time scale. The word "reasonable" is added because random time functions involving very high frequencies (shocks or discontinuities) or very low frequencies may require a different treatment. It is apparent that neither the probability representation nor any frequency representations satisfies this condition. On the other hand, it can be proved that they are insufficient. Suppose that we have two random loads producing different fatigue damage. By a suitable deformation of the time scale - which does not change the damaging effect - their probability density functions and by another deformation their power spectra can be brought to coincidence. Thus, two random loads having identical probability density functions or power spectra may have different fatigue damaging effect; which proves their insufficiency.

A representation which satisfies the required qualifications has to be based on the ordinates of the extremes (peaks and troughs) or some - preferably linear - combinations of them (since the time coordinates are, within certain limits, unnecessary data.)

Denoting the ordinates of the peaks by S_{max} or p and those of the troughs by S_{min} or t, the ascents a and the descents b are defined by

$$\left.\begin{aligned} a_n &= p_n - t_n \\ b_n &= p_n - t_{n+1} \end{aligned}\right\} \tag{11}$$

as demonstrated in Fig.2. These two quantities were initially proposed as a representation of the damaging effect. It can, however, be proved as follows that they do not provide sufficient information.

From (11) it follows that

$$t_{n+1} - t_1 = \sum_1^n (a_n - b_n) \tag{12}$$

Assuming a and b to be independent variates we have

$$\text{var}(t_{n+1} - t_1) = n[\text{var}(a) + \text{var}(b)] \tag{13}$$

Since var(a) and var(b) by definition are non-negative, $\text{var}(t_{n+1} - t_1)$ increases with n and the random load is not stationary. The same conclusion can be proved to be valid even when a and b are correlated, the only exception being that for any n

$$a_n = b_n \tag{14}$$

From (12) it then follows that $t_{n+1} = t_1$ = constant and from (11) that $p_n = a_n + t_1$. Thus, any variate a defines a stationary random load.

In the same way, the assumption

$$a_{n+1} = b_n \tag{15}$$

corresponds to the condition $p_{n+1} = p_1 = \text{constant}$.

A third alternative is obtained by

$$t_n / p_n = R = \text{constant} \tag{16}$$

Consequently, the distribution of the variate a is a proper representation for random loads having constant S_{min}, S_{max}, or $R = S_{min}/S_{max}$. The distribution of b is in all these very particular cases uniquely determined by the distribution of a.

A more general representation is based on the standardized ordinates of the peaks and, for symmetry reasons, the standardized negative ordinates of the troughs. It is postulated that the damaging effect is uniquely determined by the two distribution functions

$$F_1[(S_{max} - \overline{S}_{max})/\sigma_p] \quad \text{and} \quad F_2[(\overline{S}_{min} - S_{min})/\sigma_t] \tag{17}$$

where the means

$$\overline{S}_{max} = \overline{p} = \Sigma p_n/n \quad ; \quad \overline{S}_{min} = \overline{t} = \Sigma t_n/n \tag{18}$$

and σ_p, σ_t are the standard deviations of S_{max}, S_{min}.

Alternatively the quantities $\overline{S}_a$ and $\overline{S}_m$ defined by

$$\overline{S}_a = (\overline{S}_{max} - \overline{S}_{min})/2 \quad ; \quad \overline{S}_m = (\overline{S}_{max} + \overline{S}_{min})/2 \tag{19}$$

may be used. Then we have

$$F_1[(S_{max} - \overline{S}_m - \overline{S}_a)/\sigma_p] \quad \text{and} \quad F_2[(\overline{S}_m - S_{min} - \overline{S}_a)/\sigma_t] \tag{20}$$

and the load is, for given distribution functions, defined by the four parameters $\overline{S}_a$, $\overline{S}_m$, σ_p, and σ_t, the two first of them being stress components and the two other giving a measure of the irregularity of the load. The particular case $\sigma_p = \sigma_t = 0$ defines the constant-amplitude load.

It is important to note that the mean stress $\overline{S}_m = (\overline{p} + \overline{t})/2$ is principally different from the mean of the random time function $\overline{y(t)}$ which is defined by

$$\overline{y(t)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T y(t)\, dt$$

and which should be avoided when fatigue damage is concerned, since its value is sensitive to deformation of the time scale.

A symmetrical load, where the variates $(S_{max} - \overline{S}_m)$ and $(\overline{S}_m - S_{min})$ are identically distributed, is obviously defined by one single function F_1 and the three parameters $\overline{S}_a$, $\overline{S}_m$, and $\sigma = \sigma_p = \sigma_t$.

It should be noted that S_{max} and S_{min} are independent variates on the condition only that

$$p_o - t_o \geq 0 \tag{21}$$

where p_o is the lower bound of p and t_o is the upper bound of t. If (21) applies, a line located anywhere between p_o and t_o will have exactly the same number of crossings (n_o) as the number of extremes (n_e). In general the ratio

$$r = n_o / n_e = 1 \tag{22}$$

Another, necessary and sufficient, representation was during the conference proposed and examined by H.C.Schjelderup who uses as observed data the values

$$S_a = (p_n - t_n)/2 \quad \text{and} \quad S_m = (p_n + t_n)/2 \tag{23}$$

Since

$$S_a = S_{max} - S_m \tag{24}$$

the distributions of the standardized variates may be put in the form

$$F_3[(S_a - \overline{S}_a)/\sigma_a] = F_3[(S_{max} - S_m - \overline{S}_a)/\sigma_a]$$

and

$$F_4[(S_m - \overline{S}_m)/\sigma_m] \tag{24}$$

Comparing (20) and (25), it is found that S_m is substituted for $\overline{S}_m$. Clearly, the function F_3 differs from F_1 and so do the standard deviations σ_p and σ_a but for the case that S_m is constant.

Since by definition the variate S_a is zero-bounded, it cannot be normally distributed and it is excluded that $F_3 = F_4$ even in the symmetrical case, where the representation (20) is more convenient due to the reduction of necessary characteristics. In other cases the preference should be given to that representation which leads to the simpler distributions.

Finally, some general remarks on the relation between random-load and conventional tests seem appropriate. An ascent a and the following descent b constitute a stress cycle which will be termed <u>complete</u> if $a = b$ and incomplete if $a \neq b$. The load of constant-amplitude tests and program tests are composed of complete stress cycles, while the random load is composed of incomplete cycles.

The damaging effect of any complete stress cycle can in principle be determined by repeating it. This is excluded for an incomplete stress cycle, since it cannot be repeated. Keeping this in mind and considering the difficulties of predicting the damaging effect even of known sequences of complete stress cycles from the results of constant-amplitude tests, it seems to be a rather hopeless task to predict the damaging effect of a random, that is, an unknown sequence of incomplete stress cycles from the results of conventional fatigue tests.

A more realistic attitude would be to consider a random load of given structure (given shapes of F_1 and F_2) as a basic type of load, just as the constant-amplitude load is another and to establish by proper tests the average S-N curves and the fatigue-strength distribution $F(S)$ corresponding to the random load in question.

Taking for simplicity the symmetrical load of a given structure (F_1) and choosing $\overline{S}_a$, $\overline{S}_m$, and σ as the characteristics, then $\overline{S}_a$-N curves for properly selected values of $\overline{S}_m$ and σ (or maybe $\sigma/\overline{S}_m$) have to be determined. The curves obtained in this way will, of course, differ from those obtained by conventional fatigue tests, and it seems, for the present, excluded to deduce one from the other due to completely different character of the damage produced by these two types of load.

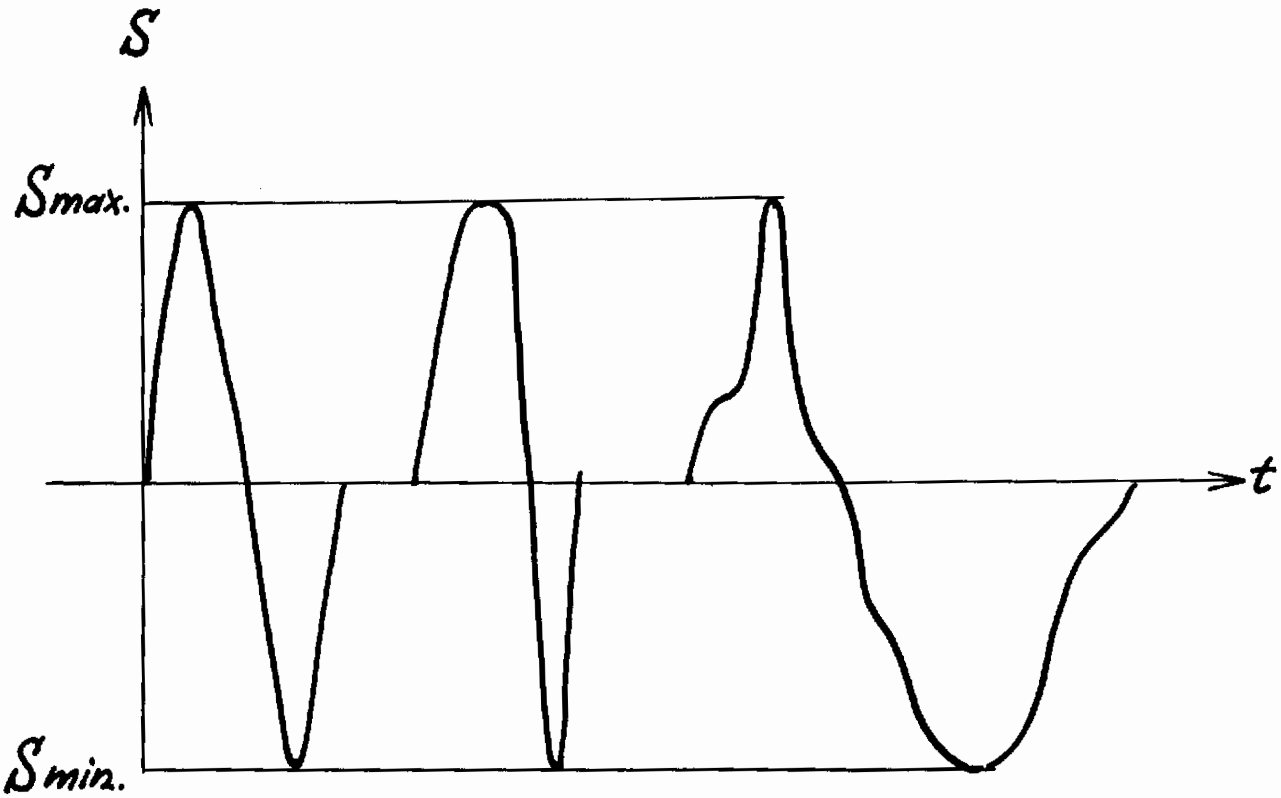

Fig. 1 - Three Stress Cycles Producing Equal Fatigue Damage

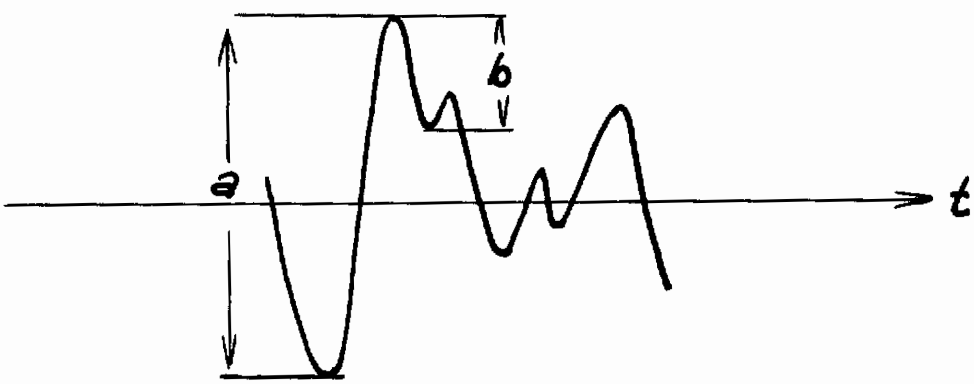

Fig. 2 - Ascents a and Descents b of a Random Time Function

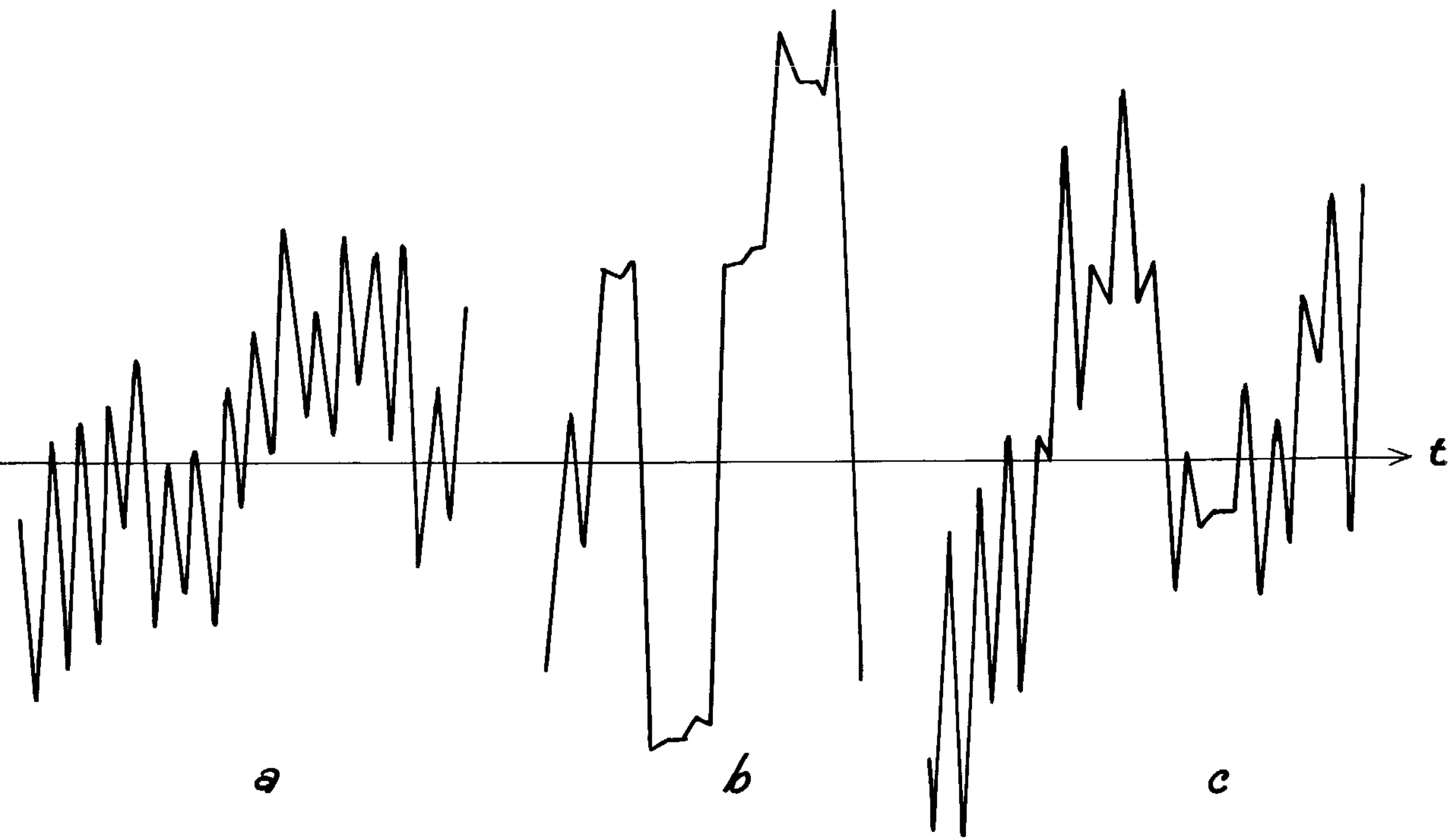

Fig. 3 - Synthetic Random Functions of Time

a Log-Normal Distributed Ascents and Descents ($\mu = 1.7$; $\sigma = 0.1$)

b Log-Normal Distributed Ascents and Descents ($\mu = 0.7$; $\sigma = 1.0$)

c Rectangularely Distributed Ascents and Descents