ON DERIVATION OF STIFFNESS MATRICES WITH C° ROTATION FIELDS FOR PLATES AND SHELLS

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The use of C° deflection (displacements and rotations) fields over triangular meshes in the middle plane of a plate for the total potential energy functional always requires the retention of the transverse shear strain energy in order to have the stiffness matrix of the supported structure non-singular when the degrees of freedom in the transverse direction are not suppressed. This situation is similar to the case of sandwich plates and shells where the transverse shear strain energy is retained because of its importance. Since the transverse shear strain energy is a function of transverse shear moduli, rather than in-plane shear modulus, it should be computed directly from the deflection fields, but not from the corresponding stress couple fields. This paper shows the method of computation of transverse shear strain energy directly from the deflection fields. Applying the method to C° deflection fields, a mathematical foundation is laid for a widely used plate and shell element by Martin, Melosh and Utku.

Introduction

After the celebrated work of Turner, Clough, Martin and Topp[1] in 1956 for solving two dimensional Elasticity problems with C° trial displacement fields and the Ritz procedure, it took about ten years to do a similar but not exactly the same thing for the plate bending problems. In 1965, Melosh reported[2] that C° displacement fields for plate bending problems were of very poor convergence, and suggested the use of hypothetical spar beams of mysteriously adjusted stiffness around the periphery of triangular elements to represent the transverse shear rigidity, and ignore the one resulting from the mathematical procedure. He mentions[2] successful and monotonically converging solutions of several plate bending problems by using the spar beam concept and the right

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triangular elements. In 1966, Utku gave[3] a mathematical derivation of stiffness matrices for thin plates and shells using C° deflection (the term deflection in this paper is used to mean both displacements and rotations) fields over arbitrary triangular meshes with or without curvature. The shell curvatures are handled by approximating the middle surface section within a triangle by a parabolic surface, and the slow convergence is overcome by replacing the portion of the transverse shear stiffness matrix of an element, which is associated with the rotations, by quantities insuring equilibrium among the forces represented by the columns of the final transverse shear stiffness matrix (although there are infinitely many ways of generating such quantities, the one named the "equilibrium algorithm" is arrived at by analogy to the behavior of clamped beams). The transverse shear stiffness matrix obtained by the "equilibrium algorithm" is identical with that of Ref. (2). The monotonic and rapid convergence characteristics of this element with various meshes are demonstrated in Ref. (3) for both plates and shells. In 1967, Utku and Melosh reported[4] that when the triangle is obtuse, the transverse shear matrix of Refs. (2) and (3) becomes indefinite, and the curved element stiffness matrix of Ref. (3) becomes indefinite when the apex of the paraboloid approximating the shell middle surface is not on the normal at the centroid of the base triangle. To prevent the indefiniteness of the transverse shear stiffness matrix, they suggested[4] to modify the transverse force portion of the matrix arbitrarily by a process they named the "constant trace scheme". In Ref. (4), extensive results are given to demonstrate the monotonic and rapid convergence of the resulting matrix in both plates and shells. In 1967, Martin arrived[5] at the same transverse shear stiffness matrix of Refs. (2) and (3) by a physical model of cover plates and spar beams with a sound engineering reasoning. However, his matrix is also indefinite when the triangle is obtuse.

Today, several computer programs, such as ELAS[6], SAMIS[7], and DYNAL[8], use the flat shell element of Ref. (4) which uses C° trial deflection fields in conjunction with the "equilibrium algorithm" and the "constant trace scheme". Because of the latter modifications, of course, one is never sure of the true class of the trial fields yielding the final stiffness matrix. In spite of this weakness, this element 1) has been the first conforming element for arbitrary triangulation in both plates and shells, 2) is still the only available element which represents the behavior of non-Kirchhoffian plates and shells by using the transverse shear moduli (rather than in-plane shear modulus), and therefore 3) is the only element suitable for sandwich plates and shells.

This work is an attempt to furnish the mathematical foundations of this widely used plate and shell element by eliminating from its definition all physical analogies and engineering approximations. For this purpose, first a method is given for the computation of transverse shear strain energy directly from the deflection fields. Then, this method is applied to C° deflection fields over triangular meshes to explain the behavior of the Martin-Melosh-Utku element.

Finally the behavior of the new element is compared with that of the Martin-Melosh-Utku element.

Computation of Transverse Shear Strain Energy Directly from the Deflection Fields

Let x,y,z denote a right-handed Cartesian coordinate system located at the centroid of a triangular thin shell element of thickness t. Let 1,2,3 denote the vertex labels of the triangle. It is assumed that the element is sufficiently small to justify the assumption that vertices 1,2 and 3 lie in the xy plane. Let $\underline{i}$, $\underline{j}$, $\underline{k}$ denote the unit vectors of the coordinate system (see Fig. 1). Let

$$\underline{d} = u\,\underline{i} + v\,\underline{j} + w\,\underline{k} \tag{1}$$

denote the displacement vector. It is assumed that the transverse shear strains $[\gamma] = [\gamma_{xz}\ \gamma_{yz}]$ and the transverse shear stresses $[\tau] = [\tau_{xz}\tau_{yz}]$ are related with

$$\begin{Bmatrix}\tau_{xz}\\ \tau_{yz}\end{Bmatrix} = \begin{bmatrix}D'_{11} & D'_{12}\\ D'_{21} & D'_{22}\end{bmatrix}\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}$$

which may be rewritten as

$$\{\tau\} = [D']\{\gamma\} \tag{2}$$

where [D'] is as displayed above. The transverse shear moduli matrix [D'] is positive definite and symmetric. The linear strain displacement relations are

$$\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix} = \begin{Bmatrix}u_{,z} + w_{,x}\\ v_{,z} + w_{,y}\end{Bmatrix} \tag{3}$$

where a comma in the subscript indicates partial differentiation with respect to the quantity following the comma. Since the thin shell or plate assumptions imply that $u_{,z}$ = const., $v_{,z}$ = const., and $w_{,z} = 0$, the rotations of middle surface normals about x and y axes, θ_x and θ_y, may be expressed as

$$\begin{Bmatrix}\theta_x\\ \theta_y\end{Bmatrix} = \begin{Bmatrix}-v_{,z}\\ u_{,z}\end{Bmatrix} \tag{4}$$

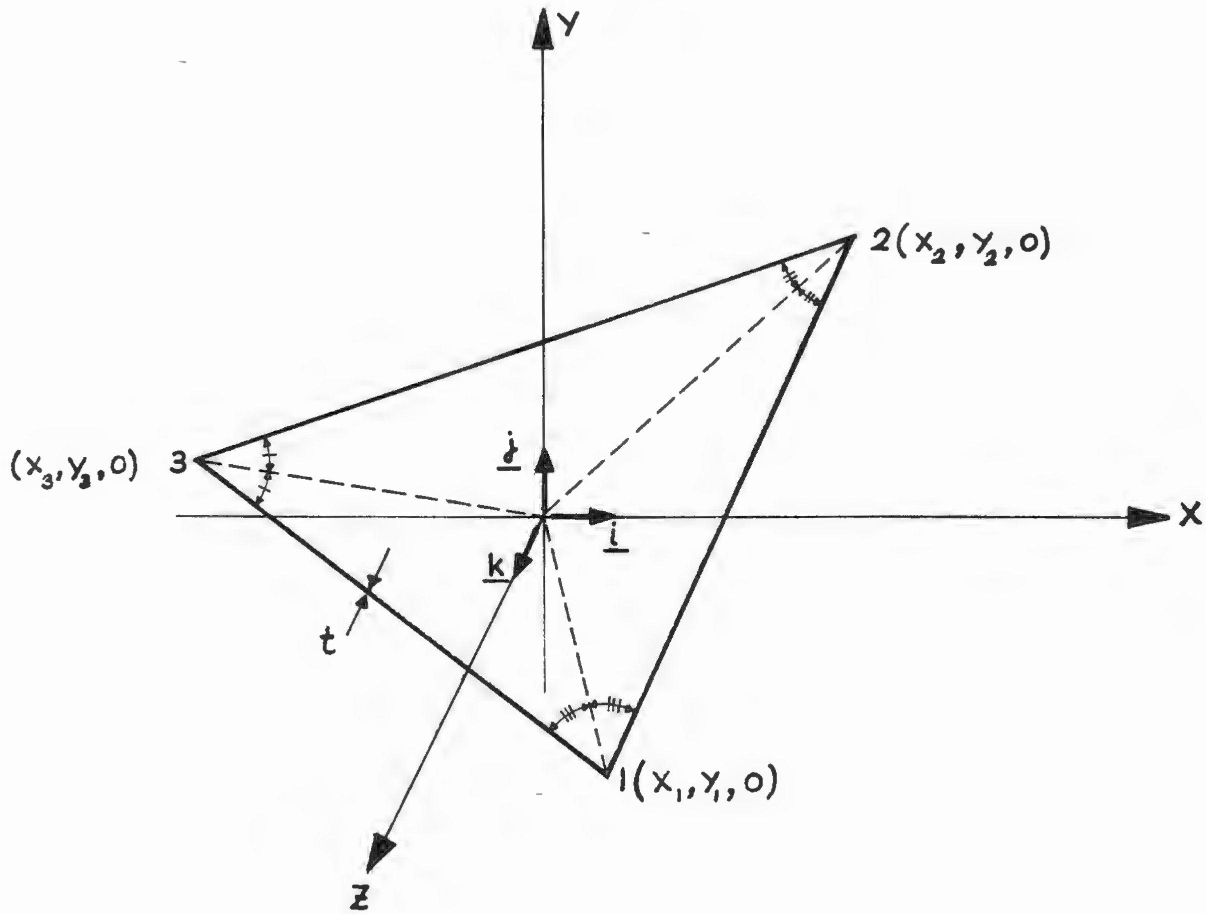

Figure 1. Definition Sketch

Substituting θ_x and θ_y from (4) into (3) one obtains

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \theta_y + w_{,x} \\ -\theta_x + w_{,y} \end{Bmatrix} \tag{5}$$

If the behavior is truly Kirchhoffian $\gamma_{xz} = \gamma_{yz} = 0$. Denoting the transverse displacement with w' in the case of Kirchhoffian behavior, one may write from (5)

$$\begin{Bmatrix} \theta_x \\ \theta_y \end{Bmatrix} = \begin{Bmatrix} w'_{,y} \\ -w'_{,x} \end{Bmatrix} \tag{6}$$

Let the difference between non-Kirchhoffian and Kirchhoffian transverse displacements be denoted by w* such that

$$w^* = w - w' \tag{7}$$

Using (6) and (7), one may rewrite (5) as

$$\begin{Bmatrix} \gamma \end{Bmatrix} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} w^*_{,x} \\ w^*_{,y} \end{Bmatrix} \tag{8}$$

Since the strain energy density for transverse shear is $\frac{1}{2}\lfloor \gamma \rfloor [D'] \{\gamma\}$ using (8), the transverse shear strain energy of the triangle may be expressed as

$$U_S = \frac{1}{2} \int_A [w^*_{,x} \; w^*_{,y}] \begin{bmatrix} D'_{11} & D'_{12} \\ D'_{21} & D'_{22} \end{bmatrix} \begin{Bmatrix} w^*_{,x} \\ w^*_{,y} \end{Bmatrix} dA \tag{9}$$

where dA is the area element, and A is the area of the triangle. If one uses (9) for the transverse shear strain energy in the total potential energy functional, for monotonic convergence, it is necessary that w* is at least of C° over the triangular mesh. In other words, one should have w* (but not its derivatives) continuous across the interelement boundaries.

Triangular Element with C° Deflection Fields for Non-Kirchhoffian Plates or Shells

Let x_i, y_i, z_i, u_i, v_i, w_i, θ_{xi}, θ_{yi}, and θ_{zi}, $i=1,2,3$ denote the values of Cartesian coordinates and the deflection components at the vertices of the triangle as referred to the x,y,z coordinate system (see Fig. 1). If one assumes that, excluding w and θ_z the deflection components vary linearly in the triangle with x, and y, the values of these components at a point with coordinates x, and y in the triangle may be expressed by linear interpolation as

$$[u \quad v \quad \theta_x \quad \theta_y] = \frac{1}{2A}\,[x \quad y \quad 1] \begin{bmatrix} \text{—}y\rightarrow \\ \text{—}x\rightarrow \\ \text{—}r\rightarrow \end{bmatrix} \begin{bmatrix} \bar{u}\downarrow & \bar{v}\downarrow & \bar{\theta}_x\downarrow & \bar{\theta}_y\downarrow \end{bmatrix} \qquad (10)$$

where $[\text{—}y\rightarrow] = [y_2-y_3 \quad y_3-y_1 \quad y_1-y_2]$, $[\text{—}x\rightarrow] = [x_3-x_2 \quad x_1-x_3 \quad x_2-x_1]$,

$[\text{—}r\rightarrow] = [x_2y_3-x_3y_2 \quad x_3y_1-x_1y_3 \quad x_1y_2-x_2y_1]$, $[\text{—}\bar{u}\rightarrow] = [u_1 \ u_2 \ u_3]$,

$[\text{—}\bar{v}\rightarrow] = [v_1 \ v_2 \ v_3]$, $[\text{—}\bar{\theta}_x\rightarrow] = [\theta_{x1} \quad \theta_{x2} \quad \theta_{x3}]$, and

$[\text{—}\bar{\theta}_y\rightarrow] = [\theta_{y1} \quad \theta_{y2} \quad \theta_{y3}]$. Now if one assumes that similar linear interpolation applies in all the triangles of the whole triangular mesh, the resulting deflection field is of C°, since the deflection components, but not their derivatives, are continuous across the interelement boundaries.

The total strain energy associated with the triangle, U, may be written as the sum of the membrane, bending, and transverse shear strain energies U_M, U_B, and U_S:

$$U = U_M + U_B + U_S \qquad (11)$$

and U_M, U_B, and U_S may be expressed as

$$U_M = \frac{1}{2}\,[\ q\ ]\,[K_M]\,\{q\} \qquad (12a)$$

$$U_B = \frac{1}{2}\,[\ q\ ]\,[K_B]\,\{q\} \qquad (12b)$$

$$U_S = \frac{1}{2}\,[\ q\ ]\,[K_S]\,\{q\} \qquad (12c)$$

where $[K_M]$, $[K_B]$, and $[K_S]$ are the membrane, the bending and the transverse shear stiffness matrices of the triangular element, and $[q] = [u_1\ u_2\ u_3 \vdots v_1\ v_2\ v_3 \vdots w_1\ w_2\ w_3 \vdots \theta_{x1}\ \theta_{x2}\ \theta_{x3} \vdots \theta_{y1}\ \theta_{y2}\ \theta_{y3}]$.

Let the tangential strains $[\varepsilon] = [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]$ and the tangential stresses $[\sigma] = [\sigma_x \sigma_y \tau_{xy}]$ be related with

$$\{\sigma\} = [D]\{\varepsilon\} \tag{13}$$

where the material matrix $[D]$ is positive definite and symmetric. Using the linear interpolation rule of (10), and stress-strain relations (13), $[K_M]$ and $[K_B]$ may be computed[3] as

$$[K_M] = \frac{t}{4A}\left[\begin{array}{c:c:c:c:c} P & R^T & 0 & 0 & 0 \\ \hdashline R & Q & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad [K_B] = \frac{t^3}{48A}\left[\begin{array}{c:c:c:c:c} 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & Q & -R^T \\ \hdashline 0 & 0 & 0 & -R & P \end{array}\right] \tag{14a,b}$$

where

$$[P] = [M]^T[D][M] \ , \qquad [R] = [M]^T[D][N] \ , \qquad [Q] = [N]^T[D][N] \tag{14c,d,e}$$

and

$$[M] = \begin{bmatrix} \longleftarrow y \longrightarrow \\ \longleftarrow 0 \longrightarrow \\ \longleftarrow x \longrightarrow \end{bmatrix}, \quad [N] = \begin{bmatrix} \longleftarrow 0 \longrightarrow \\ \longleftarrow x \longrightarrow \\ \longleftarrow y \longrightarrow \end{bmatrix}, \quad [\longleftarrow 0 \longrightarrow] = [0\ \ 0\ \ 0], \quad [0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14f,g,h,i}$$

In (14a) the contributions of local curvatures, if there are any, are ignored.

In order to compute U_S in terms of the vertex deflections using (9), one needs w' or its first derivatives as implied by (7). A consistent w' distribution in the triangle with the linear rotation distributions may be expressed as

$$w' = [x^2 \quad y^2 \quad xy \quad x \quad y \quad 1]\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{Bmatrix} \tag{15}$$

where the coefficients a_i, $i=1,\ldots,5$ may be expressed in terms of the vertex rotations as follows: Using (15) in (6) one may write

$$\theta_x = [2y \quad x \quad 1] \begin{Bmatrix} a_2 \\ a_3 \\ a_5 \end{Bmatrix} \tag{16a}$$

and

$$\theta_y = -[2x \quad y \quad 1] \begin{Bmatrix} a_1 \\ a_3 \\ a_4 \end{Bmatrix} \tag{16b}$$

and evaluating these at the vertices, after inversion one obtains:

$$\begin{Bmatrix} a_2 \\ a_3 \\ a_5 \end{Bmatrix} = \frac{1}{4A} \begin{bmatrix} \longrightarrow x \longrightarrow \\ (2) \longrightarrow y \longrightarrow \\ 4A/3 \longrightarrow 1 \longrightarrow \end{bmatrix} \begin{Bmatrix} \big| \\ \theta_x \\ \downarrow \end{Bmatrix} \tag{17a}$$

and

$$\begin{Bmatrix} a_1 \\ a_3 \\ a_4 \end{Bmatrix} = -\frac{1}{4A} \begin{bmatrix} \longrightarrow y \longrightarrow \\ (2) \longrightarrow x \longrightarrow \\ 4A/3 \longrightarrow 1 \longrightarrow \end{bmatrix} \begin{Bmatrix} \big| \\ \theta_y \\ \downarrow \end{Bmatrix} \tag{17b}$$

where $[\longrightarrow 1 \longrightarrow] = [111]$. It is observed that a_3 is computed in two different ways in (17a) and (17b). To assure single valuedness, the average of the two may be taken as a_3, so that

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \frac{1}{4A} \left[\begin{array}{c:c} \longrightarrow 0 \longrightarrow & \longrightarrow y \longrightarrow \\ \longrightarrow x \longrightarrow & \longrightarrow 0 \longrightarrow \\ \longrightarrow y \longrightarrow & \longrightarrow x \longrightarrow \\ \longrightarrow 0 \longrightarrow & 4A/3 \longrightarrow 1 \longrightarrow \\ 4A/3 \longrightarrow 1 \longrightarrow & \longrightarrow 0 \longrightarrow \end{array}\right] \begin{Bmatrix} \big| \\ \theta_x \\ \downarrow \\ \hdashline \big| \\ -\theta_y \\ \downarrow \end{Bmatrix} \tag{18}$$

Calling the value of w' at the origin w'_o, and substituting a_i, $i=1...,5$, from (18) into (15), one may write

$$w'-w'_o = \frac{1}{4A} [x^2 \quad y^2 \quad xy \quad x \quad y] \left[\begin{array}{c:c} \longrightarrow 0 \longrightarrow & \longrightarrow y \longrightarrow \\ \longrightarrow x \longrightarrow & \longrightarrow 0 \longrightarrow \\ \longrightarrow y \longrightarrow & \longrightarrow x \longrightarrow \\ \longrightarrow 0 \longrightarrow & 4A/3 \longrightarrow 1 \longrightarrow \\ 4A/3 \longrightarrow 1 \longrightarrow & \longrightarrow 0 \longrightarrow \end{array}\right] \begin{Bmatrix} \big| \\ \theta_x \\ \downarrow \\ \hdashline \big| \\ -\theta_y \\ \downarrow \end{Bmatrix} \tag{19}$$

By evaluating w' at the vertices, one obtains from (19)

$$\begin{Bmatrix} w'_1-w'_o \\ w'_2-w'_o \\ w'_3-w'_o \end{Bmatrix} = \frac{1}{4A} \begin{bmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 \\ x_3^2 & y_3^2 & x_3y_3 & x_3 & y_3 \end{bmatrix} \left[\begin{array}{c:c} \longrightarrow 0 \longrightarrow & \longrightarrow y \longrightarrow \\ \longrightarrow x \longrightarrow & \longrightarrow 0 \longrightarrow \\ \longrightarrow y \longrightarrow & \longrightarrow x \longrightarrow \\ \longrightarrow 0 \longrightarrow & 4A/3 \longrightarrow 1 \longrightarrow \\ 4A/3 \longrightarrow 1 \longrightarrow & \longrightarrow 0 \longrightarrow \end{array}\right] \begin{Bmatrix} \big| \\ \theta_x \\ \downarrow \\ \hdashline \big| \\ -\theta_y \\ \downarrow \end{Bmatrix}$$

which may be reduced to

$$\begin{Bmatrix} w_1'-w_o' \\ w_2'-w_o' \\ w_3'-w_o' \end{Bmatrix} = \frac{1}{6}\begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}\left\{\bar{\theta}_x\right\} - \frac{1}{6}\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}\left\{\bar{\theta}_y\right\} \qquad (20)$$

By using (7), this leads to

$$\left\{\bar{w}^*\right\} = \left\{\bar{w}\right\} - \frac{1}{6}\begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}\left\{\bar{\theta}_x\right\} + \frac{1}{6}\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}\left\{\bar{\theta}_y\right\} - w_o\left\{1\right\} \qquad (21)$$

where $[\longleftarrow \bar{w}^* \longrightarrow] = [w_1^* \quad w_2^* \quad w_3^*]$ and $[\longleftarrow \bar{w} \longrightarrow] = [w_1 \quad w_2 \quad w_3]$.

In order to be consistent with the interpolation rule used in obtaining the membrane and the bending stiffness matrices, one may try a C° field for w*. This means that the difference between the Kirchhoffian and the non-Kirchhoffian transverse displacements is assumed varying linearly within a triangle. By using the linear interpolation rule of (10), one may write

$$w^* = \frac{1}{2A}\,[x \quad y \quad 1]\begin{bmatrix} \longleftarrow y \longrightarrow \\ \longleftarrow x \longrightarrow \\ \longleftarrow r \longrightarrow \end{bmatrix}\left\{\bar{w}^*\right\} \qquad (22)$$

Substituting w* from (22) into (8), and using (21) one obtains

$$\{\gamma\} = \frac{1}{2A}\begin{bmatrix} \longleftarrow y \longrightarrow \\ \longleftarrow x \longrightarrow \end{bmatrix}\left[\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \;\vdots\; -\frac{1}{6}\begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \;\vdots\; \frac{1}{6}\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}\right]\begin{Bmatrix} \bar{w} \\ \bar{\theta}_x \\ \bar{\theta}_y \end{Bmatrix} \qquad (23a)$$

which can be reduced into

$$\{\gamma\} = \frac{1}{2A}\left(\frac{1}{2}\begin{bmatrix} \longleftarrow y \longrightarrow \\ \longleftarrow x \longrightarrow \end{bmatrix}\begin{bmatrix} 1 & & & \vdots & -y_1 & & & \vdots & x_1 & & \\ & 1 & & \vdots & & -y_2 & & \vdots & & x_2 & \\ & & 1 & \vdots & & & -y_3 & \vdots & & & x_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \longleftarrow y \longrightarrow & \vdots & \longleftarrow 0 \longrightarrow & \vdots & \frac{2A}{3}\longleftarrow 1 \longrightarrow \\ \longleftarrow x \longrightarrow & \vdots & -\frac{2A}{3}\longleftarrow 1 \longrightarrow & \vdots & \longleftarrow 0 \longrightarrow \end{bmatrix}\right)\begin{Bmatrix} \bar{w} \\ \bar{\theta}_x \\ \bar{\theta}_y \end{Bmatrix} \qquad (23b)$$

Note that the transverse shear strains in (23b) are expressed as the arithmetic average of two terms. The approximation of the transverse shear strains by only the second term has been studied previously under the name of "average rotations algorithm"[4].

The use of $\{\gamma\}$ from (23b) in (9) gives the transverse shear stiffness matrix $[K_s]$ of (12c) as

$$[K_S] = \frac{t}{4A}\left[\begin{array}{c:c:c:c:c} 0 & & & & \\ \hdashline 0 & 0 & & \text{Sym.} & \\ \hdashline 0 & 0 & S_{11} & & \\ \hdashline 0 & 0 & S_{21} & S_{22} & \\ \hdashline 0 & 0 & S_{31} & S_{32} & S_{33} \end{array}\right] \tag{24a}$$

where

$$[S_{11}] = \left[\begin{array}{cc} y\downarrow & x\downarrow \end{array}\right][D']\left[\begin{array}{c} \text{—}y\rightarrow \\ \text{—}x\rightarrow \end{array}\right], \qquad [S_{21}] = -\frac{1}{2}\begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix}[S_{11}], \tag{24b,c}$$

$$[S_{31}] = \frac{1}{2}\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}[S_{11}], \qquad [S_{22}] = \frac{1}{4}\begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix}[S_{11}]\begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix}, \tag{24d,e}$$

and

$$[S_{32}] = -\frac{1}{4}\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}[S_{11}]\begin{bmatrix} y_1 & & \\ & y_2 & \\ & & y_3 \end{bmatrix}, \qquad [S_{33}] = \frac{1}{4}\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}[S_{11}]\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix} \tag{24f,g}$$

The total stiffness matrix of the element may now be written as the sum of $[K_M]$, $[K_B]$ and $[K_S]$:

$$[K] = \frac{t}{4A}\left[\begin{array}{c:c:c:c:c} P & & & & \\ \hdashline R^T & Q & & & \\ \hdashline 0 & 0 & S_{11} & \text{Sym.} & \\ \hdashline 0 & 0 & S_{21} & \beta Q+S_{22} & \\ \hdashline 0 & 0 & S_{31} & -\beta R+S_{32} & \beta P+S_{33} \end{array}\right], \quad \beta = \frac{t^2}{12} \tag{25}$$

which may be used in

$$U = \frac{1}{2}\,[q]\,[K]\,\{q\} \tag{26}$$

for the computation of the total strain energy of the triangular plate or shell element in terms of the vertex deflections.

Comparison of the New Element with the Martin-Melosh-Utku Element

The membrane and the bending portions of the new element stiffness matrix are identical with those of the Martin-Melosh-Utku element. Moreover, $[S_{11}]$ submatrix in both cases are the same. The difference exists in submatrices $[S_{21}]$, $[S_{31}]$, $[S_{22}]$, $[S_{32}]$, and $[S_{33}]$ of $[K_S]$. Using superscript e to distinguish quantities belonging to the Martin-Melosh-Utku element (e stands for the "equilibrium algorithm"), for non-obtuse triangles, one may observe that

$$[S^e_{11}] = [S_{11}] \tag{27a}$$

$$[S^e_{21}] = [S_{21}] + \frac{1}{6}\left\{\begin{matrix}|\\ y\\ \downarrow\end{matrix}\right\}[a_1\ b_1\ c_1]\ , \tag{27b}$$

$$[S^e_{31}] = [S_{31}] - \frac{1}{6}\left\{\begin{matrix}|\\ x\\ \downarrow\end{matrix}\right\}[a_1\ b_1\ c_1]\ , \tag{27c}$$

$$[S^e_{22}] = [S_{22}] + \frac{1}{6}\left\{\begin{matrix}|\\ y\\ \downarrow\end{matrix}\right\}[a_2\ b_2\ c_2]\ , \tag{27d}$$

$$[S^e_{32}] = [S_{32}] - \frac{1}{6}\left\{\begin{matrix}|\\ x\\ \downarrow\end{matrix}\right\}[a_2\ b_2\ c_2]\ , \tag{27e}$$

$$[S^e_{33}] = [S_{33}] - \frac{1}{6}\left\{\begin{matrix}|\\ x\\ \downarrow\end{matrix}\right\}[a_3\ b_3\ c_3]\ , \tag{27f}$$

where

$$\begin{aligned} a_i &= (S^e_{i1})_{31} - (S^e_{i1})_{21}\ , \\ b_i &= (S^e_{i1})_{21} - (S^e_{i1})_{32}\ , \qquad i = 1,2,3 \\ c_i &= (S^e_{i1})_{32} - (S^e_{i1})_{31}\ , \end{aligned} \tag{27g}$$

the second pair of subscripts indicating the row and the column number of the entry of the submatrix to be used.

The three scalars $(S^e_{i1})_{21}$, $(S^e_{i1})_{31}$ and $(S^e_{i1})_{32}$ of (27g) are the three off-diagonal elements in the lower part of submatrix $[S_{i1}]$, i=1,2,3. For i=1, it has been shown[4] that for non-obtuse triangles these three scalars are all non-positive, and for obtuse triangles one of them is positive (for obtuse triangles the Martin-Melosh-

Utku element stiffness matrix without the "constant trace scheme" modification is indefinite). For equilateral triangles, when i=1, these three scalars are all equal, implying that $a_1 = b_1 = c_1$, and therefore $[S^e_{21}]=[S_{21}]$ and $[S^e_{31}]=[S_{31}]$. In the case of equilateral triangle, in order to have also $[S^e_{22}] = [S_{22}]$, $[S^e_{32}] = [S_{32}]$ and $[S^e_{33}] = [S_{33}]$, the orientation of the triangle relative to the x,y,z coordinate system should also be special.

The comparisons of the results by the new element and by the Martin-Melosh-Utku element for several plate problems with various meshes under a single transverse concentrated load are given in Figures 2-8, where the transverse displacement under the concentrated load load is plotted against some measure of the mesh refinement (the larger the abscissa, the finer the mesh). The results with equilateral and bilateral triangular meshes indicate that the new element converges faster (Figs. 2 and 3). In the case of right bilateral triangles the two results are almost coincident (Fig. 4). For the case of obtuse triangles, the slightly larger results by the Martin-Melosh-Utku element with the "constant trace scheme" are of the same convergence rate with those of the new element (Figs. 5 and 6).

In. Figs. 7 and 8, the better convergence characteristic of the Martin-Melosh-Utku element is obvious. The paradox between the behavior shown in Figs. 2-6, and Figs. 7-8 clearly indicates that the Martin-Melosh-Utku element is orientation dependent. It has a much higher convergence rate if the largest edge of a triangle is not quite coincident with the minimum bending direction. Presently, the mathematical explanation of this quite remarkable behavior is not complete for formal presentation.

Conclusions

From the study presented in this paper, the following conclusions may be drawn:

1. The transverse shear strain energy is a function of the quantity w*which represents the difference between the Kirchhoffian and the non-Kirchhoffian transverse displacements.

2. Conforming element representations for the non-Kirchhoffian behavior of thin plates and shells are possible by C° rotation, tangential displacement and w* fields, as shown in this work.

3. The stiffness matrices with C° deflection fields as derived in this work closely follow the behavior of the Martin-Melosh-Utku element in general.

4. The convergence characteristics of the Martin-Melosh-Utku element can be substantially improved by properly orienting the element.

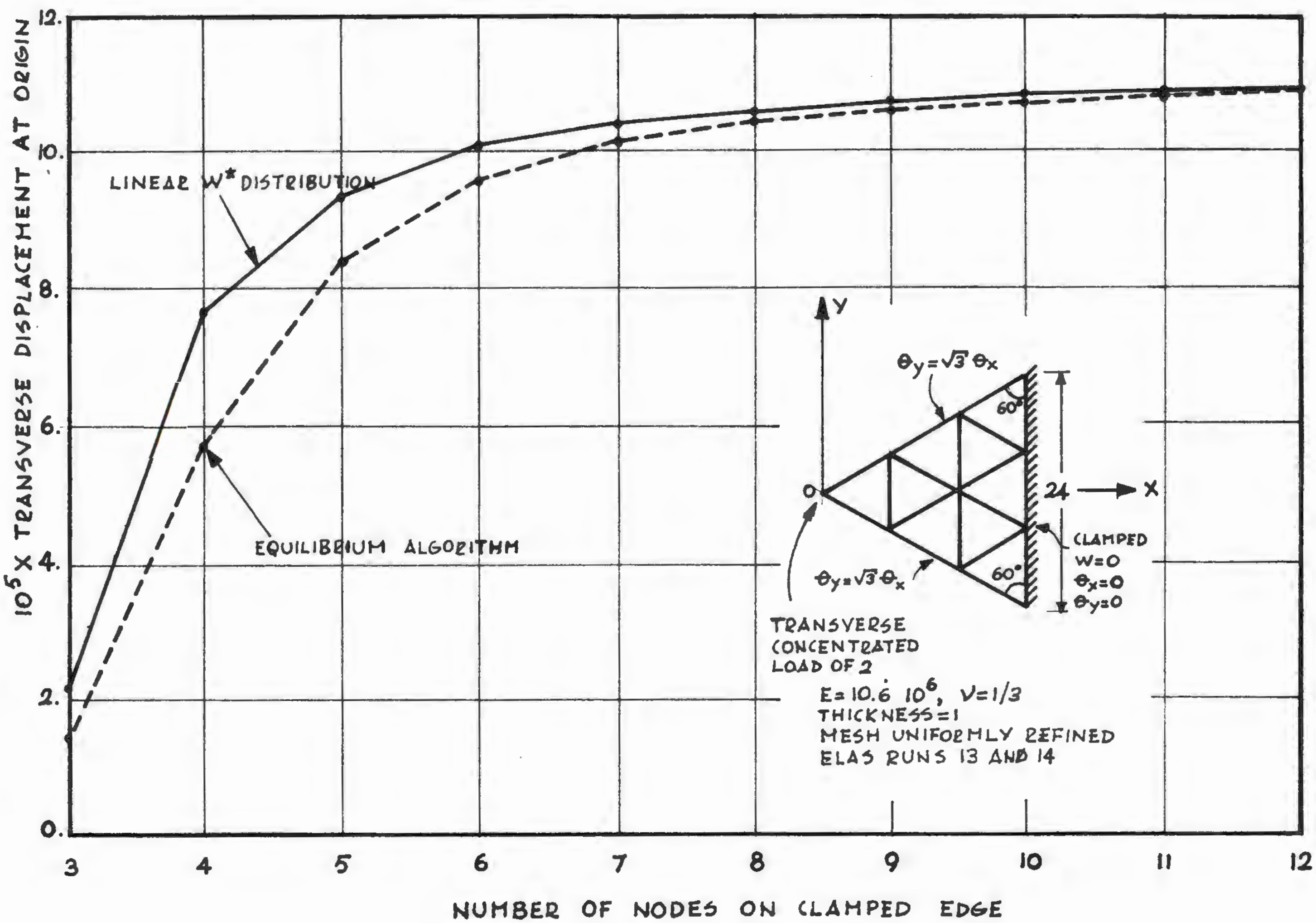

Figure 2. Convergence in Equilateral Triangular Mesh

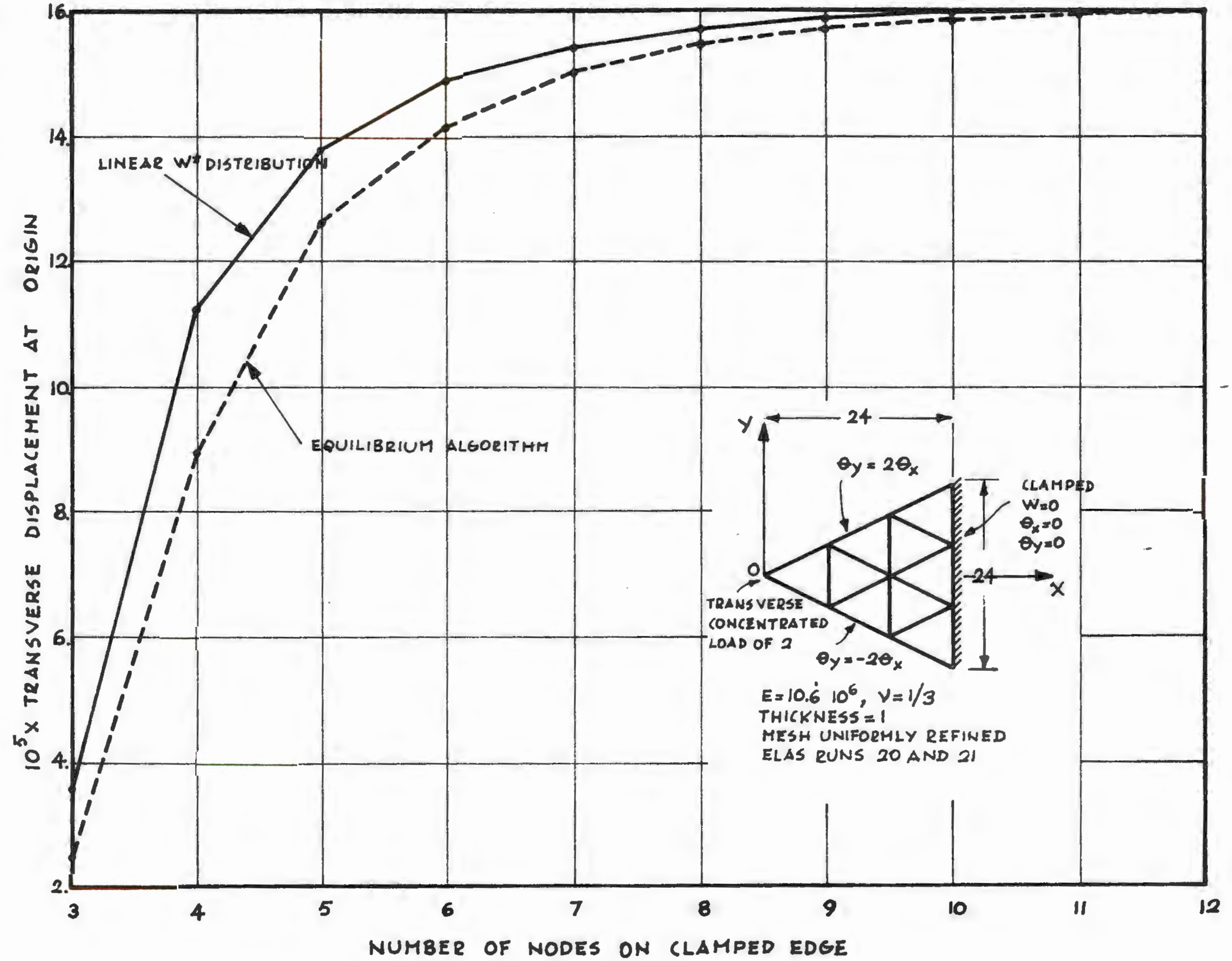

Figure 3. Convergence in Bilateral Triangular Mesh

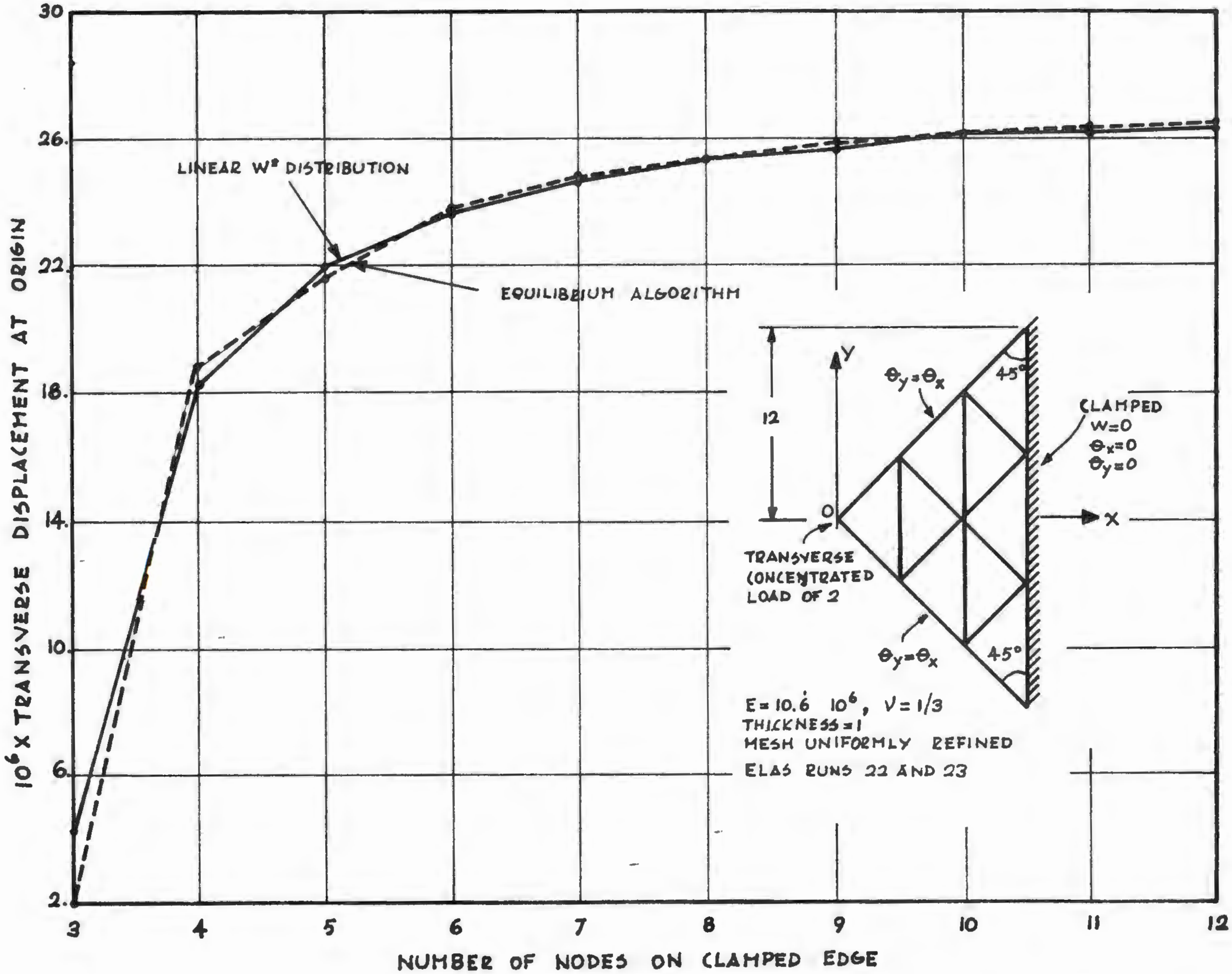

Figure 4. Convergence in Right Bilateral Triangular Mesh

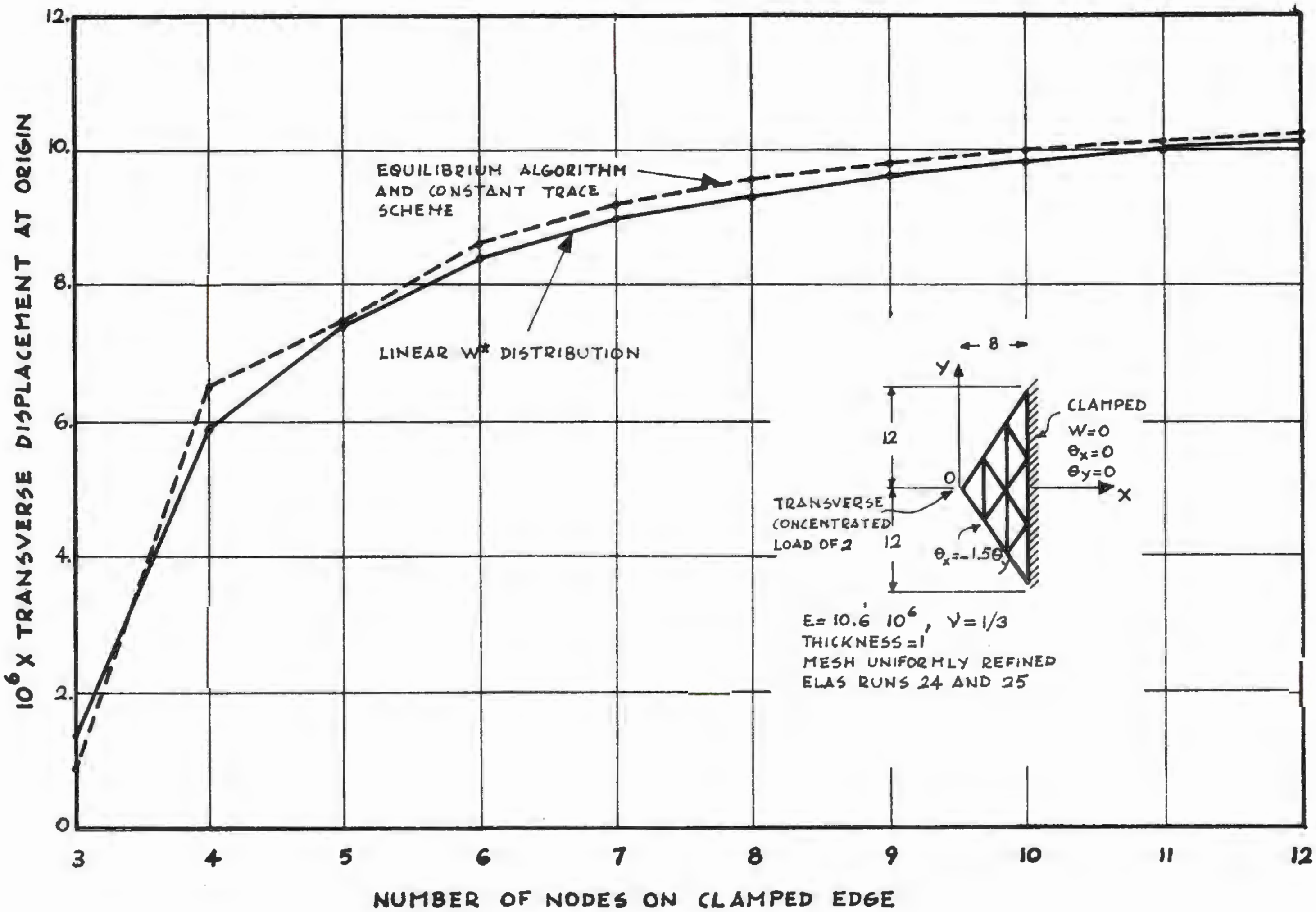

Figure 5. Convergence in Obtuse Bilateral Triangular Mesh

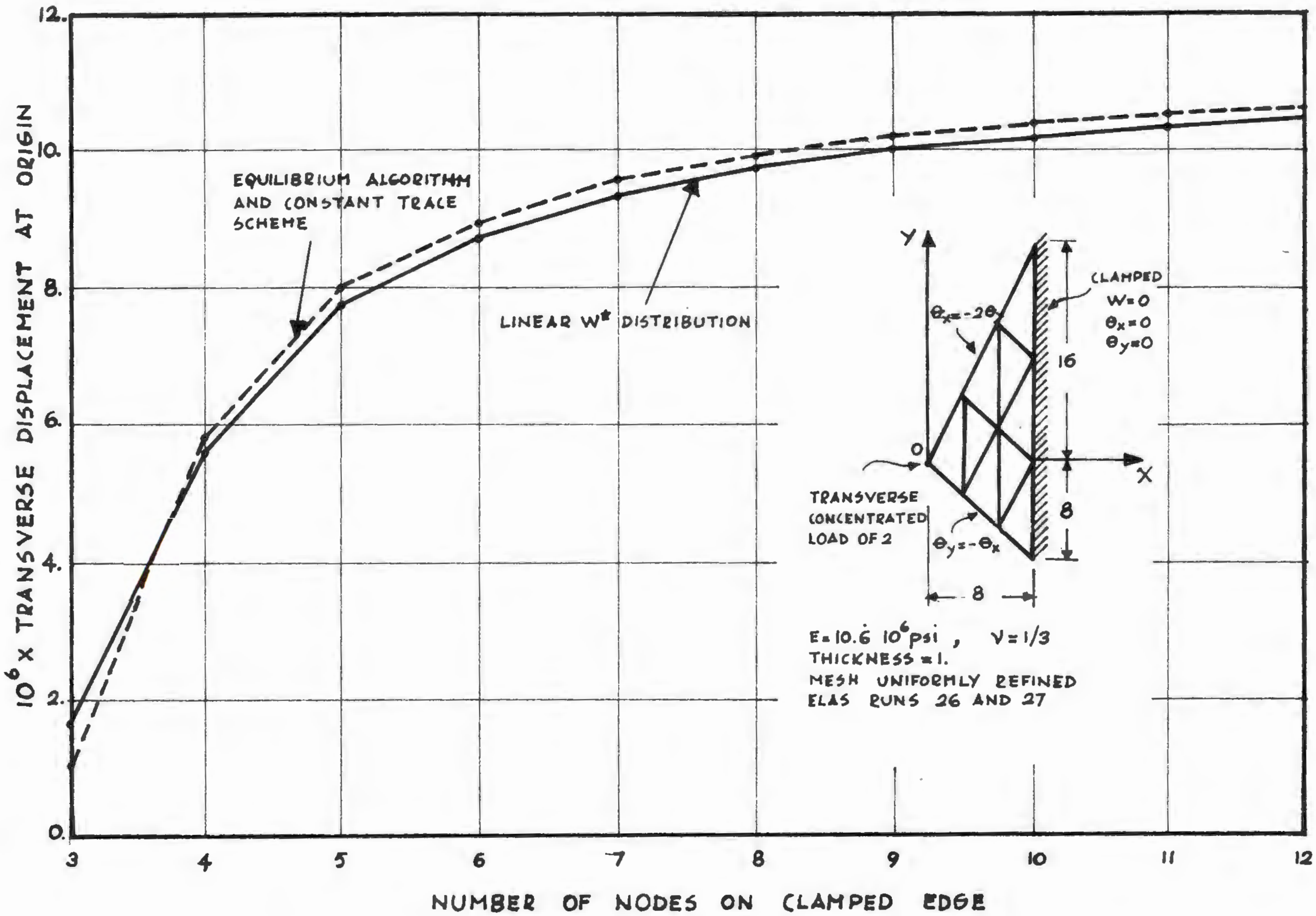

Figure 6. Convergence in Obtuse Triangular Mesh

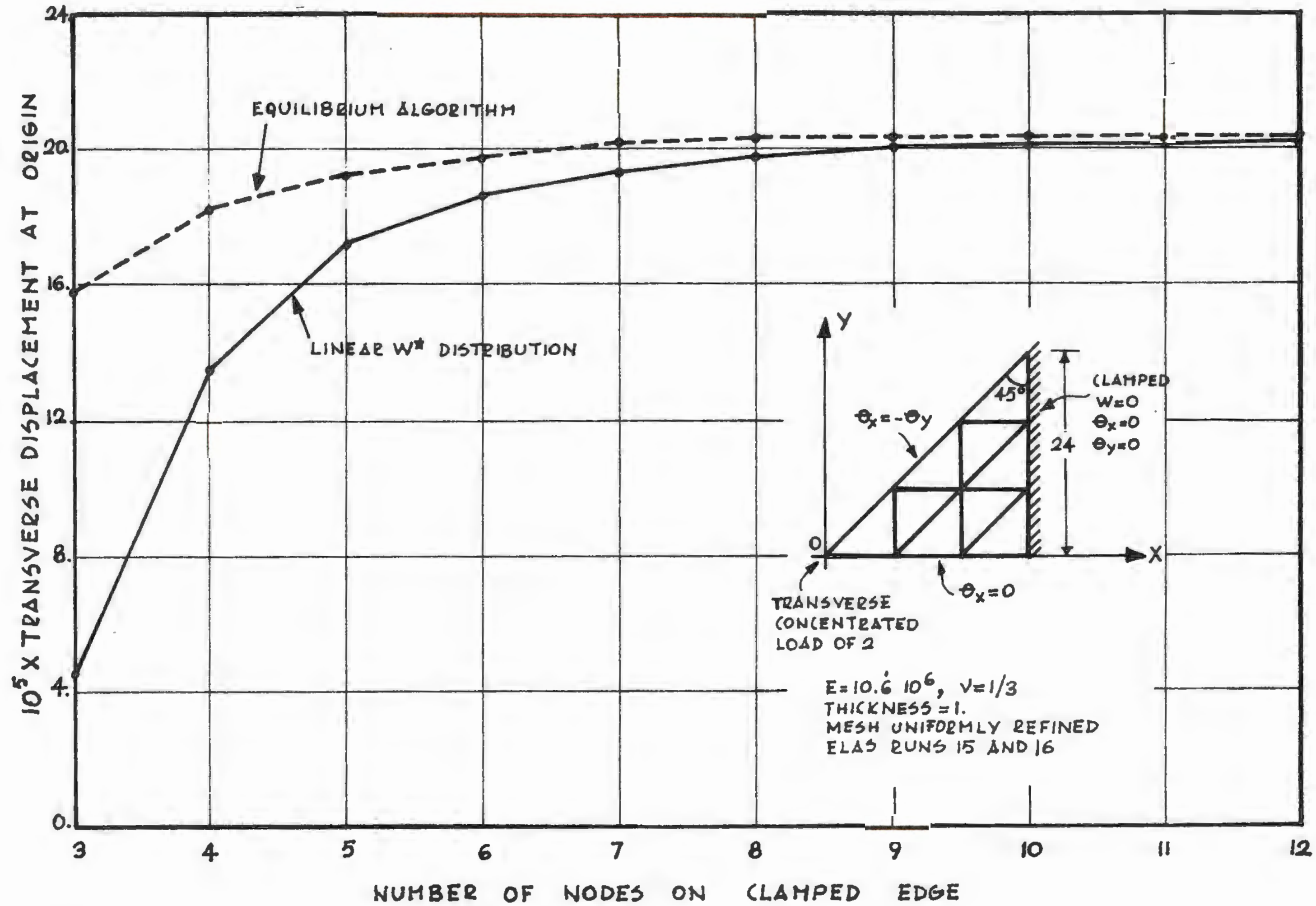

Figure 7. Convergence in Right Triangular Mesh with Proper Orientation

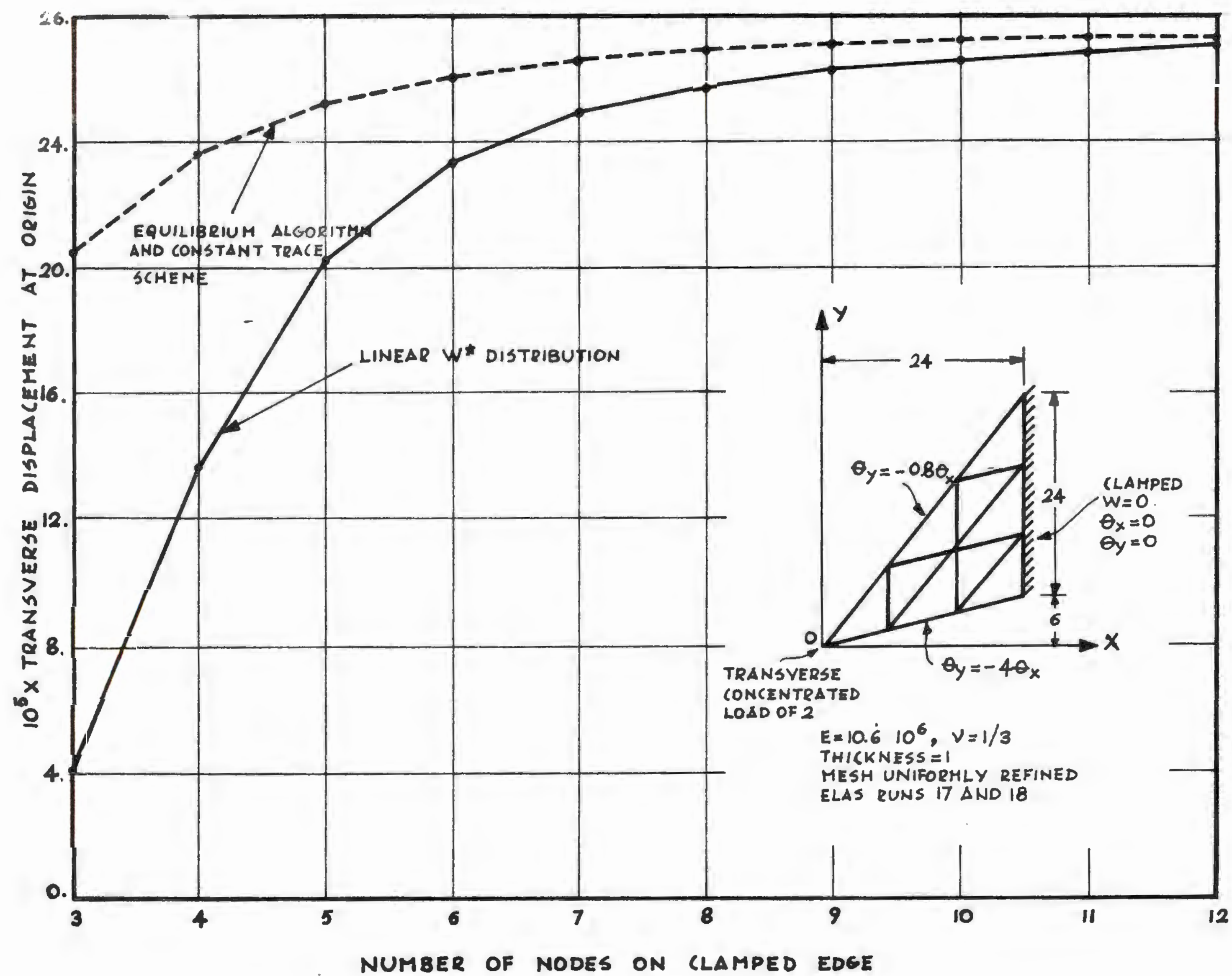

Figure 8. Convergence in Obtuse Triangular Mesh with Proper Orientation

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