SESSION 3

FINITE ELEMENT PROPERTIES

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THE GENERATION OF INTER-ELEMENT-COMPATIBLE STIFFNESS AND MASS MATRICES BY THE USE OF INTERPOLATION FORMULAS†

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This paper presents a general method of generating subelement displacement states which makes it a relatively simple task to insure completely compatible displacement states. The method involves the use of Hermite interpolation formulas of various order. The total potential energy principle is used to generate stiffness matrices based on these displacement states. The approach is presented in general orthogonal curvilinear coordinates for shell type structures. Inter-element-compatible stiffness matrices are presented for a rectangular membrane, a rectangular plate with twelve degrees of freedom, and a rectangular plate with twenty four degrees of freedom. The corresponding consistent mass matrices are presented for both plate elements. Numerical examples of static deflection analysis are presented based on both of the plate subelements and the monotonic convergence of the discretization error is confirmed. Application of the simpler plate subelement to the determination of natural modes and frequencies is also illustrated with a numerical example. All of the examples are selected so that comparison with classical results can be made.

1. INTRODUCTION

The discrete element method of structural analysis has had a long and lively history of development. The idea of representing a structural system as an assembly of simpler elements, generating a stiffness matrix for each element, and then forming a master stiffness matrix for the entire structure has been widely employed during the last decade. Early representations were based on concepts of axial load carrying members joined by panels carrying pure shear. This was followed by idealizations in which inplane two dimensional behavior was included within the discrete element (Reference 1). Also stiffness matrices for flat plates in bending have been developed (Reference 2).

In Reference 3 Melosh points out that application of the various two dimensional finite element stiffness matrices have resulted in solutions that are sometimes too stiff, sometimes too soft, sometimes exhibit monotonic convergence as the number of degrees of freedom is

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increased, sometimes show oscillatory convergence, and sometimes apparently diverge. Melosh offers a basis for the development of a discretization error-consistent set of stiffness matrices. In selecting assumed displacement states on which to base an element stiffness matrix derivation, it is necessary to require, not only that the displacements satisfy compatibility within the sub-element (intra-element compatibility) but also that it be possible to insure overall compatibility for an assemblage of sub-elements. By using the products of one dimensional interpolation formulae it will be seen that is possible to generate displacement states having the property that displacements along an edge depend only on degrees of freedom common to that edge. This than facilitates the insuring of overall compatibility by making it possible to easily satisfy inter-element compatability when assembling a collection of elements to represent an idealized structural system.

II. THE CONCEPT OF INTER-ELEMENT-COMPATIBLE SUB-ELEMENTS

Using the concept of minimum potential energy the analysis of plate and shell structures may be approximately carried out with a finite number of degrees of freedom by assuming a functional form for the displacements and seeking values of certain of the coefficients in these expressions which minimize the potential energy. The assumed expressions must, of course, be a geometrically admissible set for any choice of the undetermined coefficients. Specifically, this means that they must (1) satisfy the geometrically imposed boundary conditions and (2) describe only deformations that are physically realizable or at least do not violate any of the a priori assumptions about the behavior of the structure. When a set of assumed modes satisfies these conditions, the values obtained from the stationary conditions of potential energy will be the analysis of a structure which is stiffer* than the actual structure. As more modes and their accompanying undetermined coefficients are added to the formulation, the solution will move monotonically toward the exact solution from the stiff side. Furthermore, if the set of functions from which the modes are selected is complete, the approximate solution will converge to the exact solution.

These assumed mode methods, with all of their desirable properties, have been useless for the bulk of practical problems. This has mainly been due to the difficulty of choosing modes which satisfy complicated boundary conditions, accomodate cutouts, and produce displacements compatible with stiffening members. These difficulties can be overcome by incorporating the concepts of assumed mode analysis into a discrete element analysis.

A discrete element analysis may be thought of as an assumed mode analysis in which the assumed modes, instead of having a uniform definition over the entire structure, have a different definition in each zone or sub-element, producing a sort of patchwork quilt assumed mode. Furthermore, the local assumed modes (usually polynomials) are arranged, for convenience, so that the undetermined coefficients correspond to physical degrees of freedom at certain points (usually the corners) of the zone.

Two problems in discrete element analysis which have been most vexing are the identification of geometric admissibility conditions for the discrete element and the generation of local displacement patterns which will satisfy these conditions. With regard to the geometric admissibility conditions it is apparent the elements must be based upon modes that cause them to be geometrically compatible with adjacent elements along their entire common boundary, and satisfy everywhere, the geometric boundary conditions. This means that two dimensional elements must, along their entire common boundaries, have identical inplane

* "Stiffer" here is taken to mean having a higher potential energy than the exact solution. In the case of point loads this reduces to the usual notion of reduced displacement at the point of application.

displacements and identical transverse displacements and transverse slopes. This fact can be reasoned (assuming plane sections remain plane) on the physical grounds that to require less will allow the original structure, which was continuous, to possess gaps or to have several material points coalesce into a single point in the deformed state.

The uncertainty over the seriousness of the failure to meet these requirements has produced little light because often the incompatible elements produce reasonable answers. However, it is the authors' opinion that this situation is often due to the fact that while incompatibility error is usually stiffness decreasing, this is partially compensated for by the increased stiffness inherent in the sub-element due to the assumed displacement state within the element. It has nonetheless been reported that repetitive refinement of the mesh used in modeling a structure can cause a divergence of the analysis (Reference 3). These arguments notwithstanding, it will be assumed that if inter-element-compatible stiffness matrices were available they could be used to advantage.

Furthermore, while it is not central to the ideas presented in this paper, it should be emphasized that in doing natural frequency studies of structures, using discrete element analysis it is extremely advantageous to use mass matrices consistent with the assumed modes of the element. This point has been well made by J. Archer (Reference 4) and it is for this reason that the companion consistent mass matrices for the elements developed subsequently, will also be given. A consistent mass matrix has the advantage that it provides the exact inertia force associated with each assumed degree of freedom rather than some arbitrary mass lumping. When compatible sub-elements are used in conjunction with their consistent mass matrices to predict the natural frequencies of a structure, the frequencies so obtained will always be upper bounds to the actual frequencies. Moreover these frequencies are usually quite accurate even with relatively crude modeling of the structure.

III. THE GENERATION OF THE STIFFNESS MATRICES

The generation of compatible stiffness matrices can be reduced to straight-forward algebraic operations for what will be called the quasi-rectangular element by the use of Hermite interpolation polynomials. The quasi-rectangular element is a section of a shell described in orthogonal curvilinear coordinates, (α,β), which is bounded by $\alpha = \alpha_i$, $\alpha = \alpha_j$, $\beta = \beta_k$ and $\beta = \beta_\ell$; for example in a conical shell it is a section of a frustum which is bounded by two generators. The Hermite interpolation formulas will be described subsequently.

Strain-Displacement Relations

In orthogonal curvilinear coordinates the length of a line element on a surface is written

$$
\begin{aligned}
dS^2 &= dS_\alpha^2 + dS_\beta^2 \\
dS^2 &= A^2\, d\alpha^2 + B^2\, d\beta^2
\end{aligned}
\tag{1}
$$

where dS_α and dS_β are the lengths of line elements in the direction of the coordinate lines of principal curvature α and β, respectively; A and B are the Lamé parameters which are not constant in general i.e. $A = A(\alpha,\beta)$ and $B = B(\alpha,\beta)$.

Using this description for a line segment, Langhaar (Reference 5) derives the following strain-displacement relations for a shell

$$\begin{aligned} \epsilon_\alpha &= e_\alpha + z\kappa_\alpha \\ \epsilon_\beta &= e_\beta + z\kappa_\beta \\ \gamma_{\alpha\beta} &= e_{\alpha\beta} + z\kappa_{\alpha\beta} \end{aligned} \tag{2}$$

where,

$$\begin{aligned} e_\alpha &= \frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{\nu}{AB}\frac{\partial A}{\partial\beta} + \frac{w}{R_\alpha} + \frac{1}{2A^2}\left(\frac{\partial w}{\partial\alpha}\right)^2 \\ e_\beta &= \frac{1}{B}\frac{\partial \nu}{\partial\beta} + \frac{u}{AB}\frac{\partial B}{\partial\alpha} + \frac{w}{R_\beta} + \frac{1}{2B^2}\left(\frac{\partial w}{\partial\beta}\right)^2 \\ e_{\alpha\beta} &= \frac{1}{A}\frac{\partial \nu}{\partial\alpha} + \frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{u}{AB}\frac{\partial A}{\partial\beta} - \frac{\nu}{AB}\frac{\partial B}{\partial\alpha} + \frac{1}{AB}\frac{\partial w}{\partial\alpha}\frac{\partial w}{\partial\beta} \end{aligned} \tag{3}$$

and

$$\begin{aligned} \kappa_\alpha &= -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} \\ \kappa_\beta &= -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} \\ \kappa_{\alpha\beta} &= \frac{2}{AB}\left[\frac{1}{A}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} - \frac{\partial^2 w}{\partial\alpha\,\partial\beta}\right] \end{aligned} \tag{4}$$

u, v and w are the displacements in the $\hat{e}_\alpha$, $\hat{e}_\beta$ and $\hat{e}_n$ directions, respectively; and $\frac{1}{R_\alpha}$ and $\frac{1}{R_\beta}$ are the principal curvatures of the surface under consideration.

Langhaar incorporated several assumptions in the derivation of the above strain-displacement relations, namely

(a) normals to the middle surface of the shell remain normal throughout the deformation

(b) transverse shear deformation is neglected

(c) the normal displacement w is a function only of the middle surface coordinates, $w = w(\alpha, \beta)$

(d) the strains (Equation 2) are linear in z

(e) quadratic terms in u and ν are small compared with quadratic terms in $\frac{\partial w}{\partial \alpha}$ and $\frac{\partial w}{\partial \beta}$

(f) the bending effect is dependent only on the derivatives of w, and not on w itself

(g) the effects of u and ν on κ_α, κ_β and $\kappa_{\alpha\beta}$ are neglected.

The above assumptions permit relatively large displacements (of the order several times the thickness) but restrict the class of two dimensional structures to thin shells. We will further restrict these expressions to their linear terms subsequently when the stiffness matrices are developed.

Potential Energy

The strain energy of an isotropic linearly elastic shell expressed in curvilinear coordinates is given by

$$U = \iint_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E}{2(1-\nu^2)} \left[\epsilon_\alpha^2 + \epsilon_\beta^2 + 2\nu\, \epsilon_\alpha \epsilon_\beta + \frac{1}{2}(1-\nu)\gamma_{\alpha\beta}^2 \right] AB \; dz \; d\alpha \; d\beta \quad (5)$$

Substituting (Equation 2) for ϵ_α, ϵ_β, and $\gamma_{\alpha\beta}$ and performing the integration with respect to z, the strain energy is separated into two terms, the membrane energy U_m and the bending energy U_b,

$$U = U_m + U_b \quad (6)$$

where,

$$U_m = \frac{Eh}{2(1-\nu^2)} \iint_A \left[e_\alpha^2 + e_\beta^2 + 2\nu\, e_\alpha e_\beta + \frac{1}{2}(1-\nu) e_{\alpha\beta}^2 \right] AB \; d\alpha \; d\beta \quad (7)$$

$$U_b = \frac{Eh^3}{24(1-\nu^2)} \iint_A \left[\kappa_\alpha^2 + \kappa_\beta^2 + 2\nu\, \kappa_\alpha \kappa_\beta + \frac{1}{2}(1-\nu) \kappa_{\alpha\beta}^2 \right] AB \; d\alpha \; d\beta \quad (8)$$

The functional, U, is expressed in terms of the displacements of the middle surface, u, v, and w by means of Equations 3 and 4.

Finally, the potential energy Π_p for a shell is given by

$$\pi_p = U - W \quad (9)$$

where W is the work done by external loading. Denoting middle surface tractions in the u, v and w directions by $p_u(\alpha,\beta)$, $p_v(\alpha,\beta)$ and $p_w(\alpha,\beta)$, respectively, the external work term can be written

$$W = \iint_A \left[p_u u + p_v v + p_w w \right] AB \; d\alpha \; d\beta \quad (10)$$

If the shell structure is now imagined to be zoned off into a collection of quasi-rectangular elements each having a different assumed displacement pattern than

$$\Pi_p = \sum_{k=1}^{S} \pi_p^{(k)} \quad (11)$$

where $\pi_p^{(k)}$ is the potential energy of the k^{th} zone or element.

The Concept of Interpolation Formulas

In order to satisfy the geometric admissibility requirements it is convenient to pick element modes such that the undetermined coefficients are, or are explicitly related to, the geometrically important quantities along the edges of the element. What has often been done, for example in the case of the lateral displacement of a flat plate element, is to assume a form:

$$w(x,y) \approx \tilde{w}(x,y) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{ij} x^i y^j \tag{12}$$

and construct the algebraic equations

$$\tilde{w}(x_p, y_q) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{ij} x_p^i y_p^j$$

$$\partial\tilde{w}(x_p, y_q)/\partial x = \sum_{i=1}^{N} \sum_{j=0}^{M} i\, a_{ij} x_p^{i-1} y_q^j \qquad p,q = 1, 2 \tag{13}$$

$$\partial\tilde{w}(x_p, y_q)/\partial y = \sum_{i=0}^{N} \sum_{j=1}^{M} j\, a_{ij} x_p^i y_q^{j-1}$$

$$\vdots$$

until N·M equations are obtained. These equations may some times, but not always, be solved to express the a_{ij} in terms of $\tilde{w}(x_p, y_q)$, $\partial\tilde{w}(x_p, y_q)/\partial x$ etc. where (x_p, y_q) are the coordinates of the corners of the element. Even when this can be done the problem of interelement compatability is still unsolved because, in general, setting $\tilde{w}(x_p, y_q)$, $\partial\tilde{w}(x_j, y_q)/\partial x$ and $\partial\tilde{w}(x_p, y_q)/\partial y$ equal between adjacent elements at the corners does not insure displacement compatibility along the entire edge. The same is also true of most attempts to equate strains and other combinations of derivatives at the corners, because the modes themselves do not possess the property that the displacements and slopes along the edge depend only upon the values at the two corners associated with that edge.

A remedy to both of these difficulties is to use modes which are products of one dimensional Hermite interpolation formulas, suitably generalized for orthogonal curvilinear coordinates. These polynomials, $H_{ki}^{(N)}(s)$, have the properties:

$$(1): \quad H_{oi}^{(N)}(s_j) = \delta_{ij}$$

$$d\, H_{oi}^{(N)}(s_j)/ds = 0$$

$$d^2 H_{oi}^{(N)}(s_j)/ds^2 = 0$$

$$\vdots$$

$$d^N H_{oi}^{(N)}(s_j)/ds^N = 0$$

$$
\begin{aligned}
(2):\quad & H_{1i}^{(N)}(s_j) = 0 \\
& dH_{1i}^{(N)}(s_j)/ds = \delta_{ij} \\
& d^2H_{1i}^{(N)}(s_j)/ds^2 = 0 \\
& \vdots \\
& d^NH_{1i}^{(N)}(s_j)/ds^N = 0 \\
& \vdots \\
(N+1):\quad & H_{Ni}^{(N)}(s_j) = 0 \\
& \vdots \\
& d^NH_{Ni}^{(N)}(s_j)/ds^N = \delta_{ij}
\end{aligned}
\tag{14}
$$

where N is the number of derivatives that the set can interpolate and s_j are specific values of the argument, s, of the polynomial. For example, if $N = 0$; $i,j = 1,2$; and $s_1 = 0$, $s_2 = a$ the requirements become:

$$
\begin{aligned}
H_{01}^{(0)}(0) &= 1 \\
H_{01}^{(0)}(a) &= 0 \\
H_{02}^{(0)}(0) &= 0 \\
H_{02}^{(0)}(a) &= 1
\end{aligned}
\tag{15}
$$

A pair of polynomials, $H_{01}^{(0)}(s)$ and $H_{02}^{(0)}(s)$, having these properties is

$$
\begin{aligned}
H_{01}^{(0)}(s) &= \frac{s - s_2}{s_1 - s_2} = \frac{-(s-a)}{a} \\
H_{02}^{(0)}(s) &= \frac{s - s_1}{s_2 - s_1} = \frac{s}{a}
\end{aligned}
\tag{16}
$$

These are known as Lagrange interpolation formulas.

If $N = 1$ and $i,j = 1,2$ and $s_1 = 0$, $s_2 = a$ the requirements are

$$
\begin{aligned}
H_{01}^{(1)}(0) &= 1 & dH_{01}^{(1)}(0)/ds &= 0 \\
H_{01}^{(1)}(a) &= 0 & dH_{01}^{(1)}(a)/ds &= 0 \\
H_{02}^{(1)}(0) &= 0 & dH_{02}^{(1)}(0)/ds &= 0 \\
H_{02}^{(1)}(a) &= 1 & dH_{02}^{(1)}(a)/ds &= 0 \\
H_{11}^{(1)}(0) &= 0 & dH_{11}^{(1)}(0)/ds &= 1 \\
H_{11}^{(1)}(a) &= 0 & dH_{11}^{(1)}(a)/ds &= 0 \\
H_{12}^{(1)}(0) &= 0 & dH_{12}^{(1)}(0)/ds &= 0 \\
H_{12}^{(1)}(a) &= 0 & dH_{12}^{(1)}(a)/ds &= 1
\end{aligned}
\tag{17}
$$

A set of polynomials which satisfies these are

$$
\begin{aligned}
H_{01}^{(1)}(s) &= \left[1 - 2 \left(\frac{s - s_1}{s_1 - s_2} \right) \right] \left(\frac{s - s_2}{s_1 - s_2} \right)^2 \\
&= \frac{1}{a^3} (2s^3 - 3as^2 + a^3) \\
H_{02}^{(1)}(s) &= \left[1 - 2 \left(\frac{s - s_2}{s_2 - s_1} \right) \right] \left(\frac{s - s_1}{s_2 - s_1} \right)^2 \\
&= -\frac{1}{a^3} (2s^3 - 3as^2) \\
H_{11}^{(1)}(s) &= (s - s_1) \left(\frac{s - s_2}{s_1 - s_2} \right)^2 \\
&= \frac{1}{a^2} (s^3 - 2as^2 + a^2 s) \\
H_{12}^{(1)}(s) &= (s - s_2) \left(\frac{s - s_1}{s_2 - s_1} \right)^2 \\
&= \frac{1}{a^2} (s^3 - as^2)
\end{aligned}
\tag{18}
$$

These are known as osculatory polynomials and are pictured in Figure 2. They can be used to interpolate a function, say f(s), and its derivatives at the two points 0 and a, given the values f(0), f(a), f'(0) and f'(a) as

$$
f(s)\ \tilde{f}(s) = f(0)\, H_{01}^{(1)}(s) + f(a)\, H_{02}^{(1)}(s) + f'(0)\, H_{11}^{(1)}(s) + f'(a)\, H_{12}^{(1)}(s)
$$

These interpolation formulas can be generated for any number of points and as many derivatives as are desired (Reference 6).

If a function is constructed as an assumed mode for the membrane displacement, u(x,y), of a flat rectangular element (Figure 3), for example,

$$u(x,y) \simeq \tilde{u}(x,y) = \sum_{i=1}^{2} \sum_{j=1}^{2} H_{0i}^{(0)}(x)\, H_{0j}^{(0)}(y)\, b_{ij} \tag{19}$$

several valuable properties are obtained. First of all it is clear that $b_{ij} = \tilde{u}(x_i, y_j) \equiv u_{ij}$, the corner displacements, and furthermore that the functional form of $u(x, y_k)$ depends only upon u_{1k} and u_{2k}. This means that if two adjacent elements have the same values at their common corners, they will have the same values along that entire common boundary. This is all that is required for geometric compatibility between elements for the u and v modes. While it may be disturbing that this element deforms into a straight-sided shape with "breaks" in the slope of the grid lines (see Figure 4) it is nonetheless geometrically admissible for membrane displacement. It can be argued that the shape shown in the figure could theoretically be obtained from a real membrane by the application of external forces. Thus the approximate solution generally satisfies equilibrium only approximately along the edges of the elements as well as in the interior.

A mode shape can be obtained which does not have these "kinks" and straight edges as

$$\tilde{u}(x,y) = \sum_{i=1}^{2} \sum_{j=1}^{2} \left[H_{0i}^{(1)}(x)\, H_{0j}^{(1)}(y)\, u_{ij} + H_{1i}^{(1)}(x)\, H_{0j}^{(1)}(y)\, u_{x_{ij}} + H_{0i}^{(1)}(x)\, H_{1j}^{(1)}(y)\, u_{y_{ij}} \right] \tag{20}$$

Clearly along the edge $y = y_1$, $\tilde{u}(x, y_1)$ depends only upon u_{11}, u_{21}, $u_{x_{11}}$, and $u_{x_{12}}$ and thus if these are set equal between elements they will be compatible. It would be an arbitrary decision, but perhaps an appealing one, to set u_{y11} and u_{y12} equal between elements at these corners. This will produce a smoother deformation pattern but has fewer degrees of freedom and while the solution looks better it may in fact be worse. If a better satisfaction of equilibrium, in the sense of a lower potential energy, were obtained by setting these degrees of freedom equal then the stationary conditions of potential energy would do so automatically.

This higher order membrane element will not be developed further in this paper; it has served only as a simple example with which to develop the ideas. The bending elements developed by this method are thought to be more interesting.

The Interpolation Polynomials in Orthogonal Curvilinear Coordinates

Before proceeding with the generation of specific stiffness matrices it will be worthwhile to establish several generalizations and notations used in these developments. Let ξ and η denote the distance of the general point (α,β) in orthogonal curvilinear coordinates from the reference lines (curves) $\alpha = 0$ and $\beta = 0$, respectively. Let $\xi_{\alpha_i \beta_j}$ and $\eta_{\alpha_i \beta_j}$ denote the distance of the nodal point (α_i, β_j) from the lines $\alpha = 0$ and $\beta = 0$, respectively. Let $\xi_{\alpha_i \beta}$ represent the distance to the edge of the element, $\alpha = \alpha_i$, from the reference line $\alpha = 0$ measured along a line $\beta =$ const. Note that this is a generalization in orthogonal curvilinear coordinates of the notion of the perpendicular distance between two parallel lines. Similarly let $\eta_{\alpha\beta_j}$ be the distance of the edge $\beta = \beta_j$ from the reference line $\beta = 0$ measured along α = const. Representative examples of these quantities are shown in Figure 5 and are expressed in equation form by:

$$\xi = \int_0^{\alpha} A(\alpha,\beta)\, d\alpha \; ; \quad \beta = \text{const.} \tag{21}$$

$$\xi_{\alpha_i \beta_j} = \int_0^{\alpha_i} A(\alpha,\beta_j)\, d\alpha \tag{22}$$

$$\xi_{\alpha_i \beta} = \int_0^{\alpha_i} A(\alpha,\beta)\, d\alpha , \quad \beta = \text{const.} \tag{23}$$

and

$$\eta = \int_0^{\beta} B(\alpha,\beta)\, d\beta , \qquad \alpha = \text{const.} \tag{24}$$

$$\eta_{\alpha_i \beta_j} = \int_0^{\beta_j} B(\alpha_i,\beta)\, d\beta \tag{25}$$

$$\eta_{\alpha\beta_j} = \int_0^{\beta_j} B(\alpha,\beta)\, d\beta , \qquad \alpha = \text{const.} \tag{26}$$

Also

$$\begin{aligned} d\xi &= A\, d\alpha = dS_{\alpha} \\ d\eta &= B\, d\beta = dS_{\beta} \end{aligned} \tag{27}$$

where A and B are the Lamé parameters which are determined by the form of Equation 1 for a particular shell. For example, in the case of a flat plate A = B = 1 and $\alpha = x$, $\beta = y$ and from Equations 21 to 26 it follows that $\xi = x$ and $\eta = y$; $\xi_{\alpha_i\beta} = \xi_{\alpha_i\beta_j} = x_i$ and $\eta_{\alpha\beta_j} = \eta_{\alpha_i\beta_j} = y_j$.

This notation permits the Hermite interpolation polynomials to be written in general forms which are tabulated below for $\{H_{0i}^{(0)}\}$, $\{H_{ki}^{(1)}\}$, and $\{H_{ki}^{(2)}\}$:

$$\begin{aligned} H_{0i}^{(0)}(\xi) &= \frac{\xi - \xi_{\alpha_j\beta}}{\xi_{\alpha_i\beta} - \xi_{\alpha_j\beta}} \\ H_{0i}^{(0)}(\eta) &= \frac{\eta - \eta_{\alpha_j\beta}}{\eta_{\alpha_i\beta} - \eta_{\alpha_j\beta}} \end{aligned} \quad , \; i \neq j \tag{28}$$

$$H_{0i}^{(1)}(\xi) = \left[1 - 2(\xi - \xi_{\alpha_i\beta}) \quad dH_{0i}^{(0)}(\xi_{\alpha_i\beta})/d\xi\right]\left[H_{0i}^{(0)}(\xi)\right]^2$$

$$H_{1i}^{(1)}(\xi) = (\xi - \xi_{\alpha_i\beta})\left[H_{0i}^{(0)}(\xi)\right]^2 \tag{29}$$

and similarly for $H_{0i}^{(1)}(\eta)$ and $H_{1i}^{(1)}(\eta)$ and

$$H_{0i}^{(2)}(\xi) = \left[H_{0i}^{(0)}(\xi)\right]^3 \Big\{1 - 3\left[dH_{0i}^{(0)}(\xi_{\alpha_i\beta})/d\xi\right](\xi - \xi_{\alpha_i\beta})$$

$$+ 6\left[dH_{0i}^{(0)}(\xi_{\alpha_i\beta})d\xi\right]^2 (\xi - \xi_{\alpha_i\beta})^2\Big\} \tag{30}$$

$$H_{1i}^{(2)}(\xi) = \left[H_{0i}^{(0)}(\xi)\right] (\xi - \xi_{\alpha_i\beta})\Big\{1 - 3\left[dH_{0i}^{(0)}(\xi_{\alpha_i\beta})/d\xi\right](\xi - \xi_{\alpha_i\beta})\Big\}$$

$$H_{2i}^{(2)}(\xi) = \left[H_{0i}^{(0)}(\xi)\right]^3 \frac{(\xi - \xi_{\alpha_i\beta})^2}{2}$$

and similarly for $H_{0i}^{(2)}(\eta)$, $H_{1i}^{(2)}(\eta)$ and $H_{2i}^{(2)}(\eta)$.

The Rectangular Flat Plate Element:

The Lamé parameters for the flat plate are A = B = 1. The principal curvatures are $\frac{1}{R_x} = \frac{1}{R_y} = 0$ and the membrane and bending strain-displacement relations reduce to

$$e_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$e_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{31}$$

$$e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}$$

$$\kappa_y = -\frac{\partial^2 w}{\partial y^2} \tag{32}$$

$$\kappa_{xy} = -2\frac{\partial^2 w}{\partial x\,\partial y}$$

At this point the usual linearizations in these expressions will be made. The equations have been successfully dealt with without linearization, using the assumed modes presented here, by methods of energy search and the results of the first part of this study are reported in Reference 7.

Dropping all nonlinear terms and substituting into Equations 7 and 8 the strain energy becomes

$$U_m = \frac{K}{2} \iint_A \left[u_x^2 + v_y^2 + 2\nu\, u_x\, v_y + \frac{1}{2}(1-\nu)(u_y + v_x)^2 \right] dx\, dy \tag{33}$$

$$U_b = \frac{D}{2} \iint_A \left[w_{xx}^2 + w_{yy}^2 + 2\nu\, w_{xx}\, w_{yy} + 2(1-\nu)\, w_{xy}^2 \right] dx\, dy \tag{34}$$

where subscripts denote derivatives and

$$K = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)} \tag{35}$$

As there is no coupling between the membrane and bending terms for the linear treatment of flat elements, they may be considered separately.

For the case of the flat plate of sides a and b (Figure 3) the polynomials of Equations 28, 29 and 30 reduce to

$$\begin{aligned} H_{01}^{(0)}(x) &= -\frac{1}{a}(x-a) \\ H_{02}^{(0)}(x) &= \frac{1}{a}\, x \\ H_{01}^{(0)}(y) &= -\frac{1}{b}(y-b) \\ H_{02}^{(0)}(y) &= \frac{1}{b}\, y \end{aligned} \tag{36}$$

and

$$\begin{aligned} H_{01}^{(1)}(x) &= \frac{1}{a^3}(2x^3 - 3a x^2 + a^3) \\ H_{02}^{(1)}(x) &= -\frac{1}{a^3}(2x^3 - 3a\, x^2) \\ H_{11}^{(1)}(x) &= \frac{1}{a^2}(x^3 - 2a\, x^2 + a^2 x) \\ H_{12}^{(1)}(x) &= \frac{1}{a^2}(x^3 - a x^2) \end{aligned} \tag{37}$$

and similarly for $H_{01}^{(1)}(y)$, $H_{02}^{(1)}(y)$, $H_{11}^{(1)}(y)$ and $H_{12}^{(1)}(y)$ replacing x with y and a with b in the above.

Further

$$
\begin{aligned}
H_{01}^{(2)}(x) &= \frac{1}{a^5}\left(a^5 - 10\,a^2x^3 + 15\,a\,x^4 - 6x^5\right)\\
H_{02}^{(2)}(x) &= \frac{1}{a^5}\left(10\,a^2x^3 - 15\,a\,x^4 + 6x^5\right)\\
H_{11}^{(2)}(x) &= \frac{1}{a^4}\left(a^4x - 6\,a^2x^3 + 8\,a\,x^4 - 3x^5\right)\\
H_{12}^{(2)}(x) &= \frac{1}{a^4}\left(-4\,a^2x^3 + 7\,a\,x^4 - 3x^5\right)\\
H_{21}^{(2)}(x) &= \frac{1}{2a^3}\left(a^3x^2 - 3\,a^2x^3 + 3\,a\,x^4 - x^5\right)\\
H_{22}^{(2)}(x) &= \frac{1}{2a^3}\left(a^2x^3 - 2\,a\,x^4 + x^5\right)
\end{aligned}
\qquad (38)
$$

and again similarly for y.

Taking for the membrane displacements

$$\tilde{u}(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2} H_{0i}^{(0)}(x)\,H_{0j}^{(0)}(y)\;u_{ij} \qquad (39)$$

and

$$\tilde{v}(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2} H_{0i}^{(0)}(x)\,H_{0j}^{(0)}(y)\,v_{ij} \qquad (40)$$

constructing the total membrane potential energy for the element, taking the partial derivatives of this energy with respect to the independent degrees of freedom (u_{ij}, v_{ij}), and setting these equal to zero yields

$$\mathbf{Q}_m\;\mathbf{V} = \mathbf{P}_m \qquad (41)$$

where $\mathbf{Q}_m$ is the membrane stiffness matrix for the plate element and the vector $\mathbf{V}$ is arranged as follows:

$$\mathbf{V} = \left\{ u_{11},\, v_{11},\, u_{12},\, v_{12},\, u_{22},\, v_{22},\, u_{21},\, v_{21} \right\} \qquad (42)$$

The "load vector", $\mathbf{P}_m$, depends upon the tractions $p_u(x,y)$ and $p_v(x,y)$. The matrix $\mathbf{Q}_m$ is given in the Appendix and it is identical with that given by Melosh (Ref. 3) in connection with the direct stiffness method. Examples of the vector $\mathbf{P}_m$ are also given in the Appendix.

It should be understood that when a structure is modeled with an assembly of these membrane elements, at each node where 4 elements come together there will be only two degrees of freedom, u_{ij} and v_{ij}. This will insure the interelement compatibility necessary for monotonic convergence. It is also important to remark that while the load vector $\mathbf{P}_m$ is in terms of forces and the compatibility conditions appear to impose the same requirements as ordinary equilibrium at the node, this approach does not extend to higher order elements. In general, the resulting system of equations for the assembled structure are best viewed as the stationary conditions of the total potential energy for the entire structure wherein the unknows are the independent degrees remaining after assuring overall compatibility.

Taking for the transverse displacement

$$\tilde{w}(x,y) = \sum_{i=1}^{2} \sum_{j=1}^{2} \left[H_{0i}^{(1)}(x)\, H_{0j}^{(1)}(y)\, w_{ij} + H_{1i}^{(1)}(x)\, H_{0j}^{(1)}(y)\, w_{xij} + H_{0i}^{(1)}(x)\, H_{1j}^{(1)}(y)\, w_{yij} \right] \tag{43}$$

constructing the total bending potential energy for the element, taking the partial derivatives of this with respect to the independent degrees of freedom (w_{ij}, w_{xij}, w_{yij}) and setting these equal to zero yields

$$\mathbf{Q}_b \mathbf{W} = \mathbf{P}_b \tag{44}$$

where $\mathbf{Q}_b$ is the bending stiffness matrix for the plate element and is given in the Appendix.

The vector $\mathbf{W}$ is arranged as follows:

$$\mathbf{W} = \left\{ w_{11}, w_{x11}, w_{y11}, w_{12}, w_{x12}, w_{y12}, w_{22}, w_{x22}, w_{y22}, w_{21}, w_{x21}, w_{y21} \right\} \tag{45}$$

The load vector $\mathbf{P}_b$ depends upon $p_w(x,y)$; several examples of this vector are given in the Appendix.

This element has 12 degrees of freedom and when an assemblage of them is made to model a continuous plate structure each interior node will have only three degrees of freedom. Setting w_{ij}, w_{xij} and w_{yij} equal between all four elements at each node insures overall geometric compatibility. The geometric boundary conditions can also be satisfied exactly, if they apply along the entire edge of an element, by satisfying them at the corners.

A higher order plate element will now be developed using the hyperosculatory polynomials of Eqs. (38). Taking the transverse displacement as

$$\begin{aligned} \tilde{w}(x,y) = \sum_{i=1}^{2} \sum_{j=1}^{2} \Big[& H_{0i}^{(2)}(x)\, H_{0j}^{(2)}(y)\, w_{ij} + H_{1i}^{(2)}(x)\, H_{0j}^{(2)}(y)\, w_{xij} \\ & + H_{0i}^{(2)}(x)\, H_{1j}^{(2)}(y)\, w_{yij} + H_{2i}^{(2)}(x)\, H_{0j}^{(2)}(y)\, w_{xxij} \\ & + H_{0i}^{(2)}(x)\, H_{2j}^{(2)}(y)\, w_{yyij} + H_{1i}^{(2)}(x)\, H_{1j}^{(2)}(y)\, w_{xyij} \Big] \end{aligned} \tag{46}$$

constructing the total bending potential energy for the element, taking the partial derivatives of this with respect to the independent degrees of freedom (w_{ij}, w_{xij}, w_{yij}, w_{xxij}, w_{xyij}, w_{yyij}) and setting these equal to zero yields

$$\overline{\mathbf{Q}}_b \overline{\mathbf{W}} = \overline{\mathbf{P}}_b \tag{47}$$

where $\overline{\mathbf{Q}}_b$ is the bending stiffness matrix for this 24 degree of freedom plate element which is given in the Appendix. The vector $\overline{\mathbf{W}}$ is arranged with the same cyclic order as equations 42 and 45 with the variables ordered as w_{ij}, w_{xij}, w_{yij}, w_{xxij}, w_{xyij}, and w_{yyij}. Examples of the vector $\overline{\mathbf{p}}_b$ are also given in the Appendix.

In using this element to model a continuous plate, the compatibility requirements are

1) all four elements at a node must have identical values of w_{ij}, w_{xij}, w_{yij}, and w_{xyij} at the common corner

2) the elements joining along the edge x = const. must have identical values of w_{yyij} at their common corners

3) the elements joining along the edge y = const. must have identical values of w_{xxij} at their common corners.

These requirements are deduced by examining the dependence of $\widetilde{w}(x,y)$, $\partial \widetilde{w}(x,y)/\partial x$ and $\partial \widetilde{w}(x,y)/\partial y$ along edges y = const. and x = const. All coefficients which effect these dependencies must be equal for the elements to be compatible.

These requirements permit 8 degrees of freedom at an interior node: one value each of w_{ij}, w_{xij}, w_{yij} and w_{xyij} and two values each of w_{xxij} and w_{yyij}. If desired, these latter two may be set equal reducing the degrees of freedom to 6. While this is not required to satisfy the compatibility conditions it may be considered desirable in some cases to preclude discontinuities of curvature. Note that a discontinuous curvature is a physically realizable occurrence for example in the case of a concentrated line moment.

It is possible further to "improve" these stiffness matrices in a manner similar to that described by Pian in References 8 and 9. Noting that the essential feature which yields compatibility for the modes used here is that the form of the displacements along an edge depend only upon the displacements (and slopes) at the two corners associated with that edge, it is clear that any function which contributes nothing at the edges may be added without upsetting the compatibility. In the case of the membrane element this "augmentation function" must merely be zero on the boundary of the element. Thus for the u displacement the function

$$\tilde{u}^*(x,y) = \tilde{u}(x,y) + H_{01}^{(0)}(x)\, H_{02}^{(0)}(x)\, H_{01}^{(0)}(y)\, H_{02}^{(0)}(y) \left[\sum_{i=0}^{m} \sum_{j=0}^{m} \alpha_{ij}\, x^i\, y^j \right] \tag{48}$$

is a compatible mode. The α_{ij} are degrees of freedom in addition to those contained in $\tilde{u}(x,y)$. The v displacement modes are similar. For the transverse displacement the augmentation function must have zero derivatives as well as zero value on the boundary. Thus

$$\widetilde{w}^*(x,y) = \widetilde{w}(x,y) + H_{11}^{(1)}(x)\, H_{12}^{(1)}(x)\, H_{11}^{(1)}(y)\, H_{12}^{(1)}(y) \left[\sum_{i=0}^{m} \sum_{j=0}^{m} \alpha_{ij}\, x^i\, y^j \right] \tag{49}$$

is a compatible mode for the $\widetilde{w}$ of either equation 43 or 46. Substituting these modes into the potential energy and taking the stationary conditions yields in the case of the bending element for example

$$\begin{bmatrix} Q_b & A \\ B & C \end{bmatrix} \begin{Bmatrix} W \\ \alpha \end{Bmatrix} = \begin{Bmatrix} P_b \\ q \end{Bmatrix} \tag{50}$$

This stiffness matrix could be used as is but a more efficient method, which reduces the number of degrees of freedom required, is to solve for the α matrix and back substitute producing

$$Q_b^* W = P_b^* \tag{51}$$

where

$$Q_b^* = Q_b - AC^{-1} B \tag{52}$$

and

$$P_b^* = P_b - AC^{-1} q \tag{53}$$

Depending upon the number of terms that are taken such an element possesses a better satisfaction of equilibrium in its interior and therefore the development of the Q^* matrices is a worthwhile effort. However, it should be noted that at best the Q^* having complete equilibrium satisfaction in its interior will still represent a stiff element because of the constraints imposed by the restricted displacement pattern along the boundary. Refinements in the element can be obtained by going to higher order modes (i.e. using Reference 3);however the question of whether it is more efficient to use more low order elements or fewer high order elements is an open one. Some of the numerical results presented in a later section have a bearing on this subject. In any event the development of Q^* matrices is independent of these other questions and will not be pursued further.

The Plate Mass Matrix

Assuming D'Alembert forces in the middle surface and the absence of all other external forces, the external work associated with transverse displacement is

$$W_b = \iint_A - \rho_m \; h \, \ddot{w} \, w \; dx \, dy \tag{54}$$

where ρ_m is the mass density of the plate material which is considered constant throughout the plate element and the double dots denote the second derivative with respect to time. Making the usual assumption of sinusoidal motion, the acceleration term $\ddot{w}$ may be replaced by $-\omega^2$ w; thus

$$W_b = \iint_A \rho_m \, h \, \omega^2 \, w^2 \; dx \; dy \tag{55}$$

When the assumed modes (Equations 43 and 46) are substituted into the potential energy and the stationary conditions taken the following equations are obtained

$$\begin{aligned} Q_b W &= \omega^2 M_b W \\ \bar{Q}_b W &= \omega^2 \bar{M}_b W \end{aligned} \tag{56}$$

where M_b (or $\bar{M}_b$ in the case of Equation 46) is the consistent mass matrix for the element. The matrices M_b and $\bar{M}_b$ are given in the Appendix. In a similar manner the consistent mass matrix associated with the membrane element can be developed. This consistent mass matrix M_m is based on the inplane displacement pattern represented by Equations 39 and 40 and is given in the Appendix.

IV. NUMERICAL RESULTS

The numerical results given in this section represent a partial evaluation of the stiffness and consistent mass matrices presented. This evaluation was conducted using simple examples for which classical solutions are available. In addition to comparing the approximate solutions with classical results the convergence of the discretization error from the stiff side was also examined numerically. Numerical results based on the membrane element are not given here since this element is well known and such results are available elsewhere (for example Reference 7).

Plate Bending, (Static)

The structure used to demonstrate the bending elements is a 20 in. x 20 in. square, clamped plate subject to a uniform distributed load of 0.2 psi (see Figure 6). Only one quadrant of the plate was used because of symmetry. The plate is 0.1 in. thick and has $E = 10.92 \times 10^6$ and $\nu = 0.3$. The center deflection, $\bar{w}_c$, given by Timoshenko (Reference 10) is 0.0403 in. and is taken as the base of comparison. The results for the two bending elements for various numbers of elements follows:

Using Q_b (12-degree of freedom element)

No. of Elements	No. of Degrees of Freedom	Min π_p	w_c	$\frac{w_c - \bar{w}_c}{\bar{w}_c}$ 100
1	1	-0.0106	0.0424	-5.2%
4	8	-0.0119	0.0388	+3.7%
9	21	-0.0120	0.0390	+3.2%
16	40	-0.0121	0.0392	+2.7%

Using $\bar{Q}_b$ (24-degree of freedom element)

No. of Elements	No. of Degrees of Freedom	Min π_p	w_c	$\frac{w_c - \bar{w}_c}{\bar{w}_c}$ 100
1	5	-0.0123	0.0405	-0.5%
4	21	-0.0124	0.0402	+0.25%
4	25	-0.0124	0.0402	+0.25%

The results shown are largely self explanatory. It is however important to recognize that in assessing the monotonic convergence of the discretization error the minimum energy should be examined and not the displacement at a single point. In addition to having the advantage of monotonic convergence from the stiff side these results indicate that even with relatively coarse subdivision and few degrees of freedom accuracy adequate for many purposes can be obtained. It should be noted that strict assessment of the monotonic convergence should omit the 9-element results and consider a sequence of subdivisions (for example 1, 4, 16, 64 ... square elements) such that each new refinement includes the degrees of freedom present in the previous subdivision scheme. Also in connection with the 24-degree of freedom element results it is interesting to note that the only difference between the alternate 4-element subdivisions is that the first one shown (21 degrees of freedom) imposes continuous curvature in the direction normal to the edges between two adjacent elements. Continuous curvature normal to the edge is not required for compatibility. A 25-degree of freedom solution in which the curvatures of adjacent elements are treated as independent degrees of freedom is the last result shown.

Plate Bending, (Dynamic)

The natural frequencies of a simply supported square plate, 10 in. x 10 in. were calculated using two different subdivisions with the 12-degree of freedom element (see Figure 7). The plate was 0.1 in. thick and had $\nu = 0.3$ and $\rho_m = 0.001$ lb-sec^2/in.^{4} The exact normal modes (Reference 11) of this plate are of the form $w(x,y) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$ where a and b are the planform dimensions of the plate. The natural frequencies associated with these modes are

$$w(x,y) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

and thus for the square plate some are repeated. The approximate results are given below along with the exact results for the first 9 natural frequencies.

Using $\mathbf{Q}_b$ and $\mathbf{M}_b$ (12-degree of freedom element)

No. of Elements	No. of Degrees of freedom	ω_{11}	ω_{12}, ω_{21}	ω_{22}	ω_{31}, ω_{13}	ω_{32}, ω_{23}	ω_{33}
4	7	1094	3088	6222	----	----	-----
16	39	1069	2669	4377	5368	7169	10015
Exact		1035	2587	4138	5173	6725	9311

The approximate frequencies are all larger than the exact values and appear to converge rather rapidly.

V. CONCLUSIONS

An orderly approach to the generation of inter-element compatible discrete element stiffness matrices and the corresponding consistent mass matrices has been presented. The use of Hermite interpolation formulas in generating assumed displacement patterns is fundamental to these developments. Displacement patterns generated using products of the one dimensional interpolation formulas have the important property that displacements along an edge of a finite element depend solely upon degrees of freedom at points on that edge. Inter-element-compatible stiffness matrices and corresponding consistent mass matrices with 12 and 24 degrees of freedom have been generated and partially evaluated. The evaluation indicates that in addition to exhibiting monotonic convergence (from the stiff side) of the discretization error, good approximations of the displacement behavior can be achieved even with relatively coarse subdivision of the structure.

The extension of the procedure offered here to the generating of inter-element-compatible element stiffness matrices and the corresponding consistent mass matrices for quasi-rectangular shell elements appears to be straight forward. In doing so, however, particular care should be taken to ensure that these elements have the proper number of rigid body degrees of freedom. Matrices may also be developed for various one-dimensional stiffening elements that will be compatible among themselves and with the two dimensional elements by using interpolation formulas. While the question as to whether using fewer elements, each having more degrees of freedom, offers any advantage over using more elements, each with fewer degrees of freedom, is still open, the procedure presented lends itself to the orderly generation of higher order element matrices. Finally, it should be pointed out that the extension of ideas presented here to the generation of total potential energy functions for various finite elements including geometric nonlinearities is straight forward.

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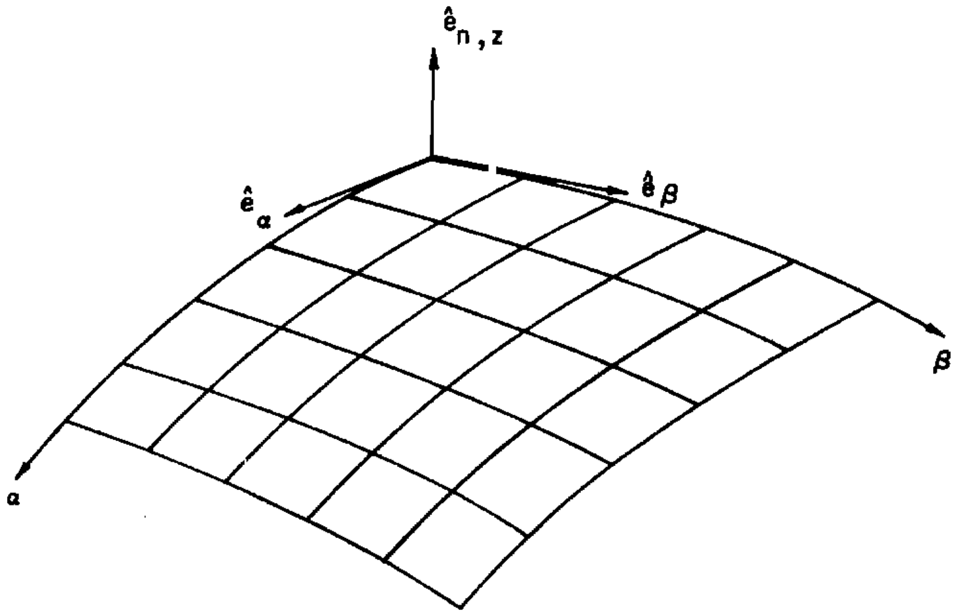


Figure 1. Surface Element Representation in Curvilinear Coordinates

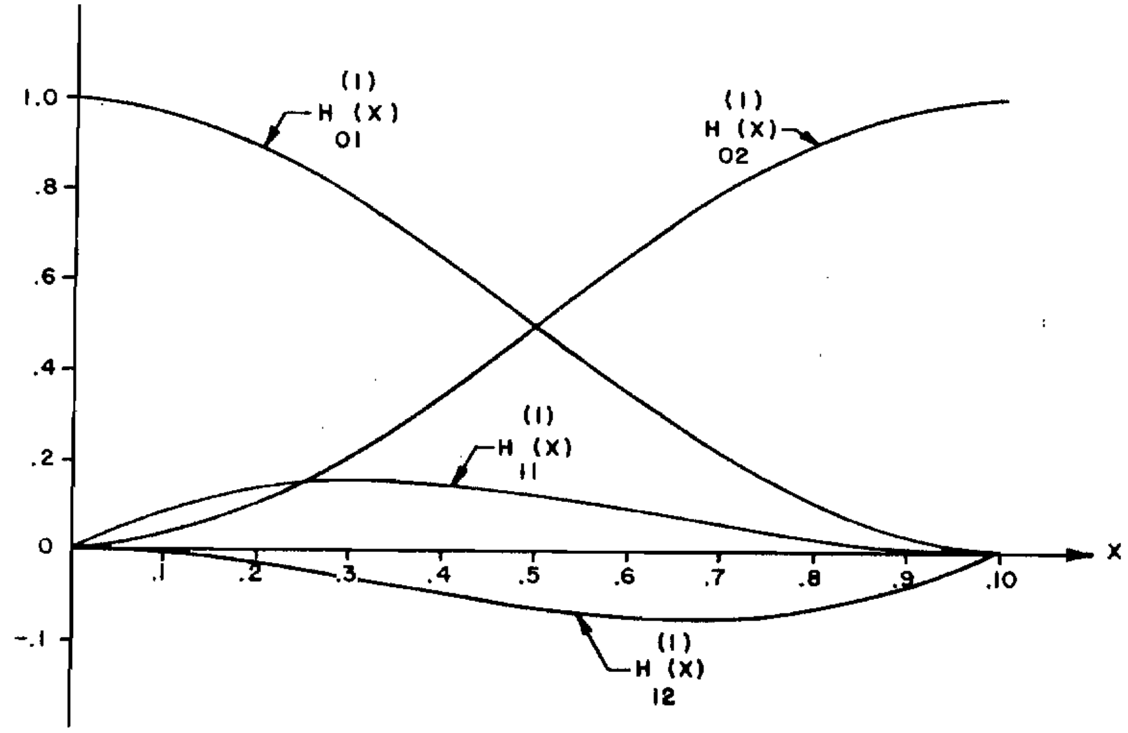


Figure 2. Osculatory Polynomials

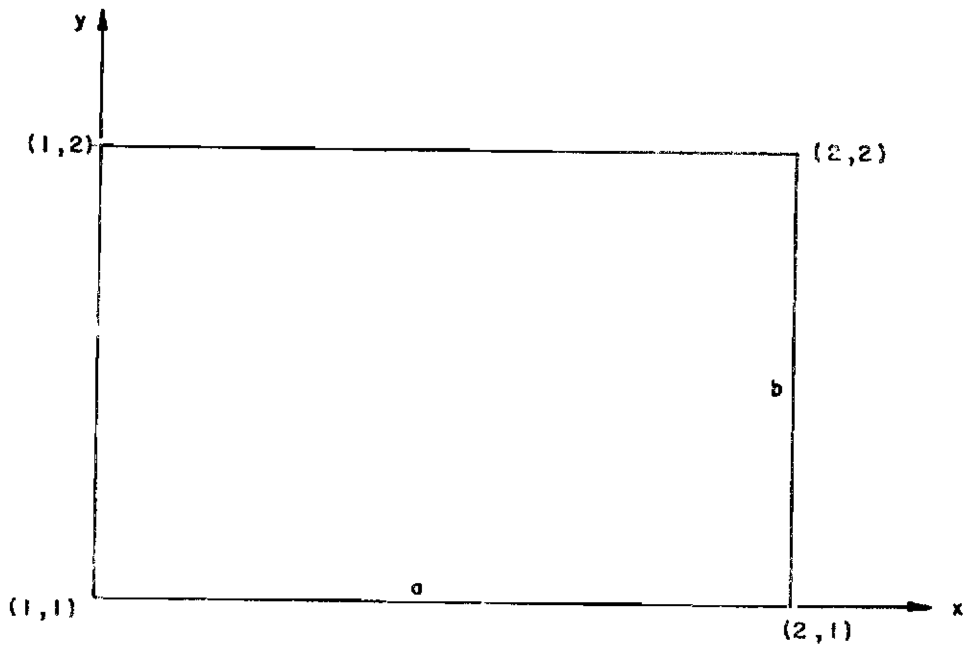


Figure 3. Flat Rectangular Plate Element

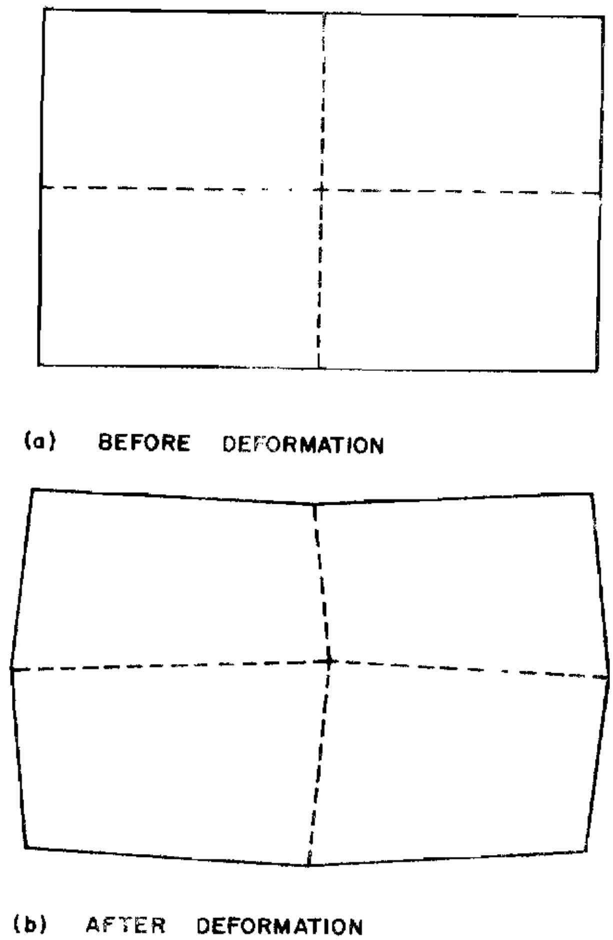


Figure 4. Admissible Membrane Displacements

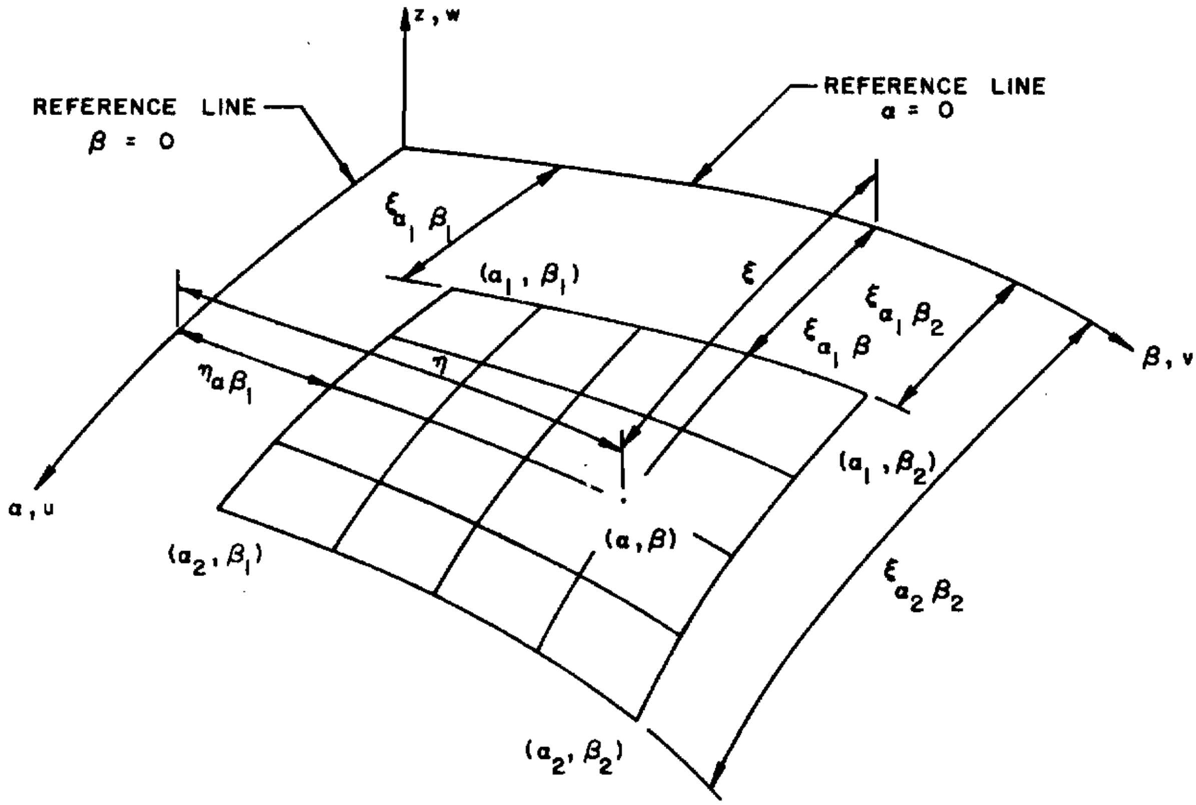


Figure 5. Middle Surface of Shell Element

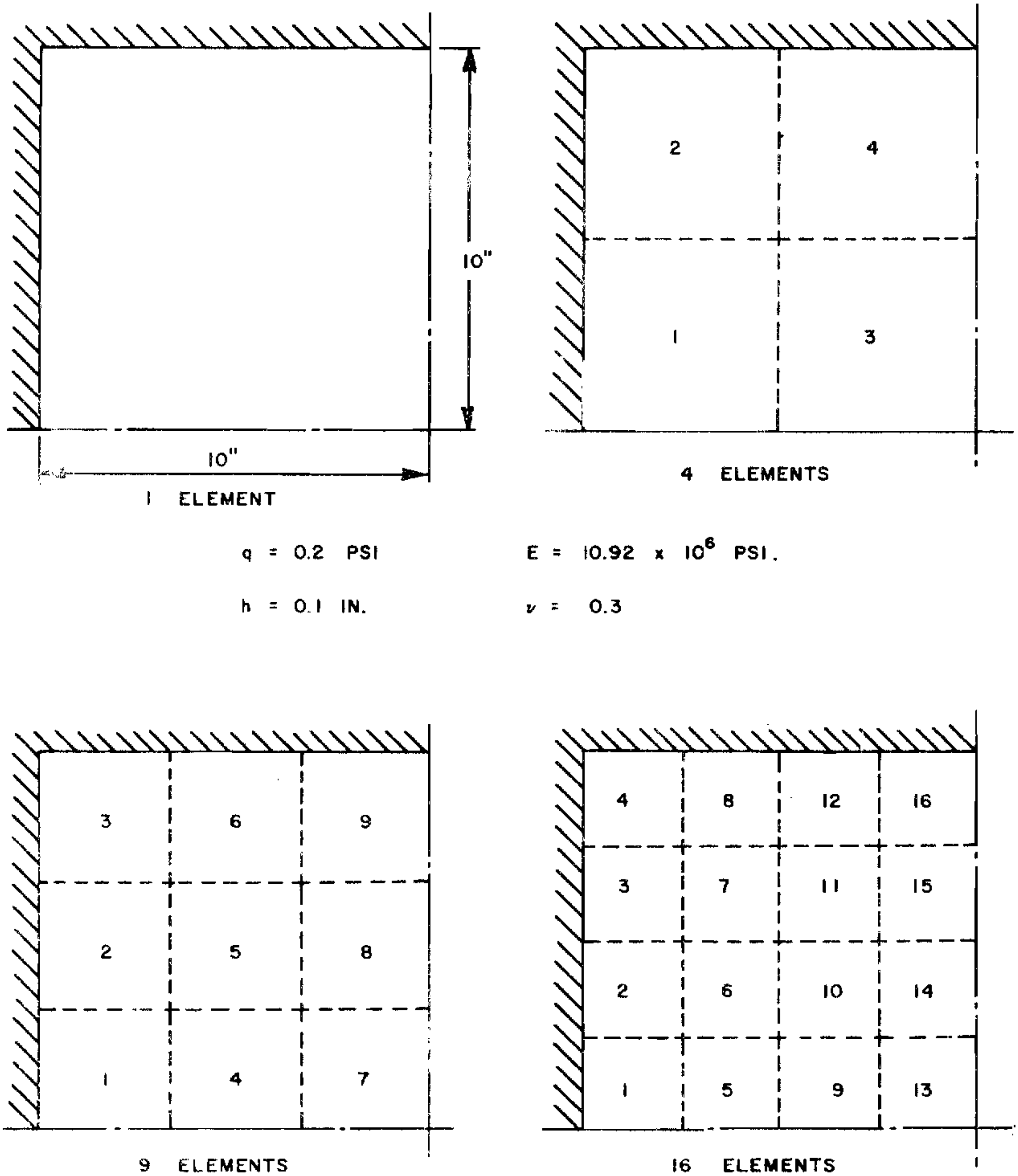
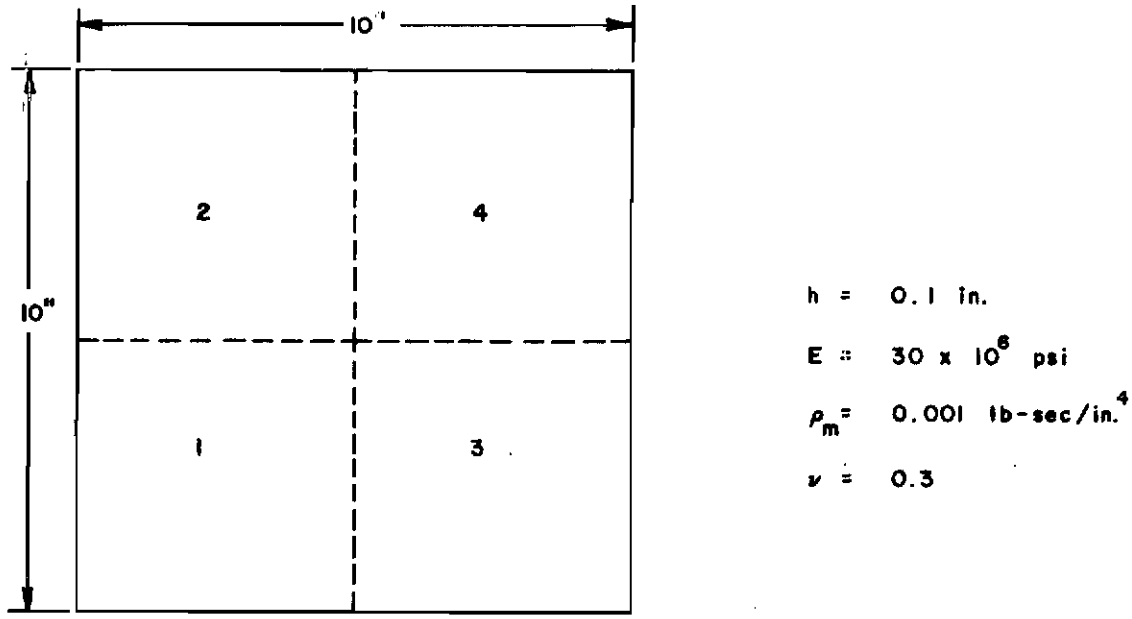


Figure 6. Clamped Plate Modeling



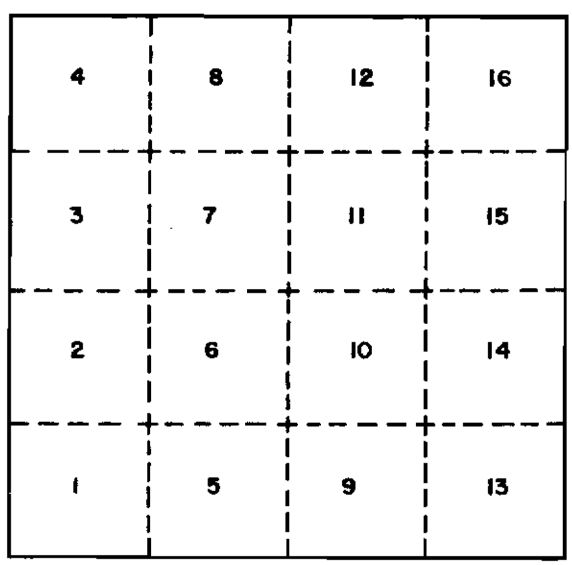


Figure 7. Simply Supported Plate Modeling

APPENDIX

I. STIFFNESS AND CONSISTENT MASS MATRICES FOR THE RECTANGULAR PLATE ELEMENT

A. Lagrangian Membrane Element (Equations 39 and 40)

Defining the following parameters,

$$\alpha_1 = \frac{1}{6}\frac{b}{a} + \frac{(1-\nu)}{12}\frac{a}{b}$$

$$\alpha_2 = \frac{1}{6}\frac{b}{a} - \frac{(1-\nu)}{6}\frac{a}{b}$$

$$\alpha_3 = -\frac{1}{3}\frac{b}{a} + \frac{(1-\nu)}{12}\frac{a}{b}$$

$$\alpha_4 = \frac{1+\nu}{8}$$

$$\alpha_5 = \frac{1-3\nu}{8}$$

$$\alpha_6 = \frac{1}{6}\frac{a}{b} + \frac{(1-\nu)}{12}\frac{b}{a}$$

$$\alpha_7 = \frac{1}{6}\frac{a}{b} - \frac{(1-\nu)}{6}\frac{b}{a}$$

$$\alpha_8 = \frac{1}{3}\frac{a}{b} + \frac{(1-\nu)}{12}\frac{b}{a}$$

the stiffness matrix $\underset{8\times8}{Q_m}$ and the consistent mass matrix $\underset{8\times8}{M_m}$ are given by

$$Q_m = K \begin{bmatrix} 2\alpha_1 & & & & & & & \\ \alpha_4 & 2\alpha_6 & & & & & \text{Symmetric} & \\ \alpha_2 & -\alpha_5 & 2\alpha_1 & & & & & \\ \alpha_5 & \alpha_8 & -\alpha_4 & 2\alpha_6 & & & & \\ -\alpha_1 & -\alpha_4 & \alpha_3 & -\alpha_5 & 2\alpha_1 & & & \\ -\alpha_4 & -\alpha_6 & \alpha_5 & \alpha_7 & \alpha_4 & 2\alpha_6 & & \\ \alpha_3 & \alpha_5 & -\alpha_1 & \alpha_4 & \alpha_2 & -\alpha_5 & 2\alpha_1 & \\ -\alpha_5 & \alpha_7 & \alpha_4 & -\alpha_6 & \alpha_5 & \alpha_8 & -\alpha_4 & 2\alpha_6 \end{bmatrix} \qquad \text{(A-1)}$$

and

$$\mathbf{M}_m = \frac{\rho_m hab}{36}\begin{bmatrix} 4 & & & & & & & \\ 0 & 4 & & & \text{Symmetric} & & & \\ -2 & 0 & 4 & & & & & \\ 0 & -2 & 0 & 4 & & & & \\ 1 & 0 & -2 & 0 & 4 & & & \\ 0 & 1 & 0 & -2 & 0 & 4 & & \\ -2 & 0 & 1 & 0 & -2 & 0 & 4 & \\ 0 & -2 & 0 & 1 & 0 & -2 & 0 & 4 \end{bmatrix} \quad \text{(A-2)}$$

B. Osculatory Bending Element (Equation 43)

The elements of the stiffness matrix $\mathbf{Q}_b = [q_{ij}]_{12\times 12}$ and the mass matrix $\mathbf{M}_b = [m_{ij}]_{12\times 12}$ can be written as

$$q_{ij} = \frac{D}{ab}\left[\gamma_{ij}^{(1)}\left(\frac{b}{a}\right)^2 + \gamma_{ij}^{(2)}\left(\frac{a}{b}\right)^2 + \gamma_{ij}^{(3)} + \gamma_{ij}^{(4)}\nu\right] a^{\lambda_{ij}}\, b^{\mu_{ij}}$$

$$m_{ij} = \frac{\rho_m hab}{1225}\gamma_{ij}^{(5)}\, a^{\lambda_{ij}}\, b^{\mu_{ij}} \quad \text{(A-3)}$$

The constants $\gamma_{ij}^{(k)}$, $k = 1,2,3,4,5$, and the exponents λ_{ij} and μ_{ij} are given in Table 1 for $i = 1,2, \ldots, 12$ and $j = 1,2, \ldots, i$. Both matrices are symmetric with respect to the indices i and j. (Example:

$$q_{39} = q_{93} = \frac{D}{ab}\left[\frac{3}{35}\left(\frac{a}{b}\right)^2 + \frac{9}{35}\left(\frac{b}{a}\right)^2 + \frac{2}{25}\right] b^2)$$

C. Hyperosculatory Bending Element

The elements of the stiffness matrix $\overline{\mathbf{Q}}_b = [\overline{q}_{ij}]_{24\times 24}$ and the mass matrix $\overline{\mathbf{M}}_b = [\overline{m}_{ij}]_{24\times 24}$ are written

$$\overline{q}_{ij} = \frac{D}{ab}\left[\gamma_{ij}^{(6)}\left(\frac{b}{a}\right)^2 + \gamma_{ij}^{(7)}\left(\frac{a}{b}\right)^2 + \gamma_{ij}^{(8)} + \gamma_{ij}^{(9)}\nu\right] a^{\lambda_{ij}}\, b^{\mu_{ij}}$$

$$\overline{m}_{ij} = \frac{\rho_m hab}{7623}\gamma_{ij}^{(10)}\, a^{\lambda_{ij}}\, b^{\mu_{ij}} \quad \text{(A-4)}$$

The constants $\gamma_{ij}^{(k)}$, $k = 6,7,8,9,10$ and the exponents λ_{ij} and μ_{ij} are given in Table 2 for $i = 1,2, \ldots, 24$ and $j = 1,2, \ldots, i$. Both matrices are symmetric in i and j. (Example:

$$\overline{q}_{16,17} = \overline{q}_{17,16} = \frac{D}{ab}\left[\frac{311}{14700}\left(\frac{b}{a}\right)^2 + \frac{123}{2156}\left(\frac{a}{b}\right)^2 + \frac{1}{140} + \frac{\nu}{60}\right] a^3 b).$$

II. LOAD VECTORS

The membrane load vectors P_m for uniformly distributed edge loadings on each of the four edges of an element (see Fig. 3) are given in Table 3, while the load vectors P_b, associated with the osculatory bending element, and $\bar{P}_b$, corresponding to the hyperosculatory bending element, are presented in Tables 4 and 5, respectively, for various types of loading.

These load vectors are the work equivalent loadings. That is, it is required that the total work done by the equivalent loading be equal to the work done by the actual load. For example, the load vector in the third column of Table 4 is obtained as follows: The work done by the distributed load q is given by $W = \int_0^a \int_0^b q\, w(x,y)\, dx\, dy$. The displacement w(x,y) is replaced by the osculatory representation $\tilde{w}$ and the integrations are performed; the first element of the load vector is that factor which is multiplied by w_{11}, the second element is that factor which appears with w_{x11} etc.

Table 1

i	j	$\gamma_{ij}^{(1)}$	$\gamma_{ij}^{(2)}$	$\gamma_{ij}^{(3)}$	$\gamma_{ij}^{(4)}$	$\gamma_{ij}^{(5)}$	λ_{ij}	μ_{ij}	i	j	$\gamma_{ij}^{(1)}$	$\gamma_{ij}^{(2)}$	$\gamma_{ij}^{(3)}$	$\gamma_{ij}^{(4)}$	$\gamma_{ij}^{(5)}$	λ_{ij}	μ_{ij}
1	1	156/35	156/35	72/25	0	169/ 1	0	0	9	4	22/35	-27/35	6/25	6/ 5	-33/ 4	0	1
2	1	78/35	22/35	6/25	6/ 5	143/ 6	1	0	9	5	11/35	-13/70	1/50	1/10	-143/ 72	1	1
2	2	52/35	4/35	8/25	0	13/ 3	2	0	9	6	-4/35	18/35	-8/25	0	3/ 2	0	2
3	1	22/35	78/35	6/25	6/ 5	143/ 6	0	1	9	7	-22/35	-78/35	-6/25	-6/ 5	-143/ 6	0	1
3	2	11/35	11/35	1/50	6/ 5	121/ 36	1	1	9	8	11/35	11/35	1/50	6/ 5	121/ 36	1	1
3	3	4/35	52/35	8/25	0	13/ 3	0	2	9	9	4/35	52/35	8/25	0	13/ 3	0	2
4	1	54/35	-156/35	-72/25	0	117/ 2	0	0	10	1	-156/35	54/35	-72/25	0	117/ 2	0	0
4	2	27/35	-22/35	-6/25	-6/ 5	33/ 4	1	0	10	2	-78/35	13/35	-6/25	0	169/ 12	1	0
4	3	13/35	-78/35	-6/25	0	169/ 12	0	1	10	3	-22/35	27/35	-6/25	-6/ 5	33/ 4	0	1
4	4	156/35	156/35	72/25	0	169/ 1	0	0	10	4	-54/35	-54/35	72/25	0	81/ 4	0	0
5	1	27/35	-22/35	-6/25	-6/ 5	33/ 4	1	0	10	5	-27/35	-13/35	6/25	0	39/ 8	1	0
5	2	18/35	-4/35	-8/25	0	3/ 2	2	0	10	6	13/35	27/35	-6/25	0	-39/ 8	0	1
5	3	13/70	-11/35	-1/50	-1/10	143/ 72	1	1	10	7	54/35	-156/35	-72/25	0	117/ 2	0	0
5	4	78/35	22/35	6/25	6/ 5	143/ 6	1	0	10	8	-27/35	22/35	6/25	6/ 5	-33/ 4	1	0
5	5	52/35	4/35	8/25	0	13/ 3	2	0	10	9	-13/35	78/35	6/25	0	-169/ 12	0	1
6	1	-13/35	78/35	6/25	0	-169/ 12	0	1	10	10	156/35	156/35	72/25	0	169/ 1	0	0
6	2	-13/70	11/35	1/50	1/10	-143/ 72	1	1	11	1	78/35	-13/35	6/25	0	-169/ 12	1	0
6	3	-3/35	26/35	-2/25	0	-13/ 4	0	2	11	2	26/35	-3/35	-2/25	0	-13/ 4	2	0
6	4	-22/35	-78/35	-6/25	-6/ 5	-143/ 6	0	1	11	3	11/35	-13/70	1/50	1/10	-143/ 72	1	1
6	5	-11/35	-11/35	-1/50	-6/ 5	-121/ 36	1	1	11	4	27/35	13/35	-6/25	0	-39/ 8	1	0
6	6	4/35	52/35	8/25	0	13/ 3	0	2	11	5	9/35	3/35	2/25	0	-9/ 8	2	0
7	1	-54/35	-54/35	72/25	0	81/ 4	0	0	11	6	-13/70	-13/70	1/50	0	169/144	1	1
7	2	-27/35	-13/35	6/25	0	39/ 8	1	0	11	7	-27/35	22/35	6/25	6/ 5	-33/ 4	1	0
7	3	-13/35	-27/35	6/25	0	39/ 8	0	1	11	8	18/35	-4/35	-8/25	0	3/ 2	2	0
7	4	-156/35	54/35	-72/25	0	117/ 2	0	0	11	9	13/70	-11/35	-1/50	-1/10	143/ 72	1	1
7	5	-78/35	13/35	-6/25	0	169/ 12	1	0	11	10	-78/35	-22/35	-6/25	-6/ 5	-143/ 6	1	0
7	6	22/35	-27/35	6/25	6/ 5	-33/ 4	0	1	11	11	52/35	4/35	8/25	0	13/ 3	2	0
7	7	156/35	156/35	72/25	0	169/ 1	0	0	12	1	-22/35	27/35	-6/25	-6/ 5	33/ 4	0	1
8	1	27/35	13/35	-6/25	0	-39/ 8	1	0	12	2	-11/35	13/70	-1/50	-1/10	143/ 72	1	1
8	2	9/35	3/35	2/25	0	-9/ 8	2	0	12	3	-4/35	18/35	-8/25	0	3/ 2	0	2
8	3	13/70	13/70	-1/50	0	-169/144	1	1	12	4	-13/35	-27/35	6/25	0	39/ 8	0	1
8	4	78/35	-13/35	6/25	0	-169/ 12	1	0	12	5	-13/70	-13/70	1/50	0	169/144	1	1
8	5	26/35	-3/35	-2/25	0	-13/ 4	2	0	12	6	3/35	9/36	2/25	0	-9/ 8	0	2
8	6	-11/35	13/70	-1/50	-1/10	143/ 72	1	1	12	7	13/35	-78/35	-6/25	0	169/ 12	0	1
8	7	-78/35	-22/35	-6/25	-6/ 5	-143/ 6	1	0	12	8	-13/70	11/35	1/50	1/10	-143/ 72	1	1
8	8	52/35	4/35	8/25	0	13/ 3	2	0	12	9	-3/35	26/35	-2/25	0	-13/ 4	0	2
9	1	13/35	27/35	-6/25	0	-39/ 8	0	1	12	10	22/35	78/35	6/25	6/ 5	143/ 6	0	1
9	2	13/70	13/70	-1/50	0	-169/144	1	1	12	11	-11/35	-11/35	-1/50	-6/ 5	-121/ 36	1	1
9	3	3/35	9/35	2/25	0	-9/ 8	0	2	12	12	4/35	52/35	8/25	0	13/ 3	0	2

Table 2

i	j	$\gamma_{ij}^{(6)}$	$\gamma_{ij}^{(7)}$	$\gamma_{ij}^{(8)}$	$\gamma_{ij}^{(9)}$	$\gamma_{ij}^{(10)}$	λ_{ij}	ν_{ij}
1	1	3620/539	3620/539	200/49	0	32761/28	0	0
2	1	1810/539	622/539	30/49	10/7	56291/280	1	0
2	2	5792/2695	416/1617	32/49	0	4706/105	2	0
3	1	622/539	1810/539	30/49	10/7	56291/280	0	1
3	2	311/539	311/539	9/98	10/7	96721/2800	1	1
3	3	416/1617	5792/2695	32/49	0	4706/105	0	2
4	1	181/1078	281/3234	5/147	0	50861/3360	2	0
4	2	181/1470	23/1078	1/21	0	4163/1120	3	0
4	3	311/10780	281/6468	1/196	1/84	87391/33600	2	1
4	4	181/5390	1/539	2/441	0	181/560	4	0
5	1	311/539	311/539	9/98	3/7	96721/2800	1	1
5	2	4976/13475	208/1617	24/245	8/35	4043/525	2	1
5	3	208/1617	4976/13475	24/245	8/35	4043/525	1	2
5	4	311/14700	23/2156	1/140	1/60	7153/11200	3	1
5	5	3328/40425	3328/40425	128/1225	0	2704/1575	2	2
6	1	281/3234	181/1078	5/147	0	50861/3360	0	2
6	2	281/6468	311/10780	1/196	1/84	87391/33600	1	2
6	3	23/1078	181/1470	1/21	0	4163/1120	0	3
6	4	281/129360	281/129360	1/3528	0	78961/403200	2	2
6	5	23/2156	311/14700	1/140	1/60	7153/11200	1	3
6	6	1/539	181/5390	2/441	0	181/560	0	4
7	1	1000/539	-3620/539	-200/49	0	4525/14	0	0
7	2	500/539	-622/539	-30/49	-10/7	1555/28	1	0
7	3	302/539	-1810/539	-30/49	0	27331/280	0	1
7	4	25/539	-281/3234	-5/147	0	1405/336	2	0
7	5	151/539	-311/539	-9/98	-3/14	46961/2800	1	1
7	6	181/3234	-181/1078	-5/147	0	32761/3360	0	2
7	7	3620/539	3620/539	200/49	0	32761/28	0	0
8	1	500/539	-622/539	-30/49	-10/7	1555/28	1	0
8	2	320/539	-416/1617	-32/49	0	260/21	2	0
8	3	151/539	-311/539	-9/98	-3/14	46961/2800	1	1
8	4	5/147	-23/1078	-1/21	0	115/112	3	0
8	5	2416/13475	-208/1617	-24/245	0	1963/525	2	1
8	6	181/6468	-311/10780	-1/196	-1/84	56291/33600	1	2
8	7	1810/539	622/539	30/49	10/7	56291/280	1	0
8	8	5792/2695	416/1617	32/49	0	4706/105	2	0
9	1	-302/539	1810/539	30/49	0	-27331/280	0	1
9	2	-151/539	311/539	9/98	3/14	-46961/2800	1	1
9	3	-38/231	3258/2695	-2/49	0	-3439/120	0	2
9	4	-151/10780	281/6468	1/196	0	-42431/33600	2	1
9	5	-19/231	2799/13475	-3/490	-1/70	-5909/1200	1	2
9	6	-26/1617	362/8085	-2/147	0	-2353/840	0	3
9	7	-622/539	-1810/539	-30/49	-10/7	-56291/280	0	1
9	8	-311/539	-311/539	-9/98	-10/7	-96721/2800	1	1
9	9	416/1617	5792/2695	32/49	0	4706/105	0	2
10	1	25/539	-281/3234	-5/147	0	1405/336	2	0
10	2	5/147	-23/1078	-1/21	0	115/112	3	0
10	3	151/10780	-281/6468	-1/196	0	42431/33600	2	1
10	4	5/539	-1/539	-2/441	0	5/56	4	0
10	5	151/14700	-23/2156	-1/140	0	3473/11200	3	1
10	6	181/129360	-281/129360	-1/3528	0	50861/403200	2	2
10	7	181/1078	281/3234	5/147	0	50861/3360	2	0
10	8	181/1470	23/1078	1/21	0	4163/1120	3	0
10	9	-311/10780	-281/6468	-1/196	-1/84	-87391/33600	2	1

i	j	$\gamma_{ij}^{(6)}$	$\gamma_{ij}^{(7)}$	$\gamma_{ij}^{(8)}$	$\gamma_{ij}^{(9)}$	$\gamma_{ij}^{(10)}$	λ_{ij}	ν_{ij}
10	10	181/5390	1/539	2/441	0	181/560	4	0
11	1	-151/539	311/539	9/98	3/14	-46961/2800	1	1
11	2	-2416/13475	208/1617	24/245	0	-1963/525	2	1
11	3	-19/231	2799/13475	-3/490	-1/70	-5909/1200	1	2
11	4	-151/14700	23/2156	1/140	0	-3473/11200	3	1
11	5	-304/5775	624/13475	-8/1225	0	-247/225	2	2
11	6	-13/1617	311/40425	-1/490	-1/210	-4043/8400	1	3
11	7	-311/539	-311/539	-9/98	-3/7	-96721/2800	1	1
11	8	-4976/13475	-208/1617	-24/245	-8/35	-4043/525	2	1
11	9	208/1617	4976/13475	24/245	8/35	4043/525	1	2
11	10	-311/14700	-23/2156	-1/140	-1/60	-7153/11200	3	1
11	11	3328/40425	3328/40425	128/1225	0	2704/1575	2	2
12	1	181/3234	-181/1078	-5/147	0	32761/3360	0	2
12	2	181/6468	-311/10780	-1/196	-1/84	56291/33600	1	2
12	3	26/1617	-362/8085	2/147	0	2353/840	0	3
12	4	181/129360	-281/129360	-1/3528	0	50861/403200	2	2
12	5	13/1617	-311/40425	1/490	1/210	4043/8400	1	3
12	6	5/3234	181/32340	1/441	0	181/672	0	4
12	7	281/3234	181/1078	5/147	0	50861/3360	0	2
12	8	281/6468	311/10780	1/196	1/84	87391/33600	1	2
12	9	-23/1078	-181/1470	-1/21	0	-4163/1120	0	3
12	10	281/129360	281/129360	1/3528	0	78961/403200	2	2
12	11	-23/2156	-311/14700	-1/140	-1/60	-7153/11200	1	3
12	12	1/539	181/5390	2/441	0	181/560	0	4
13	1	-1000/539	-1000/539	200/49	0	625/7	0	0
13	2	-500/539	-302/539	30/49	0	755/28	1	0
13	3	-302/539	-500/539	30/49	0	755/28	0	1
13	4	-25/539	-181/3234	5/147	0	905/336	2	0
13	5	-151/539	-151/539	9/98	0	22801/2800	1	1
13	6	-181/3234	-25/539	5/147	0	905/336	0	2
13	7	-3620/539	1000/539	-200/49	0	4525/14	0	0
13	8	-1810/539	302/539	-30/49	0	27331/280	1	0
13	9	622/539	-500/539	30/49	10/7	-1555/28	0	1
13	10	-181/1078	181/3234	-5/147	0	32761/3360	2	0
13	11	311/539	-151/539	9/98	3/14	-46961/2800	1	1
13	12	-281/3234	25/539	-5/147	0	1405/336	0	2
13	13	3620/539	3620/539	200/49	0	32761/28	0	0
14	1	500/539	302/539	-30/49	0	-755/28	1	0
14	2	180/539	38/231	2/49	0	-95/12	2	0
14	3	151/539	151/539	-9/98	0	-22801/2800	1	1
14	4	20/1617	26/1617	2/147	0	-65/84	3	0
14	5	1359/13475	19/231	3/490	0	-2869/1200	2	1
14	6	181/6468	151/10780	-1/196	0	-27331/33600	1	2
14	7	1810/539	-302/539	30/49	0	-27331/280	1	0
14	8	3258/2695	-38/231	-2/49	0	-3439/120	2	0
14	9	-311/539	151/539	-9/98	-3/14	46961/2800	1	1
14	10	362/8085	-26/1617	-2/147	0	-2353/840	3	0
14	11	-2799/13475	19/231	3/490	1/70	5909/1200	2	1
14	12	281/6468	-151/10780	1/196	0	-42431/33600	1	2
14	13	-1810/539	-622/539	-30/49	-10/7	-56291/280	1	0
14	14	5792/2695	416/1617	32/49	0	4706/105	2	0
15	1	302/539	500/539	-30/49	0	-755/28	0	1
15	2	151/539	151/539	-9/98	0	-22801/2800	1	1
15	3	38/231	180/539	2/49	0	-95/12	0	2

TABLE 2 (Continued)

i	j	$\gamma_{ij}^{(6)}$	$\gamma_{ij}^{(7)}$	$\gamma_{ij}^{(8)}$	$\gamma_{ij}^{(9)}$	$\gamma_{ij}^{(10)}$	λ_{ij}	μ_{ij}
15	4	151/ 10780	181/ 6468	-1/ 196	0	-27331/ 33600	2	1
15	5	19/ 231	1359/ 13475	3/ 490	0	-2866/ 1200	1	2
15	6	26/ 1617	20/ 1817	2/ 147	0	-65/ 84	0	3
15	7	622/ 539	-500/ 539	39/ 49	10/ 7	-1556/ 28	0	1
15	8	311/ 539	-151/ 539	9/ 98	3/ 14	-46961/ 2800	1	1
15	9	-416/ 1617	320/ 539	-42/ 49	0	266/ 21	0	2
15	10	311/ 10780	-181/ 6468	1/ 196	1/ 84	-56291/ 33600	2	1
15	11	-208/ 1617	2416/ 13475	-94/ 245	0	1963/ 525	1	2
15	12	23/ 1078	-5/ 147	1/ 21	0	-115/ 112	0	3
15	13	-622/ 539	-1810/ 539	-40/ 49	-10/ 7	-56295/ 280	0	1
15	14	311/ 539	311/ 539	9/ 98	10/ 7	96725/ 2800	1	1
15	15	416/ 1617	5792/ 2695	32/ 49	0	4708/ 105	0	2
16	1	-25/ 539	-181/ 3234	6/ 147	0	905/ 336	2	0
16	2	-20/ 1617	-26/ 1617	-2/ 147	0	65/ 84	3	0
16	3	-151/ 10780	-181/ 6468	1/ 196	0	27331/ 33600	2	1
16	4	5/ 3534	-5/ 3234	-1/ 441	0	25/ 336	4	0
16	5	-151/ 40425	-13/ 1617	-1/ 490	0	1963/ 8400	3	1
16	6	-181/129360	-181/129360	1/3528	0	32761/403200	2	2
16	7	-181/ 1078	181/ 3234	-5/ 147	0	32761/ 3360	2	0
16	8	-362/ 8085	26/ 1617	2/ 147	0	2353/ 840	3	0
16	9	311/ 10780	-181/ 6468	1/ 196	1/ 84	-56291/ 33600	2	1
16	10	181/ 32340	5/ 3234	1/ 441	0	181/ 672	4	0
16	11	311/ 40425	-13/ 1617	-1/ 490	-1/210	-4047/ 5400	3	1
16	12	-281/129360	181/129360	-1/3528	0	50881/403200	2	2
16	13	181/ 1078	281/ 3234	5/ 147	0	50881/ 3360	2	0
16	14	-181/ 1470	-23/ 1078	-1/ 21	0	-4163/ 1120	3	0
16	15	-311/ 10780	-281/ 6468	-1/ 196	-1/ 84	-87391/ 33800	2	1
16	16	181/ 5390	1/ 539	2/ 441	0	181/ 460	4	0
17	1	-151/ 539	-151/ 539	9/ 98	0	22801/ 2800	1	1
17	2	-1359/ 13475	-19/ 231	-3/ 490	0	2868/ 1200	2	1
17	3	-19/ 231	-1359/ 13475	-3/ 490	0	2868/ 1200	1	2
17	4	-151/ 40425	-13/ 1617	-1/ 490	0	1963/ 8400	3	1
17	5	-57/ 1925	-57/ 1925	1/2450	0	2527/ 3800	2	2
17	6	-13/ 1617	-151/ 40425	-1/ 490	0	1963/ 8400	1	3
17	7	-311/ 539	151/ 539	-9/ 98	-3/ 14	46961/ 2800	1	1
17	8	-2794/ 13475	19/ 231	3/ 490	1/ 70	5909/ 1200	2	1
17	9	208/ 1617	-2416/ 13475	24/ 245	0	-1963/ 525	1	2
17	10	-311/ 40425	13/ 1617	1/ 480	1/210	4043/ 8400	3	1
17	11	624/ 13475	-304// 5775	-8/1225	0	-247/ 225	2	2
17	12	-23/ 2156	151/ 14700	-1/ 140	0	3473/ 11200	1	3
17	13	311/ 539	311/ 539	9/ 98	3/ 7	96721/ 2800	1	1
17	14	-4976/ 13475	-208/ 1617	-24/ 245	-8/ 35	-4043/ 525	2	1
17	15	-208/ 1817	-4976/ 13475	-24/ 245	-8/ 35	-4043/ 525	1	2
17	16	311/ 14700	23/ 2156	1/ 140	1/ 60	7153/ 11200	3	1
17	17	3328/ 40425	3328/ 40425	128/1225	0	2704/ 1575	2	2
18	1	-181/ 3234	-25/ 539	5/ 147	0	905/ 336	0	2
18	2	-181/ 6468	-151/ 10780	1/ 186	0	27331/ 33600	1	2
18	3	-28/ 1617	-20/ 1617	-2/ 147	0	65/ 84	0	3
18	4	-181/129360	-181/129360	1/3528	0	32761/403200	2	2
18	5	13/ 1617	-151/ 40425	-1/ 480	0	1963/ 8400	1	3
18	6	-8/ 3234	5/ 3234	-1/ 441	0	25/ 336	1	4
18	7	-281/ 3234	25/ 539	-5/ 147	0	1405/ 336	0	2
18	8	-281/ 6468	151/ 10780	-1/ 166	0	42431/ 33600	1	2
18	10	-281/129360	181/129360	-1/3528	0 0	50861 403200	2	2
18	10	23/ 2756	-151/ 14700	1/ 140	0	-3473 11200	1	3
18	12	-1/ 539	5/ 539	-2/ 441	0	5 56	0	4
18	13	281/ 3234	181/ 1078	5/ 147	0	50861 3360	0	2
18	14	-281/ 6468	-311/ 10780	-1/ 196	-1/84	-87391 33600	1	2
18	15	-23/ 1078	-181/ 1470	-1/ 51	0	-4163/ 1120	0	3
18	16	281/129360	281/129360	1/3928	0	78961 403200	2	2
18	17	23/ 2155	311/ 14700	1/ 140	1/40	7153 11200	1	3
18	18	1/ 539	181/ 5390	2/ 440	0	181/ 560	0	4
19	1	-3620/ 539	1000/ 539	-200/ 49	0	4525/ 14	0	0
19	2	-1810/ 539	302/ 539	-30/ 49	0	27331/ 280	1	0
19	3	-622/ 539	500/ 539	-30/ 49	-10/ 7	1555/ 28	0	1
19	4	-181/ 1078	181/ 3234	-5/ 147	0	32761/ 3360	2	0
19	5	-311/ 539	151/ 539	-9/ 68	-3/14	46961/ 2800	1	1
19	6	-281/ 3234	25/ 539	-5/ 147	0	1405/ 336	0	2
19	7	-1000/ 539	-1000/ 539	200/ 49	0	625/ 7	0	0
19	8	-500/ 539	-302/ 539	30/ 49	0	755/ 28	1	0
19	9	302/ 539	500/ 539	-30/ 49	0	-755/ 28	0	1
19	10	-28/ 539	-181/ 3234	5/ 147	0	905/ 336	2	0
19	11	151/ 539	151/ 539	-9/ 98	0	-22801/ 2800	1	1
19	12	-181/ 3234	-25/ 1539	5/ 147	0	905/ 336	0	2
19	13	1000/ 539	-3620/ 539	-200/ 49	0	4525/ 14	0	0
19	14	-500/ 539	622/ 539	30/ 49	10/ 7	-1555/ 28	1	0
19	15	-302/ 539	1810/ 539	30/ 49	0	-27331/ 280	0	1
19	16	25/ 539	-281/ 3234	-5/ 147	0	1405/ 336	2	0
19	17	151/ 539	-311/ 539	-9/ 98	-3/14	46169/ 2800	1	1
19	18	181/ 3234	-181/ 1078	-5/ 147	0	32761/ 3360	0	2
19	19	3620/ 539	3620/ 539	200/ 49	0	32761/ 28	0	0
20	1	1810/ 539	-302/ 539	30/ 49	0	-27331/ 280	1	0
20	2	3258/ 8695	-38/ 231	-2/ 49	0	-3439/ 120	2	0
20	3	311/ 539	-151/ 539	9/ 98	3/14	-46961/ 2800	1	1
20	4	362/ 8085	-26/ 1617	-2/ 147	0	-2353/ 840	3	0
20	5	2799/ 13475	-19/ 231	-3/ 490	-1/70	-5909/ 1200	2	1
20	6	281/ 6468	-151/ 10780	1/ 196	0	-42431/ 33600	1	2
20	7	500/ 539	302/ 539	-30/ 49	0	-755/ 28	1	0
20	8	180/ 539	38/ 231	2/ 49	0	-95/ 12	2	0
20	9	-151/ 539	-151/ 539	9/ 98	0	22801/ 2800	1	1
20	10	26/ 1617	26/ 1617	2/ 147	0	-65/ 84	3	0
20	11	-1359/ 13475	-19/ 231	-3/ 490	0	2869/ 1200	2	1
20	12	181/ 6468	151/ 10780	-1/ 196	0	-27331/ 33800	1	2
20	13	-500/ 539	622/ 539	30/ 49	10/ 7	-1555/ 28	1	0
20	14	320/ 539	-416/ 1617	-32/ 49	0	260/ 21	2	0
20	15	151/ 539	-311/ 539	-9/ 98	-3/14	46961/ 2800	1	1
20	16	-6/ 147	23/ 1078	-1/ 21	0	-115/ 112	3	0
20	17	-5416/ 13475	208/ 1617	24/ 245	0	-1963/ 525	2	1
20	18	-181/ 6468	311/ 10780	1/ 196	1/84	-56291/ 33600	1	2
20	19	-1810/ 530	-622/ 539	-30/ 49	-10/ 7	-56291/ 280	1	0
20	20	5792/ 2695	416/ 1617	32/ 49	0	4706/ 105	2	0
21	1	-622/ 539	500/ 539	-30/ 49	-10/ 7	1555/ 28	0	1
21	2	-311/ 539	151/ 539	-9/ 98	-3/14	46961/ 2800	1	1
21	3	-416/ 1617	320/ 539	-32/ 49	0	280/ 21	0	2
21	4	-311/ 10780	181/ 6468	-1/ 196	-1/84	56291/ 33600	2	1
21	5	-208/ 1617	2416/ 13475/	-24/ 245	0	1963 525	1	2

Table 2 (continued)

i	j	$\gamma_{ij}^{(6)}$	$\gamma_{ij}^{(7)}$	$\gamma_{ij}^{(8)}$	$\gamma_{ij}^{(9)}$	$\gamma_{ij}^{(10)}$	λ_{ij}	μ_{ij}
21	7	-302/ 539	-500/ 539	10/ 49	0	755/ 28	0	1
21	8	-151/ 539	-151/ 539	9/ 98	0	22801/ 2800	1	1
21	9	38/ 231	180/ 539	2/ 49	0	-95/ 12	0	2
21	10	-151/ 10780	-181/ 6468	1/ 196	0	27331/ 33600	2	1
21	11	19/ 231	1359/ 13475	3/ 400	0	-2869/ 1200	1	2
21	12	-26/ 1617	-20/ 1617	-2/ 147	0	65/ 84	0	3
21	13	302/ 539	-1810/ 539	-10/ 49	0	27331/ 280	0	1
21	14	-151/ 539	311/ 539	9/ 98	3/ 14	-46961/ 2800	1	1
21	15	-38/ 231	3259/ 2695	-2/ 49	0	-3439/ 120	0	2
21	16	151/ 10780	-281/ 6468	-1/ 196	0	42431/ 33600	2	1
21	17	19/ 231	-2799/ 13475	3/ 400	1/ 70	5909/ 1200	1	2
21	18	26/ 1617	-362/ 8085	2/ 147	0	2353/ 840	0	3
21	19	622/ 539	1810/ 539	10/ 49	10/ 7	56291/ 280	0	1
21	20	-311/ 539	-311/ 539	-9/ 98	-10/ 7	-96721/ 2800	1	1
21	21	416/ 1617	5792/ 2695	32/ 49	0	4706/ 105	0	2
22	1	-181/ 1078	181/ 3234	-5/ 147	0	32761/ 3360	2	0
22	2	-362/ 8085	26/ 1617	2/ 147	0	2353/ 840	3	0
22	3	-311/ 10780	181/ 6468	-1/ 196	-1/ 84	56291/ 33600	2	1
22	4	181/ 32340	5/ 3234	1/ 441	0	181/ 672	4	0
22	5	-311/ 40425	13/ 1617	1/ 400	1/210	4043/ 8400	3	1
22	6	-281/129360	181/129360	-1/3528	0	50861/403200	2	2
22	7	-25/ 539	-181/ 3234	5/ 147	0	905/ 336	2	0
22	8	-20/ 1617	-26/ 1617	-2/ 147	0	65/ 84	3	0
22	9	151/ 10780	181/ 6468	-1/ 196	0	-27331/ 33600	2	1
22	10	5/ 3234	-5/ 3234	-1/ 441	0	25/ 336	4	0
22	11	151/ 40425	13/ 1617	1/ 400	0	-1963/ 8400	3	1
22	12	-181/129360	-181/129360	1/3528	0	32761/403200	2	2
22	13	25/ 539	-281/ 3234	-5/ 147	0	1405/ 336	2	0
22	14	-5/ 147	23/ 1078	1/ 21	0	-115/ 112	3	0
22	15	-151/ 10780	281/ 6468	1/ 196	0	-42431/ 33600	2	1
22	16	5/ 539	-1/ 539	-2/ 441	0	5/ 56	4	0
22	17	151/ 14700	-23/ 2156	-1/ 140	0	3473/ 11200	3	1
22	18	181/129360	-281/129360	-1/3528	0	50861/403200	2	2
22	19	181/ 1078	281/ 3234	5/ 147	0	50861/ 3360	2	0
22	20	-181/ 1470	-23/ 1078	-1/ 21	0	-4163/ 1120	3	0
22	21	311/ 10780	281/ 6468	1/ 196	1/ 84	87391/ 33600	2	1
22	22	181/ 5390	1/ 539	2/ 441	0	181/ 560	4	0
23	1	311/ 539	-151/ 539	9/ 98	3/ 14	-46961/ 2800	1	1
23	2	2794/ 13475	-19/ 231	-3/ 400	-1/ 70	-5909/ 1200	2	1
23	3	208/ 1617	-2416/ 13475	24/ 245	0	-1963/ 525	1	2
23	4	311/ 40425	-13/ 1617	-1/ 400	-1/210	-4043/ 8400	3	1
23	5	624/ 13475	-304/ 5775	-8/1225	0	-247/ 225	2	2
23	6	23/ 2156	-151/ 14700	1/ 140	0	-3473/ 11200	1	3
23	7	151/ 539	151/ 539	-9/ 98	0	-22801/ 2800	1	1
23	8	1359/ 13475	19/ 231	3/ 400	0	-2869/ 1200	2	1
23	9	-19/ 231	-1359/ 13475	-3/ 400	0	2869/ 1200	1	2
23	10	151/ 40425	13/ 1617	1/ 400	0	-1963/ 8400	3	1
23	11	-57/ 1925	-57/ 1925	1/2450	0	2527/ 3600	2	2
23	12	13/ 1617	151/ 40425	1/ 400	0	-1963/ 8400	1	3
23	13	-151/ 539	311/ 539	9/ 98	3/ 14	-46961/ 2800	1	1
23	14	2416/ 13475	-208/ 1617	-24/ 245	0	1963/ 525	2	1
23	15	19/ 231	-2799/ 13475	3/ 400	1/ 70	5909/ 1200	1	2
23	16	-151/ 14700	23/ 2156	1/ 140	0	-3473/ 11200	3	1
23	17	-304/ 5775	624/ 13475	-8/1225	0	-247/ 225	2	2
23	18	-13/ 1617	311/ 40425	-1/ 400	-1/210	-4043/ 8400	1	3
23	19	-311/ 539	-311/ 539	-9/ 98	-3/ 7	-96721/ 2800	1	1
23	20	4976/ 13475	208/ 1617	24/ 245	8/ 35	4043/ 525	2	1
23	21	-208/ 1617	-4976/ 13475	-24/ 245	-8/ 35	-4043/ 525	1	2
23	22	-311/ 14700	-23/ 2156	-1/ 140	-1/ 60	-7153/ 11200	3	1
23	23	3328/ 40425	3328/ 40425	128/1225	0	2704/ 1575	2	2
24	1	-281/ 3234	25/ 539	-5/ 147	0	1405/ 336	0	2
24	2	-281/ 6468	151/ 10780	-1/ 196	0	42431/ 33600	1	2
24	3	-23/ 1078	5/ 147	-1/ 21	0	115/ 112	0	3
24	4	-281/129360	181/129360	-1/3528	0	50861/403200	2	2
24	5	-23/ 2156	151/ 14700	-1/ 140	0	3473/ 11200	1	3
24	6	-1/ 539	5/ 539	-2/ 441	0	5/ 56	0	4
24	7	-181/ 3234	-25/ 539	5/ 147	0	905/ 336	0	2
24	8	-181/ 6468	-151/ 10780	1/ 196	0	27331/ 33600	1	2
24	9	26/ 1617	20/ 1617	2/ 147	0	-65/ 84	0	3
24	10	-181/129360	-181/129360	1/3528	0	32761/403200	2	2
24	11	13/ 1617	151/ 40425	1/ 400	0	-1963/ 8400	1	3
24	12	-5/ 3234	5/ 3234	-1/ 441	0	25/ 336	0	4
24	13	181/ 3234	-181/ 1078	-5/ 147	0	32761/ 3360	0	2
24	14	-181/ 6468	311/ 10780	1/ 196	1/ 84	-56291/ 33600	1	2
24	15	-26/ 1617	362/ 8085	-2/ 147	0	-2353/ 840	0	3
24	16	181/129360	-281/129360	-1/3528	0	50861/403200	2	2
24	17	13/ 1617	-311/ 40425	1/ 400	1/210	4043/ 8400	1	3
24	18	5/ 3234	181/ 32340	1/ 441	0	181/ 672	0	4
24	19	281/ 3234	181/ 1078	5/ 147	0	50861/ 3360	0	2
24	20	-281/ 6468	-311/ 10780	-1/ 196	-1/ 84	-87391/ 33600	1	2
24	21	23/ 1078	181/ 1470	1/ 21	0	4163/ 1120	0	3
24	22	281/129360	281/129360	1/3528	0	78961/403200	2	2
24	23	-23/ 2156	-311/ 14700	-1/ 140	-1/ 60	-7153/ 11200	1	3
24	24	1/ 539	181/ 5390	2/ 441	0	181/ 560	0	4

Table 3

	Distributed Edge Load N			
	on x=0	on x=a	on y=0	on y=b
u_{11}	bN/2	0	0	0
v_{11}	0	0	bN/2	0
u_{12}	bN/2	0	0	0
v_{12}	0	0	0	bN/2
u_{22}	0	bN/2	0	0
v_{22}	0	0	0	bN/2
u_{21}	0	bN/2	0	0
v_{21}	0	0	bN/2	0

Table 4

	Conc. Load P at (a, b)	Conc. Mom. M_x at (a, b)	Uniform-ly Dist. load q	Dist. Edge Mom. M_x on x = a	Dist. Edge Load V			
					on x = 0	on x = a	on y = 0	on y = b
w_{11}	0	0	qab/4	0	bV/2	0	aV/2	0
w_{x11}	0	0	$qa^2b/24$	0	0	0	$a^2V/12$	0
w_{y11}	0	0	$qab^2/24$	0	$b^2V/12$	0	0	0
w_{12}	0	0	qab/4	0	bV/2	0	0	aV/2
w_{x12}	0	0	$qa^2b/24$	0	0	0	0	$a^2V/12$
w_{y12}	0	0	$-qab^2/24$	0	$-b^2V/12$	0	0	0
w_{22}	P	0	qab/4	0	0	bV/2	0	aV/2
w_{x22}	0	M_x	$-qa^2b/24$	$bM_x/2$	0	0	0	$-a^2V/12$
w_{y22}	0	0	$-qab^2/24$	0	0	$-b^2V/12$	0	0
w_{21}	0	0	qab/4	0	0	bV/2	aV/2	0
w_{x21}	0	0	$-qa^2b/24$	$bM_x/2$	0	0	$-a^2V/12$	0
w_{y21}	0	0	$qab^2/24$	0	0	$b^2V/12$	0	0

Table 5

	Conc. Load P at (a, b)	Conc. Mom. M_x at (a, b)	Uniform Load q	Dist. Edge Moment M on x = 0	on x = a	on y = 0	on y = b	Dist. Edge Load V on x = 0	on x = a	on y = 0	on y = b
w_{11}	0	0	$qab/4$	0	0	0	0	$bV/2$	0	$aV/2$	0
w_{x11}	0	0	$qa^2b/20$	$bM/2$	0	0	0	0	0	$a^2V/10$	0
w_{y11}	0	0	$qab^2/20$	0	0	$aM/2$	0	$b^2V/10$	0	0	0
w_{xx11}	0	0	$qa^3b/240$	0	0	0	0	0	0	$a^3V/120$	0
w_{xy11}	0	0	$qa^2b^2/100$	$b^2M/10$	0	$a^2M/10$	0	0	0	0	0
w_{yy11}	0	0	$qab^3/240$	0	0	0	0	$b^3V/120$	0	0	0
w_{12}	0	0	$qab/4$	0	0	0	0	$bV/2$	0	0	$aV/2$
w_{x12}	0	0	$qa^2b/20$	$bM/2$	0	0	0	0	0	0	$a^2V/10$
w_{y12}	0	0	$-qab^2/20$	0	0	0	$aM/2$	-b2V/10	0	0	0
w_{xx12}	0	0	$qa^3b/240$	0	0	0	0	0	0	0	$a^3V/120$
w_{xy12}	0	0	$-qa^2b^2/100$	$-b^2M/10$	0	0	$a^2M/10$	0	0	0	0
w_{yy12}	0	0	$qab^3/240$	0	0	0	0	$b^3V/120$	0	0	0
w_{22}	P	0	$qab/4$	0	0	0	0	0	$bV/2$	0	$aV/2$
w_{x22}	0	M_x	$-qa^2b/20$	0	$bM/2$	0	0	0	0	0	$-a^2V/10$
w_{y22}	0	0	$-qab^2/20$	0	0	0	$aM/2$	0	$-b^2V/10$	0	0
w_{xx22}	0	0	$qa^3b/240$	0	0	0	0	0	0	0	$a^3V/120$
w_{xy22}	0	0	$qa^2b^2/100$	0	$-b^2M/10$	0	$-a^2M/10$	0	0	0	0
w_{yy22}	0	0	$qab^3/240$	0	0	0	0	0	$b^3V/120$	0	0
w_{21}	0	0	$qab/4$	0	0	0	0	0	$bV/2$	$aV/2$	0
w_{x21}	0	0	$-qa^2b/20$	0	$bM/2$	0	0	0	0	$-a^2V/10$	0
w_{y21}	0	0	$qab^2/20$	0	0	$aM/2$	0	0	$b^2V/10$	0	0
w_{xx21}	0	0	$qa^3b/240$	0	0	0	0	0	0	$a^3V/120$	0
w_{xy21}	0	0	$-qa^2b^2/100$	0	$b^2M/10$	$-a^2M/10$	0	0	0	0	0
w_{yy21}	0	0	$qab^3/240$	0	0	0	0	0	b V/120	0	0

Table 6

i	j	$\tilde{\gamma}^{(1)}_{ij}$	$\tilde{\gamma}^{(2)}_{ij}$	$\tilde{\gamma}^{(3)}_{ij}$	$\tilde{\gamma}^{(4)}_{ij}$	$\tilde{\gamma}^{(5)}_{ij}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$	i	j	$\tilde{\gamma}^{(1)}_{ij}$	$\tilde{\gamma}^{(2)}_{ij}$	$\tilde{\gamma}^{(3)}_{ij}$	$\tilde{\gamma}^{(4)}_{ij}$	$\tilde{\gamma}^{(5)}_{ij}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
1	1	156/ 35	156/ 35	72/ 25	-0/ 1	169/ 1	0	0	8	7	2/ 35	22/105	2/ 75	2/ 15	11/ 18	1	2
2	1	78/ 35	22/ 35	6/ 25	6/ 5	143/ 6	1	0	8	8	4/105	4/105	8/225	-0/ 1	1/ 9	2	2
2	2	52/ 35	4/ 35	8/ 25	-0/ 1	13/ 3	2	0	9	1	-54/ 35	-54/ 35	72/ 25	-0/ 1	81/ 4	0	0
3	1	22/ 35	78/ 35	6/ 25	6/ 5	143/ 6	0	1	9	2	-27/ 35	-13/ 35	6/ 25	-0/ 1	39/ 8	1	0
3	2	11/ 35	11/ 35	1/ 50	6/ 5	121/ 36	1	1	9	3	-13/ 35	-27/ 35	6/ 25	-0/ 1	39/ 8	0	1
3	3	4/ 35	52/ 35	8/ 25	-0/ 1	13/ 3	0	2	9	4	-13/ 70	-13/ 70	1/ 50	-0/ 1	169/144	1	1
4	1	11/ 35	11/ 35	1/ 50	1/ 5	121/ 36	1	1	9	5	-156/ 35	54/ 35	-72/ 25	-0/ 1	117/ 2	0	0
4	2	22/105	2/ 35	2/ 75	2/ 15	11/ 18	2	1	9	6	-78/ 35	13/ 35	-6/ 25	-0/ 1	169/ 12	1	0
4	3	2/ 35	22/105	2/ 75	2/ 15	11/ 18	1	2	9	7	22/ 35	-27/ 35	6/ 25	6/ 5	-33/ 4	0	1
4	4	4/105	4/105	8/225	-0/ 1	1/ 9	2	2	9	8	11/ 35	-13/ 70	1/ 50	1/ 10	-143/ 72	1	1
5	1	54/ 35	-156/ 35	-72/ 25	-0/ 1	117/ 2	0	0	9	9	156/ 35	156/ 35	72/ 25	-0/ 1	169/ 1	0	0
5	2	27/ 35	-22/ 35	-6/ 25	-6/ 5	33/ 4	1	0	10	1	27/ 35	13/ 35	-6/ 25	-0/ 1	-39/ 8	1	0
5	3	13/ 35	-78/ 35	-6/ 25	-0/ 1	169/ 12	0	1	10	2	9/ 35	3/ 35	2/ 25	-0/ 1	-9/ 8	2	0
5	4	13/ 70	-11/ 35	-1/ 50	-1/ 10	143/ 72	1	1	10	3	13/ 70	13/ 70	-1/ 50	-0/ 1	-169/144	1	1
5	5	156/ 35	156/ 35	72/ 25	-0/ 1	169/ 1	0	0	10	4	13/210	3/ 70	1/150	-0/ 1	-13/ 48	2	1
6	1	27/ 35	-22/ 35	-6/ 25	-6/ 5	33/ 4	1	0	10	5	78/ 35	-13/ 35	6/ 25	-0/ 1	-169/ 12	1	0
6	2	18/ 35	-4/ 35	-8/ 25	-0/ 1	3/ 2	2	0	10	6	26/ 35	-3/ 35	-2/ 25	-0/ 1	-13/ 4	2	0
6	3	13/ 70	-11/ 35	-1/ 50	-1/ 10	143/ 72	1	1	10	7	-11/ 35	13/ 70	-1/ 50	-1/ 10	143/ 72	1	1
6	4	13/105	-2/ 35	-2/ 75	-0/ 1	13/ 36	2	1	10	8	-11/105	3/ 70	1/150	1/ 30	11/ 24	2	1
6	5	78/ 35	22/ 35	6/ 25	6/ 5	143/ 6	1	0	10	9	-78/ 35	-22/ 35	-6/ 25	-6/ 5	-143/ 6	1	0
6	6	52/ 35	4/ 35	8/ 25	-0/ 1	13/ 3	2	0	10	10	52/ 35	4/ 35	8/ 25	-0/ 1	13/ 3	2	0
7	1	-13/ 35	78/ 35	6/ 25	-0/ 1	-169/ 12	0	1	11	1	13/ 35	27/ 35	-6/ 25	-0/ 1	-39/ 8	0	1
7	2	-13/ 70	11/ 35	1/ 50	1/ 10	-143/ 72	1	1	11	2	13/ 70	13/ 70	-1/ 50	-0/ 1	-169/144	1	1
7	3	-3/ 35	26/ 35	-2/ 25	-0/ 1	-13/ 4	0	2	11	3	3/ 35	9/ 35	2/ 25	-0/ 1	-9/ 8	0	2
7	4	-3/ 70	11/105	-1/150	-1/ 30	-11/ 24	1	2	11	4	3/ 70	13/210	1/150	-0/ 1	-13/ 48	1	2
7	5	-22/ 35	-78/ 35	-6/ 25	-6/ 5	-143/ 6	0	1	11	5	22/ 35	-27/ 35	6/ 25	6/ 5	-33/ 4	0	1
7	6	-11/ 35	-11/ 35	-1/ 50	-6/ 5	-121/ 36	1	1	11	6	11/ 35	-13/ 70	1/ 50	1/ 10	-143/ 72	1	1
7	7	4/ 35	52/ 35	8/ 25	-0/ 1	13/ 3	0	2	11	7	-4/ 35	18/ 35	-8/ 25	-0/ 1	3/ 2	0	2
8	1	-13/ 70	11/ 35	1/ 50	1/ 10	-143/ 72	1	1	11	8	-2/ 35	13/105	-2/ 75	-0/ 1	13/ 36	1	2
8	2	-13/105	2/ 35	2/ 75	-0/ 1	-13/ 36	2	1	11	9	-22/ 35	-78/ 35	-6/ 25	-6/ 5	-143/ 6	0	1
8	3	-3/ 70	11/105	-1/150	-1/ 30	-11/ 24	1	2	11	10	11/ 35	11/ 35	1/ 50	6/ 5	121/ 36	1	1
8	4	-1/ 35	2/105	-2/225	-0/ 1	-1/ 12	2	2	11	11	4/ 35	52/ 35	8/ 25	-0/ 1	13/ 3	0	2
8	5	-11/ 35	-11/ 35	-1/ 50	-1/ 5	-121/ 36	1	1	12	1	-13/ 70	-13/ 70	1/ 50	-0/ 1	169/144	1	1
8	6	-22/105	-2/ 35	-2/ 75	-2/ 15	-11/ 18	2	1	12	2	-13/210	-3/ 70	-1/150	-0/ 1	13/ 48	2	1

Table 6 (continued)

i	j	$\tilde{\gamma}_{ij}^{(1)}$	$\tilde{\gamma}_{ij}^{(2)}$	$\tilde{\gamma}_{ij}^{(3)}$	$\tilde{\gamma}_{ij}^{(4)}$	$\tilde{\gamma}_{ij}^{(5)}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$	i	j	$\tilde{\gamma}_{ij}^{(1)}$	$\tilde{\gamma}_{ij}^{(2)}$	$\tilde{\gamma}_{ij}^{(3)}$	$\tilde{\gamma}_{ij}^{(4)}$	$\tilde{\gamma}_{ij}^{(5)}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
12	3	-3/ 70	-13/210	-1/150	-0/ 1	13/ 48	1	2	14	12	-13/105	2/ 35	2/ 75	-0/ 1	-13/ 36	2	1
12	4	-1/ 70	-1/ 70	1/450	-0/ 1	1/ 16	2	2	14	13	-78/ 35	-22/ 35	-6/ 25	-6/ 5	-143/ 6	1	0
12	5	-11/ 35	13/ 70	-1/ 50	-1/ 10	143/ 72	1	1	14	14	52/ 35	4/ 35	8/ 25	-0/ 1	13/ 3	2	0
12	6	-11/105	3/ 70	1/150	1/ 30	11/ 24	2	1	15	1	-22/ 35	27/ 35	-6/ 25	-6/ 5	33/ 4	0	1
12	7	2/ 35	-13/105	2/ 75	-0/ 1	-13/ 36	1	2	15	2	-11/ 35	13/ 70	-1/ 50	-1/ 10	143/ 72	1	1
12	8	2/105	-1/ 35	-2/225	-0/ 1	-1/ 12	2	2	15	3	-4/ 35	18/ 35	-8/ 25	-0/ 1	3/ 2	0	2
12	9	11/ 35	11/ 35	1/ 50	1/ 5	121/ 36	1	1	15	4	-2/ 35	13/105	-2/ 75	-0/ 1	13/ 36	1	2
12	10	-22/105	-2/ 35	-2/ 75	-2/ 15	-11/ 18	2	1	15	5	-13/ 35	-27/ 35	6/ 25	-0/ 1	39/ 8	0	1
12	11	-2/ 35	-22/105	-2/ 75	-2/ 15	-11/ 18	1	2	15	6	-13/ 70	-13/ 70	1/ 50	-0/ 1	169/144	1	1
12	12	4/105	4/105	8/225	-0/ 1	1/ 9	2	2	15	7	3/ 35	9/ 35	2/ 25	-0/ 1	-9/ 8	0	2
13	1	-156/ 35	54/ 35	-72/ 25	-0/ 1	117/ 2	0	0	15	8	3/ 70	13/210	1/150	-0/ 1	-13/ 48	1	2
13	2	-78/ 35	13/ 35	-6/ 25	-0/ 1	169/ 12	1	0	15	9	13/ 35	-78/ 35	-6/ 25	-0/ 1	169/ 12	0	1
13	3	-22/ 35	27/ 35	-6/ 25	-6/ 5	33/ 4	0	1	15	10	-13/ 70	11/ 35	1/ 50	1/ 10	-143/ 72	1	1
13	4	-11/ 35	13/ 70	-1/ 50	-1/ 10	143/ 72	1	1	15	11	-3/ 35	26/ 35	-2/ 25	-0/ 1	-13/ 4	0	2
13	5	-54/ 35	-54/ 35	72/ 25	-0/ 1	81/ 4	0	0	15	12	3/ 70	-11/105	1/150	1/ 30	11/ 24	1	2
13	6	-27/ 35	-13/ 35	6/ 25	-0/ 1	39/ 8	1	0	15	13	22/ 35	78/ 35	6/ 25	6/ 5	143/ 6	0	1
13	7	13/ 35	27/ 35	-6/ 25	-0/ 1	-39/ 8	0	1	15	14	-11/ 35	-11/ 35	-1/ 50	-6/ 5	-121/ 36	1	1
13	8	13/ 70	13/ 70	-1/ 50	-0/ 1	-169/144	1	1	15	15	4/ 35	52/ 35	8/ 25	-0/ 1	13/ 3	0	2
13	9	54/ 35	-156/ 35	-72/ 25	-0/ 1	117/ 2	0	0	16	1	11/ 35	-13/ 70	1/ 50	1/ 10	-143/ 72	1	1
13	10	-27/ 35	22/ 35	6/ 25	6/ 5	-33/ 4	1	0	16	2	11/105	-3/ 70	-1/150	-1/ 30	-11/ 24	2	1
13	11	-13/ 35	78/ 35	6/ 25	-0/ 1	-169/ 12	0	1	16	3	2/ 35	-13/105	2/ 75	-0/ 1	-13/ 36	1	2
13	12	13/ 70	-11/ 35	-1/ 50	-1/ 10	143/ 72	1	1	16	4	2/105	-1/ 35	-2/225	-0/ 1	-1/ 12	2	2
13	13	156/ 35	156/ 35	72/ 25	-0/ 1	169/ 1	0	0	16	5	13/ 70	13/ 70	-1/ 50	-0/ 1	-169/144	1	1
14	1	78/ 35	-13/ 35	6/ 25	-0/ 1	-169/ 12	1	0	16	6	13/210	3/ 70	1/150	-0/ 1	-13/ 48	2	1
14	2	26/ 35	-3/ 35	-2/ 25	-0/ 1	-13/ 4	2	0	16	7	-3/ 70	-13/210	-1/150	-0/ 1	13/ 48	1	2
14	3	11/ 35	-13/ 70	1/ 50	1/ 10	-143/ 72	1	1	16	8	-1/ 70	-1/ 70	1/450	-0/ 1	1/ 16	2	2
14	4	11/105	-3/ 70	-1/150	-1/ 30	-11/ 24	2	1	16	9	-13/ 70	11/ 35	1/ 50	1/ 10	-143/ 72	1	1
14	5	27/ 35	13/ 35	-6/ 25	-0/ 1	-39/ 8	1	0	16	10	13/105	-2/ 35	-2/ 75	-0/ 1	13/ 36	2	1
14	6	9/ 35	3/ 35	2/ 25	-0/ 1	-9/ 8	2	0	16	11	3/ 70	-11/105	1/150	1/ 30	11/ 24	1	2
14	7	-13/ 70	-13/ 70	1/ 50	-0/ 1	169/144	1	1	16	12	-1/ 35	2/105	-2/225	-0/ 1	-1/ 12	2	2
14	8	-13/210	-3/ 70	-1/150	-0/ 1	13/ 48	2	1	16	13	-11/ 35	-11/ 35	-1/ 50	-1/ 5	-121/ 36	1	1
14	9	-27/ 35	22/ 35	6/ 25	6/ 5	-33/ 4	1	0	16	14	22/105	2/ 35	2/ 75	2/ 15	11/ 18	2	1
14	10	18/ 35	-4/ 35	-8/ 25	-0/ 1	3/ 2	2	0	16	15	-2/ 35	-22/105	-2/ 75	-2/ 15	-11/ 18	1	2
14	11	13/ 70	-11/ 35	-1/ 50	-1/ 10	143/ 72	1	1	16	16	4/105	4/105	8/225	-0/ 1	1/ 9	2	2

Table 7

	Conc. Load P at(a,b)	Conc. Mom. M_x at(a,b)	Uniformly Dist. load q	Dist. Edge Mom. M_x on x = a	Dist. Edge Load V on x = 0	on x = a	on y = 0	on y = b
w_{11}	0	0	$qab/4$	0	$bV/2$	0	$aV/2$	0
w_{x11}	0	0	$qa^2b/24$	0	0	0	$a^2V/12$	0
w_{y11}	0	0	$qab^2/24$	0	$b^2V/12$	0	0	0
w_{xy11}	0	0	$qa^2b^2/144$	0	0	0	0	0
w_{12}	0	0	$qab/4$	0	$bV/2$	0	0	$aV/2$
w_{x12}	0	0	$qa^2b/24$	0	0	0	0	$a^2V/12$
w_{y12}	0	0	$-qab^2/24$	0	$-b^2V/12$	0	0	0
w_{xy12}	0	0	$-qa^2b^2/144$	0	0	0	0	0
w_{22}	P	0	$qab/4$	0	0	$bV/2$	0	$aV/2$
w_{x22}	0	M_x	$-qa^2b/24$	$bM_x/2$	0	0	0	$-a^2V/12$
w_{y22}	0	0	$-qab^2/24$	0	0	$-b^2V/12$	0	0
w_{xy22}	0	0	$qa^2b^2/144$	$-b^2M_x/12$	0	0	0	0
w_{21}	0	0	$qab/4$	0	0	$bV/2$	$aV/2$	0
w_{x21}	0	0	$-qa^2b/24$	$bM_x/2$	0	0	$-a^2V/12$	0
w_{y21}	0	0	$qab^2/24$	0	0	$b^2V/12$	0	0
w_{xy21}	0	0	$-qa^2b^2/144$	$b^2M_x/12$	0	0	0	0

Table 8

i	j	$\tilde{\gamma}_{ij}^{(6)}$	$\tilde{\gamma}_{ij}^{(7)}$	$\tilde{\gamma}_{ij}^{(8)}$	$\tilde{\gamma}_{ij}^{(9)}$	$\tilde{\gamma}_{ij}^{(10)}$	$\tilde{\lambda}_{ij}$	$\tilde{\nu}_{ij}$
1	1	3620/ 539	3620/ 539	200/ 49	0/ 1	32761/ 196	0	0
2	1	1810/ 539	622/ 539	30/ 49	10/ 7	56291/ 1960	1	0
2	2	5792/ 2695	416/ 1617	32/ 49	0/ 1	4706/ 735	2	0
3	1	622/ 539	1810/ 539	30/ 49	10/ 7	56291/ 1960	0	1
3	2	311/ 539	311/ 539	9/ 98	10/ 7	96721/ 19600	1	1
3	3	416/ 1617	5792/ 2695	32/ 49	0/ 1	4706/ 735	0	2
4	1	311/ 539	311/ 539	9/ 98	3/ 7	96721/ 19600	1	1
4	2	4976/ 13475	208/ 1617	24/ 245	8/ 35	4043/ 3675	2	1
4	3	208/ 1617	4976/ 13475	24/ 245	8/ 35	4043/ 3675	1	2
4	4	3328/ 40425	3328/ 40425	128/ 1225	0/ 1	2704/ 11025	2	2
5	1	181/ 1078	281/ 3234	5/ 147	0/ 1	50861/ 23520	2	0
5	2	181/ 1470	23/ 1078	1/ 21	0/ 1	4163/ 7840	3	0
5	3	311/ 10780	281/ 6468	1/ 196	1/ 84	87391/ 235200	2	1
5	4	311/ 14700	23/ 2156	1/ 140	1/ 60	7153/ 78400	3	1
5	5	181/ 5390	1/ 539	2/ 441	0/ 1	181/ 3920	4	0
6	1	281/ 3234	181/ 1078	5/ 147	0/ 1	50861/ 23520	0	2
6	2	281/ 6468	311/ 10780	1/ 196	1/ 84	87391/ 235200	1	2
6	3	23/ 1078	181/ 1470	1/ 21	0/ 1	4163/ 7840	0	3
6	4	23/ 2156	311/ 14700	1/ 140	1/ 60	7153/ 78400	1	3
6	5	281/ 129360	281/ 129360	1/ 3528	0/ 1	78961/2822400	2	2
6	6	1/ 539	181/ 5390	2/ 441	0/ 1	181/ 3920	0	4
7	1	311/ 10780	281/ 6468	1/ 196	1/ 84	87391/ 235200	2	1
7	2	311/ 14700	23/ 2156	1/ 140	1/ 60	7153/ 78400	3	1
7	3	52/ 8085	1124/ 40425	4/ 735	0/ 1	3653/ 44100	2	2
7	4	52/ 11025	92/ 13475	4/ 525	0/ 1	299/ 14700	3	2
7	5	311/ 53900	1/ 1078	1/ 1470	1/ 630	311/ 39200	4	1
7	6	23/ 43120	281/ 176400	1/ 2520	0/ 1	6463/ 940800	2	3
7	7	52/ 40425	4/ 13475	8/ 11025	0/ 1	13/ 7350	4	2
8	1	281/ 6468	311/ 10780	1/ 196	1/ 84	87391/ 235200	1	2
8	2	1124/ 40425	52/ 8085	4/ 735	0/ 1	3653/ 44100	2	2
8	3	23/ 2156	311/ 14700	1/ 140	1/ 60	7153/ 78400	1	3
8	4	92/ 13475	52/ 11025	4/ 525	0/ 1	299/ 14700	2	3
8	5	281/ 176400	23/ 43120	1/ 2520	0/ 1	6463/ 940800	3	2
8	6	1/ 1078	311/ 53900	1/ 1470	1/ 630	311/ 39200	1	4
8	7	23/ 58800	23/ 58800	1/ 1800	0/ 1	529/ 313600	3	3
8	8	8/ 13475	52/ 40425	8/ 11025	0/ 1	13/ 7350	2	4
9	1	281/ 129360	281/ 129360	1/ 3528	0/ 1	78961/2822400	2	2
9	2	281/ 176400	23/ 43120	1/ 2520	0/ 1	6463/ 940800	3	2
9	3	23/ 43120	281/ 176400	1/ 2520	0/ 1	6463/ 940800	2	3
9	4	23/ 58800	23/ 58800	1/ 1800	0/ 1	529/ 313600	3	3
9	5	281/ 646800	1/ 21560	1/ 26460	0/ 1	281/ 470400	4	2
9	6	1/ 21560	281/ 646800	1/ 26460	0/ 1	281/ 470400	2	4
9	7	23/ 215600	1/ 29400	1/ 18900	0/ 1	23/ 156800	4	3
9	8	1/ 29400	23/ 215600	1/ 18900	0/ 1	23/ 156800	3	4
9	9	1/ 107800	1/ 107800	1/198450	0/ 1	1/ 78400	4	4
10	1	1000/ 539	-3620/ 539	-200/ 49	0/ 1	4525/ 98	0	0
10	2	500/ 539	-622/ 539	-30/ 49	-10/ 7	1555/ 196	1	0
10	3	402/ 539	-1810/ 539	-30/ 49	0/ 1	27331/ 1960	0	1
10	4	151/ 539	-311/ 539	-9/ 98	-3/ 14	46961/ 19600	1	1
10	5	25/ 539	-281/ 3234	-5/ 147	0/ 1	1405/ 2352	2	0
10	6	181/ 3234	-181/ 1078	-5/ 147	0/ 1	32761/ 23520	0	2
10	7	151/ 10780	-281/ 6468	-1/ 196	0/ 1	42431/ 235200	2	1
10	8	181/ 6468	-311/ 10780	-1/ 196	-1/ 84	56291/ 235200	1	2
10	9	181/ 129360	-281/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
10	10	3620/ 539	3620/ 539	200/ 49	0/ 1	32761/ 196	0	0
11	1	400/ 539	-622/ 539	-30/ 49	-10/ 7	1555/ 196	1	0
11	2	320/ 539	-416/ 1617	-32/ 49	0/ 1	260/ 147	2	0
11	3	151/ 539	-311/ 539	-9/ 98	-3/ 14	46961/ 19600	1	1
11	4	2416/ 13475	-208/ 1617	-24/ 245	0/ 1	1963/ 3675	2	1
11	5	5/ 147	-23/ 1078	-1/ 21	0/ 1	115/ 784	3	0
11	6	181/ 6468	-311/ 10780	-1/ 196	-1/ 84	56291/ 235200	1	2
11	7	151/ 14700	-23/ 2156	-1/ 140	0/ 1	3473/ 78400	3	1
11	8	724/ 40425	-52/ 8085	-4/ 735	0/ 1	2353/ 44100	2	2
11	9	181/ 176400	-23/ 43120	-1/ 2520	0/ 1	4163/ 940800	3	2
11	10	1810/ 539	622/ 539	30/ 49	10/ 7	56291/ 1960	1	0
11	11	5792/ 2695	416/ 1617	32/ 49	0/ 1	4706/ 735	2	0
12	1	-302/ 539	1810/ 539	30/ 49	0/ 1	-27331/ 1960	0	1
12	2	-151/ 539	311/ 539	9/ 98	3/ 14	-46961/ 19600	1	1
12	3	-38/ 231	3256/ 2695	-2/ 49	0/ 1	-3439/ 840	0	2
12	4	-19/ 231	2799/ 13475	-3/ 490	-1/ 70	-5909/ 8400	1	2
12	5	-151/ 10780	281/ 6468	1/ 196	0/ 1	-42431/ 235200	2	1
12	6	-26/ 1617	362/ 8085	-2/ 147	0/ 1	-2353/ 5880	0	3
12	7	-19/ 4620	843/ 53900	-1/ 2940	0/ 1	-5339/ 100800	2	2
12	8	-13/ 1617	311/ 40425	-1/ 490	-1/ 210	-4043/ 58800	1	3
12	9	-13/ 32340	281/ 485100	-1/ 8820	0/ 1	-3653/ 705600	2	3
12	10	-622/ 539	-1810/ 539	-30/ 49	-10/ 7	-56291/ 1960	0	1
12	11	-311/ 539	-311/ 539	-9/ 98	-10/ 7	-96721/ 19600	1	1
12	12	416/ 1617	5792/ 2695	32/ 49	0/ 1	4706/ 735	0	2
13	1	-151/ 539	311/ 539	9/ 98	3/ 14	-46961/ 19600	1	1
13	2	-2416/ 13475	208/ 1617	24/ 245	0/ 1	-1963/ 3675	2	1
13	3	-19/ 231	2799/ 13475	-3/ 490	-1/ 70	-5909/ 8400	1	2
13	4	-304/ 5775	624/ 13475	-8/ 1225	0/ 1	-247/ 1575	2	2
13	5	-151/ 14700	23/ 2156	1/ 140	0/ 1	-3473/ 78400	3	1
13	6	-13/ 1617	311/ 40425	-1/ 490	-1/ 210	-4043/ 58800	1	3
13	7	-19/ 6300	207/ 53900	-1/ 2100	0/ 1	-437/ 33600	3	2
13	8	-208/ 40425	208/ 121275	-8/ 3675	0/ 1	-169/ 11025	2	3
13	9	-13/ 44100	23/ 161700	-1/ 6300	0/ 1	-299/ 235200	3	3
13	10	-311/ 539	-311/ 539	-9/ 98	-3/ 7	-96721/ 19600	1	1
13	11	-4976/ 13475	-208/ 1617	-24/ 245	-8/ 35	-4043/ 3675	2	1
13	12	208/ 1617	4976/ 13475	24/ 245	8/ 35	4043/ 3675	1	2
13	13	3328/ 40425	3328/ 40425	128/ 1225	0/ 1	2704/ 11025	2	2
14	1	25/ 539	-281/ 3234	-5/ 147	0/ 1	1405/ 2352	2	0
14	2	5/ 147	-23/ 1078	-1/ 21	0/ 1	115/ 784	3	0
14	3	151/ 10780	-281/ 6468	-1/ 196	0/ 1	42431/ 235200	2	1
14	4	151/ 14700	-23/ 2156	-1/ 140	0/ 1	3473/ 78400	3	1
14	5	5/ 539	-1/ 539	-2/ 441	0/ 1	5/ 392	4	0
14	6	181/ 129360	-281/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
14	7	151/ 53900	-1/ 1078	-1/ 1470	0/ 1	151/ 39200	4	1
14	8	181/ 176400	-23/ 43120	-1/ 2520	0/ 1	4163/ 940800	3	2
14	9	181/ 646800	-1/ 21560	-1/ 26460	0/ 1	181/ 470400	4	2
14	10	181/ 1078	281/ 3234	5/ 147	0/ 1	50861/ 23520	2	0
14	11	181/ 1470	23/ 1078	1/ 21	0/ 1	4163/ 7840	3	0
14	12	-311/ 10780	-281/ 6468	-1/ 196	-1/ 84	-87391/ 235200	2	1
14	13	-311/ 14700	-23/ 2156	-1/ 140	-1/ 60	-7153/ 78400	3	1
14	14	181/ 5390	1/ 539	2/ 441	0/ 1	181/ 3920	4	0
15	1	181/ 3234	-181/ 1078	-5/ 147	0/ 1	32761/ 23520	0	2
15	2	181/ 6468	-311/ 10780	-1/ 196	-1/ 84	56291/ 235200	1	2
15	3	26/ 1617	-362/ 8085	2/ 147	0/ 1	2353/ 5880	0	3
15	4	13/ 1617	-311/ 40425	1/ 490	1/ 210	4043/ 58800	1	3
15	5	181/ 129360	-281/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2

Table 8 (continued)

i	j	$\tilde{\gamma}_{ij}^{(6)}$	$\tilde{\gamma}_{ij}^{(7)}$	$\tilde{\gamma}_{ij}^{(8)}$	$\tilde{\gamma}_{ij}^{(9)}$	$\tilde{\gamma}_{ij}^{(10)}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
15	6	5/ 3234	181/ 32340	1/ 441	0/ 1	181/ 4704	0	4
15	7	13/ 32340	-281/ 485100	1/ 8820	0/ 1	3653/ 705600	2	3
15	8	5/ 6468	311/ 323400	1/ 2940	1/ 1260	311/ 47040	1	4
15	9	1/ 25872	281/ 3880800	1/ 52920	0/ 1	281/ 564480	2	4
15	10	281/ 3234	181/ 1078	5/ 147	0/ 1	50861/ 23520	0	2
15	11	281/ 6468	311/ 10780	1/ 196	1/ 84	87391/ 235200	1	2
15	12	-23/ 1078	-181/ 1470	-1/ 21	0/ 1	-4163/ 7840	0	3
15	13	-23/ 2156	-311/ 14700	-1/ 140	-1/ 60	-7153/ 78400	1	3
15	14	281/ 129360	281/ 129360	1/ 3528	0/ 1	78961/ 2822400	2	2
15	15	1/ 539	181/ 5390	2/ 441	0/ 1	181/ 3920	0	4
16	1	-151/ 10780	281/ 6468	1/ 196	0/ 1	-42431/ 235200	2	1
16	2	-151/ 14700	23/ 2156	1/ 140	0/ 1	-3473/ 78400	3	1
16	3	-19/ 4620	844/ 53900	-1/ 2940	0/ 1	-5339/ 100800	2	2
16	4	-19/ 6300	207/ 53900	-1/ 2100	0/ 1	-437/ 33600	3	2
16	5	-151/ 53900	1/ 1078	1/ 1470	0/ 1	-151/ 39200	4	1
16	6	-13/ 32340	281/ 485100	-1/ 8820	0/ 1	-3653/ 705600	2	3
16	7	-19/ 23100	9/ 26950	-1/ 22050	0/ 1	-19/ 16800	4	2
16	8	-13/ 44100	23/ 161700	-1/ 6300	0/ 1	-299/ 235200	3	3
16	9	-13/ 161700	1/ 30850	-1/ 66150	0/ 1	-13/ 117600	4	3
16	10	-311/ 10780	-281/ 6468	-1/ 196	-1/ 84	-87391/ 235200	2	1
16	11	-311/ 14700	-23/ 2156	-1/ 140	-1/ 60	-7153/ 78400	3	1
16	12	52/ 8085	1124/ 40425	4/ 735	0/ 1	3653/ 44100	2	2
16	13	52/ 11025	92/ 13475	4/ 525	0/ 1	299/ 14700	3	2
16	14	-311/ 53900	-1/ 1078	-1/ 1470	-1/ 630	-311/ 39200	4	1
16	15	-23/ 43120	-281/ 176400	-1/ 2520	0/ 1	-6463/ 940800	2	3
16	16	52/ 40425	8/ 13475	8/ 11025	0/ 1	13/ 7350	4	2
17	1	181/ 6468	-311/ 10780	-1/ 196	-1/ 84	56291/ 235200	1	2
17	2	724/ 40425	-52/ 8085	-4/ 735	0/ 1	2353/ 44100	2	2
17	3	13/ 1617	-311/ 40425	1/ 490	1/ 210	4043/ 58800	1	3
17	4	208/ 40425	-208/ 121275	8/ 3675	0/ 1	169/ 11025	2	3
17	5	181/ 176400	-23/ 43120	-1/ 2520	0/ 1	4163/ 940800	3	2
17	6	5/ 6468	311/ 323400	1/ 2940	1/ 1260	311/ 47040	1	4
17	7	13/ 44100	-23/ 161700	1/ 6300	0/ 1	299/ 235200	3	3
17	8	4/ 8085	26/ 121275	4/ 11025	0/ 1	13/ 8820	2	4
17	9	1/ 35280	23/ 1293600	1/ 37800	0/ 1	23/ 188160	3	4
17	10	281/ 6468	311/ 10780	1/ 196	1/ 84	87391/ 235200	1	2
17	11	1124/ 40425	52/ 8085	4/ 735	0/ 1	3653/ 44100	2	2
17	12	-23/ 2156	-311/ 14700	-1/ 140	-1/ 60	-7153/ 78400	1	3
17	13	-92/ 13475	-52/ 11025	-4/ 525	0/ 1	-299/ 14700	2	3
17	14	281/ 176400	23/ 43120	1/ 2520	0/ 1	6463/ 940800	3	2
17	15	1/ 1078	311/ 53900	1/ 1470	1/ 630	311/ 39200	1	4
17	16	-23/ 58800	-23/ 58800	-1/ 1800	0/ 1	-529/ 313600	3	3
17	17	8/ 13475	52/ 40425	8/ 11025	0/ 1	13/ 7350	2	4
18	1	181/ 129360	-281/ 129360	-1/ 3528	0/ 1	50861/ 2822400	2	2
18	2	181/ 176400	-23/ 43120	-1/ 2520	0/ 1	4163/ 940800	3	2
18	3	13/ 32340	-281/ 485100	1/ 8820	0/ 1	3653/ 705600	2	3
18	4	13/ 44100	-23/ 161700	1/ 6300	0/ 1	299/ 235200	3	3
18	5	181/ 646800	-1/ 21560	-1/ 26460	0/ 1	181/ 470400	4	2
18	6	1/ 25872	281/ 3880800	1/ 52920	0/ 1	281/ 564480	2	4
18	7	13/ 161700	-1/ 80850	1/ 66150	0/ 1	13/ 117600	4	3
18	8	1/ 35280	23/ 1293600	1/ 37800	0/ 1	23/ 188160	3	4
18	9	1/ 129360	1/ 646800	1/ 344900	0/ 1	1/ 94080	4	4
18	10	281/ 129360	281/ 129360	1/ 3528	0/ 1	78961/ 2822400	2	2
18	11	281/ 176400	23/ 43120	1/ 2520	0/ 1	6463/ 940800	3	2
18	12	-23/ 43120	-281/ 176400	-1/ 2520	0/ 1	-6463/ 940800	2	3

i	j	$\tilde{\gamma}_{ij}^{(6)}$	$\tilde{\gamma}_{ij}^{(7)}$	$\tilde{\gamma}_{ij}^{(8)}$	$\tilde{\gamma}_{ij}^{(9)}$	$\tilde{\gamma}_{ij}^{(10)}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
18	13	-23/ 58800	-23/ 58800	-1/ 1800	0/ 1	-529/ 313600	3	3
18	14	281/ 646800	1/ 21560	1/ 26460	0/ 1	281/ 470400	4	2
18	15	1/ 21560	281/ 646800	1/ 26460	0/ 1	281/ 470400	2	4
18	16	-23/ 215600	-1/ 29400	-1/ 18900	0/ 1	-23/ 156800	4	3
18	17	1/ 29400	23/ 215600	1/ 18900	0/ 1	23/ 156800	3	4
18	18	1/ 107800	1/ 107800	1/ 198450	0/ 1	1/ 78400	4	4
19	1	-1000/ 539	-1000/ 539	200/ 49	0/ 1	625/ 49	0	0
19	2	-500/ 539	-302/ 539	30/ 49	0/ 1	755/ 196	1	0
19	3	-302/ 539	-500/ 539	30/ 49	0/ 1	755/ 196	0	1
19	4	-151/ 539	-151/ 539	9/ 98	0/ 1	22801/ 19600	1	1
19	5	-25/ 539	-181/ 3234	5/ 147	0/ 1	905/ 2352	2	0
19	6	-181/ 3234	-25/ 539	5/ 147	0/ 1	905/ 2352	0	2
19	7	-151/ 10780	-181/ 6468	1/ 196	0/ 1	27331/ 235200	2	1
19	8	-181/ 6468	-151/ 10780	1/ 196	0/ 1	27331/ 235200	1	2
19	9	-181/ 129360	-181/ 129360	1/ 3528	0/ 1	32761/ 2822400	2	2
19	10	-3620/ 539	1000/ 539	-200/ 49	0/ 1	4525/ 98	0	0
19	11	-1810/ 539	302/ 539	-30/ 49	0/ 1	27331/ 1960	1	0
19	12	622/ 539	-500/ 539	30/ 49	10/ 7	-1555/ 196	0	1
19	13	311/ 539	-151/ 539	0/ 98	3/ 14	-46961/ 19600	1	1
19	14	-181/ 1078	181/ 3234	-5/ 147	0/ 1	32761/ 23520	2	0
19	15	-281/ 3234	25/ 539	-5/ 147	0/ 1	1405/ 2352	0	2
19	16	311/ 10780	-181/ 6468	1/ 196	1/ 84	-56291/ 235200	2	1
19	17	-281/ 6468	151/ 10780	-1/ 196	0/ 1	42431/ 235200	1	2
19	18	-281/ 129360	181/ 129360	-1/ 3528	0/ 1	50861/ 2822400	2	2
19	19	3620/ 539	3620/ 539	280/ 49	0/ 1	32761/ 196	0	0
20	1	500/ 539	302/ 539	-30/ 49	0/ 1	-755/ 196	1	0
20	2	180/ 539	38/ 231	2/ 49	0/ 1	-95/ 84	2	0
20	3	151/ 539	151/ 539	-9/ 98	0/ 1	-22801/ 19600	1	1
20	4	1359/ 13475	19/ 231	3/ 490	0/ 1	-2869/ 8400	2	1
20	5	20/ 1617	26/ 1617	2/ 147	0/ 1	-65/ 588	3	0
20	6	181/ 6468	151/ 10780	-1/ 196	0/ 1	-27331/ 235200	1	2
20	7	151/ 40425	13/ 1617	1/ 490	0/ 1	-1963/ 58800	3	1
20	8	543/ 53900	19/ 4620	1/ 2940	0/ 1	-3439/ 100800	2	2
20	9	181/ 485100	13/ 32340	1/ 8820	0/ 1	-2353/ 705600	3	2
20	10	1810/ 539	-302/ 539	30/ 49	0/ 1	-27331/ 1960	1	0
20	11	3258/ 2695	-38/ 231	-2/ 49	0/ 1	-3439/ 840	2	0
20	12	-311/ 539	151/ 539	-9/ 98	-3/ 14	46961/ 19600	1	1
20	13	-2799/ 13475	19/ 231	3/ 490	1/ 70	5909/ 8400	2	1
20	14	362/ 8085	-26/ 1617	-2/ 147	0/ 1	-2353/ 5880	3	0
20	15	281/ 6468	-151/ 10780	1/ 196	0/ 1	-42431/ 235200	1	2
20	16	-311/ 40425	13/ 1617	1/ 490	1/ 210	4043/ 58800	3	1
20	17	843/ 53900	-19/ 4620	-1/ 2940	0/ 1	-5339/ 100800	2	2
20	18	281/ 485100	-13/ 32340	-1/ 8820	0/ 1	-3653/ 705600	3	2
20	19	-1810/ 539	-622/ 539	-30/ 49	-10/ 7	-56291/ 1960	1	0
20	20	5792/ 2695	416/ 1617	32/ 49	0/ 1	4706/ 735	2	0
21	1	302/ 539	500/ 539	-30/ 49	0/ 1	-755/ 196	0	1
21	2	151/ 539	151/ 539	-9/ 98	0/ 1	-22801/ 19600	1	1
21	3	38/ 231	180/ 539	2/ 49	0/ 1	-95/ 84	0	2
21	4	19/ 231	1359/ 13475	3/ 490	0/ 1	-2869/ 8400	1	2
21	5	151/ 10780	181/ 6468	-1/ 196	0/ 1	-27331/ 235200	2	1
21	6	26/ 1617	20/ 1617	2/ 147	0/ 1	-65/ 588	0	3
21	7	19/ 4620	543/ 53900	1/ 2940	0/ 1	-3439/ 100800	2	2
21	8	13/ 1617	151/ 40425	1/ 490	0/ 1	-1963/ 58800	1	3
21	9	13/ 32340	181/ 485100	1/ 8820	0/ 1	-2353/ 705600	2	3
21	10	622/ 539	-500/ 539	30/ 49	10/ 7	-1555/ 196	0	1

Table 8 (continued)

i	j	$\tilde{\gamma}_{ij}^{(6)}$	$\tilde{\gamma}_{ij}^{(7)}$	$\tilde{\gamma}_{ij}^{(8)}$	$\tilde{\gamma}_{ij}^{(9)}$	$\tilde{\gamma}_{ij}^{(10)}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
21	11	311/ 539	-151/ 539	9/ 98	3/ 14	-46961/ 19600	1	1
21	12	-416/ 1617	320/ 539	-32/ 49	0/ 1	260/ 147	0	2
21	13	-208/ 1617	2416/ 13475	-24/ 245	0/ 1	1963/ 3675	1	2
21	14	311/ 10780	-181/ 6468	1/ 196	1/ 84	-56291/ 235200	2	1
21	15	23/ 1078	-5/ 147	1/ 21	0/ 1	-115/ 784	0	3
21	16	-52/ 8085	724/ 40425	-4/ 735	0/ 1	2353/ 44100	2	2
21	17	23/ 2156	-151/ 14700	1/ 140	0/ 1	-3473/ 78400	1	3
21	18	23/ 43120	-181/ 176400	1/ 2520	0/ 1	-4163/ 940800	2	3
21	19	-622/ 539	-1810/ 539	-30/ 49	-10/ 7	-56291/ 1960	0	1
21	20	311/ 539	311/ 539	9/ 98	10/ 7	96721/ 19600	1	1
21	21	416/ 1617	5792/ 2695	32/ 49	0/ 1	4706/ 735	0	2
22	1	-151/ 539	-151/ 539	9/ 98	0/ 1	22601/ 19600	1	1
22	2	-1359/ 13475	-19/ 231	-3/ 490	0/ 1	2869/ 8400	2	1
22	3	-19/ 231	-1359/ 13475	-3/ 490	0/ 1	2869/ 8400	1	2
22	4	-57/ 1925	-57/ 1925	1/ 2450	0/ 1	361/ 3600	2	2
22	5	-151/ 40425	-13/ 1617	-1/ 490	0/ 1	1963/ 58800	3	1
22	6	-13/ 1617	-151/ 40425	-1/ 490	0/ 1	1963/ 58800	1	3
22	7	-19/ 17325	-34/ 13475	1/ 7350	0/ 1	247/ 25200	3	2
22	8	-39/ 13475	-19/ 17325	1/ 7350	0/ 1	247/ 25200	2	3
22	9	-13/ 121275	-13/ 121275	1/ 22050	0/ 1	169/ 176400	3	3
22	10	-311/ 539	151/ 539	-9/ 98	-3/ 14	46961/ 19600	1	1
22	11	-2799/ 13475	19/ 231	3/ 490	1/ 70	5909/ 8400	2	1
22	12	208/ 1617	-2416/ 13475	24/ 245	0/ 1	-1963/ 3675	1	2
22	13	624/ 13475	-304/ 6775	-8/ 1225	0/ 1	-247/ 1575	2	2
22	14	-311/ 40425	13/ 1617	1/ 490	1/ 210	4043/ 58800	3	1
22	15	-23/ 2156	151/ 14700	-1/ 140	0/ 1	3473/ 78400	1	3
22	16	208/ 121275	-208/ 40425	-8/ 3675	0/ 1	-169/ 11025	3	2
22	17	-207/ 53900	19/ 6300	1/ 2100	0/ 1	437/ 33600	2	3
22	18	-23/ 161700	13/ 44100	1/ 6300	0/ 1	299/ 235200	3	3
22	19	311/ 539	311/ 539	9/ 98	3/ 7	46721/ 19600	1	1
22	20	-4476/ 13475	-208/ 1617	-24/ 245	-8/ 35	-4043/ 3675	2	1
22	21	-208/ 1617	-4976/ 13475	-24/ 245	-8/ 35	-4043/ 3675	1	2
22	22	3328/ 40425	3328/ 40425	128/ 1225	0/ 1	2704/ 11025	2	2
23	1	-25/ 539	-181/ 3234	5/ 147	0/ 1	905/ 2352	2	0
23	2	-20/ 1617	-26/ 1617	-2/ 147	0/ 1	65/ 588	3	0
23	3	-151/ 10780	-181/ 6468	1/ 196	0/ 1	27331/ 235200	2	1
23	4	-151/ 40425	-13/ 1617	-1/ 490	0/ 1	1963/ 58800	3	1
23	5	5/ 3234	-5/ 3234	-1/ 441	0/ 1	25/ 2352	4	0
23	6	-181/ 129360	-181/ 129360	1/ 3528	0/ 1	32761/2822400	2	2
23	7	151/ 323400	-5/ 6468	-1/ 2940	0/ 1	151/ 47040	4	1
23	8	-181/ 485100	-13/ 32340	-1/ 8820	0/ 1	2353/ 705600	3	2
23	9	181/3880800	-1/ 25872	-1/ 52920	0/ 1	181/ 564480	4	2
23	10	-181/ 1078	181/ 3234	-5/ 147	0/ 1	32761/ 23520	2	0
23	11	-362/ 8085	26/ 1617	2/ 147	0/ 1	2353/ 5880	3	0
23	12	311/ 10780	-181/ 6468	1/ 196	1/ 84	-56291/ 235200	2	1
23	13	311/ 40425	-13/ 1617	-1/ 490	-1/ 210	-4043/ 58800	3	1
23	14	181/ 32340	5/ 3234	1/ 441	0/ 1	181/ 4704	4	0
23	15	-281/ 129360	181/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
23	16	-311/ 323400	-5/ 6468	-1/ 2940	-1/1260	-311/ 47040	4	1
23	17	-281/ 485100	13/ 32340	1/ 8820	0/ 1	3653/ 705600	3	2
23	18	281/3880800	1/ 25872	1/ 52920	0/ 1	281/ 564480	4	2
23	19	181/ 1078	281/ 3234	5/ 147	0/ 1	50861/ 23520	2	0
23	20	-181/ 1470	-23/ 1078	-1/ 21	0/ 1	-4163/ 7840	3	0
23	21	-311/ 10780	-281/ 6468	-1/ 196	-1/ 84	-87391/ 235200	2	1
23	22	311/ 14700	23/ 2156	1/ 140	1/ 60	7153/ 78400	3	1
23	23	181/ 5390	1/ 539	2/ 441	0/ 1	181/ 3920	0	0
24	1	-181/ 3234	-25/ 539	5/ 147	0/ 1	905/ 2352	0	2
24	2	-181/ 6468	-151/ 10780	1/ 196	0/ 1	27331/ 235200	1	2
24	3	-26/ 1617	-20/ 1617	-2/ 147	0/ 1	65/ 588	0	3
24	4	-13/ 1617	-151/ 40425	-1/ 490	0/ 1	1963/ 58800	1	3
24	5	-181/ 129360	-181/ 129360	1/ 3528	0/ 1	32761/2822400	2	2
24	6	-5/ 3234	5/ 3234	-1/ 441	0/ 1	25/ 2352	0	4
24	7	-13/ 32340	-181/ 485100	-1/ 8820	0/ 1	2353/ 705600	2	3
24	8	-5/ 6468	151/ 323400	-1/ 2940	0/ 1	151/ 47040	1	4
24	9	-1/ 25872	181/3880800	-1/ 52920	0/ 1	181/ 564480	2	4
24	10	-281/ 3234	25/ 539	-5/ 147	0/ 1	1405/ 2352	0	2
24	11	-281/ 6468	151/ 10780	-1/ 196	0/ 1	42431/ 235200	1	2
24	12	23/ 1078	-5/ 147	1/ 21	0/ 1	-115/ 784	0	3
24	13	23/ 2156	-151/ 14700	1/ 140	0/ 1	-3473/ 78400	1	3
24	14	-281/ 129360	181/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
24	15	-1/ 539	5/ 539	-2/ 441	0/ 1	5/ 392	0	4
24	16	23/ 43120	-181/ 176400	1/ 2520	0/ 1	-4163/ 940800	2	3
24	17	-1/ 1078	151/ 53900	-1/ 1470	0/ 1	151/ 39200	1	4
24	18	-1/ 21560	181/ 646800	-1/ 26460	0/ 1	181/ 470400	2	4
24	19	281/ 3234	181/ 1078	5/ 147	0/ 1	50861/ 23520	0	2
24	20	-281/ 6468	-311/ 10780	-1/ 196	-1/ 84	-87391/ 235200	1	2
24	21	-23/ 1078	-181/ 1470	-1/ 21	0/ 1	-4163/ 7840	0	3
24	22	23/ 2156	311/ 14700	1/ 140	1/ 60	7153/ 78400	1	3
24	23	281/ 129360	281/ 129360	1/ 3528	0/ 1	78961/2822400	2	2
24	24	1/ 539	181/ 5390	2/ 441	0/ 1	181/ 3920	0	4
25	1	151/ 10780	181/ 6468	-1/ 196	0/ 1	-27331/ 235200	2	1
25	2	151/ 40425	13/ 1617	1/ 490	0/ 1	-1963/ 58800	3	1
25	3	19/ 4620	543/ 53900	1/ 2940	0/ 1	-3439/ 100800	2	2
25	4	19/ 17325	34/ 13475	-1/ 7350	0/ 1	-247/ 25200	3	2
25	5	-151/ 323400	5/ 6468	1/ 2940	0/ 1	-151/ 47040	4	1
25	6	13/ 32340	181/ 485100	1/ 8820	0/ 1	-2353/ 705600	2	3
25	7	-19/ 138600	3/ 10780	-1/ 44100	0/ 1	-19/ 20160	4	2
25	8	13/ 121275	13/ 121275	-1/ 22050	0/ 1	-169/ 176400	3	3
25	9	-13/ 970200	1/ 97020	-1/132300	0/ 1	-13/ 141120	4	3
25	10	311/ 10780	-181/ 6468	1/ 196	1/ 84	-56291/ 235200	2	1
25	11	311/ 40425	-13/ 1617	-1/ 490	-1/ 210	-4043/ 58800	3	1
25	12	-52/ 8085	724/ 40425	-4/ 735	0/ 1	2353/ 44100	2	2
25	13	-208/ 121275	208/ 40425	8/ 3675	0/ 1	169/ 11025	3	2
25	14	-311/ 323400	-5/ 6468	-1/ 2940	-1/1260	-311/ 47040	4	1
25	15	23/ 43120	-181/ 176400	1/ 2520	0/ 1	-4163/ 940800	2	3
25	16	26/ 121275	4/ 8085	4/ 11025	0/ 1	13/ 8820	4	2
25	17	23/ 161700	-13/ 44100	-1/ 6300	0/ 1	-299/ 235200	3	3
25	18	-23/1293600	-1/ 35280	-1/ 37800	0/ 1	-23/ 188160	4	3
25	19	-311/ 10780	-281/ 6468	-1/ 196	-1/ 84	-87391/ 235200	2	1
25	20	311/ 14700	23/ 2156	1/ 140	1/ 60	7153/ 78400	3	1
25	21	52/ 8085	1124/ 40425	4/ 735	0/ 1	3653/ 44100	2	2
25	22	-52/ 11025	-92/ 13475	-4/ 525	0/ 1	-299/ 14700	3	2
25	23	-311/ 53900	-1/ 1078	-1/ 1470	-1/ 630	-311/ 39200	4	1
25	24	-23/ 43120	-281/ 176400	-1/ 2520	0/ 1	-6463/ 940800	2	3
25	25	52/ 40425	8/ 13475	8/ 11025	0/ 1	13/ 7350	4	2
26	1	181/ 6468	151/ 10780	-1/ 196	0/ 1	-27331/ 235200	1	2
26	2	543/ 53900	19/ 4620	1/ 2940	0/ 1	-3439/ 100800	2	2
26	3	13/ 1617	151/ 40425	1/ 490	0/ 1	-1963/ 58800	1	3
26	4	39/ 13475	19/ 17325	-1/ 7350	0/ 1	-247/ 25200	2	3
26	5	181/ 485100	13/ 32340	1/ 8820	0/ 1	-2353/ 705600	3	2

Table 8 (continued)

i	j	$\tilde{\gamma}_{ij}^{(6)}$	$\tilde{\gamma}_{ij}^{(7)}$	$\tilde{\gamma}_{ij}^{(8)}$	$\tilde{\gamma}_{ij}^{(9)}$	$\tilde{\gamma}_{ij}^{(10)}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
26	6	5/ 6468	-151/ 323400	1/ 2940	0/ 1	-151/ 47040	1	4
26	7	13/ 121275	13/ 121275	-1/ 22050	0/ 1	-169/ 176400	3	3
26	8	3/ 10780	-19/ 138600	-1/ 44100	0/ 1	-19/ 20160	2	4
26	9	1/ 97020	-13/ 970200	-1/132300	0/ 1	-13/ 141120	3	4
26	10	281/ 6468	-151/ 10780	1/ 196	0/ 1	-42431/ 235200	1	2
26	11	843/ 53900	-19/ 4620	-1/ 2940	0/ 1	-5339/ 100800	2	2
26	12	-23/ 2156	151/ 14700	-1/ 180	0/ 1	3473/ 78400	1	3
26	13	-207/ 53900	19/ 6300	1/ 2100	0/ 1	437/ 33600	2	3
26	14	281/ 485100	-13/ 32340	-1/ 8820	0/ 1	-3653/ 705600	3	2
26	15	1/ 1078	-151/ 53900	1/ 1470	0/ 1	-151/ 39200	1	4
26	16	-23/ 161700	13/ 44100	1/ 6300	0/ 1	299/ 235200	3	3
26	17	9/ 26950	-19/ 23100	-1/ 22050	0/ 1	-19/ 16800	2	4
26	18	1/ 80850	-13/ 161700	-1/ 66150	0/ 1	-13/ 117600	3	4
26	19	-281/ 6468	-311/ 10780	-1/ 196	-1/ 84	-87391/ 235200	1	2
26	20	1124/ 40425	52/ 8085	8/ 735	0/ 1	3653/ 44100	2	2
26	21	23/ 2156	311/ 14700	1/ 180	1/ 60	7153/ 78400	1	3
26	22	-92/ 13475	-52/ 11025	-8/ 525	0/ 1	-299/ 14700	2	3
26	23	-281/ 176400	-23/ 43120	-1/ 2520	0/ 1	-6463/ 940800	3	2
26	24	-1/ 1078	-311/ 53900	-1/ 1470	-1/ 630	-311/ 39200	1	4
26	25	23/ 58800	23/ 58800	1/ 1800	0/ 1	529/ 313600	3	3
26	26	8/ 13475	52/ 40425	8/ 11025	0/ 1	13/ 7350	2	4
27	1	-181/ 129360	-181/ 129360	1/ 3528	0/ 1	32761/2822400	2	2
27	2	-181/ 485100	-13/ 32340	-1/ 8820	0/ 1	2353/ 705600	3	2
27	3	-13/ 32340	-181/ 485100	-1/ 8820	0/ 1	2353/ 705600	2	3
27	4	-13/ 121275	-13/ 121275	1/ 22050	0/ 1	169/ 176400	3	3
27	5	181/3880800	-1/ 25872	-1/ 52920	0/ 1	181/ 564480	4	2
27	6	-1/ 25872	181/3880800	-1/ 52920	0/ 1	181/ 564480	2	4
27	7	13/ 970200	-1/ 97020	1/132300	0/ 1	13/ 141120	4	3
27	8	-1/ 97020	13/ 970200	1/132300	0/ 1	13/ 141120	3	4
27	9	1/ 776160	1/ 776160	1/793800	0/ 1	1/ 112896	4	4
27	10	-281/ 129360	181/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
27	11	-281/ 485100	13/ 32340	1/ 8820	0/ 1	3653/ 705600	3	2
27	12	23/ 43120	-181/ 176400	1/ 2520	0/ 1	-4163/ 940800	2	3
27	13	23/ 161700	-13/ 44100	-1/ 6300	0/ 1	-299/ 235200	3	3
27	14	281/3880800	1/ 25872	1/ 52920	0/ 1	281/ 564480	4	2
27	15	-1/ 21560	181/ 646800	-1/ 26460	0/ 1	181/ 470400	2	4
27	16	-23/1293600	-1/ 35280	-1/ 37800	0/ 1	-23/ 188160	4	3
27	17	-1/ 80850	13/ 161700	1/ 66150	0/ 1	13/ 117600	3	4
27	18	1/ 646800	1/ 129360	1/396900	0/ 1	1/ 94080	4	4
27	19	281/ 129360	281/ 129360	1/ 3528	0/ 1	78961/2822400	2	2
27	20	-281/ 176400	-23/ 43120	-1/ 2520	0/ 1	-6463/ 940800	3	2
27	21	-23/ 43120	-281/ 176400	-1/ 2520	0/ 1	-6463/ 940800	2	3
27	22	23/ 58800	23/ 58800	1/ 1800	0/ 1	529/ 313600	3	3
27	23	281/ 646800	1/ 21560	1/ 26460	0/ 1	281/ 470400	4	2
27	24	1/ 21560	281/ 646800	1/ 26460	0/ 1	281/ 470400	2	4
27	25	-23/ 215600	-1/ 29400	-1/ 18900	0/ 1	-23/ 156800	4	3
27	26	-1/ 29400	-23/ 215600	-1/ 18900	0/ 1	-23/ 156800	3	4
27	27	1/ 107800	1/ 107800	1/188450	0/ 1	1/ 78400	4	4
28	1	-3620/ 539	1000/ 539	-200/ 49	0/ 1	4525/ 98	0	0
28	2	-1810/ 539	302/ 539	-30/ 49	0/ 1	27331/ 1960	1	0
28	3	-622/ 539	500/ 539	-30/ 49	-10/ 7	1555/ 196	0	1
28	4	-311/ 539	151/ 539	-9/ 98	-3/ 14	46961/ 19600	1	1
28	5	-181/ 1078	181/ 3234	-5/ 147	0/ 1	32761/ 23520	2	0
28	6	-281/ 3234	25/ 539	-5/ 147	0/ 1	1405/ 2352	0	2
28	7	-311/ 10780	181/ 6468	-1/ 196	-1/ 84	56291/ 235200	2	1

i	j	$\tilde{\gamma}_{ij}^{(6)}$	$\tilde{\gamma}_{ij}^{(7)}$	$\tilde{\gamma}_{ij}^{(8)}$	$\tilde{\gamma}_{ij}^{(9)}$	$\tilde{\gamma}_{ij}^{(10)}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
28	8	-281/ 6468	151/ 10780	-1/ 196	0/ 1	42431/ 235200	1	2
28	9	-281/ 129360	181/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
28	10	-1000/ 539	-1000/ 539	200/ 49	0/ 1	625/ 49	0	0
28	11	-500/ 539	-302/ 539	30/ 49	0/ 1	755/ 196	1	0
28	12	302/ 539	500/ 539	-30/ 49	0/ 1	-755/ 196	0	1
28	13	151/ 539	151/ 539	-9/ 98	0/ 1	-22801/ 19600	1	1
28	14	-25/ 539	-181/ 3234	5/ 147	0/ 1	905/ 2352	2	0
28	15	-181/ 3234	-25/ 539	5/ 147	0/ 1	905/ 2352	0	2
28	16	151/ 10780	181/ 6468	-1/ 196	0/ 1	-27331/ 235200	2	1
28	17	-181/ 6468	-151/ 10780	1/ 196	0/ 1	27331/ 235200	1	2
28	18	-181/ 129360	-181/ 129360	1/ 3528	0/ 1	32761/2822400	2	2
28	19	1000/ 539	-3620/ 539	-200/ 49	0/ 1	4525/ 98	0	0
28	20	-500/ 539	622/ 539	30/ 49	10/ 7	-1555/ 196	1	0
28	21	-302/ 539	1810/ 539	30/ 49	0/ 1	-27331/ 1960	0	1
28	22	151/ 539	-311/ 539	-9/ 98	-3/ 14	46961/ 19600	1	1
28	23	25/ 539	-281/ 3234	-5/ 147	0/ 1	1405/ 2352	2	0
28	24	181/ 3234	-181/ 1078	-5/ 147	0/ 1	32761/ 23520	0	2
28	25	-151/ 10780	281/ 6468	1/ 196	0/ 1	-42431/ 235200	2	1
28	26	-181/ 6468	311/ 10780	1/ 196	1/ 84	-56291/ 235200	1	2
28	27	181/ 129360	-281/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
28	28	3620/ 539	3620/ 539	200/ 49	0/ 1	32761/ 196	0	0
29	1	1810/ 539	-302/ 539	30/ 49	0/ 1	-27331/ 1960	1	0
29	2	3258/ 2695	-38/ 231	-2/ 49	0/ 1	-3439/ 840	2	0
29	3	311/ 539	-151/ 539	9/ 98	3/ 14	-46961/ 19600	1	1
29	4	2799/ 13475	-19/ 231	-3/ 490	-1/ 70	-5909/ 8400	2	1
29	5	362/ 8085	-26/ 1617	-2/ 147	0/ 1	-2353/ 5880	3	0
29	6	281/ 6468	-151/ 10780	1/ 196	0/ 1	-42431/ 235200	1	2
29	7	311/ 40425	-13/ 1617	-1/ 490	-1/ 210	-4043/ 58800	3	1
29	8	843/ 53900	-19/ 4620	-1/ 2940	0/ 1	-5339/ 100800	2	2
29	9	281/ 485100	-13/ 32340	-1/ 8820	0/ 1	-3653/ 705600	3	2
29	10	500/ 539	302/ 539	-30/ 49	0/ 1	-755/ 196	1	0
29	11	180/ 539	38/ 231	2/ 49	0/ 1	-95/ 84	2	0
29	12	-151/ 539	-151/ 539	9/ 98	0/ 1	22801/ 19600	1	1
29	13	-1359/ 13475	-19/ 231	-3/ 490	0/ 1	2869/ 8400	2	1
29	14	20/ 1617	26/ 1617	2/ 147	0/ 1	-65/ 588	3	0
29	15	181/ 6468	151/ 10780	-1/ 196	0/ 1	-27331/ 235200	1	2
29	16	-151/ 40425	-13/ 1617	-1/ 490	0/ 1	1963/ 58800	3	1
29	17	543/ 53900	19/ 4620	1/ 2940	0/ 1	-3439/ 100800	2	2
29	18	181/ 485100	13/ 32340	1/ 8820	0/ 1	-2353/ 705600	3	2
29	19	-500/ 539	622/ 539	30/ 49	10/ 7	-1555/ 196	1	0
29	20	320/ 539	-416/ 1617	-32/ 49	0/ 1	260/ 147	2	0
29	21	151/ 539	-311/ 539	-9/ 98	-3/ 14	46961/ 19600	1	1
29	22	-2416/ 13475	208/ 1617	24/ 245	0/ 1	-1963/ 3675	2	1
29	23	-5/ 147	23/ 1078	1/ 21	0/ 1	-115/ 784	3	0
29	24	-181/ 6468	311/ 10780	1/ 196	1/ 84	-56291/ 235200	1	2
29	25	151/ 14700	-23/ 2156	-1/ 180	0/ 1	3473/ 78400	3	1
29	26	724/ 40425	-52/ 8085	-8/ 735	0/ 1	2353/ 44100	2	2
29	27	-181/ 176400	23/ 43120	1/ 2520	0/ 1	-4163/ 940800	3	2
29	28	-1810/ 539	-622/ 539	-30/ 49	-10/ 7	-56291/ 1960	1	0
29	29	5792/ 2695	416/ 1617	32/ 49	0/ 1	4706/ 735	2	0
30	1	-622/ 539	500/ 539	-30/ 49	-10/ 7	1555/ 196	0	1
30	2	-311/ 539	151/ 539	-9/ 98	-3/ 14	46961/ 19600	1	1
30	3	-416/ 1617	320/ 539	-32/ 49	0/ 1	260/ 147	0	2
30	4	-208/ 1617	2416/ 13475	-24/ 245	0/ 1	1963/ 3675	1	2
30	5	-311/ 10780	181/ 6468	-1/ 196	-1/ 84	56291/ 235200	2	1

Table 8 (continued)

i	j	$\tilde{\gamma}_{ij}^{(6)}$	$\tilde{\gamma}_{ij}^{(7)}$	$\tilde{\gamma}_{ij}^{(8)}$	$\tilde{\gamma}_{ij}^{(9)}$	$\tilde{\gamma}_{ij}^{(10)}$	$\tilde{\lambda}_{ij}$	$\tilde{\nu}_{ij}$
30	6	-23/ 1078	5/ 147	-1/ 21	0/ 1	115/ 784	0	3
30	7	-52/ 8085	724/ 40425	-4/ 735	0/ 1	2353/ 44100	2	2
30	8	-23/ 2156	151/ 14700	-1/ 140	0/ 1	3473/ 78400	1	3
30	9	-23/ 43120	181/ 176400	-1/ 2520	0/ 1	4163/ 940800	2	3
30	10	-302/ 539	-500/ 539	30/ 49	0/ 1	755/ 196	0	1
30	11	-151/ 539	-151/ 539	0/ 98	0/ 1	22801/ 19600	1	1
30	12	36/ 231	180/ 539	2/ 49	0/ 1	-95/ 84	0	2
30	13	19/ 231	1354/ 13475	3/ 490	0/ 1	-2869/ 8400	1	2
30	14	-151/ 10780	-181/ 6468	1/ 196	0/ 1	27331/ 235200	2	1
30	15	-26/ 1617	-20/ 1617	-2/ 147	0/ 1	65/ 588	0	3
30	16	19/ 4620	543/ 53900	1/ 2940	0/ 1	-3439/ 100800	2	2
30	17	-13/ 1617	-151/ 40425	-1/ 490	0/ 1	1963/ 58800	1	3
30	18	-13/ 32340	-181/ 485100	-1/ 8820	0/ 1	2353/ 705600	2	3
30	19	302/ 539	-1810/ 539	-30/ 49	0/ 1	27331/ 1960	0	1
30	20	-151/ 539	311/ 539	9/ 98	3/ 14	-46961/ 19600	1	1
30	21	-38/ 231	3254/ 2695	-2/ 49	0/ 1	-3439/ 840	0	2
30	22	19/ 231	-2794/ 13475	3/ 490	1/ 70	5909/ 8400	1	2
30	23	151/ 10780	-281/ 6468	-1/ 196	0/ 1	42431/ 235200	2	1
30	24	26/ 1617	-362/ 8085	2/ 147	0/ 1	2353/ 5880	0	3
30	25	-19/ 4620	643/ 53900	-1/ 2940	0/ 1	-5339/ 100800	2	2
30	26	-13/ 1617	311/ 40425	-1/ 490	-1/ 210	-4043/ 58800	1	3
30	27	13/ 32340	-281/ 485100	1/ 8820	0/ 1	3653/ 705600	2	3
30	28	622/ 539	1810/ 539	30/ 49	10/ 7	56291/ 1960	0	1
30	29	-311/ 539	-311/ 539	-9/ 98	-10/ 7	-96721/ 19600	1	1
30	30	416/ 1617	5792/ 2695	32/ 49	0/ 1	4706/ 735	0	2
31	1	311/ 539	-151/ 539	9/ 98	3/ 14	-46961/ 19600	1	1
31	2	2799/ 13475	-19/ 231	-3/ 490	-1/ 70	-5909/ 8400	2	1
31	3	208/ 1617	-2416/ 13475	24/ 245	0/ 1	-1963/ 3675	1	2
31	4	624/ 13475	-304/ 5775	-8/ 1225	0/ 1	-247/ 1575	2	2
31	5	311/ 40425	-13/ 1617	-1/ 490	-1/ 210	-4043/ 58800	3	1
31	6	23/ 2156	-151/ 14700	1/ 140	0/ 1	-3473/ 78400	1	3
31	7	208/ 121275	-208/ 40425	-8/ 3675	0/ 1	-169/ 11025	3	2
31	8	207/ 53900	-19/ 6300	-1/ 2100	0/ 1	-437/ 33600	2	3
31	9	23/ 161700	-13/ 4410	-1/ 6300	0/ 1	-299/ 235200	3	3
31	10	151/ 539	151/ 539	-9/ 98	0/ 1	-22801/ 19600	1	1
31	11	1359/ 13475	19/ 231	3/ 490	0/ 1	-2869/ 8400	2	1
31	12	-19/ 231	-1359/ 13475	-3/ 490	0/ 1	2869/ 8400	1	2
31	13	-57/ 1925	-57/ 1925	1/ 2450	0/ 1	361/ 3600	2	2
31	14	151/ 40425	13/ 1617	1/ 490	0/ 1	-1963/ 58800	3	1
31	15	13/ 1617	151/ 40425	1/ 490	0/ 1	-1963/ 58800	1	3
31	16	-19/ 17325	-39/ 13475	1/ 7350	0/ 1	247/ 25200	3	2
31	17	39/ 13475	19/ 17325	-1/ 7350	0/ 1	-247/ 25200	2	3
31	18	13/ 121275	13/ 121275	-1/ 22050	0/ 1	-169/ 176400	3	3
31	19	-151/ 539	311/ 539	9/ 98	3/ 14	-46961/ 19600	1	1
31	20	2416/ 13475	-208/ 1617	-24/ 245	0/ 1	1963/ 3675	2	1
31	21	19/ 231	-2799/ 13475	3/ 490	1/ 70	5909/ 8400	1	2
31	22	-304/ 5775	624/ 13475	-8/ 1225	0/ 1	-247/ 1575	2	2
31	23	-151/ 14700	23/ 2156	1/ 140	0/ 1	-3473/ 78400	3	1
31	24	-13/ 1617	311/ 40425	-1/ 490	-1/ 210	-4043/ 58800	1	3
31	25	19/ 6300	-207/ 5390	1/ 2100	0/ 1	437/ 33600	3	2
31	26	208/ 40425	-208/ 121275	8/ 3675	0/ 1	169/ 11025	2	3
31	27	-13/ 44100	23/ 161700	-1/ 6300	0/ 1	-299/ 235200	3	3
31	28	-311/ 539	-311/ 539	-9/ 98	-3/ 7	-96721/ 19600	1	1
31	29	4976/ 13475	208/ 1617	24/ 245	8/ 35	4043/ 3675	2	1
31	30	-208/ 1617	-4976/ 13475	-24/ 245	-8/ 35	-4043/ 3675	1	2

i	j	$\tilde{\gamma}_{ij}^{(6)}$	$\tilde{\gamma}_{ij}^{(7)}$	$\tilde{\gamma}_{ij}^{(8)}$	$\tilde{\gamma}_{ij}^{(9)}$	$\tilde{\gamma}_{ij}^{(10)}$	$\tilde{\lambda}_{ij}$	$\tilde{\nu}_{ij}$
31	31	3328/ 40425	3328/ 40425	128/ 1225	0/ 1	2704/ 11025	2	2
32	1	-181/ 1078	181/ 3234	-5/ 147	0/ 1	32761/ 23520	2	0
32	2	-362/ 8085	26/ 1617	2/ 147	0/ 1	2353/ 5880	3	0
32	3	-311/ 10780	181/ 6468	-1/ 196	-1/ 84	56291/ 235200	2	1
32	4	-311/ 40425	13/ 1617	1/ 490	1/ 210	4043/ 58800	3	1
32	5	181/ 32340	5/ 3234	1/ 441	0/ 1	181/ 4704	4	0
32	6	-281/ 129360	181/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
32	7	311/ 323400	5/ 6468	1/ 2940	1/1260	311/ 47040	4	1
32	8	-281/ 485100	13/ 32340	1/ 8820	0/ 1	3653/ 705600	3	2
32	9	281/3880800	1/ 25872	1/ 52920	0/ 1	281/ 564480	4	2
32	10	-25/ 539	-181/ 3234	5/ 147	0/ 1	905/ 2352	2	0
32	11	-20/ 1617	-26/ 1617	-2/ 147	0/ 1	65/ 588	3	0
32	12	151/ 10780	181/ 6468	-1/ 196	0/ 1	-27331/ 235200	2	1
32	13	151/ 40425	13/ 1617	1/ 490	0/ 1	-1963/ 58800	3	1
32	14	5/ 3234	-5/ 3234	-1/ 441	0/ 1	25/ 2352	4	0
32	15	-181/ 129360	-181/ 129360	1/ 3528	0/ 1	32761/2822400	2	2
32	16	-151/ 323400	5/ 6468	1/ 2940	0/ 1	-151/ 47040	4	1
32	17	-181/ 485100	-13/ 32340	-1/ 8820	0/ 1	2353/ 705600	3	2
32	18	181/3880800	-1/ 25872	-1/ 52920	0/ 1	181/ 564480	4	2
32	19	25/ 539	-281/ 3234	-5/ 147	0/ 1	1405/ 2352	2	0
32	20	-5/ 147	23/ 1078	1/ 21	0/ 1	-115/ 784	3	0
32	21	-151/ 10780	281/ 6468	1/ 196	0/ 1	-42431/ 235200	2	1
32	22	151/ 14700	-23/ 2156	-1/ 140	0/ 1	3473/ 78400	3	1
32	23	5/ 539	-1/ 539	-2/ 441	0/ 1	5/ 392	4	0
32	24	181/ 129360	-281/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
32	25	-151/ 53900	1/ 1078	1/ 1470	0/ 1	-151/ 39200	4	1
32	26	-181/ 176400	23/ 43120	1/ 2520	0/ 1	-4163/ 940800	3	2
32	27	181/ 646800	-1/ 21560	-1/ 26460	0/ 1	181/ 470400	4	2
32	28	181/ 1078	281/ 3234	5/ 147	0/ 1	50861/ 23520	2	0
32	29	-181/ 1470	-23/ 1078	-1/ 21	0/ 1	-4163/ 7840	3	0
32	30	311/ 10780	281/ 6468	1/ 196	1/ 84	87391/ 235200	2	1
32	31	-311/ 14700	-23/ 2156	-1/ 140	-1/ 60	-7153/ 78400	3	1
32	32	181/ 5390	1/ 539	2/ 441	0/ 1	181/ 3920	4	0
33	1	-281/ 3234	25/ 539	-5/ 147	0/ 1	1405/ 2352	0	2
33	2	-281/ 6468	151/ 10780	-1/ 196	0/ 1	42431/ 235200	1	2
33	3	-23/ 1078	5/ 147	-1/ 21	0/ 1	115/ 784	0	3
33	4	-23/ 2156	151/ 14700	-1/ 140	0/ 1	3473/ 78400	1	3
33	5	-281/ 129360	181/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
33	6	-1/ 539	5/ 539	-2/ 441	0/ 1	5/ 392	0	4
33	7	-23/ 43120	181/ 176400	-1/ 2520	0/ 1	4163/ 940800	2	3
33	8	-1/ 1078	151/ 53900	-1/ 1470	0/ 1	151/ 39200	1	4
33	9	-1/ 21560	181/ 646800	-1/ 26460	0/ 1	181/ 470400	2	4
33	10	-181/ 3234	-25/ 539	5/ 147	0/ 1	905/ 2352	0	2
33	11	-181/ 6468	-151/ 10780	1/ 196	0/ 1	27331/ 235200	1	2
33	12	26/ 1617	20/ 1617	2/ 147	0/ 1	-65/ 588	0	3
33	13	13/ 1617	151/ 40425	1/ 490	0/ 1	-1963/ 58800	1	3
33	14	-181/ 129360	-181/ 129360	1/ 3528	0/ 1	32761/2822400	2	2
33	15	-5/ 3234	5/ 3234	-1/ 441	0/ 1	25/ 2352	0	4
33	16	13/ 32340	181/ 485100	1/ 8820	0/ 1	-2353/ 705600	2	3
33	17	-5/ 6468	151/ 323400	-1/ 2940	0/ 1	151/ 47040	1	4
33	18	-1/ 25872	181/3880800	-1/ 52920	0/ 1	181/ 564480	2	4
33	19	181/ 3234	-181/ 1078	-5/ 147	0/ 1	32761/ 23520	0	2
33	20	-181/ 6468	311/ 10780	1/ 196	1/ 84	-56291/ 235200	1	2
33	21	-26/ 1617	362/ 8085	-2/ 147	0/ 1	-2353/ 5880	0	3
33	22	13/ 1617	-311/ 40425	1/ 490	1/ 210	4043/ 58800	1	3

Table 8 (continued)

i	j	$\tilde{\gamma}^{(6)}_{ij}$	$\tilde{\gamma}^{(7)}_{ij}$	$\tilde{\gamma}^{(8)}_{ij}$	$\tilde{\gamma}^{(9)}_{ij}$	$\tilde{\gamma}^{(10)}_{ij}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
33	23	181/ 129360	-281/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
33	24	5/ 3234	181/ 32340	1/ 441	0/ 1	181/ 4704	0	4
33	25	-13/ 32340	281/ 485100	-1/ 8820	0/ 1	-3653/ 705600	2	3
33	26	-5/ 6468	-311/ 323400	-1/ 2940	-1/1260	-311/ 47040	1	4
33	27	1/ 25872	281/3880800	1/ 52920	0/ 1	281/ 564480	2	4
33	28	281/ 3234	181/ 1078	5/ 147	0/ 1	50861/ 23520	0	2
33	29	-281/ 6468	-311/ 10780	-1/ 196	-1/ 84	-87391/ 235200	1	2
33	30	23/ 1078	181/ 1470	1/ 21	0/ 1	4163/ 7840	0	3
33	31	-23/ 2156	-311/ 14700	-1/ 140	-1/ 60	-7153/ 78400	1	3
33	32	281/ 129360	281/ 129360	1/ 3528	0/ 1	78961/2822400	2	2
33	33	1/ 539	181/ 5390	2/ 441	0/ 1	181/ 3920	0	4
34	1	-311/ 10780	181/ 6468	-1/ 196	-1/ 84	56291/ 235200	2	1
34	2	-311/ 40425	13/ 1617	1/ 490	1/ 210	4043/ 58800	3	1
34	3	-52/ 8085	724/ 40425	-4/ 735	0/ 1	2353/ 44100	2	2
34	4	-208/ 121275	208/ 40425	8/ 3675	0/ 1	169/ 11025	3	2
34	5	311/ 323400	5/ 6468	1/ 2940	1/1260	311/ 47040	4	1
34	6	-23/ 43120	181/ 176400	-1/ 2520	0/ 1	4163/ 940800	2	3
34	7	26/ 121275	4/ 8085	4/ 11025	0/ 1	13/ 8820	4	2
34	8	-23/ 161700	13/ 44100	1/ 6300	0/ 1	299/ 235200	3	3
34	9	23/1293600	1/ 35280	1/ 37800	0/ 1	23/ 188160	4	3
34	10	-151/ 10780	-181/ 6468	1/ 196	0/ 1	27331/ 235200	2	1
34	11	-151/ 40425	-13/ 1617	-1/ 490	0/ 1	1963/ 58800	3	1
34	12	19/ 4620	543/ 53900	1/ 2940	0/ 1	-3439/ 100800	2	2
34	13	19/ 17325	39/ 13475	-1/ 7350	0/ 1	-247/ 25200	3	2
34	14	151/ 323400	-5/ 6468	-1/ 2940	0/ 1	151/ 47040	4	1
34	15	-13/ 32340	-181/ 485100	-1/ 8820	0/ 1	2353/ 705600	2	3
34	16	-19/ 138600	3/ 10780	-1/ 44100	0/ 1	-19/ 20160	4	2
34	17	-13/ 121275	-13/ 121275	1/ 22050	0/ 1	169/ 176400	3	3
34	18	13/ 970200	-1/ 97020	1/132300	0/ 1	13/ 141120	4	3
34	19	151/ 10780	-281/ 6468	-1/ 196	0/ 1	42431/ 235200	2	1
34	20	-151/ 14700	23/ 2156	1/ 140	0/ 1	-3473/ 78400	3	1
34	21	-19/ 4620	843/ 53900	-1/ 2940	0/ 1	-5339/ 100800	2	2
34	22	19/ 6300	-207/ 53900	1/ 2100	0/ 1	437/ 33600	3	2
34	23	151/ 53400	-1/ 1078	-1/ 1470	0/ 1	151/ 39200	4	1
34	24	13/ 32340	-281/ 485100	1/ 8820	0/ 1	3653/ 705600	2	3
34	25	-19/ 23100	9/ 26950	-1/ 22050	0/ 1	-19/ 16800	4	2
34	26	-13/ 44100	23/ 161700	-1/ 6300	0/ 1	-299/ 235200	3	3
34	27	13/ 161700	-1/ 80850	1/ 66150	0/ 1	13/ 117600	4	3
34	28	311/ 10780	281/ 6468	1/ 196	1/ 84	87391/ 235200	2	1
34	29	-311/ 14700	-23/ 2156	-1/ 140	-1/ 60	-7153/ 78400	3	1
34	30	52/ 8085	1124/ 40425	4/ 735	0/ 1	3653/ 44100	2	2
34	31	-52/ 11025	-92/ 13475	-4/ 525	0/ 1	-299/ 14700	3	2
34	32	311/ 53400	1/ 1078	1/ 1470	1/ 630	311/ 39200	4	1
34	33	23/ 43120	281/ 176400	1/ 2520	0/ 1	6463/ 940800	2	3
34	34	52/ 40425	8/ 13475	8/ 11025	0/ 1	13/ 7350	4	2
35	1	281/ 6468	-151/ 10780	1/ 196	0/ 1	-42431/ 235200	1	2
35	2	843/ 53400	-19/ 4620	-1/ 2940	0/ 1	-5339/ 100800	2	2
35	3	23/ 2156	-151/ 14700	1/ 140	0/ 1	-3473/ 78400	1	3
35	4	207/ 53400	-19/ 6300	-1/ 2100	0/ 1	-437/ 33600	2	3
35	5	281/ 485100	-13/ 32340	-1/ 8820	0/ 1	-3653/ 705600	3	2
35	6	1/ 1078	-151/ 53900	1/ 1470	0/ 1	-151/ 39200	1	4
35	7	23/ 161700	-13/ 44100	-1/ 6300	0/ 1	-299/ 235200	3	3
35	8	9/ 26950	-19/ 23100	-1/ 22050	0/ 1	-19/ 16800	2	4
35	9	1/ 80850	-13/ 161700	-1/ 66150	0/ 1	-13/ 117600	3	4
35	10	181/ 6468	151/ 10780	-1/ 196	0/ 1	-27331/ 235200	1	2
35	11	543/ 53900	19/ 4620	1/ 2940	0/ 1	-3439/ 100800	2	2
35	12	-13/ 1617	-151/ 40425	-1/ 490	0/ 1	1963/ 58800	1	3
35	13	-39/ 13475	-19/ 17325	1/ 7350	0/ 1	247/ 25200	2	3

i	j	$\tilde{\gamma}^{(6)}_{ij}$	$\tilde{\gamma}^{(7)}_{ij}$	$\tilde{\gamma}^{(8)}_{ij}$	$\tilde{\gamma}^{(9)}_{ij}$	$\tilde{\gamma}^{(10)}_{ij}$	$\tilde{\lambda}_{ij}$	$\tilde{\mu}_{ij}$
35	14	181/ 485100	13/ 32340	1/ 8820	0/ 1	-2353/ 705600	3	2
35	15	5/ 6468	-151/ 323400	1/ 2940	0/ 1	-151/ 47040	1	4
35	16	-13/ 121275	-13/ 121275	1/ 22050	0/ 1	169/ 176400	3	3
35	17	3/ 10780	-19/ 138600	-1/ 44100	0/ 1	-19/ 20160	2	4
35	18	1/ 97020	-13/ 970200	-1/132300	0/ 1	-13/ 141120	3	4
35	19	-181/ 6468	311/ 10780	1/ 196	1/ 84	-56291/ 235200	1	2
35	20	724/ 40425	-52/ 8085	-4/ 735	0/ 1	2353/ 44100	2	2
35	21	13/ 1617	-311/ 40425	1/ 490	1/ 210	4043/ 58800	1	3
35	22	-208/ 40425	208/ 121275	-8/ 3675	0/ 1	-169/ 11025	2	3
35	23	-181/ 176400	23/ 43120	1/ 2520	0/ 1	-4163/ 940800	3	2
35	24	-5/ 6468	-311/ 323400	-1/ 2940	-1/1260	-311/ 47040	1	4
35	25	13/ 44100	-23/ 161700	1/ 6300	0/ 1	299/ 235200	3	3
35	26	4/ 8085	26/ 121275	4/ 11025	0/ 1	13/ 8820	2	4
35	27	-1/ 35280	-23/1293600	-1/ 37800	0/ 1	-23/ 188160	3	4
35	28	-281/ 6468	-311/ 10780	-1/ 196	-1/ 84	-87391/ 235200	1	2
35	29	1124/ 40425	52/ 8085	4/ 735	0/ 1	3653/ 44100	2	2
35	30	-23/ 2156	-311/ 14700	-1/ 140	-1/ 60	-7153/ 78400	1	3
35	31	92/ 13475	52/ 11025	4/ 525	0/ 1	299/ 14700	2	3
35	32	-281/ 176400	-23/ 43120	-1/ 2520	0/ 1	-6463/ 940800	3	2
35	33	-1/ 1078	-311/ 53900	-1/ 1470	-1/ 630	-311/ 39200	1	4
35	34	-23/ 58800	-23/ 58800	-1/ 1800	0/ 1	-529/ 313600	3	3
35	35	8/ 13475	52/ 40425	8/ 11025	0/ 1	13/ 7350	2	4
36	1	-281/ 129360	181/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
36	2	-281/ 485100	13/ 32340	1/ 8820	0/ 1	3653/ 705600	3	2
36	3	-23/ 43120	181/ 176400	-1/ 2520	0/ 1	4163/ 940800	2	3
36	4	-23/ 161700	13/ 44100	1/ 6300	0/ 1	299/ 235200	3	3
36	5	281/3880800	1/ 25872	1/ 52920	0/ 1	281/ 564480	4	2
36	6	-1/ 21560	181/ 646800	-1/ 26460	0/ 1	181/ 470400	2	4
36	7	23/1293600	1/ 35280	1/ 37800	0/ 1	23/ 188160	4	3
36	8	-1/ 80850	13/ 161700	1/ 66150	0/ 1	13/ 117600	3	4
36	9	1/ 646800	1/ 129360	1/396900	0/ 1	1/ 94080	4	4
36	10	-181/ 129360	-181/ 129360	1/ 3528	0/ 1	32761/2822400	2	2
36	11	-181/ 485100	-13/ 32340	-1/ 8820	0/ 1	2353/ 705600	3	2
36	12	13/ 32340	181/ 485100	1/ 8820	0/ 1	-2353/ 705600	2	3
36	13	13/ 121275	13/ 121275	-1/ 22050	0/ 1	-169/ 176400	3	3
36	14	181/3880800	-1/ 25872	-1/ 52920	0/ 1	181/ 564480	4	2
36	15	-1/ 25872	181/3880800	-1/ 52920	0/ 1	181/ 564480	2	4
36	16	-13/ 970200	1/ 97020	-1/132300	0/ 1	-13/ 141120	4	3
36	17	-1/ 97020	13/ 970200	1/132300	0/ 1	13/ 141120	3	4
36	18	1/ 776160	1/ 776160	1/793800	0/ 1	1/ 112896	4	4
36	19	181/ 129360	-281/ 129360	-1/ 3528	0/ 1	50861/2822400	2	2
36	20	-181/ 176400	23/ 43120	1/ 2520	0/ 1	-4163/ 940800	3	2
36	21	-13/ 32340	281/ 485100	-1/ 8820	0/ 1	-3653/ 705600	2	3
36	22	13/ 44100	-23/ 161700	1/ 6300	0/ 1	299/ 235200	3	3
36	23	181/ 646800	-1/ 21560	-1/ 26460	0/ 1	181/ 470400	4	2
36	24	1/ 25872	281/3880800	1/ 52920	0/ 1	281/ 564480	2	4
36	25	-13/ 161700	1/ 80850	-1/ 66150	0/ 1	-13/ 117600	4	3
36	26	-1/ 35280	-23/1293600	-1/ 37800	0/ 1	-23/ 188160	3	4
36	27	1/ 129360	1/ 646800	1/396900	0/ 1	1/ 94080	4	4
36	28	281/ 129360	281/ 129360	1/ 3528	0/ 1	78961/2822400	2	2
36	29	-281/ 176400	-23/ 43120	-1/ 2520	0/ 1	-6463/ 940800	3	2
36	30	23/ 43120	281/ 176400	1/ 2520	0/ 1	6463/ 940800	2	3
36	31	-23/ 58800	-23/ 58800	-1/ 1800	0/ 1	-529/ 313600	3	3
36	32	281/ 646800	1/ 21560	1/ 26460	0/ 1	281/ 470400	4	2
36	33	1/ 21560	281/ 646800	1/ 26460	0/ 1	281/ 470400	2	4
36	34	23/ 215600	1/ 29400	1/ 18900	0/ 1	23/ 156800	4	3
36	35	-1/ 29400	-23/ 215600	-1/ 18900	0/ 1	-23/ 156800	3	4
36	36	1/ 107800	1/ 107800	1/149450	0/ 1	1/ 78400	4	4

Table 9

	Conc. Load P at(a,b)	Conc. Mom. M_x at(a,b)	Uniform Load q	Dist. Edge Moment M				Dist. Edge Load V			
				on x = 0	on x = a	on y = 0	on y = b	on x = 0	on x = a	on y = 0	on y = b
w_{11}	0	0	$qab/4$	0	0	0	0	$bV/2$	0	$aV/2$	0
w_{x11}	0	0	$qa^2b/20$	$bM/2$	0	0	0	0	0	$a^2V/10$	0
w_{y11}	0	0	$qab^2/20$	0	0	$aM/2$	0	$b^2V/12$	0	0	0
w_{xy11}	0	0	$qa^2b^2/100$	$b^2M/10$	0	$a^2M/10$	0	0	0	0	0
w_{xx11}	0	0	$qa^3b/240$	0	0	0	0	0	0	$a^3V/120$	0
w_{yy11}	0	0	$qab^3/240$	0	0	0	0	$b^3V/120$	0	0	0
w_{xxy11}	0	0	$qa^3b^2/1200$	0	0	$a^3M/120$	0	0	0	0	0
w_{xyy11}	0	0	$qa^2b^3/1200$	$b^3M/120$	0	0	0	0	0	0	0
w_{xxyy11}	0	0	$qa^3b^3/14400$	0	0	0	0	0	0	0	0
w_{12}	0	0	$qab/4$	0	0	0	0	$bV/2$	0	0	$aV/2$
w_{x12}	0	0	$qa^2b/20$	$bM/2$	0	0	0	0	0	0	$a^2V/10$
w_{y12}	0	0	$-qab^2/20$	0	0	0	$aM/2$	$-b^2V/10$	0	0	0
w_{xy12}	0	0	$-qa^2b^2/100$	$-b^2M/10$	0	0	$a^2M/10$	0	0	0	0
w_{xx12}	0	0	$qa^3b/240$	0	0	0	0	0	0	0	$a^3V/120$
w_{yy12}	0	0	$qab^3/240$	0	0	0	0	$b^3V/120$	0	0	0
w_{xxy12}	0	0	$-qa^3b^2/1200$	0	0	0	$a^3M/120$	0	0	0	0
w_{xyy12}	0	0	$qa^2b^3/1200$	$b^3M/120$	0	0	0	0	0	0	0
w_{xxyy12}	0	0	$qa^3b^3/14400$	0	0	0	0	0	0	0	0

Table 9 (continued)

	Conc. Load P at(a,b)	Conc. Mom. M_x at(a,b)	Uniform Load q	Dist. Edge Moment M on x = 0	on x = a	on y = 0	on y = b	Dist. Edge Load V on x = 0	on x = a	on y = 0	on y = b
w_{22}	P	0	$qab/4$	0	0	0	0	0	$bV/2$	0	$aV/2$
w_{x22}	0	M_x	$-qa^2b/20$	0	$bM/2$	0	0	0	0	0	$-a^2V/10$
w_{y22}	0	0	$-qab^2/20$	0	0	0	$aM/2$	0	$-b^2V/10$	0	0
w_{xy22}	0	0	$qa^2b^2/100$	0	$-b^2M/10$	0	$-a^2M/10$	0	0	0	0
w_{xx22}	0	0	$qa^3b/240$	0	0	0	0	0	0	0	$a^3V/120$
w_{yy22}	0	0	$qab^3/240$	0	0	0	0	0	$b^3V/120$	0	0
w_{xxy22}	0	0	$-qa^3b^2/1200$	0	0	0	$a^3M/120$	0	0	0	0
w_{xyy22}	0	0	$-qa^2b^3/1200$	0	$b^3M/120$	0	0	0	0	0	0
w_{xxyy22}	0	0	$qa^3b^3/14400$	0	0	0	0	0	0	0	0
w_{21}	0	0	$qab/4$	0	0	0	0	0	$bV/2$	$aV/2$	0
w_{x21}	0	0	$-qa^2b/20$	0	$bM/2$	0	0	0	0	$-a^2V/10$	0
w_{y21}	0	0	$qab^2/20$	0	0	$aM/2$	0	0	$b^2V/10$	0	0
w_{xy21}	0	0	$-qa^2b^2/100$	0	$b^2M/10$	$-a^2M/10$	0	0	0	0	0
w_{xx21}	0	0	$qa^3b/240$	0	0	0	0	0	0	$a^3V/120$	0
w_{yy21}	0	0	$qab^3/240$	0	0	0	0	0	$b\ V/120$	0	0
w_{xxy21}	0	0	$qa^3b^2/1200$	0	0	$a^3M/120$	0	0	0	0	0
w_{xyy21}	0	0	$-qa^2b^3/1200$	0	$b^3M/120$	0	0	0	0	0	0
w_{xxyy21}	0	0	$qa^3b^3/14400$	0	0	0	0	0	0	0	0

ADDENDUM

During the period between the submission of the manuscript of this paper and its presentation at the conference, certain deficiencies in the bending elements were uncovered in the course of continuing numerical experimentation and application. These deficiencies are due to the fact that the assumed displacement states while geometrically admissible are incomplete. This means that certain simple, displacement states (e.g. $w(x,y) = xy$) are missing from the representation while higher order states (e.g. $w(x,y) = x^3y^3$) are included. In addition to giving the elements an unnecessarily high stiffness, this situation can affect the monotonicity of convergence.

In the interest of accuracy and completeness this addendum was prepared to present improved versions of the compatible elements.

Taking for the displacement

$$\tilde{w}(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2}\Big[H_{0i}^{(1)}(x)\,H_{0j}^{(1)}(y)\,w_{ij} + H_{1i}^{(1)}(x)\,H_{0j}^{(1)}(y)\,w_{ij} + H_{0i}^{(1)}(x)\,H_{1i}^{(1)}(y)\,w_{yij} + H_{1i}^{(1)}(x)\,H_{1j}^{(1)}(y)\,w_{xyij}\Big] \tag{57}$$

instead of Equation 43, a 16-degree of freedom element is obtained (as opposed to 12 with the previous displacement state). This assumed displacement mode is capable of representing exactly any displacement of the form

$$w(x,y) = \sum_{r=0}^{3}\sum_{s=0}^{3} \alpha_{rs}\, x^r\, y^s \tag{58}$$

whereas the 12-degree of freedom element is not. Constructing the total potential energy for the element, taking the partial derivatives of this with respect to the independent degrees of freedom (w_{ij}, w_{xij}, w_{yij}, w_{xyij}) and setting these equal to zero yields

$$\tilde{Q}_b\,\tilde{W} = \tilde{P}_b$$

where $\tilde{Q}_b$ is the new bending stiffness matrix (16 x 16) for the plate element. The elements of this matrix are given by the formula:

$$\tilde{q}_{ij} = \frac{D}{ab}\left[\tilde{\gamma}_{ij}^{(1)}\left(\frac{b}{a}\right)^2 + \tilde{\gamma}_{ij}^{(2)}\left(\frac{a}{b}\right)^2 + \tilde{\gamma}_{ij}^{(3)} + \tilde{\gamma}_{ij}^{(4)}\nu\right] a^{\tilde{\lambda}_{ij}}\; b^{\tilde{\mu}_{ij}} \tag{60}$$

and the consistent mass matrix, $\tilde{M}_b$, by

$$\tilde{m}_{ij} = \frac{\rho_m\, h\, a b}{1225}\,\tilde{\gamma}_{ij}\; a^{\tilde{\lambda}_{ij}}\; b^{\tilde{\mu}_{ij}} \tag{61}$$

where $\tilde{\gamma}_{ij}^{(k)}$, k = 1, ..., 5 and $\tilde{\lambda}_{ij}$ and $\tilde{\mu}_{ij}$ are given in Table 6 (in this addendum). The vector $\tilde{W}$ is arranged with the same cyclic order as before but with the variables ordered as w_{ij}, w_{xij}, w_{yij}, w_{xyij}. Examples of the "load vector", $\tilde{P}_b$, are given in Table 7.

In order to assemble these elements into a compatible collection, each interior node will have four degrees of freedom, w_{ij}, w_{xij}, w_{yij}, and w_{xyij}.

For the hyperosculatory element the complete mode is:

$$\tilde{w}(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2}\Big[H_{0i}^{(2)}(x)H_{0j}^{(2)}(y)w_{ij} + H_{1i}^{(2)}(x)H_{0j}^{(2)}(y)w_{xij} + H_{0i}^{(2)}(x)H_{1j}^{(2)}(y)w_{yij}$$

$$+ H_{1i}^{(2)}(x)H_{1j}^{(2)}(y)w_{xyij} + H_{2i}^{(2)}(x)H_{0j}^{(2)}(y)w_{xxij} + H_{0i}^{(2)}(x)H_{2j}^{(2)}(y)w_{yyij}$$

$$+ H_{2i}^{(2)}(x)H_{1j}^{(2)}(y)w_{xxyij} + H_{1i}^{(2)}(x)H_{2j}^{(2)}(y)w_{xyyij} + H_{2i}^{(2)}(x)H_{2j}^{(2)}(y)w_{xxyyij}\Big]$$

This displacement mode can represent any displacement state of the form:

$$w(x,y) = \sum_{r=0}^{5}\sum_{s=0}^{5} a_{rs}\, x^r y^s$$

The stationary condition for the element potential energy using this mode is:

$$\hat{Q}_b \hat{W} = \hat{P}_b \tag{63}$$

where $\hat{Q}_b$ is the new hyperosculatory bending stiffness matrix (36 x 36). The elements of this matrix are given by the formula

$$\hat{q}_{ij} = \frac{D}{ab}\left[\tilde{\gamma}_{ij}^{(6)}\left(\frac{b}{a}\right)^2 + \tilde{\gamma}_{ij}^{(7)}\left(\frac{a}{b}\right)^2 + \tilde{\gamma}_{ij}^{(8)} + \tilde{\gamma}_{ij}^{(9)}\,\nu\right] a^{\tilde{\lambda}_{ij}}\, b^{\tilde{\mu}_{ij}} \tag{64}$$

and of the consistent mass matrix, M_t, by

$$\hat{m}_{ij} = \frac{\rho_m\, h\, ab}{1089}\,\tilde{\gamma}_{ij}^{(10)}\, a^{\tilde{\lambda}_{ij}}\, b^{\tilde{\mu}_{ij}} \tag{65}$$

where $\tilde{\gamma}_{ij}^{(k)}$, k = 6, ..., 10, and $\tilde{\lambda}_{ij}$ and $\tilde{\mu}_{ij}$, are given in Table 8. The vector $\hat{W}$ is arranged with the same cyclic order as before but with the variables arranged as w_{ij}, w_{xij}, w_{yij}, w_{xyij}, w_{xxij}, w_{yyij}, w_{xxyij}, w_{xyyij}, and w_{xxyyij}. Examples of the vector $\hat{P}_b$ are given in Table 9.

In using this element to model a continuous plate, the compatibility requirements are:

1) all four elements at a node must have identical values of w_{ij}, w_{xij}, w_{yij}, and w_{xyij} at the common corner

2) the elements joining along the edge x = const. must have identical values of w_{yyij} and w_{xyyij} at their common corners.

3) the elements joining the edge y = const. must have identical values of w_{xxij} and w_{xxyij} at their common corners.

These requirements are deduced by examining the dependence of $\tilde{w}(x,y)$, $\partial\tilde{w}(x,y)/\partial x$ and $\partial\tilde{w}(x,y)/\partial y$ along edges y = const. and x = const. All coefficients which affect these dependencies must be equal for the elements to be compatible.

These requirements permit 16 degrees of freedom at an interior node: one value each of w_{ij}, w_{xij}, w_{yij} and w_{xyij} and two values each of w_{xxij}, w_{yyij}, w_{xxyij} and w_{xyyij} and four values of w_{xxyyij}. If desired, these may all be set equal reducing the degrees of freedom to 9. While this is not required to satisfy the compatibility conditions it may be considered desirable in some cases to preclude discontinuities of curvature.

The versions of these elements given in this addendum were applied to the same problems that are given in the body of the paper. For the clamped plate the new static results are:

Using $\tilde{Q}_b$ (16-degree of freedom element)

No. of Elements	No. of Degrees of Freedom	min π_p (in.-lb)	w_c (in.)
1	1	-0.010598	0.042393
4	9	-0.012282	0.040475
9	25	-0.012415	0.040482
16	49	-0.012440	0.040487
Using $\hat{Q}_b$ (36-degree of freedom element)			
1	9	-0.012449	0.040488
4	36	-0.012451	0.040490
4	49	-0.012451	0.040490

For the simply supported plate the frequency results are:

Using $\tilde{Q}_b$ and $\tilde{M}_b$ (16-degree of freedom element)

No. of Elements	No. of Degrees of Freedom	ω_{11}	ω_{12}, ω_{21}	ω_{22}	ω_{31}, ω_{13}	ω_{32}, ω_{23}	ω_{33}
4	16	1037	2777	4399	6222	7702	10745
16	64	1035	2593	4147	5251	6791	9410
Exact		1035	2587	4138	5173	6725	9311

In addition to the observations made in the body of the paper it may also be noted that these results exhibit extremely rapid convergence.

It develops that monotonicity of convergence of the previous element was not always assured for repetitive refinements of the gridwork. Results obtained using the 12-degree of freedom element for natural frequency studies of a cantilever plate failed to converge monotonically for some frequencies. The 16-degree of freedom element has resolved the difficulty.