SESSION 1. STRUCTURAL WEIGHT OPTIMIZATION

Session Chairman

M. J. Turner

The Boeing Co.
Seattle, Washington

APPLICATION OF THE CREATED RESPONSE SURFACE TECHNIQUE TO STRUCTURAL OPTIMIZATION

Pedro V. Marcal*
Brown University, Providence, Rhode Island

Ronald A. Gellatly**
Textron's Bell Aerosystems, Buffalo, New York

The created response surface technique was applied to structural optimization with multiload conditions. The constrained optimization problem was converted to an unconstrained problem by the use of penalty functions which varied inversely with the distance of the design point from a constraint. Response surfaces were also introduced for optimization with discrete variables. The Rosenbrock Method of unconstrained optimization was used. The original method was modified to take advantage of the interaction between the optimization procedure and the response characteristics of the structure. This, when coupled with the r extrapolation procedure, resulted in considerable savings in computing time. A pilot computer program was developed, and case studies of truss-type structures were made.

---

* Assistant Professor of Engineering, formerly Senior Research Engineer, Textron's Bell Aerosystems.

** Assistant Chief Engineer, Advanced Structural Design Technology.

SECTION I

INTRODUCTION

The basic approach to the problem of elastic structural optimization as a constrained minimization was presented by Schmit (Reference 1) in 1960. The concepts, which are based upon the combination of methods of automatic structural analysis and operations research, are now well known. The original methods, which are only strictly usable for small scale structures, have been subjected to considerable development and are now applicable to realistic large scale structures (References 2 and 3).

In order to solve the constrained optimization problem, a design space approach has been developed. The coordinates of the space are the variables in the structure to be optimized. Within the space, a number of surfaces are defined to correspond to constraint conditions and a merit function. The merit function is frequently the weight of the structure but other criteria may also be used. Using the nonlinear mathematical programming techniques reviewed later, the minimum value of the merit function is sought, subject to the prescribed constraints on the response characteristics of the system. Generally, the cross-sectional areas and geometric configuration form the design variables while the constraints are limits on stresses and displacements. When appropriate, additional classes of design variables and constraints may be used.

Although this approach has been developed and used for large scale optimizations, there are certain limits to the scope and range of economic applicability. In effect, these limitations restrict the use of the constrained optimization approach to cases in which the merit criterion is a linear function of the design variables. This limit generally implies that only material thicknesses and cross-sectional areas can be treated as variables and that all configuration and geometry must be fixed. For nonlinear merit functions, the search methods are less efficient, and computational expense can become prohibitively large.

To overcome these restrictions, a modified approach, the unconstrained approach is necessary to the problem. Since the natural expression of the structural optimization problem is a constrained minimization, the representation in an unconstrained form requires some mathematical reformulation. To accomplish this, so-called penalty functions are defined that account for nonsatisfaction of the constraint conditions. The penalty function, which is continuous with continuous derivatives, is weighted and added to the basic merit or objective function to form a "created response surface". For each value of the weighting factor, a complete continuous response surface exists whose minimum may be found using unconstrained search techniques. By successive reductions of the weighting factor, a series of minima is determined whose limit, when the weighting factor vanishes, provides the constrained minimum of the basic merit function only. This approach was originally proposed by Carroll (Reference 4).

By using the reformulated unconstrained minimization approach, a great degree of generality is achieved in the definable classes of design variables. The only restriction that is still applicable is that the variables be continuous functions. In many classes of structures, e.g. space frames fabricated from standard sections, the assumption of continuous variation of property is inappropriate. A method for considering discrete variation is required.

Toakley (Reference 5) in a paper dealing with minimum weight plastic design, considered discrete sections using three different methods. Since the methods of limit analysis were used, the problem reduced to a purely linear one. The methods used are not applicable to the present totally nonlinear regimes. Within the framework of the created response surface, a technique has been developed that will permit the use of discrete variables. Effectively, the introduction of discrete sections into the design process is equivalent to the specification of a multiplicity of equality constraints. That is, a variable must be equal to one of a prescribed list of values. To effect this an additional penalty function is introduced into the unconstrained minimization process. The new function will vanish only when equality conditions are satisfied and will be nonzero at all other points. By weighting this function in an appropriate manner the optimization process can be made to converge on the discrete values.

Melosh and Luik in Reference 6 have developed an efficient reanalysis method using self-equilibrating stress systems. This was combined with an allocation procedure to optimize trusses with discrete or continuous section properties.

More recently, Schmit et al. in Reference 7 have applied the unconstrained minimization method to study integrally stiffened cylindrical shells.

A small scale optimization program has been developed using the unconstrained approach with the capability of including discrete sections. Results obtained using this program are included along with a description of the unconstrained techniques.

Before proceeding to a detailed presentation of the unconstrained minimization methods, a brief discussion of the constrained problem is given.

SECTION II

OPTIMIZATION

CONSTRAINED OPTIMIZATION

The general constrained optimization problem may be expressed in a strictly mathematical form.

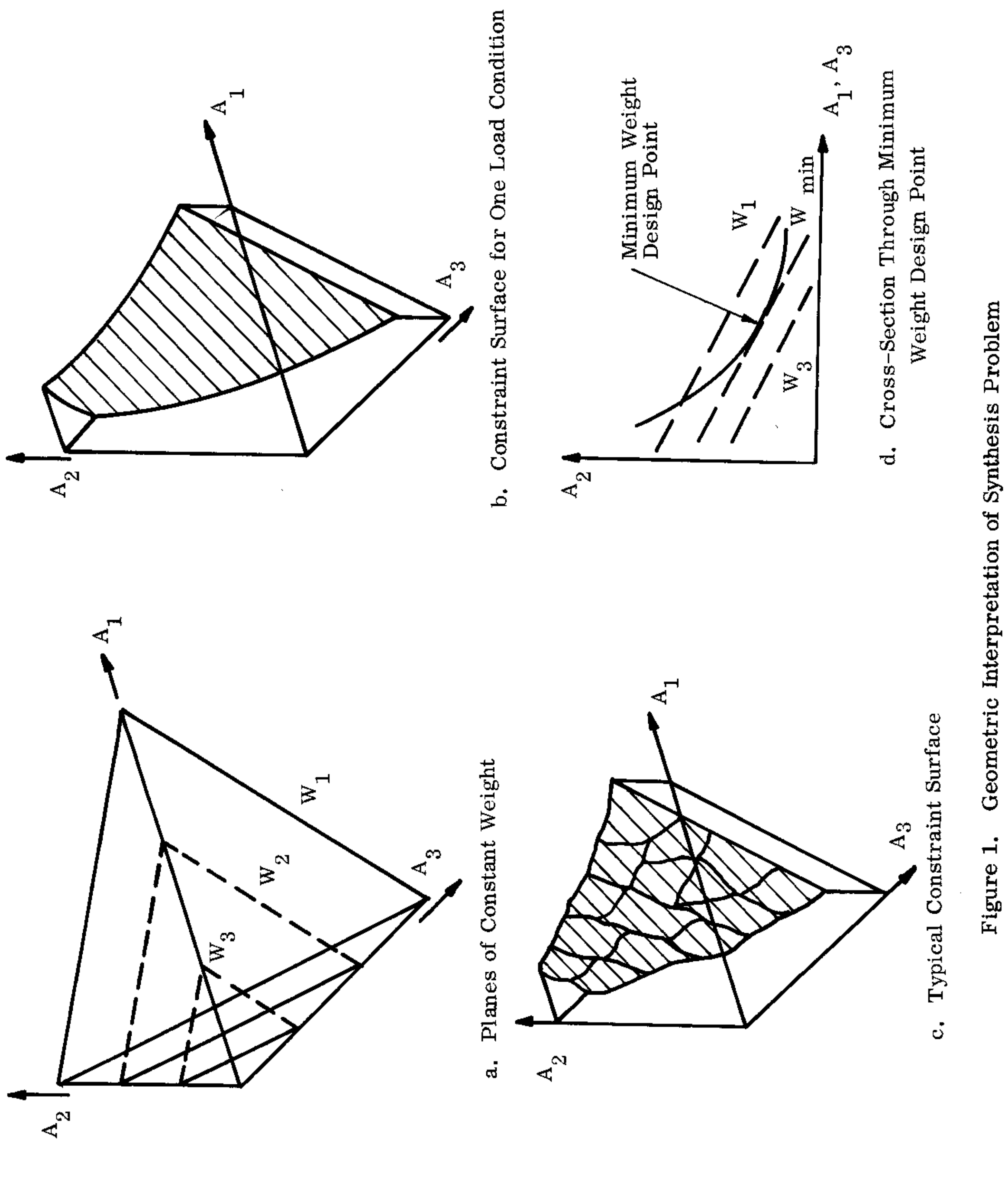
It is required to find the values of the variables $a_1\ a_2 \ldots a_n$ such that the function $f\ (a_1\ a_2 \ldots a_n)$ has a minimum value subject to satisfaction of the p constraint conditions.

$$g_i\ (a_1, a_2 \cdots a_n) + k_i \geq 0 \qquad i = 1, 2, \cdots p \tag{1}$$

The function f may be the weight of the structure and the functions g_i the stresses and displacements which must be less than prescribed values k_i.

In order to discuss the method of solution, it is convenient to use a simple geometric representation of the problem. A design space is defined in which each dimension represents a variable. For graphical purposes it is necessary to limit this to two or three dimensional space; but in the general problem with n variables, an n-dimensional hyperspace must be defined. Since mathematical processes developed for the two or three dimensional spaces are applicable to the higher order spaces, only the simple three dimensional case need be discussed. If attention is restricted to designs in which the configuration and geometry are fixed and only one cross-sectional dimension for each element is variable, the weight is a linear function of these variables. Then all designs of a given weight lie on a plane in three dimensions (Figure 1a). In higher order spaces, the function is still linear and corresponds to a hyperplane. For every weight, such a linear function exists and hence all possible weights can be represented by a family of parallel lines or planes.

In general the constraint conditions, Equation 1, are nonlinear functions. The equality conditions will be represented by curved surfaces in the design space shown in Figure 1b. For each of the p constraints, a surface will exist. These surfaces are generally convex when viewed from the origin. The dominant portions combine to form a composite constraint surface as indicated in Figure 1c. This surface then provides the boundary between the regions of the space in which a design is acceptable (inequality of Equation 1 is satisfied) and is unacceptable (inequality not satisfied). Since the nearer a weight surface is to the origin, the lower the weight; the minimum weight acceptable design will occur when a weight surface touches the



a. Planes of Constant Weight

b. Constraint Surface for One Load Condition

c. Typical Constraint Surface

d. Cross-Section Through Minimum Weight Design Point

Figure 1. Geometric Interpretation of Synthesis Problem

composite constraint at one point only (Figure 1d). The processes of determining this osculatory point can now be reduced to the definition of a suitable travel path through the design space from some arbitrary starting point until the optimum is reached. Each step along this path involves a change in the variables and hence a redesign of the structure. Analyses are performed at each step to provide information for the size and direction of future steps. In one approach (Reference 2) two modes of travel are used to determine the optimum point, steepest descent and side step. In the steepest descent mode, the weight is reduced in the most rapid fashion until a constraint surface is met. The object of the next stage of redesign, the side step, is to move to a point, as far as possible, away from the constraint surface from which a new steepest descent may be initiated. This step is performed at constant weight. Details of this approach, including methods of determining constraint derivatives, are presented in Reference 2.

If additional variables are introduced to make the merit function nonlinear, it can be seen that the above method is no longer strictly applicable. Travel orthogonal to the merit function would be along a curved path, requiring continuous recomputation of the directions. A modified version of this approach has been developed for nonlinear merit functions (Reference 3) but it has become apparent that the unconstrained formulation is more efficient and has greater potential.

UNCONSTRAINED OPTIMIZATION

The constrained minimization requires the determination of the lowest value of some function for which no mathematical minimum may be found, within a region bounded by prescribed constraint conditions, In the unconstrained approach a combined continuous merit function is defined to replace the original separate merit and constraint functions. With this new single function, both the basic merit function and the constraints are approximated within the feasible region. By suitable adjustment of an arbitrary weighting factor, the combined function can be made to approximate the basic merit as closely as desired until in the limit the desired minimum is found. To perform this transformation a so-called penalty function is created. This penalty, which is weighted by an arbitrary scalar, and combined with the basic merit function, accounts for the nonsatisfaction of the constraint conditions. In a given structural problem,

$DV_i \equiv$ design variables $i = 1, 2, \ldots \ell$

$BF_{jk} \equiv$ jth response characteristic* of the structure under the kth loading condition $j = 1, 2, \ldots m$, $k = 1, 2, \ldots n$

U, L = Superscripts denoting upper and lower limiting values (i.e. constraint conditions)

* Response characteristic is taken here to mean critical stress, displacement or any other measurable response of the structure to the loading system.

Then a penalty function P_c due to the constraint conditions on both design variables and response functions is defined by

$$P_c = \sum_{i=1}^{\ell} (DV_i^U - DV_i^L) \left[\frac{1}{DV_i^U - DV_i} + \frac{1}{DV_i - DV_i^L} \right]$$
$$+ \sum_{j=1}^{m} \sum_{k=1}^{m} (BF_{jk}^U - BF_{jk}^L) \left[\frac{1}{BF_{jk}^U - BF_{jk}} + \frac{1}{BF_{jk} - BF_{jk}^L} \right] \qquad (2)$$

Since the various design variables and behavior functions can take on values of differing orders of magnitude, it is necessary to introduce scaling factors to ensure that each component of the summation has an approximately equal contribution to the total penalty. The total permissible range, e.g. $(DV_i^U - DV_i^L)$, for each characteristic is used as this factor. It can be seen that this penalty function will tend to infinity when any constraint condition is satisfied but each term diminishes rapidly at some distance from a constraint. To obtain the total function that will be used in the unconstrained optimization, the above penalty is weighted by an arbitrary factor and added to the basic merit function W.

The total function is then written as

$$F = W + r_c P_c \qquad (3)$$

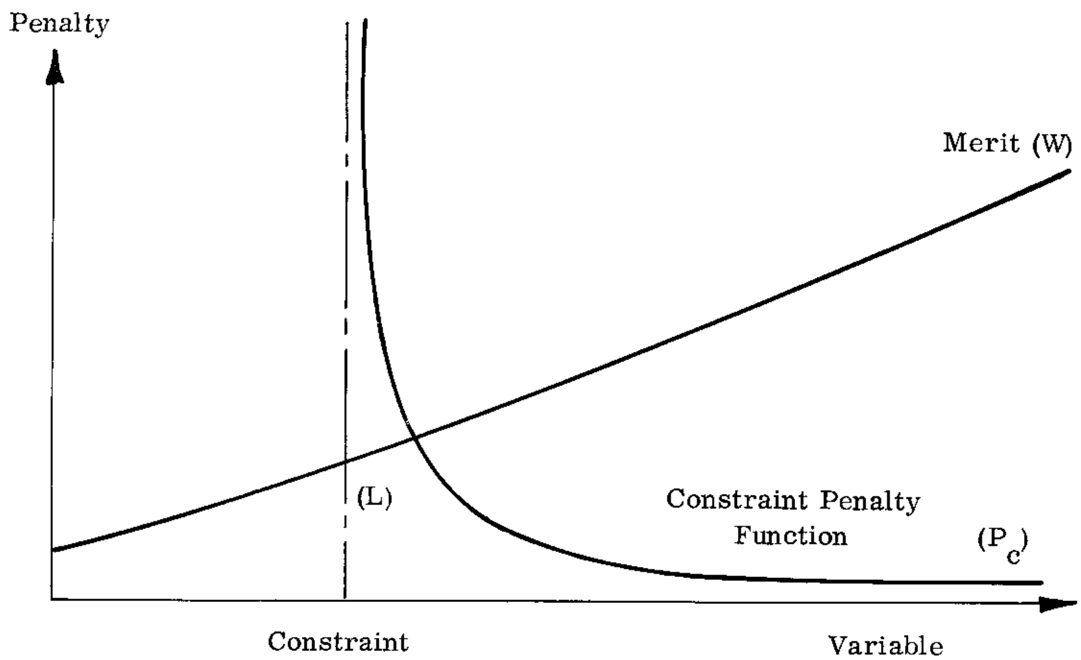


Figure 2. Unconstrained Minimization Penalty Functions

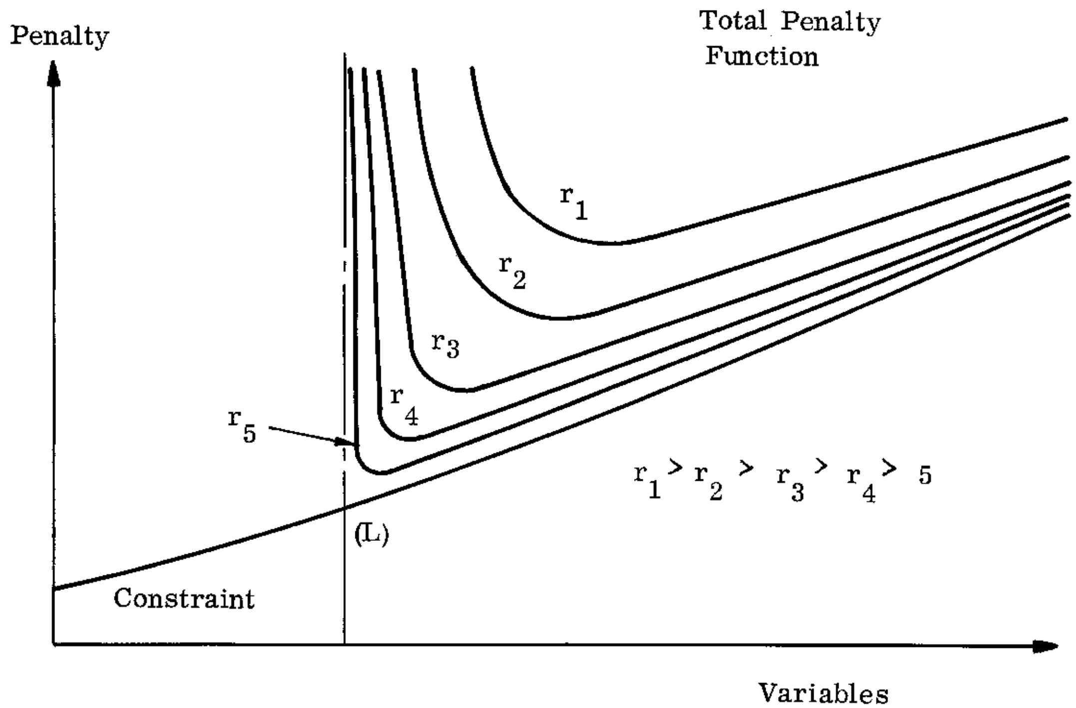


Figure 3. Created Response Surfaces

The effect of this summation is to create a family of response surfaces, which will lie in the nonviolated region of the design space and will be bounded by the composite constraint surface. This approach, basically attributable to Carroll (Reference 4), is known as the Created Response Surface Technique.

The physical meaning of these surfaces may be presented graphically as in Figures 2 and 3. In the figures the horizontal axis represents the design variables whereas the vertical direction represents the total value of the penalty function Equation 3. The basic merit function W is some simple (monotonic) function of the design variables and it has a lowest value of L corresponding to some constraint condition. The basic penalty P_c in Equation 2 has the general form of a rectangular hyperbola having both he constraint condition and the horizontal axis as aymptotes. By introducing the weighting factor r_c, a family of hyperbolas is created. Combining these hyperbolas with W, Equaion 3 yields the series of curves (surfaces) of Figure 3. Each curve of the family exhibits a unique minimum. With decreasing values of r_c these minima approach the lowest value L until, in the limit as r_c tends to zero, the value L is actually achieved. The successive minima of the created response surfaces are determined by unconstrained search techniques.

With this approach, the precise nature of the basic merit function (linear or nonlinear) is immaterial to the optimization process. Hence, the method has an almost unlimited range of applications.

Before proceeding to discussion of the search techniques used, it is appropriate to introduce the discrete variables problem at this stage.

OPTIMIZATION WITH DISCRETE VARIABLES

In the vast majority of structure, fabrication requirements will demand that some components be standard sections for which the properties cannot be regarded as continuously variable.

To produce many components (e.g. rolled steel sections) in sizes other than the standard ranges are normally prohibitively expensive. Therefore, although a true optimum structure (on a weight or other basis) may require nonstandard components, economics will dictate the use of the cheaper standard size.

The original formulation of the optimization problem has only inequality constraints, Equation 1. In the discrete variable problem, equality conditions are required. The approach to the incorporation of discrete variables is based upon Fiacco and McCormick's (Reference 8) extension of the created response surface technique to include equality constraints.

In order to incorporate the discrete variables an additional penalty function P_d is introduced to account for nonsatisfaction of the specified discrete sizes.

The new penalty function has the form

$$P_d = \sum_{i=1}^{\ell} \prod_{j=1}^{d} \frac{DV_i - DS_{ij}}{DS_{ij}} \tag{4}$$

where DV_i = current value of one of the ℓ design variables

DS_{ij} = the jth allowable discrete value for the ith design variable

and π indicates a finite product over j.

In this function, the product term will ensure that each individual term of the summation will vanish when the appropriate design variable assumes any one of its prescribed discrete values. As for the constraint penalty, this discrete penalty is weighted and added to the combined function, Equation 3, to form a new total penalty.

$$F = W + r_c P_c + r_d P_d \tag{5}$$

The weighting factor r_c was decreased in value as the optimization proceeded. Because of the different nature of P_d from P_c, the weighting factor r_d is increased successively. This forces the design to assume the discrete values at which the penalty P_d vanishes. The effect of including the discrete penalty function can be illustrated graphically in two ways. The combined merit and

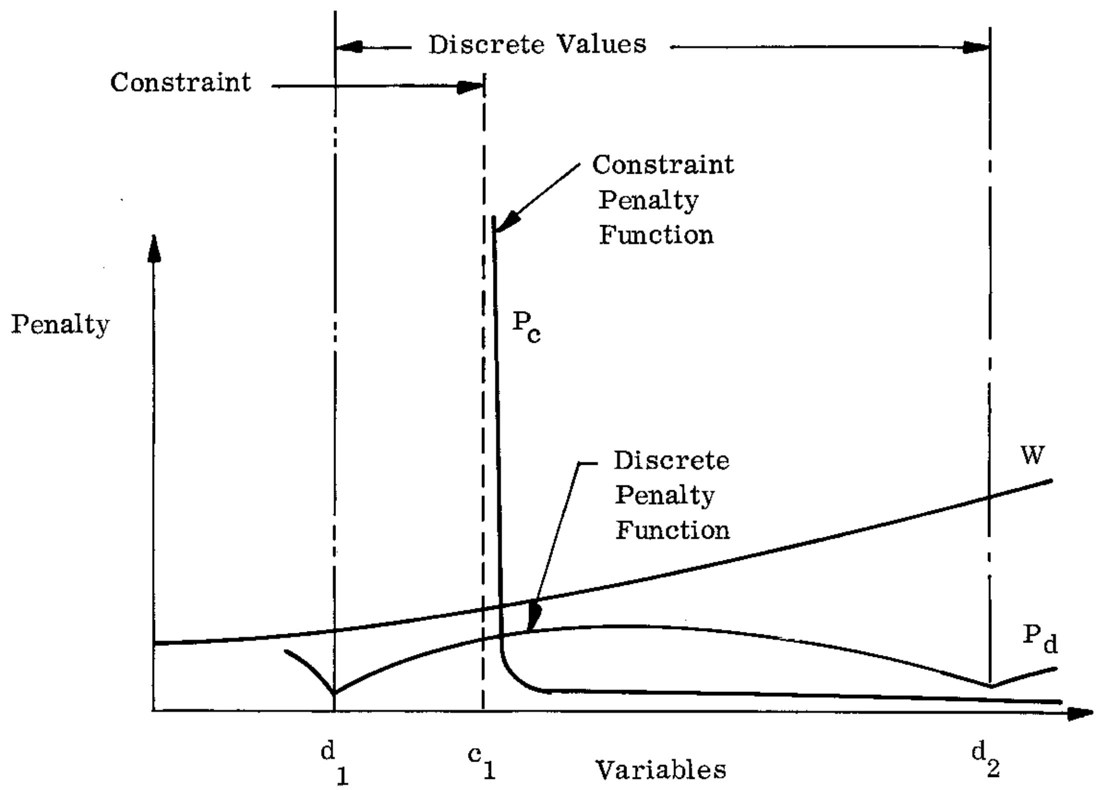


Figure 4. Discrete Optimization Penalty Functions

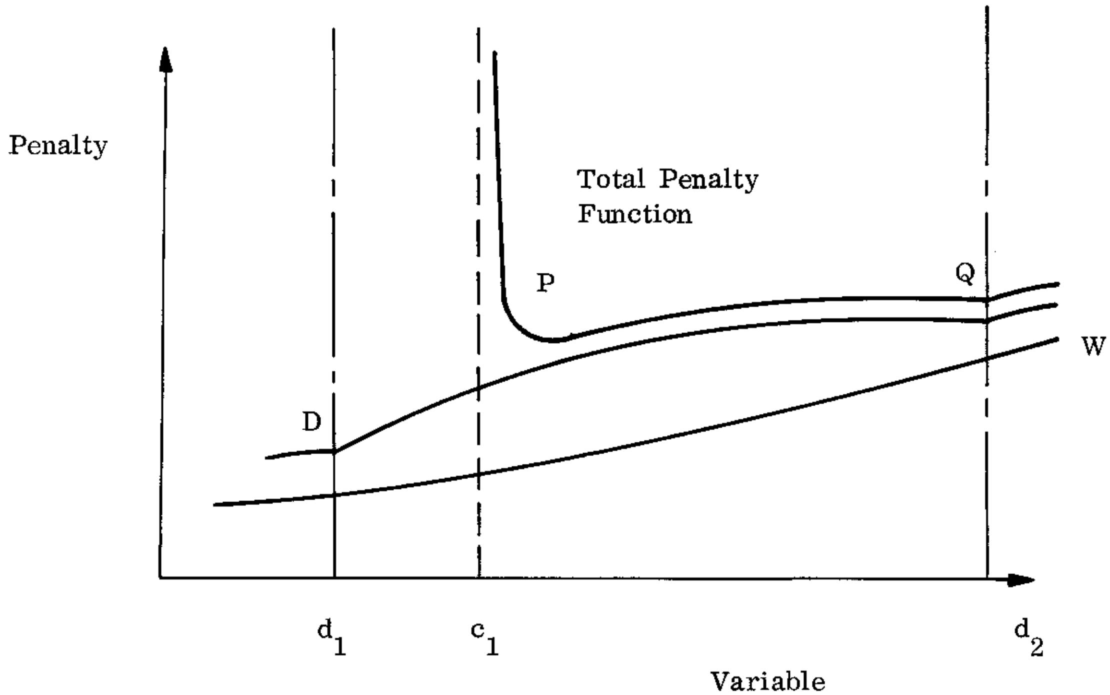


Figure 5. Created Response Surface for Discrete Optimization

constraint penalty functions are shown in Figure 4. The function P_d has an oscillatory form, as indicated in Figure 4, having minima at the discrete values d_1, d_2, d_3, etc. Combining all functions, a total response surface such as depicted in Figure 5 results. Under normal conditions the minimum will occur at a point such as D. The penalty P_d does not take into account the existence of main constraint conditions. As a result situations can arise in which the design reaches the point P, and attempts to converge on the unacceptable discrete value d_1. The constraint C_1 prevents this, but a nondiscrete minimum is reached near P. To correct this situation, the discrete value d_1 is eliminated and the design will usually then slip into the minimum at Q as desired. An alternative view of the problem is obtained in Figure 6. The coordinates here are the design variables. The lines C_1 and C_2 represent the active constraint conditions at whose intersection the true minimum lies. For a particular value of r_c and with r_d set to zero, contour lines for the created response surface are shown. The minimum lies at F_5, some distance from the true minimum. As r_c is reduced the contours approach the constraints more closely. To include the penalty P_d a rectangular grid is superimposed. The lines correspond to the specified discrete values of the variables. The function P_d adds a peak on each rectangle and is zero at the intersection points. When r_d becomes very large, the design is forced very strongly toward these intersections.

In principle, it is possible to start with any feasible design and then to decrease the weighting function r_c and increase the weighting function r_d simultaneously. But in practice, it is difficult both to estimate the initial values of r_c and r_d and their relative rates of change with progress in the minimization. In any case, it was argued that the discrete minimum is always near the continuous minimum so that the continuous minimum design is always taken as the starting point for a discrete design. This proximity of the discrete minimum design to the continuous minimum design can best be seen from the contour plots used previously to interpret the created response surface function. For a well-behaved created response surface, function F, such as that shown in Figure 6, the discrete minimum is at one of the intersections of the rectangular grid that encloses the continuous minimum.

In a practical problem, although it might be possible theoretically to select some initial values for both r_c and r_d and manipulate both simultaneously in concert, it has been found convenient to ignore the discrete requirements at first (i.e. $r_d = 0$) until an optimum design has been found and then proceed from that point with r_d increasing in value.

MINIMIZATION PROCEDURES

To determine the unconstrained minimum of a function, a number of methods have been developed in the general field of operations research.

The choice of the minimization procedure for the current work was influenced to a large extent by the observation that the gradient of the function F is not easily calculated and that the created response surface varies very rapidly near the constraints. This reduced the choice of methods to one that did not require the calculation of the derivatives in closed form. Fletcher (Reference 9) compared the more promising methods and results and suggests that the methods of Powell (Reference 10) or Rosenbrock (Reference 11) would be suitable for minimization with large numbers of variables. The Rosenbrock method was finally selected because experience with the reliability of the procedure has been obtained on some earlier problems.

In Rosenbrock's method some initial point in the n-variable space is selected. n orthonormal directions, p_1, p_2 - - - p_n are defined. For the first step, these directions will normally be parallel to the coordinate directions. Travel along the first direction is initiated and continues in the direction of decreasing value of the response function until a minimum (or approximation thereto) is found. The total distance of travel is designated by the scalar α_1. New travel along the second direction now takes place, using the previous minimum as a starting point and

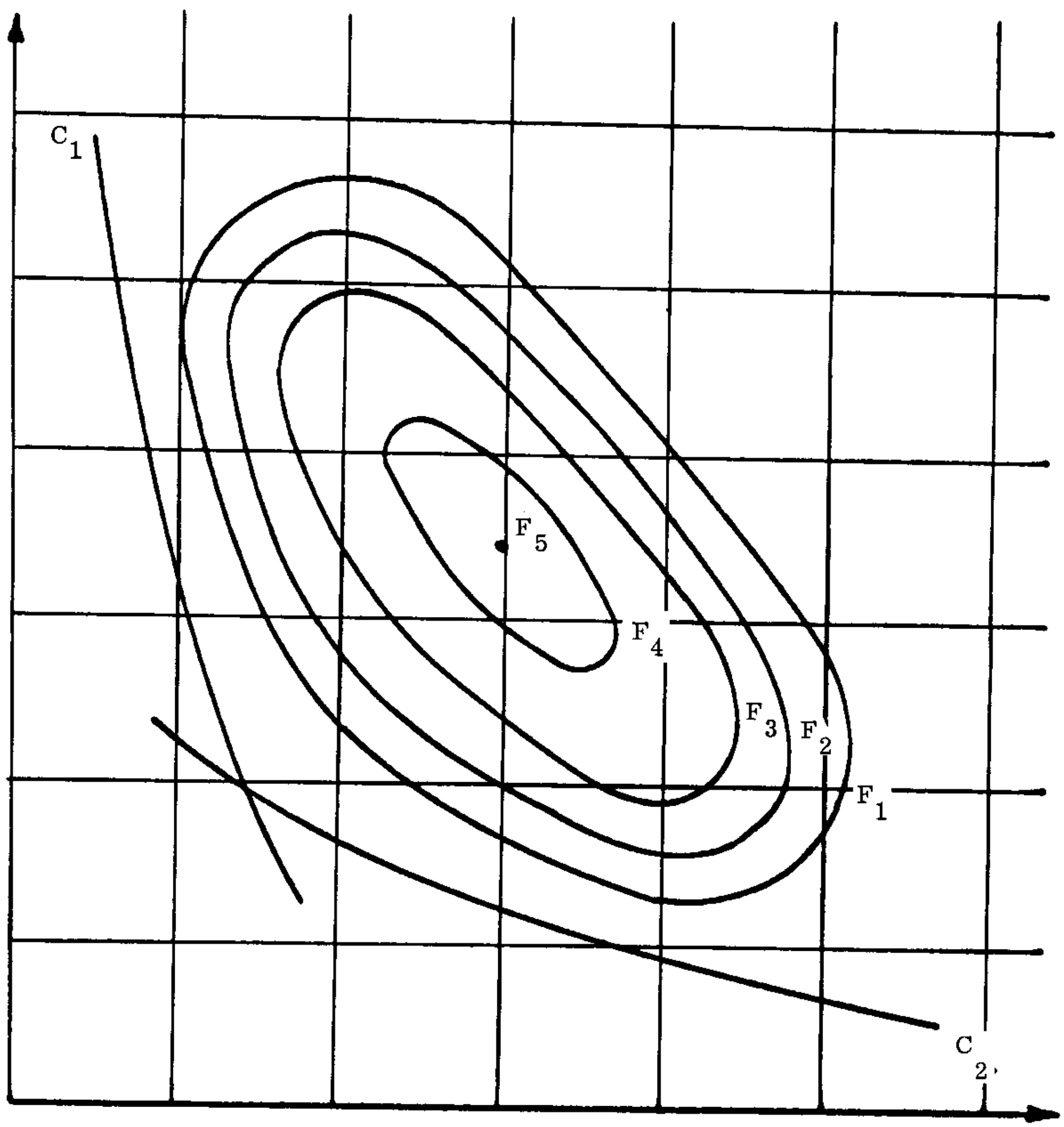


Figure 6. Contour Plot of Created Response Surface

this continues until a minimum is found. This is travel distance α_2. The process is repeated for each of the n-directions in turn. From the results of these explorations a new set of vector directions $q_1, q_2 \ldots q_n$ are then constructed so that

$$\begin{aligned} q_1 &= \alpha_1 p_1 + \alpha_2 p_2 + - - - - - + \alpha_n p_n \\ q_2 &= \qquad\quad \alpha_2 p_2 + - - - - - + \alpha_n p_n \\ &\vdots \\ q_n &= \qquad\qquad\qquad\qquad\qquad \alpha_n p_n \end{aligned} \tag{6}$$

The set of vector directions $q_1, q_2 \ldots q_n$ is then orthonormalized by the Schmit process to give a new set of travel vectors $p_1, p_2 \ldots p_n$. New travel may now be initiated and the entire process repeated until the optimum is found. In physical terms, this can be regarded as a linear extrapolation process. The first generated vector q_1, which is subsequently normalized to provide the first travel direction p_1, is the vector joining the initial and final points of the exploration. It was along this direction that the greatest reduction of the merit function was achieved; hence, it is intended that further travel along this direction would accomplish further meaningful reductions in the function.

In the original work, Rosenbrock proposed a method of determining the distance of travel α_i to minimize approximately the merit function. An arbitrary step length ϵ was first tried. If this reduced the value of the function, a new step length $\beta + \epsilon$ was taken, where $\beta +$ is some scalar greater than unity. If the step did not effect a reduction, a new step $-\beta - \epsilon$ was taken, with $0 < \beta - < 1$. In practice, suitable values selected for $\beta +$ and $\beta -$ are 3.0 and 0.5 respectively. No attempt was made to find the exact minimum in the chosen direction. Interpolation or other procedures were ruled out on the grounds that the additional computation costs were unnecessary, since the exact value of the minimum at that stage has little influence on the later search procedure. In fact, with experience, it has been found that it is sufficient to work with a maximum of two successful steps along any travel direction.

SECTION III

COMPUTER PROGRAM

In order to evaluate quantitatively the effectiveness of the techniques discussed in the previous section, a pilot computer program for the optimization of small scale structures has been developed.

To minimize the effort involved in the development of a pilot program a relatively inefficient analysis module was selected from a previously coded computer program. This analysis was limited to approximately 50 degrees of freedom and originally contained only an axial force member in the element library. This was considered sufficient for initial research.

INITIAL DESIGN PARAMETERS

The reciprocal nature of the penalty function requires that the constraints of the problem are never violated. Therefore, it is necessary to pick starting values of the design parameters that do not violate any of the constraints. This is achieved by means of a fully stressed design that, if necessary, is scaled up so that the displacement constraints are satisfied. A fully stressed design for multiple loads is that design in which every element is subjected to its maximum allowable stress in at least one load condition. The design is accomplished by assuming that the stress in each element is affected only by its size and that there is no cross effect due to static indeterminacy between the elements. The element sizes are then proportioned to give a fully stressed design. Because there is a cross effect, the fully stressed design is not achieved immediately. However, repeated applications of this procedure usually lead to rapid convergence. In practice, because of the need to keep away from the constraints, lower values of the stress constraints are used for the design, typically 0.95 times the allowable stresses.

BASIC METHOD

Two slightly different methods can be adopted for reducing the scaling factors r_c, during the course of the optimization. The first approach is to keep r_c constant and find the minimum of the function F. Then reduce r_c and find the next minimum, etc. The second approach is to reduce r_c with the progress of the minimization, because it is pointless to find the exact minimum for every value of r_c since the exact minimum is only required for the final value of r_c. Because previous workers had used the first approach, it was decided to explore the second approach.

IMPROVED METHOD

After coding of the unconstrained minimization using the created response surface approach with Rosenbrock's search, a number of simple problems were used to check the operation of the program. Although the methods used yielded minima, it was felt that considerable improvements in operational efficiency could be produced. To this end three modifications in principle were introduced, all of which produced significant reductions in computational expense.

1. Extrapolations Techniques. When using the created response surface approach as discussed previously, the weighting factor r_c for the penalty function P_c is assumed to have some artibrary initial value. This factor has been selected to make the value of the penalty initially equal to the value of the merit function at the starting point. The factor r_c is then reduced stepwise until further reduction produces no appreciable decrease in the merit function W.

Such a process may require a considerable number of iterations. To accelerate the procedure, an extrapolation method was developed. The extrapolation procedure was based on the assumption that each component of the minimum of the function F(DV, r_c) can be expanded in a power series in $r_c^{1/2}$. A two term series was used

$$DV(r_c) = a_0 + a_1 r_c^{1/2} \tag{7}$$

The initial weighting factor r_c was chosen so that the penalty function was equal to the weight. Three values of r_c were used, the value of r_c being reduced by a factor of 4 after each minimization. Substitution of the values of a parameter at a consecutive minimum for two values of r_c defined the constants a_0 and a_1. Equation 7 was then used to estimate the values of the design parameters (DV_i) for the next weighting factor r_c. Equation 7 also was used to extrapolate to the minimum as the weighting factor r_c tends to zero since by Equation (7)

$$DV(r_c) = a_0 \quad \text{as} \quad r_c \to 0 \tag{8}$$

Finally, we note that, for the extrapolation process, Fiacco and McCormick stressed the importance of an accurate evaluation of the minimum for each value of r_c.

2. Truncated Search. It was noted that during the search in a chosen direction of the Rosenbrock method, the third step after two successes usually proved unproductive. An examination of Rosenbrock's original results showed an improvement in the minimization of a function for a fixed number of trials as the factor $\beta+$ was increased from 1 to 5. It was conjectured that the improvement was brought about by the reduction of the number of trials required in the search along each of the chosen directions. It was therefore decided to truncate the search after two successful steps. Similarly, it was observed that a step, following two failures and a success, also proved unproductive and the search was also truncated after the successful step.

3. Linearization Methods. Much time in any optimization process is taken up by the repeated complete reanalyses that are necessary after every change of design parameters. Although the response characteristics (stresses, displacements) are strictly nonlinear functions of all the possible design variables, it is practical to regard the rates of change of these responses to be sensibly constant for small changes in the design parameters. That is, the continuously curved response characteristics may be assumed to be stepwise linear. This may lead to slight inaccuracies, especially when step sizes are large. On the other hand, on search procedures great accuracy is not necessary (and may be needlessly expensive to obtain) in early stages when remote from the optimum. Near the optimum the step sizes generally become very small and the linearization concept is highly accurate.

SECTION IV

EXAMPLE PROBLEMS

INITIAL ATTEMPT

Two problems were investigated, the first that of the three bar truss studied by Schmit and Mallett (Reference 12) and shown in Figure 7. Table 1 gives the three sets of applied loads. A fully stressed design was first found and the value of r_c was selected so that the ratio of the penalty function to the weight was one to ten. A function minimization was then carried out with the r_c being reduced to 0.7 of its value after each stage. The result of 8.717 lb was obtained by Schmit and Mallett and 8.720 lb by Gellatly et. al. (Reference 2). The second problem was that of the 25-bar truss.

The next example chosen is that of the 25-bar truss studied by Fox and Schmit (Reference 13). Fox and Schmit included buckling constraints and used tube diameter and thickness as the design variables. The cross-sectional area of the circular tubes are used here as the design variables. The Euler criteria are included as stress limits for each member. The stress limits unless modified by the Euler buckling criteria are specified to be ±40,000 psi. The displacement limits for each displacement component of any node in the truss are ± 0.35 in. The truss is shown on Figure 8. Symmetry is imposed about the XZ and YZ plane

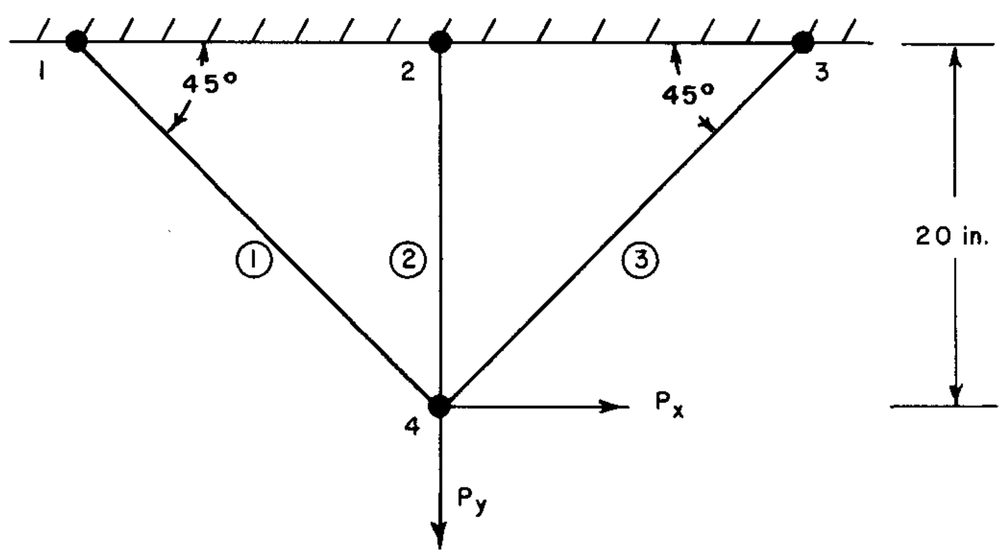


Figure 7. Three-Bar Truss

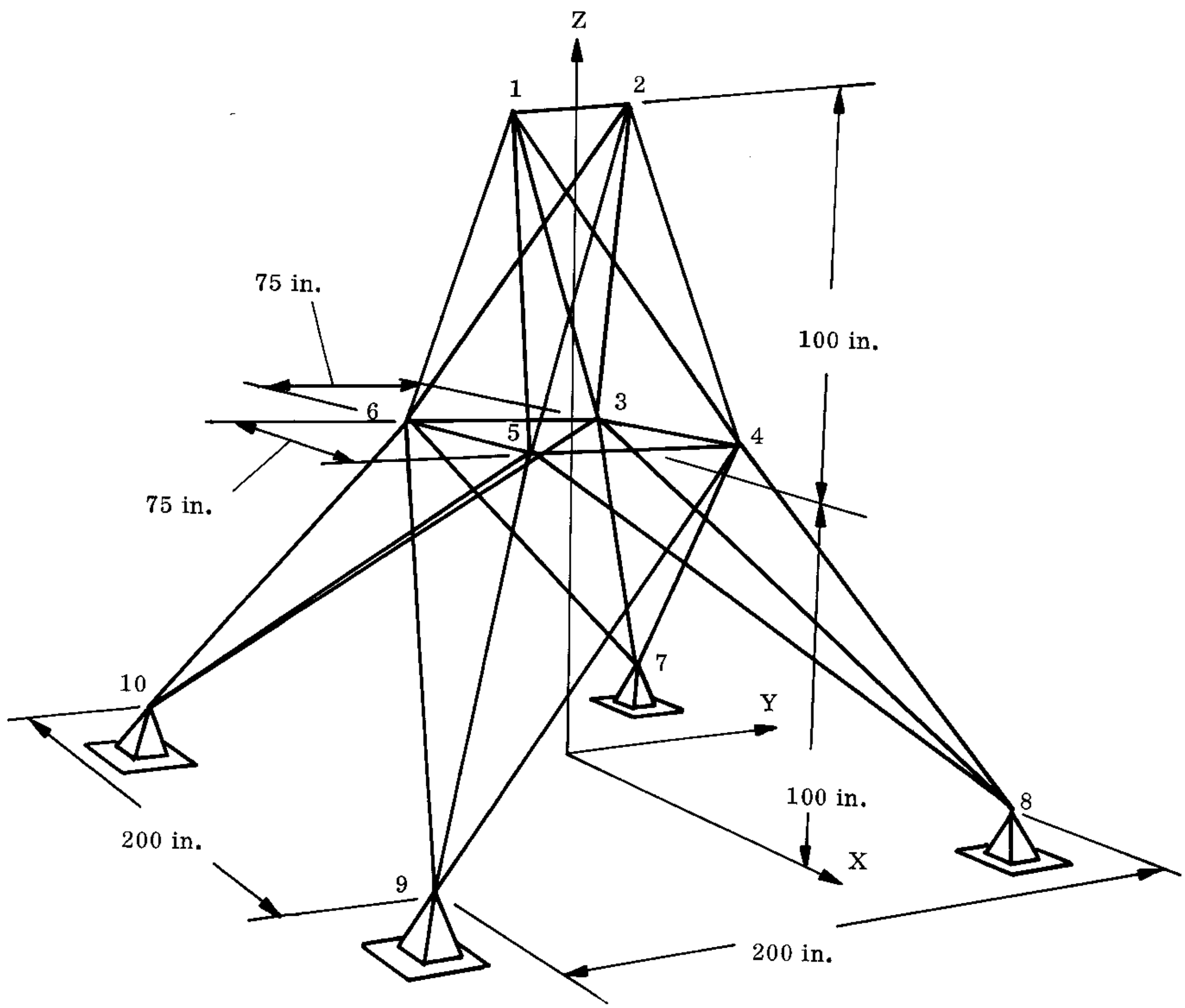


Figure 8. Transmission Tower

TABLE I

LOADS ON THREE-BAR TRUSS

Load Condition	P_x	P_y	$(E\alpha\Delta T)_1$	$(E\alpha\Delta T)_2$	$(E\alpha\Delta T)_3$
1	10^5 lb	0	6500 psi	13,000 psi	19,500 psi
2	-9.5459×10^4 lb	-9.5459×10^4 lb	19,500 psi	13,000 psi	6500 psi
3	8.195×10^4 lb	-5.7358×10^4 lb	0	0	0

TABLE 2

LOADS ON 25-BAR TRUSS

Load Condition	Nodal Point	Direction of Load		
		X	Y	Z
1	1	1000	10,000	-5,000
	2	0	10,000	-5,000
	3	500	0	0
	6	500	0	0
2	1	0	20,000	-5,000
	2	0	-20,000	-5,000

by means of a linking feature that forms the following groups of cross-sectional areas:

$$A_1 , A_2 = A_3 = A_4 = A_5 , A_6 = A_7 = A_8 = A_9 , A_{10} = A_{11} , A_{12} = A_{13} ,$$

$$A_{14} = A_{15} = A_{16} = A_{17} , A_{18} = A_{19} = A_{20} = A_{21} \text{ and } A_{22} = A_{23} = A_{24} = A_{25} .$$

The truss is only loaded at two nodal points. The imposed loading on the truss is given in Table 2. This second example was constructed so that the displacement constraints were active. When the same procedure that was applied to the three-bar truss was applied to the second example, the function F proved difficult to minimize. A minimum was only obtained after 30 Rosenbrock stages and required about 20 minutes of IBM 7090 time. Minimum weight obtained was 555 lb. This may be compared with a weight of 570 lb by Fox and Schmit (Reference 13) and 551 lb by Gellatly (Reference 3). The differences between first and the last weight are though to be due to the different accuracy criteria.

However, it is relevant to note that progress from the 570-lb weight to 555-lb took about a half of the computing time.

IMPROVEMENT OF BASIC METHOD

The introduction of the improved procedures resulted in a significant reduction of the computing time required to minimize the twenty-five bar truss.

1. Extrapolation Technique. The extrapolation technique resulted in an extrapolated minimum weight of 556 lb (minimum weight at last value of r_c = 558 lb). The number of stages in the Rosenbrock minimization and the computing time were reduced by a factor of two.

2. Truncated Search. The truncated search procedures brought the average number of steps in each search direction from four to three, which saved about a quarter of the computing time and the minimum weight was not affected.

3. Linearized Procedures. Before the start of each stage, the rate of change of the stresses and displacements with respect to the step length in each of the chosen directions were evaluated by a finite difference procedure. Since an average of three function evaluations are required in each search direction, this linearized approach reduced the number of stress analyses required at each stage by a third of the original number and reflected itself in a halving of the computing time. The final optimization with all the improvements incorporated required 3.6 minutes of IBM 7090 time. The progress in reducing the computing times is summarized in Table 3.

Finally, it is estimated that a further halving of the computing time can be achieved by using the well-known techniques of elastic redesign (see for instance Gellatly and Gallagher, Reference 14) which make use of the total differential of the equations of equilibrium. The estimated time of 1.8 minutes IBM 7090 is of the same order as the time of 1.2 minutes UNIVAC 1107 estimated by Fox and Schmit for the solution of this problem by the integrated approach to structural synthesis. However, it should be noted that this suggested improvement requires a doubling in the size of the computer store, which may in fact be too stringent a requirement.

The second interesting observation of the minimization procedure is that the search along the P_1 direction provides an order of magnitude improvement over the search along the other directions. It is as if the only purpose of the search along the other directions is to explore and set up a new best direction for P_1.

DISCRETE DESIGN

1. Example 1, Three-Bar Truss.

The same three-bar truss was used as a first example for discrete design. The same load conditions were used. Four discrete designs were produced. Sections were allowed to vary in increments of 0.2, 0.3, 0.4 and 0.5 in.^{2}, in turn and starting from its lowest permissible value of 0.001. Sufficient sections were specified so that the discrete design did not take on the highest specified value.

To achieve the design with discrete variations of 0.5 in.^{2}, it was necessary to force back element 3 and then element 1 in turn.

TABLE 3

IMPROVEMENTS IN COMPUTING TIME

Method	Min. Weight lb	Comp. Time IBM 7090 min.	Stages
Initial Method	555	20	30
Extrapolation	556	11	15
Extrapolation and Truncated Search	556	8	14
Extrapolation, Truncated Search and Linearized Change	555	3.6	14

TABLE 4

DISCRETE DESIGN OF THREE-BAR TRUSS

Increments in Areas, $in.^2$	% Penalty of Discrete Design	Minimum Weight lb	Cross - Sectional Area $in.^2$		
			Element 1	Element 2	Element 3
Continuous	0	8.717	1.113	0.577	1.544
0.2	1	8.791	1.195	0.400	1.600
0.3	11	9.782	1.200	0.600	1.800
0.4	1	8.791	1.195	0.400	1.600
0.5	28	11.008	1.500	0.500	2.000

The results are presented in Table 4. It can be seen that the design for sections with increments in area of 0.4 $in.^2$ is the same as that for 0.2 $in.^2$. Thus the weight of the discrete designs do not increase monotonically but are somewhat dependent on the distance of travel required to reach a discrete design from the minimum continuous design.

2. Example 2, Twenty-five Bar Truss.

The twenty-five bar truss shown in Figure 8 was used as a second example of discrete design. Two discrete designs were effected. Sections were allowed to vary in increments of 0.4 and 0.8 $in.^2$.

The results of the discrete designs are shown in Tables 5 and 6. Table 5 gives the weight of the optimum designs. It also gives details on which specified sections had to be removed from the list in order to achieve the discrete design. Table 6 gives the cross-sectional areas obtained in the optimum design. It will be noted that in the cross-sectional areas, A_1 moves a considerable distance away from its value at the continuous minimum. It was this possibility that prevented the use of a simpler search through all the combinations of the discrete sections adjacent to the continuous design parameter values. This large change appears to be due to the design being insensitive to the design parameter A_1. Because improvements in the minimization procedure were made at the same time as the above results were being obtained, it is difficult to quote typical computing times for the analysis. However, it was observed that on average 12-15 Rosenbrock stages were required to reach a discrete design in the twenty-five-bar truss.

TABLE 5

RESULTS OF DISCRETE OPTIMIZATION, 25-BAR TRUSS

Increments in Sectional Areas, $in.^2$	Weight of Discrete Design, lb	Number of Sections to Be Discretized	Element No. of Which Deletion Is Made	Element Size Deleted from List, $in.^2$
Continuous	554.9			
0.4	595.4	19	21	1.6
		8	5	2.0
		1	1	1.2
		1	1	1.6
0.8	642.4	11	22	2.4
		1	1	0.0111(min.)

TABLE 6

OPTIMAL CROSS-SECTION AREAS FOR DISCRETE DESIGN

Increments in Specified Sections, in.^{2}	Area for Elements, in.^{2}							
	A_1	A_2	A_6	A_{10}	A_{12}	A_{14}	A_{18}	A_{22}
Continuous	0.189	2.142	2.488	0.038	0.084	0.694	1.866	2.771
0.4	2.009	2.400	2.400	0.011	0.011	0.800	2.000	2.800
0.8	1.56	2.400	2.400	0.011	0.011	0.800	2.400	3.200

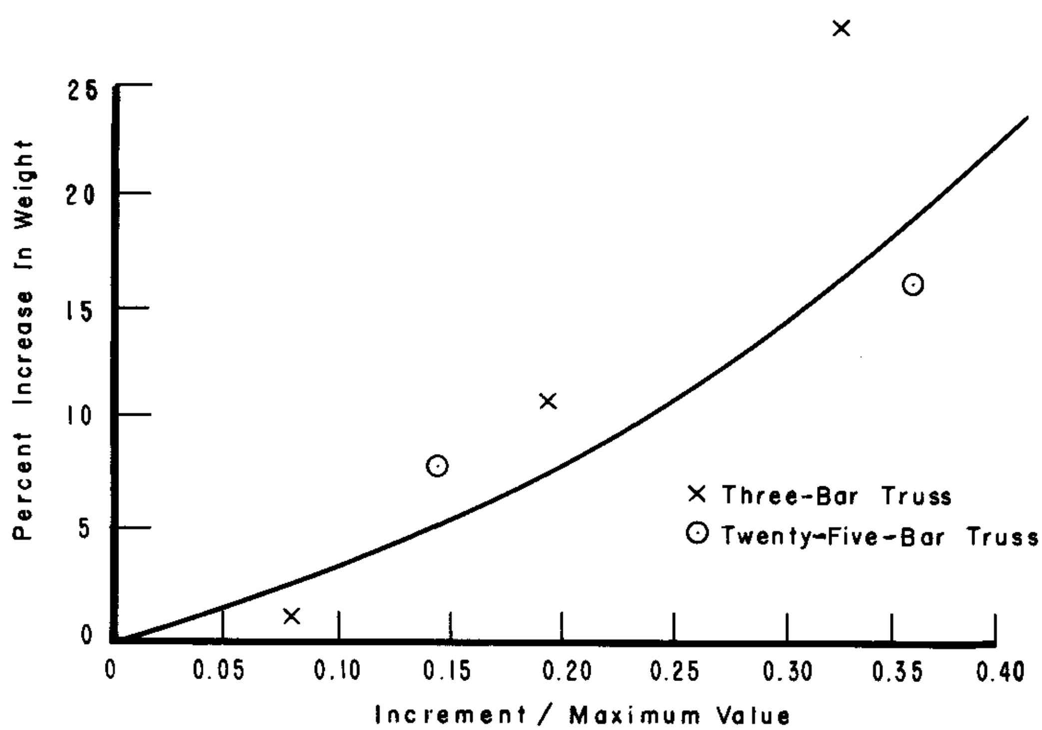


Figure 9. Penalty Due to Discrete Design

Finally, Figure 9 shows the change in weight of the otpimum design with increase in the size of the increments in the specified cross-sectional areas. The increase in weight is shown as a percentage of the continuous design while the size of the increment is shown as a fraction of the maximum design section used in the continuous design.

Although the results do not increase regularly, the trend is as expected, i.e., the optimum weight increases in the size of increments in the specified cross-sectional area.

CONCLUSIONS

The Created Response Surface Technique was successfully applied to some problems of structural optimization with multi-load conditions. The original method was modified to take advantage of the interaction between the optimization procedure and the response of the structure to changes in sectional areas. This, when coupled with the r extrapolation procedure, resulted in considerable savings in computing time.

Discrete design was effected by introducing a penalty function which only vanished when the variables were at the specified discrete values.

ACKNOWLEDGMENT

The work reported in this paper was performed under NASA Contract NASw-1344 with Dr. H. S. Wolko and Dr. F. J. Stebbins acting as technical monitors. The authors are grateful to Drs. Wolko and Stebbins for useful discussions.

SECTION VII

REFERENCES

1. Schmit, L. A., "Structural Design by Systematic Synthesis," Proc. of the Second Conference on Electronic Computation, ASCE, September 1960.

2. Gellatly, R. A., Gallagher, R. H. and Luberacki, W. A., Development of a Procedure for Automated Synthesis of Minimum Weight Structures, FDL-TDR-64-141, November 1964.

3. Gellatly, R. A., "Development of Procedures for Large Scale Minimum Weight Structural Design," AFFDL-TR-66-180, December 1966.

4. Carroll, C. W., "The Created Response Surface Technique for Optimizing Nonlinear Restrained Systems," Operations Research, Vol. 9, No. 2, 1961.

5. Toakley, A. R., "Studies in Minimum Weight Rigid Plastic Design with Particular Reference to Discrete Section," Private Communication, May 1966.

6. Melosh, L. J. and Luik, R., Approximate Multiple Configuration Analysis and Allocation for Least Weight Structural Design, AFFDL-TR-67-59, Air Force Flight Dynamics Laboratory, WPAFB, Ohio, 1967.

7. Schmit, L. A., Jr., Morrow, W. M., II and Kicher, T. P., "Structural Synthesis Capability for Integrally Stiffened Cylindrical Shells," Proc. AIAA/ASME 9th Structures, Structural Dynamics and Materials Conference, Paper 68-327, 1968.

8. Fiacco, A. V. and McCormick, "The Sequential Unconstrained Minimization Technique for Convex Programming with Equality Constraints," Research Analysis Corporation, RAC-TP-155, April 1965.

9. Fletcher, R., "Function Minimization Without Evaluating Derivative - A Review," Computer Journal, Vol. 8, 1965.

10. Powell, M. J. D., "An Efficient Method for Finding the Minimum of a Function of Several Variables Without Calculating Derivatives," Computer Journal, Vol. 7, 1964.

REFERENCES (CONTD)

11. Rosenbrock, H. H., "An Automatic Method for Finding the Greatest or Least Value of a Function," Computer Journal, Vol. 3, 1960.

12. Schmit, L. A. and Mallett, R. H., "Structural Synthesis and Design Parameter Hierarchy," Journal Structure Division, ASCE Vol. 89, August 1963.

13. Fox, R. L. and Schmit, L. A., "Advances in the Integrated Approach to Structural Synthesis," AIAA/ASME Materials Conference, 1964.

14. Gellatly, R. A. and Gallagher, R. H., "A Procedure for Automated Minimum Weight Structural Design," Part 1, The Aeronautical Quarterly, Vol. XVII, Part 3, August 1966.

15. Gellatly, R. A. and Marcal, P. V., "Investigation of Advanced Spacecraft Structural Design Technology," Bell Aerosystems Report No. 2356-950001, Report to NASA under Contract NASw-1344, July 1967.