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FOREWORD

This report has been prepared by the Martin Marietta Corporation, Orlando Division, Orlando, Florida, under Air Force Contract No. AF33 (657)10315. This contract was sponsored by Technical Area 750A, Mechanics of Flight. The work was accomplished under Project 1467, Structural Analysis Methods, Task 146702, Thermoelastic Structural Analysis Methods, and was administered under the Structures Division, AF Flight Dynamics Laboratory, Deputy Commander/Technology, Aeronautical Systems Division, with Mr. T. N. Bernstein acting as Project Engineer.

The studies began in January 1963 and were concluded in August 1963 by the Research Division of Martin Orlando. Mathematical analysis and development was conducted by Mr. J. G. Torian under the supervision of Dr. J. M. Spurlock, Manager of the Aerosciences Research Laboratory. The digital programming was performed by Mr. G. K. Bennett under the supervision of Mr. M. Robinson of the Digital Computer Laboratory. Mr. F. A. Phillips, also of the Digital Computer Laboratory, prepared the eigenvalue and Bessel function subroutines.

This is the final report on Contract No. AF33(657)10315. The contractor's report number is OR 3351. Any questions pertaining to the study or use of the computer program should be directed to ASRMS-13.

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ABSTRACT

This study was undertaken to develop an exact mathematical formulation of transient heat conduction to reduce the amount of idealization required by currently employed finite difference techniques. A two-dimensional analysis is developed beginning with the differential equations for conduction of heat in a segment of a hollow cylinder in polar coordinates. An exact solution with constant thermal diffusivity and conditions of prescribed surface temperature, convection, and direct heat input is written in an infinite series of Bessel and trigonometric functions. Provisions are made for incorporating a coordinate varying, arbitrary, initial temperature distribution as well as coordinate and time varying arbitrary surface conditions. A solution that incorporates the effect of temperature dependent thermophysical properties is developed by combining the exact solution with a finite difference approach. The solution has been programmed for an IBM 7090 or 7094 digital computer.

The geometry of the problem is indicative of wing leading edges such as found on the ASSET vehicle. It is anticipated that this geometry will remain applicable to many future vehicles.

PUBLICATION REVIEW

This technical documentary report has been reviewed and is approved.

FOR THE COMMANDER:

WILLIAM A. SLOAN, JR.
Colonel, USAF
Chief, Structures Division
AF Flight Dynamics Laboratory

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NOMENCLATURE

Symbol Text	Symbol FORTRAN	Meaning
α	ALP	Thermal diffusivity
β	BOO	Eigenvalue in Bessel function argument (radial position)
γ	G	Eigenvalue in trigonometric argument (angular position) transient case
θ	T	Time (Capitals used for time function)
ϕ	PHI	Angle (Capitals used for angular function)
λ		Dummy variable in Duhammel relation
σ	G	Eigenvalue in trigonometric function argument (angular position) steady-state case
ϵ	EPS	Eigenvalue in hyperbolic function argument (radial position) steady-state case
ρ	PEO	Density
h		Heat transfer coefficient
k_{ij}	AKij	Boundary condition constants at ϕ
ℓ_{ij}	ALij	Boundary condition constants at r
r	R	Radius (Capital used for radius function in text)
k	EKO	Thermal conductivity
C_p	CEO	Specific heat
B		Units of heat
F		Units of temperature
L		Units of length

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Symbol Text	Symbol FORTRAN	Meaning
$A_{\gamma\beta}$	A	Coefficient in initial temperature distribution expansion
B_{γ}	BZ	Coefficient in trigonometric expansion
$C_{\gamma\beta}$	C	Coefficient in Bessel series expansion
$G_{\gamma\beta}$	GO	Coefficient in coordinate varying surface temperature distribution expansion
T		Temperature
T_c		Convection sink temperature
T_s		Temperature component, steady-state effects
T_i		Temperature component, initial temperature distribution effects
T_{si}		Temperature component, partial coordinate varying temperature effects
T_{λ}		Temperature component, coordinate varying and dummy time variable expression
$T_{s\lambda}$		Temperature component, steady-state with dummy time variable
$T_{si\lambda}$		Temperature component, partial coordinate varying effects with dummy time variable
$\overline{\overline{T}}$		Temperature difference above 500°F used in sample problem
T_{aw}		Adiabatic wall temperature
T_{r1}		Temperature at r_1 face
T_{r2}		Temperature at r_2 face
$T_{\phi 1}$		Temperature at ϕ_1 face
$T_{\phi 2}$		Temperature at ϕ_2 face
T_{θ}		Temperature component, coordinate and time varying temperature effects

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Symbol Text	Symbol FORTRAN	Meaning
I_{ij}		General notation for boundary condition indicator
L_{σ}	RL	Arbitrary coefficient
M_{ϵ}	GM	Arbitrary coefficient
N_i	WNi	Arbitrary coefficient
P_{σ}	P	Arbitrary coefficient
Q_{ϵ}	Q	Arbitrary coefficient
$S_{1\epsilon, 2\epsilon}$	SA, SB	Arbitrary coefficients
W_{ϵ}	W	Arbitrary coefficient

Note: Capital letters with asterisk (*) are used for arbitrary constants in the development of particular solutions to distinguish from the same letter used in the general solution. In general, there is no direct relationship between similar letters used in this manner in the two solutions.

Prime denotes differentiation.

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SECTION I - INTRODUCTION

The differential equation for two-dimensional conduction of heat in a cylinder with constant thermal diffusivity written in polar coordinates is

$$\frac{\partial T}{\partial \theta} = \alpha \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right]. \tag{1}$$

This report develops an exact solution to this differential equation as applicable to a segment of a hollow circular cylinder. Various cases are solved to provide for arbitrary initial temperature distribution and arbitrary time and coordinate varying conditions at the boundaries. A solution that incorporates the effects of temperature dependent thermophysical properties is developed by combining the exact solution with a finite difference approach.

The solution has been programmed for an IBM 7090 or 7094 computer. The program provides for input of the arbitrary functions as coefficients of polynomials up to 7th degree or as tabular data.

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SECTION II – MATHEMATICAL FORMULATION (CONSTANT THERMOPHYSICAL PROPERTIES)

A. SIMPLIFICATION OF GENERAL PROBLEM

The general solution must satisfy the initial condition of a coordinate varying temperature distribution and the boundary conditions of coordinate and time varying temperatures or heat flux at the surfaces. The solution will be developed in such a way that the arbitrary coordinate and time varying functions of temperature may be input as a surface temperature, or convection at the surface into a medium at a temperature, defined by the arbitrary function. "Temperature at the Surface" refers to either of these modes.

If $T(\theta, r, \phi)$ represents the temperature at (r, ϕ) at time θ, the initial and boundary conditions may be expressed as

$$T(\theta, r, \phi) = h(r, \phi) \qquad \theta = 0 \qquad (2)$$

$$k_{11} \frac{\partial T}{\partial \phi} + k_{12} T = k_{13} F_{\phi 1}(r) g(\theta) \qquad \phi = \phi_1 \qquad (3)$$

$$k_{21} \frac{\partial T}{\partial \phi} + k_{22} T = k_{23} F_{\phi 2}(r) g(\theta) \qquad \phi = \phi_2 \qquad (4)$$

$$\ell_{11} \frac{\partial T}{\partial r} + \ell_{12} T = \ell_{13} F_{r1}(\phi) g(\theta) \qquad r = r_1 \qquad (5)$$

$$\ell_{21} \frac{\partial T}{\partial r} + \ell_{22} T = \ell_{23} F_{r2}(\phi) g(\theta) \qquad r = r_2 \qquad (6)$$

where k_{ij}'s and ℓ_{ij}'s are constants. Selection of various values of these constants alters the mode of heat transfer from the body to the prescribed arbitrary function at the boundary. Application of these constants to impose surface temperature, convection, heat flux, or insulation are discussed in the section on interpretation of boundary conditions (Section III). The mathematical model is shown in Figure 1.

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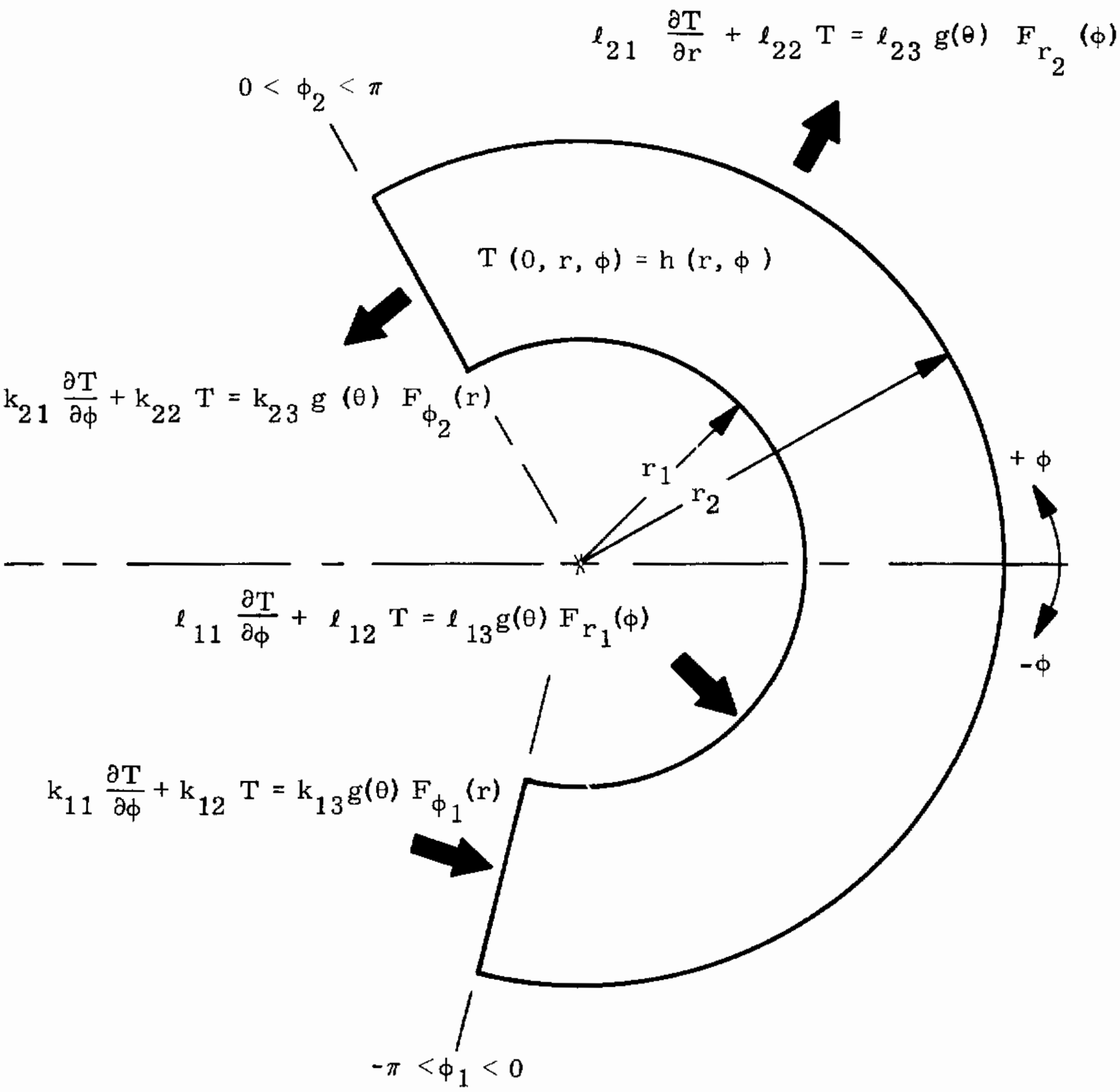

Figure 1. Mathematical Model

Conditions (2) through (6) can be satisfied by

$$T = T (\theta, r, \phi) = T_i (\theta, r, \phi) + \int_0^\theta \left\{g(\lambda)\right\} \left\{\frac{\partial}{\partial \theta} \left[T_s (r, \phi) - T_{si} (\theta - \lambda, r, \phi)\right]\right\} d\lambda$$

$$= T_i (\theta, r, \phi) + T_\theta (\theta, r, \phi) \qquad (7)$$

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(Refer to Paragraph 1.14 of Reference 1 for discussion of simplification of general problems of conduction.)

Here $T_i(\theta, r, \phi)$ is the solution to the problem with an arbitrary initial temperature distribution and homogeneous boundary conditions. This solution is referred to as Case I and is treated in detail in subsequent paragraphs. The second term in the second member of (7) is the application of Duhammel's theorem to impose time variant boundary conditions. The theorem as applied here is that if $g(\lambda)\left[T_S(r, \phi) - T_{Si}(\theta, r, \phi)\right]$ represents the temperature at (r, ϕ) at time θ in the body with initial temperature equal to zero, while its surface temperature is a function of λ, r, and ϕ, then the solution of the problem in which the initial temperature is zero, and the surface temperature is a function of θ, r, and ϕ, is given by the integral expression in (7). This integral expression, stated more simply as $T_\theta(\theta, r, \phi)$, is referred to as Case IV.

$T_S(r, \phi)$ and $T_{Si}(\theta, r, \phi)$ are required to develop Case IV. $T_S(r, \phi)$ is the steady-state solution to a problem with coordinate varying surface condition. It is treated as Case II in this report. $T_{Si}(\theta, r, \phi)$ is the transient solution to a problem with an initial temperature distribution equal to the solution to Case II and homogeneous boundary conditions. This is Case III. Case II minus Case III constitutes the solution to a problem with zero initial temperature and coordinate varying boundary conditions as required to develop Case IV. Rather than handle this case as a specific problem the results of Cases II and III are directly applied to Case IV.

As implied by (7), the sum of the solutions to Case I and Case IV constitute the solution to the general problem defined by (1) through (6).

B. CASE I - TRANSIENT SOLUTION WITH ARBITRARY INITIAL TEMPERATURE DISTRIBUTION AND HOMOGENEOUS BOUNDARY CONDITIONS

1. PARTICULAR SOLUTION

The partial differential equation to be solved is

$$\frac{\partial T}{\partial \theta} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] . \qquad (1)$$

We assume a product solution of the form

$$T(\theta, r, \phi) = \Theta(\theta)R(r)\Phi(\phi). \qquad (8)$$

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Substituting (8) in (1) we obtain

$$R\Phi\theta' = \alpha\left[\theta\Phi R'' + \frac{1}{r^2}\theta R\Phi'' + \frac{1}{r}\theta\Phi R'\right].$$

Rearranging

$$\frac{\theta'}{\alpha\theta} = \frac{R''}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} + \frac{1}{r}\frac{R'}{R},$$

which leads to

$$\frac{\theta'}{\alpha\theta} = \frac{1}{R}\left[R'' + \frac{1}{r}R'\right] + \frac{1}{r^2}\frac{\Phi''}{\Phi} = -\beta^2 \qquad (9)$$

where β is a constant.

From the first and third members of (9)

$$\theta' + \alpha\beta^2\theta = 0. \qquad (10)$$

By the methods of ordinary differential equations, (10) is satisfied by

$$\theta = K^* e^{-\alpha\beta^2\theta} .$$

From the second and third members of (9)

$$\frac{r^2}{R}\left[R'' + \frac{1}{r}R'\right] + r^2\beta^2 = -\frac{\Phi''}{\Phi} = \gamma^2 \qquad (11)$$

where γ is any constant.

From the second and third members of (11)

$$\Phi'' + \gamma^2\Phi = 0, \qquad (12)$$

which is satisfied by

$$\Phi = A^* \sin\gamma\phi + B^* \cos\gamma\phi. \qquad (13)$$

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From the first and third members of (11)

$$r^2 R'' + r R' + (r^2 \beta^2 - \gamma^2) R = 0, \tag{14}$$

which is a Bessel equation of order γ, satisfied by

$$R = C^* J_\gamma (\beta r) + D^* J_{-\gamma} (\beta r). \tag{15}$$

Accordingly, (1) is satisfied by

$$T(\theta, r, \phi) = K^* \left[A^* \sin \gamma\phi + B^* \cos \gamma\phi\right] \left[C^* J_\gamma (\beta r) + D^* J_{-\gamma} (\beta r)\right] e^{-\alpha\beta^2 \theta} \tag{16}$$

2. GENERAL SOLUTION

We consider here the general solution to Equation (1) with the initial condition

$$T_i (\theta, r, \phi) = h (r, \phi) \qquad \theta = 0 \tag{2}$$

and the boundary conditions

$$k_{11} \frac{\partial T_i}{\partial \phi} + k_{12} T_i = 0 \qquad \phi = \phi_1 \tag{17}$$

$$k_{21} \frac{\partial T_i}{\partial \phi} + k_{22} T_i = 0 \qquad \phi = \phi_2 \tag{18}$$

$$\ell_{11} \frac{\partial T_i}{\partial r} + \ell_{12} T_i = 0 \qquad r = r_1 \tag{19}$$

$$\ell_{21} \frac{\partial T_i}{\partial r} + \ell_{22} T_i = 0 \qquad r = r_2 \tag{20}$$

Note that (12) may be written

$$\frac{\partial}{\partial \phi}\left(\frac{\partial T_i}{\partial \phi}\right) + \gamma^2 T_i = 0. \tag{21}$$

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Here, (17), (18), and (21) constitute a Sturm-Liouville System (Reference 9, pages 254 to 268) with a weight function equal to unity.

Similarly, (14) may be written

$$\frac{\partial}{\partial r}\left(r \frac{\partial T_i}{\partial r}\right) + (r\beta^2 - \frac{1}{r}\gamma^2)\, T_i = 0 . \tag{22}$$

Here (19), (20), and (22) constitute a Sturm-Liouville System with a weight function equal to $1/r$.

Accordingly, the General Solution may be handled as a Sturm-Liouville System, which is a particular type of eigenvalue problem. The Sturm-Liouville System satisfies the orthogonality properties and can be made orthonormal by the application of a normalizing factor in the solution.

Since the boundary and initial conditions establish a relationship between the arbitrary constants in (16), and since the differential Equation (1) is linear, we may rearrange the constants and write as a series expansion

$$T_i(\theta, r, \phi) = \sum_{\gamma}^{\infty}\sum_{\beta}^{\infty} A_{\gamma\beta}\left[B_\gamma \sin\gamma\phi + \cos\gamma\phi\right]\left[C_{\gamma\beta} J_\gamma(\beta r) + J_{-\gamma}(\beta r)\right] e^{-\alpha\beta^2\theta} \tag{23}$$

The boundary conditions (17) and (18) require that

$$k_{11}\left[B_\gamma \gamma \cos\gamma\phi_1 - \gamma \sin\gamma\phi_1\right] + k_{12}\left[B_\gamma \sin\gamma\phi_1 + \cos\gamma\phi_1\right] = 0 \tag{24}$$

$$k_{21}\left[B_\gamma \gamma \cos\gamma\phi_1 - \gamma \sin\gamma\phi_1\right] + k_{22}\left[B_\gamma \sin\gamma\phi_2 + \cos\gamma\phi_1\right] = 0 \tag{25}$$

The equation for the eigenvalues γ is obtained by eliminating B_γ from (24) and (25) which results in

$$\begin{vmatrix} \left[k_{11}\gamma\cos\gamma\phi_1 + k_{12}\sin\gamma\phi_1\right] & \left[k_{12}\cos\gamma\phi_1 - k_{11}\gamma\sin\gamma\phi_1\right] \\ \left[k_{21}\gamma\cos\gamma\phi_2 + k_{22}\sin\gamma\phi_2\right] & \left[k_{22}\cos\gamma\phi_2 - k_{21}\gamma\sin\gamma\phi_2\right] \end{vmatrix} = 0 \tag{26}$$

where γ's are the positive non-zero roots of (26). Also from the boundary conditions (24) and (25)

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$$B_\gamma = \frac{-k_{12}\cos\gamma\phi_1 + k_{11}\gamma\sin\gamma\phi_1}{k_{11}\gamma\cos\gamma\phi_1 + k_{12}\sin\gamma\phi_1} \quad (27)$$

$$= \frac{-k_{22}\cos\gamma\phi_2 + k_{21}\gamma\sin\gamma\phi_2}{k_{21}\gamma\cos\gamma\phi_2 + k_{22}\sin\gamma\phi_1} . \quad (28)$$

Similarly, from (19) and (20)

$$\ell_{11}\left[C_{\gamma\beta} J'_\gamma(\beta r_1) + J'_{-\gamma}(\beta r_1)\right] + \ell_{12}\left[C_{\gamma\beta} J_\gamma(\beta r_1) + J_{-\gamma}(\beta r_1)\right] = 0 \quad (29)^*$$

$$\ell_{21}\left[C_{\gamma\beta} J'_\gamma(\beta r_2) + J'_{-\gamma}(\beta r_2)\right] + \ell_{22}\left[C_{\gamma\beta} J_\gamma(\beta r_2) + J_{-\gamma}(\beta r_2)\right] = 0 \quad (30)$$

The equation for the eigenvalues β is obtained by elimination of $C_{\gamma\beta}$ from (29) and (30), which results in

$$\begin{vmatrix} \left[\ell_{11} J'_\gamma(\beta r_1) + \ell_{12} J_\gamma(\beta r_1)\right] & \left[\ell_{11} J'_{-\gamma}(\beta r_1) + \ell_{12} J_{-\gamma}(\beta r_1)\right] \\ \left[\ell_{21} J'_\gamma(\beta r_2) + \ell_{22} J_\gamma(\beta r_2)\right] & \left[\ell_{21} J'_{-\gamma}(\beta r_2) + \ell_{22} J_{-\gamma}(\beta r_2)\right] \end{vmatrix} = 0 \quad (31)$$

where β's are the positive non-zero roots of (31). Note that for every γ there is a corresponding set of β's.

Also from (29) and (30)

$$C_{\gamma\beta} = \frac{-\ell_{11} J'_{-\gamma}(\beta r_1) - \ell_{12} J_{-\gamma}(\beta r_1)}{\ell_{11} J'_\gamma(\beta r_1) + \ell_{12} J_\gamma(\beta r_1)} \quad (32)$$

$$= \frac{-\ell_{21} J'_{-\gamma}(\beta r_2) - \ell_{22} J_{-\gamma}(\beta r_2)}{\ell_{21} J'_{-\gamma}(\beta r_2) + \ell_{22} J_\gamma(\beta r_2)} . \quad (33)$$

---

*Take caution in the notation $J'_\gamma(\beta r_1)$, etc; here $J'_\gamma(\beta r_1) = \frac{\partial}{\partial r}\left[J_\gamma(\beta r)\right]_{r = r_1}$, etc. Some authors use the notation $\beta J'_\gamma(\beta r_1)$.

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We expand the initial temperature distribution in the form

$$h(r, \phi) = \sum_{\gamma}^{\infty} \left\{ \left[\sum_{\beta}^{\infty} A_{\gamma\beta} \left[C_{\gamma\beta} J_{\gamma}(\beta r) + J_{-\gamma}(\beta r) \right] \right] \left[B_{\gamma} \sin \gamma\phi + \cos \gamma\phi \right] \right\}$$

where

$$\sum_{\beta}^{\infty} A_{\gamma\beta} \left[C_{\gamma\beta} (\beta r) + J_{-\gamma} (\beta r) \right] = \frac{\int_{\phi 1}^{\phi 2} h(r, \phi) \left[B_{\gamma} \sin \gamma\phi + \cos \gamma\phi \right] d\phi}{\int_{\phi 1}^{\phi 2} \left[B_{\gamma} \sin \gamma\phi + \cos \gamma\phi \right]^2 d\phi}$$

which leads to

$$A_{\gamma\beta} = \frac{\int_{\phi 1}^{\phi 2} \int_{r1}^{r2} h(r, \phi) \left[B_{\gamma} \sin \gamma\phi + \cos \gamma\phi \right] \left[C_{\gamma\beta} (\beta r) + J_{-\gamma} (\beta r) \right] r \, dr \, d\phi}{\int_{r1}^{r2} r \left[C_{\gamma\beta} J_{\gamma} (\beta r) + J_{-\gamma} (\beta r) \right]^2 dr \int_{\phi 1}^{\phi 2} \left[B_{\gamma} \sin \gamma\phi + \cos \gamma\phi \right]^2 d\phi} \tag{34}$$

The terms in the denominator of Equation (34) do not contain the arbitrary functions and as such may be integrated directly. The integration is carried out in Appendix A.

The general solution for Case I is

$$T_i(\theta, r, \phi) = \sum_{\gamma}^{\infty} \sum_{\beta}^{\infty} A_{\gamma\beta} \left[B_{\gamma} \sin \gamma\phi + \cos \gamma\phi \right] \left[C_{\gamma\beta} J_{\gamma} (\beta r) + J_{-\gamma} (\beta r) \right] e^{-\alpha\beta^2\theta} \tag{23}$$

where $A_{\gamma\beta}$, B_{γ} and $C_{\gamma\beta}$ are defined by (34), (27) and (32) respectively. γ's and β's are defined by (26) and (31).

C. CASE II - STEADY-STATE SOLUTION WITH ARBITRARY COORDINATE VARYING BOUNDARY CONDITIONS

1. PARTICULAR SOLUTION

The differential equation to be solved is

$$\frac{\partial^2 T_s}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 T_s}{\partial \phi^2} + \frac{1}{r} \frac{\partial T_s}{\partial r} = 0. \tag{35}$$

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We assume a product solution of the form

$$T_s(r, \phi) = R(r)\,\Phi(\phi). \qquad (36)$$

Substituting (36) in (35) we obtain

$$\Phi R'' + \frac{1}{r^2} R\Phi'' + \frac{1}{r}\Phi R' = 0. \qquad (37)$$

Rearranging

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\Phi''}{\Phi} = \delta^2 \qquad (38)$$

where δ is any constant.

From the second and third members of (38)

$$\Phi'' + \delta^2 \Phi = 0. \qquad (39)$$

By the methods of ordinary differential equations, (39) is satisfied by

$$\Phi = K^* \sin \delta\Phi + L^* \cos \delta\phi \qquad (40)$$

and we set

$$s = \ln r. \qquad (41)$$

Substituting (41) in the first and third members of (38)

$$R''_s - \delta^2 R_s = 0, \qquad (42)$$

which is satisfied by

$$R_s = M^* \sinh \delta s + N^* \cosh \delta s. \qquad (43)$$

Accordingly, (35) is satisfied by

$$T_s(r, \phi) = \left[K^* \sin \delta\phi + L^* \cos \delta\phi\right]\left[M^* \sinh(\delta \ln r) + N^* \cosh(\delta \ln r)\right] \qquad (44)$$

Returning to (38) we rewrite as

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\Phi'}{\Phi} = -\epsilon^2 \qquad (45)$$

where ϵ is any constant.

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From the second and third members of (45)

$$\Phi' - \epsilon^2 \Phi = 0, \tag{46}$$

which by the methods of ordinary differential equations is satisfied by

$$\Phi = S^* \sinh \epsilon\phi + T^* \cosh \epsilon\phi. \tag{47}$$

From the first and third members of (45)

$$r^2 R'' + r R' + \epsilon^2 R = 0. \tag{48}$$

Setting s = ln r, (48) becomes

$$R_s'' + \epsilon^2 R_s = 0. \tag{49}$$

This is satisfied by

$$R_s = M^* \sin \epsilon s + N^* \cos \epsilon s; \tag{50}$$

that is,

$$R = M^* \sin \epsilon(\ln r) + N^* \cos \epsilon(\ln r). \tag{51}$$

Accordingly, (35) may also be satisfied by

$$T_s(r, \phi) = \left[M^* \sin \epsilon(\ln r) + N^* \cos \epsilon(\ln r)\right]\left[S^* \sinh(\epsilon\phi) + T^* \cosh(\epsilon\phi)\right] \tag{52}$$

Note that (44) and (52), both of which satisfy (35), differ as a result of the change of sign of the constant term chosen in (38) and (45) respectively. The resulting difference is an interchange in r and ϕ in the particular solutions. Both of these particular solutions will be used in the general solutions that follow. General solutions that require expansion of arbitrary functions in ϕ will utilize (44) in such a way that the expansion will be periodic in ϕ. General solutions that require expansion of arbitrary functions in r will use (52) to obtain an expansion periodic in r.

2. GENERAL SOLUTION

The boundary conditions to be satisfied are

$$k_{11} \frac{\partial T}{\partial \phi} + k_{12} T = k_{13} F_{\phi 1}(r) \qquad \phi = \phi_1 \tag{53}$$

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$$k_{21} \frac{\partial T}{\partial \phi} + k_{22} T = k_{23} F_{\phi 2}(r) \qquad \phi = \phi_2 \tag{54}$$

$$\ell_{11} \frac{\partial T}{\partial r} + \ell_{12} T = \ell_{13} F_{r1}(\phi) \qquad r = r_1 \tag{55}$$

$$\ell_{21} \frac{\partial T}{\partial r} + \ell_{22} T = \ell_{23} F_{r2}(\phi) \qquad r = r_2 \tag{56}$$

The general solution will be developed as the sum of four solutions in the form

$$T_s(r, \phi) = T_{sr2}(r, \phi) + T_{sr1}(r, \phi) + T_{s\phi 1}(r, \phi) + T_{s\phi 2}(r, \phi). \tag{57}$$

The four solutions will be referred to as Cases IIa, b, c, and d and will be obtained by alternately assigning the boundary conditions (53) through (56) with the remaining boundaries homogeneous.

a. Case IIa, $T_{sr2}(r, \phi)$

The boundary conditions to be satisfied are

$$k_{11} \frac{\partial T_s}{\partial \phi} + k_{12} T_s = 0 \qquad \phi = \phi_1 \tag{58}$$

$$k_{21} \frac{\partial T_s}{\partial \phi} + k_{22} T_s = 0 \qquad \phi = \phi_2 \tag{59}$$

$$\ell_{11} \frac{\partial T_s}{\partial r} + \ell_{12} T_s = 0 \qquad r = r_1 \tag{60}$$

$$\ell_{21} \frac{\partial T_s}{\partial r} + \ell_{22} T_s = \ell_{23} F_{r2}(\phi) \qquad r = r_2 \tag{61}$$

We use the form of the particular solution (44). Rearranging the arbitrary constants and expanding as a series solution we obtain

$$T_{sr2}(r, \phi) = \sum_{\sigma}^{\infty} L_\sigma \left[Z_\sigma \sin \sigma\phi + \cos \sigma\phi\right]\left[N_{2\sigma} \sinh(\sigma \ln r) + \cosh(\sigma \ln r)\right]. \tag{62}$$

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The boundary conditions (58) and (59) require

$$k_{11}\left[Z_\sigma \sigma \cos \sigma\phi_1 - \sigma \sin \sigma\phi_1\right] + k_{12}\left[Z_\sigma \sin \sigma\phi_1 + \cos \sigma\phi_1\right] = 0 \tag{63}$$

$$k_{21}\left[Z_\sigma \sigma \cos \sigma\phi_2 - \sigma \sin \sigma\phi_2\right] + k_{22}\left[Z_\sigma \sin \sigma\phi_2 + \cos \sigma\phi_2\right] = 0 \tag{64}$$

Eliminating Z_σ from (63) and (64) we obtain the equation of the eigenvalues σ in the form

$$\begin{vmatrix} \left[k_{11}\,\sigma \cos \sigma\phi_1 + k_{12} \sin \sigma\phi_1\right] & \left[k_{12} \cos \sigma\phi_1 - k_{11}\,\sigma \sin \sigma\phi_1\right] \\ \left[k_{21}\,\sigma \cos \sigma\phi_2 + k_{22} \sin \sigma\phi_2\right] & \left[k_{22} \cos \sigma\phi_2 - k_{21}\,\sigma \sin \sigma\phi_2\right] \end{vmatrix} = 0. \tag{65}$$

Also from (63) and (64)

$$Z_\sigma = \frac{-k_{12} \cos \sigma\phi_1 + k_{11}\,\sigma \sin \sigma\phi_1}{k_{11}\,\sigma \cos \sigma\phi_1 + k_{12} \sin \sigma\phi_1} \tag{66}$$

$$\frac{-k_{22} \cos \sigma\phi_2 + k_{21}\,\sigma \sin \sigma\phi_2}{k_{21}\,\sigma \cos \sigma\phi_2 + k_{22} \sin \sigma\phi_2} \tag{67}$$

Condition (60) requires that

$$\ell_{11} \frac{1}{r_1}\left[N_{2\sigma}\,\sigma \cosh(\sigma \ln r_1) + \sigma \sinh(\sigma \ln r_1)\right]$$

$$+ \ell_{12}\left[N_{2\sigma} \sinh(\sigma \ln r_1) + \cosh(\sigma \ln r_1)\right] = 0, \tag{68}$$

which gives

$$N_{2\sigma} = \frac{-\ell_{11}\frac{\sigma}{r_1} \sinh(\sigma \ln r_1) - \ell_{12} \cosh(\sigma \ln r_1)}{\ell_{11}\frac{\sigma}{r_1} \cosh(\sigma \ln r_1) + \ell_{12} \sinh(\sigma \ln r_1)} \tag{69}$$

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Expanding condition (61)

$$\ell_{23} F_{r2}(\phi) = \sum_{\sigma}^{\infty} L_{\sigma} \frac{\ell_{21}}{r_2} \Big\{ \left[N_{2\sigma}\, \sigma \cosh(\sigma \ln r_2) + \sigma \sinh(\sigma \ln r_2)\right]$$

$$\times \left[Z_{\sigma} \sin \sigma\phi + \cos \sigma\phi\right]$$

$$+ \ell_{22} \left[N_{2\sigma} \sinh(\sigma \ln r_2) + \cosh(\sigma \ln r_2)\right]$$

$$\times \left[Z_{\sigma} \sin \sigma\phi + \cos \sigma\phi\right] \Big\}. \quad (70)$$

This leads to

$$L_{\sigma} = \frac{\ell_{23} \int_{\phi 1}^{\phi 2} F_{r2}(\phi) \left[Z_{\sigma} \sin \sigma\phi + \cos \sigma\phi\right] d\phi}{Y_{2\sigma} \int_{\phi 1}^{\phi 2} \left[Z_{\sigma} \sin \sigma\phi + \cos \sigma\phi\right]^2 d\phi} \quad (71)$$

where

$$Y_{2\sigma} = \left(\frac{\ell_{21} N_{2\sigma} \sigma}{r_2} + \ell_{22}\right) \cosh(\sigma \ln r_2) + \left(\frac{\ell_{21} \sigma}{r_2} + \ell_{22} N_{2\sigma}\right) \sinh(\sigma \ln r_2). \quad (72)$$

The integral term in the denominator of (71) may be integrated directly. This is in the same form as the trigonometric function in (34). See Appendix A.

b. Case IIb, $T_{sr1}(r, \phi)$

The boundary conditions to be satisfied are

$$k_{11} \frac{\partial T_s}{\partial \phi} + k_{12} T_s = 0 \qquad \phi = \phi_1 \quad (73)$$

$$k_{21} \frac{\partial T_s}{\partial \phi} + k_{22} T_s = 0 \qquad \phi = \phi_2 \quad (74)$$

$$\ell_{11} \frac{\partial T_s}{\partial r} + \ell_{12} T_s = \ell_{13} F_{r1}(\phi) \qquad r = r_1 \quad (75)$$

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$$\ell_{21} \frac{\partial T_s}{\partial r} + \ell_{22} T_s = 0 \qquad r = r_2 \tag{76}$$

We use the form of the particular solution (44). By methods similar to Case IIa, the general solution may be written as

$$T_{sr1}(r, \phi) = \sum_{\sigma}^{\infty} P_\sigma \left[Z_\sigma \sin \sigma\phi + \cos \sigma\phi\right]$$
$$\times \left[N_{1\sigma} \sinh(\sigma \ln r) + \cosh(\sigma \ln r)\right] . \tag{77}$$

Where σ's are the real positive roots of (65), Z_σ is defined by (66) or (67),

$$P_\sigma = \frac{\ell_{13} \int_{\phi 1}^{\phi 2} F_{r1}(\phi) \left[Z_\sigma \sin \sigma\phi + \cos \sigma\phi\right] d\phi}{Y_{1\sigma} \int_{\phi 1}^{\phi 2} \left[Z_\sigma \sin \sigma\phi + \cos \sigma\phi\right]^2 d\phi} , \tag{78}$$

$$Y_{1\sigma} = \left(\frac{\ell_{11} N_{1\sigma} \sigma}{r_1} + \ell_{12}\right) \cosh(\sigma \ln r_1) + \left(\frac{\ell_{11} \sigma}{r_1} + \ell_{12} N_{1\sigma}\right) \sinh(\sigma \ln r_1), \tag{79}$$

and

$$N_{1\sigma} = \frac{-\ell_{21} \frac{\sigma}{r_2} \sinh(\sigma \ln r_2) - \ell_{22} \cosh(\sigma \ln r_2)}{\ell_{21} \frac{\sigma}{r_2} \cosh(\sigma \ln r_2) + \ell_{22} \sinh(\sigma \ln r_2)} . \tag{80}$$

The integral term in the denominator of (78) may be integrated directly. This is in the same form as the trigonometric function in (34). See Appendix A.

c. Case IIc, $T_{s\phi 1}(r, \phi)$

The boundary conditions to be satisfied are

$$k_{11} \frac{\partial T_s}{\partial \phi} + k_{12} T_s = k_{13} F_{\phi 1}(r) \qquad \phi = \phi_1 \tag{81}$$

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$$k_{21}\frac{\partial T_s}{\partial \phi} + k_{22} T_s = 0 \qquad \phi = \phi_2 \tag{82}$$

$$\ell_{11}\frac{\partial T_s}{\partial r} + \ell_{12} T_s = 0 \qquad r = r_1 \tag{83}$$

$$\ell_{21}\frac{\partial T_s}{\partial r} + \ell_{22} T_s = 0 \qquad r = r_2 \tag{84}$$

We use the form of the particular solution (52). By methods similar to previous cases the general solution is

$$T_{s\phi 1}(r, \phi) = \sum_{\epsilon}^{\infty} Q_\epsilon \left[M_\epsilon \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right] \times \left[S_{1\epsilon} \sinh \epsilon\phi + \cosh \epsilon\phi\right] \tag{85}$$

where ϵ's are the positive roots of

$$\begin{vmatrix} \left[\frac{\ell_{11}\epsilon}{r_1}\cos(\epsilon \ln r_1) + \ell_{12}\sin(\epsilon \ln r_1)\right] & \left[\ell_{12}\cos(\epsilon \ln r_1) - \frac{\ell_{11}\epsilon}{r_1}\sin(\epsilon \ln r_1)\right] \\ \left[\frac{\ell_{21}\epsilon}{r_2}\cos(\epsilon \ln r_2) + \ell_{21}\sin(\epsilon \ln r_2)\right] & \left[\ell_{22}\cos(\epsilon \ln r_2) - \frac{\ell_{21}\epsilon}{r_2}\sin(\epsilon \ln r_2)\right] \end{vmatrix} = 0, \tag{86}$$

$$M_\epsilon = \frac{\frac{\ell_{11}\epsilon}{r_1}\sin(\epsilon \ln r_1) - \ell_{12}\cos(\epsilon \ln r_1)}{\frac{\ell_{11}\epsilon}{r_1}\cos(\epsilon \ln r_1) + \ell_{12}\sin(\epsilon \ln r_2)}, \tag{87}$$

$$Q_\epsilon = \frac{k_{13}\int_{r1}^{r2} F_{\phi 1}(r)\left[M_\epsilon \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right]\frac{dr}{r}}{\left[(k_{11}\epsilon + k_{12}s_{1\epsilon})\sinh \epsilon\phi_1 + (k_{11}\epsilon s_{1\epsilon} + k_{12})\cosh \epsilon\phi_1\right]} \times \frac{1}{\int_{r1}^{r2}\left[M_\epsilon \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right]^2 \frac{dr}{r}}, \tag{88}$$ (88)*

*See Appendix A for integration of the denominator.

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and

$$S_{1\epsilon} = \frac{-\epsilon\, k_{21} \sinh \epsilon\phi_2 - k_{22} \cosh \epsilon\phi_2}{\epsilon\, k_{21} \cosh \epsilon\phi_2 + k_{22} \sinh \epsilon\phi_2} \quad . \tag{89}$$

d. Case IId, $T_{s\phi 2}(r, \phi)$

The boundary conditions to be satisfied are

$$k_{11} \frac{\partial T_s}{\partial \phi} + k_{12} T_s = 0 \qquad \phi = \phi_1 \tag{90}$$

$$k_{21} \frac{\partial T_s}{\partial \phi} + k_{22} T_s = k_{23} F_{\phi 1}(r) \qquad \phi = \phi_2 \tag{91}$$

$$\ell_{11} \frac{\partial T_s}{\partial r} + \ell_{12} T_s = 0 \qquad r = r_1 \tag{92}$$

$$\ell_{21} \frac{\partial T_s}{\partial r} + \ell_{22} T_s = 0 \qquad r = r_2 \tag{93}$$

We use the form of the particular solution (52). By methods similar to previous cases the general solution is

$$T_{s\phi 2}(r, \phi) = \sum_{\epsilon}^{\infty} W_\epsilon \left[M_\epsilon \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right]\left[S_{2\epsilon} \sinh \epsilon\phi + \cosh \epsilon\phi\right] \tag{94}$$

where ϵ's are the real positive roots of (86), M_ϵ is defined by (87),

$$W_\epsilon = \frac{k_{23} \int_{r1}^{r2} F_{\phi 2}(r) \left[M_\epsilon \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right] \frac{dr}{r}}{\left[(k_{21}\,\epsilon + k_{22} S_{2\epsilon}) \sinh \epsilon\phi_2 + (k_{21}\,\epsilon S_{2\epsilon} + k_{22}) \cosh \epsilon\phi_2\right]} \times \frac{1}{\int_{r1}^{r2} \left[M_\epsilon \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right]^2 \frac{dr}{r}} \ , \tag{95}^*$$

*See Appendix A for integration of the denominator.

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and

$$S_{2\epsilon} = \frac{-\epsilon\, k_{11} \sinh \epsilon\phi_1 - k_{12} \cosh \epsilon\phi_1}{\epsilon\, k_{11} \cosh \epsilon\phi_1 + k_{12} \sinh \epsilon\phi_1} \quad . \tag{96}$$

Returning to (57) we now write

$$\begin{aligned} T_s(r,\phi) = {} & \sum_{\sigma}^{\infty} \left[L_\sigma \; Z_\sigma \sin \sigma\phi + \cos \sigma\phi\right] \left[N_{2\sigma} \sinh(\sigma \ln r) + \cosh(\sigma \ln r)\right] \\ & + \sum_{\sigma}^{\infty} P_\sigma \left[Z_\sigma \sin \sigma\phi + \cos \sigma\phi\right] \left[N_{1\sigma} \sinh(\sigma \ln r) + \cosh(\sigma \ln r)\right] \\ & + \sum_{\epsilon}^{\infty} Q_\epsilon \left[M_\epsilon \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right] \left[S_{1\epsilon} \sinh \epsilon\phi + \cosh \epsilon\phi\right] \\ & + \sum_{\epsilon}^{\infty} W_\epsilon \left[M_\epsilon \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right] \left[S_{2\epsilon} \sinh \epsilon\phi + \cosh \epsilon\phi\right] \end{aligned} \tag{97}$$

where σ's, ϵ's, L_σ, P_σ, Q_ϵ, W_ϵ, Z_σ, $N_{2\sigma}$, $N_{1\sigma}$, M_ϵ, $S_{1\epsilon}$, and $S_{2\epsilon}$ are defined by (65), (86), (71), (78), (88), (95), (66), (69), (80), (87), (89), and (96) respectively.

D. CASE III – TRANSIENT SOLUTION WITH INITIAL TEMPERATURE DISTRIBUTION EQUAL TO THE SOLUTION OF CASE II AND HOMOGENEOUS BOUNDARY CONDITIONS

This problem is similar to Case I except the initial temperature distribution is equal to $T_s(r, \phi)$ rather than $h(r, \phi)$.

The boundary conditions to be satisfied are

$$k_{11} \frac{\partial T_{si}}{\partial \phi} + k_{12} T_{si} = 0 \qquad \phi = \phi_1 \tag{98}$$

$$k_{21} \frac{\partial T_{si}}{\partial \phi} + k_{22} T_{si} = 0 \qquad \phi = \phi_2 \tag{99}$$

$$\ell_{11} \frac{\partial T_{si}}{\partial r} + \ell_{12} T_{si} = 0 \qquad r = r_1 \tag{100}$$

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$$\ell_{21} \frac{\partial T_{si}}{\partial r} + \ell_{22} T_{si} = 0 \qquad r = r_2 \tag{101}$$

with

$$T_{si} = T_s (r, \phi) \text{ initially.} \tag{102}$$

The solution may be written directly by the method of Case I as

$$T_{si} (\theta, r, \phi) = \sum_{\gamma}^{\infty} \sum_{\beta}^{\infty} G_{\gamma\beta} \left[B_\gamma \sin \gamma\phi + \cos \gamma\phi\right] \left[C_{\gamma\beta} J_\gamma (\beta r) + J_{-\gamma} (\beta r)\right] e^{-\alpha\beta^2\theta} \tag{103}$$

where

$$G_{\gamma\beta} = \frac{\int_{\phi 1}^{\phi 2} \int_{r 1}^{r 2} T_s (r, \phi) \left[B_\gamma \sin \gamma\phi + \cos \gamma\phi\right] \left[C_{\gamma\beta} J_\gamma (\beta r) + J_{-\gamma} (\beta r)\right] r \, dr \, d\phi}{\int_{\phi 1}^{\phi 2} \left[B_\gamma \sin \gamma\phi + \cos \gamma\phi\right]^2 d\phi \int_{r 1}^{r 2} r \left[C_{\gamma\beta} J_\gamma (\beta r) + J_{-\gamma} (\beta r)\right]^2 dr} \tag{104}$$

and γ's, β's, $T_s (r, \phi)$, B_γ, and $C_{\gamma\beta}$ are defined by (26), (31), (97), (27), and (32) respectively.

E. CASE IV – TRANSIENT SOLUTION WITH INITIAL TEMPERATURE DISTRIBUTION EQUAL TO ZERO AND ARBITRARY COORDINATE AND TIME VARYING CONDITIONS AT THE BOUNDARIES

The boundary conditions to be satisfied are

$$k_{11} \frac{\partial T_\theta}{\partial \phi} + k_{12} T_\theta = k_{13} F_{\phi 1} (r) g (\theta) \qquad \phi = \phi_1 \tag{105}$$

$$k_{21} \frac{\partial T_\theta}{\partial \phi} + k_{22} T_\theta = k_{23} F_{\phi 2} (r) g (\theta) \qquad \phi = \phi_2 \tag{106}$$

$$\ell_{11} \frac{\partial T_\theta}{\partial r} + \ell_{12} T_\theta = \ell_{13} F_{r1} (\phi) g (\theta) \qquad r = r_1 \tag{107}$$

$$\ell_{21} \frac{\partial T_\theta}{\partial r} + \ell_{22} T_\theta = \ell_{23} F_{r2} (\phi) g (\theta) \qquad r = r_2 \tag{108}$$

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We introduce the dummy variable λ and proceed toward a solution with zero initial temperature and boundary conditions

$$k_{11}\frac{\partial T_\lambda}{\partial \phi}+k_{12}T_\lambda = k_{13}F_{\phi 1}(r)\,g(\lambda) \qquad \phi = \phi_1 \tag{109}$$

$$k_{21}\frac{\partial T_\lambda}{\partial \phi}+k_{22}T_\lambda = k_{23}F_{\phi 2}(r)\,g(\lambda) \qquad \phi = \phi_2 \tag{110}$$

$$\ell_{11}\frac{\partial T_\lambda}{\partial r}+\ell_{12}T_\lambda = \ell_{13}F_{r1}(\phi)\,g(\lambda) \qquad r = r_1 \tag{111}$$

$$\ell_{21}\frac{\partial T_\lambda}{\partial r}+\ell_{22}T_\lambda = \ell_{23}F_{r2}(\phi)\,g(\lambda) \qquad r = r_2 \tag{112}$$

We set

$$T_\lambda = T_{s\lambda}(r, \phi, \lambda) - T_{si\lambda}(\theta, r, \phi, \lambda) \tag{113}$$

where

$$\frac{\partial^2 T_{s\lambda}}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 T_{s\lambda}}{\partial \phi^2}+\frac{1}{r}\frac{\partial T_{s\lambda}}{\partial \phi} = 0, \tag{114}$$

with

$$k_{11}\frac{\partial T_{s\lambda}}{\partial \phi}+k_{12}T_{s\lambda} = k_{13}F_{\phi 1}(r)\,g(\lambda) \tag{115}$$

$$k_{21}\frac{\partial T_{s\lambda}}{\partial \phi}+k_{22}T_{s\lambda} = k_{13}F_{\phi 2}(r)\,g(\lambda) \tag{116}$$

$$\ell_{11}\frac{\partial T_{s\lambda}}{\partial r}+\ell_{12}T_{s\lambda} = \ell_{13}F_{r1}(\phi)\,g(\lambda) \tag{117}$$

$$\ell_{21}\frac{\partial T_{s\lambda}}{\partial r}+\ell_{22}T_{s\lambda} = \ell_{23}F_{r2}(\phi)\,g(\lambda) \tag{118}$$

at the boundaries.

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Also

$$\frac{\partial^2 T_{si\lambda}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 T_{si\lambda}}{\partial \phi^2} + \frac{1}{r} \frac{\partial T_{si\lambda}}{\partial r} = \frac{1}{\alpha} \frac{\partial T_{si\lambda}}{\partial \theta} \tag{119}$$

with

$$T_{si\lambda}(\theta, r, \phi, \lambda) = T_{s\lambda}(r, \phi, \lambda) \quad \text{at} \quad \theta = 0 \tag{120}$$

as the initial condition, and

$$k_{11} \frac{\partial T_{si\lambda}}{\partial \phi} + k_{12} T_{si\lambda} = 0 \qquad \phi = \phi_1 \tag{121}$$

$$k_{21} \frac{\partial T_{si\lambda}}{\partial \phi} + k_{22} T_{si\lambda} = 0 \qquad \phi = \phi_2 \tag{122}$$

$$\ell_{11} \frac{\partial T_{si\lambda}}{\partial r} + \ell_{12} T_{si\lambda} = 0 \qquad r = r_1 \tag{123}$$

$$\ell_{21} \frac{\partial T_{si\lambda}}{\partial r} + \ell_{22} T_{si\lambda} = 0 \qquad r = r_2 \tag{124}$$

at the boundaries.

Note that (114) through (118) are similar to (35) and (53) through (56) (Case II) except that the arbitrary functions are multiplied by g (λ). Returning to (97) we note that the arbitrary coefficients L_σ, P_σ, Q_ϵ and W_ϵ are the only part of the solution containing the arbitrary function. Accordingly, the system defined by (114) through (118) has a solution similar to (97) except that each term in the series is multiplied by g(λ). We may then write

$$T_{s\lambda}(r, \phi, \lambda) = g(\lambda)\, T_s(r, \phi). \tag{125}$$

By comparison with Case III, which is similar, we can express the solution to the system defined by (119) through (124) as

$$T_{si\lambda}(\theta, r, \phi, \lambda) = g(\lambda)\, T_{si}(\theta, r, \phi). \tag{126}$$

Then from (113), (125), and (126)

$$T_\lambda = g(\lambda) \left[T_s(r, \phi) - T_{si}(\theta, r, \phi) \right] \tag{127}$$

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where (127) represents the temperature at (r, ϕ) at time θ in the body with initial temperature equal to zero, while its surface temperature is a function of r, ϕ, and λ. Applying Duhammel's theorem to (127) as discussed in Section IIA, we obtain

$$T_\theta(\theta, r, \phi) = \int_0^\theta g(\lambda) \frac{\partial}{\partial\theta} \left[T_s(r, \phi) - T_{s1}(\theta - \lambda, r, \phi)\right] d\lambda, \qquad (128)$$

which is the integral expression that was presented directly in (7).

By differentiation of (97) and (103) with respect to θ after substituting θ-λ for θ we obtain the solution in the form

$$T_\theta(\theta, r, \phi) = \sum_\gamma^\infty \sum_\beta^\infty G_{\gamma\beta} \left[B_\gamma \sin\gamma\phi + \cos\gamma\phi\right] \left[C_{\gamma\beta} J_\gamma(\beta r) + J_{-\gamma}(\beta r)\right]$$

$$\times \left[\alpha\beta^2\right] \int_0^\theta g(\lambda)\, e^{-\left[\alpha\beta^2(\theta-\lambda)\right]} d\lambda \qquad (129)$$

where γ's, β's, $G_{\gamma\beta}$, B_γ, and $C_{\gamma\beta}$ are defined by (26), (31), (104), (27) and (32) respectively.

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SECTION III - INTERPRETATION OF BOUNDARY CONDITIONS

The mathematical formulation of the previous section has been developed with general boundary conditions in which the selection of various indices is used to apply the solution to various modes of heat transfer in the physical problem. This section deals with the physical significance and dimensional requirements of the various indices and arbitrary functions. To retain a general applicability of the results, the system is developed in general units of heat, length, time, and temperature. Angular units are in radians except as noted in Section VI, Utilization of Program.

Since the results of this section are a key to program utilization, a summary is included in Table I, located in Section VI. Table I is developed at this point, however, to avoid excessive detail in the utilization section and to provide information required for solution of the variable thermophysical property portion of the program developed in Section IV.

Returning to the boundary conditions, Equations (3) through (6), we rewrite a typical boundary condition as

$$I_{i1} \frac{\partial T}{\partial x} + I_{i2} T = I_{i3} F(y, \theta) \qquad x = x_i \tag{130}$$

where T is the temperature, x is the coordinate normal to the boundary, and y is the tangential coordinate. Here the I_{ij}'s are equivalent to the k_{ij} and ℓ_{ij} indices of (3) through (6). The general notation of (130) will be used to explain the application of the indices to incorporate various modes of heat transfer at the surfaces.

A. PRESCRIBED SURFACE TEMPERATURE

If we set $I_{i1} = 0$ and $I_{i2} = I_{i3} = 1.0$, (130) becomes

$$T = F(y, \theta) \qquad x = x_i \tag{131}$$

That is, the temperature at surface x_i is a function of time and the coordinate tangent to the surface.

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B. INSULATED SURFACE

If we set $I_{i1} = 1.0$ and $I_{i2} = I_{i3} = 0$, (130) becomes

$$\frac{\partial T}{\partial x} = 0 \qquad x = x_i \tag{132}$$

That is, the rate of change of temperature in the direction normal to x at x_i is equal to zero.

Note that in some cases setting I_{i2} equal to zero does not yield the correct result for a problem in which a face is insulated. Physically, the distinction is: If I_{i2} is positive (see Part D of this section), the final temperature in a solid with an initial temperature distribution of $h(r, \phi)$ and zero at the boundaries (Case I) will be zero no matter how small I_{i2}. If all faces are insulated the final temperature will be an average value of $h(r, \phi)$. Mathematically, the eigenvalue equations have a zero root. The series will converge to the difference between the actual temperature and the average temperature of the initial temperature distribution. In some cases this average temperature could be incorporated by addition of a constant term analogous to the first term in a Fourier cosine series. A term of this type has been included in the steady-state portion of the program (Case II) to handle the special case of I_{i2} equal to zero on three sides. However, the transient part of the solution necessarily contains Bessel functions of the first kind of non-integer order. Since zero is an integer, the mathematics will not handle a case where the eigenvalue equation, (26), has a zero root.

The boundary condition combinations in which the eigenvalue equations will have a zero root are:

(1) If both k_{12} and k_{22} are equal to zero;

(2) If both ℓ_{12} and ℓ_{22} are equal to zero.

The latter case results in a zero root in (86). This set of roots, however, is used only in the steady-state portion to input coordinate varying conditions at the ϕ_1 and ϕ_2 boundaries. Accordingly, if non-coordinate varying functions are input at these boundaries, a valid solution will result. That is, ℓ_{12} and ℓ_{22} may both be set equal to zero provided k_{13} and k_{23} are equal to zero.

C. PRESCRIBED HEAT FLUX AT THE SURFACE*

If we set $I_{i1} = 1.0$, $I_{i2} = 0$ and $I_{i3} = \frac{1}{k}$, Equation (130) becomes

$$\frac{\partial T}{\partial x} = \frac{1}{k} F(y, \theta) \qquad x = x_i \tag{133}$$

*Refer to Part B concerning $I_{i2} = 0$.

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That is, the rate of change of temperature in the direction normal to the surface is proportional to a function of time and the coordinate tangent to that surface. This is a prescribed heat flux of F (y, θ) at surface x_i. The direction of heat flow is in the direction of decreasing x. Setting $I_{i1} = -1.0$ reverses the direction of flow.* Here, k is the thermal conductivity of the body and has the units

$$k \sim \frac{B}{L\ \theta\ F} \tag{134}$$

From (133) and (134) F(y, θ) must have the units

$$F\ (y,\ \theta) \sim \frac{B}{\theta\ L^2} \tag{135}$$

if x has the unit of length (L), such as in the r-direction of Equations (5) and (6).

F (y, θ) must have the units

$$F\ (y,\ \theta) \sim \frac{B}{\theta\ L} \tag{136}$$

if x is dimensionless, such as in the ϕ direction of Equations (3) and (4). In application (136) is realized by setting

$$F\ (y,\ \theta) = \frac{r_2 - r_1}{\ln \frac{r_2}{r_1}}\ f\,(y,\ \theta) \tag{137}$$

where f (y, θ) has the units

$$f\ (y,\ \theta) \sim \frac{B}{\theta\ L^2} \tag{138}$$

which is similar to (135); and

$$\frac{r_2 - r_1}{\ln \frac{r_2}{r_1}} \sim L \tag{139}$$

*A sign convention is introduced in Table I; a positive F (y, θ) always means flux into the body.

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incorporates a log mean value of r such that the differentiation in (133) is with respect to the variable x in radians. Note that in (137) f (y, θ) is the prescribed heat flux at the surface.

The log mean r may also be incorporated by setting

$$I_{i3} = \frac{r_2 - r_1}{\ln \frac{r_2}{r_1}} \left(\frac{1}{k}\right) \tag{140}$$

rather than $\frac{1}{k}$ at the ϕ boundaries.

D. LINEAR HEAT TRANSFER AT THE SURFACE (CONVECTION)

If we set $I_{i1} = 1.0$ and $I_{i2} = I_{i3} = \frac{h}{k}$, (130) becomes

$$\frac{\partial T}{\partial x} = \frac{h}{k}\left[F\,(y, \theta) - T\right] \qquad x = x_i \tag{141}$$

which is equivalent to

$$q = h\,(T_c - T). \tag{142}$$

Equation (142) defines heat transfer normal to the surface x_i by convection between the surface at temperature T and a medium at $T_c = F\,(y, \theta)$. Here, h is the convective heat transfer coefficient, and k is the thermal conductivity of the body. The properties h and k must have compatible units so that

$$\frac{h}{k} \sim \frac{\left(\frac{B}{\theta\, L^2\, F}\right)}{\left(\frac{B}{\theta\, L\, F}\right)} \sim \frac{1}{L} \tag{143}$$

and F (y, θ) has the units F.

As discussed in the previous section, (143) holds only when x has the units of L. As before, at the ϕ boundaries a log mean value of r must be incorporated. In practice this is realized by setting

$$I_{i2} = I_{i3} = \frac{r_2 - r_1}{\ln \frac{r_2}{r_1}} \left(\frac{h}{k}\right)$$

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Note that the log mean r cannot be incorporated in the arbitrary function as suggested in the prescribed heat flux case.

The direction of heat transfer is controlled by the relationship of T_c to T. The direction can be reversed by changing the sign of I_{i1}. As previously mentioned, a sign convention compatible with the structure of the solution is introduced in Table I. This sign convention is developed to ensure the correct heat flow direction in the application of convective boundary conditions.

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SECTION IV – MATHEMATICAL FORMULATION (VARIABLE THERMOPHYSICAL PROPERTIES)

The incorporation of the effects of temperature-dependent thermophysical properties requires:

(1) Thermal diffusivity (α) varies as a function of temperature where

$$\alpha = \frac{k}{\rho C_p} \quad . \tag{144}$$

(2) The boundary condition indicators I_{i2}'s and I_{i3}'s vary as a function of temperature as affected by k.

The resulting differential equation is non-linear and as such does not lend itself to the principle of superposition of solutions as required by this problem. An exact mathematical solution cannot be developed in which each component temperature is aware of the temperature of the body as affected by other components. Case I, for example, is aware only of the effects of initial temperature distribution and would incorporate the effects of variable thermophysical properties at temperatures associated with its contribution to the overall temperature of the body. A reiteration is required to inform Case I of what Case IV is doing, and conversely.

The above mentioned effects are incorporated by recycling the solution for constant thermophysical properties and varying the thermophysical properties in response to the temperature of the preceding cycle. Basically, this is achieved by inputing ρ, C_p, and k as functions of temperature and defining an initial α_0 and k_0. α_0 is a "first guess" at the α associated with the initial temperature distribution; k_0 is the value of k used in determining I_{i2}'s and I_{i3}'s. The initial temperature distribution is calculated first. The properties ρ, C_p, and k are evaluated at the average temperature and used to calculate a new α and correct the I_{i2}'s and I_{i3}'s as affected by a change in k. The temperature distribution at time θ_1 is then calculated. The properties ρ, C_p, and k are re-evaluated from the temperature distribution at time θ_1. A new α and corrected I_{i2}'s and I_{i3}'s are evaluated and the temperature distribution at time θ_1 is inserted as the initial temperature distribution for calculation of the solution at time θ_2. The program continues in this manner for defined increments of time up to θ_n.

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SECTION V – PROGRAMMING

The application of the mathematical formulation of the previous sections is practical only when programmed on a high-speed digital computer; therefore, a computer program has been developed for use on an IBM 7090 or 7094 computer. The flow diagram for the program is shown in Figure 2. Information required for use of the program is included in Section VI. The FORTRAN program listing is given in Appendix B.

The program provides for input of the arbitrary functions as polynomials or as tabulated data. All integrations are carried out numerically by Simpson's rule. Numerical integration is used so that the arbitrary coefficients may be evaluated from arbitrary functions defined by tabulated data or coefficients of polynomials. The handling of tabulated data is required to transfer the solution of Case II into the final solution and to recycle in the variable thermophysical solution as well as to handle tabulated inputs. Actually, special cases of input functions can be integrated by exact methods and programmed algebraically; however, because of the requirement to accept tabulated data, the numerical integration is used throughout the program. The increased complexity of incorporating the special cases in a more exact form is not justified by the moderate increase in accuracy.

The program also provides for an input of the desired accuracy. In theory this would allow for extreme accuracy by increasing the number of terms in the series. In actual practice, however, the storage capacity of the machine limits the number of terms that can be summed. Machine overflow could also occur in the hyperbolic functions for large eigenvalues. To provide the greatest flexibility regarding accuracy, the printout will inform the user in the event the series does not converge to the input accuracy requirement. If this occurs at the maximum number of eigenvalues, the accuracy requirement must be reduced. This scheme provides the user with a knowledge of the maximum accuracy he can obtain for his problem.

A special subroutine DESI is used at the beginning of the program. This subroutine is used for defining input and output tape designations to provide flexibility for use on various systems. As shown in Appendix B, the subroutine is set up for input on tape 2 and output on tape 10 as used at Martin Orlando. If other tape designations are used cards 00000004 and 00000005 may be revised and the subroutine recompiled.

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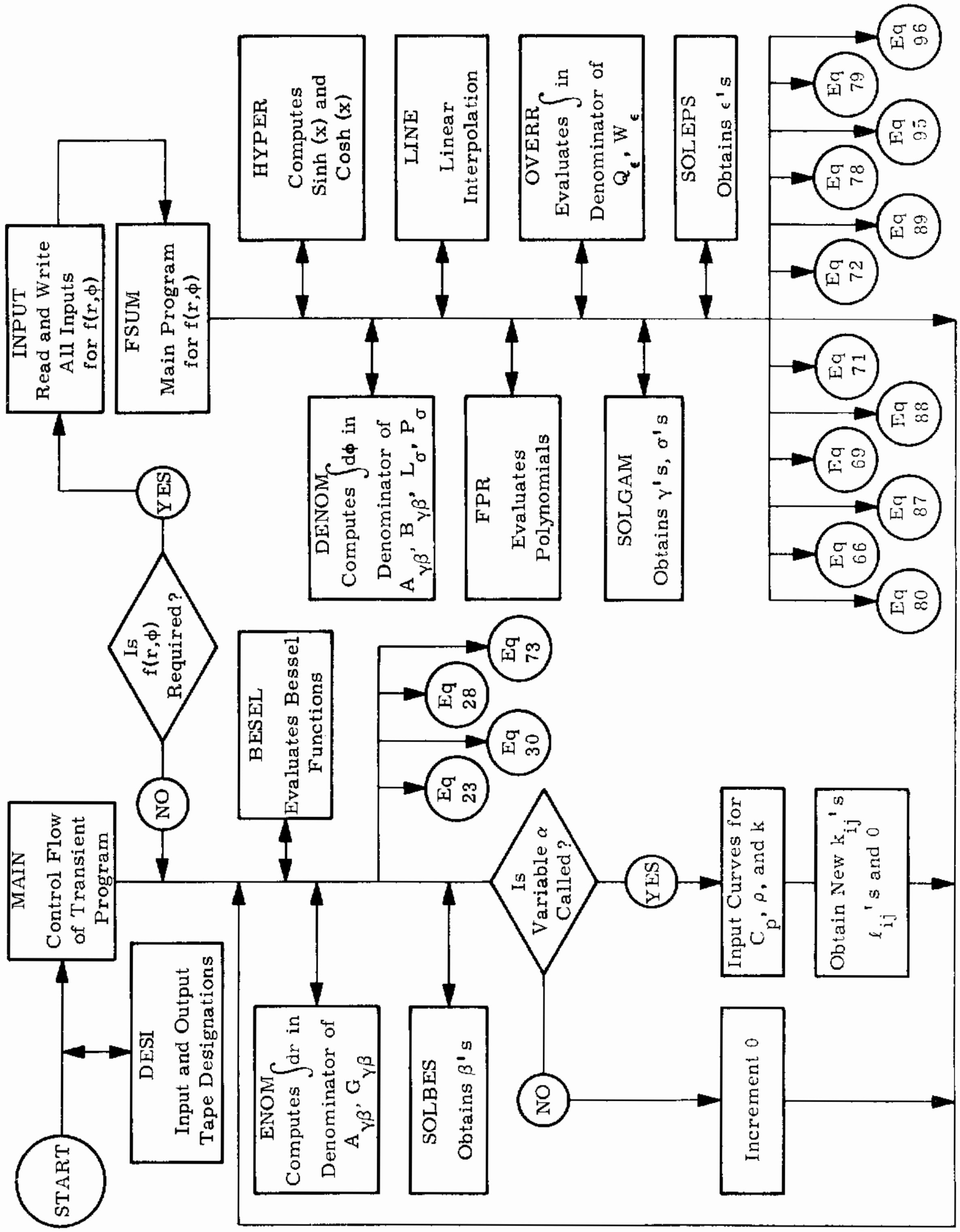

Figure 2. Digital Program Flow Diagram

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SECTION VI - UTILIZATION OF PROGRAM

A. RESTRICTIONS

1. COMBINATIONS OF BOUNDARY CONDITIONS

See Table I for boundary condition indicators required to obtain various heat transfer modes at the surface.

a. Do not set both k_{12} and k_{22} equal to zero.

b. If both ℓ_{12} and ℓ_{22} are equal to zero, then k_{13} and k_{23} must also be equal to zero.

2. GEOMETRY

a. Angular Boundaries

(1) $-\pi < \phi_1 < \phi_2 < \pi$

(2) $\phi_1 \neq -\phi_2 \pm 2$ degrees

(3) $\phi_2 - \phi_1 \neq n\,(90 \pm 2$ degrees) when n is an integer

b. Radial Boundaries

(1) $r_1 \neq 0$

(2) $r_1 \neq 1 \pm 0.01$

(3) $r_2 \neq 1 \pm 0.01$

B. DESCRIPTION OF OUTPUT

(1) Eigenvalues γ, β, and ϵ will be printed and noted as such for reference. σ is equal to γ.

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TABLE I

Boundary Condition Indices for Various Heat Transfer Modes

Heat Transfer Mode	Boundary Equation	k_{11}*	k_{12}	k_{13}	k_{21}	k_{22}	k_{23}	ℓ_{11}*	ℓ_{12}	ℓ_{13}	ℓ_{21}	ℓ_{22}	ℓ_{23}	Units of F (r), F (ϕ)
At $\phi = \phi_1$														
Surface Temp	$T = F_{\phi_1}(r)\, g(\theta)$	0	1	1	-	-	-	-	-	-	-	-	-	F
Heat Flux	$q = F_{\phi_1}(r)\, g(\theta)$	-1	0	$\frac{r_2 - r_1}{\ln \frac{r_2}{r_1}}(\frac{1}{k})$	-	-	-	-	-	-	-	-	-	$B/L^2\,\theta$
Con-vection	$q = h\left[F_{\phi_1}(r)\, g(\theta) - T\right]$	-1	$\frac{r_2 - r_1}{\ln \frac{r_2}{r_1}}(\frac{h}{k})$	$\frac{r_2 - r_1}{\ln \frac{r_2}{r_1}}(\frac{h}{k})$	-	-	-	-	-	-	-	-	-	F
Insulated Surface	$q = 0$	1	0	0	-	-	-	-	-	-	-	-	-	-
At $\phi = \phi_2$														
Surface Temp	$T = F_{\phi_2}(r)\, g(\theta)$	-	-	-	0	1	1	-	-	-	-	-	-	F
Heat Flux	$q = F_{\phi_2}(r)\, g(\theta)$	-	-	-	1	0	$\frac{r_2 - r_1}{\ln \frac{r_2}{r_1}}(\frac{1}{k})$	-	-	-	-	-	-	$B/L^2\,\theta$
Con-vection	$q = h\left[F_{\phi_2}(r)\, g(\theta) - T\right]$	-	-	-	1	$\frac{r_2 - r_1}{\ln \frac{r_2}{r_1}}(\frac{h}{k})$	$\frac{r_2 - r_1}{\ln \frac{r_2}{r_1}}(\frac{h}{k})$	-	-	-	-	-	-	F
Insulated Surface	$q = 0$	-	-	-	1	0	0	-	-	-	-	-	-	-
At $r = r_1$														
Surface Temp	$T = F_{r_1}(\phi)\, g(\theta)$	-	-	-	-	-	-	0	1	1	-	-	-	F
Heat Flux	$q = F_{r_1}(\phi)\, g(\theta)$	-	-	-	-	-	-	-1	0	1/k	-	-	-	$B/L^2\,\theta$
Con-vection	$q = h\left[F_{r_1}(\phi)\, g(\theta) - T\right]$	-	-	-	-	-	-	-1	h/k	h/k	-	-	-	F
Insulated Surface	$q = 0$	-	-	-	-	-	-	1	0	0	-	-	-	-
At $r = r_2$														
Surface Temp	$T = F_{r_2}(\phi)\, g(\theta)$	-	-	-	-	-	-	-	-	-	0	1.0	1.0	F
Heat Flux	$q = F_{r_2}(\phi)\, g(\theta)$	-	-	-	-	-	-	-	-	-	1	0	1/k	$B/L^2\,\theta$
Con-vection	$q = h\left[F_{r_2}(\phi)\, g(\theta) - T\right]$	-	-	-	-	-	-	-	-	-	1	h/k	h/k	F
Insulated Surface	$q = 0$	-	-	-	-	-	-	-	-	-	1	0	0	-

*Negative sign on k_{11} and ℓ_{11} for convective input and heat flux input; with signs as noted here positive F(r) or F(ϕ) give heat flow into surface for heat flux input. (Negative F(r) and F(ϕ) give heat flow out.) The convective input sign convention used here is necessary to ensure correct heat flow direction in the various solutions of the general solution.

(2) The input indicators, k_{ij} and ℓ_{ij}, will be printed and noted as such.

(3) A lattice network of 21 by 6 temperature points representing the steady-state solution is printed for reference to check convergence of that portion of the solution. Corresponding values of r and ϕ are noted.

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(4) If the first series defined as

$$\sum_{\sigma}^{\infty} L_{\sigma} \left[Z_{\sigma} \sin \sigma\phi + \cos \sigma\phi\right] \left[N_{2\sigma} \sinh (\sigma \ln r) + \cosh (\sigma \ln r)\right]$$

$$+ \sum_{\sigma}^{\infty} P_{\sigma} \left[Z_{\sigma} \sin \sigma\phi + \cos \sigma\phi\right] \left[N_{1\sigma} \sinh (\sigma \ln r) + \cosh (\sigma \ln r)\right]$$

does not converge within a given tolerance, "0" will be printed at its corresponding lattice point and the following message will appear in the output:

NO CONVERGENCE FOR FIRST SUMMATION PHI = X, R = Y.
ZERO WILL BE PRINTED AT THIS LATTICE POINT.

(5) If the second series defined as

$$\sum_{\epsilon}^{\infty} Q_{\epsilon} \left[M_{\epsilon} \sin (\epsilon \ln r) + \cos (\epsilon \ln r)\right] \left[S_{1\epsilon} \sinh \epsilon\phi + \cosh \epsilon\phi\right]$$

$$+ \sum_{\epsilon}^{\infty} W_{\epsilon} \left[M_{\epsilon} \sin (\epsilon \ln r) + \cos (\epsilon \ln r)\right] \left[S_{2\epsilon} \sinh \epsilon\phi + \cosh \epsilon\phi\right]$$

does not converge within a given tolerance, "0" will be printed at its corresponding lattice point and the following message will appear in the output:

NO CONVERGENCE FOR SECOND SUMMATION PHI = X, R = Y.
ZERO WILL BE PRINTED AT THIS LATTICE POINT.

(6) If $\ell_{12} + \ell_{22} + k_{12} + k_{22} = 0$, the following message will be printed:

INFINITE STEADY STATE SOLUTION

(7) A lattice network of temperatures at the time requested will be printed. The lattice is defined by selection of $1 < MM < 20$, where $\varphi_2 - \varphi_1$ is divided into MM evenly spaced points, and $1 < NN < 5$, where $r_2 - r_1$ is divided into NN evenly spaced points. Note that the number of points printed is (MM + 1) (NN + 1). The corresponding values of r and ϕ are noted for columns and rows. The corresponding time is printed directly preceding the temperature lattice.

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(8) If the series defined by

$$\sum_{\gamma}^{\infty}\sum_{\beta}^{\infty} A_{\gamma\beta}\left[B_{\gamma}\sin\gamma\phi+\cos\gamma\phi\right]\left[C_{\gamma\beta}J_{\gamma}(\beta r)+J_{-\gamma}(\beta r)\right]e^{-\alpha\beta^{2}\theta}$$

$$+\sum_{\gamma}^{\infty}\sum_{\beta}^{\infty} G_{\gamma\beta}\left[B_{\gamma}\sin\gamma\phi+\cos\gamma\phi\right]\left[C_{\gamma\beta}J_{\gamma}(\beta r)+J_{-\gamma}(\beta r)\right]\left[\alpha\beta^{2}\right]\int_{0}^{\theta} g(\lambda)e^{-\left[\alpha\beta^{2}(\theta-\lambda)\right]}d\lambda$$

does not converge within a given tolerance, "0" will be printed at its corresponding lattice point and the following message will appear in the output:

NO CONVERGENCE WITHIN TOLERANCE PHI = X, R = Y.

C. INPUT DATA FORMAT

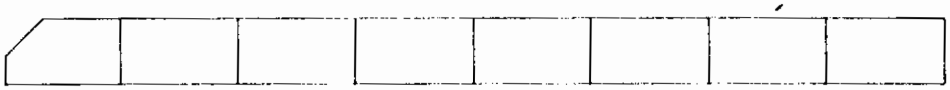
1. DATA ENTRY

a. All inputs, except those underlined, are read into the computer as floating point numbers; they occupy a 10-column field. These numbers and their associated decimal points may be entered anywhere in the 10-column field. Decimal points must be present.

b. All underlined inputs are read into the computer as integers. These numbers, without the decimal point, are entered to the far right of each 10-column field.

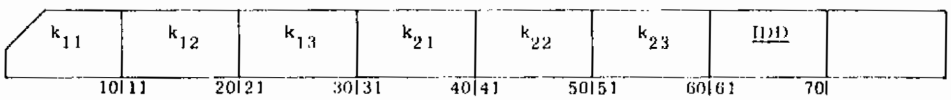
2. INPUT DATA REQUIREMENTS

The input data necessary for one run consists of the following information. The complete sequence of cards can be repeated for stacked runs.



Card 0

(This is a 72-column comment card for identification of the problem.)



Card 1

where

$k_{11}, k_{12}, \ldots, k_{23}$ = input constants (see Table I).

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IDD = indicator

(1) If IDD = 1, program will choose variable thermal diffusivity.

(2) If IDD = 0, program will choose constant thermal diffusivity.

Card 2

ℓ_{11}	ℓ_{12}	ℓ_{13}	ℓ_{21}	ℓ_{22}	ℓ_{23}	TUL1	
10|11	20|21	30|31	40|41	50|51	60|61	70|	

where

$\ell_{11}, \ell_{12}, \ldots, \ell_{23}$ = input constants (see Table I).

TUL1 = tolerance chosen for the series defined by (23) and (129).

The following method is used to determine convergence:

$$\left[\left|T_{\alpha 1}\right| + \left|T_{\alpha 2}\right| \cdots + \left|T_{\alpha\beta}\right|\right] + \left[\left|T_{1\beta}\right| + \left|T_{2\beta}\right| \cdots + \left|T_{(\alpha-1)(\beta-1)}\right|\right] \leq \text{TUL 1}$$

where $T_{\alpha\beta}$'s denote terms in series corresponding to successive eigenvalues.

Card 3

ϕ_1	ϕ_2	r_1	r_2	MM	NN	IPT	
10|11	20|21	30|31	40|41	50|51	60|61	70|	

where

ϕ_1 = boundary angle, input in degrees

ϕ_2 = boundary angle, input in degrees

r_1 = inside radius

r_2 = outside radius

MM = number of equal intervals of $\phi_2 - \phi_1$ printed out. $20 \geq \text{MM} \geq 1$

NN = number of equal intervals of $r_2 - r_1$ printed out. $5 \geq \text{NN} \geq 1$

IPT = 0 or blank. No intermediate print (yes = 1).

Card 4

IND	MI	MI1	NNU	MC			
10|11	20|21	30|31	40|41	50|51	60|61	70|	

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Contrails

where

IND = indicator

(1) Set IND = 1, initial temperature distribution to be read in as coefficient of polynomial up to 7th degree.

(2) Set IND = 0, initial temperature distribution to be read in as data points.

MI = degree of polynomial defining h_r (r) if IND = 1. Set equal to zero if IND = 0.

MI1 = degree of polynomial defining ($h_\phi(\phi)$ if IND = 1. Set equal to zero if IND = 0.

NNU = estimated number of eigenvalues needed for convergence of f (r, ϕ). The input number is algebraically added to 25. In most cases, NNU = -10 is sufficient to handle convergence. If this is not sufficient, input NNU = 0 in column 10 and 25 eigenvalues will be generated. Do not enter NNU as a positive number.

MC = indicator

If

MC = 1, $h(\phi, r) = h_r(r) + h_\phi(\phi)$

MC = 2, $h(\phi, r) = h_r(r) \times h_\phi(\phi)$

MC = 3, $h(\phi, r) = h_r(r) / h_\phi(\phi)$

MC = 4, $h(\phi, r) = h_\phi(\phi) / h_r(r)$

MMC	NNC						
10	11 20	21 30	31 40	41 50	51 60	61 70	

Card 5

where

MMC = number of ϕ's. $1 \leq$ MMC ≤ 21. to be read in as data points for h (r, ϕ)

NNC = number of r's. $1 \leq$ NNC ≤ 6. to be read in as data points for h (r, ϕ)

r_1 r_2

ϕ_1 . . . . .
. . . . .
. . . . .
ϕ_2 . . . . .

here:
MMC = 4
NNC = 4

Note: Card 5 is required only if IND = 0; omit if IND = 1.

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If IND = 1

a_0	a_1	a_2	a_3	a_4	a_5	a_6		Card 6
a_7								Card 6a if required
10	11 20	21 30	31 40	41 50	51 60	61 70		

where

$$hr(r) = a_o r^{MI} + a_1 r^{MI-1} + \ldots + a_{(MI-1)} r^{MI-(MI-1)} + a_{MI}$$

Coefficients of $h_\phi(\phi)$

b_0	b_1	b_2	b_3	b_4	b_5	b_6		Card 7
b_7								Card 7a if required
10	11 20	21 30	31 40	41 50	51 60	61 70		

where the coefficients of $h_\phi(\phi)$ will be listed in the following manner:

$$h_\phi(\phi) = b_0\phi^{MI} + b_1\phi^{MI1-1} + \ldots + b_{(MI1-1)}\phi^{MI1-(MI1-1)} + b_{MI1}$$

Note that ϕ is in radians.

If IND = 0, read in temperature for $h(r,\phi)$ corresponding to the lattice points chosen.

$h(r_1, \phi_1)$	-	-	-	-	$h(r_2, \phi_1)$			Card 6
-	-	-	-	-	-	-		Total of MMC Cards
.							.	
$h(r_1, \phi_2)$	-	-	-	-	$h(r_2, \phi_2)$			

← Total of NNC 10-Column Fields →

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Omit Card 7 if IND = 0.

1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	
IN	MA							Card 8

where

IN = indicator

(1) If IN = 1, read in coefficients for g (λ). The degree of the polynomial may be as large as 7.

(2) If IN = 2, read in θ, g(λ) pairs. θ must be ascending in order but not necessarily equally spaced – maximum of 50 pairs.

MA = degree of polynomial defining g(λ) if IN = 1

= number of θ, g(λ) pairs if IN = 2.

If

IN = 1

1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	
c_0	c_1	c_2	c_3	c_4	c_5	c_6		Card 9
c_7								Card 9a if required

The coefficients are listed in the same manner as $h_r(r)$ with MA corresponding to MI and c corresponding to a.

If

IN = 2

1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	
θ_1	$g(\lambda_1)$	θ_2	$g(\lambda_2)$	θ_3	$g(\lambda_3)$			Card 9
θ_{MA}	$g(\lambda_{MA})$							Card 9a as required

1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	
THETA	DEL	EDEL						Card 10

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where

(1) If IDD = 0

THETA = initial time

DEL = increments of time

EDEL = final time

(2) If IDD = 1

Place 0 in column 10.

BB	CC	ALP	TT				
10	11 20	21 30	31 40	41 50	51 60	61 70	

Card 11

where

BB = constant placed in front of (23). If BB = 0, then corresponding series is zero.

CC = constant placed in front of (129). If CC = 0, then corresponding series is zero.

Otherwise set BB and CC equal to 1.

ALP = thermal diffusivity

TT = datum temperature. Program will call $f(r, \phi)$ + TT.

Cards 12 through 20 will be present in the input data, if, and only if CC $\neq$ 0.

TOL	TOL1						
10	11 20	21 30	31 40	41 50	51 60	61 70	

Card 12

where

TOL = tolerance for first summation defined in Section VI B (4).

TOL1 = tolerance for first summation defined in Section VI B (5).

NI	NU	MU					
10	11 20	21 30	31 40	41 50	51 60	61 70	

Card 13

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NI = indicator

(1) If NI = 1, read in coefficients of polynomial defining $F_{\phi 1}(r)$. The polynomial may be as large as seventh degree.

(2) If NI = 2, read in data points for $F_{\phi 1}(r)$ – maximum of 50 points.

NU = number of intervals chosen for integration. If NU = 0 or if the field is left blank, the program will choose 150 intervals. (100 intervals are more than sufficient in most cases.)

MU = degree of polynomial defining $F_{\phi 1}(r)$ if NI = 1

= number of r, $F_{\phi 1}(r)$ pairs if NI = 2.

If NI = 1

a_0	a_1	a_2	a_3	a_4	a_5	a_6		Card 14
a_7								Card 14a if required
10	11 20	21 30	31 40	41 50	51 60	61 70		

where

$$F_{\phi 1}(r) = a_0 r^{MU} + a_1 r^{MU-1} + \ldots . a_{MU-1} r + a_{MU}$$

If NI = 2

r_1	$F_{\phi 1}(r_1)$							Card 14
r_2	$F_{\phi 1}(r_2)$							Card 14a if required
10	11 20	21 30	31 40	41 50	51 60	61 70		

where number of point pairs input equals MU. r's must be in ascending order but not necessarily equally spaced. (maximum of 100 points)

Card 15

Same as 13 except that indicators refer to $F_{\phi 2}(r)$.

Card 16

Same as 14 except that input defines $F_{\phi 2}(r)$.

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Card 17

Same as 13 except that indicators refer to $F_{r2}(\phi)$.

Card 18

Same as 14 except that input defines $F_{r2}(\phi)$. Note that ϕ is in radians.

Card 19

Same as 13 except that indicators refer to $F_{r1}(\phi)$.

Card 20

Same as 14 except that input defines $F_{r1}(\phi)$. Note that ϕ is in radians.

Cards 21 through 25 will be present in the input data, if, and only if IDD = 1.

MUD	MUD1	MUD2	EEK					Card 21
10	11 20	21 30	31 40	41 50	51 60	61 70		

where

MUD = number of pairs of points to be read in for curve ρ versus T.

MUD1 = number of pairs of points to be read in for curve C_p versus T.

MUD2 = number of pairs of points to be read in for curve k versus T.

EEK = k initial, k_0.

T_1	ρ_1	T_2	ρ_2	T_3	ρ_3			Card 22
T_{MUD}	ρ_{MUD}							Card 22a if required
10	11 20	21 30	31 40	41 50	51 60	61 70		

where

(T, ρ) are values for the curve ρ. T must be ascending in order but not necessarily equally spaced – maximum of 50 pairs.

T_1	C_{p1}	T_2	C_{p2}	T_3	C_{p3}			Card 23
T_{MUD1}	C_{pMUD1}							Card 23a if required
10	11 20	21 30	31 40	41 50	51 60	61 70		

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where

(T, C_p) are values for the curve C_p. T must be ascending in order but not necessarily equally spaced – maximum of 50 pairs.

1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–	
T_1	K_1	T_2	K_2	T_3	K_3			Card 24
T_{MUD2}	K_{MUD2}							Card 24a if required

where

(T, k) are values for the curve k. T must be ascending in order but not necessarily equally spaced.

1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–	
THETA	DEL	EDEL						Card 25

where

THETA = initial time for theta

DEL = increments of theta

EDEL = final time for theta

D. MACHINE REQUIREMENTS

Standard FORTRAN Monitor.

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Contrails

SECTION VII - PROGRAM EVALUATION AND SAMPLE PROBLEM

A. EVALUATION METHODS

A computer program of this type can be evaluated by two methods. The first method checks the problem against itself by altering inputs that should not change the resulting temperature distribution and observing the results. The second method compares the results with an existing program. Both of these methods were used to check out and evaluate the program. In the latter case the problem used for comparison with an existing computer program is also used as a sample problem to aid in understanding the use of the computer program.

The first phase of the evaluation was conducted extensively on the steady-state portion of the solution; time span of the contract and the large number of combinations of boundary condition and arbitrary functions did not permit as extensive a cross check on the complete transient solution. The similarity of the mathematical approach to the complete solution and the steady-state portion as well as the common usage of various subroutines does, however, reduce the requirements for a more extensive cross check on the complete solution. It is felt that the comparison problem and the cross checking provide a sufficient confidence level.

In particular the cross checking was conducted by:

(1) Inputting several problems in which $\phi_2 - \phi_1$ and boundary conditions were held constant and φ_1 was varied. This effectively rotates a given physical problem from one end of the 270 degree range of the solution to the other and provides an excellent check on the phase shift characteristics of the eigenvalues.

(2) Inputting several problems in which ${}^{r_2}/r_1$ and boundary conditions were held constant and r_1 was varied. This effectively moves a given physical problem in the radial direction and demonstrates the non-dimensionality of the solution, and also checks the phase shift characteristics of the eigenvalues.

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(3) Inputting the same arbitrary functions as tabulated data in one problem and as a polynomial in another. This is basically a mechanical check on the programming.

(4) Inputting direct heat into a face and then inputting the resulting surface temperature as the boundary condition for a second problem. This provides a check on the mathematics associated with the modes of input at the boundaries.

(5) Inputting a constant and equal temperature into all four faces of the steady-state solution. This provides an accuracy check in that the solution is known physically to be equal to the input constant temperature. This problem, although physically simple, is actually a difficult problem for the mathematics to handle. The solution requires that the various harmonics in the series build a square wave at each face tapering to zero at the opposite face and a summing of the four solutions to obtain the constant input temperature.

A review of the intermediate and final printout of these types of problems revealed that the solution is insensitive to orientation of the problem and mode of input for the same physical problem. In cases regarding orientation the intermediate printout reveals that the solution uses the same series term by term with the eigenvalues serving to shift the solution to the input physical boundaries.

B. SAMPLE PROBLEM- CONSTANT THERMOPHYSICAL PROPERTIES

The sample problem, under conditions of constant thermophysical properties, is defined as follows:

Geometry:

$r_1 = 0.5$ ft

$r_2 = 0.9$ ft

$$\left.\begin{array}{l}\phi_1 = -20 \text{ degrees} \\ \phi_2 = 135 \text{ degrees}\end{array}\right\} \phi_2 - \phi_1 = 155 \text{ degrees}$$

Material:

Beryllium

$$\left.\begin{array}{l}\alpha = 0.000303 \text{ ft}^2/\text{sec} \\ k = 0.0133 \text{ Btu/sec-ft-°F}\end{array}\right\} \text{Evaluated at 1500°F}$$

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Boundary conditions:

$$T_{\phi 1} = 500°F \qquad \text{at } \phi_1$$

$$T_{\phi 2} = 500°F \qquad \text{at } \phi_2$$

$$q_{r2} = hA\,(T_{aw} - T_{r2}) \qquad \text{at } r_2$$

$$q_{r1} = 0 \qquad \text{at } r_1$$

where

$$h = 0.01 \text{ Btu/sec-ft}^2\text{-°F}$$

$$T_{aw} = 500 + (-27.5\phi^2 + 55.2\phi + 522.6)\, g(\theta)$$

where $g(\theta)$ is defined by tabular data as

$$g(0) = 0.5$$
$$g(100) = 2.0$$
$$g(200) = 3.0$$
$$g(250) = 3.0$$
$$g(300) = 2.5$$
$$g(400) = 2.0$$
$$g(600) = 1.5$$

Initial conditions:

$$T_i = -56.95\phi^2 + 1143\phi - 125r + 659.3$$

The sample problem is illustrated on Figure 3. Note that in this particular problem we set

$$T = \overline{T} + 500°F$$

and solve for $\overline{T}$ with zero temperature at ϕ_1 and ϕ_2. Then

$$\overline{T}_{aw} = (-27.5\phi^2 + 55.2\phi + 522.6)\, g(\theta)$$
$$= F_{r2}(\phi)\, g(\theta)$$

We can rearrange T_i in the form

$$T_i = (-56.95\phi^2 + 114.3\phi + 46.8) + (-125r + 112.5) + 500.$$

Then

$$\overline{T}_i = (-56.95\phi^2 + 114.3\phi + 46.8) + (-125r + 112.5)$$
$$= h_\phi(\phi) + h_r(r) = h(r_1\phi).$$

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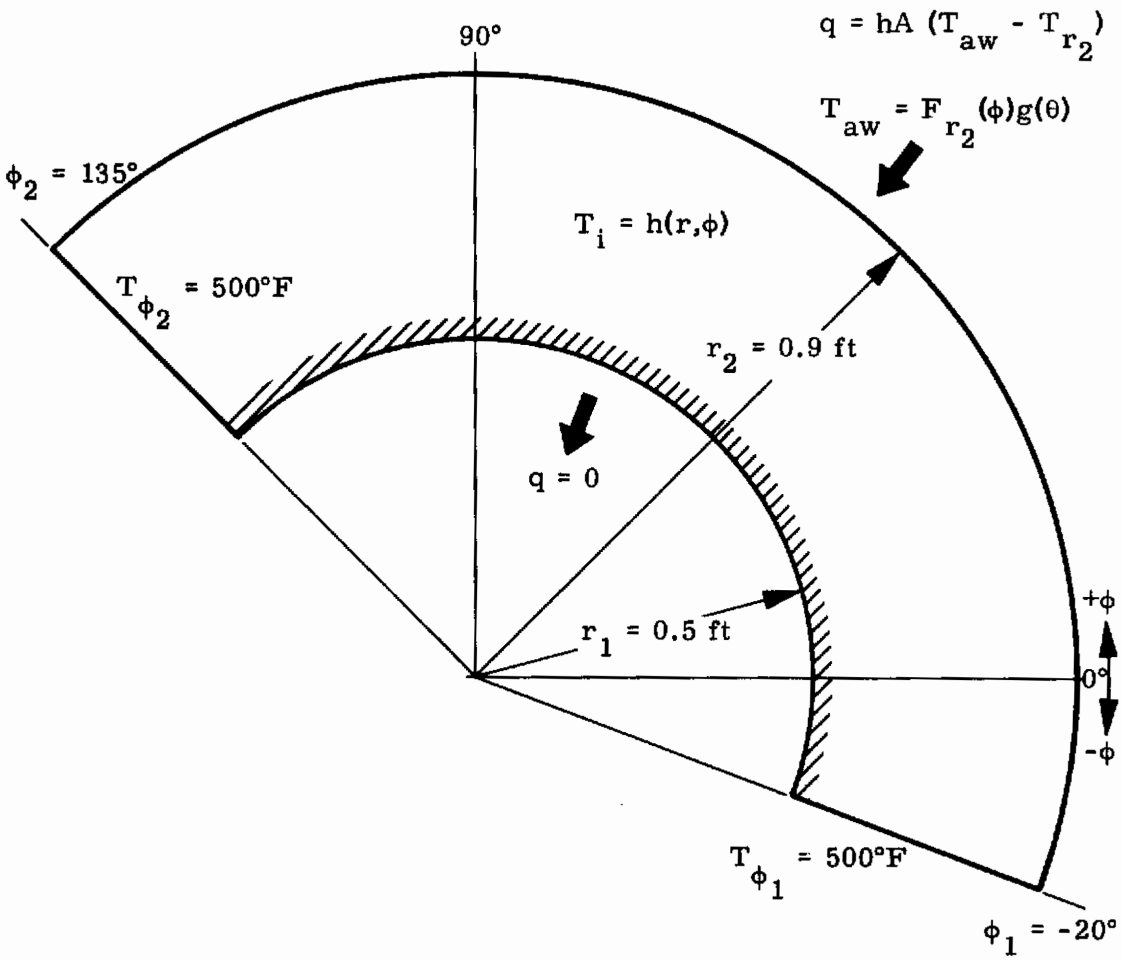

Figure 3. Sample Problem Geometry

By reference to Table I and Section VI the inputs to obtain $\widetilde{T}$ may be coded as shown in Table II. Printout is requested for 0, 200, 400, and 600 seconds with temperature data at 36 points representing five equally spaced divisions of $\phi_2 - \phi_1$ and five equally spaced divisions of $r_2 - r_1$. Tolerance requested is 100 on (23) and (129) and 25 for (97).

Printout of the steady-state portion of the solution is shown on Table III. Printouts of the temperature distributions at 0, 200, 400, and 600 seconds are given on Tables IV through VII. Figures 4 through 7 are illustrations of the temperature distributions. The total run time on a 7094 Computer was 0.056 hours.

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TABLE II

Sample Problem – Constant Thermophysical Properties – FORTRAN Coding Form

FORM 2265 APR63 MARTIN ORLANDO

FORTRAN CODING FORM

TITLE SAMPLE PROBLEM – CONSTANT THERMOPHYSICAL PROPERTIES ANALYST

JOB NO. DATE SHEET I OF I

REMARKS	1–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80
CARD 0	SAMPLE PROB CONST THERM PROP							
1	0	1	0	0	1	0	0	
2	1	0	0	1	0.828	0.828	100	
3	-20	135	0.5	0.9	5	5	0	
4	1	1	2	-10	1			
6	-125	112.5						
7	-56.95	114.3	46.8					
8	2	7						
9	0	.5	100	2.0	200	3.0		
9a	250	3.0	300	2.5	400	2.0		
9b	600	1.5						
10	0	200	600					
11	1	1	.000303	0				
12	25	25						
13	1	10	0					
14	0							
15	1	10	0					
16	0							
17	1	100	2					
18	-27.5	55.2	522.6					
19	1	10	0					
20	0							

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TABLE III

Temperature Distribution Printout – Sample Problem –
Constant Thermophysical Properties, Steady-State Solution

K11= 0. K12= 1.0000000E 00 K13= 0. K21= 0. K22= 1.0000000E 00 K23= 0.
L11= 1.0000000E 00 L12= 0. L13= 0. L21= 1.0000000E 00 L22= 8.2799999E-01 L23= 8.2799999E-01

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
127.25		50.20	51.75	55.91	62.47	71.63	90.68
119.50		97.93	100.72	108.16	119.73	138.53	162.68
111.75		141.16	144.71	154.02	168.18	186.50	209.83
104.00		178.66	182.46	192.27	206.69	221.67	239.90
96.25		209.95	213.67	223.06	236.34	249.79	264.71
88.50		235.11	238.62	247.32	259.23	272.85	289.24
80.75		254.43	257.77	266.01	277.16	289.71	302.43
73.00		268.15	271.44	279.59	290.75	303.67	317.48
65.25		276.40	279.70	287.97	299.52	311.77	325.65
57.50		279.15	282.47	290.82	301.02	313.34	325.91
49.75		276.40	279.70	287.97	299.52	311.76	325.65
42.00		268.15	271.44	279.59	290.75	303.67	317.48
34.25		254.42	257.77	266.01	277.15	289.71	302.43
26.50		235.11	238.62	247.32	259.23	272.84	289.24
18.75		209.95	213.67	223.06	236.34	249.79	264.71
11.00		178.66	182.46	192.27	206.69	221.67	239.90
3.25		141.16	144.71	154.02	168.18	186.49	209.83
-4.50		97.93	100.72	108.16	119.73	138.53	162.68
-12.25		50.20	51.75	55.91	62.47	71.63	90.68
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE IV

Temperature Distribution Printout - Sample Problem - Constant Thermophysical Properties, at Time Equals Zero

THETA= 0.

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
104.00		100.35	97.69	90.62	81.53	73.09	67.16
73.00		160.99	156.53	144.73	129.61	115.72	106.13
42.00		159.28	154.63	142.38	126.78	112.57	103.00
11.00		97.59	94.62	86.82	76.93	67.99	62.09
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

TABLE V

Temperature Distribution Printout - Sample Problem - Constant Thermophysical Properties, at Time Equals 200 Seconds

THETA= 2.00000000E 02

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
104.00		99.90	112.36	142.29	175.08	195.89	195.02
73.00		139.24	180.91	229.32	282.35	316.03	314.66
42.00		138.15	179.81	228.18	281.19	314.88	313.56
11.00		98.14	110.57	140.45	173.20	194.03	193.24
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE VI

Temperature Distribution Printout – Sample Problem –
Constant Thermophysical Properties, at Time Equals 400 Seconds

THETA= 4.00000000E 02

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
104.00		195.31	206.32	232.15	259.25	274.12	267.96
73.00		315.65	333.44	375.23	419.08	443.13	433.19
42.00		315.17	332.96	374.73	418.58	442.64	432.71
11.00		194.55	205.54	231.35	258.45	273.32	267.20
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

TABLE VII

Temperature Distribution Printout – Sample Problem –
Constant Thermophysical Properties, at Time Equals 600 Seconds

THETA= 6.00000000E 02

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
104.00		233.70	241.54	259.43	277.13	284.63	275.49
73.00		377.96	390.69	419.60	448.23	460.37	445.58
42.00		377.76	390.44	419.38	448.02	460.16	445.38
11.00		233.37	241.20	259.09	276.78	284.29	275.16
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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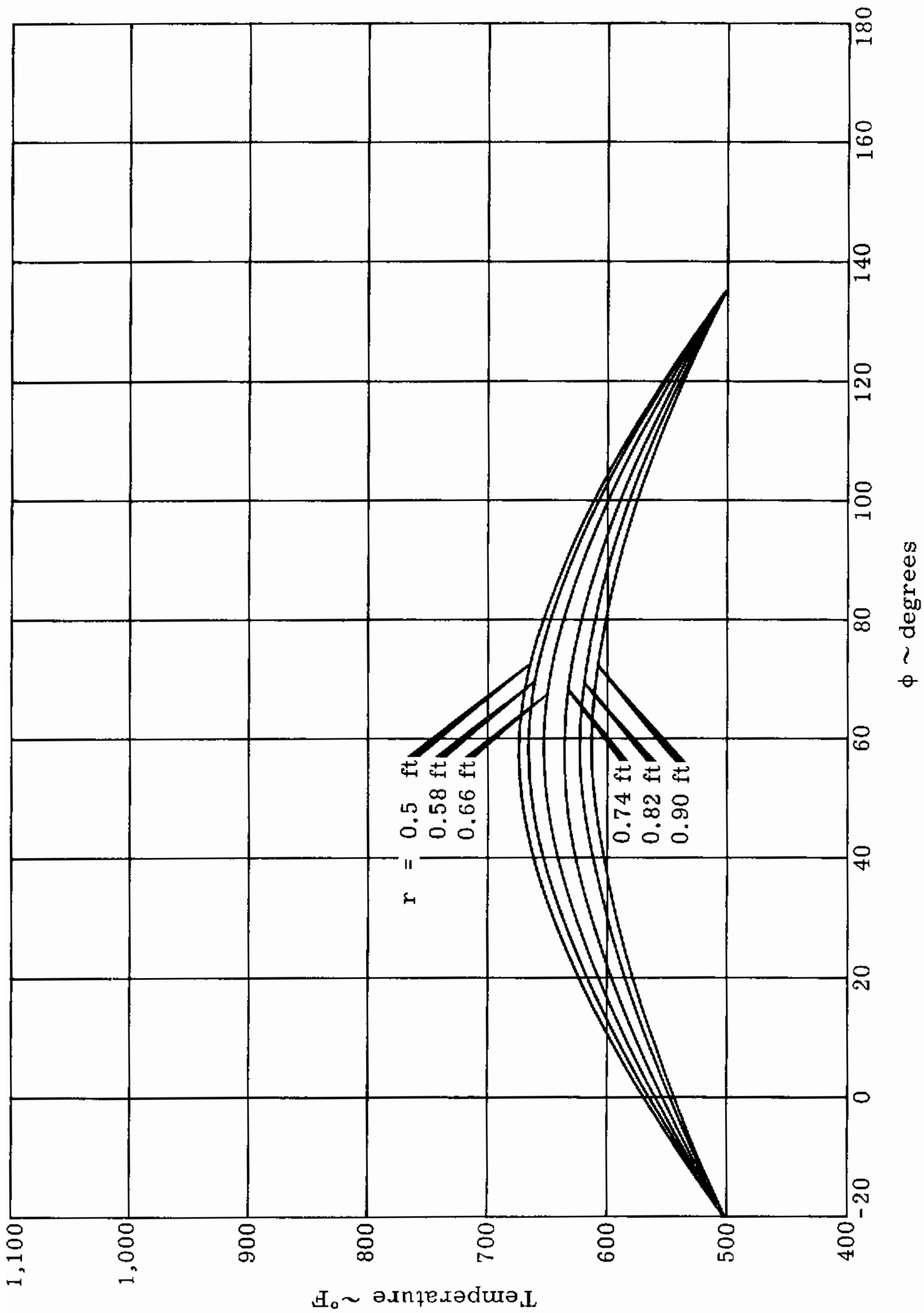

Figure 4. Temperature Distribution – Sample Problem – Constant Thermophysical Properties, $\theta = 0$

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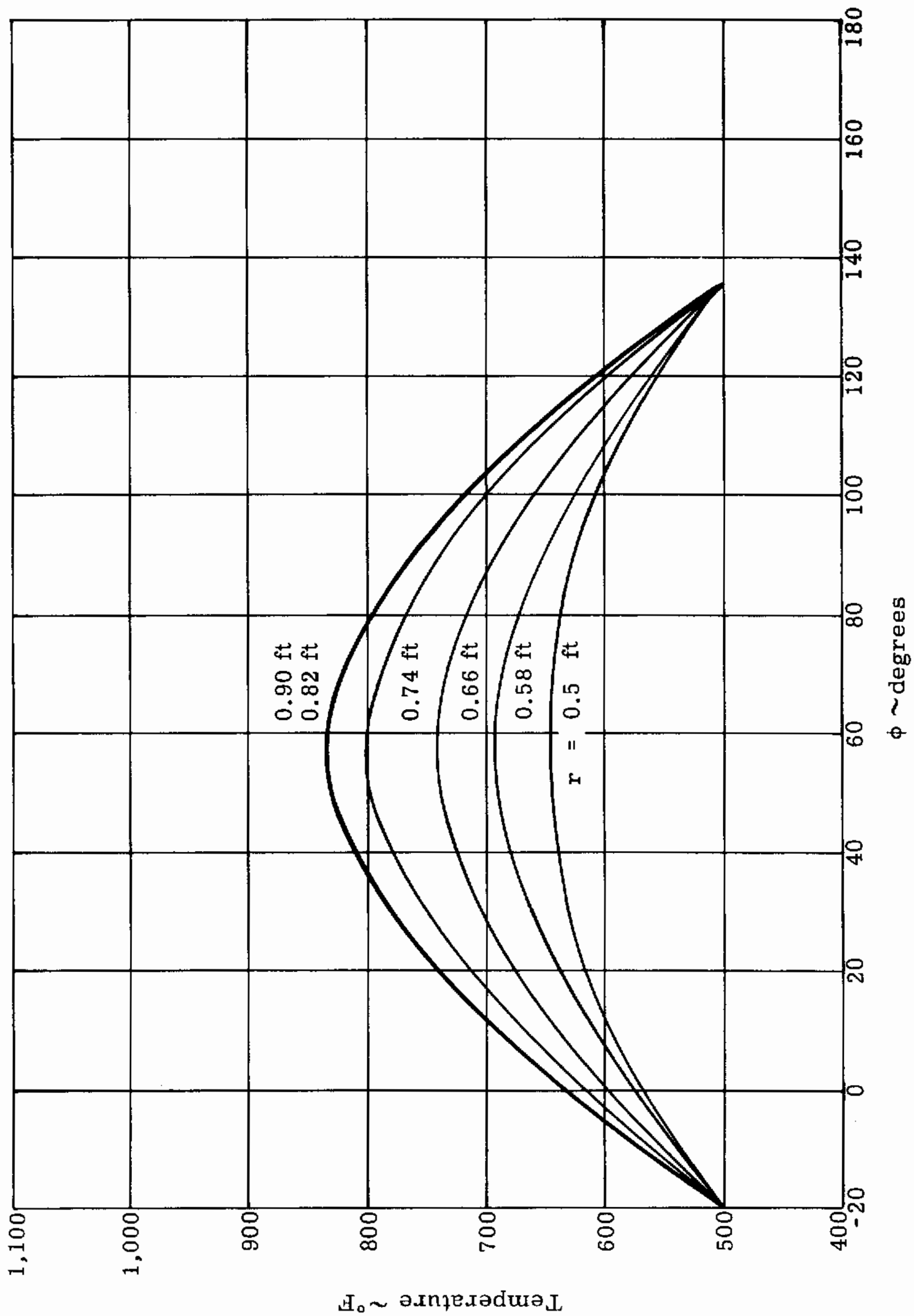

Figure 5. Temperature Distribution – Sample Problem – Constant Thermophysical Properties, $\theta = 200$

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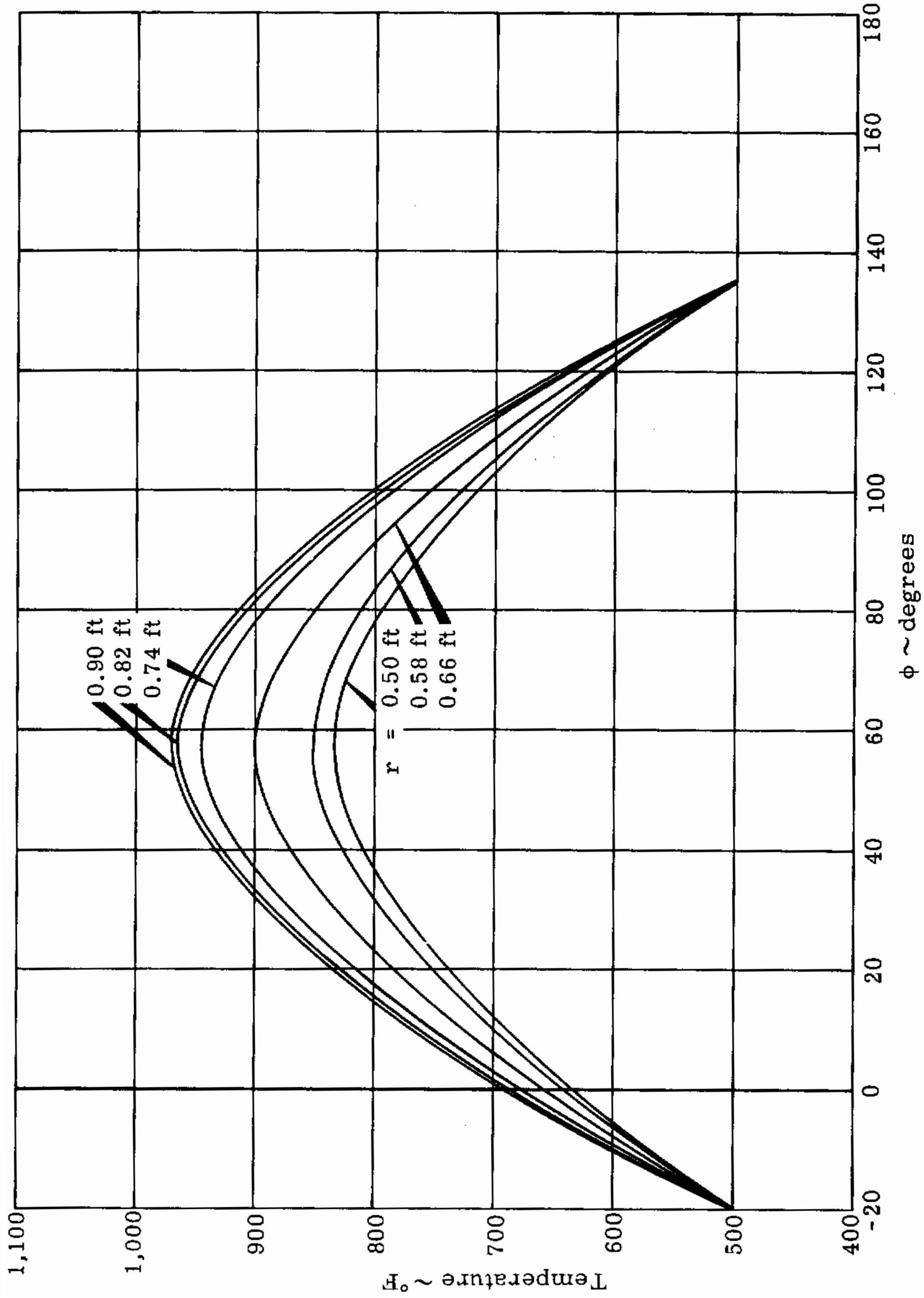

Figure 6. Temperature Distribution - Sample Problem - Constant Thermophysical Properties, $\theta = 400$

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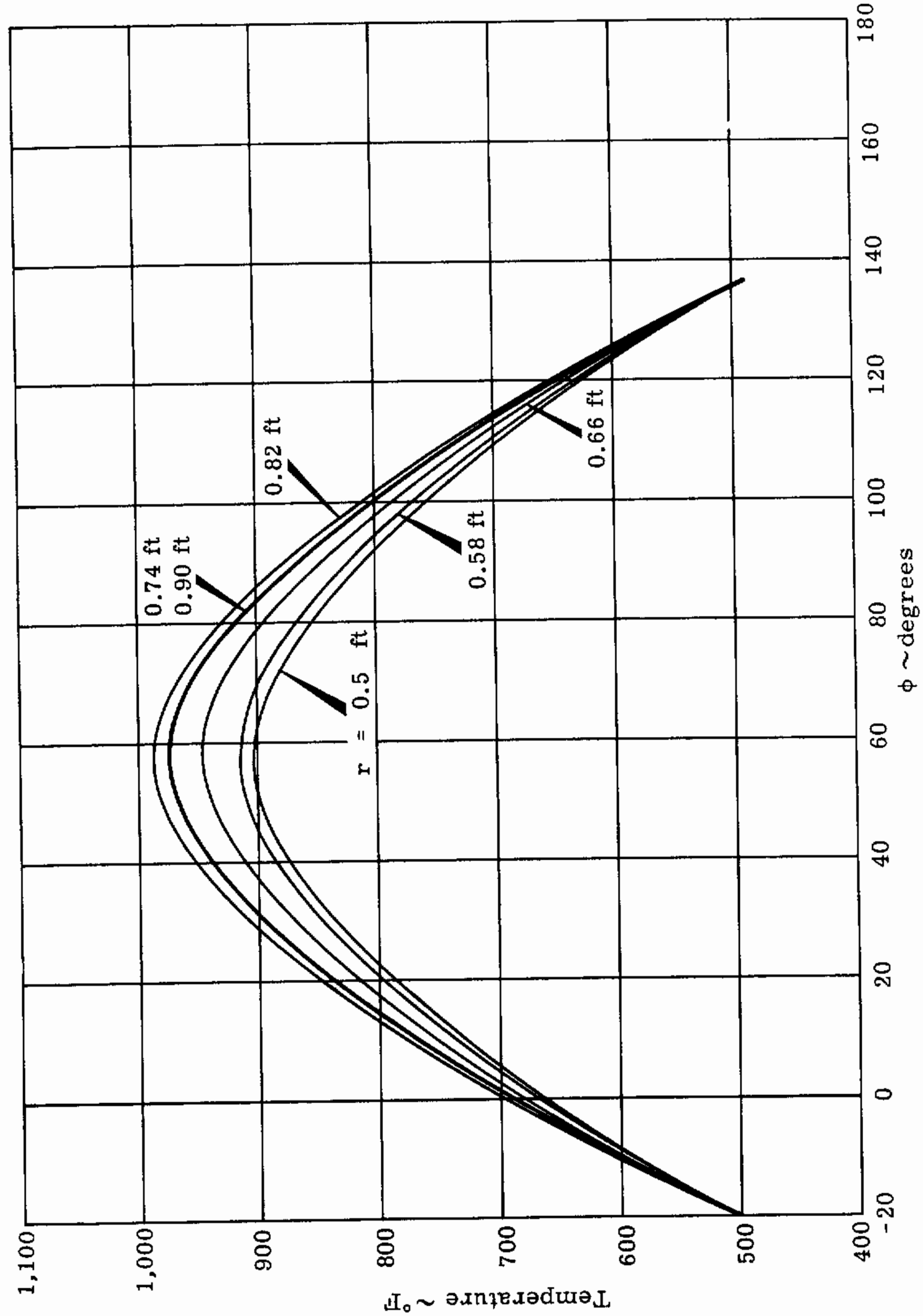

Figure 7. Temperature Distribution – Sample Problem – Constant Thermophysical Properties, $\theta = 600$

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C. SAMPLE PROBLEM-VARIABLE THERMOPHYSICAL PROPERTIES

The sample problem, under conditions of variable thermophysical properties, is of the same physical configuration as the previous constant thermophysical property problem, but in this case C_p, ρ, and k vary with temperature as shown in Figure 8. The coding of this problem is shown in Table VIII. Note that ρ, C_p, and k have been input at values shifted 500°F to obtain a solution in $\overline{T}$. Before

$$\overline{T} = T - 500$$

Printout of the solution is shown in Tables IX through XVI. Note that a steady-state solution appears between the temperature distribution for each time requested. Figures 9 through 11 are illustrations of the temperature distributions. The temperature distribution at time equal zero is the same as for the constant thermophysical property case (Figure 4). The total run time on a 7094 Computer was 0.180 hours.

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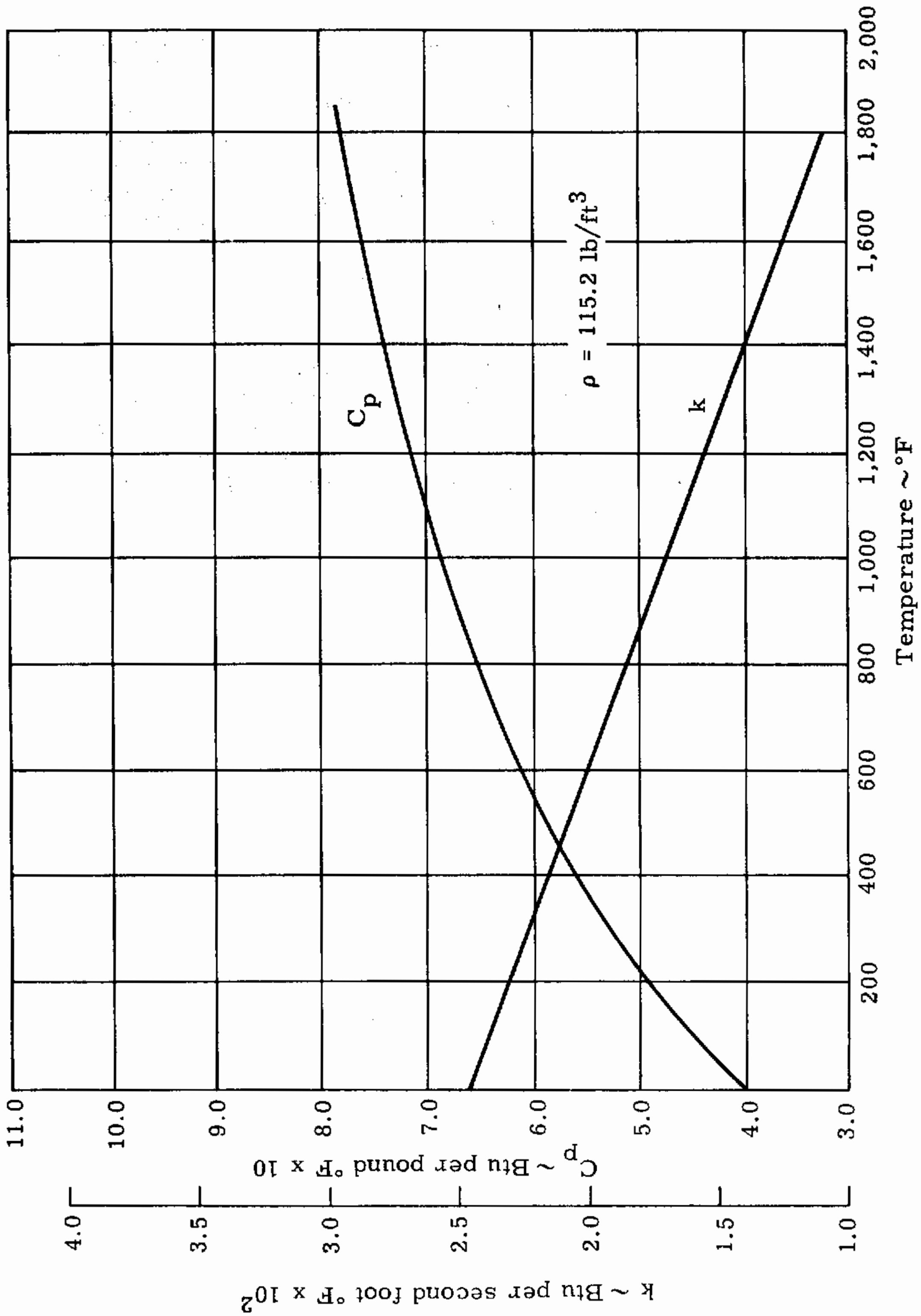

Figure 8. Thermophysical Properties – Beryllium

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TABLE VIII

Sample Problem – Variable Thermophysical Properties – FORTRAN Coding Form

FORM 2265 APR63 MARTIN

FORTRAN CODING FORM

TITLE SAMPLE PROBLEM – VARIABLE THERMOPHYSICAL PROPERTIES ANALYST

JOB NO. DATE SHEET 1 OF 1

REMARKS	1–10	11–20	21–30	31–40	41–50	51–60	61–70
CARD 0	SAMPLE PROB	VARI THERM	PROP				
1	0	1	0	0	1	0	1
2	1	0	0	1	0.647	0.647	100
3	-20	135	0.5	0.9	5	5	0
4	1	1	2	-10	1		
6	-125	112.5					
7	-56.95	114.3	46.8				
8	2	7					
9	0	.5	100	2.0	200	3.0	
9a	250	3.0	300	2.5	400	2.0	
9b	600	1.5					
10	0						
11	1	1	.00036	0			
12	25	25					
13	1	10	0				
14	0						
15	1	10	0				
16	0						
17	1	100	2				
18	-27.5	55.2	522.6				
19	1	10	0				
20	0						
21	3	6	3	.0170			
22	0	115.2	750	115.2	1500	115.2	
23	0	.585	300	.655	600	.703	
23a	900	.741	1200	.777	1500	.790	
24	0	.0222	750	.0152	1500	.00985	
25	0	200	600				

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TABLE IX

Temperature Distribution Printout - Sample Problem - Variable
Thermophysical Properties, Steady-State Solution, First Time Step

K11= 0. L11= 1.0000000E 00 K12= 1.0000000E 00 L12= 0. K13= 0. L13= 0. K21= 0. L21= 1.0000000E 00 K22= 1.0000000E 00 L22= 6.4700000E-01 K23= 0. L23= 6.4700000E-01

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
127.25		43.90	45.20	48.69	54.17	61.78	71.91
119.50		85.71	88.07	94.33	104.04	119.62	139.58
111.75		123.73	126.74	134.64	146.61	162.06	181.65
104.00		156.86	160.13	168.54	180.88	196.58	209.60
96.25		184.66	187.90	196.07	207.62	221.55	232.78
88.50		207.13	210.22	217.91	228.47	240.61	253.25
80.75		224.44	227.43	234.80	244.81	256.17	267.84
73.00		236.78	239.73	247.06	257.13	268.84	281.42
65.25		244.20	247.17	254.61	265.01	276.21	288.89
57.50		246.67	249.66	257.17	267.75	277.78	289.43
49.75		244.20	247.17	254.61	265.01	276.21	288.89
42.00		236.78	239.73	247.06	257.13	268.84	281.42
34.25		224.44	227.43	234.80	244.81	256.17	267.84
26.50		207.13	210.22	217.91	228.47	240.61	253.25
18.75		184.66	187.90	196.07	207.62	221.55	232.77
11.00		156.86	160.13	168.54	180.88	196.58	209.60
3.25		123.73	126.74	134.64	146.61	162.06	181.64
-4.50		85.71	88.07	94.33	104.04	119.62	139.58
-12.25		43.90	45.20	48.69	54.17	61.78	71.91
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE X

Temperature Distribution Printout – Sample Problem –
Variable Thermophysical Properties, at Time Equals Zero

THETA= 0.

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
104.00		100.56	97.80	90.51	81.29	73.05	67.79
73.00		161.32	156.70	144.56	129.25	115.65	107.13
42.00		159.60	154.79	142.22	126.42	112.50	103.97
11.00		97.78	94.71	86.72	76.72	67.96	62.68
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE XI

Temperature Distribution Printout - Sample Problem - Variable Thermophysical Properties, Steady-State Solution, Second Time Step

K11= 0. K12= 7.8985648E-01 K13= 0. K21= 0. K22= 7.8985648E-01 K23= 0.
L11= 1.0000000E 00 L12= 0. L13= 0. L21= 1.0000000E 00 L22= 5.1103714E-01 L23= 5.1103714E-01

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
127.25		38.12	39.21	42.14	46.71	53.05	61.44
119.50		74.48	76.46	81.73	89.87	100.96	119.33
111.75		107.63	110.19	116.88	126.98	140.35	156.42
104.00		136.65	139.44	146.62	157.15	170.51	186.48
96.25		161.10	163.89	170.96	180.95	193.00	206.46
88.50		180.93	183.64	190.37	199.63	210.33	221.56
80.75		196.27	198.90	205.42	214.30	224.44	234.95
73.00		207.22	209.84	216.35	225.30	235.76	247.05
65.25		213.81	216.45	223.06	232.31	243.42	253.78
57.50		216.01	218.67	225.34	234.74	246.17	254.46
49.75		213.81	216.45	223.06	232.31	243.42	253.78
42.00		207.22	209.84	216.35	225.30	235.76	247.05
34.25		196.27	198.90	205.42	214.30	224.44	234.95
26.50		180.93	183.64	190.37	199.63	210.33	221.56
18.75		161.10	163.89	170.95	180.95	193.00	206.46
11.00		136.65	139.44	146.62	157.15	170.51	186.48
3.25		107.63	110.19	116.87	126.98	140.35	156.42
-4.50		74.48	76.46	81.73	89.87	100.96	119.33
-12.25		38.12	39.21	42.14	46.71	53.05	61.44
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE XII

Temperature Distribution Printout - Sample Problem -
Variable Thermophysical Properties, at Time Equals 200 Seconds

THETA= 2.00000000E 02

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
104.00		96.94	104.38	122.22	141.98	155.06	155.71
73.00		155.54	167.55	196.37	228.29	249.45	250.52
42.00		153.93	165.89	194.65	226.51	247.65	248.77
11.00		94.32	101.70	119.44	139.10	152.16	152.67
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE XIII

Temperature Distribution Printout – Sample Problem – Variable Thermophysical Properties, Steady-State Solution, Third Time Step

K11= 0. K12= 8.0349697E-01 K13= 0. K21= 0. K22= 8.0349697E-01 K23= 0.
L11= 1.0000000E 00 L12= 0. L13= 0. L21= 1.0000000E 00 L22= 5.1986254E-01 L23= 5.1986254E-01

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
127.25		38.53	39.63	42.60	47.24	53.66	62.17
119.50		75.27	77.28	82.62	90.87	102.10	120.73
111.75		108.77	111.36	118.13	128.36	141.91	158.19
104.00		138.08	140.91	148.18	158.83	172.35	188.52
96.25		162.78	165.60	172.74	182.84	195.03	205.13
88.50		182.80	185.53	192.34	201.69	212.49	223.83
80.75		198.28	200.94	207.52	216.48	226.71	237.31
73.00		209.33	211.98	218.54	227.58	238.13	249.52
65.25		215.98	218.65	225.32	234.65	245.87	256.30
57.50		218.20	220.88	227.62	237.10	248.64	256.97
49.75		215.98	218.65	225.32	234.65	245.87	256.30
42.00		209.33	211.98	218.54	227.58	238.13	249.52
34.25		198.28	200.94	207.52	216.48	226.71	237.30
26.50		182.80	185.53	192.34	201.69	212.49	223.83
18.75		162.77	165.60	172.74	182.84	195.03	205.13
11.00		138.08	140.91	148.18	158.83	172.35	188.52
3.25		108.77	111.36	118.13	128.36	141.91	158.19
-4.50		75.27	77.28	82.62	90.87	102.10	120.73
-12.25		38.52	39.63	42.60	47.24	53.66	62.17
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE XIV

Temperature Distribution Printout – Sample Problem –
Variable Thermophysical Properties, at Time Equals 400 Seconds

THETA= 4.00000000E 02

		0.500 R1	0.580	0.660	0.740	0.820	0.900 R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
104.00		165.64	172.01	186.92	202.74	212.01	209.75
73.00		264.91	275.16	299.14	324.63	339.60	336.01
42.00		261.09	271.25	295.06	320.42	335.33	331.84
11.00		159.45	165.69	18C.32	195.92	205.12	203.00
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE XV

Temperature Distribution Printout – Sample Problem – Variable Thermophysical Properties, Steady-State Solution, Fourth Time Step

K11= 0. K12= 8.2256744E-01 K13= 0. K21= 0. K22= 8.2256744E-01 K23= 0.
L11= 1.0000000E 00 L12= 0. L13= 0. L21= 1.0000000E 00 L22= 5.3220113E-01 L23= 5.3220113E-01

		0.500	0.560	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
127.25		39.09	40.22	43.23	47.96	54.50	63.17
119.50		76.37	78.41	83.85	92.24	103.68	122.68
111.75		110.34	112.97	119.86	130.27	144.06	160.63
104.00		140.06	142.93	150.32	161.14	174.89	186.51
96.25		165.08	167.95	175.20	185.45	197.82	208.04
88.50		185.37	188.14	195.04	204.52	215.47	226.94
80.75		201.05	203.74	210.40	219.48	229.83	240.55
73.00		212.24	214.91	221.56	230.71	241.39	252.91
65.25		218.97	221.67	228.42	237.87	249.23	259.77
57.50		221.22	223.93	230.75	240.35	249.66	260.43
49.75		218.97	221.67	228.42	237.87	249.23	259.77
42.00		212.24	214.91	221.56	230.71	241.39	252.91
34.25		201.05	203.74	210.40	219.48	229.83	240.55
26.50		185.37	188.14	195.04	204.52	215.47	226.94
18.75		165.08	167.95	175.20	185.45	197.82	208.04
11.00		140.06	142.93	150.32	161.14	174.89	186.51
3.25		110.34	112.97	119.86	130.27	144.05	160.63
-4.50		76.37	78.41	83.85	92.24	103.68	122.68
-12.25		39.09	40.22	43.23	47.96	54.50	63.17
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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TABLE XVI

Temperature Distribution Printout – Sample Problem –
Variable Thermophysical Properties, at Time Equals 600 Seconds

THETA= 6.00000000E 02

		0.500	0.580	0.660	0.740	0.820	0.900
		R1					R2
135.00	PHI 2	0.00	0.00	0.00	0.00	0.00	0.00
104.00		202.53	206.26	214.46	222.28	225.17	220.16
73.00		323.73	329.67	342.77	355.26	359.89	351.89
42.00		318.82	324.65	337.53	349.84	354.41	346.53
11.00		194.59	198.13	205.99	213.50	216.30	211.49
-20.00	PHI 1	0.00	0.00	0.00	0.00	0.00	0.00

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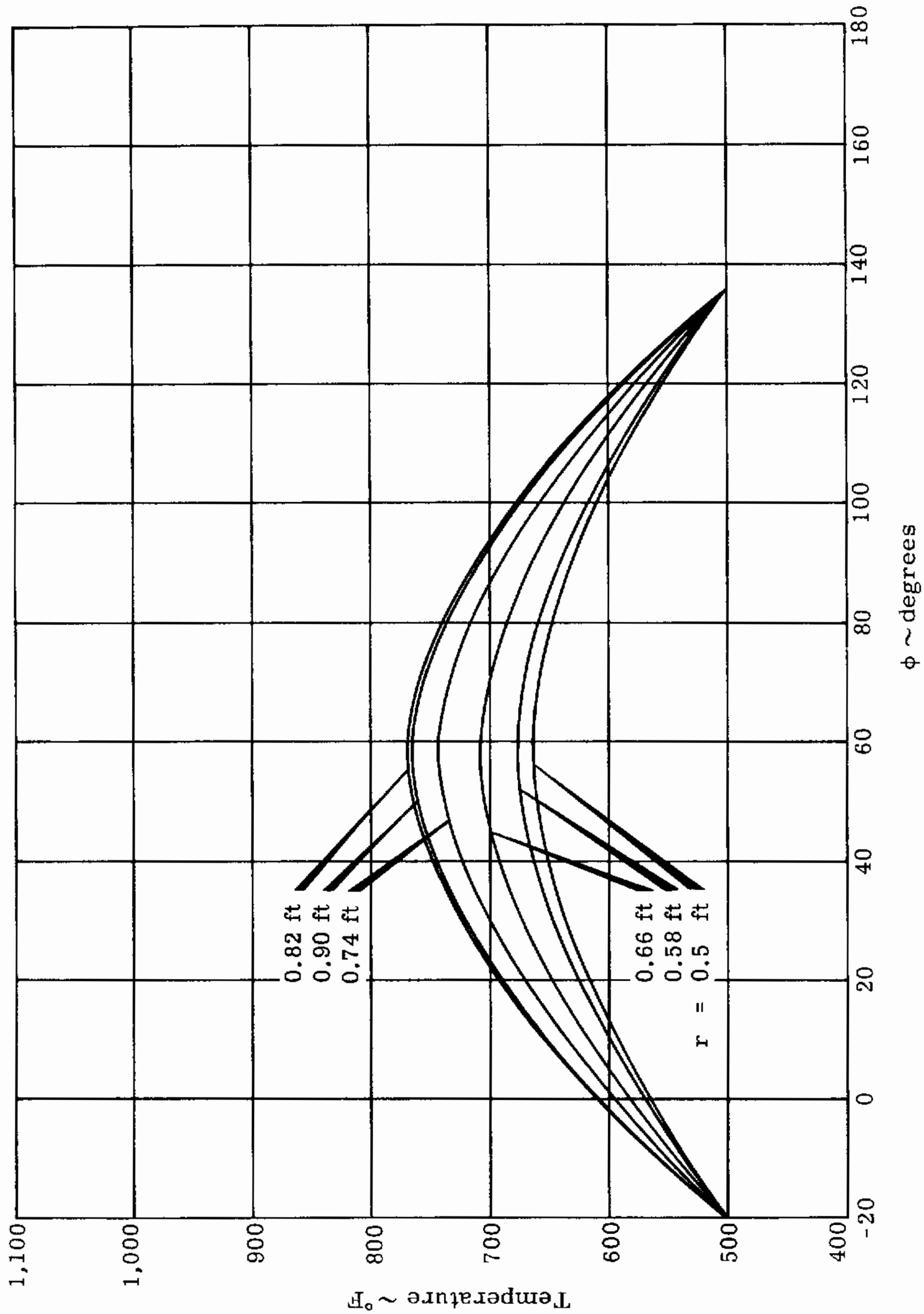

Figure 9. Temperature Distribution – Sample Problem Variable Thermophysical Properties, $\theta = 200$

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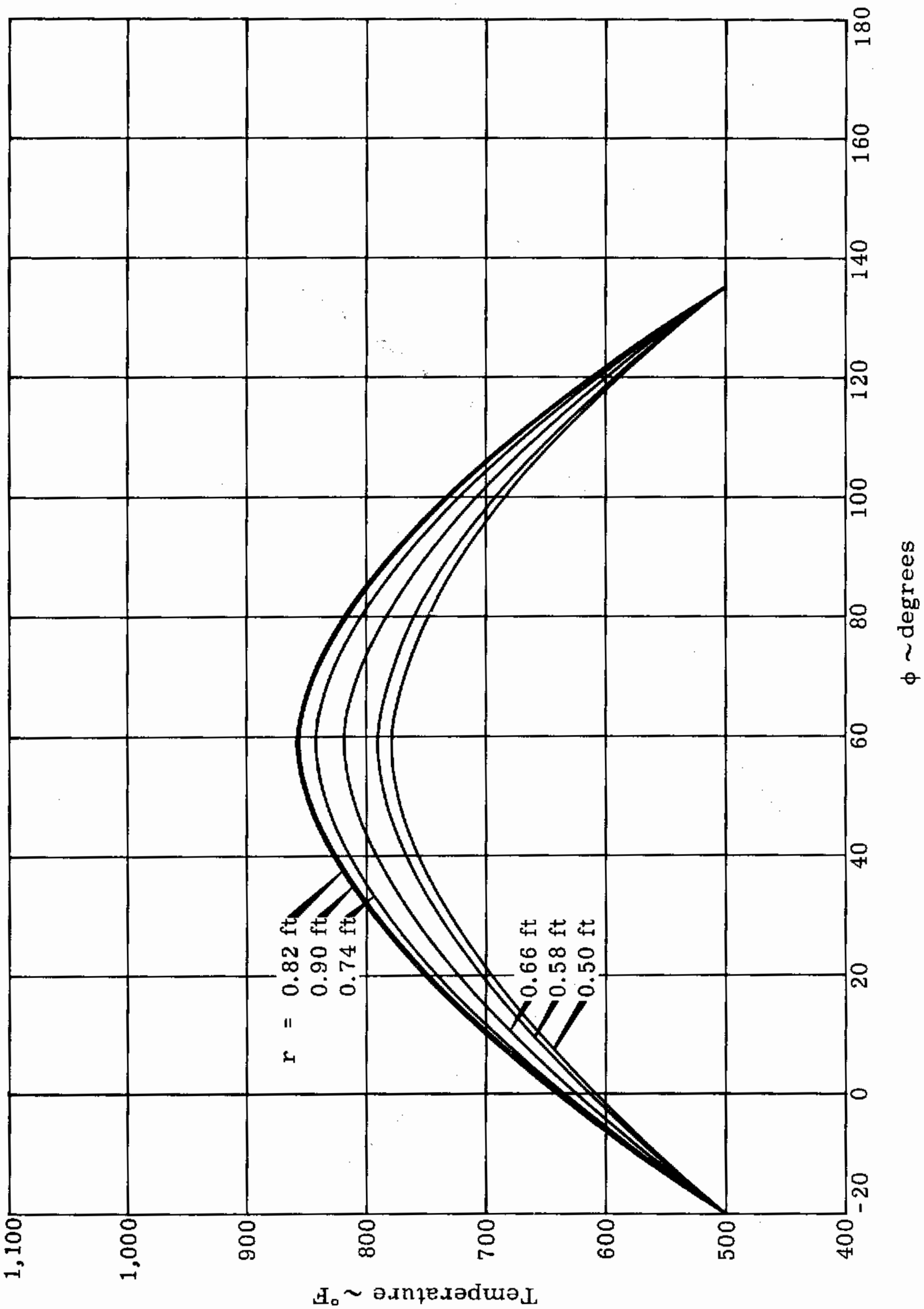

Figure 10. Temperature Distribution – Sample Problem, Variable Thermophysical Properties, $\theta = 400$

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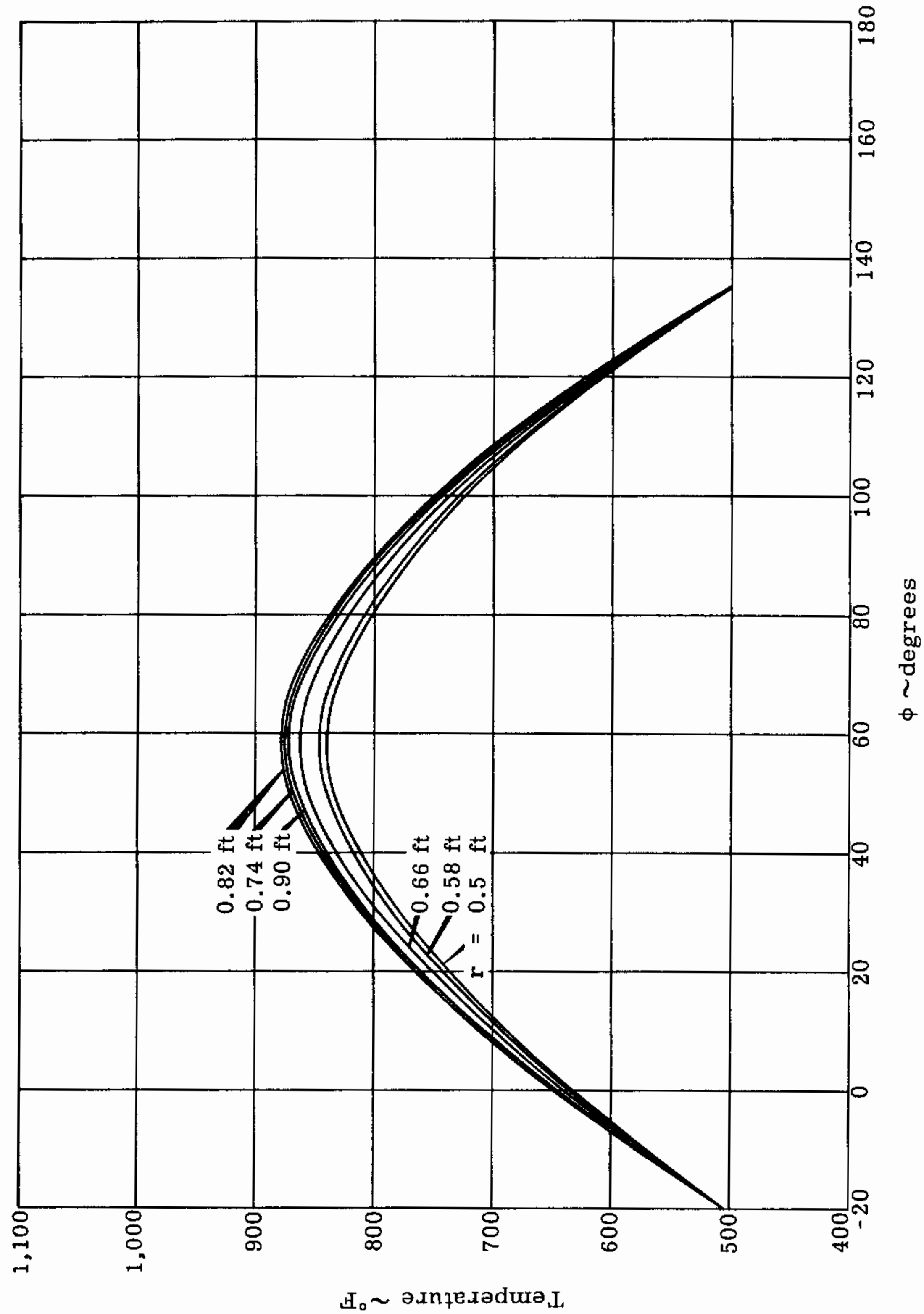

Figure 11. Temperature Distribution – Sample Problem, Variable Thermophysical Properties, $\theta = 600$

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D. DISCUSSION OF RESULTS

The results obtained by the exact solution for the variable thermophysical property sample problem have been compared with a finite difference transient heat transfer program used at Martin Orlando. This program computes three dimensional transient heat transfer in a rectangular parallelepiped. The body is divided into nodes by cuts parallel to the three planes in a rectilinear coordinate system. The distances between slices may be unequal. For each node the initial temperature, density, thermal conductivity, specific heat, and heat flow due to a heat source or sink are given. For each exposed face of a node, the heat transfer coefficient, the adiabatic wall temperature, the radiation heat sink temperature, the shape factor, the emissivity, the heat flow due to solar radiation, and the heat flow due to compartment heating are input. These are all read in as tables and the table entries are specified for each node.

The orientation of the finite difference nodes used in the comparison model are shown on Figure 12. Since the sample problem is symmetrical there is no flow of heat across the centerline. Accordingly, in the comparison model, the centerline was treated as an insulated surface and only half of the problem is considered in the analysis. The finite difference calculations were performed at 10 second intervals. The total run time on a 7094 computer was 0.055 hours.

A plot of the circumferential temperature distributions at a radius of 0.74 feet is shown on Figure 13. The data is presented for both computer methods for 0, 200, 400, and 600 seconds. As indicated by Figure 13, the two programs result in average temperatures at the times represented that are approximately equal. (The average temperatures are implied by the respective areas under the curves.) The two programs, however, show a marked difference in gradients through the body. The differences in gradients are attributed to:

(1) Inherent inaccuracy of each of the programs.

(2) Inability to approach a high degree of similarity between the two models.

No attempt is made in this report to further evaluate or isolate the cause of these differences.

For the sample problem the finite difference solution required 0.055 hours of machine time. The exact solution required 0.056 hours for the constant thermophysical property case and 0.180 hours for the variable thermophysical property case. The finite difference method was calcula-

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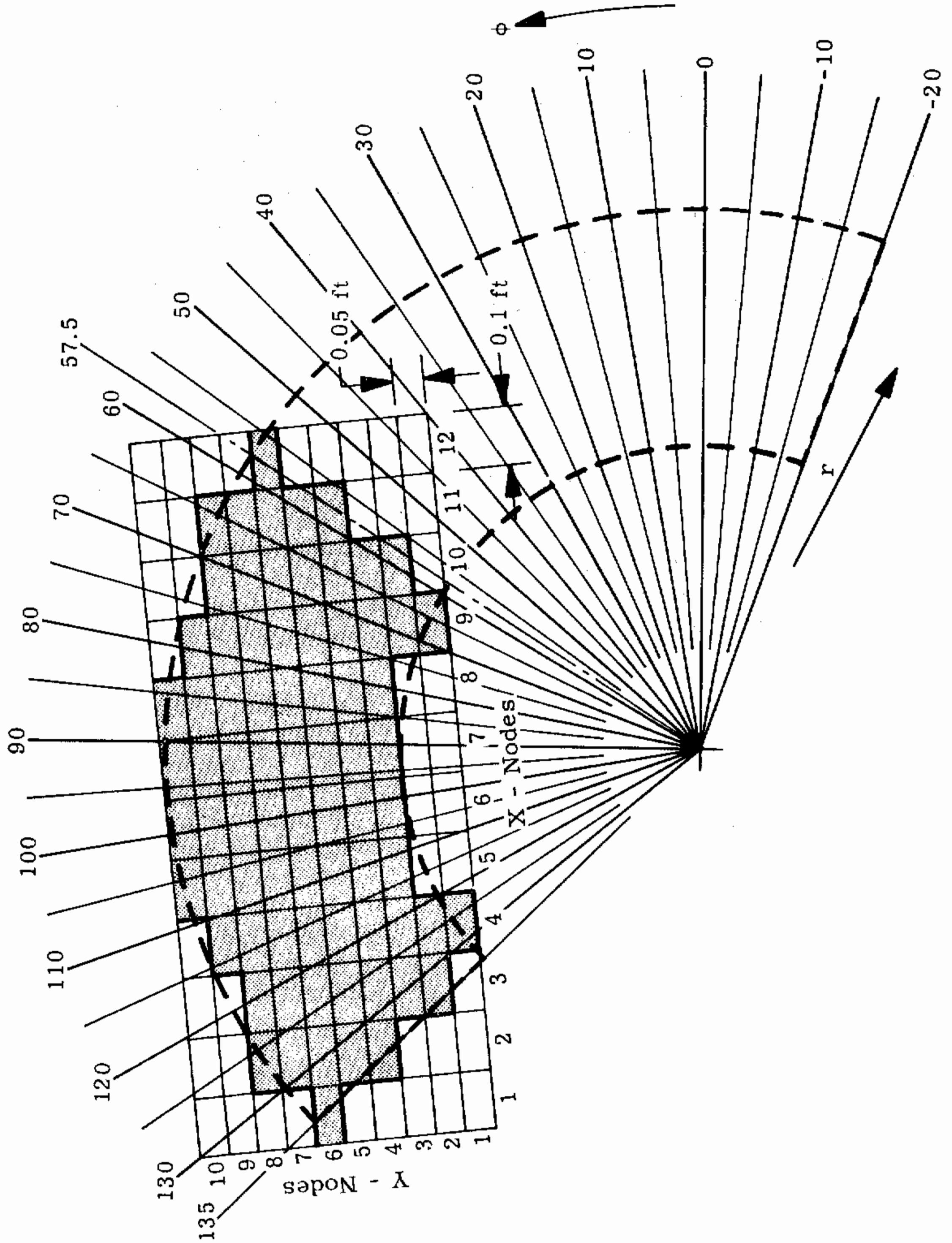

Figure 12. Orientation of Finite Difference Model Nodes

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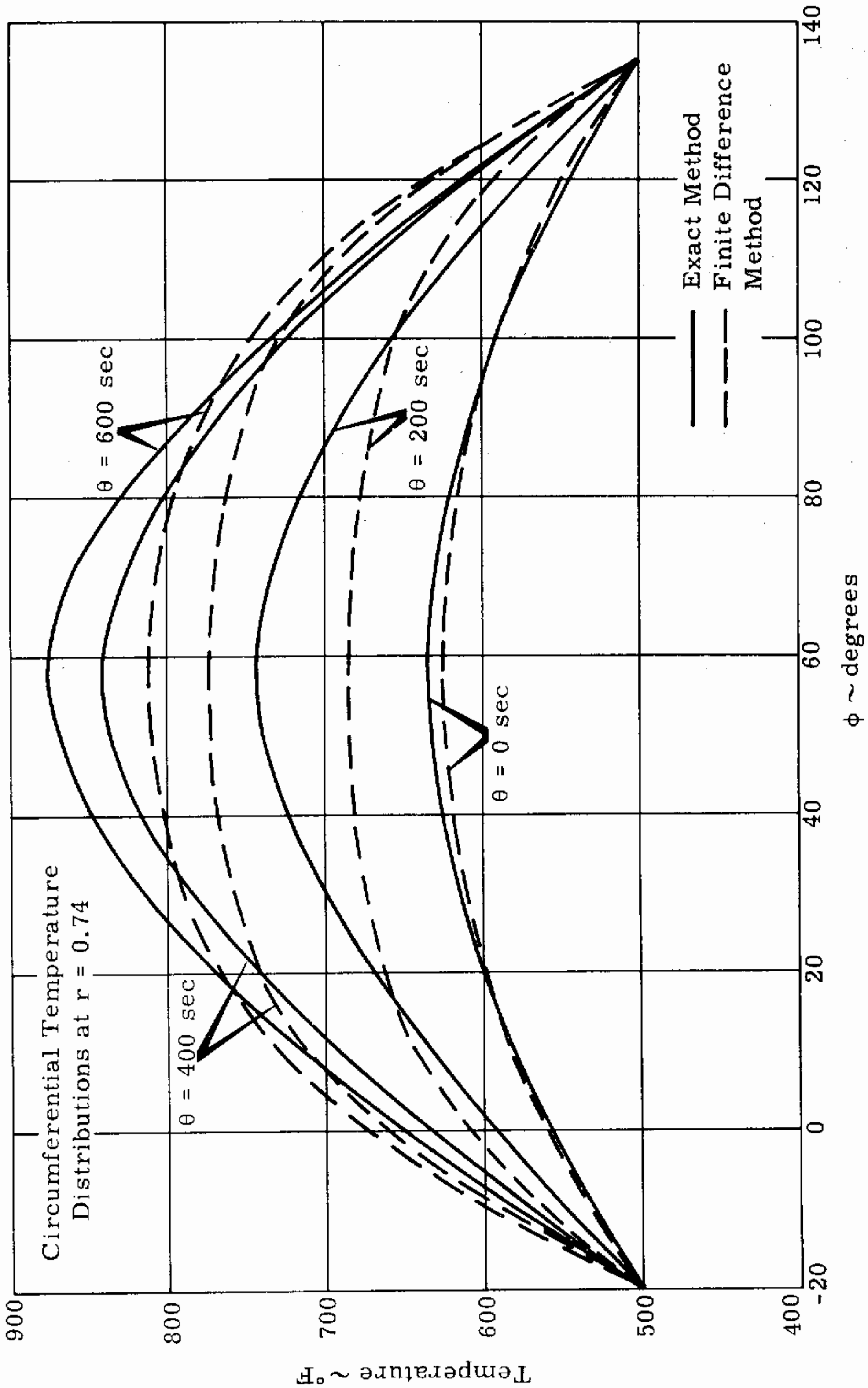

Figure 13. Circumferential Temperature Distributions

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ted in 10 second intervals; the constant thermophysical property exact method computes only at the time printout is required. This implies that the exact method (constant thermophysical properties) offers a reduced run time advantage over the finite difference method for cases in which data is required after extended exposure.

For example, if the requested printout times had been zero, 400, 800, and 1200 seconds, the finite difference method would have required approximately twice as long to run; the exact method run time would have remained the same. If printout had been requested at 600 seconds only, the exact method run time would have been reduced, but the finite difference method run time would have remained the same.

The above comments do not apply when comparing run times of the variable thermophysical property solution with the finite difference method. Here the exact solution also requires a time stepping. The exact method would result in lower run time than the finite difference method only in long run time cases in which the thermophysical property variation allows long time steps.

Note that the preceding exact solutions, in their present state-of-the-art, are being compared with a refined (third revision) finite difference program. Experience gained in this initial programming of the exact solutions can be used to significantly reduce run times in any subsequent programmings.

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SECTION VIII - CONCLUSIONS AND RECOMMENDATIONS

In addition to increased accuracy through lack of idealization of the mathematical model, the exact solution approach to the problem offers the following advantages:

(1) Ease of loading;

(2) Ability of the exact solution to obtain solutions at a given time without incrementing from zero time.

Both of these advantages are of value in parametric studies and in the determination of quasi-steady-state solutions to long flight time vehicles subject to periodic driving functions (such as skipglide vehicles).

Concerning the programming of similar problems, the following suggestions are made based on experience with this problem.

(1) Considerable effort was expended in developing and programming convergence criteria to avoid calculation of needless terms as well as to afford the user with a knowledge of the accuracy of his particular problem. Since storage must be provided for the maximum number of terms in the series, the calculation of a set number of terms and printout of each term, if required for accuracy knowledge, may prove to be the most practical approach. This scheme should be investigated.

(2) Due to the large storage requirements of this program, it is recommended that development of a three-dimensional solution to the same basic problem consider separation of the various cases involved in the superposition into separate programs.

(3) The present program applies the same time function to all four faces. Considerable practical application advantages could be obtained by modifying the program to accept a different time function at each face.

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APPENDIX A

EVALUATION OF INTEGRALS NOT CONTAINING THE ARBITRARY FUNCTION

Set

$$D = \int_{\phi 1}^{\phi 2} \left[B_{\gamma\beta} \sin \gamma\phi + \cos \gamma\phi \right]^2 d\phi$$

$$= B_{\gamma\beta} \int_{\phi 1}^{\phi 2} \sin^2 \gamma\phi \, d\phi + 2B_{\gamma\beta} \int_{\phi 1}^{\phi 2} \sin \gamma\phi \cos \gamma\phi \, d\phi$$

$$+ \int_{\phi 1}^{\phi 2} \cos^2 \gamma\phi \, d\phi$$

$$= B^2_{\gamma\beta} \left[\frac{\phi}{2} - \frac{\sin 2\gamma\phi}{4\gamma} \right]_{\phi 1}^{\phi 2} + \left[\frac{\phi}{2} + \frac{\sin 2\gamma\phi}{4\gamma} \right]_{\phi 1}^{\phi 2}$$

$$+ 2B_{\gamma\beta} \left[\frac{1}{2\gamma} \sin^2 \gamma\phi \right]_{\phi 1}^{\phi 2}$$

$$D = \left[\frac{\phi}{2} (B^2_{\gamma\beta} + 1) - \frac{1}{4\gamma} (B^2_{\gamma\beta} - 1) \sin 2\gamma\phi + \frac{B_{\gamma\beta}}{\gamma} \sin^2 \gamma\phi \right]_{\phi 1}^{\phi 2}$$

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Set

$$E = \int_{r1}^{r2} r \left[C_{\gamma\beta} J_{\gamma}(\beta r) + J_{-\gamma}(\beta r)\right]^2 dr$$

$$= C_{\gamma\beta}^2 \int_{r1}^{r2} r J_{\gamma}^2(\beta r)\, dr + 2C_{\gamma\beta} \int_{r1}^{r2} r J_{\gamma}(\beta r) J_{-\gamma}(\beta r)\, dr$$

$$+ \int_{r1}^{r2} r J_{-\gamma}^2(\beta r)\, dr$$

$$E = \left\{ \frac{C_{\gamma\beta}^2 r^2}{2} \left[J_{\gamma}^2(\beta r) - J_{\gamma-1}(\beta r) J_{\gamma+1}(\beta r)\right]\right.$$

$$+ \frac{C_{\gamma\beta} r^2}{2} \left[2 J_{\gamma}(\beta r) J_{-\gamma}(\beta r) + J_{\gamma-1}(\beta r) J_{-\gamma-1}(\beta r)\right.$$

$$\left. + J_{\gamma+1}(\beta r) J_{-\gamma+1}(\beta r)\right]$$

$$\left. + \frac{r^2}{2} \left[J_{-\gamma}^2(\beta r) - J_{-\gamma+1}(\beta r) J_{-\gamma-1}(\beta r)\right] \right\} \Bigg|_{r1}^{r2}$$

Set

$$H = \int_{r1}^{r2} \frac{1}{r} \left[M_{\epsilon} \sin(\epsilon \ln r) + \cos(\epsilon \ln r)\right]^2 dr$$

$$= \int_{r1}^{r2} \frac{1}{r} \left[M_{\epsilon}^2 \sin^2(\epsilon \ln r)\right] dr + \int_{r1}^{r2} \frac{1}{r} \left[\cos^2(\epsilon \ln r)\right] dr$$

$$+ \int_{r1}^{r2} \frac{1}{r} \left[2 M_{\epsilon} \cos(\epsilon \ln r) \sin(\epsilon \ln r)\right] dr$$

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$$H = \frac{M_\epsilon^2}{\epsilon}\left[\frac{\epsilon \ln r}{2} - \frac{\sin 2(\epsilon \ln r)}{4}\right]_{r1}^{r2} + \frac{1}{\epsilon}\left[\frac{\epsilon \ln r}{2} + \frac{\sin 2(\epsilon \ln r)}{4}\right]_{r1}^{r2}$$

$$+ \frac{2 M_\epsilon}{\epsilon}\left[\frac{1}{2}\sin^2(\epsilon \ln r)\right]_{r1}^{r2} .$$

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APPENDIX B

FORTRAN LANGUAGE OF PROGRAM

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```
      SUBROUTINE DESI(MTAPE,NTAPE)
C THIS ROUTINE IS FOR DEFINING YOUR INPUT AND OUTPUT TAPE DESINATIONS.
C NTAPE IS YOR INPUT TAPE UNIT. MTAPE IS YOUR OUTPUT TAPE UNIT.
      NTAPE=2
      MTAPE=10
      RETURN
      END


C MAIN PROGRAM FOR CONSTANT AND VARIABLE DIFFUSIVITY.
C PROGRAMMED BY KEM BENNETT.
      DIMENSION A(50),B(50),C(50),Z(50),RL(50),P(50),Y2(50),Y1(50),WN2(5
     10),WN1(50),BZ(50),Q(50),W(50),SA(50),SB(50),GM(50),PHI(21),R(6),F(
     221,6),                         GG(50),EE(50)
     3,EB(100),                        D(50),U(50),X(50),        CE(25,
     425),GO(25,25),AC(25,25),         ZOR(160),EQ(25,25),PHO(21),V(21,6)
     5,ZOM(200),E2(6),S(30),H(25,25),ROF(25,25),ROTS(50),PP(50),CP(50),
     6EK(50)
      DIMENSION DUM(150),ANS(50),CARD(15)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23,MU,IU,KU,JU,NU,NUA,NUB,NUC,RL,P,Y2,Y1,WN2,
     2WN1,BZ,Q,W,SA,SB,GM,G,EPS,A,B,C,F,R,PHI,Z,NI,NI1,NI2,NI3,TOL,TOL1,
     3IPT,NNU,T,MM,NN,V,MURLA,GG,NTAPE,MTAPE
      CALL DESI(MTAPE,NTAPE)
      WRITE OUTPUT TAPE MTAPE,909
   94 READ INPUT TAPE NTAPE,181,(CARD(I),I=1,12)
      WRITE OUTPUT TAPE MTAPE,181,(CARD(I),I=1,12)
  181 FORMAT(12A6)
      READ INPUT TAPE NTAPE,609,AK11,AK12,AK13,AK21,AK22,AK23,IDD
  609 FORMAT(6F10.0,I10)
    1 FORMAT(7F10.0)
      READ INPUT TAPE NTAPE,1,AL11,AL12,AL13,AL21,AL22,AL23,TUL1
      READ INPUT TAPE NTAPE,2,P1,P2,R1,R2,MM,NN,IPT
    2 FORMAT(4F10.0,3I10)
      P1=P1*.01745329
```


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```
      P2=P2*.01745329
      READ INPUT TAPE NTAPE,3,IND,MI,MI1,NNU,MC
      IF(IND)4,5,4
    5 READ INPUT TAPE NTAPE,3,MMC,NNC
      READ INPUT TAPE NTAPE,1,((H(I,J),J=1,NNC),I=1,MMC)
      GO TO 24
    3 FORMAT(7I10)
    4 MIP1=MI+1
      MI1P1=MI1+1
      READ INPUT TAPE NTAPE,1,(O(K),K=1,MIP1)
      READ INPUT TAPE NTAPE,1,(U(K),K=1,MI1P1)
   24 READ INPUT TAPE NTAPE,3,IN,MA
      IF(IN-1)102,103,102
  102 MA1=MA*2
      READ INPUT TAPE NTAPE,444,(X(K),K=1,MA1)
  444 FORMAT(6F10.0)
      GO TO 6
  103 MAP1=MA+1
      READ INPUT TAPE NTAPE,1,(X(K),K=1,MAP1)
    6 READ INPUT TAPE NTAPE,1,THETA,DEL,EDEL
      READ INPUT TAPE NTAPE,1,BB,CC,ALP,TT
      KEM3=0
      NBU=MM+1
      NCU=NN+1
      KBJ=1
      MMP1=MM+1
      NNP1=NN+1
      MURLA=1
      APPLE=0.
      SBAR=0.0
      T=THETA
      MCO=0
      NCO=0
      KKK=0
      TTT=0.0
      JUMP=0
      MOM=0
      NNU=NNU+25
```


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```
 855 CALL SOLGAM(NNU,GG,ND)
     WRITE OUTPUT TAPE MTAPE,222
 222 FORMAT(1HO)
     IF(CC)104,105,104
 104 IF(JUMP-1)953,954,953
 954 CALL FSUM
     GO TO 105
 953 CALL INPUT
C THE FOLLOWING WILL DETERMINE THE NUMBER OF LATTICE PTS. IN QUESTION.
 105 M=MM
     N=NN
     NP1=N+1
     MP1=M+1
     NM1=N-1
     FN=N
     R(1)=R1
     R(N+1)=R2
     DO 65 J=1,NM1
     FJ=J
  65 R(J+1)=R1  +(R2-R1)*FJ/FN
     MM1=M-1
     FM=M
     PHI(1)=P2
     PHI(M+1)=P1
     DO 66 K=1,MM1
     FK=K
  66 PHI(K+1)=P2-(P2-P1)*FK/FM
     ENCU=NCU-1
     ENBU=NBU-1
     XBN=(P2-P1)/ENBU
     XCN=(R2-R1)/ENCU
     MMP1=MM+1
     NNP1=NN+1
     XAN=(P2-P1)/20.
     XDN=(R2-R1)/5.
 857 DO 716 I=2,NNU
     G=GG(I)
     C1=COSF(G*P1)
```


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```
      S1=SINF(G*P1)
      BZ(I)=(-AK12*C1+AK11*G*S1)/(AK11*G*C1+AK12*S1)
      CALL SOLBES(G,NNU,ROTS)
      WRITE OUTPUT TAPE MTAPE,228,GG(I)
  228 FORMAT(11H FOR GAMMA=E20.8///)
      DO 716 J=1,NNU
      ROF(I,J)=ROTS(J)
  716 WRITE OUTPUT TAPE MTAPE,224,ROF(I,J)
  224 FORMAT(6H BETA=E20.8)
  806 DO 400 K=1,MMP1
      DO 399 L=1,NNP1
C START METHOD FOR CONVERGENCE OF DOUBLE SERIES.
  808 SUM1=0.0
      SUMX=0.0
      SUM2=0.0
      SUM3=0.0
      NO=2
      NOP1=NO+1
   51 I=2
   52 DO 50 J=1,NO
      G=GG(I)
      ZB=BZ(I)
      BOO=ROF(I,J)
      KIS=G
      GI=KIS
      GF=G-GI
      AP=GF
      AN=1.-GF
      NP=KIS
      NNN=-(KIS+1)
      IF(J-NO)10,20,20
   10 IF(I-NOP1)15,20,20
   15 IF(NO-2)36,36,50
   36 NA=1
   20 IF(MCO-I)480,422,422
  422 IF(NCO-J)480,111,111
  480 X1=BOO*R1
      CALL BESEL(C,X1,AP,NP,ANS,BOO,  LLL)
```


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```
      BE=ANS(1)
      BEP=BOO*ANS(2)-G*ANS(1)/R1
      CALL BESEL(O,X1,AN,NNN,ANS,BOO,    LLL)
      BE1=ANS(1)
      BE1P=-G * ANS(1)/R1-BOO*ANS(2)
      CE(I,J)=-(AL11*BE1P+AL12*BE1)/(AL11*BEP+AL12*BE)
      IF(IPT)22,25,22
   22 WRITE OUTPUT TAPE MTAPE,321,CE(I,J),BE,BE1,BEP,BE1P
  321 FORMAT(3H C=E20.8,4E20.8)
   25 MF=0
      CIJ=CE(I,J)
   11 IF(CC)12,13,12
   13 GO(I,J)=0.0
      GO TO 41
   12 CALL DENOM(ZB,D1)
      CALL ENOM(CIJ,BOO,G,EN)
      DO 43 NE=1,6
      ANN1= NE-1
      RE=R1+XDN*(ANN1)
      X1=BOO*RE
      CALL BESEL(O,X1,AP,NP,ANS,BOO,  LLL)
      BEE=ANS(1)
      CALL BESEL(O,X1,AN,NNN,ANS,BOO,    LLL)
      BET=ANS(1)
      DO 44 ME=1,21
      AMM1=ME-1
      PE=P1+XAN*(AMM1)
      S(ME)=(F(ME,NE)+TT)*(BZ(I)*SINF(G*PE)+COSF(G*PE))*(CIJ    *BEE+BET)
     1*RE
   44 CONTINUE
      IF(IPT)701,43,701
  701 WRITE OUTPUT TAPE MTAPE,702,S(ME),BEE,BET
  702 FORMAT(4H SE=E20.8,2E20.8)
   43 CALL SIMP(21,XAN,S(1),E2(NE))
      CALL SIMP(6,XDN,E2(1),E3)
      GO(I,J)=E3/(D1*EN)
      IF(IPT)23,29,23
   23 WRITE OUTPUT TAPE MTAPE,26,GO(I,J),EN,E3
```


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```
 26 FORMAT(3H G=E20.8,4H EN=E20.8,4H E3=E20.8)
 29 MF=1
 41 IF(BB)16,17,16
 17 AC(I,J)=0.0
    GO TO 111
 16 IF(MF)18,19,18
 19 CALL DENOM(ZB,D1)
    CALL ENOM(CIJ,BOO,G,EN)
 18 IF(IND)28,21,28
 28 DO 34 NE=1,NCU
    ANM1=NE-1
    XE=R1+XCN*(ANM1)
    X1=BOO*XE
    CALL BESEL(0,X1,AP,NP,ANS,BOO,  LLL)
    BEE=ANS(1)
    CALL BESEL(0,X1,AN,NNN,ANS,BOO,   LLL)
    BET=ANS(1)
    CALL FPR(MI,XE,O,PLL)
    DO 35 ME=1,NBU
    AMM1=ME-1
    XXE=P1+XBN*(AMM1)
    CALL FPR(MI1,XXE,U,POLY)
    GO TO(30,31,32,33),MC
 30 BOB=PLL+POLY
    GO TO 35
 31 BOB=PLL*POLY
    GO TO 35
 32 BOB=PLL/POLY
    GO TO 35
 33 BOB=POLY/PLL
 35 ZOM(ME)=BOB*(BZ(I)*SINF(G*XXE)+COSF(G*XXE))*(CIJ*BEE+BET)*XE
    IF(IPT)700,34,700
700 WRITE OUTPUT TAPE MTAPE,703,ZOM(NE),BEE,BET,BEEP,BETP
703 FORMAT(5H ZOM=E20.8,5H BEE=E16.7,5H BET=E16.7,6H BEEP=E16.7,6H BET
   1P=E16.7)
 34 CALL SIMP(NBU,XBN,ZOM(1),EB(NE))
    CALL SIMP(NBU,XCN,EB(1),E4)
    IF(IPT)704,705,704
```


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```
704 WRITE OUTPUT TAPE MTAPE,706,E4
706 FORMAT(4H E4=E20.8)
705 GO TO 401
 21 FNC=NNC-1
    FMC=MMC-1
    XXN=(R2-R1)/FNC
    XXM=(P2-P1)/FMC
    DO 199 NOT=1,NNC
    ANN1=NOT-1
    RO=R1+XXN*ANN1
    X1=BOO*RO
    CALL BESEL(0,X1,AP,NP,ANS,BOO,  LLL)
    BE=ANS(1)
    CALL BESEL(0,X1,AN,NNN,ANS,BOO,   LLL)
    BET=ANS(1)
    DO 117 MOT=1,MMC
    AMM1=MOT-1
    PO=P1+XXM*AMM1
    ZOM(MOT)=(H(MOT,NOT)*(BZ(I)*SINF(G*PO)+COSF(G*PO))*(CIJ*BEE+BET)
   1*RO)
117 CONTINUE
199 CALL SIMP(MMC,XXM,ZOM(1),EB(NOT))
    CALL SIMP(NNC,XXN,EB(1),E4)
401 AC(I,J)=E4/(D1*EN)
    IF(IPT)707,111,707
707 WRITE OUTPUT TAPE MTAPE,709,AC(I,J)
709 FORMAT(3H A=E20.8)
111 IF(T)504,505,504
505 GINT=0.
    GO TO 506
504 IF(CC)812,506,812
812 TUB2=T/150.
    DO 813 MR=1,151
    AKM1=MR-1
    XER=TUB2*AKM1
    XEP=TUB2*(AKM1)+APPLE
    GO TO(119,110),IN
110 CALL LINE(MA,XEP,X(1),POLY)
    GO TO 200
```


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```
119 CALL FPR(MA,XEP,X,POLY)
200 ZOR(MR)=POLY*EXPF(-ALP*BOO**2*(T-XER))
813 CONTINUE
    CALL SIMP(151,TUB2,ZOR,GINT)
506 RR=R(L)
    JAR=J
    CIJ=CE(I,J)
    X1=BOO*RR
    CALL BESEL(O,X1,AP,NP,ANS,BOO,  LLL)
    BER=ANS(1)
    CALL BESEL(O,X1,AN,NNN,ANS,BOO,    LLL)
    BAR=ANS(1)
    CO=BZ(I)*SINF(G*PHI(K))+COSF(G*PHI(K))
    EQ(I,J)=BB*AC(I,J)*CO*(CIJ*BER+BAR)*EXPF(-ALP*BOO**2*T) +CC*GO
   1(I,J)*CO*(CIJ*BER+BAR)*(ALP*BOO**2)*GINT
    IF(IPT)800,801,800
800 WRITE OUTPUT TAPE MTAPE,802,CO,EQ(I,J)
802 FORMAT(4H CO=E20.8,4H EQ=E20.8)
801 IF(NA-1)9,7,9
  7 NA=0
    SUM1=EQ(I,J)+SUM1
    GO TO 50
  9 SUM2=SUM2 +EQ(I,J)
    SUM3=SUM3+ABSF(EQ(I,J))
 50 CONTINUE
 60 I=I+1
    IF(I-(NO+1))52,52,280
280 SUMX=SUM1+SUM2
    IF(SUM3-TUL1)281,281,100
100 IF(I-NNU)101,101,500
500 WRITE OUTPUT TAPE MTAPE,502,PHI(K),R(L)
502 FORMAT(37H NO CONVERGENCE WITHIN TOLERANCE PHI=E20.8, 3H R=E20.8)
830 MCO=I-1
    NCO=JAR
832 KKK=1
432 GO TO 399
101 NO=NO+1
    NOP1=NO+1
```


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```
  423 SUM1=SUM1+SUM2
      SUM2=0.
      SUM3=0.0
      GO TO 51
  281 V(K,L)=SUMX
      IF(KKK-1)427,600,427
  427 IF(MCO-I)834,833,833
  833 IF(NCO-J)834,600,600
  834 MCO=I-1
      NCO=JAR
  600 IF(IPT)666,399,666
  666 WRITE OUTPUT TAPE MTAPE,900,PHI(K),R(L)
  900 FORMAT(33H CONVERGENCE WAS REACHED FOR PHI=E20.8,3H R=E20.8)
  399 CONTINUE
  400 CONTINUE
      IF(KEM3-1)62,63,62
   63 T=TTT
   62 CALL OOUPU
      IF(IDD)608,660,608
  608 IF(KEM3)777,72,777
   72 READ INPUT TAPE NTAPE,741,MUD,MUD1,MUD2,EEK
  741 FORMAT(3I10,F10.0)
      MUDD=MUD*2
      READ INPUT TAPE NTAPE,444,(PP(K),K=1,MUDD)
      MUDD1=MUD1*2
      READ INPUT TAPE NTAPE,444,(CP(K),K=1,MUDD1)
      MUD8=MUD2*2
      READ INPUT TAPE NTAPE,444,(EK(I),I=1,MUD8)
      READ INPUT TAPE NTAPE,1,THETA,DEL,EDEL
      KEM3=1
      T=THETA
  777 DO 776 J=1,MMP1
      DO 776 K=1,NNP1
      H(J,K)=V(J,K)
      V(J,K)=0.0
  776 SBAR=H(J,K)+SBAR
      SAM=MMP1*NNP1
      TBAR=SBAR/SAM
```


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```
      SBAR=0.0
      MF=0
      IND=0
      MCO=0
      NCO=0
      KKK=0
      NNC=NN+1
      MMC=MM+1
      JUMP=1
   73 CALL LINE(MUD,TBAR,PP(1),PEO)
      CALL LINE(MUD1,TBAR,CP(1),CEO)
      CALL LINE(MUD2,TBAR,EK(1),EKO)
      ALP=EKO/(PEO*CEO)
      ERP=EEK
      EBB=ERP/EKO
      AK12=AK12*EBB
      AK13=AK13*EBB
      AK22=AK22*EBB
      AK23=AK23*EBB
      AL12=AL12*EBB
      AL13=AL13*EBB
      AL22=AL22*EBB
      AL23=AL23*EBB
      EEK=EKO
      IF(MOM-1)541,540,540
  540 APPLE=APPLE+DEL
  541 TTT=TTT+DEL
      MOM=1
      IF(TTT-EDEL)604,604,46
  604 T=DEL
      GO TO 855
  660 DO 731 K=1,MMP1
      DO 731 L=1,NNP1
  731 V(K,L)=0.0
      IF(T-EDEL)48,46,46
   48 T=T+DEL
      GO TO 806
   46 WRITE OUTPUT TAPE  MTAPE,909
```


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```
909 FORMAT(1H1)
    GO TO 94
    END
```


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```
      SUBROUTINE INPUT
      DIMENSION A(50),B(50),C(50),Z(50),RL(50),P(50),Y2(50),Y1(50),WN2(5
     10),WN1(50),BZ(50),Q(50),W(50),SA(50),SB(50),GM(50),PHI(21),R(6),F(
     221,6),V(21,6),GG(50)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23,  MU,IU,KU,JU,NU,NUA,NUB,NUC,RL,P,Y2,Y1,WN2,
     2WN1,BZ,Q,W,SA,SB,GM,G,EPS,A,B,C,F,R,PHI,Z,NI,NI1,NI2,NI3,TOL,TOL1,
     3IPT,NNU,T,MM,NN,V,MURLA,GG,NTAPE,MTAPE
      READ INPUT TAPE NTAPE,11,TOL,TOL1
C READING IN FP1(R)
   11 FORMAT(7F10.0)
    6 FORMAT(3I10)
   55 FORMAT(6F10.0)
      READ INPUT TAPE NTAPE,6,NI,NU,MU
      IF(NI-3)42,14,14
   42 IF(NI-1)88,13,99
   14 IF(NI-4)704,704,704
   13 MUP1=MU+1
      READ INPUT TAPE NTAPE,11,(A(K),K=1,MUP1)
      GO TO 66
   99 MUL2=MU*2
      READ INPUT TAPE NTAPE,55,(A(K),K=1,MUL2)
C READING IN FP2(R)
   66 READ INPUT TAPE NTAPE,6,NI1,NUA,IU
      IF(NI1-3)16,18,18
   16 IF(NI1-1)88,17,19
   18 IF(NI1-4)704,704,704
   17 IUP1=IU+1
      READ INPUT TAPE NTAPE,11,(B(K),K=1,IUP1)
      GO TO 67
   19 IUL2=IU*2
      READ INPUT TAPE NTAPE,55,(B(K),K=1,IUL2)
C READING IN FR2(P)
   67 READ INPUT TAPE NTAPE,6,NI2,NUB,KU
      IF(NI2-3)20,22,22
   20 IF(NI2-1)88,21,23
   22 IF(NI2-4)704,704,704
   21 KUP1=KU+1
```


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```
      READ INPUT TAPE NTAPE,11,(C(K),K=1,KUP1)
      GO TO 68
   23 KUL2=KU*2
      READ INPUT TAPE NTAPE,55,(C(K),K=1,KUL2)
C READING IN FR1(P)
   68 READ INPUT TAPE NTAPE,6,NI3,NUC,JU
      IF(NI3-3)30,32,32
   30 IF(NI3-1)88,31,33
   32 IF(NI3-4)704,704,704
   31 JUP1=JU+1
      READ INPUT TAPE NTAPE,11,(Z(K),K=1,JUP1)
      GO TO 1
   33 JUL2=JU*2
      READ INPUT TAPE NTAPE,55,(Z(K),K=1,JUL2)
      GO TO 1
  704 WRITE OUTPUT TAPE MTAPE,95
   95 FORMAT(44H YOU HAVE USED AN UNDEFINED INPUT INDICATOR.//)
      GO TO 12
   88 WRITE.OUTPUT TAPE MTAPE,69
   69 FORMAT(53H HOW DID YOU GET A NI LESS THAN ONE- PLEASE TRY AGAIN//)
   12 CALL OUTPUT
      GO TO 94
    1 CALL FSUM
      WRITE OUTPUT TAPE MTAPE,6000
 6000 FORMAT(1HO)
   94 RETURN
      END
```


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```
      SUBROUTINE OOUPU
C THIS SUBROUTINE IS FOR OUTPUT IN THE CONSTANT AND VARIABLE PROBLEM.
      DIMENSION A(50),B(50),C(50),Z(50),RL(50),P(50),Y2(50),Y1(50),WN2(5
     10),WN1(50),BZ(50),Q(50),W(50),SA(50),SB(50),GM(50),PHI(21),R(6),F(
     221,6),PHO(21),V(21,6),GG(50)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23,MU,IU,KU,JU,NU,NUA,NUB,NUC,RL,P,Y2,Y1,WN2,
     2WN1,BZ,Q,W,SA,SB,GM,G,EPS,A,B,C,F,R,PHI,Z,NI,NI1,NI2,NI3,TOL,TOL1,
     3IPT,NNU,T,MM,NN,V,MURLA,GG,NTAPE,MTAPE
      M=MM
      N=NN
      NP1=N+1
      MP1=M+1
      IF(MURLA-1)18,9,18
    9 DO 12 J=1,MP1
   12 PHO(J)=PHI(J)*57.29578
      MURLA=2
   18 WRITE OUTPUT TAPE MTAPE,1
    1 FORMAT(1H1)
      WRITE OUTPUT TAPE MTAPE,400,T
  400 FORMAT(7H THETA=1PE20.8)
      DO 70 I=1,10
   70 WRITE OUTPUT TAPE MTAPE,6000
      WRITE OUTPUT TAPE MTAPE,106,(R(J),J=1,NP1)
      WRITE OUTPUT TAPE MTAPE,6000
      GO TO(8,2,3,4,5),N
  106 FORMAT(20X,6F14.3)
    8 WRITE OUTPUT TAPE MTAPE,2001
      GO TO 10
    2 WRITE OUTPUT TAPE MTAPE,2002
      GO TO 10
    3 WRITE OUTPUT TAPE MTAPE,2003
      GO TO 10
    4 WRITE OUTPUT TAPE MTAPE,2004
      GO TO 10
    5 WRITE OUTPUT TAPE MTAPE,2005
   10 WRITE OUTPUT TAPE MTAPE,8000
      WRITE OUTPUT TAPE MTAPE,3000,PHO(1),(V(1,L),L=1,NP1)
```


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```
      WRITE OUTPUT TAPF MTAPE,6000
      DO 100 J=2,M
      WRITE OUTPUT TAPE MTAPE,4000,PHO(J),(V(J,K),K=1,NP1)
  100 WRITE OUTPUT TAPE MTAPE,6000
      WRITE OUTPUT TAPE MTAPE,5000,PHO(MP1),(V(MP1,I),I=1,NP1)
 7000 FORMAT(1P7E16.7)
 2002 FORMAT(30X,3H R1,25X,3H R2)
 2001 FORMAT(30X,3H R1,11X,3H R2)
 2003 FORMAT(30X,3H R1,39X,3H R2)
 2004 FORMAT(30X,3H R1,53X,3H R2)
 2005 FORMAT(30X,3H R1,67X,3H R2)
 3000 FORMAT(F13.2,8H    PHI 2,6F14.2)
 4000 FORMAT(F13.2,8X,6F14.2)
 5000 FORMAT(F13.2,8H    PHI 1,6F14.2)
 6000 FORMAT(1H )
 8000 FORMAT(1H0)
  401 RETURN
      END
```


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```
      SUBROUTINE FSUM
C STEADY STATE SOLUTION. MAIN SUBROUTINE.
C PROGRAMMED BY KEM BENNETT.
      DIMENSION A(50),B(50),C(50),Z(50),RL(50),P(50),Y2(50),Y1(50),WN2(5
     10),WN1(50),BZ(50),Q(50),W(50),SA(50),SB(50),GM(50),PHI(21),R(6),F(
     221,6),Y(300),ZOD(300),EM(300),GG(50),EE(50),TM(50),UM(50),HS4A(50)
     3,HC4A(50),HS3A(50),HC3A(50),V(21,6)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23,MU,IU,KU,JU,NU,NUA,NUB,NUC,RL,P,Y2,Y1,WN2,
     2WN1,BZ,Q,W,SA,SB,GM,G,EPS,A,B,C,F,R,PHI,Z,NI,NI1,NI2,NI3,TOL,TOL1,
     3IPT,NNU,T,MM,NN,V,MURLA,GG,NTAPE,MTAPE
      M=20
      N=5
      XX=0.0
      DO 291  K=1,20
      DO 291  J=1,6
  291 F(K,J)=0.0
      IF(NU)242,243,242
  243 NU=150
  242 IF(NUA)244,245,244
  245 NUA=150
  244 IF(NUB)246,247,246
  247 NUB=150
  246 IF(NUC)990,251,990
  251 NUC=150
  990 IF(AK12+AK22+AL12+AL22)950,951,950
  951 WRITE OUTPUT TAPE MTAPE,952
  952 FORMAT(31H INFINTTE STEADY STATE SOLUTION///)
      GO TO 18
  950 IF(AK12+AK22+AL12)953,954,953
  954 DVEP=1./(P2-P1)
      ENUB=NUB
      NUBP1=NUB+1
      XIP=(P2-P1)/ENUB
      GO TO(956,957,704,704,704),NI2
  956 KEM=1
      GO TO 958
```


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```
957 KEM=2
958 DO 955 K=1,NUBP1
    AKM1=K-1
    XEP=P1+XIP*(AKM1)
    GO TO(959,960),KEM
959 CALL FPR(KU,XEP,C,POLY)
    GO TO 955
960 CALL LINE(KU,XEP,C,POLY)
955 Y(K)=POLY*OVEP
    CALL SIMP(NUBP1,XIP,Y,XX)
    GO TO 507
953 IF(AK12+AK22+AL22)961,962,961
962 OVEP=1./(P2-P1)
    NUCP1=NUC+1
    ENUC=NUC
    GO TO(963,964,704,704,704),NI3
963 KOU=1
    GO TO 965
964 KOU=2
965 XIP=(P2-P1)/ENUC
    DO 966 K=1,NUCP1
    AKM1=K-1
    XEP=P1+XIP*(AKM1)
    GO TO(967,968),KOU
967 CALL FPR(JU,XEP,Z,POLY)
    GO TO 966
968 CALL LINE(JU,XIP,Z(1),POLY)
966 Y(K)=POLY*OVEP
    CALL SIMP(NUCP1,XIP,Y,XX)
    GO TO 507
961 IF(AK12+AL12+AL22)969,970,969
970 OOR=1./(R2-R1)
    NUAP1=NUA+1
    ENUA=NUA
    GO TO(971,972,704,704,704),NI1
971 KPU=1
    GO TO 976
972 KPU=2
```


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```
976 XIP=(R2-R1)/ENUA
    DO 977 K=1,NUAP1
    AKM1=K-1
    XEP=R1+XIP*AKM1
    GO TO(974,975),KPU
974 CALL FPR(IU,XEP,B,POLY)
    GO TO 977
975 CALL LINE(IU,XEP,B(1),POLY)
977 Y(K)=POLY*OOR
    CALL SIMP(NUAP1,XIP,Y,XX)
    GO TO 507
969 IF(AK22+AK12+AL22)507,981,507
981 ENU=NU
    NUP1=NU+1
    OOR=1./(R2-R1)
    GO TO(982,983,704,704,704),NI
982 KIM=1
    GO TO 989
983 KIM=2
989 XIP=(R2-R1)/ENU
    DO 987 K=1,NUP1
    AKM1=K-1
    XEP=R1+XIP*(AKM1)
    GO TO(985,986),KIM
985 CALL FPR(MU,XEP,A,POLY)
    GO TO 987
986 CALL LINE(MU,XEP,A(1),POLY)
987 Y(K)=POLY*OOR
    CALL SIMP(NUP1,XIP,Y,XX)
507 CALL SOLEPS(NNU,EE,ND)
    DO 88 I=2,NNU
 88 WRITE OUTPUT TAPE MTAPE,42,EE(I)
C THE FOLLOWING WILL DETERMINE THE NUMBER OF LATTICE PTS. IN QUESTION.
992 NP1=N+1
    MP1=M+1
    NM1=N-1
    FN=N
    R(1)=R1
```


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```
      R(N+1)=R2
      DO 65 J=1,NM1
      FJ=J
   65 R(J+1)=R(1)+(R2-R1)*FJ/FN
      MM1=M-1
      FM=M
      PHI(1)=P2
      PHI(M+1)=P1
      DO 66 K=1,MM1
      FK=K
   66 PHI(K+1)=P2-(P2-P1)*FK/FM
      GO TO(62,63,704,704,704),NI2
   62 KEM=1
      GO TO 47
   63 KEM=2
   47 ENUB=NUB
      NUBP1=NUB+1
      XBN=(P2-P1)/ENUB
      GO TO(77,99,704,704,704),NI3
   77 KOU=1
      GO TO 1400
   99 KOU=2
 1400 NUCP1=NUC+1
      ENUC=NUC
      XIN=(P2-P1)/ENUC
      GO TO(702,703,704,704,704),NI
  702 KIM=1
      GO TO 113
  703 KIM=2
  113 ENU=NU
      NUP1=NU+1
      XEN=(R2-R1)/ENU
      GO TO(708,709,704,704,704),NI1
  708 KPU=1
      GO TO 51
  709 KPU=2
   51 ENUA=NUA
  203 NUAP1=NUA+1
```


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```
      XCN=(R2-R1)/ENUA
      DO 20 L=1,MP1
      DO 20 J=1,NP1
      BSUM=0.0
      ASUM=0.0
      DO 215 I=2,NNU
C SOLVING ALL EQUATIONS DEPENDENT ON SIGMA.
      G=GG(I)
      G1=G*P1
      C1=COSF(G1)
      S1=SINF(G1)
      BZ(I)=(-AK12*C1+AK11*G*S1)/(AK11*G*C1+AK12*S1)
      X=G*LOGF(R2)
      CALL HYPER(X,HS2,HC2)
      X=G*LOGF(R1)
      CALL HYPER(X,HS1,HC1)
  902 WN1(I)=(-AL21*G/R2*HS2-AL22*HC2)/(AL21*G/R2*HC2+AL22*HS2)
      Y1(I)=(AL11*WN1(I)*G/R1+AL12)*HC1+(AL11*G/R1+AL12*WN1(I))*HS1
  904 WN2(I)=(-AL11*G/R1*HS1-AL12*HC1)/(AL11*G/R1*HC1+AL12*HS1)
      Y2(I)=(AL21*WN2(I)*G/R2+AL22)*HC2+(AL21*G/R2+AL22*WN2(I))*HS2
  609 IF(IPT)55,56,55
   55 WRITE OUTPUT TAPE MTAPE,803,I
      WRITE OUTPUT TAPE MTAPE,91,BZ(I),Y2(I),Y1(I)
      WRITE OUTPUT TAPE MTAPE,8000
      WRITE OUTPUT TAPE MTAPE,901,WN2(I),WN1(I)
      WRITE OUTPUT TAPE MTAPE,8000
   56 DO 813 K=1,NUBP1
      AKM1=K-1
      XE=P1+XBN*(AKM1)
      GO TO(10,33),KEM
   10 CALL FPR(KU,XE,C,POLY)
      GO TO 34
   33 CALL LINE(KU,XE,C(1),POLY)
   34 ZOO(K)=BZ(I)*SINF(G*XE)+COSF(G*XE)
      Y(K)=POLY*ZOO(K)
  813 CONTINUE
      CALL SIMP(NUBP1,XBN,Y,E1)
      ZB=BZ(I)
```


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```
      CALL DENOM(ZB,D)
      RL(I)=AL23*E1/(Y2(I)*D)
      IF(IPT)200,46,200
  200 WRITE OUTPUT TAPE MTAPE,1000,RL(I),D,E1
   46 DO 812 K=1,NUCP1
      AKM1=K-1
      XE= P1+XIN*(AKM1)
      GO TO(8,61),KOU
    8 CALL FPR(JU,XE,Z,POLY)
      GO TO 30
   61 CALL LINE(JU,XE,Z(1),POLY)
   30 ZOO(K)=BZ(I)*SINF(G*XE)+COSF(G*XE)
  812 Y(K)=POLY*ZOO(K)
      CALL SIMP(NUCP1,XIN,Y,E)
      P(I)=AL13*E/(Y1(I)*D)
      IF(IPT)202,796,202
  202 WRITE OUTPUT TAPE MTAPE,1001,P(I),E,D
C THE FOLLOWING SUMS THE FIRST HALF OF F49 DEPENDING ON THE SIGMAS
C REQUIRED FOR CONVERGENCE.
  796 S1=SINF(G*PHI(L))
      C1=COSF(G*PHI(L))
      X=G*LOGF(R(J))
      CALL HYPER(X,HS,HC)
      IF(AL13)81,80,81
   80 SEC=0.0
      GO TO 87
   81 SEC=P(I)*(BZ(I)*S1+C1)*(WN1(I)*HS+HC)
   87 IF(AL23)85,84,85
   84 FST=0.0
      GO TO 90
   85 FST=RL(I)*(BZ(I)*S1+C1)*(WN2(I)*HS+HC)
   90 IF(I-5)410,411,411
  410 TM(I)=FST+SEC
      IF(IPT)209,210,209
  209 WRITE OUTPUT TAPE MTAPE,444,TM(I)
  210 GO TO 470
  411 TM(I)=FST+SEC
      IF(IPT)211,212,211
```


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```
  211 WRITE OUTPUT TAPE MTAPE,444,TM(I)
  212 ASUM=ASUM+TM(I)
  444 FORMAT(4H TM=E20.8)
      CONVE=ABSF(TM(I-3))+ABSF(TM(I-2))+ABSF(TM(I-1))+ABSF(TM(I))
      IF(IPT)213,214,213
  213 WRITE OUTPUT TAPE MTAPE,305,CONVE
  305 FORMAT(7H CONVE=E20.8)
  214 IF(CONVE-TOL)280,280,9
    9 IF(I-NNU)795,6,6
    6 WRITE OUTPUT TAPE MTAPE,17,PHI(L),R(J)
      GO TO 20
  470 ASUM=TM(I)+ASUM
  795 IF(IPT)3,215,3
    3 WRITE OUTPUT TAPE MTAPE,660,ASUM,S1,C1,HS,HC,FST,SEC
  215 CONTINUE
C THE FOLLOWING SUMS THE LAST HALF OF F49 DEPENDING ON THE EPSILIONS
C REQUIRED FOR CONVERGENCE.
  280 IF(IPT)216,2,216
  216 WRITE OUTPUT TAPE MTAPE,660,ASUM,S1,C1,HS,HC,FST,SEC
      WRITE OUTPUT TAPE MTAPE,303,I,PHI(L),R(J)
  303 FORMAT(31H CONVERGENCE WAS REACHED FOR I=I10  ,5H PHI=E15.8 ,3H R=
     1E15.8///)
    2 DO 5 I=2,NNU
C SOLVING THE EQUATIONS DEPENDENT ON EPSILON.
      EPS=EE(I)
      X=EPS*P2
      CALL HYPER(X,HS4A(I),HC4A(I))
      X=EPS*P1
      CALL HYPER(X,HS3A(I),HC3A(I))
      ER1=EPS*LOGF(R1)
      S3=SINF(ER1)
      C3=COSF(ER1)
      GM(I)=(AL11*EPS/R1*S3-AL12*C3)/(AL11*EPS/R1*C3+AL12*S3)
      SA(I)=(-EPS*AK21*HS4A(I)-AK22*HC4A(I))/(EPS*AK21*HC4A(I)+AK22*HS4A
     1(I))
      SB(I)=(-EPS*AK11*HS3A(I)-AK12*HC3A(I))/(EPS*AK11*HC3A(I)+AK12*HS3A
     1(I))
  600 IF(IPT)57,54,57
```


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```
 57 WRITE OUTPUT TAPE MTAPE,803,I
    WRITE OUTPUT TAPE MTAPE,801,SA(I),SB(I),GM(I)
    WRITE OUTPUT TAPE MTAPE,800C
 54 DO 513 K=1,NUP1
    AKM1=K-1
    XE=R1+XEN*(AKM1)
    GO TO(35,36),KIM
 35  CALL FPR(MU,XE,A,POLY)
    GO TO 16
 36 CALL LINE(MU,XE,A(1),POLY)
 16 EM(K)=(GM(I)*SINF(EPS*LOGF(XE))+COSF(EPS*LOGF(XE)))/XE
513 Y(K)=POLY*EM(K)
    CALL SIMP(NUP1,XEN,Y,OR1)
    GM1=GM(I)
    CALL OVERR(GM1,OVR)
    IF(IPT)204,205,204
204 WRITE OUTPUT TAPE MTAPE,306,OVR,OR1
205 Q(I)=(AK13*OR1)/(((AK11*EPS+AK12*SA(I))*HS3A(I)+(AK11*EPS*SA(I)+AK
   112)*HC3A(I))*OVR)
    IF(IPT)206,207,206
206 WRITE OUTPUT TAPE MTAPE,1002,Q(I)
207 DO 800 K=1,NUAP1
    AKM1=K-1
    XE=R1+XCN*AKM1
    GO TO(98,89),KPU
 98 CALL FPR(IU,XE,B,POLY)
    GO TO 790
 89 CALL LINE(IU,XE,B(1),POLY)
790 EM(K)=(GM(I)*SINF(EPS*LOGF(XE))+COSF(EPS*LOGF(XE)))/XE
800 Y(K)=POLY*EM(K)
    CALL SIMP(NUAP1,XCN,Y,OR2)
    W(I)=AK23*OR2/(((AK21*EPS+AK22*SB(I))*HS4A(I)+(AK21*EPS*SB(I)+AK22
   1)*HC4A(I))*OVR)
    IF(IPT)208,14,208
208 WRITE OUTPUT TAPE MTAPE,1003,W(I),OR2
 14 EPS=EE(I)
    X=EPS*PHI(L)
    ELN=EPS*LOGF(R(J))
```


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```
      S3=SINF(ELN)
      C3=COSF(ELN)
      CALL HYPER(X,HS,HC)
      IF(AK13)21,22,21
   22 CAT=0.0
      GO TO 23
   21 CAT=Q(I)*(GM(I)*S3+C3)*(SA(I)*HS+HC)
   23 IF(AK23)28,29,28
   29 DOG=0.0
      GO TO 24
   28 DOG=W(I)*(GM(I)*S3+C3)*(SB(I)*HS+HC)
   24 IF(I-5)412,413,413
  412 UM(I)=DOG+CAT
      IF(IPT)230,231,230
  230 WRITE OUTPUT TAPE MTAPE,660,BSUM,DOG,CAT,HS,HC
      WRITE OUTPUT TAPE MTAPE,222,UM(I)
  222 FORMAT(4H UM=E20.8)
  231 GO TO 471
  413 UM(I)=DOG+CAT
      BSUM=UM(I)+BSUM
      IF(IPT)217,218,217
  217 WRITE OUTPUT TAPE MTAPE,660,BSUM,DOG,CAT,HS,HC
  218 CONV= ABSF(UM(I-3))+ABSF(UM(I-2))+ABSF(UM(I-1))+ABSF(UM(I))
      IF(IPT)219,220,219
  219 WRITE OUTPUT TAPE MTAPE,221,CONV,UM(I)
  221 FORMAT(6H CONV=E20.8,4H UM=E20.8)
  220 IF(CONV-TOL1)4,4,11
   11 IF(I-NNU)5,180,180
  471 BSUM=UM(I)+BSUM
    5 CONTINUE
    4 IF(IPT)480,481,480
  480 WRITE OUTPUT TAPE MTAPE,661,I,PHI(L),R(J)
  661 FORMAT(48H CONVERGENCE WAS REACHED FOR SECOND SUMMATION I=I10, 5H
     1PHI=E20.8, 3H R=E20.8///)
  481 F(L,J)=ASUM+BSUM+XX
      GO TO 20
  704 WRITE OUTPUT TAPE MTAPE,95
   95 FORMAT(44H YOU HAVE USED AN UNDEFINED INPUT INDICATOR.//)
```


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```
      GO TO 20
  180 WRITE OUTPUT TAPE MTAPE,302,PHI(L),R(J)
  302 FORMAT(41H NO CONVERGENCE FOR SECOND SUMMATION PHI=E20.8, 3H R=E20
     1.8,  44H ZERO WILL BE PRINTED AT THIS LATTICE POINT.///)
   20 CONTINUE
   18 CALL OUTPUT
   17 FORMAT(40H NO CONVERGENCE FOR FIRST SUMMATION PHI=E20.8, 3H R=E20.
     18,   44H ZERO WILL BE PRINTED AT THIS LATTICE POINT.///)
   41 FORMAT(7H GAMMA=1PE20.8)
   42 FORMAT(5H EPS= E20.8)
   60 FORMAT(1H0)
   91 FORMAT(3H Z=1PE16.7, 5H  Y2=E16.7, 5H  Y1=E16.7)
 1000 FORMAT(3H L=E20.8, 3H D=E20.8, 4H E1=E20.8)
 1001 FORMAT(3H P=E20.8, 3H E=E20.8, 3H D=E20.8)
 1002 FORMAT(3H Q=E20.8)
 1003 FORMAT(3H W=E20.8, 5H OR2=E20.8)
  660 FORMAT(1P7E16.7)
  801 FORMAT(4H S1= 1PE16.7, 5H  S2=  E16.7, 4H  M= E16.7)
  803 FORMAT(7H FOR I=I10)
 901  FORMAT(4H N2=1PE16.7, 4H N1= E16.7)
 8000 FORMAT(1H0)
  306 FORMAT(5H OVR=20.8, 5H OR1=E20.8)
      RETURN
      END
```


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```
      SUBROUTINE DENOM(ZB,ANS)
C EVALUATION OF INTEGRAL USED IN DENOMINATOR OF AL,
C P,G, AND A.
C PROGRAMMED BY KEM BENNETT 3/5/63.
      DIMENSION A(50),B(50),C(50),Z(50),RL(50),P(50),Y2(50),Y1(50),WN2(5
     10),WN1(50),BZ(50),Q(50),W(50),SA(50),SB(50),GM(50),PHI(21),R(6),F(
     221,6),Y(300),ZOO(300),EM(300),GG(50),EE(50),TM(50),UM(50),HS4A(50)
     3,HC4A(50),HS3A(50),HC3A(50),V(21,6)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23,MU,IU,KU,JU,NU,NUA,NUB,NUC,RL,P,Y2,Y1,WN2,
     2WN1,BZ,Q,W,SA,SB,GM,G,EPS,A,B,C,F,R,PHI,Z,NI,NI1,NI2,NI3,TOL,TOL1,
     3IPT,NNU,T,MM,NN,V,MURLA,GG,NTAPE,MTAPE
      ANS=(P2/2.*(ZB**2+1.)-1./(4.*G)*(ZB**2-1.)*SINF(2.*G*P2)+ZB/
     1G*SINF(G*P2)**2)-P1/2.*(ZB**2+1.)+1./(4.*G)*(ZB**2-1.)*SINF(2.*G*P
     21)-ZB/G*SINF(G*P1)**2
      RETURN
      END
```


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Contrails


```
      SUBROUTINE SIMP(NT,DX,V,AREA)
C     EQUALLY SPACED POINTS
C     EVEN NO. OF INTERVALS---SIMPSONS RULE
C     ODD NO.--- FIRST N-1 INTERVALS-SIMPSONS LAST INTERVAL-TRAPEZOID
      DIMENSION  V(1)
      AREA=0.0
      IF(NT-1)47,50,47
   47 IF((NT/2)*2-NT)28,29,28
   29 NOD=1
      N1=NT-1
      GO TO 32
   28 N1=NT
      NOD=0
   32 AREA1=0.0
      IF(N1-1)35,35,445
  445 J=N1-1
      DO 33 I=2,J,2
   33 AREA=V(I)+AREA
      IF(N1-3)446,446,7447
 7447 J=N1-2
      DO 34 I=3,J,2
   34 AREA1=V(I)+AREA1
  446 AREA =.33333333*(V(1)+V(N1)+2.*AREA1+4.*AREA)
      IF(NOD)35,36,35
   35 AREA= AREA +.5*(V(N1)+V(N1+1))
   36 AREA= DX*AREA
   50 RETURN
      END
```


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Contrails


```
      SUBROUTINE OVERR(GMEER,OVR)
C SUBROUTINE OVERR COMPUTES DENOMINATOR OF Q(I) + W(I).
C PROGRAMMED BY KEM BENNETT
      DIMENSION A(50),B(50),C(50),Z(50),RL(50),P(50),Y2(50),Y1(50),WN2(5
     10),WN1(50),BZ(50),Q(50),W(50),SA(50),SB(50),GM(50)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23,MU,IU,KU,JU,NU,NUA,NUB,NUC,RL,P,Y2,Y1,WN2,
     2WN1,BZ,Q,W,SA,SB,GM,G,EPS
      AR1=EPS*LOGF(R1)
      AR2=EPS*LOGF(R2)
      OVR=AR2/2.*(GMEER**2/EPS+1./EPS)-1./4.*(GMEER**2/EPS-1.)*SINF(2.*A
     1R2)+GMEER/EPS*SINF(AR2)**2-(AR1/2.*(GMEER**2/EPS+1./EPS)-1./4.*(
     2GMEER**2/EPS-1.)*SINF(2.*AR1)+GMEER/EPS*SINF(AR1)**2)
      RETURN
      END
```


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```
      SUBROUTINE HYPER(X,HS,HC)
C EVALUATION OF HYPERBOLIC SINE AND COSINE.
C PROGRAMMED BY KEM BENNETT 2/28/63.
      EX=EXPF(X)
      HS=0.5*(EX-1./EX)
      HC=0.5*(EX+1./EX)
      RETURN
      END
```


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```
      SUBROUTINE LINE(N,X,A,Y)
      DIMENSION A(1)
      IF (N) 100,100,101
  101 NA=2*N
      DO 102 J=3,NA,2
      I=J
      IF (X-A(I)) 103,103,102
  102 CONTINUE
  103 Y=A(I-1)+(X-A(I-2))* ((A(I+1)-A(I-1))/(A(I)-A(I-2)))
      GO TO 104
  100 Y=0.
  104 RETURN
      END
```


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Contrails

```
      SUBROUTINE FPR(MU,XE,A,POLY)
       DIMENSION A(8)
      IF(MU)700,6,700
    6 POLY=A(1)
      GO TO 14
  700 POLY=A(1)
      DO 516 J=1,MU
  516 POLY=POLY*XE+A(J+1)
   14 RETURN
      END
```


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Contrails

```
      SUBROUTINE SOLGAM(I,ROOT,NTRUBL)
      DIMENSION ROOT(100)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23
      IGT=1
      RINC=3.1415927/(P2-P1)
      ROOT(1)=0.
      J=2
      G=.01
    7 G1=G*P1
      G2=G*P2
      C1=COSF(G1)
      C2=COSF(G2)
      S1=SINF(G1)
      S2=SINF(G2)
      T1=AK11*C1
      T2=AK11*S1
      T3=AK12*C1
      T4=AK12*S1
      T5=AK21*C2
      T6=AK21*S2
      T7=AK22*C2
      T8=AK22*S2
      D1=G*T1+T4
      D2=T3-G*T2
      D3=G*T5+T8
      D4=T7-G*T6
      DP1=T1+P1*D2
      DP2=-T2-P1*D1
      DP3=T5+P2*D4
      DP4=-T6-P2*D3
      FG=D1*D4-D2*D3
      IF(FG)6,11,6
    6 GO TO (8,9,14),IGT
    8 IGT=2
    5 G=G+RINC/10.
      FL=FG
      GO TO 7
```


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```
  9 IF(FL*FG)12,5,5
 11 ROOT(J)=G
    J=J+1
    G=G+RINC
    IGT=3
    IF(J-I-1)7,25,25
 12 IGT=3
 14 FPG=DP1*D4-DP2*D3+D1*DP4-D2*DP3
    G2G=G-FG/FPG
    G=G2G
    TOL=.00001*ABSF(FPG)
    IF(ABSF(FG)-TOL)15,15,7
 15 ROOT(J)=G
    IF(J-I)16,25,25
 16 J=J+1
    G=G+RINC
    GO TO 7
 25 RETURN
    END
```


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```
      SUBROUTINE GAMMA(X,Y)                                           00001016
      DIMENSION A(10)                                                 00001017
      A(1) = .83333333E-01                                            00001018
      A(2) =-.27777778E-02                                            00001019
      A(3) = .79365079E-03                                            00001020
      A(4) =-.59523810E-03                                            00001021
      A(5) = .84175084E-03                                            00001022
      A(6) =-.19175269E-02                                            00001023
      A(7) = .64102564E-02                                            00001024
      A(8) =-.29550654E-01                                            00001025
      A(9) = .17964437E 00                                            00001026
      A(10)=-.13924322E 01                                            00001027
      IF(X)10,13,30                                                   00001028
   10 INT=X                                                           00001029
      XINT=INT                                                        00001030
      XF=XINT-X                                                       00001031
      AX=X                                                            00001032
      XM=1.                                                           00001033
      J=3-INT                                                         00001034
      DO 15 I=1,J                                                     00001035
      XM=XM*AX                                                        00001036
   15 AX=AX+1.                                                        00001037
      IF(ABSF(XM)-1.0E-30)11,11,29                                    00001038
   11 IE=INT/2                                                        00001039
      IF(INT-2*IE)13,12,13                                            00001040
   12 Y=-1.0E 30                                                      00001041
      GO TO 50                                                        00001042
   13 Y=1.0E 30                                                       00001043
      GO TO 50                                                        00001044
   29 AX=3.-XF                                                        00001045
      GO TO 38                                                        00001046
   30 IF(X-1.)34,37,36                                                00001047
   34 IF(X-1.0E-30)13,13,37                                           00001048
   37 AX=X+2.                                                         00001049
      XM=X*(X+1.)                                                     00001050
      GO TO 38                                                        00001051
   36 IF(X-2.)39,39,14                                                00001052
   39 AX=X+1.                                                         00001053
```


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Contrails

```
      XM=X
      GO TO 38
   14 AX=X
      XM=1.
      IF(X-30.)38,13,13
   38 C=1./AX**2
      T1=A(1)/AX
      F=T1
      DO 35 I=2,10
      TN=T1*A(I)/A(I-1)*C
      IF(ABSF(TN)-ABSF(T1))32,32,33
   32 F=F+TN
      T1=TN
      IF(ABSF(TN)-.2E-08)33,33,35
   35 CONTINUE
   33 EX=(AX-.5)*LOGF(AX)-AX+F+.91893853
      Y=EXPF(EX)
      Y=Y/XM
   50 RETURN
      END
```


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Contrails

```
      SUBROUTINE BESEL(J,X,A,N,ANS,BOO,LY)                              00001074
      DIMENSION ANS(1)                                                  00001075
      BETE=BOO                                                          00001076
      RORD=N                                                            00001077
      GAMB=A+RORD                                                       00001078
      RARG=X/BETE                                                       00001079
      CALL BES(BETE,RARG,GAMB,YJ,YJP)                                   00001080
      ANS(1)=YJ                                                         00001081
      IF(GAMB)10,20,20                                                  00001082
   10 ANS(2)=(GAMB*YJ-RARG*YJP)/X                                       00001083
      GO TO 30                                                          00001084
   20 ANS(2)=(GAMB*YJ+RARG*YJP)/X                                       00001085
   30 RETURN                                                            00001086
      END                                                               00001087
```


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Contrails

```
      SUBROUTINE BES(A,X,Y,B,BP)
    2 FORMAT(1H ,4E20.8)
D     P=0.
D     BJ0=0.
D     BJ1=0.
D     ARG=0.
      PI=3.1415927
      INEG=0
      P=Y
      ARG =.5*A*X
      IF(ARG-.5)90,91,91
   90 IF(P)91,11,11
   91 CONTINUE
      IF(P)5,11,4
    5 INEG=-1
      I=P
      PINT=I
      NIT=1-I
      PF=1.+P-PINT
      P=PF
      GO TO 11
    4 IF(ABSF(P-1.)-1.)11,11,6
    6 INEG=1
      I=P
      PINT=I
      NIT=I-1
      PF=P-PINT
      P=PF
   11 CONTINUE
D     P=P
      ARG=.5*A*X
      IF(ARG+P)15,10,15
   10 B=1.
      BP=0.
      GO TO 50
   15 IF(ARG)17,16,17
   16 ER=0.
      GO TO 25
```


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Contrails

```
17 IF(ARG-4.)20,30,30                                   00001126
20 ER=EXPF(P*LOGF(ARG))                                 00001127
25 AD=P+2.                                              00001128
   AP=P+1.                                              00001129
   A1=ARG**2                                            00001130
   AK=1.                                                00001131
   CALL GAMMA(AP,G)                                     00001132
   T1=ER/G                                              00001133
   B=T1                                                 00001134
   BP=T1*P                                              00001135
   DO 28 I=1,100                                        00001136
   TN=-T1/AK*A1/AP                                      00001137
   B=B+TN                                               00001138
   TNP=TN*AD                                            00001139
   BP=BP+TNP                                            00001140
   IF(ABSF(TN)-.1E-07)26,26,27                          00001141
26 IF(ABSF(TNP)-.1E-07)29,29,27                         00001142
27 T1=TN                                                00001143
   AK=AK+1.                                             00001144
   AD=AD+2.                                             00001145
28 AP=AP+1.                                             00001146
29 BP=BP/X                                              00001147
   IF(INEG)60,50,60                                     00001148
60 BJO=B                                                00001149
D     BJO=BJO                                           00001150
   BJ1=(B*P-X*BP)/(A*X)                                 00001151
D     BJ1=BJ1                                           00001152
   GO TO 50                                             00001153
30 F1=SQRTF(PI*ARG)                                     00001154
   IDER=0                                               00001155
   A1=-(A*X)**2                                         00001156
   R=4.*P**2                                            00001157
   F2=2.*ARG-(2.*P+1.)*PI/4.                            00001158
31 CF2=COSF(F2)                                         00001159
   SF2=SINF(F2)                                         00001160
   T1= (R-1.)*(R-9.)/(128.*A1)                          00001161
   T1M=T1                                               00001162
   T2=(1.-R)/(8.*A*X)                                   00001163
```


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```
      T2M=T2                                                            00001164
      S1=1.+T1                                                          00001165
      S2=T2                                                             00001166
      Z=2.                                                              00001167
      J=1                                                               00001168
      DO 36 I=1,100                                                     00001169
      ICNT=I                                                            00001170
      GO TO(40,45,40),J                                                 00001171
   40 TM1=(R-(4.*Z-3.)**2)*(R-(4.*Z-1.)**2)/((2.*Z-1.)*Z*128.*A1)       00001172
      IF(ABSF(TM1)-ABSF(T1M))43,51,51                                   00001173
   51 IF(ABSF(TM1)-1.)43,41,41                                          00001174
   41 GO TO(42,42,37),J                                                 00001175
   42 J=2                                                               00001176
      GO TO 44                                                          00001177
   43 T1=T1*TM1                                                         00001178
      T1M=TM1                                                           00001179
      S1=S1+T1                                                          00001180
      IF(ABSF(T1)-.1E-07)41,41,44                                       00001181
   44 GO TO(45,45,36),J                                                 00001182
   45 TM2=(R-(4.*Z-3.)**2)*(R-(4.*Z-5.)**2)/((2.*Z-1.)*(Z-1.)*128.*A1)  00001183
      IF(ABSF(TM2)-ABSF(T2M))48,49,49                                   00001184
   49 IF(ABSF(TM2)-1.)48,46,46                                          00001185
   46 GO TO(47,37,47),J                                                 00001186
   47 J=3                                                               00001187
      GO TO 36                                                          00001188
   48 T2=T2*TM2                                                         00001189
      T2M=TM2                                                           00001190
      S2=S2+T2                                                          00001191
      IF(ABSF(T2)-.1E-07)46,46,36                                       00001192
   36 Z=Z+1.                                                            00001193
   37 IF(IDER)39,38,39                                                  00001194
   38 B=(CF2*S1+SF2*S2)/F1                                              00001195
      F2=F2-PI/2.                                                       00001196
      R=4.*(P+1.)**2                                                    00001197
      IDER=1                                                            00001198
      GO TO 31                                                          00001199
   39 BD=(CF2*S1+SF2*S2)/F1                                             00001200
      BP=P*B/X-A*BD                                                     00001201
```


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```
         BJO=B
D        BJO=BJO
         BJ1=BD
D        BJ1=BJ1
     50  IF(INEG)65,70,80
     65  DO 61  I=1,NIT
D        BR=P*BJO/ARG-BJ1
D        BJ1=BJO
D        BJO=BR
D    61  P=P-1.
         B=BJO
         BP=P*B/X-A*BJ1
         GO TO 70
D    80  P=P+1.
         DO 81  I=1,NIT
D        BR=(P*BJ1)/ARG-BJO
D        BJO=BJ1
D        BJ1=BR
D    81  P=P+1.
         B=BJ1
         BP=A*BJO-P*B/X
     70  RETURN
         END
```


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Contrails

```
      SUBROUTINE ENOM(CIJ,B,H,EN)
      DIMENSION DUM(150),ANS(50)
      COMMON P2,P1,R1,R2
      BOO=B
      G=H
      KA=0
    8 KIS=G
      GI=KIS
      GF=G-GI
      AP=GF
      AN=1.-GF
      NP=KIS
      NNN=-(KIS+1)
      KA=KA+1
      GO TO(1,2,3),KA
    1 X1=B*R2
      CALL BESEL(0,X1,AP,NP,ANS,BOO,  LLL)
      AR2=ANS(1)
      CALL BESEL(0,X1,AN,NNN,ANS,BOO,    LLL)
      DR2=ANS(1)
      X1=B*R1
      CALL BESEL(0,X1,AP,NP,ANS,BOO.  LLL)
      AR1=ANS(1)
      CALL BESEL(0,X1,AN,NNN,ANS,BOO,    LLL)
      DR1=ANS(1)
      G=H-1.
      GO TO 8
    2 X1=B*R2
      CALL BESEL(0,X1,AP,NP,ANS,BOO,  LLL)
      BR2=ANS(1)
      CALL BESEL(0,X1,AN,NNN,ANS,BOO,    LLL)
      ER2=ANS(1)
      X1=B*R1
      CALL BESEL(0,X1,AP,NP,ANS,BOO,  LLL)
      BR1=ANS(1)
      CALL BESEL(0,X1,AN,NNN,ANS,BOO,    LLL)
      ER1=ANS(1)
      G=H+1.
```


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Contrails

```
      GO TO 8
    3 X1=B*R2
      CALL BESEL(O,X1,AP,NP,ANS,BOO,  LLL)
      CR2=ANS(1)
      CALL BESEL(O,X1,AN,NNN,ANS,BOO,    LLL)
      FR2=ANS(1)
      X1=B*R1
      CALL BESEL(O,X1,AP,NP,ANS,BOO,  LLL)
      CR1=ANS(1)
      CALL BESEL(O,X1,AN,NNN,ANS,BOO,    LLL)
      FR1=ANS(1)
      T1R1=(CIJ**2*R1**2/2.)*(AR1**2-BR1*CR1)
      T1R2=(CIJ**2*R2**2/2.)*(AR2**2-BR2*CR2)
      T2R1=(CIJ*R1**2/2.)*(2.*AR1*DR1+BR1*FR1+CR1*ER1)
      T2R2=(CIJ*R2**2/2.)*(2.*AR2*DR2+BR2*FR2+CR2*ER2)
      T3R1=(R1**2/2.)*(DR1**2-FR1*FR1)
      T3R2=(R2**2/2.)*(DR2**2-ER2*FR2)
      EN=T1R2+T2R2+T3R2-T1R1-T2R1-T3R1
      RETURN
      END
```


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```
      SUBROUTINE SOLEPS(I,ROOT,NTRUBL)
      DIMENSION ROOT(100)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23
      A11=AL11/R1
      A12=AL12
      A21=AL21/R2
      A22=AL22
      PE1=LOGF(R1)
      PE2=LOGF(R2)
      IGT=1
      RINC=3.1415927/(PE2-PE1)
      ROOT(1)=0.
      J=2
      G=.01
    7 G1=G*PE1
      G2=G*PE2
      C1=COSF(G1)
      C2=COSF(G2)
      S1=SINF(G1)
      S2=SINF(G2)
      T1=A11*C1
      T2=A11*S1
      T3=A12*C1
      T4=A12*S1
      T5=A21*C2
      T6=A21*S2
      T7=A22*C2
      T8=A22*S2
      D1=G*T1+T4
      D2=T3-G*T2
      D3=G*T5+T8
      D4=T7-G*T6
      DP1=T1+PE1*D2
      DP2=-T2-PE1*D1
      DP3=T5+PE2*D4
      DP4=-T6-PE2*D3
      FG=D1*D4-D2*D3
```


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```
      IF(FG)6,11,6                                          00001321
    6 GO TO(8,9,14),IGT                                     00001322
    8 IGT=2                                                 00001323
    5 G=G+RINC/10.                                          00001324
      FL=FG                                                 00001325
      GO TO 7                                               00001326
    9 IF(FL*FG)12,5,5                                       00001327
   11 ROOT(J)=G                                             00001328
      J=J+1                                                 00001329
      G=G+RINC                                              00001330
      IGT=3                                                 00001331
      IF(J-I-1)7,25,25                                      00001332
   12 IGT=3                                                 00001333
   14 FPG=DP1*D4-DP2*D3+D1*DP4-D2*DP3                       00001334
      G2G=G-FG/FPG                                          00001335
      G=G2G                                                 00001336
      TOL=.00001*ABSF(FPG)                                  00001337
      IF(ABSF(FG)-TOL)15,15,7                               00001338
   15 ROOT(J)=G                                             00001339
      IF(J-I)16,25,25                                       00001340
   16 J=J+1                                                 00001341
      G=G+RINC                                              00001342
      GO TO 7                                               00001343
   25 RETURN                                                00001344
      END                                                   00001345
```


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```
      SUBROUTINE SOLBES(GAM,NROOT,ROOT)
      DIMENSION DUM(150),ANS(50)
      DIMENSION ROOT(1)
      DIMENSION FG(50), FGP(50)
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,
     1AL13,AL21,AL22,AL23
      K=GAM
      G=GAM
      GINT=K
      GF=GAM-GINT
      AP=GF
      AN=1.-GF
      NP=K
      NN=-(K+1)
      RI  =3.1415927/(R2-R1)
      RI04=RI/4.
      IF1=0
      IF2=0
      IF3=0
      BO=.5*G
      BINC=.125
      B=BO
      I=1
   10 X1=B*R1
      X2=B*R2
      CALL BESEL(0,X1,AP,NP,ANS,B    ,L)
      Y=ANS(1)
      YP=B*ANS(2)-GAM*ANS(1)/R1
      T1=AL11*YP+AL12*Y
      T1P=AL12*R1/B*YP+AL11*(G**2/(B*R1)-B*R1)*Y
      CALL BESEL(0,X2,AP,NP,ANS,B    ,L)
      Y=ANS(1)
      YP=B*ANS(2)-GAM*ANS(1)/R2
      T3=AL21*YP+AL22*Y
      T3P=AL22*R2/B*YP+AL21*(G**2/(B*R2)-B*R2)*Y
      CALL BESEL(0,X1,AN,NN,ANS,B    ,L)
      Y=ANS(1)
      YP=-GAM*ANS(1)/R1-B*ANS(2)
```


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```
      T2=AL11*YP+AL12*Y
      T2P=AL11*(G**2/(B*R1)-B*R1)*Y+AL12*R1/B*YP
      CALL BESEL(0,X2,AN,NN,ANS,B     ,L)
      Y=ANS(1)
      YP=-GAM*ANS(1)/R2-B*ANS(2)
      T4=AL21*YP+AL22*Y
      T4P=AL21*(G**2/(B*R2)-B*R2)*Y+AL22*R2/B*YP
      F=T1*T4-T2*T3
      FP=T1*T4P-T2*T3P+T1P*T4-T2P*T3
   18 CONTINUE
      IF(IF1)20,20,30
   20 IF(IF2)21,21,22
   21 IF2=1
   19 BS=B
      FS=F
      FPS=FP
      B=B+BINC
      GO TO 10
   22 IF(SIGNF(1.,F)-SIGNF(1.,FS))23,24,23
   23 B=(B*FS-F*BS)/(FS-F)
      IF1=1
      GO TO 10
   24 IF(SIGNF(1.,FP)-SIGNF(1.,FPS))25,26,25
   25 IF3=1
      GO TO 19
   26 IF(IF3)27,27,19
   27 BINC=2.*BINC
      IF(BINC-RIO4)19,19,28
   28 BINC=RIO4
      GO TO 19
   30 B2=B-F/FP
      B=B2
      TOL=.00001*ABSF(FP)
      IF(ABSF(F)-TOL)31,31,10
   31 ROOT(I)=B
      FG(I)=F
      FGP(I)=FP
      IF(I-NROOT)32,50,50
```


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```
32 IF(I-2)33,34,34
33 BINC=.125
   B=B+BINC
   I=I+1
   IF1=0
   IF2=0
   IF3=0
   GO TO 10
34 IF(ROOT(I)-ROOT(I-1)-.8*RI)33,35,35
35 B=B+RI
   I=I+1
   GO TO 10
50 RETURN
   END
```


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```
      SUBROUTINE OUTPUT                                                 00001436
      DIMENSION A(50),B(50),C(50),Z(50),RL(50),P(50),Y2(50),Y1(50),WN2(500001437
     10),WN1(50),BZ(50),Q(50),W(50),SA(50),SB(50),GM(50),PHI(21),R(6),F(00001438
     221,6),PHY(21),V(21,6),GG(50)                                      00001439
      COMMON P2,P1,R1,R2,M,N,AK11,AK12,AK13,AK21,AK22,AK23,AL11,AL12,   00001440
     1AL13,AL21,AL22,AL23,MU,IU,KU,JU,NU,NUA,NUB,NUC,RL,P,Y2,Y1,WN2,    00001441
     2WN1,BZ,Q,W,SA,SB,GM,G,EPS,A,B,C,F,R,PHI,Z,NI,NI1,NI2,NI3,TOL,TOL1,00001442
     3IPT,NNU,T,MM,NN,V,MURLA,GG,NTAPE,MTAPE                            00001443
      NP1=N+1                                                           00001444
      MP1=M+1                                                           00001445
      DO 12 J=1,MP1                                                     00001446
   12 PHY(J)=PHI(J)*57.29578                                            00001447
   18 WRITE OUTPUT TAPE MTAPE,1                                         00001448
    1 FORMAT(1H1)                                                       00001449
   54 WRITE OUTPUT TAPE MTAPE,804,AK11,AK12,AK13,AK21,AK22,AK23         00001450
      WRITE OUTPUT TAPE MTAPE,805,AL11,AL12,AL13,AL21,AL22,AL23         00001562
  804 FORMAT(5H K11= 1PE14.7, 6H  K12= E14.7,  6H  K13= E14.7,  6H  K21=00001452
     1E14.7,  6H  K22= E14.7,  6H  K23= E14.7)                          00001453
  805 FORMAT(5H L11= 1PE14.7, 6H  L12= E14.7,  6H  L13= E14.7,  6H  L21=00001454
     1E14.7,  6H  L22= E14.7,  6H  L23= E14.7)                          00001455
      DO 70 I=1,10                                                      00001456
   70 WRITE OUTPUT TAPE MTAPE,6000                                      00001457
      WRITE OUTPUT TAPE MTAPE,106,(R(J),J=1,NP1)                        00001458
      WRITE OUTPUT TAPE MTAPE,6000                                      00001459
      GO TO(8,2,3,4,5),N                                                00001460
  106 FORMAT(20X,6F14.3)                                                00001461
    8 WRITE OUTPUT TAPE MTAPE,2001                                      00001462
      GO TO 10                                                          00001463
    2 WRITE OUTPUT TAPE MTAPE,2002                                      00001464
      GO TO 10                                                          00001465
    3 WRITE OUTPUT TAPE MTAPE,2003                                      00001466
      GO TO 10                                                          00001467
    4 WRITE OUTPUT TAPE MTAPE,2004                                      00001468
      GO TO 10                                                          00001469
    5 WRITE OUTPUT TAPE MTAPE,2005                                      00001470
   10 WRITE OUTPUT TAPE MTAPE,8000                                      00001471
      WRITE OUTPUT TAPE MTAPE,3000,PHY(1),(F(1,L),L=1,NP1)              00001472
      WRITE OUTPUT TAPE MTAPE,6000                                      00001473
```


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```
      DO 100 J=2,M
      WRITE OUTPUT TAPE MTAPE,4000,PHY(J),(F(J,K),K=1,NP1)
  100 WRITE OUTPUT TAPE MTAPE,6000
      WRITE OUTPUT TAPE MTAPE,5000,PHY(MP1),(F(MP1,I),I=1,NP1)
 7000 FORMAT(1P7E16.7)
 2002 FORMAT(30X,3H R1,25X,3H R2)
 2001 FORMAT(30X,3H R1,11X,3H R2)
 2003 FORMAT(30X,3H R1,39X,3H R2)
 2004 FORMAT(30X,3H R1,53X,3H R2)
 2005 FORMAT(30X,3H R1,67X,3H R2)
 3000 FORMAT(F13.2,8H   PHI 2,6F14.2)
 4000 FORMAT(F13.2,8X,6F14.2)
 5000 FORMAT(F13.2,8H   PHI 1,6F14.2)
 6000 FORMAT(1H )
 8000 FORMAT(1H0)
      RETURN
      END
```


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Contrails

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