AFFDL-TR-69-84

APPROXIMATE SOLUTION OF PLANE ORTHOTROPIC ELASTICITY PROBLEMS

JOEL SCHIMKE

KEVIN THOMAS

JOHN GARRISON

FOREWORD

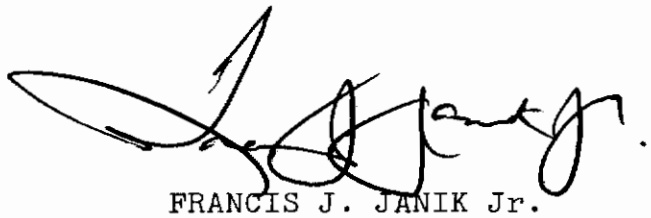
This report was prepared by Bell Aerosystems Company, under United States Air Force Contract No. AF 33(615)-68-C-1560.

The contract was sponsored by the Air Force Flight Dynamics Laboratory of the Directorate of Laboratories Air Force Systems Command, Wright-Patterson Air Force Base, Ohio. The work was initiated under Project No. 1467, "Structural Analysis Methods," Task No. 146702, "Thermoelastic Stress Analysis Methods". The work was administered under the direction of Mr. T. N. Bernstein (FDTR), Project Engineer.

This report covers work conducted from April 1968 to September 1969. The report was submitted in October 1969.

The technical director and principal investigator for this project was Dr. Joel Schimke. Mr. Richard Dworak served as program manager. Major analytical contributions to the project were made by Dr. Kevin Thomas and Dr. John Garrison. Additional support was received from Mr. Gary Worner, Mr. Bart Nigro, Dr. Ronald Gellatly, and Dr. Robert Mallett. The computer programming effort was headed by Mr. John Carlsen with major support from Miss Monica Ostanski. Additional programming assistance was received from Mr. Richard Lindner, Mr. William Garth, Mrs. Maria Roberts, Mr. Pasqual Battaglia, Mrs. Constance Ryan, and Mr. Alan Bauer. This report was ably typed by Miss Adele Dworzanski with major assistance from Mrs. Catherine Muir.

This technical report has been reviewed and is approved.



FRANCIS J. JANIK Jr.
Chief Solid Mechanics Branch
Structures Division
AF Flight Dynamics Laboratory

ABSTRACT

The purpose of this study is to compare and evaluate a number of approximate methods for the solution of plane orthotropic elasticity problems. The methods under consideration fall into three main categories. The first category is perturbation methods, which are based on the assumption that the behavior of the orthotropic material under a particular loading is not greatly different than the behavior of some nearby isotropic material under the same loading. The second category is the so-called boundary residual methods. In these techniques the assumed families of functions are required to satisfy all the equations of elasticity internally but none of the boundary conditions. The boundary conditions are then satisfied approximately in accordance with the particular boundary residual criterion under consideration. The two criteria that are examined in this report are the integral least squares and boundary Reissner energy methods. The third category of approximate methods is that for which the assumed functions may have to satisfy some, but not all, of the equations of elasticity internally and certain of the boundary conditions. The other equations of elasticity and the other boundary conditions (if any) are then satisfied approximately in accordance with the particular method. The methods of this category which are considered are the potential, complementary, and Reissner energy methods. It is also shown that potential, complementary, and Reissner energy methods are completely equivalent to the Galerkin method as applied to the displacement, stress, and combined stress-displacement formulations of elasticity, respectively.

In order to facilitate evaluation of the various methods, four standard sample problems, for which exact solutions are known, were selected to test these methods. These sample problems involve an infinite sheet with a circular hole under different stress and displacement boundary conditions. In addition six different orthotropic materials, each exhibiting distinctly different characteristics, were selected for use in these sample problems. Numerical results for the different approximate methods, sample cases, and materials are compared to each other and the exact solution. Conclusions as to the utility of these approximate methods under different circumstances are drawn based on these numerical results and also upon the nature and amount of prerequisite analytical and programming work.

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

TABLE OF CONTENTS (continued)

TABLE OF CONTENTS (continued)

TABLE OF CONTENTS (continued)

LIST OF FIGURES

LIST OF FIGURES (continued)

LIST OF FIGURES (continued)

LIST OF TABLES

LIST OF TABLES (continued)

SECTION I

INTRODUCTION

In recent years there has been a great increase in the variety of composite materials available for practical use, and a corresponding increase in applications. Many of these composites,e.g., plywood, are inhomogeneous in the large. However, very often equivalent homogeneous elastic constants can be determined, either experimentally or analytically, for such materials. Having determined the equivalent homogeneous material corresponding to the particular composite, the analysis of problems involving the composite falls into the category of the classical theory of elasticity of anisotropic bodies. The general three-dimensional theory is quite involved. However, certain special cases are amenable to mathematical and numerical treatment. In this study we are particularly concerned with planar (plane stress or strain) problems for orthotropic materials. These problems are the "next step" in difficulty above planar isotropic elasticity problems, but the increase in mathematical and numerical difficulty is, in some respects, formidable.

Because of the increased usage of plane orthotropic sheets it is important to improve the analysis capability for these problems. A method such as finite differences can be applied to an orthotropic problem about as easily as an isotropic problem. However, even if such a method yields good results in the istoropic case there is no guarantee that this will continue to be true in the corresponding problem for an orthotropic material. (Many of the idiosyncracies of orthotropic materials will be brought out in our analytical and numerical results.) Very often (especially if the boundary of the region is irregular or sharp variations in the loading or boundary conditions are present) straightforward finite difference methods are inadequate.

One of the more successful modern methods of structural analysis is the so-called "finite element" method, which, in a sense, is a hybrid method - a cross between finite difference and classical Ritz-type variational analyses. Many workers are busy experimenting with finite element methods for the analysis of orthotropic sheets. However, it does seem that some of the

traditional engineering analysis methods have been neglected in connection with such analyses. These are the so-called weighted residual methods. They include the Ritz-method used with the variational principles of elasticity, least squares, collocation, and other less well known techniques of this class. It is the purpose of this study to investigate a number of these methods to determine their worth as analysis tools for plane orthotropic elasticity problems.

In Section II we present the basic equations of elasticity in both rectangular and polar coordinates, especially as they apply to orthotropic materials. We also present and discuss three variational formulations of the equations of elasticity; the potential, complementary, and Reissner energy formulations. In Section III exact solutions for plane orthotropic elasticity problems are discussed. Special attention is given to problems involving a circular hole in an infinite sheet. Such problems are among the few nontrivial problems in orthotropic elasticity for which exact solutions have been obtained. The reason for considering exact solutions is two-fold:

1) In order to evaluate approximate solutions obtained by the various weighted residual methods it is important to have a standard of comparison. The exact solutions serve this purpose.

2) For certain of the weighted residual methods it is necessary to have available families of functions which satisfy the field equations of orthotropic elasticity exactly. The analysis of Section III shows how such families may be constructed.

Also, in this section extensive numerical results from the exact solution program are presented in the form of contour plots. These plots give a clear picture of the wide variety of behavior one encounters by varying material properties while maintaining the same loading and geometry.

In Section IV we obtain approximate solutions by means of a perturbation technique. This method is based on the assumption that the material under consideration is not strongly orthotropic, and therefore that the solution of the same problem for some "nearby" isotropic material furnishes a good first approximation to the given problem. This technique is not a weighted residual method. However, in common with weighted residual methods, the perturbation method has been somewhat neglected for the analysis of plane orthotropic elasticity problems.

In Section V we consider two weighted residual methods for which the assumed form of the solution is chosen to satisfy the field equations of elasticity exactly. The unknown coefficients of this assumed form are then selected in accordance with the particular weighted residual criterion so that the boundary residual is minimized in some sense. The particular methods we consider are integral least squares and the boundary Reissner energy method. The applications of the latter method in any context have been few, so the results of this section are especially interesting.

In Section VI we examine Ritz-type methods used in connection with three variational principles of elasticity; the potential, complementary, and Reissner energy principles. The assumed form of the solution is not required to satisfy the field equations of elasticity. However, in the case of the potential and complementary energy principles, the assumed form may have to satisfy certain boundary conditions. Applications of the Reissner energy principle in Ritz fashion as discussed in this section have been few so the results are interesting from this point of view.

In Section VII we present extensive numerical results obtained by means of the approximate methods discussed in Sections IV, V, and VI. The results from a particular method are compared to those from the other approximate methods and also with the exact solution. We conclude this section with evaluations of the various approximate methods.

In Appendices I - VI various points touched upon in Sections I - VI are discussed in detail. In Appendices VII and VIII certain algebraic detail arising in the Reissner energy formulation is presented. Appendix IX is a user's manual for the seven computer programs developed in this study. There are six programs which generate approximate solutions in accordance with the formulations discussed in Sections IV - VI. The other program computes exact solutions based on the analysis of Section III.

SECTION II

BASIC GOVERNING EQUATIONS

1. CONSTITUTIVE EQUATIONS

For the developments of Articles 2 and 3 of this section and also for later sections, it is essential to have the constitutive (stress-strain) relations in both rectangular and polar coordinates. In this article we list these relations in rectangular coordinates for a general anisotropic material in three dimensions. We then specialize these equations to plane stress and plane strain problems and develop the appropriate transformations to polar coordinates. Finally we discuss these relations for plane stress problems in orthotropic materials.

1a. Constitutive Equations in Rectangular Coordinates

Our notation for elastic constants conforms to what appears to be a fairly well accepted standard. See Hearmon [1] for a detailed explanation of this notation. In terms of matrices the stress-strain relationship for an anisotropic material can be written

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \left[\begin{array}{ccc:ccc} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ \hdashline S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{array}\right] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix} \qquad (2.1.1)$$

The symmetric 6x6 matrix in (2.1.1) is known as the elastic compliance matrix. In the most general case the compliance matrix has 21 independent elements. For orthotropic materials

the elements in the upper right and lower left parts of the compliance matrix (set off by the dotted lines) are zero, so there are only 9 independent elements. (2.2.1) and the inverse relation can be written in shorthand notation as

$$\begin{aligned} \{\varepsilon\} &= [S_6] \{\sigma\} \\ \{\sigma\} &= [C_6] \{\varepsilon\} \end{aligned} , \qquad (2.1.2)$$

where $[C_6] = [S_6]^{-1}$. $[C_6]$ is known as the elastic stiffness matrix. The subscript 6 is used to distinguish between these 6x6 elastic matrices and 3x3 versions of them which are used for planar problems. The relations (2.1.2) written in tensor notation are given in Appendix I.

Inasmuch as we treat only planar problems in this study we do not have to deal with the full relations (2.1.2). For plane stress ($\sigma_z = \tau_{yz} = \tau_{zx} = 0$) we can obtain from (2.1.1) the following:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad (2.1.3)$$

To obtain the stresses in terms of the strains for plane stress we must invert the 3x3 matrix of (2.1.3). We define $[B] \equiv [S_3]^{-1}$. It should be noted that although the elements of $[S_3]$ can be identified with elements of $[S_6]$, no such identification is possible between the elements of $[B]$ and $[C_6]$. However, for plane strain ($\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$) there is a correspondence and we can write

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (2.1.4)$$

To write the strains in terms of the stresses $[C_3]^{-1}$ must be obtained. In the rest of this report we consider only plane stress problems, but it should be apparent that with an appropriate redefinition of the elements of the elastic matrices all analyses presented here are equally applicable to plane strain problems. Also, in the rest of this report the subscript 3 in $[S_3]$ and $[C_3]$ is omitted.

1b. Transformations From Rectangular to Polar Coordinates

Later in this report it is necessary to know how the elements of the planar

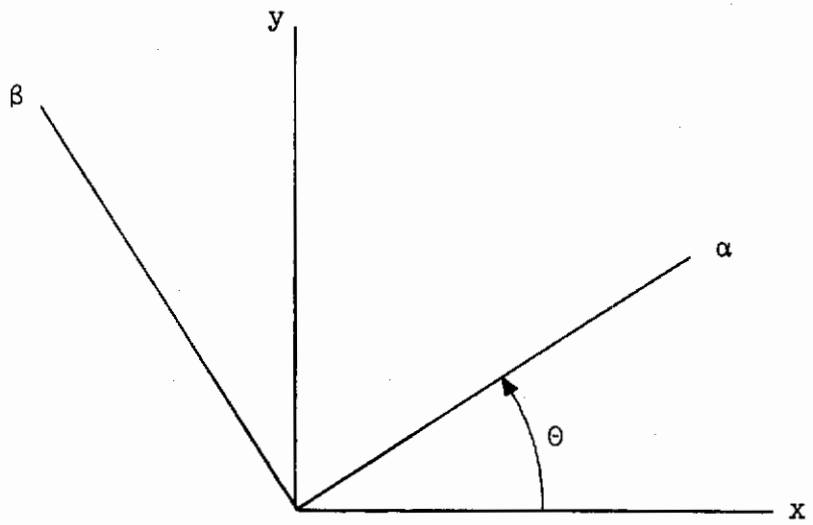


Figure 1. Rotation of Coordinates

compliance and stiffness matrices transform under a rotation of the coordinate system (as indicated in Figure 1). It is convenient to perform this analysis here. Under such a rotation we have

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & (\cos^2\theta-\sin^2\theta) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad (2.1.5)$$

We denote the 3x3 transformation matrix by $[T]$. The components of strain transform just as in (2.1.5) providing that $\{\varepsilon_x, \varepsilon_y, \gamma_{xy}/2\}$ is used as the column vector in the transformation. Inasmuch as we have redefined this strain "vector" we must modify (2.1.3) as follows:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ \frac{S_{16}}{2} & \frac{S_{26}}{2} & \frac{S_{66}}{2} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{2.1.6}$$

We abbreviate (2.1.6) as $\{\tilde{\varepsilon}_{xy}\} = [\tilde{S}^{xy}]\{\sigma_{xy}\}$, where the tildes denote the redefined quantities in (2.1.6). Then

$$\begin{aligned} \{\tilde{\varepsilon}_{\alpha\beta}\} &= [T]\{\tilde{\varepsilon}_{xy}\} = [T][\tilde{S}^{xy}]\{\sigma_{xy}\} \\ &= [T][\tilde{S}^{xy}][T]^{-1}\{\sigma_{\alpha\beta}\} \equiv [\tilde{S}^{\alpha\beta}]\{\sigma_{\alpha\beta}\}, \end{aligned} \tag{2.1.7}$$

where

$$T^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix} \tag{2.1.8}$$

The elements of $[\tilde{S}^{\alpha\beta}]$ are computed by the straightforward matrix multiplications indicated in (2.1.7). After obtaining $[\tilde{S}^{\alpha\beta}]$ we can revert to the conventional form of the stress-strain relationship, e.g., (2.1.3), with a symmetric compliance matrix, by multiplying the last row of $[\tilde{S}^{\alpha\beta}]$ by 2. The elements of the symmetric compliance matrix $[S^{\alpha\beta}]$ can then be written as follows:

$$
\begin{aligned}
S_{11}^{\alpha\beta} &= H_1\cos 4\theta + H_2\cos 2\theta + H_3\sin 4\theta + H_4\sin 2\theta + S_{11}^{o} \\
S_{22}^{\alpha\beta} &= H_1\cos 4\theta - H_2\cos 2\theta + H_3\sin 4\theta - H_4\sin 2\theta + S_{22}^{o} \\
S_{12}^{\alpha\beta} &= -H_1\cos 4\theta - H_3\sin 4\theta + S_{12}^{o} \\
S_{66}^{\alpha\beta} &= -4H_1\cos 4\theta - 4H_3\sin 4\theta + S_{66}^{o} \\
S_{16}^{\alpha\beta} &= -2H_1\sin 4\theta - H_2\sin 2\theta + 2H_3\cos 4\theta + H_4\cos 2\theta \\
S_{26}^{\alpha\beta} &= 2H_1\sin 4\theta - H_2\sin 2\theta - 2H_3\cos 4\theta + H_4\cos 2\theta \quad ,
\end{aligned}
\tag{2.1.9}
$$

where

$$
\begin{aligned}
H_1 &= \tfrac{1}{8}(S_{11} + S_{22} - 2S_{12} - S_{66}) \\
H_2 &= \tfrac{1}{2}(S_{11} - S_{22}) \\
H_3 &= \tfrac{1}{4}(S_{16} - S_{26}) \\
H_4 &= \tfrac{1}{2}(S_{16} + S_{26}) \\
S_{11}^{o} = S_{22}^{o} &= \tfrac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66}) \\
S_{12}^{o} &= \tfrac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66}) \\
S_{66}^{o} &= \tfrac{1}{2}(S_{11} + S_{22} - 2S_{12} + S_{66}) = 2(S_{11}^{o} - S_{12}^{o})
\end{aligned}
\tag{2.1.10}
$$

As the material approaches isotropy H_1, H_2, H_3, and H_4 approach zero, as expected. In Appendix I we show that the elements of the isotropic compliance matrix "nearest" to a given anisotropic compliance matrix are in fact the S_{11}^{o}, S_{22}^{o}, S_{12}^{o}, and S_{66}^{o} of (2.1.10). Thus, H_1, H_2, H_3, and H_4 are measures of the anisotropy of the material. If one or more of the H's vanish the material has a special type of anisotropy. If $H_3 = H_4 = 0$ the material is symmetric about the x and y axes, i.e., it is orthotropic. If, in addition $H_2 = 0$, then the material is also symmetric about the rays $\theta = \pi/4, 3\pi/4$. The physical significance of the special type of orthotropy, $H_1 \neq 0$, $H_2 = H_3 = H_4 = 0$, is not apparent.

It is convenient to define "normalized" measures of these four types of anisotropy as

$$\varepsilon_1 = \frac{\sqrt{8}H_1}{D} , \quad \varepsilon_2 = \frac{\sqrt{2}H_2}{D} , \quad \varepsilon_3 = \frac{\sqrt{8}H_3}{D} , \quad \varepsilon_4 = \frac{\sqrt{2}H_4}{D} , \tag{2.1.11}$$

where

$$D = \left[3(S^o_{11})^2 - 2S^o_{11}\, S^o_{12} + 3(S^o_{12})^2\right]^{\frac{1}{2}}$$

Now by substituting the $S^{\alpha\beta}_{ij}$ of (2.1.9) for the S_{ij} of (2.1.10) it can be shown that the S^o_{ij} are independent of the orientation of the coordinate system and are therefore fundamental material invariants characterizing the "nearest" isotropic material. However, the H_i (or ε_i), which are measures of the deviation from this isotropic material, do depend on the coordinate system. It would be desirable to have measures of anisotropy which are independent of the coordinate system. Three such measures are

$$\begin{aligned} \varepsilon_{13} &= \left[\varepsilon_1^2 + \varepsilon_3^2\right]^{\frac{1}{2}} \\ \varepsilon_{24} &= \left[\varepsilon_2^2 + \varepsilon_4^2\right]^{\frac{1}{2}} \\ \varepsilon &= \left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2\right]^{\frac{1}{2}} = \left[\varepsilon_{13}^2 + \varepsilon_{24}^2\right]^{\frac{1}{2}} \end{aligned} \tag{2.1.12}$$

The definitions (2.1.11) and (2.1.12) arise naturally from the analysis of Appendix I. These measures of anisotropy and the concept of the "nearest" elastic compliance matrix $[S^o]$ play an important role in our perturbation analyses for mildly orthotropic materials in Section IV, For more strongly anisotropic materials the ε's are generally less useful as measures of anisotropy. For such materials the character and type of anisotropy is better measured by the ratios of the constant coefficients (which depend on the material) of the generalized biharmonic in rectangular coordinates. The constant coefficients of the two partial differential equations of the displacement formulation could also be used for this purpose. The generalized biharmonic and the partial differential equations of the displacement formulation are developed in Article 2a of this section. In Article 3 of Section III the effect of various types of material orthotropy is illustrated in contour plots of the stresses and displacements for selected sample problems.

The inverse of relations (2.1.9) - (2.1.10) is as follows:

$$
\begin{aligned}
B_{11}^{\alpha\beta} &= \tilde{H}_1\cos 4\theta + \tilde{H}_2\cos 2\theta + \tilde{H}_3\sin 4\theta + \tilde{H}_4\sin 2\theta + B_{11}^{o} \\
B_{22}^{\alpha\beta} &= \tilde{H}_1\cos 4\theta - \tilde{H}_2\cos 2\theta + \tilde{H}_3\sin 4\theta - \tilde{H}_4\sin 2\theta + B_{22}^{o} \\
B_{12}^{\alpha\beta} &= -\tilde{H}_1\cos 4\theta - \tilde{H}_3\sin 4\theta + B_{12}^{o} \\
B_{66}^{\alpha\beta} &= -\tilde{H}_1\cos 4\theta - \tilde{H}_3\sin 4\theta + B_{66}^{o} \\
B_{16}^{\alpha\beta} &= -\tilde{H}_1\sin 4\theta - \frac{\tilde{H}_2}{2}\sin 2\theta + \tilde{H}_3\cos 4\theta + \frac{\tilde{H}_4}{2}\cos 2\theta \\
B_{26}^{\alpha\beta} &= \tilde{H}_1\sin 4\theta - \frac{\tilde{H}_2}{2}\sin 2\theta - \tilde{H}_3\cos 4\theta + \frac{\tilde{H}_4}{2}\cos 2\theta \quad ,
\end{aligned}
\tag{2.1.13}
$$

where

$$
\begin{aligned}
\tilde{H}_1 &= \tfrac{1}{8}(B_{11} + B_{22} - 2B_{12} - 4B_{66}) \\
\tilde{H}_2 &= \tfrac{1}{2}(B_{11} - B_{22}) \\
\tilde{H}_3 &= \tfrac{1}{2}(B_{16} - B_{26}) \\
\tilde{H}_4 &= (B_{12} + B_{26}) \\
B_{11}^{o} = B_{22}^{o} &= \tfrac{1}{8}(3B_{11} + 3B_{22} + 2B_{12} + 4B_{66}) \\
B_{12}^{o} &= \tfrac{1}{8}(B_{11} + B_{22} + 6B_{12} - 4B_{66}) \\
B_{66}^{o} &= \tfrac{1}{8}(B_{11} + B_{22} - 2B_{12} + 4B_{66}) = \tfrac{1}{2}(B_{11}^{o} - B_{12}^{o})
\end{aligned}
\tag{2.1.14}
$$

We note that the isotropic material corresponding to $[B^{o}]$ is not in general the same as that corresponding to $[S^{o}]$. In other words the matrix product $[B^{o}][S^{o}]$ does not reduce to the identity matrix. Likewise, the special anisotropy corresponding to the vanishing of a particular H_i is generally different from that resulting from the vanishing of $\tilde{H}_i$ (for the same i) in (2.1.10). We can define measures of orthotropy $\tilde{\varepsilon}$, $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, $\tilde{\varepsilon}_3$, $\tilde{\varepsilon}_4$, $\tilde{\varepsilon}_{13}$ and $\tilde{\varepsilon}_{24}$ just as in (2.1.11) and (2.1.12) with the H's replaced by $\tilde{H}$'s and the S_{ij}^{o} replaced by B_{ij}^{o}.

1c. Specialization to Orthotropic Materials

From this point on we confine our attention to orthotropic materials (elastic symmetry about x-y, y-z, and z-x planes). For this case (2.1.3) becomes

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} , \tag{2.1.15}$$

i.e., for the orthotropic plane stress (or strain) problem there are four independent elastic constants. The elements of the compliance matrix in (2.1.15) can be written in terms of the Young's moduli, the shear modulus, and the Poisson's ratios as follows:

$$\begin{aligned} S_{11} &= \frac{1}{E_x} , \quad S_{12} = \frac{-\nu_{xy}}{E_x} = \frac{-\nu_{yx}}{E_y} , \\ S_{22} &= \frac{1}{E_y} , \quad S_{33} = \frac{1}{G_{xy}} \end{aligned} \tag{2.1.16}$$

Although there are four independent elastic constants for planar problems involving orthotropic materials, these constants are not completely arbitrary. There is the condition of thermodynamic admissibility, i.e., the positive definiteness of the stiffness and compliance matrices, that must be satisfied. Lempriere [2] has considered this question for orthotropic materials in three dimensions. For plane stress problems the relevant conditions are

$$\begin{aligned} &E_x, E_y, G_{xy} > 0 \\ &|\nu_{xy}| < (E_x/E_y)^{\frac{1}{2}} , \quad |\nu_{yx}| < (E_y/E_x)^{\frac{1}{2}} \end{aligned} \tag{2.1.17}$$

The inverse of (2.1.15) is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.1.18}$$

where

$$B_{11} = \frac{S_{22}}{(S_{11}\, S_{22} - S_{12}^2)}$$

$$B_{12} = \frac{-S_{12}}{(S_{11}\, S_{22} - S_{12}^2)} \tag{2.1.19}$$

$$B_{22} = \frac{S_{11}}{(S_{11}\, S_{22} - S_{12}^2)}$$

$$B_{66} = \frac{1}{S_{66}}$$

These are the B_{ij} that are to be used in (2.1.14) for orthotropic materials.

2. EQUILIBRIUM, STRAIN-DISPLACEMENT, AND RELATED EQUATIONS

In this article we list for planar problems in rectangular coordinates the equilibrium and strain-displacement relations, and also stress and displacement boundary conditions. Using these results and the stress-strain relationship for orthotropic materials we formulate the problem entirely in terms of stress variables and also in terms of displacement variables. We then develop the appropriate transformations of these equations to polar coordinates.

2a. Equations in Rectangular Coordinates

In the general three-dimensional theory of elasticity there are fifteen basic governing differential equations: three equilibrium equations, six strain-displacement equations, and six stress-strain equations. For planar problems only eight of these are relevant. In addition to the stress-strain equations already

introduced there are two equilibrium equations,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad , \tag{2.2.1}$$

and three strain-displacement relations,

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned} \tag{2.2.2}$$

In order to complete the formulation boundary conditions must be specified. Stress boundary conditions can be expressed in the following form:

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{Bmatrix} \cos(x,n) \\ \cos(y,n) \end{Bmatrix} \quad , \tag{2.2.3}$$

where X and Y are the applied stresses in the x and y directions and n is the external normal. Displacement boundary conditions are simply

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} \tag{2.2.4}$$

where bars denote prescribed quantities.

It is often convenient to formulate the problem entirely in terms of stresses or displacements. Using (2.2.2) and the stress-strain relationships (2.1.18) we can write the equilibrium equations in terms of the displacements.

$$B_{11}\frac{\partial^2 u}{\partial x^2} + B_{66}\frac{\partial^2 u}{\partial y^2} + (B_{12} + B_{66})\frac{\partial^2 v}{\partial x \partial y} = 0$$

$$B_{22}\frac{\partial^2 v}{\partial y^2} + B_{66}\frac{\partial^2 v}{\partial x^2} + (B_{12} + B_{66})\frac{\partial^2 u}{\partial x \partial y} = 0 \qquad (2.2.5)$$

If stresses are prescribed on the boundary then (2.2.3) must be expressed in terms of displacements using the stress-strain and strain-displacement relations. (2.2.5) and the boundary conditions constitute a complete statement of the problem.

In order to formulate the problem in terms of stresses we first obtain the compatibility equation by eliminating the displacements from the strain-displacement relations (2.2.2). The result is

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad (2.2.6)$$

Now using the stress-strain relations (2.1.15) we can rewrite (2.2.6) in terms of stresses.

$$S_{22}\frac{\partial^2 \sigma_y}{\partial x^2} + S_{12}\left(\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial x^2}\right) - S_{66}\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + S_{11}\frac{\partial^2 \sigma_x}{\partial y^2} = 0 \qquad (2.2.7)$$

(2.2.7) along with the two equilibrium equations (2.2.1) and the boundary conditions expressed in terms of stresses form a complete statement of the problem. It is customary to satisfy the equilibrium equations identically by the introduction of a stress function Ψ as follows:

$$\sigma_x = \frac{\partial^2 \Psi}{\partial y^2}$$

$$\sigma_y = \frac{\partial^2 \Psi}{\partial x^2} \qquad (2.2.8)$$

$$\tau_{xy} = -\frac{\partial^2 \Psi}{\partial x \partial y}$$

Using (2.2.8) in (2.2.7) we obtain

$$S_{22}\frac{\partial^4\Psi}{\partial x^4} + (2S_{12} + S_{66})\frac{\partial^4\Psi}{\partial x^2\partial y^2} + S_{11}\frac{\partial^4\Psi}{\partial y^4} = 0 \qquad (2.2.9)$$

In the isotropic case (2.2.9) reduces to the biharmonic equation, $\nabla^4\Psi = 0$. We refer to (2.2.9) as the generalized biharmonic. If stresses are prescribed on the boundary, (2.2.8) can be substituted into (2.2.3) in order to express the boundary conditions in terms of derivatives of Ψ. If displacements are prescribed the boundary conditions would in general involve both derivatives and integrals of Ψ (resulting from the integration of the stress-displacement relations). Such boundary conditions are usually awkward to handle when using approximate methods. However, in Article 3 we show that by using a variational formulation in terms of stress variables (complementary energy principle) these difficulties in treating displacement boundary conditions are eliminated. Also, in Section III it is shown that in order for the stress function to satisfy the generalized biharmonic exactly it must have a certain form. Knowledge of this form permits a reformulation of displacement boundary conditions in terms of certain auxiliary variables. In this new form displacement boundary conditions are as easily handled as stress boundary conditions.

Another way of formulating the problem is to eliminate the strains from the stress-strain relationship (2.1.18) using the strain-displacement equations (2.2.2). This result along with the equilibrium equations (2.2.1) and the boundary conditions (2.2.3) and (2.2.4) constitute the combined stress-displacement formulation. The combined formulation has the disadvantage of containing more unknowns than either the stress or displacement formulations. However, it does have the advantage that stress and displacement variables are obtained to the same degree of accuracy. Also, the boundary conditions are always expressible in a simple form.

2b. Equations in Polar Coordinates

Inasmuch as the same problems for this study are most conveniently formulated in polar coordinates it is essential that we obtain the polar version of the equations of Article 2a. The equilibrium equations in the r and θ directions are

$$\begin{aligned}\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{r} &= 0\\ \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} &= 0\end{aligned} \qquad (2.2.10)$$

The strain-displacement relations are

$$\begin{aligned} \varepsilon_r &= \frac{\partial u}{\partial r} \\ \varepsilon_\theta &= \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad , \end{aligned} \tag{2.2.11}$$

where u and v are the displacements in the radial and circumferential directions respectively. In situations where there is possibility of confusion between this notation and that for rectangular coordinates we will append subscripts r and θ. The boundary conditions have the same form as for rectangular coordinates (2.2.3) and (2.2.4) with x and y replaced by r and θ.

In order to formulate the problem entirely in terms of displacements we proceed just as in Article 2a. We write the stresses in terms of the displacements using (2.2.11) and (2.1.13). (Recall that $\tilde{H}_3 = \tilde{H}_4 = 0$ for orthotropic materials.) We then substitute these expressions for the stresses into the equilibrium equations (2.2.10). The result is

$$\begin{aligned} &(B_{11}^o + \tilde{H}_2\cos 2\theta + \tilde{H}_1\cos 4\theta)\frac{\partial^2 u}{\partial r^2} + (B_{11}^o - 3\tilde{H}_1\cos 4\theta)\frac{\partial}{\partial r}\left(\frac{u}{r}\right) \\ &+ (B_{66}^o - \tilde{H}_1\cos 4\theta)\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - (\tilde{H}_2\sin 2\theta + 2\tilde{H}_1\sin 4\theta)\frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta} \\ &+ (4\tilde{H}_1\sin 4\theta)\frac{1}{r^2}\frac{\partial u}{\partial \theta} - \left(\frac{\tilde{H}_2}{2}\sin 2\theta + \tilde{H}_1\sin 4\theta\right)\frac{\partial^2 v}{\partial r^2} \\ &+ \left(\frac{\tilde{H}_2}{2}\sin 2\theta + 3\tilde{H}_1\sin 4\theta\right)\frac{\partial}{\partial r}\left(\frac{v}{r}\right) - \left(\frac{\tilde{H}_2}{2}\sin 2\theta - \tilde{H}_1\sin 4\theta\right)\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \\ &+ (B_{12}^o + B_{66}^o - 2\tilde{H}_1\cos 4\theta)\frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} - (B_{11}^o + B_{66}^o - 4\tilde{H}_1\cos 4\theta)\frac{1}{r^2}\frac{\partial v}{\partial \theta} = 0 \end{aligned}$$

$$(B_{11}^{o} - \tilde{H}_2\cos 2\theta + \tilde{H}_1\cos 4\theta)\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + (B_{66}^{o} - \tilde{H}_2\cos 2\theta + 3\tilde{H}_1\cos 4\theta)\frac{\partial}{\partial r}\left(\frac{v}{r}\right)$$

$$+ (B_{66}^{o} - \tilde{H}_1\cos 4\theta)\frac{\partial^2 v}{\partial r^2} - (\tilde{H}_2\sin 2\theta - 2\tilde{H}_1\sin 4\theta)\frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta}$$

$$+ (2\tilde{H}_2\sin 2\theta - 4\tilde{H}_1\sin 4\theta)\frac{1}{r^2}\frac{\partial v}{\partial\theta} - \left(\frac{\tilde{H}_2}{2}\sin 2\theta - \tilde{H}_1\sin 4\theta\right)\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$- (\tfrac{3}{2}\tilde{H}_2\sin 2\theta - 3\tilde{H}_1\sin 4\theta)\frac{\partial}{\partial r}\left(\frac{u}{r}\right) - \left(\frac{\tilde{H}_2}{2}\sin 2\theta + \tilde{H}_1\sin 4\theta\right)\frac{\partial^2 u}{\partial r^2}$$

$$+ (B_{12}^{o} + B_{66}^{o} - 2\tilde{H}_1\cos 4\theta)\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}$$

$$+ (B_{11}^{o} + B_{66}^{o} - 2\tilde{H}_2\cos 2\theta + 4\tilde{H}_1\cos 4\theta)\frac{1}{r^2}\frac{\partial u}{\partial\theta} = 0 \qquad (2.2.12)$$

For isotropic materials $\tilde{H}_1 = \tilde{H}_2 = 0$, and (2.2.12) simplifies considerably.

We now formulate the problem in terms of stress variables. In polar coordinates the compatibility equation is

$$r\left[\frac{\partial^2}{\partial r^2}(r\varepsilon_\theta) - \frac{\partial\varepsilon_r}{\partial r}\right] = \frac{\partial^2}{\partial r\partial\theta}(r\gamma_{r\theta}) - \frac{\partial^2\varepsilon_r}{\partial\theta^2} \qquad (2.2.13)$$

To obtain the generalized biharmonic in polar coordinates from (2.2.13) we would first express the strains in terms of the stresses using (2.1.9), and then express the stresses in terms of the stress function Ψ by using the relations,

$$\sigma_r = \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\Psi}{\partial r^2} \qquad (2.2.14)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)$$

Alternatively we could start with the generalized biharmonic in rectangular coordinates (2.2.9) and transform derivatives with respect to x and y to derivatives with respect to r and θ. In any case the result is[1]

$$
\begin{aligned}
&(S^{o}_{11} - H_2\cos 2\theta + H_1\cos 4\theta)\frac{\partial^4\psi}{\partial r^4} \\
&+ (2S^{o}_{11} - 6H_1\cos 4\theta)\left(\frac{1}{r}\frac{\partial^3\psi}{\partial r^3} + \frac{1}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2}\right) \\
&- (S^{o}_{11} - 3H_2\cos 2\theta - 15H_1\cos 4\theta)\left(\frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3}\frac{\partial\psi}{\partial r} + \frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2}\right) \\
&+ (4S^{o}_{11} - 8H_2\cos 2\theta - 44H_1\cos 4\theta)\frac{1}{r^4}\frac{\partial^2\psi}{\partial\theta^2} \\
&+ (S^{o}_{11} + H_2\cos 2\theta + H_1\cos 4\theta)\frac{1}{r^4}\frac{\partial^4\psi}{\partial\theta^4} \\
&+ (2H_2\sin 2\theta - 4H_1\sin 4\theta)\frac{1}{r}\frac{\partial^4\psi}{\partial r^3\partial\theta} \\
&+ (2H_2\sin 2\theta + 4H_1\sin 4\theta)\left(\frac{1}{r^3}\frac{\partial^4\psi}{\partial r\partial\theta^3} - \frac{3}{r^4}\frac{\partial^3\psi}{\partial\theta^3}\right) \\
&- (4H_2\sin 2\theta + 56H_1\sin 4\theta)\frac{1}{r^3}\frac{\partial^2\psi}{\partial r\partial\theta} \\
&+ (24H_1\sin 4\theta)\left(\frac{1}{r^2}\frac{\partial^3\psi}{\partial r^2\partial\theta} + \frac{2}{r^4}\frac{\partial\psi}{\partial\theta}\right) = 0
\end{aligned}
\qquad (2.2.15)
$$

(2.2.15) is the generalized biharmonic in polar coordinates. For the isotropic case it simplifies considerably.

[1] The derivation of this equation was supplied by James Rudd of the Flight Dynamics Laboratory at WPAFB.

We are now in a position to make certain observations about the nature of the response to various types of boundary loadings (including imposed displacements) for the sample problems of this study. The geometry of these sample problems is either an annulus or an infinite sheet with a circular hole. Since both the material and geometry are symmetric about the x and y axes it is clear that boundary loadings which are symmetric about the x (y) axis produce responses for which u, σ_r, σ_θ, Ψ are symmetric and v, $\tau_{r\theta}$ are antisymmetric about the x (y) axis.

Under fairly mild restrictions the applied boundary loadings can be expressed as a Fourier series. Since the problem is linear we can consider the effect of each term of the Fourier series separately, and then superpose these individual effects to obtain the complete solution. As an example let us consider a problem for which Ψ is symmetric about the x-axis. For a fixed value of $r = r_F$ (not necessarily on the boundary) we can write

$$\Psi(r_F,\theta) = \sum_{j=0}^{\infty} a_j \cos j\theta \qquad (2.2.16)$$

If the exact solution were known then the Fourier coefficients a_j could be determined in the usual manner by integrating numerically (or analytically if possible). Since (2.2.16) holds for any r_F (with the a_j depending on the particular r_F of course) we can write in general

$$\Psi(r,\theta) = \sum_{j=0}^{\infty} a_j(r) \cos j\theta \qquad (2.2.17)$$

A possible approach to obtaining a solution to the generalized biharmonic (2.2.15) for loading symmetric about the x-axis is to assume a solution of the form (2.2.17). Then substituting this assumed form into (2.2.15) and making use of trigonometric identities such as

$$\begin{aligned} \cos j\theta \cos 4\theta &= \tfrac{1}{2}\left[\cos(j+4)\theta + \cos(j-4)\theta\right] \\ \sin j\theta \sin 4\theta &= -\tfrac{1}{2}\left[\cos(j+4)\theta - \cos(j-4)\theta\right], \end{aligned} \qquad (2.2.18)$$

all θ-terms can be expressed as an infinite sum of cosine harmonics. Equating to zero the coefficient of each of these cosine harmonics yields an infinite system of coupled ordinary differential equations for the $a_j(r)$. Assuming that the stress or displacement boundary conditions have been expressed in terms of Ψ, the values of the a_j on the boundary are known and are the boundary conditions for the system of ordinary differential equations. In order to obtain approximate solutions by this technique it is of course necessary to truncate the infinite system and to solve the resulting finite system in some manner, probably numerically. We do not pursue this approach in the present study. We can, however, make certain important observations from the preceding considerations:

1) Because of the nature of the θ-coefficients in (2.2.15) it can be seen that in the general problem, assuming solutions of the form,

$$\Psi(r,\theta) = \sum_{j=0}^{\infty} a_j(r)\cos j\theta + \sum_{j=1}^{\infty} b_j(r)\sin j\theta \quad ,$$

the infinite system of ordinary differential equations for the $a_j(r)$ would be uncoupled from that for the $b_j(r)$.

2) Further, it is seen that because the coefficients in (2.2.15) contain only even harmonics, the problem for the $a_j(r)$ and $b_j(r)$ with j even uncouples from the problem with j odd. Thus, in the general case we have four separate infinite systems of ordinary differential equations to solve. They are the systems corresponding to

   i) even cosine harmonics

   ii) odd cosine harmonics

   iii) even sine harmonics

   iv) odd sine harmonics

3) In the general anisotropic problem a typical θ-coefficient in the generalization of (2.2.15) would contain both even sine and cosine harmonics. Because of this the infinite system corresponding to the even (odd) cosine harmonics is no longer uncoupled from that corresponding to the even (odd) sine harmonics. However, the problem for the even harmonics is still uncoupled from that for the odd harmonics.

In this discussion of the nature of the response to various boundary loadings we have made the generalized biharmonic (2.2.15) the basis of our arguments. Inasmuch as the nature of the Θ-coefficients in the equations of the displacement formulation (2.2.12) are similar to those of the generalized biharmonic, we could have arrived at the same conclusions from (2.2.12).

The preceding considerations will be useful in selecting families of functions to use in our approximate methods. This is especially true for Ritz-type approximations used in connection with the variational formulations discussed in the following article.

3. VARIATIONAL FORMULATIONS

In this article we develop the variational versions of the stress, displacement, and combined stress-displacement formulations of Article 2. These variational formulations are the complementary, potential, and Reissner energy principles respectively. In addition we show that by imposing certain restrictions on the admissible functions the Reissner energy formulation specializes to the potential or complementary energy formulation. Finally we show how the Reissner energy formulation can be reduced to a variational theorem involving only integrals over the boundary of the region.

3a. Potential, Complementary, and Reissner Energy Principles

There are two scalar point functions which play a central role in the variational formulation of the equations of elasticity. One is the strain energy density $U(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy})$, which is defined such that

$$\sigma_x = \frac{\partial U}{\partial \varepsilon_x}, \quad \sigma_y = \frac{\partial U}{\partial \varepsilon_y}, \quad \ldots\ldots\ldots, \quad \tau_{xy} = \frac{\partial U}{\partial \gamma_{xy}} \qquad (2.3.1)$$

The other is the complementary energy density $W(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy})$ defined such that

$$\varepsilon_x = \frac{\partial W}{\partial \sigma_x}, \quad \varepsilon_y = \frac{\partial W}{\partial \sigma_y}, \quad \ldots\ldots\ldots, \quad \gamma_{xy} = \frac{\partial W}{\partial \tau_{xy}} \qquad (2.3.2)$$

For materials which obey a linear stress strain relationship we have for plane stress problems

$$U = \frac{1}{2} \left\{\varepsilon_{xy}\right\}^T \left[B^{xy}\right] \left\{\varepsilon_{xy}\right\}$$
$$W = \frac{1}{2} \left\{\sigma_{xy}\right\}^T \left[S^{xy}\right] \left\{\sigma_{xy}\right\} \qquad (2.3.3)$$

If the components of stress and strain in x-y coordinates are expressed in terms of the components of stress and strain in an α-β system (see Figure 1) then after some algebra (2.3.3) can be expressed in the form

$$U = \frac{1}{2} \left\{\varepsilon_{\alpha\beta}\right\}^T \left[B^{\alpha\beta}\right] \left\{\varepsilon_{\alpha\beta}\right\}$$
$$W = \frac{1}{2} \left\{\sigma_{\alpha\beta}\right\}^T \left[S^{\alpha\beta}\right] \left\{\sigma_{\alpha\beta}\right\} \qquad (2.3.4)$$

and the relations (2.3.1) and (2.3.2) continue to hold with x and y replaced by α and β respectively. For orthotropic materials there are some zero elements in the elastic matrices of (2.3.3) but not in (2.3.4). In addition the elements of the elastic matrices of (2.3.4) (given in (2.1.9) and (2.1.13)) are more involved than those of (2.3.3). Also, when $\{\varepsilon_{\alpha\beta}\}$ is expressed in terms of displacements using (2.2.11) and when $\{\sigma_{\alpha\beta}\}$ is expressed in terms of a stress function using (2.2.14) these vectors are not as simple as their x-y counterparts. Thus, it is clear that the expanded forms of (2.3.4) are much more complicated than for (2.3.3).

For planar problems, assuming body forces are zero, the potential energy functional Φ_P can be written

$$\Phi_P = \int_A U \, dA - \int_{C_s} \vec{\vec{F}} \cdot \vec{u} \, ds \quad , \qquad (2.3.5)$$

where C_s is the part of the boundary on which stresses (as opposed to displacements) are prescribed. $\vec{\vec{F}}$ is the applied stress vector and $\vec{u}$ the resulting displacement vector. In (2.3.5) U is to be expressed in terms of displacement variables by using strain-displacement relations such as (2.2.2) or (2.2.11). In (2.3.5) we have really written the potential energy functional

per unit plate thickness in the z-direction. To obtain the total potential energy it is necessary to multiply by the thickness, but this has no effect on the results, so we shall hereafter consider the plate to be of unit thickness so that (2.3.5) is the total potential energy. The principle of minimum potential energy for the small displacement theory of elasticity can be stated as follows:

> Of all displacement states which satisfy the prescribed displacement conditions on C_d, that which minimizes Φ_P satisfies the equilibrium conditions and is the actual displacement state.

In other words, the stationary conditions of the potential energy functional (2.3.5) are the equilibrium equations expressed in terms of displacement variables, e.g., (2.2.5) or (2.2.12), and the boundary conditions on C_s, e.g., (2.2.3) expressed in terms of displacement variables.

For planar problems, assuming body forces are zero, the complementary energy functional Φ_C can be written

$$\Phi_C = -\int_A W \, dA + \int_{C_d} \vec{F} \cdot \vec{\bar{u}} \, ds \qquad (2.3.6)$$

where C_d is the part of the boundary on which displacements are prescribed. $\vec{\bar{u}}$ is the imposed displacement vector and $\vec{F}$ the vector of the resulting boundary stresses. The principle of maximum complementary energy can be stated as follows:

> Of all stress states which satisfy the prescribed stress conditions on C_s and internal equilibrium, that which maximizes Φ_C satisfies the compatibility conditions and is the actual stress state.

A convenient way to ensure that all stress states satisfy internal equilibrium is to express the stresses in terms of a stress function Ψ as in (2.2.8) or (2.2.14). When this is done the stationary conditions of the complementary energy functional (2.3.6) are the compatibility equation expressed in terms of Ψ, e.g., (2.2.9) or (2.2.15), and the boundary conditions on C_d, e.g., (2.2.4) expressed in terms of Ψ. As discussed at the end of Article 2a of this section such boundary conditions will generally involve integrals of Ψ, which are inconvenient when

obtaining approximate solutions from the differential equation point of view. However, when approximate solutions are obtained via the Ritz method, i.e., by maximizing the complementary energy functional over a given family of stress functions (which need only satisfy stress boundary conditions), the whole question of displacement boundary conditions is avoided. The displacement conditions are approximately satisfied in the maximization process, just as the generalized biharmonic is approximately satisfied.

There is, of course, a similar advantage in obtaining an approximate solution for the displacement state through the use of the Ritz technique in connection with the potential energy functional. By so doing one is able to avoid dealing with stress boundary conditions. However, since it is always possible to express these stress boundary conditions in terms of the displacements and their derivatives, the advantage of the Ritz method over a numerical solution (such as finite differences) of the differential equation formulation may not be as decisive.

For planar problems, assuming body forces are zero, the Reissner[2] energy functional Φ_R can be written

$$\Phi_R = \int_A \left[\{\sigma\}^T \{\varepsilon\} - W \right] dA - \int_{C_s} \vec{\bar{F}} \bullet \vec{u} \, ds - \int_{C_d} (\vec{u} - \vec{\bar{u}}) \bullet \vec{F} \, ds \quad , \tag{2.3.7}$$

where $\{\sigma\}$ and $\{\varepsilon\}$ are column vectors as, for example, in (2.3.3) and (2.3.4). The quantities in the line integrals have

[2] Reissner readily acknowledges the priority of Hellinger [3], who apparently was the first to enunciate the energy theorem associated with Reissner's name. It has also been observed that from the standpoint of the calculus of variations this energy theorem is just a statement of the canonical form of the problem, Courant-Hilbert [4], pp. 238-40. Accordingly, Reissner has proposed that no one's name be attached to the theorem and that it be referred to as "energy theorem for stresses and displacements". Some workers have adopted the designation "Reissner-Hellinger theorem". However, in view of the growing importance of this theorem in applications, largely due to the impetus given it by Reissner, it does not seem inappropriate to retain the designation "Reissner energy."

appeared before in either (2.3.5) or (2.3.6). The elements of $\{\varepsilon\}$ are to be expressed in terms of the displacements using the strain-displacement relationships. Having done this it is possible to use the divergence theorem to transform the integral involving $\{\sigma\}^T\{\varepsilon\}$ to give an alternate form of the Reissner energy functional. It is

$$\Phi_R = \int_A \left[- \vec{u} \bullet D\{\sigma\} - W \right] dA + \int_{C_s} (\vec{\bar{F}} - \vec{F}) \bullet \vec{u}\, ds + \int_{C_d} \vec{\bar{F}} \bullet \vec{u}\, ds \quad , \tag{2.3.8}$$

where the vector $\overrightarrow{D\{\sigma\}}$ represents the equilibrium equations in the two coordinate directions. For example, in rectangular coordinates

$$\overrightarrow{D\{\sigma\}} = \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) \hat{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) \hat{j} \tag{2.3.9}$$

The Reissner energy principle can be stated as follows:

> Of all stress and displacement states, that which renders Φ_R stationary satisfies the equilibrium and stress-displacement relations and is the actual state of stress and displacement.

In other words, the Euler equations of Φ_R are the equilibrium and stress-displacement relations, and all boundary conditions are natural boundary conditions. It is noteworthy that no restrictions (other than suitable continuity) are placed on the stress and displacement states in the statement of the Reissner energy principle. This is a definite advantage of the Reissner formulation when being used in Ritz fashion to obtain approximate solutions.

Reissner [5] and [6] has established several interesting and useful properties of the Reissner energy principle. If, in the statement of the principle, the stress states satisfy internal equilibrium and the stress boundary conditions, then the Reissner principle is equivalent to the principle of maximum complementary energy. This is easily seen from the form of Φ_R given in (2.3.8). Since internal equilibrium is satisfied the first term of the area integral vanishes, and since boundary conditions are satisfied the integral over C_s vanishes, and the remaining terms of Φ_R are identical to those of Φ_C in (2.3.6).

On the other hand if, in the statement of the Reissner energy principle, the stress and displacement states are such that the stress-displacement relations are satisfied internally and the displacement boundary conditions are satisfied, then the Reissner principle is equivalent to the principle of minimum potential energy. In order to show the equivalence it is necessary to express the stresses in terms of the displacements. (This is possible since the stress and displacement states are no longer independent.) Having done this the Reissner energy functional of (2.3.7) can be expressed entirely in terms of displacements and is identical to the potential energy functional of (2.3.5).

From the preceding considerations it is clear that for the exact solution, $\Phi_P = \Phi_C = \Phi_R$. Since Φ_P is a minimum, and Φ_C a maximum for the exact solution, $\tilde{\Phi}_P$ and $\tilde{\Phi}_C$ (computed from approximate solutions) will satisfy the inequality,

$$\tilde{\Phi}_C \leq \Phi_R \leq \tilde{\Phi}_P \tag{2.3.10}$$

Now the exact value of Φ_R is usually unknown so the value of $\tilde{\Phi}_C$ alone, or $\tilde{\Phi}_P$ alone, does not yield any information as to the accuracy of the approximate solution. However, the nearness of $\tilde{\Phi}_C$ to $\tilde{\Phi}_P$ does provide some measure of the goodness of the approximation. Such ideas play a central role in the method of the hypercircle. See Courant-Hilbert [4], pp. 252-257, or Synge [7] for details.

Another special case of the Reissner energy principle can be obtained by requiring that the stress and displacement states satisfy the equilibrium and stress-displacement relations in the interior of the body, but not necessarily any of the boundary conditions. In order to discover the implications of this we introduce a third form of the Reissner energy functional obtained by adding the forms (2.3.7) and (2.3.8) and then dividing by two. The result is

$$\Phi_R = \int_A \left[\frac{1}{2} \{\sigma\}^T \{\varepsilon\} - W - \frac{1}{2} \vec{u} \cdot D \overrightarrow{\{\sigma\}} \right] dA$$

$$+ \frac{1}{2} \int_{C_s} (\vec{F} - 2\vec{\bar{F}}) \cdot \vec{u} \, ds - \frac{1}{2} \int_{C_d} (\vec{u} - 2\vec{\bar{u}}) \cdot \vec{F} \, ds \tag{2.3.11}$$

Since equilibrium is satisfied internally the last term of the surface integral vanishes. Since the stress-displacement relations are satisfied we can use (2.3.2) to rewrite the first two terms of the surface integral as follows:

$$\frac{1}{2}\left\{\sigma\right\}^{T}\left\{\varepsilon\right\} - W = \frac{1}{2}\left\{\sigma\right\}^{T}\left\{\frac{\partial W}{\partial \sigma}\right\} - W \qquad (2.3.12)$$

For the materials we are considering (linear-stress strain relationship) W is a homogeneous function of degree 2, which means that (2.3.12) vanishes, so the Reissner energy principle involves only the line integrals of (2.3.11). It can be shown that the stationary conditions of Φ_R in the form (2.3.11) with the previously mentioned restrictions on the stress and displacement states are in fact the stress and displacement boundary conditions. We defer this proof until Section V where the use of this boundary principle in Ritz fashion is discussed in detail.

SECTION III

EXACT SOLUTIONS

1. GENERAL THEORY

In this article we discuss the essential features of a method for determining exact solutions of plane orthotropic elasticity problems. This method utilizes the theory of several complex variables to obtain general solutions of the generalized biharmonic. These general solutions are then used to obtain exact solutions for special problems, and they will also be needed in the approximate techniques of Section V where it is necessary to use families of functions which satisfy the field equations exactly.

1a. Complex Variable Solution of Generalized Biharmonic

The theory that is presented in this Section is for orthotropic materials only. However, the ideas are readily extended to the general plane anisotropic elasticity problem. Full details are given in Lekhnitskii [8] and [9] and Savin[10]. We have already obtained the generalized biharmonic for orthotropic materials in Article 2a of Section II. Repeated here it is

$$S_{22}\frac{\partial^4\Psi}{\partial x^4} + (2S_{12} + S_{66})\frac{\partial^4\Psi}{\partial x^2\partial y^2} + S_{11}\frac{\partial^4\Psi}{\partial y^4} = 0 \qquad (3.1.1)$$

The general solution of (3.1.1) can be written

$$\Psi = \sum_{i=1}^{4} \Psi_i(x + \mu_i y) \quad , \qquad (3.1.2)$$

where each of the Ψ_i is an arbitrary analytic function. The μ_i are solutions of

$$S_{11}\mu^4 + (2S_{12} + S_{66})\mu^2 + S_{22} = 0 \qquad (3.1.3)$$

That is,

$$\mu = \pm\left[\frac{-(2S_{12} + S_{66}) \pm \sqrt{(2S_{12} + S_{66})^2 - 4S_{11}S_{22}}}{2S_{11}}\right]^{\frac{1}{2}} \qquad (3.1.4)$$

The four roots are obtained from the four different combinations of the two ± signs in (3.1.4). It will be demonstrated shortly that it is not necessary to consider cases for which there are double roots or real roots. Purely imaginary roots are of the form,

$$\mu_1 = ai,\ \mu_2 = bi,\ \mu_3 = -ai,\ \mu_4 = -bi, \tag{3.1.5}$$

while complex roots have the form,

$$\mu_1 = a + bi,\ \mu_2 = -a + bi,\ \mu_3 = a - bi,\ \mu_4 = -a - bi, \tag{3.1.6}$$

where a and b are, by convention, positive. Thus, in either case, $\mu_3 = \bar{\mu}_1$ and $\mu_4 = \bar{\mu}_2$. We then note that the equation

$$(\mu - \mu_1)(\mu - \bar{\mu}_1)(\mu - \mu_2)(\mu - \bar{\mu}_2) = 0 \tag{3.1.7}$$

is equivalent to (3.1.3). Now if the roots are purely imaginary $\bar{\mu}_1 = -\mu_1$ and $\bar{\mu}_2 = -\mu_2$. If they are complex $\bar{u}_1 = -\mu_2$ and $\bar{\mu}_2 = -\mu_1$. In either case (3.1.7) becomes

$$(\mu^2 - \mu_1^2)(\mu^2 - \mu_2^2) = 0 \tag{3.1.8}$$

By comparing coefficients in (3.1.8) and (3.1.3) we conclude

$$\begin{aligned} \mu_1^2 + \mu_2^2 &= -(2S_{12} + S_{66})/S_{11} \\ \mu_1^2\,\mu_2^2 &= S_{22}/S_{11} \end{aligned} \tag{3.1.9}$$

We will find relations (3.1.8) and (3.1.9) useful later in this article.

We note that if $(2S_{12} + S_{66})^2 = 4S_{11}\,S_{22}$ in (3.1.4) then there are only two distinct roots, so the general solution (3.1.2) is no longer valid. However, in this case (3.1.1) can be rewritten

$$\left(\sqrt{S_{22}}\,\frac{\partial^2}{\partial x^2} + \sqrt{S_{11}}\,\frac{\partial^2}{\partial y^2}\right)^2 \Psi = 0 \tag{3.1.10}$$

By a simple stretching transformation of the x and/or y coordinates (3.1.10) can be reduced to the biharmonic, $\nabla^4 \psi = 0$. Thus, when the generalized biharmonic can be written in the form (3.1.10) the problem is equivalent to an isotropic problem (for the stretched version of the region of the original problem). For this reason we exclude this special case from our considerations in the rest of this report.

We shall now demonstrate the impossibility of real values of μ. In order to do this we use the conditions of thermodynamic admissibility listed in (2.1.17). A necessary condition for μ to be real is that

$\sqrt{(2S_{12} + S_{66})^2 - 4S_{11} S_{22}}$ be real in (3.1.4), i.e.,

$$4S_{12}^2 + 4S_{12} S_{66} + S_{66}^2 - 4S_{11} S_{22} \geq 0 \qquad (3.1.11)$$

A further necessary condition is that $-(2S_{12} + S_{66}) \geq 0$, i.e.,

$$2S_{12} \leq -S_{66} \qquad (3.1.12)$$

According to (2.1.16) and (2.1.17) S_{66} is always positive, so (3.1.12) implies that S_{12} is negative. Also, from (2.1.16) and (2.1.17) we obtain

$$S_{12}^2 < S_{11} S_{22} \qquad (3.1.13)$$

Using this in (3.1.11) we obtain

$$4S_{12} S_{66} + S_{66}^2 > 0 \text{ , or}$$

$$4S_{12} > -S_{66} \qquad (3.1.14)$$

Clearly (3.1.14) contradicts (3.1.12) thus proving the impossibility of real values of μ.

We now introduce the complex variables

$$z_1 = x + \mu_1 y \quad , \quad z_2 = x + \mu_2 y \tag{3.1.15}$$

Since the stress function is real we must have

$\Psi_3 = \bar{\Psi}_1(\bar{z}_1)$ and $\Psi_4 = \bar{\Psi}_2(\bar{z}_2)$ in (3.1.2), where bars denote complex conjugates. (3.1.2) can then be written

$$\Psi = 2\mathrm{Re}\left[\Psi_1(z_1) + \Psi_2(z_2)\right] \tag{3.1.16}$$

Next we introduce

$$\begin{aligned} \Phi_1(z_1) &= \frac{\partial \Psi_1}{\partial z_1} \quad , \quad \Phi_2(z_2) = \frac{\partial \Psi_2}{\partial z_2} \\ \Phi_1'(z_1) &= \frac{\partial \Phi_1}{\partial z_1} \quad , \quad \Phi_2'(z_2) = \frac{\partial \Phi_2}{\partial z_2} \end{aligned} \tag{3.1.17}$$

Relations (2.2.8) written in terms of Φ_1' and Φ_2' are

$$\begin{aligned} \sigma_x &= \frac{\partial^2 \Psi}{\partial y^2} = 2\mathrm{Re}\left[\mu_1^2\ \Phi_1'(z_1) + \mu_2^2\ \Phi_2'(z_2)\right] \\ \sigma_y &= \frac{\partial^2 \Psi}{\partial x^2} = 2\mathrm{Re}\left[\Phi_1'(z_1) + \Phi_2'(z_2)\right] \\ \tau_{xy} &= -\frac{\partial^2 \Psi}{\partial x \partial y} = -2\mathrm{Re}\left[\mu_1\ \Phi_1'(z_1) + \mu_2\ \Phi_2'(z_2)\right] \end{aligned} \tag{3.1.18}$$

Having expressed the stresses in the form (3.1.18) it is especially easy to show that the stress-displacement relations ((2.2.3) with the strains expressed in terms of Φ_1' and Φ_2' using (2.1.15) and (3.1.18)) are satisfied by displacements expressed as

$$\begin{aligned} u &= 2\mathrm{Re}\left[p_1\ \Phi_1(z_1) + p_2\ \Phi_2(z_2)\right] - \omega y + u_o \\ v &= 2\mathrm{Re}\left[q_1\ \Phi_1(z_1) + q_2\ \Phi_2(z_2)\right] + \omega x + v_o \end{aligned} \tag{3.1.19}$$

where u_o, v_o and ω are rigid body terms, and

$$p_1 = \mu_1^2 S_{11} + S_{12} \quad , \quad p_2 = \mu_2^2 S_{11} + S_{12}$$
$$q_1 = \frac{S_{22}}{\mu_1} + \mu_1 S_{12} \quad , \quad q_2 = \frac{S_{22}}{\mu_2} + \mu_2 S_{12} \tag{3.1.20}$$

1b. Boundary Conditions

Stress boundary conditions were given in (2.2.3). Repeated here they are

$$\sigma_x \cos(x,n) + \tau_{xy} \cos(y,n) = X(s)$$
$$\tau_{xy} \cos(x,n) + \sigma_y \cos(y,n) = Y(s), \tag{3.1.21}$$

where X and Y are the prescribed boundary stresses in the coordinate directions, n is the external normal, and s is arc length measured form some arbitrary starting point. Now if the boundary of the region is traversed in the direction shown in Figure 2, the following relations hold:

$$\frac{dx}{ds} = -\cos(y,n) \quad , \quad \frac{dy}{ds} = \cos(x,n) \tag{3.1.22}$$

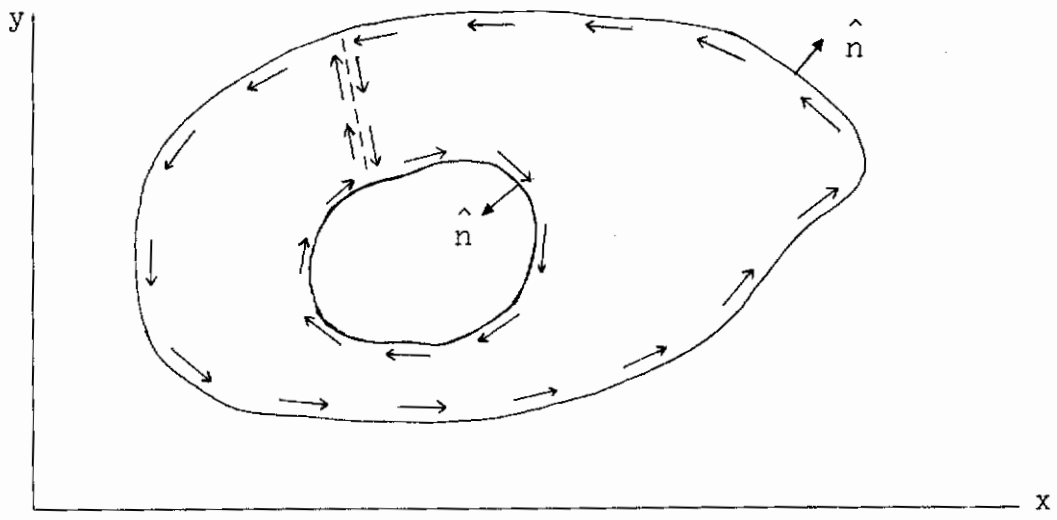


Figure 2. Arbitrary Planar Region

Using (3.1.22) in (3.1.21) and expressing the stresses in terms of Ψ we have

$$\frac{\partial^2\Psi}{\partial y^2}\,dy + \frac{\partial^2\Psi}{\partial x\partial y}\,dx = X(s)\,ds$$

$$\frac{\partial^2\Psi}{\partial x\partial y}\,dy + \frac{\partial^2\Psi}{\partial x^2}\,dx = -Y(s)\,ds \qquad (3.1.23)$$

Integrating (3.1.23) yields

$$\frac{\partial\Psi}{\partial y} = \int_{s_o}^{s} X(s)\,ds + c_1$$

$$\frac{\partial\Psi}{\partial x} = \int_{s_o}^{s} -Y(s)\,ds + c_2 \qquad (3.1.24)$$

In terms of Φ_1 and Φ_2 defined in (3.1.17), (3.1.24) becomes

$$2\mathrm{Re}\left\{\mu_1\,\Phi_1(z_1) + \mu_2\,\Phi_2(z_2)\right\} = \int_{s_o}^{s} X(s)\,ds + c_1$$

$$2\mathrm{Re}\left\{\Phi_1(z_1) + \Phi_2(z_2)\right\} = -\int_{s_o}^{s} Y(s)ds + c_2 \qquad (3.1.25)$$

Since X(s) and Y(s) are known functions of s the right sides of (3.1.25) are in principle known functions of s. Thus, relations (3.1.25) are the boundary conditions that must be satisfied by an appropriate choice of analytic functions Φ_1 and Φ_2 when stresses are prescribed.

It is important to note that when dealing with a doubly connected region (as in Figure 2) that if the applied stresses on a given closed curve are not self equilibrating then the integrals in (3.1.24) and (3.1.25) around the closed curve are non-zero. This means that Ψ, Φ_1, and Φ_2 are not single-valued. In order to remedy this the region can be made singly connected by the introduction of a fictitious slit as indicated in Figure 2. By so doing Ψ, Φ_1, and Φ_2 become single-valued within the region of interest.

When displacements, $\bar{u}(s)$ and $\bar{v}(s)$, are prescribed the boundary conditions for Φ_1 and Φ_2 are obtained directly from (3.1.19). They are

$$2\mathrm{Re}\left[p_1\,\Phi_1(z_1) + p_2\,\Phi_2(z_2)\right] - \omega y + u_o = \bar{u}(s)$$
$$2\mathrm{Re}\left[q_1\,\Phi_1(z_1) + q_2\,\Phi_2(z_2)\right] + \omega x + v_o = \bar{v}(s) \qquad (3.1.26)$$

In (3.1.26) the constants u_o, v_o, and ω, corresponding to rigid body motion, are free parameters which, along with Φ_1 and Φ_2, are to be selected to satisfy these relations. It is important to note that when the problem is formulated in terms of the auxiliary variables Φ_1 and Φ_2 instead of the stress function Ψ, displacement boundary conditions are no more difficult to treat than stress boundary conditions. This is in contrast to the stress function formulation (generalized biharmonic (3.1.1) for which it is generally impossible to conveniently express displacement boundary conditions in terms of Ψ.

2. SOLUTIONS FOR AN INFINITE REGION WITH A CIRCULAR HOLE

In this article we describe how exact solutions may be obtained for an infinite sheet with a circular hole loaded arbitrarily along the circular boundary (with the restriction that this loading can be analyzed into a Fourier series) and with the possibility of uniform stresses at infinity. Results such as these can be obtained by means of conformal mapping techniques. (See Lekhnitskii [8], [9] or Savin [10] for details.) The main difficulty in determining exact solutions by this method lies in discovering mapping functions, $\zeta = f_1(z_1)$ and $\zeta = f_2(z_2)$, which map the given region in the $z_1(\equiv x + \mu_1 y)$ and $z_2(\equiv x + \mu_2 y)$ planes onto the exterior (or interior) of the unit circle in the ζ-plane. For problems involving an elliptical or circular hole (in the physical plane) the corresponding regions in the z_1 and z_2-planes are the exteriors of ellipses. The function which maps the exterior of an ellipse onto the exterior of the unit circle is well known.[3] We will not go through the conformal mapping procedure here, but we will content ourselves that the solutions so obtained are in fact exact solutions to the sample problems of this report.

[3] The problem of mapping the interior of an ellipse onto the interior of the unit circle is difficult. Consequently, exact solutions for the interior of an ellipse or circle do not appear in the classical literature of plane anisotropic elasticity problems. For a discussion of the use of conformal mapping to solve problems involving an interior of an ellipse in isotropic elasticity see Sokolnikoff [11], p. 296.

2a. Solutions for General Loadings

In this article and throughout the rest of this report we regard, without loss of generality, the radius of the hole to be unity. On the assumption that the stresses, $X(\theta)$ and $Y(\theta)$, prescribed on the circular boundary can be expressed as Fourier series, the line integrals appearing in (3.1.24) and (3.1.25) can be written

$$\int_{s_o}^{s} -Y(s)\,ds = \int_{\theta_o}^{\theta} Y(\theta)\,d\theta$$

$$= \frac{P_y}{2\pi}\theta + \alpha_o + \sum_{k=1}^{\infty} (\alpha_k e^{ik\theta} + \bar{\alpha}_k e^{-ik\theta})$$

$$\int_{s_o}^{s} X(s)\,ds = -\int_{\theta_o}^{\theta} X(\theta)\,d\theta$$

$$= \frac{-P_x}{2\pi}\theta + \beta_o + \sum_{k=1}^{\infty} (\beta_k e^{ik\theta} + \bar{\beta}_k e^{-ik\theta}) \quad , \tag{3.2.1}$$

where we have arbitrarily taken θ_o to be zero. Note that according to the convention regarding the positive direction along the boundary (see Figure 2), we have $ds = -d\theta$ in the integrals of (3.2.1). In (3.2.1) P_x and P_y are the resultant forces in the coordinate directions, and

$$\alpha_o = -\sum_{k=1}^{\infty} (\alpha_k + \bar{\alpha}_k)$$

$$\beta_o = -\sum_{k=1}^{\infty} (\beta_k + \bar{\beta}_k) \tag{3.2.2}$$

If displacements $\bar{u}(\theta)$ and $\bar{v}(\theta)$ are prescribed on the circular boundary we similarly assume that they can be expressed as Fourier series.

$$\bar{u}(\theta) = \gamma_o + \sum_{k=1}^{\infty} (\gamma_k e^{ik\theta} + \bar{\gamma}_k e^{-ik\theta})$$

$$\bar{v}(\theta) = \delta_o + \sum_{k=1}^{\infty} (\delta_k e^{ik\theta} + \bar{\delta}_k e^{-ik\theta}) \tag{3.2.3}$$

Next we introduce two complex functions of z_1 and z_2.[4]

$$\zeta_1 = \frac{z_1 + \sqrt{z_1^2 - (1 + \mu_1^2)}}{1 - i\mu_1}$$

$$\zeta_2 = \frac{z_2 + \sqrt{z_2^2 - (1 + \mu_2^2)}}{1 - i\mu_2} \qquad (3.2.4)$$

The square roots in (3.2.4) can be defined so that their branch cuts do not lie in the region of interest. Then if the proper branches are used in (3.2.4) ζ_1 and ζ_2 reduce to $e^{i\theta}$ on the boundary of the circle ($x = \cos\theta$, $y = \sin\theta$) in the physical plane and $|\zeta_{1,2}| \to \infty$ as $|z_{1,2}| \to \infty$. Details of the appropriate definitions of the square roots and choice of branches is given in Appendix II.

Since ζ_1 and ζ_2 are analytic throughout the region of interest we can look for solutions of the form,

$$\Phi_1(z_1) = A_o + A_L z_1 + A \ln \zeta_1 + \sum_{k=1}^{\infty} A_k \zeta_1^{-k}$$

$$\Phi_2(z_2) = B_o + B_L z_2 + B \ln \zeta_2 + \sum_{k=1}^{\infty} B_k \zeta_2^{-k} \qquad (3.2.5)$$

[4] These functions are the inverses of the mapping functions $z_{1,2} = \frac{1}{2}\left[(1 - i\mu_{1,2})\zeta + (1 + i\mu_{1,2})\zeta^{-1}\right]$, which map the exterior of the unit circle in the ζ-plane onto the exteriors of the ellipses, $z_{1,2} = \cos\theta + \mu_{1,2}\sin\theta$, in the $z_{1,2}$-planes. It should be noted that this mapping is dependent on the convention adopted in (3.1.5) and (3.1.6), i.e., the imaginary parts of μ_1 and μ_2 are positive. If μ_1 and μ_2 are defined to have negative imaginary parts then the mapping just given maps the interior of the unit circle in the ζ-plane onto the exteriors of the ellipses in the z_1 and z_2-planes.

The constants A_o and B_o in (3.2.5) have no effect on the stresses. These constants can contribute to rigid body displacements. However, in the formulas for displacements (3.1.19) rigid body motion has already been accounted for in the terms, u_o, v_o, ωx, and ωy. Thus, we set $A_o = B_o = 0$ without loss of generality. The terms linear in z_1 and z_2 are necessary when there is uniform stress at infinity. The terms with $\ln \zeta_1$ and $\ln \zeta_2$ are present whenever the resultant of the applied stresses on the circular boundary is non-zero.

We now determine the values of the coefficients, A_L and B_L, associated with the linear z_1 and z_2-terms of (3.2.5). Suppose that at infinity there is uniform tension (of magnitude σ_∞), and the direction of the tension is at an angle ϕ with respect to the x-axis. The stresses at infinity are then

$$\sigma_x = \sigma_\infty \cos^2\phi$$

$$\sigma_y = \sigma_\infty \sin^2\phi \qquad (3.2.6)$$

$$\tau_{xy} = \sigma_\infty \sin\phi \cos\phi$$

At infinity the linear z_1 and z_2-terms are the only relevant terms of (3.2.5). Since the coefficients, A_L and B_L, are in general complex there are really four constants to be determined. Thus, the three conditions (3.2.6) do not uniquely determine A_L and B_L. A fourth condition involving displacements must be specified. Since we have been excluding rigid body motion terms from Φ_1 and Φ_2, an appropriate restriction on the displacements at infinity is

$$\frac{1}{2}\left(\frac{\partial v^*}{\partial x} - \frac{\partial u^*}{\partial y}\right) = 0 \quad , \qquad (3.2.7)$$

where u* and v* are the parts of u and v which depend on Φ_1 and Φ_2 in formulas (3.1.19). Substituting the assumed forms of Φ_1 and Φ_2 (3.2.5) into formulas (3.1.18) and (3.1.19), and then requiring satisfaction of the conditions (3.2.6) and (3.2.7) at infinity yields the following four equations:

$$\mu_1^2 A_L + \bar{\mu}_1^2 \bar{A}_L + \mu_2^2 B_L + \bar{\mu}_2^2 \bar{B}_L = \sigma_\infty \cos^2\phi$$

$$A_L + \bar{A}_L + B_L + \bar{B}_L = \sigma_\infty \sin^2\phi$$

$$\mu_1 A_L + \bar{\mu}_1 \bar{A}_L + \mu_2 B_L + \bar{\mu}_2 \bar{B}_L = -\sigma_\infty \sin\phi \cos\phi$$

$$(q_1 - \mu_1 p_1)A_L + (\bar{q}_1 - \bar{\mu}_1 \bar{p}_1)\bar{A}_L + (q_2 - \mu_2 p_2)B_L + (\bar{q}_2 - \bar{\mu}_2 \bar{p}_2)\bar{B}_L = 0 \qquad (3.2.8)$$

Using the definitions of p_1, p_2, q_1, and q_2 in (3.1.20) we can write

$$\begin{aligned} q_1 - \mu_1 p_1 &= -2S_{11}\,\mu_1^3 + \frac{1}{\mu_1}(S_{22} + S_{11}\,\mu_1^4) \\ &= -2S_{11}\,\mu_1^3 + S_{11}(\mu_1^2 + \mu_2^2)\mu_1 \quad , \end{aligned} \qquad (3.2.9)$$

where relations (3.1.3) (with μ_1 substituted for μ) and the first of (3.1.9) were used to obtain the final form of (3.2.9). After expressing the other coefficients in the last of (3.2.8) in a similar form we divide out S_{11}. We then multiply the third of (3.2.8) by $(\mu_1^2 + \mu_2^2)$ and subtract from the last equation. The resulting equation along with the first three can be written

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \mu_1 & \bar{\mu}_1 & \mu_2 & \bar{\mu}_2 \\ \mu_1^2 & \bar{\mu}_1^2 & \mu_2^2 & \bar{\mu}_2^2 \\ \mu_1^3 & \bar{\mu}_1^3 & \mu_2^3 & \bar{\mu}_2^3 \end{bmatrix} \begin{Bmatrix} A_L \\ \bar{A}_L \\ B_L \\ \bar{B}_L \end{Bmatrix} = \begin{Bmatrix} \sigma_\infty \sin^2\phi \\ -\sigma_\infty \sin\phi \cos\phi \\ \sigma_\infty \cos^2\phi \\ -\frac{\sigma_\infty}{2}(\mu_1^2 + \mu_2^2)\sin\phi \cos\phi \end{Bmatrix} , \qquad (3.2.10)$$

where the matrix of coefficients is a Vandermonde matrix. The solution to this system is

$$A_L = \frac{\sigma_\infty\left[-2\mu_1\mu_2^2 \sin^2\phi + (\mu_2^2 - \mu_1^2)\cos\phi \sin\phi + 2\mu_1 \cos^2\phi\right]}{4\mu_1(\mu_1^2 - \mu_2^2)}$$

$$B_L = \frac{\sigma_\infty\left[-2\mu_2\mu_1^2 \sin^2\phi + (\mu_1^2 - \mu_2^2)\cos\phi \sin\phi + 2\mu_2 \cos^2\phi\right]}{4\mu_2(\mu_2^2 - \mu_1^2)} \qquad (3.2.11)$$

Having obtained A_L and B_L by imposing the "boundary" conditions at infinity, we are ready to determine the other constants in (3.2.5) by requiring that the stress boundary conditions (3.1.25) or the displacement boundary conditions (3.1.26) be satisfied on the circular boundary. First let us consider the case of stress boundary conditions. We substitute (3.2.5) into the left sides of (3.1.25) and use (3.2.1) on the right sides. Making use of the fact that on the circular boundary

$$\zeta_1 = \zeta_2 = e^{i\theta}$$
$$z_{1,2} = \frac{1}{2}\left[(1 - i\mu_{1,2})e^{i\theta} + (1 + i\mu_{1,2})e^{-i\theta}\right] \qquad (3.2.12)$$

(3.1.25) becomes

$$\mathrm{Re}\Big\{A_L\mu_1\left[(1 - i\mu_1)e^{i\theta} + (1 + i\mu_1)e^{-i\theta}\right] + 2A\mu_1 i\theta + 2\mu_1 \sum_{k=1}^{\infty} A_k e^{-ik\theta}$$
$$+B_L\mu_2\left[(1 - i\mu_2)e^{i\theta} + (1 + i\mu_2)e^{-i\theta}\right] + 2B\mu_2 i\theta + 2\mu_2 \sum_{k=1}^{\infty} B_k e^{-ik\theta}\Big\}$$
$$= \left\{\frac{-P_x}{2\pi}\theta + \beta_o + \sum_{k=1}^{\infty}(\beta_k e^{ik\theta} + \bar{\beta}_k e^{-ik\theta}) + c_1\right\}$$

$$\mathrm{Re}\Big\{A_L\left[(1 - i\mu_1)e^{i\theta} + (1 + i\mu_1)e^{-i\theta}\right] + 2Ai\theta + 2\sum_{k=1}^{\infty} A_k e^{-ik\theta}$$
$$+B_L\left[(1 - i\mu_2)e^{i\theta} + (1 + i\mu_2)e^{-i\theta}\right] + 2Bi\theta + 2\sum_{k=1}^{\infty} B_k e^{-ik\theta}\Big\}$$
$$= \left\{\frac{P_y}{2\pi}\theta + \alpha_o + \sum_{k=1}^{\infty}(\alpha_k e^{ik\theta} + \bar{\alpha}_k e^{-ik\theta}) + c_2\right\} \qquad (3.2.13)$$

By comparing terms on both sides of equations (3.2.13) we can determine equations for the unknown coefficients, A, B, A_k, and B_k. For example for k>1 we have

$$\begin{aligned} \mu_1 A_k + \mu_2 B_k &= \bar{\beta}_k \\ A_k + B_k &= \bar{\alpha}_k \quad , \end{aligned} \tag{3.2.14}$$

which gives

$$\begin{aligned} A_k &= \frac{\bar{\beta}_k - \mu_2 \bar{\alpha}_k}{\mu_1 - \mu_2} \\ B_k &= \frac{\bar{\beta}_k - \mu_1 \bar{\alpha}_k}{\mu_2 - \mu_1} \end{aligned} \tag{3.2.15}$$

For k=1 we obtain

$$\begin{aligned} &\mu_1 A_1 + \mu_2 B_1 = \bar{\beta}_1 \\ &\quad - \tfrac{1}{2}\left\{\mu_1(1 + i\mu_1)A_L + \bar{\mu}_1(1 + i\bar{\mu}_1)\bar{A}_L + \mu_2(1 + i\mu_2)B_L \right. \\ &\qquad\qquad \left. + \bar{\mu}_2(1 + i\bar{\mu}_2)\bar{B}_L\right\} \\ &\quad = \bar{\beta}_1 - \frac{\sigma_\infty}{2}\left\{i\cos^2\phi - \sin\phi\cos\phi\right\} \\ &A_1 + B_1 = \bar{\alpha}_1 \\ &\quad - \tfrac{1}{2}\left\{(1 + i\mu_1)A_L + (1 + i\bar{\mu}_1)\bar{A}_L + (1 + i\mu_2)B_L + (1 + i\bar{\mu}_2)\bar{B}_L\right\} \\ &\quad = \bar{\alpha}_1 - \frac{\sigma_\infty}{2}\left\{\sin^2\phi - i\sin\phi\cos\phi\right\} \quad , \end{aligned} \tag{3.2.16}$$

where we have used system (3.2.10) to obtain the final form on the right sides. Thus,

$$A_1 = \frac{\bar{\beta}_1 - \mu_2\bar{\alpha}_1}{\mu_1 - \mu_2} - \frac{\sigma_\infty}{2(\mu_1 - \mu_2)}\left\{(i\cos^2\phi - \mu_2\sin^2\phi)\right.$$

$$\left. - (1 - i\mu_2)\sin\phi\cos\phi\right\}$$

$$B_1 = \frac{\bar{\beta}_1 - \mu_1\bar{\alpha}_1}{\mu_2 - \mu_1} - \frac{\sigma_\infty}{2(\mu_2 - \mu_1)}\left\{(i\cos^2\phi - \mu_1\sin^2\phi)\right.$$

$$\left. - (1 - i\mu_1)\sin\phi\cos\phi\right\} \qquad (3.2.17)$$

Finally we must determine the constants A and B in (3.2.11). From (3.2.13) we obtain the two equations,

$$\begin{aligned} \mu_1 A - \bar{\mu}_1\bar{A} + \mu_2 B - \bar{\mu}_2\bar{B} &= -\frac{P_x}{2\pi i} \\ A - \bar{A} + B - \bar{B} &= \frac{P_y}{2\pi i} \end{aligned} \qquad (3.2.18)$$

Unlike the equations for the A_k and B_k, e.g., (3.2.14) and (3.2.16), system (3.2.18) involves not only A and B but their conjugates as well. Thus, two more conditions must be imposed in order to completely determine A and B. Two such conditions are the requirements that the displacements be single-valued. We impose the conditions $u(0) = u(2\pi)$ and $v(0) = v(2\pi)$ on the circular boundary. Using (3.2.5) in (3.1.19) these two conditions become

$$\begin{aligned} p_1 A - \bar{p}_1\bar{A} + p_2 B - \bar{p}_2\bar{B} &= 0 \\ q_1 A - \bar{q}_1\bar{A} + q_2 B - \bar{q}_2\bar{B} &= 0 \end{aligned} \qquad (3.2.19)$$

Making use of the definitions of p_1, p_2, q_1, and q_2 given in (3.1.20) and after some rearranging we can write equations (3.2.18) and (3.2.19) as follows:

$$\begin{bmatrix} \mu_1^{-1} & \bar{\mu}_1^{-1} & \mu_2^{-1} & \bar{\mu}_2^{-1} \\ 1 & 1 & 1 & 1 \\ \mu_1 & \bar{\mu}_1 & \mu_2 & \bar{\mu}_2 \\ \mu_1^2 & \bar{\mu}_1^2 & \mu_2^2 & \bar{\mu}_2^2 \end{bmatrix} \begin{Bmatrix} A \\ -\bar{A} \\ B \\ -\bar{B} \end{Bmatrix} = \begin{Bmatrix} \frac{S_{12}}{S_{22}} \frac{P_x}{2\pi i} \\ \frac{P_y}{2\pi i} \\ -\frac{P_x}{2\pi i} \\ -\frac{S_{12}}{S_{11}} \frac{P_y}{2\pi i} \end{Bmatrix} \qquad (3.2.20)$$

This system is similar to (3.2.10). The solution is

$$A = \frac{1}{4(\mu_2^2 - \mu_1^2)\pi i}\left[\mu_1\left(1 + \mu_2^2 \frac{S_{12}}{S_{22}}\right)P_x + \left(\mu_2^2 + \frac{S_{12}}{S_{11}}\right)P_y\right]$$

$$B = \frac{1}{4(\mu_1^2 - \mu_2^2)\pi i}\left[\mu_2\left(1 + \mu_1^2 \frac{S_{12}}{S_{22}}\right)P_x + \left(\mu_1^2 + \frac{S_{12}}{S_{11}}\right)P_y\right] \qquad (3.2.21)$$

Now let us consider the problem when displacements are prescribed on the boundary of the hole. We assume that the imposed displacements can be expressed as a Fourier series as in (3.2.3). Using these Fourier series on the right side of (3.1.26) and the assumed form of Φ_1 and Φ_2 (3.2.5) on the left, we arrive at a set of relations similar to (3.2.13).

$$\mathrm{Re}\left\{ A_L p_1 \left[(1 - i\mu_1)e^{i\theta} + (1 + i\mu_1)e^{-i\theta}\right] + 2Ap_1 i\theta + 2p_1 \sum_{k=1}^{\infty} A_k e^{-ik\theta} \right.$$

$$\left. + B_L p_2 \left[(1 - i\mu_2)e^{i\theta} + (1 + i\mu_2)e^{-i\theta}\right] + 2Bp_2 i\theta + 2p_2 \sum_{k=1}^{\infty} B_k e^{-ik\theta} \right\}$$

$$= \left\{ \gamma_o + \sum_{k=1}^{\infty} (\gamma_k e^{ik\theta} + \bar{\gamma}_k e^{-ik\theta}) \right\} + \frac{\omega}{2i}(e^{i\theta} - e^{-i\theta}) - u_o$$

$$\mathrm{Re}\left\{ A_L q_1 \left[(1 - i\mu_1)e^{i\theta} + (1 + i\mu_1)e^{-i\theta}\right] + 2Aq_1 i\theta + 2q_1 \sum_{k=1}^{\infty} A_k e^{-ik\theta} \right.$$

$$\left. + B_L q_2 \left[(1 - i\mu_2)e^{i\theta} + (1 + i\mu_2)e^{-i\theta}\right] + 2Bq_2 i\theta + 2q_2 \sum_{k=1}^{\infty} B_k e^{-ik\theta} \right\}$$

$$= \left\{ \delta_o + \sum_{k=1}^{\infty} (\delta_k e^{ik\theta} + \bar{\delta}_k e^{-ik\theta}) \right\} - \frac{\omega}{2}(e^{i\theta} + e^{-i\theta}) - v_o \qquad (3.2.22)$$

The constants u_o, v_o, and ω, corresponding to rigid body motion, cannot be determined from equations (3.2.22). In order to obtain these constants three additional conditions on the displacements must be imposed. Thus, we shall obtain formulas for the A's and B's of (3.2.22) in terms of these rigid body constants. Having these A's and B's we can write expressions for the displacements, and then impose the three displacement conditions to obtain u_o, v_o, and ω.

On the left sides of both equations of (3.2.22) there are two terms linear in θ. There are no such terms on the right, as there were for prescribed stresses, and we conclude $A = B = 0$. On the right side of each equation there are two constants γ_o, u_o and δ_o, v_o. It may be that γ_o and δ_o, corresponding to uniform displacements of the circular boundary, are the result of rigid body motion of the infinite sheet. In this situation $\gamma_o = u_o$ and $\delta_o = v_o$ and the constant terms in (3.2.22) cancel. However, consider a situation for which the stresses and displacements vanish at infinity. Then $u_o = v_o = \omega = 0$. However, it is still possible to prescribe constant displacements of the circular boundary giving $\gamma_o \neq 0$, and/or $\delta_o \neq 0$. Clearly it is impossible to satisfy (3.2.22) since there are no other constant terms in these equations. Thus, it appears that our assumed form of Φ_1 and Φ_2 is not general enough to handle this problem. This is not surprising since the same problem for isotropic media cannot be solved in terms of simple functions. Thus, we exclude from consideration those problems for which $\gamma_o \neq u_o$ and $\delta_o \neq v_o$. Just as before we obtain for $k>1$

$$A_k = (q_2\bar{\gamma}_k - p_2\bar{\delta}_k)/D \quad ,$$
$$B_k = (p_1\bar{\delta}_k - q_1\bar{\gamma}_k)/D \quad , \qquad (3.2.23)$$

where

$$D = p_1 q_2 - p_2 q_1 \qquad (3.2.24)$$

For k=1 we have

$$A_1 = \Big\{ (q_2\bar{\gamma}_1 - p_2\bar{\delta}_1) + \frac{\omega}{2}(p_2 + iq_2)$$

$$+ \frac{\sigma_\infty}{2}\Big[(ip_2S_{22} - q_2S_{12})\sin^2\phi + (ip_2S_{12} - q_2S_{11})\cos^2\phi$$

$$+ \frac{S_{66}}{2}(p_2 - iq_2)\sin\phi\cos\phi\Big]\Big\}/D$$

$$B_1 = \Big\{ (p_1\bar{\delta}_1 - q_1\bar{\gamma}_1) - \frac{\omega}{2}(p_1 + iq_1)$$

$$- \frac{\sigma_\infty}{2}\Big[(ip_1S_{22} - q_1S_{12})\sin^2\phi + (ip_1S_{12} - q_1S_{11})\cos^2\phi$$

$$+ \frac{S_{66}}{2}(p_1 - iq_1)\sin\phi\cos\phi\Big]\Big\}/D \qquad (3.2.25)$$

We now have formulas for Φ_1 and Φ_2 for the solution of the general problem involving an arbitrarily loaded circular hole in an infinite sheet with uniform stresses at infinity. To obtain stresses and displacements these expressions for Φ_1 and Φ_2 are used in formulas (3.1.18) and (3.1.19). We note that for use in (3.1.18)

$$\Phi' \equiv \frac{d\Phi}{dz} = \frac{d\Phi}{d\zeta}\frac{d\zeta}{dz} = \frac{\zeta}{\sqrt{z^2 - (1 + \mu^2)}}\frac{d\Phi}{d\zeta} \qquad (3.2.26)$$

2b. Solutions for Four Sample Problems

We now specialize the results of Article 2a to the four sample problems to be considered in this study.

Case I: Uniform pressure (of unit magnitude) on the circular boundary.

$$X(\theta) = \cos\theta \quad , \quad Y(\theta) = \sin\theta$$

$$-\int_o^\theta X(\theta)d\theta = -\int_o^\theta \cos\theta\, d\theta = -\sin\theta \tag{3.2.27}$$

$$\int_o^\theta Y(\theta)d\theta = \int_o^\theta \sin\theta\, d\theta = 1 - \cos\theta$$

Then, from (3.2.1), we conclude

$$\beta_1 = \frac{i}{2} \quad , \quad \alpha_1 = -\frac{1}{2} \quad , \quad \alpha_o = 1 \quad , \tag{3.2.28}$$

and all other α's, β's, P_x and P_y are zero. Then, from (3.2.17) we obtain

$$A_1 = \frac{(-i + \mu_2)}{2(\mu_1 - \mu_2)} \quad , \quad B_1 = \frac{(-i + \mu_1)}{2(\mu_2 - \mu_1)} \quad , \tag{3.2.29}$$

with all the other A's and B's of (3.2.5) equal to zero.

Case II: Uniform tension (of unit magnitude) at infinity and zero stress on circular boundary.

$$X(\theta) = Y(\theta) = 0$$

Thus, all of the α's, β's, P_x, and P_y of (3.2.1) are zero. The only non-zero terms in the expressions for Φ_1 and Φ_2 of (3.2.5) are those corresponding to A_L, B_L, A_1, and B_1. Formulas for these constants are given in (3.2.11) and (3.2.17) with $\sigma_\infty = 1$ and $\bar{\alpha}_1 = \bar{\beta}_1 = 0$.

Case III: Uniform **radial** expansion (of unit magnitude) of circular boundary.

In (3.2.3) we have

$$\gamma_1 = \frac{1}{2} \quad , \quad \delta_1 = \frac{1}{2i} \tag{3.2.30}$$

and all other γ's and δ's are zero. Formulas (3.2.23) then give

$$A_1 = \frac{(q_2 - ip_2)}{2(p_1 q_2 - p_2 q_1)}$$

$$B_1 = \frac{-(q_1 - ip_1)}{2(p_1 q_2 - p_2 q_1)} \qquad (3.2.31)$$

with all other A's and B's of (3.2.5) equal to zero.

Case IV: Uniform tension (of unit magnitude) at infinity and zero displacement of circular boundary

Since the displacement is prescribed to be zero on the circular boundary, all of the γ's and δ's of (3.2.3) are zero. The only non-zero terms in the expressions for Φ_1 and Φ_2 of (3.2.5) are those corresponding to A_L, B_L, A_1, and B_1. Formulas for these constants are given in (3.2.11) and (3.2.25) with $\sigma_\infty = 1$ and $\omega = 0$.

3. CONTOUR PLOTS OF SOLUTIONS TO THE FOUR SAMPLE PROBLEMS

In this article we present, in the form of contour plots, stresses and displacements for the four sample problems of this study. These stresses and displacements were obtained from a computer program based on the formulation of Article 2 of this Section. This computer program is described in Appendix IX. The results from this program were then used as input to a contour plotting program which generated the plots of Figures 3-12.

3a. Discussion of Numerical Results

For the purposes of the graphical presentation of this article and also for the approximate numerical results of Section VII we have selected six different orthotropic materials which illustrate a variety of behavior possible in problems involving such materials. The six materials are listed in Table I.

Table I. Properties of Sample Materials

Material Number	E_x	E_y	ν_{xy}	G_{xy}	μ_1	μ_2
1	3.86	1.00	.598	.743	1.28i	1.54i
2	1.50	1.00	.375	.400	.796i	1.54i
3	24.60	1.00	.298	.750	.885i	5.61i
4	20.00	1.00	1.000	10.000	1.49+1.49i	-1.49+1.49i
5	1.00	1.00	.774	2.040	.874+.485i	-.874+.485i
6	1.00	1.00	.250	.050	.227i	4.41i

Note that all materials have been "normalized" so that E_y is unity. This facilitates comparison of the behavior of these materials. Three of the materials of Table I; (1), (3) and (5), correspond to layups (such as plywood) of real orthotropic materials. The other three are hypothetical, but thermodynamically admissible, materials which illustrate distinct types of material orthotropy. In problems with stress boundary conditions only, the response to a given loading depends only upon the constant coefficients in the generalized biharmonic (3.1.1). These coefficients for the six sample materials are listed in Table II. Materials (3) - (6) correspond to various

Table II. Coefficients of Generalized Biharmonic for Sample Materials

Material Number	S_{22}	$2S_{12}+S_{66}$	S_{11}
1	1.00	1.04	.259
2	1.00	2.00	.667
3	1.00	1.31	.041
4	1.00	0.00	.050
5	1.00	- 1.06	1.000
6	1.00	19.50	1.000

extreme cases of these coefficients which result in quite different types of behavior as seen in Figures 3, 5, 7, 9 and 11. Materials (3), (4) and (6) are extreme in that one or more coefficients of the generalized biharmonic are quite small compared to the others. Material (5) is extreme in that it deviates from an isotropic material even more than one for which the coefficient $(2S_{12}+S_{66})$ vanishes. Material (2) is a mildly orthotropic material suitable for use with the perturbation techniques of Section IV. Material (1) might be considered

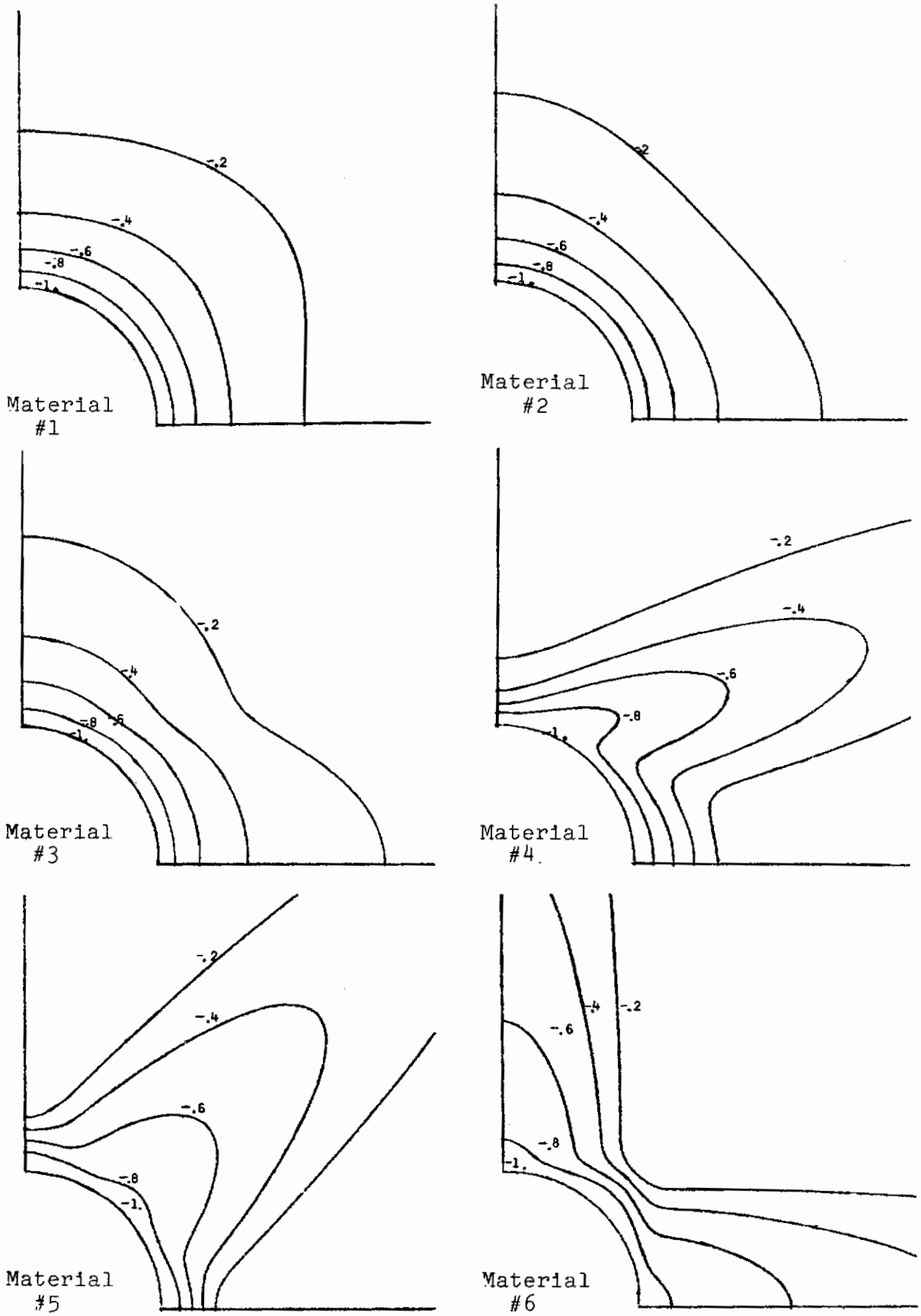


Figure 3. Contour Plots of σ_r (Case I) for Various Materials

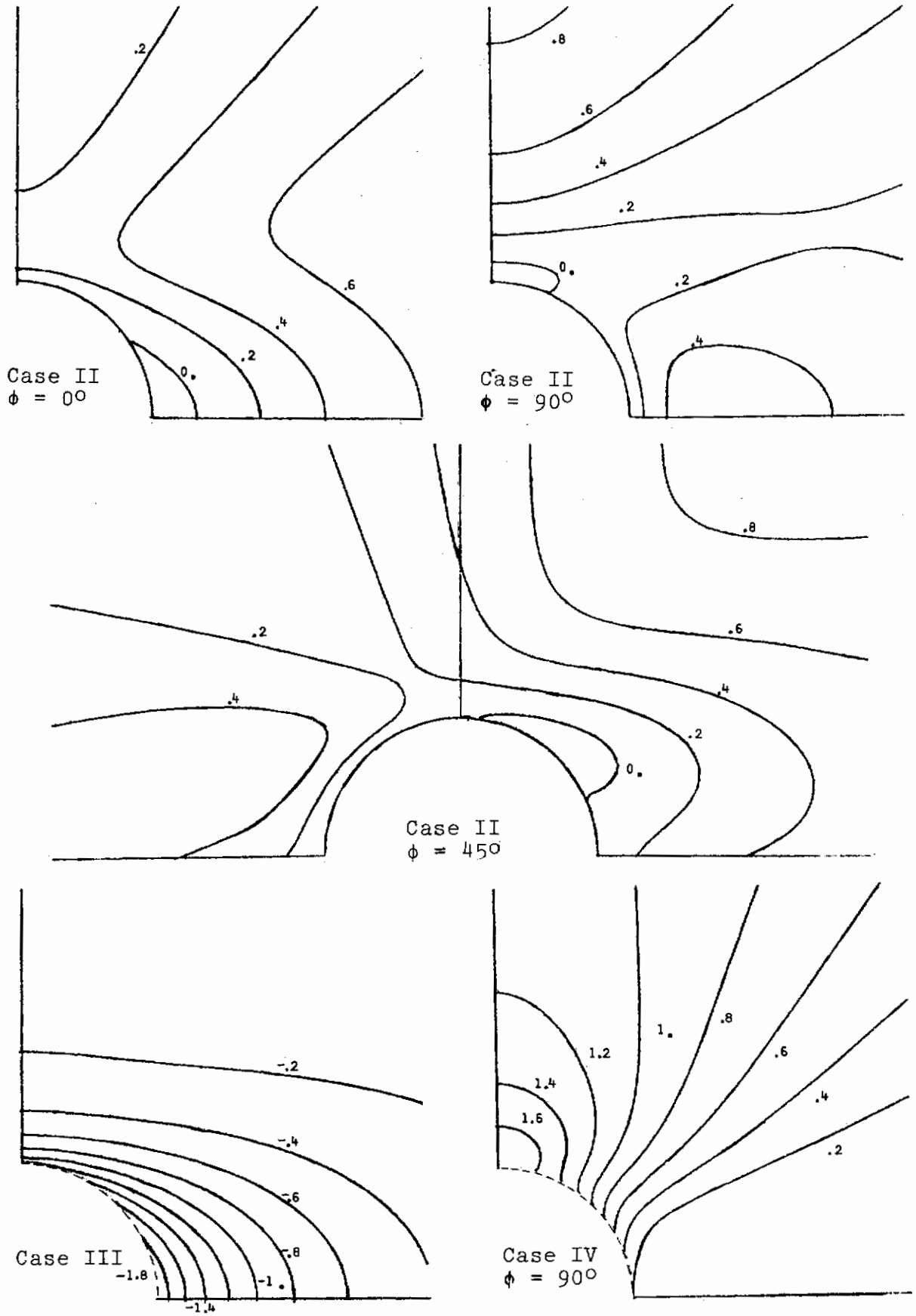


Figure 4. Contour Plots of σ_r (Material 1) for Cases II, III, and IV

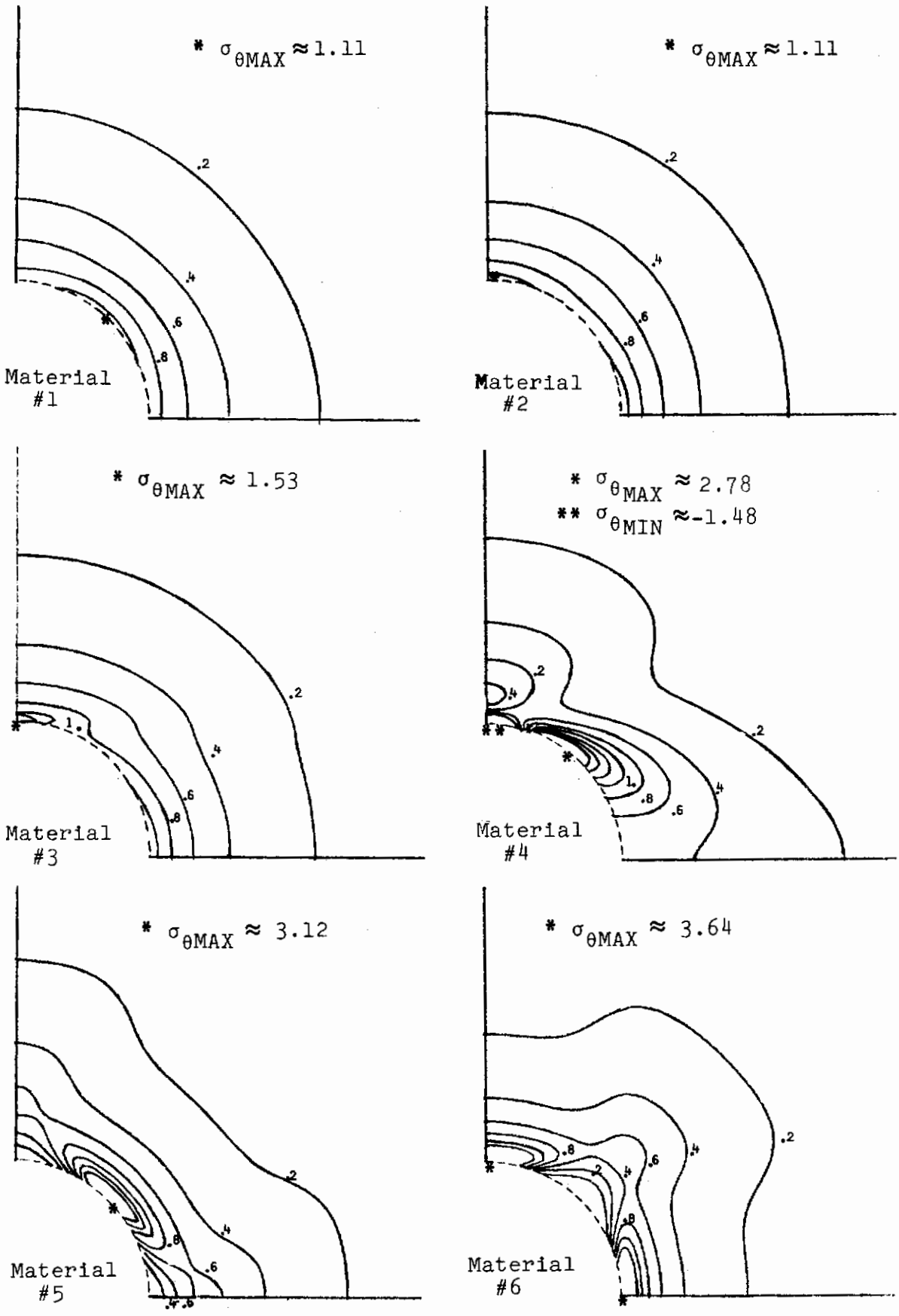


Figure 5. Contour Plots of σ_θ (Case I) for Various Materials

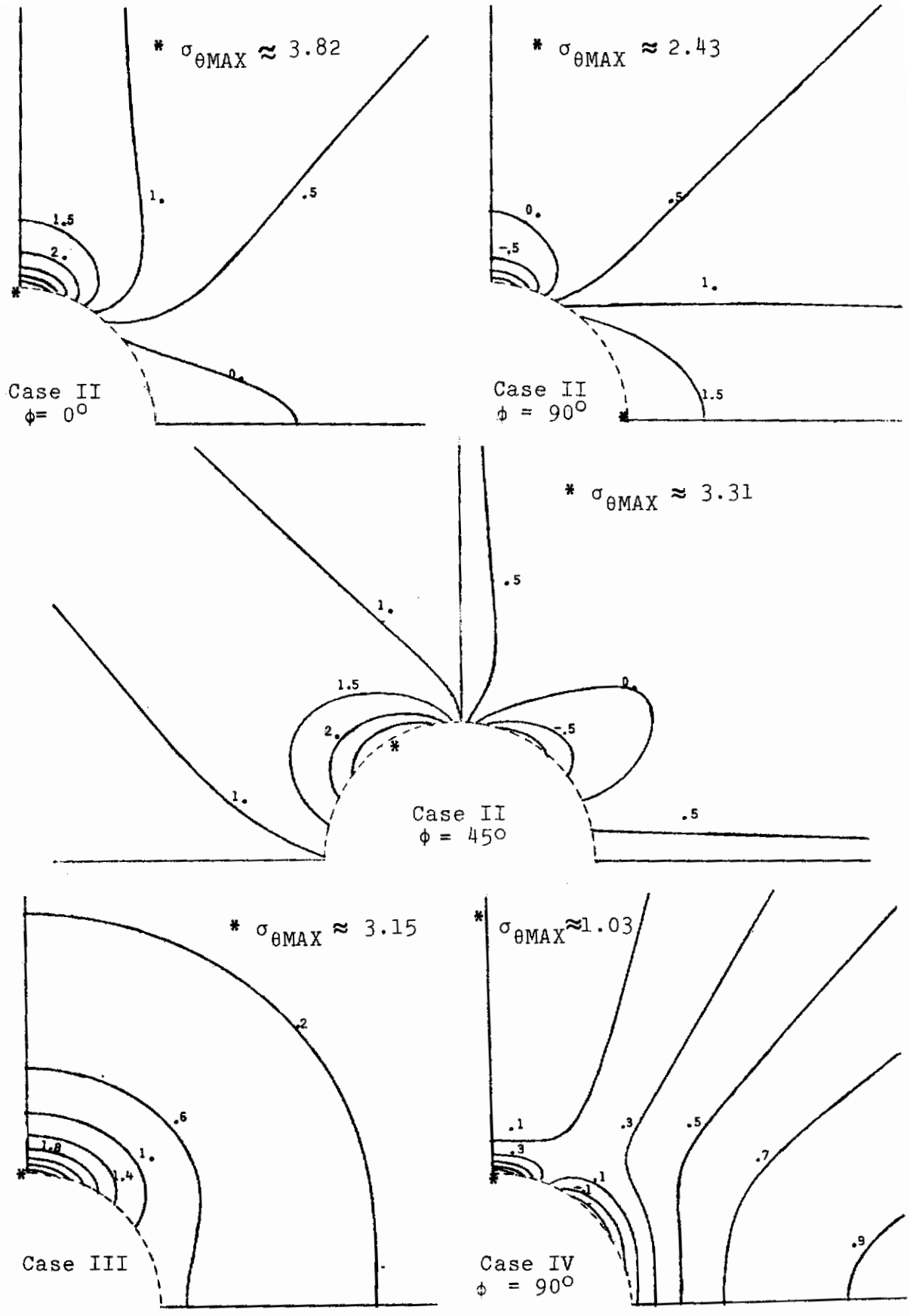


Figure 6. Contour Plots of σ_θ (Material 1) for Cases II, III, and IV

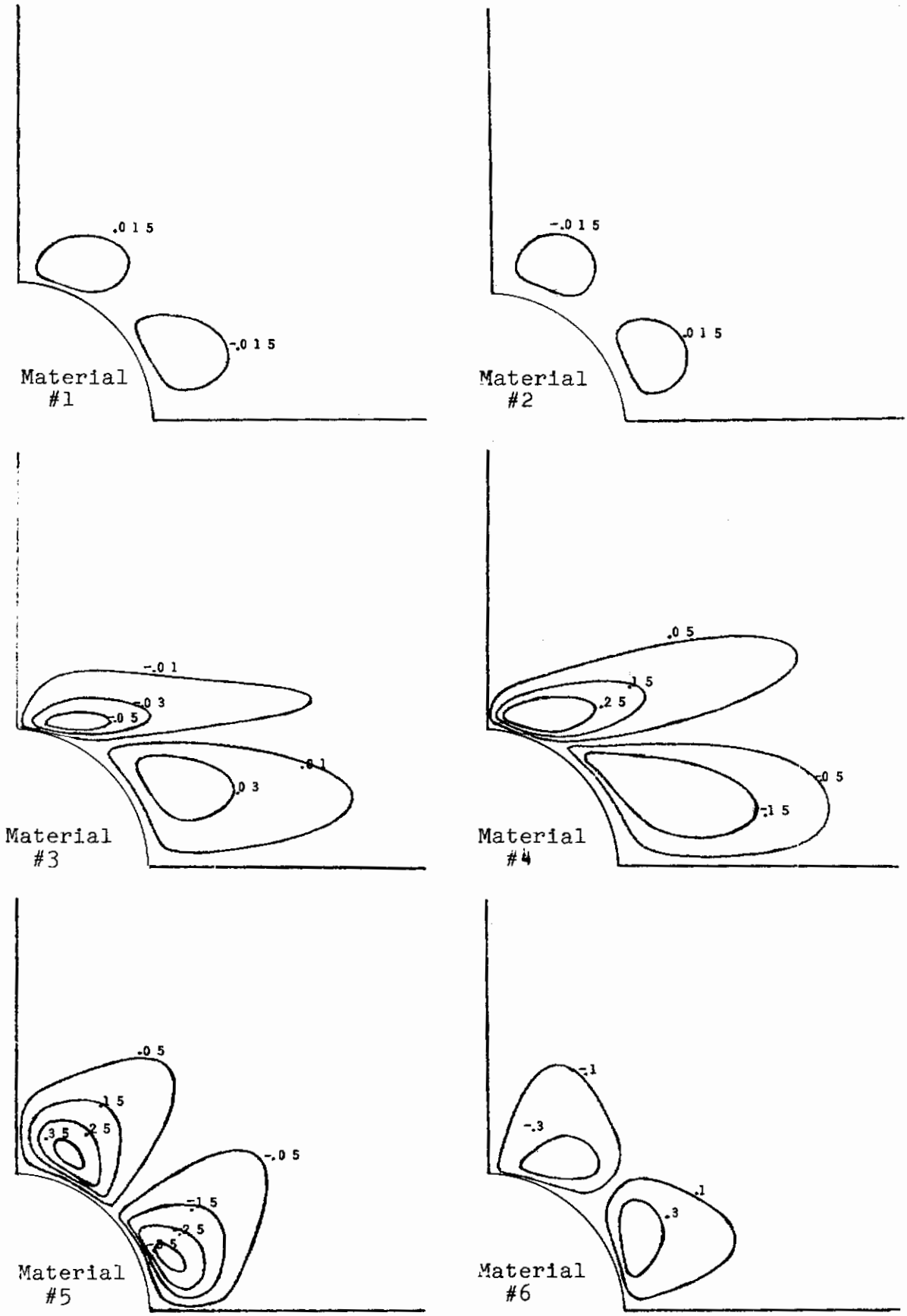


Figure 7. Contour Plots of $\tau_{r\theta}$ (Case I) for Various Materials

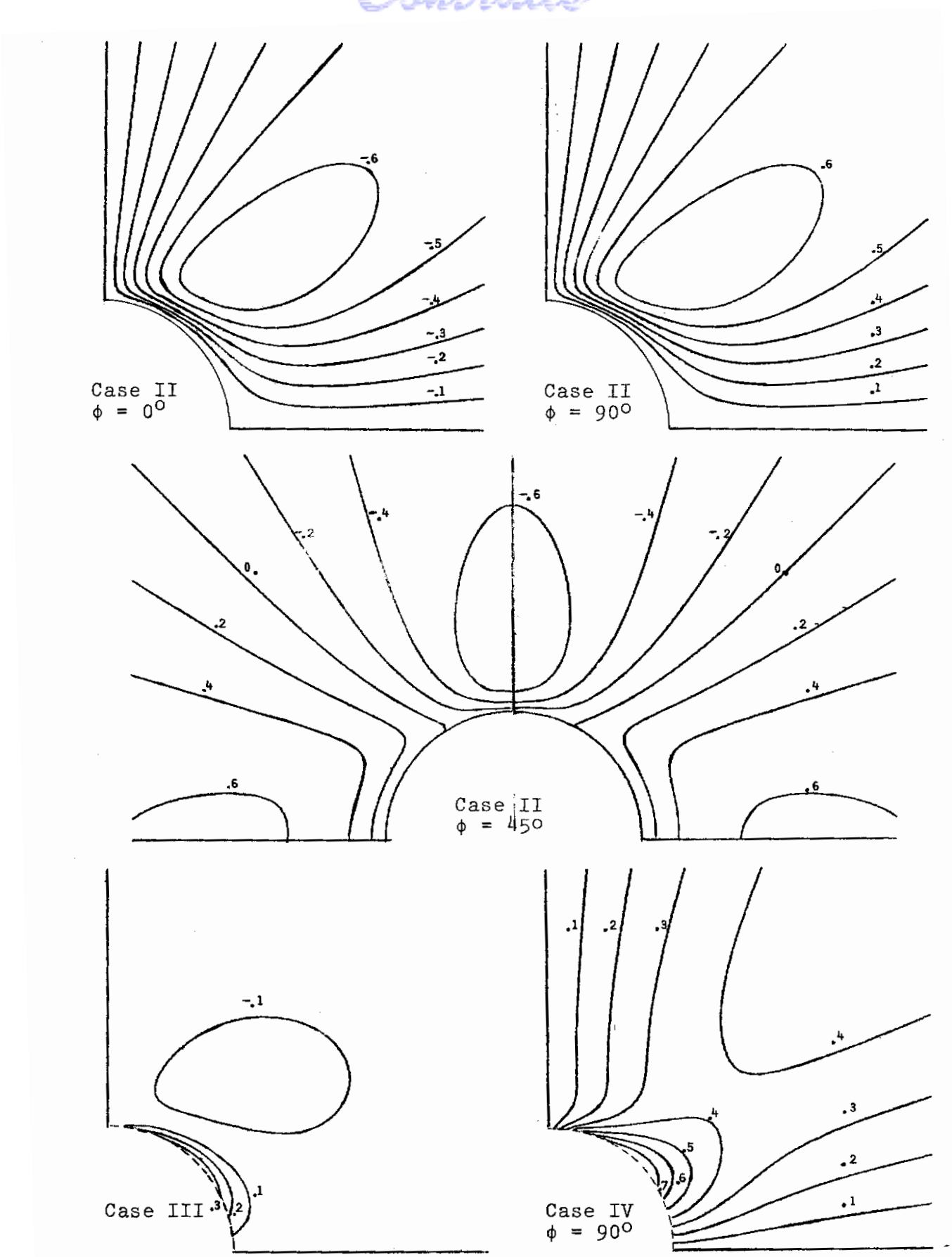


Figure 8. Contour Plots of $\tau_{r\theta}$ (Material 1) for Cases II, III, and IV

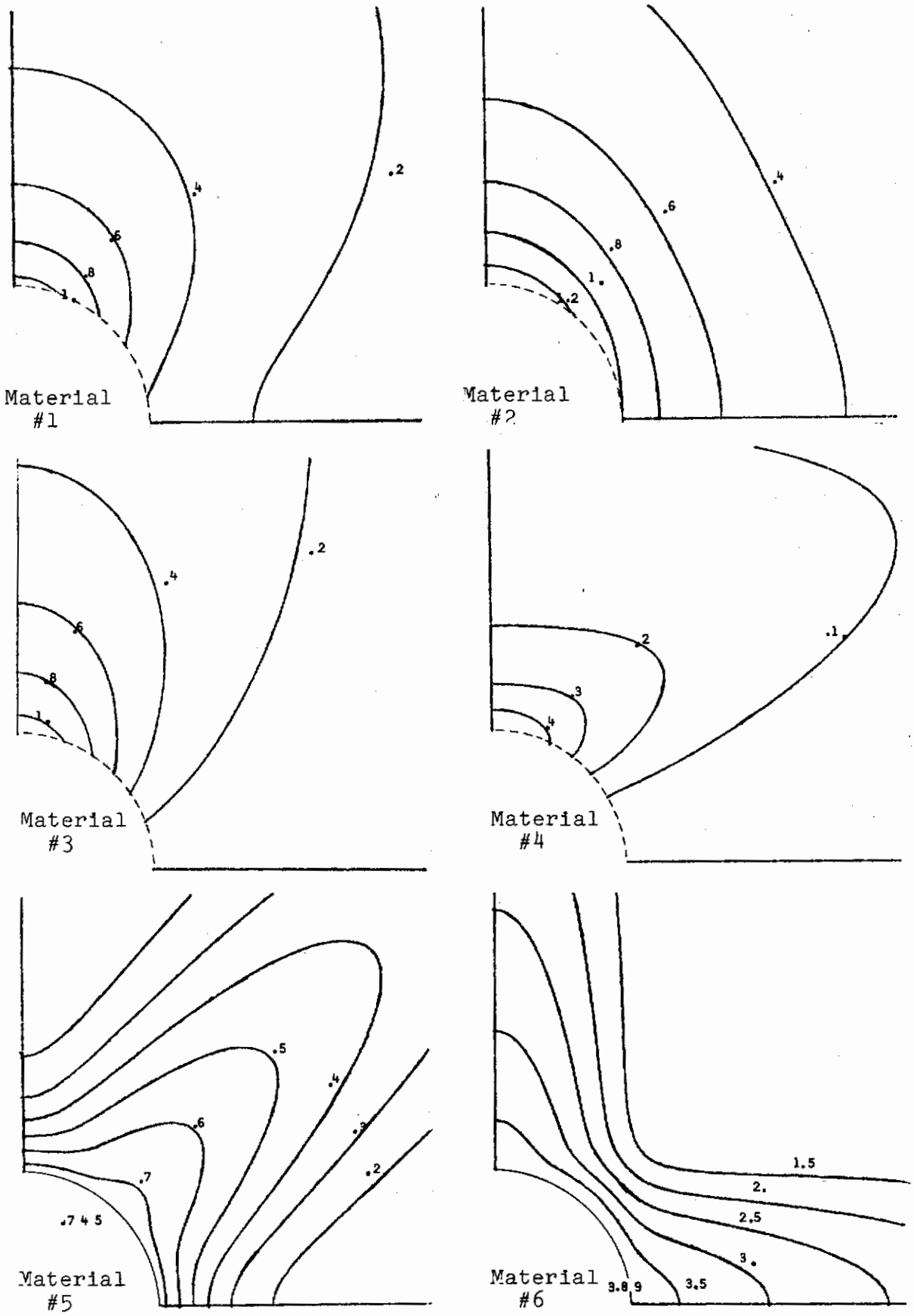


Figure 9. Contour Plots of u_r (Case I) for Various Materials

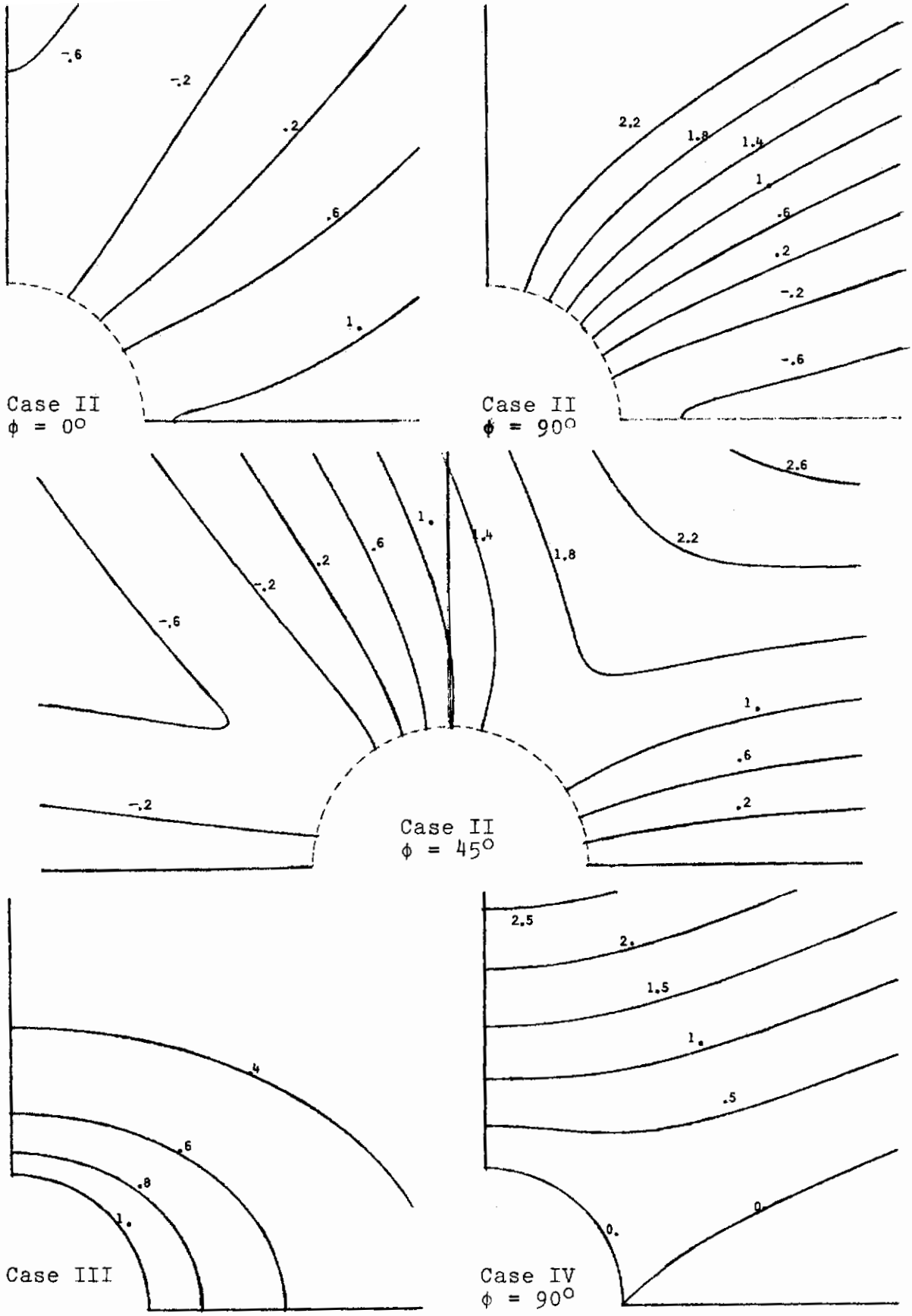


Figure 10. Contour Plots of u_r (Material 1) for Cases II, III, and IV

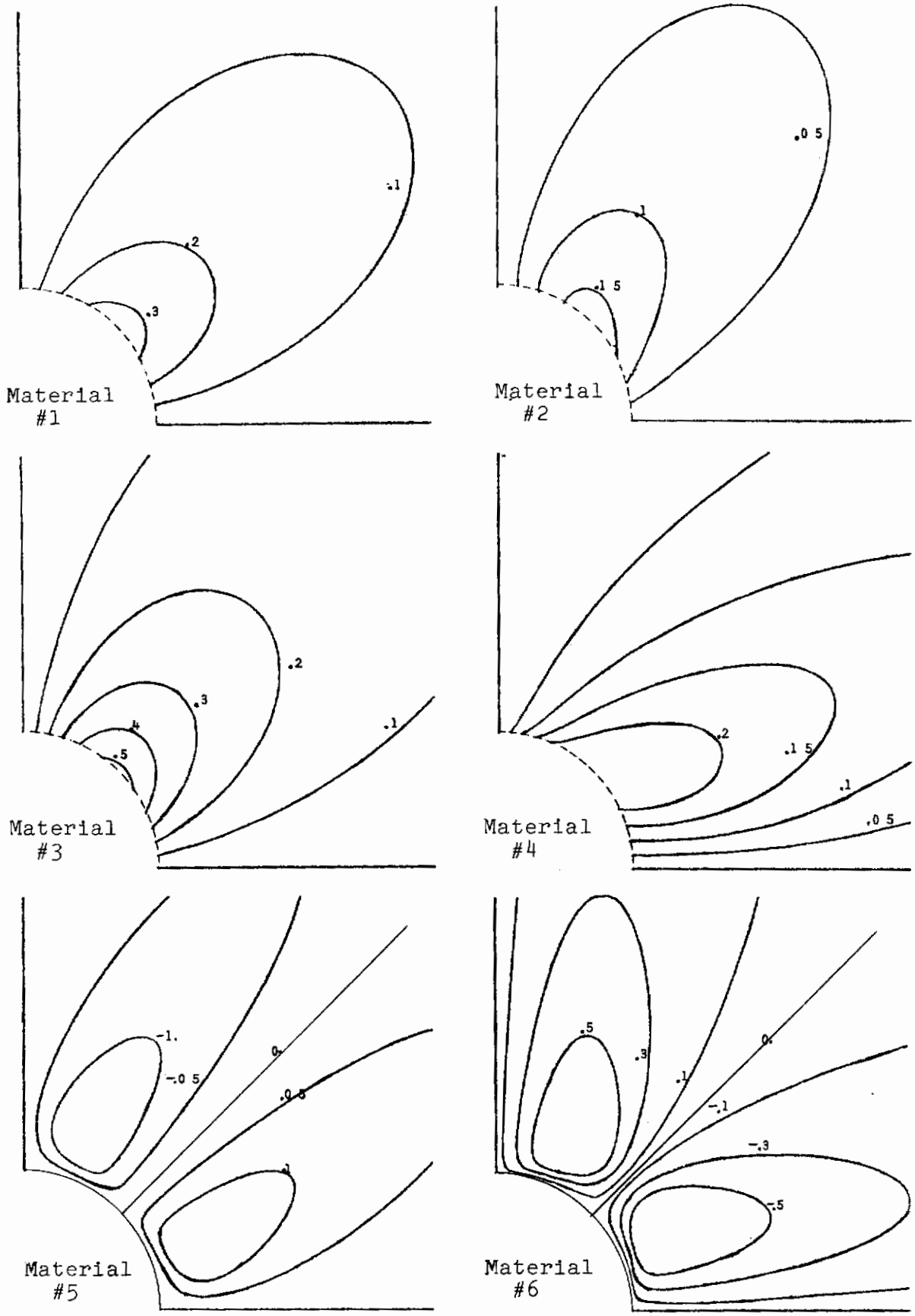


Figure 11. Contour Plots of u_θ (Case I) for Various Materials

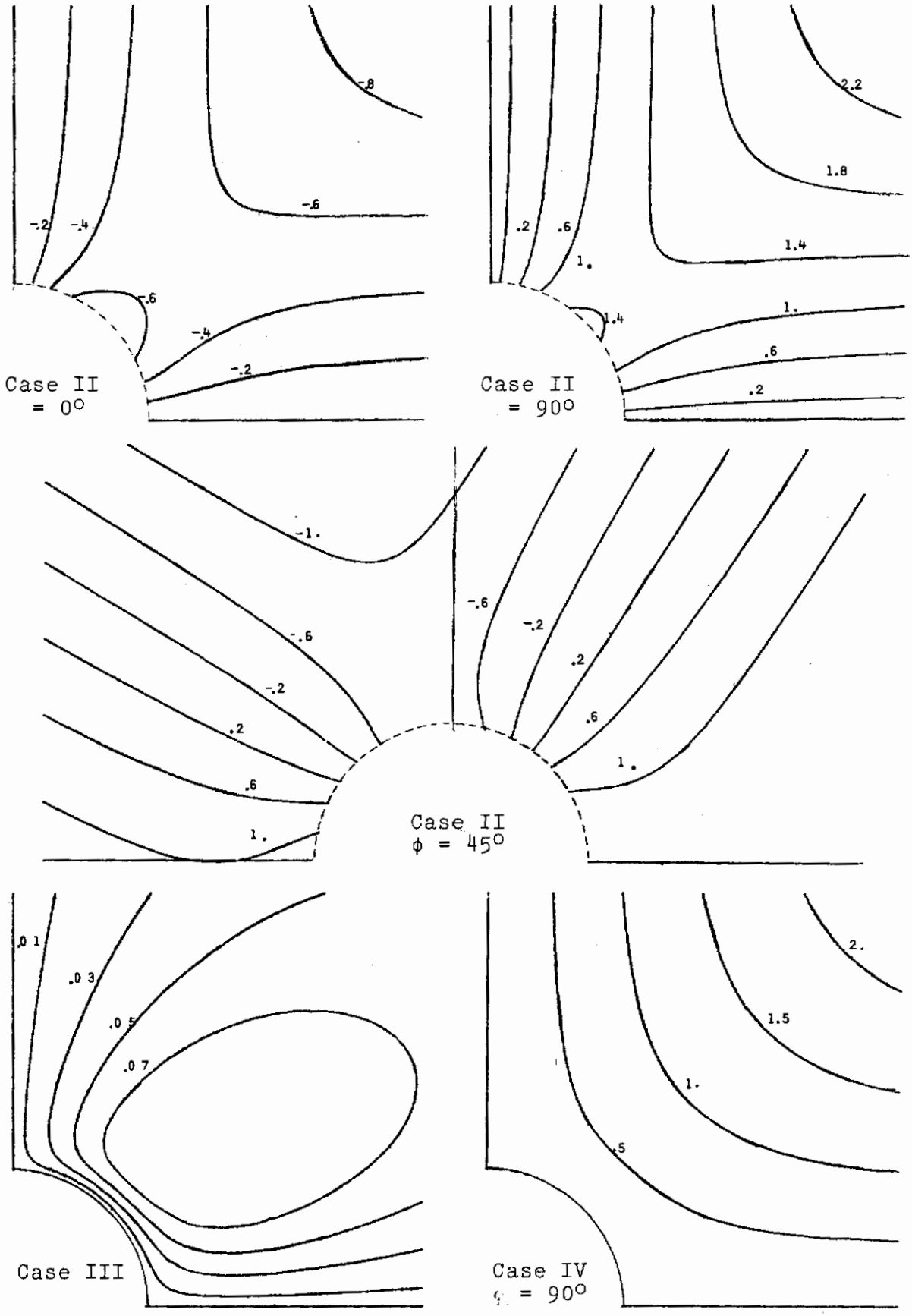


Figure 12. Contour Plots of u_θ (Material 1) for Cases II, III, and IV

typical in that it is distinctly orthotropic but does not yield extreme values for the coefficients of the generalized biharmonic. We have chosen this material to illustrate the effect of different loadings as shown in Figures 4, 6, 8, 10 and 12.

When displacement conditions are prescribed the stresses (and displacements) will in general depend on all four of the orthotropic material constants. In such problems there may be materials for which the response is quite different in character than for any of those listed in Table I. For example, for a given value of the coefficient $(2S_{12} + S_{66})$ materials corresponding to extreme values of S_{12} and S_{66} could be considered. Alternatively, one could study the effect of extreme values of the coefficients in the displacement formulation (2.2.5). We shall not pursue this question here. Along these same lines it should be noted that when $E_x = E_y$ (Materials (5) and (6)) Case III is equivalent to Case I in that uniform radial expansion is produced by σ_r = constant, $\tau_{r\theta} = 0$ on the boundary. This can be seen from the contour plots of u_r and u_θ for Materials (5) and (6) (Figures 9 and 11). This can also be seen (much less easily) from formula (3.1.19) by substituting the appropriate expressions for Φ_1 and Φ_2 (evaluated on the boundary) and then transforming to polar displacement components.

The contour plots of Figures 3-12 are all in the first quadrant, except for the one corresponding to Case II, $\phi = 45^\circ$. The circular boundary is indicated by a dotted line except in those cases for which this boundary is a contour. In order to avoid crowding, some of the contours have not been labeled. However, whenever such a label has been omitted the appropriate contour level can be deduced from neighboring contours since the contour levels follow a regular pattern. For the problems illustrated in the following contour plots the circumferential stress σ_θ is (with the exception of Case IV) the most critical stress. Thus, we have indicated the approximate position and magnitude of this critical stress on the contour plots of σ_θ. In this connection we note that for an isotropic material the maximum value of σ_θ is 1 in Case I and 3 in Case II. It can be seen in Figures 5 and 6 that (with the exception of Case II, $\phi = 90^\circ$) the maximum circumferential stress in Cases I and II is greater than that of the corresponding isotropic case. A similar comparison for Cases III and IV is not as simple since the isotropic stresses depend on Poisson's ratio.

From the contour plots of the preceding pages it is evident that stress concentrations can be produced or intensified when an isotropic material is replaced by an orthotropic material in a given problem. The most dramatic case of this in these contour plots is the stress concentration in σ_θ at $\theta=0^\circ$ (or $\theta=90^\circ$) for Case I and Material (6). (See Figure 5.) In Figures 13 and 14 we have plotted the variation of σ_θ along the ray $\theta=0$ and along the boundary. In Figure 13 we have also plotted the deviation from the isotropic solution and the stress function Ψ, from which σ_θ can be derived $\left(\sigma_\theta = \frac{\partial^2 \Psi}{\partial r^2}\right)$. In Figure 14 $\Psi=0$ and the isotropic solution is $\sigma_\theta=1$, so the stress function and deviation from the isotropic solution have not been plotted. In these figures the horizontal scale has been stretched by a factor of four in order to show the curves more clearly. Had the curves been plotted without stretching σ_θ and $\sigma_{\theta D}$ would appear to have practically vertical slopes at $\theta=0,90^\circ$. The true slopes are about 40 (versus 2 in the isotropic case), which is indicative of the extremely localized character of the stress concentration. We note in passing that $\sigma_{\theta D}$ approaches zero as r^{-4} while σ_θ varies as r^{-2} for large r. This is in contrast to σ_r, σ_{rD}, and $\tau_{r\theta}$ $(=\tau_{r\theta D})$ which all vary as r^{-2} for large r. Ψ behaves as $-\log r$ and Ψ_D as a constant as $r\to\infty$.

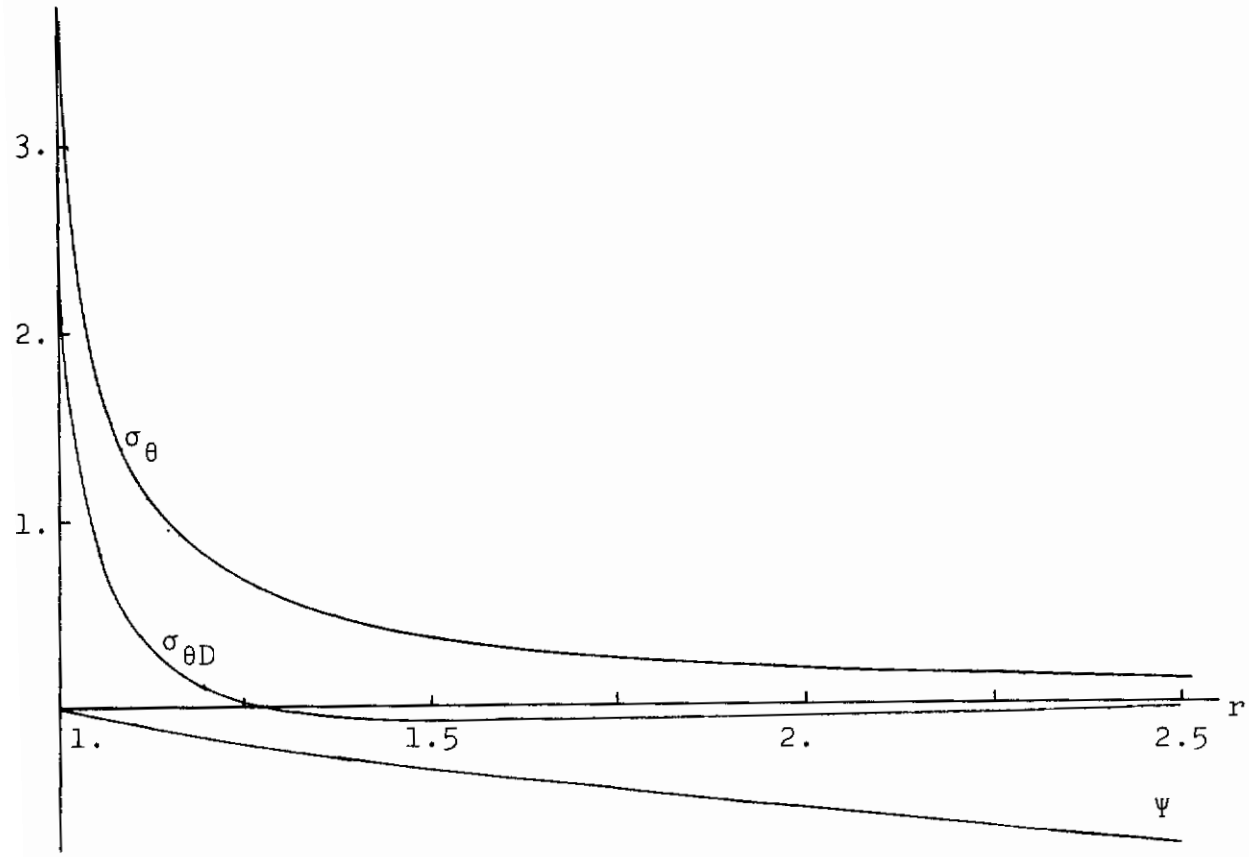


Figure 13. Variation of σ_θ, $\sigma_{\theta D}$, and Ψ Along the Ray $\theta=0$

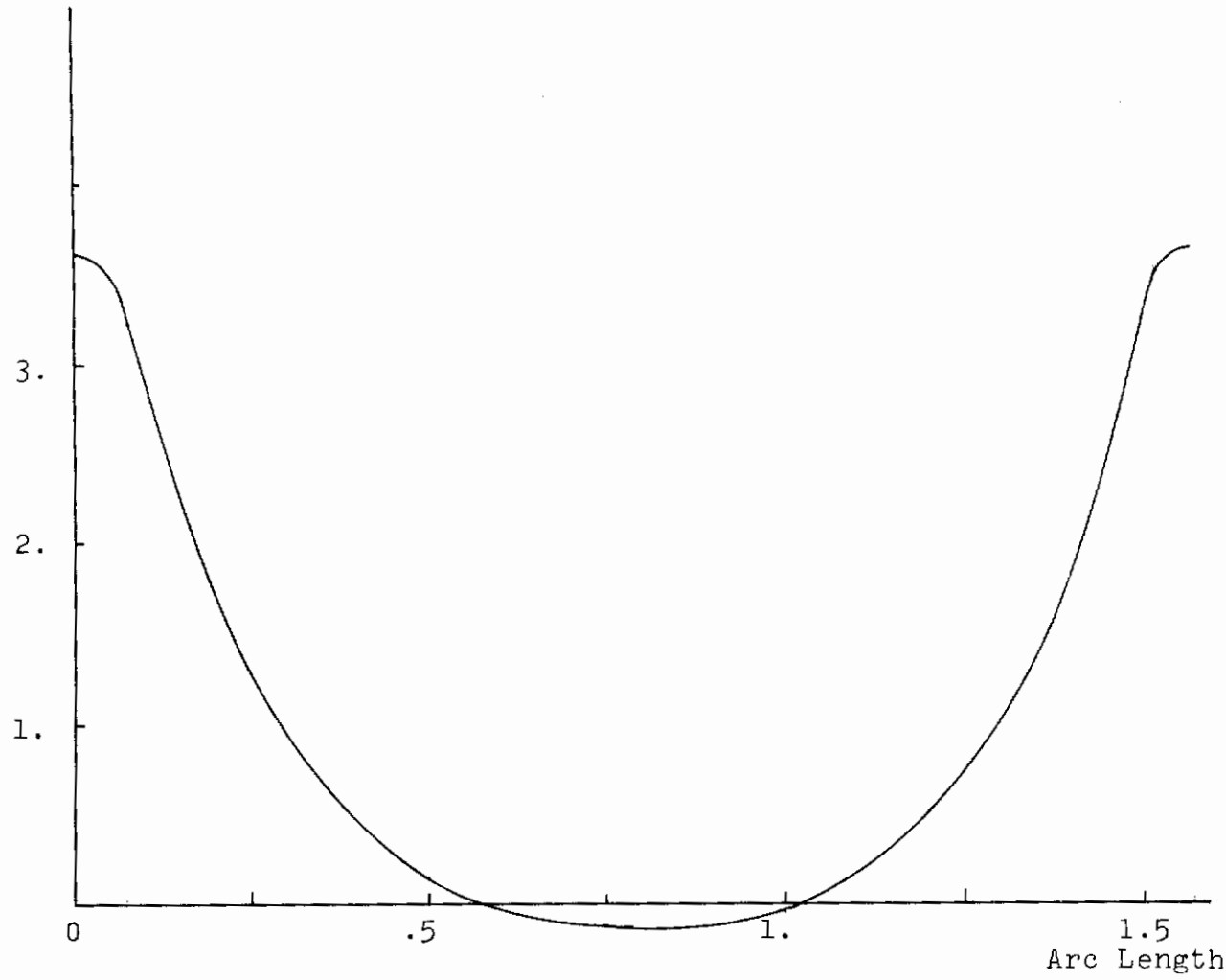


Figure 14. Variation of σ_θ Along the Boundary

SECTION IV

PERTURBATION SOLUTIONS

1. GENERAL THEORY

If a material is only mildly orthotropic (in a sense to be defined shortly) then one can usually expect that the behavior of this material under loading will not be much different than the behavior of some "near" isotropic material under the same loading. This suggests looking for a solution of the orthotropic problem in the form of a regular perturbation series, the first term of which is the solution of an isotropic problem (which presumably can be much more easily solved) and the expansion parameter corresponds to a measure of the deviation of the orthotropic elastic matrix from its "nearest" isotropic counterpart. In this article we develop a perturbation analysis for plane orthotropic elasticity problems based on the generalized biharmonic.

1a. Decomposition of the Generalized Biharmonic

Our starting point is the generalized biharmonic (2.2.9). Rewritten here it is

$$S_{22}\frac{\partial^4\Psi}{\partial x^4} + (2S_{12} + S_{66})\frac{\partial^4\Psi}{\partial x^2\partial y^2} + S_{11}\frac{\partial^4\Psi}{\partial y^4} = 0 \qquad (4.1.1)$$

Sokolnikoff [12] has proposed two different decompositions of (4.1.1) in order to apply regular perturbation methods for mildly orthotropic materials. For the first he defines

$$\varepsilon_1' = 1 - \frac{2S_{22}}{(2S_{12} + S_{66})}\ ,\quad \varepsilon_2' = 1 - \frac{2S_{11}}{(2S_{12} + S_{66})}\ , \qquad (4.1.2)$$

which is then used to rewrite (4.1.1) as

$$(1-\varepsilon_1')\frac{\partial^4\Psi}{\partial x^4} + 2\frac{\partial^4\Psi}{\partial x^2\partial y^2} + (1-\varepsilon_2')\frac{\partial^4\Psi}{\partial y^4} = 0 \qquad (4.1.3)$$

For mildly orthotropic materials ε_1' and ε_2' are small compared to unity and can be used as expansion parameters in a regular perturbation series. (Details of this perturbation procedure will be given after alternate decompositions of (4.1.1) are presented.)

For Sokolnikoff's second decomposition, the generalized biharmonic is first written in the form

$$\mu_1^2\mu_2^2 \frac{\partial^4\Psi}{\partial x^4} - (\mu_1^2 + \mu_2^2) \frac{\partial^4\Psi}{\partial x^2\partial y^2} + \frac{\partial^4\Psi}{\partial y^4} = 0, \qquad (4.1.4)$$

where μ_1 and μ_2 are the roots of the characteristic equation corresponding to (4.1.1). (See Article 1a of Section III.) Now for an isotropic material $\mu_1 = \mu_2 = i$, so it is natural to write

$$\mu_1 = i\,(1 + \varepsilon_1''), \quad \mu_2 = i\,(1 + \varepsilon_2''), \qquad (4.1.5)$$

where ε_1'' and ε_2'' are in general complex. For a mildly orthotropic material $|\varepsilon_1''|, |\varepsilon_2''| << 1$. When (4.1.5) is substituted into (4.1.4) the generalized biharmonic is in a form analogous to (4.1.3).

We now give a third decomposition of the generalized biharmonic which is the one that is actually used to obtain perturbation solutions to the four sample problems being considered in this study. Our approach is similar to that of Yih-O-Tu [13]. We first write

$$[S] = [S^O] + [S^R] \qquad (4.1.6)$$

Here $[S^O]$ is an isotropic compliance matrix which is, in some sense, near $[S]$. For mildly orthotropic materials this implies that the elements of $[S^R]$ are small compared to those of $[S^O]$. For the time being we do not restrict $[S^O]$ to be the "nearest" isotropic matrix in the sense of Appendix I. Next we express the generalized biharmonic (4.1.1) in terms of the S_{ij}^O and S_{ij}^R. The result, after dividing through by $S_{11}^O (= S_{22}^O)$, is

$$\left[1 + \frac{S_{22}^R}{S_{11}^o}\right]\frac{\partial^4\Psi}{\partial x^4} + \left[2 + \frac{(2S_{12}^R + S_{66}^R)}{S_{11}^o}\right]\frac{\partial^4\Psi}{\partial x^2\partial y^2} + \left[1 + \frac{S_{11}^R}{S_{11}^o}\right]\frac{\partial^4\Psi}{\partial y^4} = 0 \tag{4.1.7}$$

It is convenient for the following analysis to use different independent variables, z and $\bar{z}$, defined in the usual manner, i.e.,

$$z = x + iy, \quad \bar{z} = x - iy \tag{4.1.8}$$

Derivatives with respect to x and y are transformed to derivatives with respect to z and $\bar{z}$ by a straightforward application of the chain rule. The results that will be needed are

$$\begin{aligned}
\frac{\partial\Psi}{\partial x} &= \frac{\partial\Psi}{\partial z} + \frac{\partial\Psi}{\partial\bar{z}} \\
\frac{\partial\Psi}{\partial y} &= i\left(\frac{\partial\Psi}{\partial z} - \frac{\partial\Psi}{\partial\bar{z}}\right) \\
\frac{\partial^2\Psi}{\partial x^2} &= \frac{\partial^2\Psi}{\partial z^2} + 2\frac{\partial^2\Psi}{\partial z\partial\bar{z}} + \frac{\partial^2\Psi}{\partial\bar{z}^2} \\
\frac{\partial^2\Psi}{\partial y^2} &= -\frac{\partial^2\Psi}{\partial z^2} + 2\frac{\partial^2\Psi}{\partial z\partial\bar{z}} - \frac{\partial^2\Psi}{\partial\bar{z}^2} \\
\frac{\partial^2\Psi}{\partial x\partial y} &= i\left(\frac{\partial^2\Psi}{\partial z^2} - \frac{\partial^2\Psi}{\partial\bar{z}^2}\right) \\
\frac{\partial^4\Psi}{\partial x^4} &= \frac{\partial^4\Psi}{\partial z^4} + 6\frac{\partial^4\Psi}{\partial z^2\partial\bar{z}^2} + \frac{\partial^4\Psi}{\partial\bar{z}^4} + 4\left(\frac{\partial^4\Psi}{\partial z^3\partial\bar{z}} + \frac{\partial^4\Psi}{\partial z\partial\bar{z}^3}\right) \\
\frac{\partial^4\Psi}{\partial y^4} &= \frac{\partial^4\Psi}{\partial z^4} + 6\frac{\partial^4\Psi}{\partial z^2\partial\bar{z}^2} + \frac{\partial^4\Psi}{\partial\bar{z}^4} - 4\left(\frac{\partial^4\Psi}{\partial z^3\partial\bar{z}} + \frac{\partial^4\Psi}{\partial z\partial\bar{z}^3}\right) \\
\frac{\partial^4\Psi}{\partial x^2\partial y^2} &= -\frac{\partial^4\Psi}{\partial z^4} + 2\frac{\partial^4\Psi}{\partial z^2\partial\bar{z}^2} - \frac{\partial^4\Psi}{\partial\bar{z}^4}
\end{aligned} \tag{4.1.9}$$

Now using (4.1.9) we can write (4.1.7) as

$$\frac{\partial^4\Psi}{\partial z^2\partial\bar{z}^2} = \frac{1}{2S^o_{11}}\left\{-H^R_o\frac{\partial^4\Psi}{\partial z^2\partial\bar{z}^2} - H^R_1\left(\frac{\partial^4\Psi}{\partial z^4} + \frac{\partial^4\Psi}{\partial\bar{z}^4}\right) + H^R_2\left(\frac{\partial^4\Psi}{\partial z^3\partial\bar{z}} + \frac{\partial^4\Psi}{\partial z\partial\bar{z}^3}\right)\right\}, \tag{4.1.10}$$

where

$$H^R_o = \tfrac{1}{4}\left[3S^R_{11} + 3S^R_{22} + 2S^R_{12} + S^R_{66}\right]$$

$$H^R_1 = \tfrac{1}{8}\left[S^R_{11} + S^R_{22} - 2S^R_{12} - S^R_{66}\right] = \tfrac{1}{8}\left[S_{11} + S_{22} - 2S_{12} - S_{66}\right] = H_1$$

$$H^R_2 = \tfrac{1}{2}\left[S^R_{11} - S^R_{22}\right] = \tfrac{1}{2}\left[S_{11} - S_{22}\right] = H_2 \tag{4.1.11}$$

We note that $\frac{\partial^4\Psi}{\partial z^2\partial\bar{z}^2}$ appearing on the left of (4.1.10) is equivalent to $\nabla^4/16$ in (x,y) coordinates. The final form of H^R_1 and H^R_2 in (4.1.11) is obtained by writing $S^R_{ij} = S_{ij} - S^o_{ij}$ and noting that the contributions of the S^o_{ij} terms vanish. If S^o_{11} is chosen to be the 1-1 element of the nearest isotropic matrix in the sense of Appendix I, then it is easily checked that $H^R_o = 0$. If S^o_{11} is chosen differently then H^R_o can be expressed as a linear combination of H_1 and H_2. Thus, defining $[S^o]$ as in Appendix I has a definite advantage since the right side of (4.1.10) simplifies somewhat. Further support for this choice of $[S^o]$ is given later in this article. Thus, we proceed on the assumption that $[S^o]$ has been so chosen. We then define

$$\hat{\varepsilon}_1 = \frac{-H_1}{2S^o_{11}}, \quad \hat{\varepsilon}_2 = \frac{H_2}{2S^o_{11}}, \tag{4.1.12}$$

Definitions (4.1.12), which are used only in this section, differ slightly from ε_1 and ε_2 defined in (2.1.11). Using (4.1.12) we rewrite (4.1.10) as

$$\frac{\partial^4\Psi}{\partial z^2\partial\bar{z}^2} = \hat{\varepsilon}_1\left(\frac{\partial^4\Psi}{\partial z^4} + \frac{\partial^4\Psi}{\partial\bar{z}^4}\right) + \hat{\varepsilon}_2\left(\frac{\partial^4\Psi}{\partial z^3\partial\bar{z}} + \frac{\partial^4\Psi}{\partial z\partial\bar{z}^3}\right) \tag{4.1.13}$$

For any of the definitions of the two perturbation parameters (4.1.2), (4.1.5), or (4.1.12) we look for solutions of the form

$$\Psi = \sum_{i,j=0}^{\infty} \varepsilon_1^i\,\varepsilon_2^j\,\Psi_{ij} \tag{4.1.14}$$

(4.1.14) is then substituted into the corresponding decomposition of the generalized biharmonic and the coefficient of $\varepsilon^i\varepsilon^j$ (for each i-j combination) is equated to zero yielding a partial differential equation. The first of these (corresponding to i=j=0) is just the biharmonic. The other partial differential equations are of the form,

$$\nabla^4\Psi_{ij} = f(\Psi_{00},\Psi_{01},\Psi_{10},\Psi_{11},\ldots,\Psi_{i-1,j-1}), \tag{4.1.15}$$

where the right side of (4.1.15) is known since the $\Psi_{mn}(m<i,\ n<j)$ have been obtained in previous steps. For most problems (including the sample problems of this study) ε_1 and ε_2 do not appear in the prescribed boundary conditions. In such problems Ψ_{00} is determined so that these prescribed boundary conditions are satisfied, and all of the other Ψ_{ij} satisfy homogeneous boundary conditions.

Before proceeding further we discuss the relative merits of the three decompositions of the generalized biharmonic. Expansions of the form (4.1.14) are more useful if the perturbation parameters correspond to physically meaningful measures of orthotropy. The perturbation parameters defined in (4.1.2) and (4.1.12) do have this property. However, those

defined in (4.1.5) are related to the elastic constants in a rather complicated way and therefore do not have an easily interpretable physical meaning.

A more important consideration is how well the exact solution is approximated in a given number of terms for each of the three approaches. To decide rigorously which of the three is best in this respect would be difficult. However, we can give some qualitative arguments backed up by a few numerical experiments. We first observe that the forms of the generalized biharmonic in (4.1.3) and (4.1.4) are equivalent to special cases of (4.1.6)-(4.1.7). For example, (4.1.6)-(4.1.7) reduces to (4.1.3) if we choose

$$S^o_{11} = S^o_{22} = (S_{66} + 2S_{12})/2, \quad S^o_{12} = S_{12} + \gamma,$$
$$S^o_{66} = 2(S^o_{11} - S^o_{12}) = S_{66} - 2\gamma, \tag{4.1.16}$$

where γ is an arbitrary parameter, which has no effect on the determination of stresses if the boundary conditions are stress conditions. Using (4.1.16) and (4.1.2) we obtain

$$S^R_{11} \equiv S_{11} - S^o_{11} = -\epsilon_2' S^o_{11}, \quad S^R_{22} \equiv S_{22} - S^o_{11} = -\epsilon_1' S^o_{11}$$
$$S^R_{12} \equiv S_{12} - S^o_{12} = -\gamma \quad , \quad S^R_{66} \equiv S_{66} - S^o_{66} = -2\gamma \tag{4.1.17}$$

When (4.1.17) is used in (4.1.7), (4.1.3) is reproduced. Similarly (4.1.6)-(4.1.7) are equivalent to (4.1.4) if we choose

$$S^o_{11} = S^o_{22} = S_{11}, \quad S^o_{12} = S_{11}/2 + \gamma, \quad S^o_{66} = S_{11} - 2\gamma \tag{4.1.18}$$

Then using (4.1.5) we obtain

$$S_{11}^R = 0$$

$$S_{22}^R = \left[2\left(\varepsilon_1'' + \varepsilon_2''\right) + \left(\varepsilon_1''^2 + 4\varepsilon_1''\varepsilon_2'' + \varepsilon_2''^2\right) + 2\varepsilon_1''\varepsilon_2''\left(\varepsilon_1'' + \varepsilon_2''\right) + \left(\varepsilon_1''\varepsilon_2''\right)^2\right] S_{11}^o$$

$$2S_{12}^R + S_{66}^R = \left[2\left(\varepsilon_1'' + \varepsilon_2''\right) + \left(\varepsilon_1''^2 + \varepsilon_2''^2\right)\right] S_{11}^o \qquad (4.1.19)$$

When (4.1.19) is used in (4.1.7), the decomposition corresponding (4.1.4)-(4.1.5) is reproduced.

Since all three decompositions of the generalized biharmonic can be cast in the form (4.1.10), we use (4.1.10) to compare the different versions. According to (4.1.11) H_1^R and H_2^R depend only on the S_{ij}. Thus, the terms on the right of (4.1.10) involving H_1^R and H_2^R are the same for all three versions except for the factor $(2S_{11}^o)$. For mildly orthotropic materials S_{11}^o will not differ greatly among the three versions, and so there is little to choose between the three versions on the basis of the terms involving H_1^R and H_2^R. However, the term on the right of (4.1.10) involving H_o^R does depend critically on the choice of S_{11}^o. As mentioned previously, when S_{11}^o is the 1-1 element of the nearest isotropic matrix (in the sense of Appendix I) we have $H_o^R = 0$. For S_{11}^o as in (4.1.16) we have $H_o^R = 6H_1$, and for S_{11}^o as in (4.1.18) we have $H_o^R = -2H_1 - 2H_2$. Now as long as the boundary conditions are stress conditions the effect of the choice of S_{11}^o appears only on the right side of (4.1.10). (As will be seen in Article 2 of this section, in the case of displacement boundary conditions S_{11}^o and S_{12}^o appear in the solution in a fairly complicated way so the following qualitative arguments may not be as plausible as in the case of prescribed stresses.) The magnitude of any Ψ_{ij}

(and the resulting term of the perturbation series) is proportional to the right side of (4.1.10) corresponding to the particular i-j combination. Intuitively one might expect that the right side can be minimized in some sense by choosing S^o_{11} such that $H^R_o = 0$. Numerical results for sample problem I (uniform pressure on a hole in an infinite sheet) with the perturbation parameters defined as in (4.1.12) showed that solutions with S^o_{11} chosen as in (A1-11) were distinctly better than those with S^o_{11} chosen as in (4.1.16) or (4.1.18). It is recognized that use of perturbation parameters defined as in (4.1.2) and (4.1.5) would have some effect on the Ψ_{ij} (except for Ψ_{oo}). However, it is felt that the important factor is in the choice of S^o_{11}. (For displacement boundary conditions S^o_{12} would also have an influence, but we have not studied this effect.) On the basis of these qualitative arguments and the limited numerical comparisons it seems that choosing $[S^o]$ as in (A1-11) has definite advantages over the other choices as far as the convergence of the perturbation series is concerned.

Finally we observe that when the problem is formulated in terms of the complex variables z and $\bar{z}$ the algebra is somewhat simpler when S^o_{11} is chosen so that H^R_o vanishes and the perturbation parameters are defined as in (4.1.12). Formulation of the problem in terms of z and $\bar{z}$ is advantageous since the highly developed theory of complex variable solutions to isotropic plane elasticity problems can be utilized. Thus, on all three bases of comparison; correspondence between perturbation parameters and physical quantities, convergence of perturbation series, and algebraic simplicity; the choice of $[S^o]$ as in (A1-11) and $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ as in (4.1.12) appears to be as good or better than the other choices under consideration.

1b. Perturbation Analysis for a Particular Decomposition

Having given some justification for the formulation (4.1.12)-(4.1.13) let us proceed with the analysis. We list the governing differential equations for $\Psi_{ij}(i + j \leq 2)$.

$$\frac{\partial^4 \Psi_{00}}{\partial z^2 \partial \bar{z}^2} = 0$$

$$\frac{\partial^4 \Psi_{10}}{\partial z^2 \partial \bar{z}^2} = \frac{\partial^4 \Psi_{00}}{\partial z^4} + \frac{\partial^4 \Psi_{00}}{\partial \bar{z}^4}$$

$$\frac{\partial^4 \Psi_{01}}{\partial z^2 \partial \bar{z}^2} = \frac{\partial^4 \Psi_{00}}{\partial z^3 \partial \bar{z}} + \frac{\partial^4 \Psi_{00}}{\partial z \partial \bar{z}^3}$$

$$\frac{\partial^4 \Psi_{20}}{\partial z^2 \partial \bar{z}^2} = \frac{\partial^4 \Psi_{10}}{\partial z^4} + \frac{\partial^4 \Psi_{10}}{\partial \bar{z}^4} \qquad (4.1.20)$$

$$\frac{\partial^4 \Psi_{11}}{\partial z^2 \partial \bar{z}^2} = \frac{\partial^4 \Psi_{01}}{\partial z^4} + \frac{\partial^4 \Psi_{01}}{\partial \bar{z}^4} + \frac{\partial^4 \Psi_{10}}{\partial z^3 \partial \bar{z}} + \frac{\partial^4 \Psi_{10}}{\partial z \partial \bar{z}^3}$$

$$\frac{\partial^4 \Psi_{02}}{\partial z^2 \partial \bar{z}^2} = \frac{\partial^4 \Psi_{01}}{\partial z^3 \partial \bar{z}} + \frac{\partial^4 \Psi_{01}}{\partial z \partial \bar{z}^3}$$

Equations for higher terms in the series follow similarly. If $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are of the same order of magnitude it is necessary to determine all the Ψ_{ij} for $i+j \leq n$ in order to obtain a consistent n^{th} order approximation. As n increases the Ψ_{ij} proliferate. In practice one would seldom consider n>2. Yih-O-Tu [13] has avoided this proliferation by expanding the solution as

$$\Psi = \sum_{i=0}^{\infty} \hat{\epsilon}^i \Psi_i \qquad (4.1.21)$$

instead of (4.1.14). In effect (4.1.21) implies that

$$\hat{\epsilon}_1 = k_1 \hat{\epsilon} \quad , \quad \hat{\epsilon}_2 = k_2 \hat{\epsilon} \quad , \qquad (4.1.22)$$

where $\hat{\epsilon}$ is a new perturbation parameter which is a measure of the total orthotropy, e.g., as in (A1-16), and where k_1 and k_2 are constants of at most unit order. However, the decrease in the number of the Ψ_i (for a given order of approximation) occurs at the expense of more complicated right sides in the governing partial differential equations, and the total amount of work in either approach is the same. The expansion in two perturbation parameters has the advantage that for problems with stress boundary conditions the Ψ_{ij} are independent of the elastic constants. In the expansion (4.1.21) the Ψ_i are functions of k_1 and k_2, which do depend on the elastic constants. As far as numerical results are concerned the two types of expansions are equivalent, but the expansion in two perturbation parameters is in a more readily interpretable form.

The solutions of the partial differential equations in (4.1.20) can be written

$$\Psi_{ij} = \Psi_{Hij} + \Psi_{Pij} , \tag{4.1.23}$$

i.e.,the sum of a homogeneous and a particular solution. Ψ_{Hij} satisfies the biharmonic equation and has the form

$$\Psi_{Hij} = \bar{z}\phi_{ij}(z) + z\bar{\phi}_{ij}(\bar{z}) + \omega_{ij}(z) + \bar{\omega}_{ij}(\bar{z}), \tag{4.1.24}$$

where $\phi_{ij}(z)$ and $\omega_{ij}(z)$ are analytic functions. When the biharmonic is written in terms of z and $\bar{z}$, ie., as in the first of (4.1.20), the representation (4.1.24) is immediate. This representation is developed from a different point of view by Sokolnikoff [11]. Because of the simplicity of the left sides of (4.1.20) the determination of Ψ_{Pij} is straightforward. It is just a matter of integrating the right sides with respect to ż and $\bar{z}$. The results are

$$\Psi_{P10} = \frac{\bar{z}^3}{6}\phi''_{00} + \frac{\bar{z}^2}{2}\omega''_{00} + \frac{z^3}{6}\bar{\phi}''_{00} + \frac{z^2}{2}\bar{\omega}''_{00}$$

$$\Psi_{P01} = \frac{\bar{z}^2}{2}\phi'_{00} + \frac{z^2}{2}\bar{\phi}'_{00}$$

$$\Psi_{P20} = \left[\frac{\bar{z}^3}{6}\phi''_{10} + \frac{\bar{z}^2}{2}\omega''_{10} + \frac{z^3}{6}\bar{\phi}''_{10} + \frac{z^2}{2}\bar{\omega}''_{10}\right] + \left[\frac{\bar{z}^5}{120}\phi'''''_{00} + \frac{\bar{z}^4}{24}\omega'''''_{00} + \frac{z^5}{120}\bar{\phi}'''''_{00} + \frac{z^4}{24}\bar{\omega}'''''_{00}\right]$$

$$\Psi_{P11} = \left[\frac{\bar{z}^3}{6}\phi''_{01} + \frac{\bar{z}^2}{2}\omega''_{01} + \frac{z^3}{6}\bar{\phi}''_{01} + \frac{z^2}{2}\bar{\omega}''_{01} + \frac{\bar{z}^2}{2}\phi'_{10} + \frac{z^2}{2}\bar{\phi}'_{10}\right] + \left[2\left(\frac{\bar{z}^4}{24}\phi'''_{00} + \frac{z^4}{24}\bar{\phi}'''_{00}\right) + \frac{\bar{z}^3}{6}\omega'''_{00} + \frac{z^3}{6}\bar{\omega}'''_{00} + \frac{\bar{z}^2}{2}\phi'_{00} + \frac{z^2}{2}\bar{\phi}'_{00}\right]$$

$$\Psi_{P02} = \left[\frac{\bar{z}^2}{2}\phi'_{01} + \frac{z^2}{2}\bar{\phi}'_{01}\right] + \left[\frac{\bar{z}^3}{6}\phi''_{00} + \frac{z^3}{6}\bar{\phi}''_{00}\right]$$

(4.1.25)

Yih-O-Tu [13] has established a general formula for the Ψ_{Pi} (he uses only one expansion parameter) of his analysis. However, in practice it is unlikely that one would ever consider terms of higher than second order in the expansion parameters, so we have not developed a general formula for the Ψ_{Pij}.

Having obtained Ψ (in the form of a perturbation series) as a function of z and $\bar{z}$ the determination of stresses using (2.2.8) and (4.1.9) is straightforward. The corresponding displacements are obtained by integrating the stress-displacement relations. These results can be written in a more compact and computationally more convenient form if formulas for the "complex displacements" $u_{ij} + iv_{ij}$ are listed. They are

$$u_{00} + iv_{00} = 2\left(3S_{11}^{o} + S_{12}^{o}\right)\phi_{00} + 2\left(S_{12}^{o} - S_{11}^{o}\right)\left[z\bar{\phi}_{00}' + \bar{\omega}_{00}'\right]$$

$$u_{10} + iv_{10} = 2\left(3S_{11}^{o} + S_{12}^{o}\right)\phi_{10} + 2\left(S_{12}^{o} - S_{11}^{o}\right)\left[z\bar{\phi}_{10}' + \bar{\omega}_{10}'\right]$$
$$+ S_{11}^{o}\left[3\bar{z}^{2}\phi_{00}'' + 6\bar{z}\omega_{00}'' - \frac{z^{3}}{3}\bar{\phi}_{00}''' - z^{2}\bar{\omega}_{00}'''\right]$$
$$+ S_{12}^{o}\left[\bar{z}^{2}\phi_{00}'' + 2\bar{z}\omega_{00}'' + \frac{z^{3}}{3}\bar{\phi}_{00}''' + z^{2}\bar{\omega}_{00}'''\right]$$

$$u_{01} + iv_{01} = 2\left(3S_{11}^{o} + S_{12}^{o}\right)\phi_{01} + 2\left(S_{12}^{o} - S_{11}^{o}\right)\left[z\bar{\phi}_{01}' + \bar{\omega}_{01}'\right]$$
$$+ S_{11}^{o}\left[2\bar{z}\phi_{00}' - 4\omega_{00}' + 4\bar{\phi}_{00} - z^{2}\bar{\phi}_{00}''\right]$$
$$+ S_{12}^{o}\left[2\bar{z}\phi_{00}' + z^{2}\bar{\phi}_{00}''\right]$$

$$u_{20} + iv_{20} = 2\left(3S_{11}^{o} + S_{12}^{o}\right)\phi_{20} + 2\left(S_{12}^{o} - S_{11}^{o}\right)\left[z\bar{\phi}_{20}' + \bar{\omega}_{20}'\right]$$
$$+ S_{11}^{o}\left\{\left[3\bar{z}^{2}\phi_{10}'' + 6\bar{z}\omega_{10}'' - \frac{z^{3}}{3}\bar{\phi}_{10}''' - z^{2}\bar{\omega}_{10}'''\right]\right.$$
$$\left.+\left[-8\phi_{00} + \frac{\bar{z}^{4}}{4}\phi_{00}'''' + \bar{z}^{3}\omega_{00}'''' + 8z\bar{\phi}_{00}' + 8\bar{\omega}_{00}' - \frac{z^{5}}{60}\bar{\phi}_{00}'''' - \frac{z^{4}}{12}\bar{\omega}_{00}''''\right]\right\}$$
$$+ S_{12}^{o}\left\{\left[\bar{z}^{2}\phi_{10}'' + 2\bar{z}\omega_{10}'' + \frac{z^{3}}{3}\bar{\phi}_{10}''' + z^{2}\bar{\omega}_{10}'''\right]\right.$$
$$\left.+\left[\frac{\bar{z}^{4}}{12}\phi_{00}'''' + \frac{\bar{z}^{3}}{3}\omega_{00}'''' + \frac{z^{5}}{60}\bar{\phi}_{00}'''' + \frac{z^{4}}{12}\bar{\omega}_{00}''''\right]\right\}$$

$$u_{02} + iv_{02} = 2\left(3S_{11}^{o} + S_{12}^{o}\right)\phi_{02} + 2\left(S_{12}^{o} - S_{11}^{o}\right)\left[z\bar{\phi}_{02}' + \bar{\omega}_{02}'\right]$$
$$+ S_{11}^{o}\left\{\left[2\bar{z}\phi_{01}' - 4\omega_{01}' + 4\bar{\phi}_{01} - z^{2}\bar{\phi}_{01}''\right]\right.$$
$$\left.+\left[-8\phi_{00} + \bar{z}^{2}\phi_{00}'' + 4z\bar{\phi}_{00}' - \frac{z^{3}}{3}\bar{\phi}_{00}'''\right]\right\}$$
$$+ S_{12}^{o}\left\{\left[2\bar{z}\phi_{01}' + z^{2}\bar{\phi}_{01}''\right] + \left[\bar{z}^{2}\phi_{00}'' + \frac{z^{3}}{3}\bar{\phi}_{00}'''\right]\right\}$$

$$
\begin{aligned}
u_{11} + iv_{11} = {} & 2\left(3S_{11}^{o} + S_{12}^{o}\right)\phi_{11} + 2\left(S_{12}^{o} - S_{11}^{o}\right)\left[z\bar{\phi}_{11}' + \bar{\omega}_{11}'\right] \\
& + S_{11}^{o}\left\{\left[3\bar{z}^2\phi_{01}'' + 6\bar{z}\omega_{01}'' - \frac{z^3}{3}\bar{\phi}_{01}''' - z^2\bar{\omega}_{01}'''\right]\right. \\
& + \left[2\bar{z}\phi_{10}' - 4\omega_{10}' + 4\bar{\phi}_{10} - z^2\bar{\phi}_{10}''\right] \\
& + \left[-2\bar{z}\phi_{00}' - 8\omega_{00}' + \frac{4}{3}\bar{z}^3\phi_{00}''' + \bar{z}^2\omega_{00}'''\right. \\
& \left.\left. + 8\bar{\phi}_{00} + z^2\bar{\phi}_{00}'' + 4z\bar{\omega}_{00}'' - \frac{z^4}{6}\bar{\phi}_{00}'''' - \frac{z^3}{3}\bar{\omega}_{00}''''\right]\right\} \\
& + S_{12}^{o}\left\{\left[\bar{z}^2\phi_{01}'' + 2\bar{z}\omega_{01}'' + \frac{z^3}{3}\bar{\phi}_{01}''' + z^2\bar{\omega}_{01}'''\right] + \left[2\bar{z}\phi_{10}' + z^2\bar{\phi}_{10}''\right]\right. \\
& \left.\left. + \left[2\bar{z}\phi_{00}' + \frac{2}{3}\bar{z}^3\phi_{00}''' + \bar{z}^2\omega_{00}''' + z^2\bar{\phi}_{00}'' + \frac{z^4}{6}\bar{\phi}_{00}'''' + \frac{z^3}{3}\bar{\omega}_{00}''''\right]\right\}
\end{aligned}
\tag{4.1.26}
$$

Expressions for u_{ij} and v_{ij} are obtained by taking the real and imaginary parts of the right sides of formulas (4.1.26). It can be verified that all three stress-displacement relations are satisfied term by term when u_{ij} and v_{ij} are as in (4.1.26).

It should be emphasized that the formulas for stresses and displacements (in terms of the ϕ_{ij} and ω_{ij}) developed in this section are suitable for the perturbation analysis of any plane orthotropic elasticity problem. Of course there still remains the task of determining the ϕ_{ij} and ω_{ij} for a particular geometry and loading. We will do this for our four sample problems in Article 2 of this section. If displacements are prescribed on the boundary of the region then we must choose ϕ_{ij} and ω_{ij} such that

$$
\begin{aligned}
& u_{00} + iv_{00} = \bar{u} + i\bar{v} \\
& u_{ij} + iv_{ij} = 0 \qquad , \; i + j > 0
\end{aligned}
\tag{4.1.27}
$$

on the boundary, where $\bar{u}$ and $\bar{v}$ are the prescribed displacements. In the case of prescribed stresses we can use (3.1.24) to arrive at the following form of the stress boundary conditions:

$$2\frac{\partial \Psi_{00}}{\partial \bar{z}} = \int_{s_o}^{s} \left[i\,X(s) - Y(s) \right] ds + c$$

$$\frac{\partial \Psi_{ij}}{\partial \bar{z}} = 0 \quad , \; i + j > 0 \qquad (4.1.28)$$

In (4.1.28) the integral is along the boundary of the region in the sense indicated in Figure 2 of Section III, and s_o is an arbitrary starting point. The Ψ_{ij} are to be expressed in terms of the ϕ_{ij} and ω_{ij} in accordance with the formulae of this article. Then the ϕ_{ij} and ω_{ij} are to be determined so that conditions (4.1.28) are satisfied. The constant c in the first of (4.1.28) has no effect on the stress state and can be chosen in any convenient way.

2. SOLUTIONS FOR AN INFINITE REGION WITH A CIRCULAR HOLE

In this article we discuss the perturbation analysis of problems involving an infinite region bounded by a circular hole under fairly general loadings. In order to illustrate the procedure we develop general formulae for the ϕ_{ij} and ω'_{ij} (ω_{ij} itself is not needed) for the case of prescribed stresses on the circular boundary. General formulae for displacement boundary conditions can be established in a similar manner. However, such formulae are algebraically more involved, and since exact solutions to these problems are available it is doubtful that the results of this perturbation analysis would be of much use in general. Thus, we content ourselves (as far as displacement boundary conditions are concerned) with listing specific results for the sample problems under consideration in this study.

2a. Solutions for General Loadings

We consider arbitrary self-equilibrating loadings on the circular boundary, assuming that they can be analyzed into a Fourier series. We also include the possibility of uniform stresses at infinity. Under these conditions ϕ_{00} and ω_{00}, which correspond to an isotropic problem, are of the form

$$\phi_{00} = \sum_{i=-1}^{\infty} a_i z^{-i}$$

$$\omega_{00} = \sum_{i=-1}^{\infty} b_i z^{-i}, \qquad (4.2.1)$$

where a_i and b_i are in general complex. For non-self-equilibrating loads on the circular boundary it is necessary to include a log z-term in the forms of ϕ_{00} and ω_{00}, but this complicates the following analysis so we specifically exclude such loadings. If there are prescribed stresses at infinity a_{-1} and b_{-1} are determined first. Since a_{-1} and b_{-1} are complex, four conditions must be imposed at infinity; three stress conditions and one displacement condition, just as in the exact solutions of Section III. (See (3.2.6) and (3.2.7).) Having a_{-1} and b_{-1} and assuming the prescribed stresses on the circular boundary, $X(\theta)$ and $Y(\theta)$, have been expressed as a Fourier series it is straightforward to determine the other a_i and b_i by imposing the first of (4.1.28). (See Sokolnikoff [11] , p. 286 for a detailed discussion of the analysis of this isotropic problem.)

Since the problem is linear superposition is valid so it is only necessary to consider ϕ_{00} and ω'_{00} of the form

$$\phi_{00} = az^m \quad , \quad \omega'_{00} = bz^n, \qquad (4.2.2)$$

where m and n are integers less than or equal to plus one, and a and b are complex constants. Then by imposing the second of (4.1.28) and also the condition that the ϕ_{ij} and ω'_{ij} $(i+j>0)$ not contribute to the stresses or displacements at infinity, one finds for these ϕ_{ij} and ω'_{ij} corresponding to (4.2.2) the following:

$$\phi_{10} = \frac{-m(m-1)}{2} a z^{(m-4)} - n b z^{(n-2)}$$

$$\omega'_{10} = \frac{m(m-1)(m-5)}{3} a z^{(m-6)} + \frac{n(n-3)}{2} b z^{(n-4)}$$

$$\phi_{01} = -m a z^{(m-2)}$$

$$\omega'_{01} = \frac{m(m-3)}{2} a z^{(m-4)}$$

$$\phi_{02} = \frac{m(m-3)}{2} a z^{(m-4)}$$

$$\omega'_{02} = \frac{-m(m-4)(m-5)}{6} a z^{(m-6)}$$

$$\phi_{20} = \frac{-m(m-1)(m-6)(m-7)}{8} a z^{(m-8)} - \frac{n(n-4)(n-5)}{6} b z^{(n-6)}$$

$$\omega'_{20} = \frac{m(m-1)(m-7)(m-8)(m-9)}{30} a z^{(m-10)} + \frac{n(n-5)(n-6)(n-7)}{24} b z^{(n-8)}$$

$$\phi_{11} = \frac{m(m+2)(m-5)}{6} a z^{(m-6)} + \frac{n(n-3)}{2} b z^{(n-4)} - m a z^{(m-2)}$$

$$\omega'_{11} = \frac{-(m+1)(m-6)(m-7)}{12} a z^{(m-8)} - \frac{n(n-4)(n-5)}{6} b z^{(n-6)} + \frac{m(m-3)}{2} a z^{(m-4)}$$

(4.2.3)

If ϕ_{00} and ω'_{00} are sums of terms like those in (4.2.2) the corresponding ϕ_{ij} and ω_{ij} $(i+j>0)$ are obtained by adding the results of (4.2.3) arising from the individual terms of these sums. As mentioned previously, formulae analogous to (4.2.3) for displacement boundary conditions can be developed, but they involve the elastic constants S°_{11} and S°_{12} and are more complicated so we do not list them here.

2b. Solutions to the Four Sample Problems

The results for Cases I and II, for which stresses are prescribed, can be obtained from the formulae (4.2.3) once ϕ_{00} and ω'_{00} have been determined. For displacement boundary conditions, Cases III and IV, the results have been obtained from similar formulae not listed here.

CASE I: Uniform pressure (of unit magnitude) on the circular boundary.

$$X(\theta) = \cos\theta\ , \qquad Y(\theta) = \sin\theta$$

$$\phi_{00} = 0\ , \qquad \omega'_{00} = \frac{-1}{2z}$$

$$\phi_{10} = \frac{-1}{2z^3}\ , \qquad \omega'_{10} = \frac{-1}{z^5}$$

$$\phi_{01} = 0\ , \qquad \omega'_{01} = 0 \qquad (4.2.4)$$

$$\phi_{20} = \frac{-5}{2z^7}\ , \qquad \omega'_{20} = \frac{-7}{z^9}$$

$$\phi_{11} = \frac{-1}{z^5}\ , \qquad \omega'_{11} = \frac{-5}{2z^7}$$

$$\phi_{02} = 0\ , \qquad \omega'_{02} = 0$$

CASE II: Uniform tension (of unit magnitude) at infinity at an angle ϕ with respect to the x-axis and zero stress on the circular boundary.

$$X(\theta) = 0 \quad , \qquad Y(\theta) = 0$$

$$\phi_{00} = \frac{e^{2i\phi}}{4z} + \frac{z}{8}$$

$$\omega'_{00} = \frac{e^{2i\phi}}{4z^3} - \frac{1}{4z} - \frac{e^{-2i\phi}}{4}z$$

$$\phi_{10} = \frac{e^{2i\phi}}{2z^5} - \frac{1}{4z^3}$$

$$\omega'_{10} = \frac{5e^{2i\phi}}{4z^7} - \frac{1}{2z^5} + \frac{e^{2i\phi}}{4}\, z$$

$$\phi_{01} = \frac{e^{2i\phi}}{4z^3} + \left(\frac{\sin 2\phi}{8}\right) iz \qquad (4.2.5)$$

$$\omega'_{01} = \frac{e^{2i\phi}}{2z^5} - \frac{z}{8}$$

$$\phi_{20} = \frac{7e^{2i\phi}}{2z^9} - \frac{5}{4z^7}$$

$$\omega'_{20} = \frac{21e^{2i\phi}}{2z^{11}} - \frac{7}{2z^9} - \frac{e^{-2i\phi}}{4}\, z$$

$$\phi_{11} = \frac{5e^{2i\phi}}{2z^7} - \frac{1}{2z^5} + \frac{e^{2i\phi}}{4z^3} + \left(\frac{\sin 2\phi}{2}\right) iz$$

$$\omega'_{11} = \frac{7e^{2i\phi}}{z^9} - \frac{5}{4z^7} + \frac{e^{2i\phi}}{2z^5}$$

$$\phi_{02} = \frac{e^{2i\phi}}{2z^5}$$

$$\omega'_{02} = \frac{5e^{2i\phi}}{4z^7} + \left(\frac{\sin 2\phi}{8}\right) iz$$

In order to express the ϕ_{ij} and ω_{ij} in a more compact form for Cases III and IV we introduce the following definitions:

$$
\begin{aligned}
T &= (3S^{o}_{11} + S^{o}_{12})^{-1} \\
V &= (S^{o}_{12} - S^{o}_{11})^{-1} \\
W &= S^{o}_{11}\, T \\
X &= S^{o}_{12}\, T \\
Y &= S^{o}_{11}\, V \\
Z &= S^{o}_{12}\, V
\end{aligned}
\qquad (4.2.6)
$$

CASE III: Uniform radial expansion (of unit magnitude) of circular boundary.

$$\bar{u} + i\bar{v} = e^{i\theta}$$

$$
\begin{aligned}
\phi_{00} &= 0 \quad , & \omega'_{00} &= \frac{V}{2z} \\
\phi_{10} &= \frac{V}{2z^3} \quad , & \omega'_{10} &= \frac{V}{z^5} \\
\phi_{01} &= \frac{VW}{z} \quad , & \omega'_{01} &= \frac{VW}{z^3} \\
\phi_{20} &= \frac{5V}{2z^7} \quad , & \omega'_{20} &= \frac{7V}{z^9} - \frac{2VY}{z} \\
\phi_{02} &= \frac{VW}{z^3} \quad , & \omega'_{02} &= \frac{2VW}{z^5} - \frac{2TY^2}{z} \\
\phi_{11} &= \frac{V(5W+X)}{z^5} + \frac{2VW}{z} \quad , & \omega'_{11} &= \frac{5V(5W+X)}{2z^7} + \frac{2VW}{z^3}
\end{aligned}
\qquad (4.2.7)
$$

CASE IV: Uniform tension (of unit magnitude) at infinity at an angle ϕ with respect to the x-axis and zero displacement of the circular boundary.

$$\bar{u} + i\bar{v} = 0$$

$$\phi_{00} = \frac{(X-W)e^{2i\phi}}{4z} + \frac{z}{8}$$

$$\omega'_{00} = \frac{(X-W)e^{2i\phi}}{4z^3} - \frac{(Y+Z)}{4z} - \frac{e^{-2i\phi}}{4}\,z$$

$$\phi_{10} = \frac{(X-W)e^{2i\phi}}{2z^5} - \frac{(Y+Z)}{4z^3} + \frac{We^{-2i\phi}}{z}$$

$$\omega'_{10} = \frac{5(X-W)e^{2i\phi}}{4z^7} - \frac{(Y+Z)}{2z^5} + \frac{We^{-2i\phi}}{z^3} + \frac{e^{2i\phi}}{4}\,z \qquad (4.2.8)$$

$$\phi_{01} = \frac{(X-W)e^{2i\phi}}{4z^3} - \frac{WZ}{z} + \left(\frac{\sin 2\phi}{8}\right) iz$$

$$\omega'_{01} = \frac{(X-W)e^{2i\phi}}{2z^5} - \frac{WZ}{z^3} - Y\left[(W+X)e^{2i\phi} - \left(\frac{\sin 2\phi}{2}\right)1\right]\frac{1}{z} - \frac{z}{8}$$

$$\phi_{20} = \frac{7(X-W)e^{2i\phi}}{2z^9} - \frac{5(Y+Z)}{4z^7} + \frac{2We^{-2i\phi}}{z^5} + \frac{W(X-W)e^{2i\phi}}{z}$$

$$\omega'_{20} = \frac{21(X-W)e^{2i\phi}}{2z^{11}} - \frac{7(Y+Z)}{2z^9} + \frac{5We^{-2i\phi}}{z^7}$$

$$+ \frac{W(X-W)e^{2i\phi}}{z^3} + \frac{Y(Y+Z)}{z} - \frac{e^{-2i\phi}}{4}\,z$$

$$\phi_{02} = \frac{(X-W)e^{2i\phi}}{2z^5} - \frac{WZ}{z^3} + W\left[(XZ-4WZ-WY)e^{2i\phi} + Y\sin 2\phi i\right]\frac{1}{z}$$

$$\omega'_{02} = \frac{5(X-W)e^{2i\phi}}{4z^7} - \frac{2WZ}{z^5} + W\left[(XZ-4WZ-WY)e^{2i\phi} + Y\sin 2\phi i\right]\frac{1}{z^3}$$

$$+ \frac{2WZY}{z} + \left(\frac{\sin 2\phi}{8}\right)iz$$

$$\phi_{11} = \frac{5(X-W)e^{2i\phi}}{2z^7} - \frac{(3WY+8WZ+XZ)}{2z^5}$$

$$-\left[\frac{(3WY+6WZ-XZ)e^{2i\phi}}{4} - We^{-2i\phi} - Y\left(\frac{\sin 2\phi}{2}\right)i\right]\frac{1}{z^3}$$

$$- \frac{W(Y+Z)}{z} + \left(\frac{\sin 2\phi}{2}\right)iz$$

$$\omega'_{11} = \frac{7(X-W)e^{2i\phi}}{z^9} - \frac{5(3WY+8WZ+XZ)}{4z^7}$$

$$- \left[\frac{(3WY+6WZ-XZ)e^{2i\phi}}{2} - 2We^{-2i\phi} - Y\sin 2\phi\, i\right]\frac{1}{z^5}$$

$$- \frac{W(Y+Z)}{z^3} + \left[2Y\sin 2\phi i + W(Y+Z)\left(e^{-2i\phi} - 2e^{2i\phi}\right)\right]\frac{1}{z}$$

SECTION V

BOUNDARY RESIDUAL METHODS

1. INTRODUCTION

In this section we are concerned with weighted residual approximate methods for which the family of assumed functions satisfies the field equations exactly and where the boundary conditions are satisfied in some approximate sense. The way in which the boundary conditions are approximately satisfied depends on the particular weighted residual method employed. Here we are primarily concerned with two of these methods; integral least squares and the use of the boundary Reissner energy principle, which was established in (2.3.1)ff. Other candidates are point least squares, collocation, subdomain, minimax, and Trefftz methods. For more details on these methods we refer the reader to Crandall [14] and Hopper, et. al. [15]. In this section these methods are mentioned only insofar as they are related to the integral least squares or boundary Reissner energy methods.

In order to be able to use boundary residual methods one must have at his disposal families of exact solutions to the governing field equations. As discussed in Section II (equations (3.1.2)ff.) exact solutions to the generalized biharmonic are readily available. Thus, from this standpoint boundary residual methods are well suited for the determination of approximate solutions to orthotropic plane elasticity problems.

It should also be noted that boundary methods are particularly useful when the boundaries are complex or irregular. This is in contrast to the variational methods to be discussed in Section VI which involve surface integrals. If the region of integration is not simple these integrals may be difficult to perform analytically. One might attempt to evaluate such integrals numerically. However, at present the theory of numerical quadrature for multiple integrals is in a rather primitive state. On the other hand boundary methods involve at most line integrals, for which there exists a highly developed theory of numerical quadrature, and such integrals can be evaluated along irregular boundaries almost as easily as they can along simple boundaries.

1a. Integral Least Squares on the Boundary

In Section III, on exact solutions, we showed that the field equations of elasticity are satisfied exactly when the stresses and displacements are expressed in terms of the analytic functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ in accordance with (3.1.18) and (3.1.19). Recall that $z_{1,2} = x + \mu_{1,2}y$, where μ is complex. For the approximate methods of this section we assume

$$\Phi_1(z_1) = \sum_{i=1}^{n} a_i f_i(z_1)$$

$$\Phi_2(z_2) = \sum_{i=1}^{n} b_i g_i(z_2) \quad , \tag{5.1.1}$$

where f_i and g_i are analytic functions to be chosen in a convenient fashion and hopefully also in a way such that a good approximation to the true solution is contained within these families. a_i and b_i are complex constants to be determined in accordance with the particular boundary residual method under consideration.

In what follows it is assumed that the stresses and displacements have been expressed in terms of the a_i, b_i, f_i, and g_i using (5.1.1) in (3.1.18) and (3.1.19). Now we form the least squares functional,

$$I_{LS} = W \int_{C_d} \left\{ (\vec{u} - \vec{\bar{u}}) \bullet (\vec{u} - \vec{\bar{u}}) \right\} ds + \int_{C_s} \left\{ (\vec{F} - \vec{\bar{F}}) \bullet (\vec{F} - \vec{\bar{F}}) \right\} ds, \tag{5.1.2}$$

where $\vec{\bar{u}}$ and $\vec{\bar{F}}$ are the prescribed displacement and stress vectors. $\vec{u}$ and $\vec{F}$ are obtained from the assumed forms of Φ_1 and Φ_2. W is a weighting function which is almost always necessary when dealing with mixed boundary value problems. The reason for

this is that the stresses are generally several orders of magnitude larger than the displacements, and use of (5.1.2) without a weighting factor would result in the integral along C_s having a disproportionately strong influence on the determination of the generalized coefficients. To avoid this W is chosen large enough to give the integral over C_d a reasonable influence.

I_{LS} is a quadratic function of the 2n generalized coefficients $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. Since a_i and b_i are complex there are really 4n real coefficients to be determined. When employing the least squares principle we minimize I_{LS} over these 4n real variables. This is a standard minimization problem, ie., I_{LS} is differentiated with respect to each of the 4n variables, and the result of each differentiation is set to zero. This yields a symmetric system of 4n linear algebraic equations to be solved. When the solution of this system is obtained the corresponding expressions for stresses and displacements are immediate.

Conceptually the method of integral least squares is quite straightforward. In practice certain problems present themselves. One such problem is the critical selection of the assumed forms of Φ_1 and Φ_2. A poor choice makes it impossible to obtain good approximate solutions. Discussion of this point will be given in Article 2 of this section. Having chosen the forms of Φ_1 and Φ_2 it is next necessary to perform the relevant line integrations in (5.1.2). For certain choices of the assumed forms of Φ_1 and Φ_2 and for simple boundaries it may be possible to perform the integrations analytically. Generally, however, it is expedient, if not necessary, to determine the integrals numerically. The most common numerical quadrature formulas are of the Newton-Cotes or Gauss families. These formulas have the general form,

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \quad , \qquad (5.1.3)$$

where w_i are weights and x_i are abscissae which depend on the particular quadrature formula. If the interval of integration is too long to give the desired accuracy, the interval is broken into subintervals, and formula (5.1.3) is used for each subinterval. The results for each subinterval are then summed to obtain an approximate value for the total integral. In any case when the integrals of (5.1.2) are evaluated by means of a quadrature formula of the form (5.1.3) the method of integral least squares on the boundary is equivalent to a particular weighted point least squares method on the boundary. There is a family of formulas of the form (5.1.3) for which all of the w_i are equal. These are known as Gauss-Chebyshev quadrature formulas. This special property of the Gauss-Chebyshev family is interesting in the present application since the method of integral least squares is then equivalent to the method of unweighted point least squares providing the square of the boundary residual is evaluated at the abscissae of the particular Gauss-Chebyshev formula. However, these abscissae are not equally spaced, so we cannot obtain equivalence with the most common type of unweighted point least squares. It should be mentioned that the method of integral least squares and unweighted point least squares on a uniform mesh become equivalent when the quadratures are done using the trapezoidal rule, the simplest of the Newton-Cotes formulas.

Gaussian quadrature formulas usually give more accurate results than Newton-Cotes formulas (for equal numbers of points in the interval). Since the functions that we must integrate are obtained from relatively involved formulas it is important to keep the number of function evaluations (at the abscissae, x_i, of the quadrature formula (5.1.3)) to a minimum. This consideration points in favor of using a Gaussian quadrature formula for our work. Of these formulas the Gauss-Chebyshev is computationally the simplest since all the weights are the same. For the reasons cited in this and the preceding paragraph it was decided to use a Gauss-Chebyshev formula, in particular a five point formula, for all of the numerical quadratures required in this study. (See Ralston [16] for a discussion of numerical quadrature, including Gauss-Chebyshev quadrature.)

Finally we give a brief discussion of a serious computational difficulty commonly occurring with least squares methods. This is the problem of the conditioning of the coefficient matrix. (See Appendix III for a discussion of

matrix conditioning in the general case.) Perhaps the most elementary type of least squares analysis is the determination of the "best" fit of a given curve over a finite interval using monomials as fitting functions. In this case it can be shown (see Ralston [16] p. 232ff.) that the coefficient matrix is approximately a Hilbert matrix, which is notorious for being ill-conditioned. In more involved least squares analyses, such as ours, it is not such a simple matter to demonstrate analytically the ill-conditioning of the coefficient matrix. However, numerical results generally do indicate a poorly conditioned coefficient matrix. It is to be emphasized that a poorly conditioned coefficient matrix does not imply that it is impossible to obtain a good fit with the assumed family of fitting functions. If enough significance could be carried in the computations the conditioning of the coefficient matrix becomes irrelevant, and a good fit is obtained, providing the assumed family of fitting functions has been suitably chosen. However, in view of the limitations of existing computing equipment, matrix conditioning is an important consideration. From this standpoint it is interesting to compare various weighted residual methods (using the same family of assumed functions). This will be done in Section VII for boundary least squares and boundary Reissner energy.

Often by a suitable rearrangement of the assumed family of functions it is possible to improve the conditioning of the coefficient matrix. For example, the conditioning of the coefficient matrix for a least squares fit using monomials as fitting functions is often improved if a family of Chebyshev polynomials (of the same degree), but not necessarily orthogonal over the fitting interval, is used instead. See [17], p. 263 for some discussion of this. An even better way of improving matrix conditioning is to use orthogonal functions so that the coefficient matrix is diagonal. This is especially useful when fitting with polynomial or trigonometric functions, since there are well known sets of orthogonal polynomials and trigonometric functions. However, in our work the required fitting functions are peculiar to the particular application. Thus, it would be necessary to develop our own orthogonal functions, which would require considerable effort. In addition different sets of orthogonal functions would be required depending on the type of boundary conditions. For this reason we have not tried to use orthogonal functions in our numerical work.

1b. Reissner Energy on the Boundary

Just as in the boundary least squares analysis we assume Φ_1 and Φ_2 to be of the form (5.1.1). We then use the resulting stresses and displacements in the Reissner energy functional (2.3.11). As explained there the volume integral vanishes when the field equations are exactly satisfied. Thus, (2.3.11) becomes

$$\Phi_{RB} = \frac{1}{2}\int_{C_s} (\vec{F} - 2\vec{\bar{F}})\bullet \vec{u}\, ds - \frac{1}{2}\int_{C_d} (\vec{u} - 2\vec{\bar{u}})\bullet \vec{F}\, ds \qquad (5.1.4)$$

Like the least squares functional, I_{LS}, Φ_{RB} is a quadratic function of the generalized coordinates a_i and b_i. In accordance with the Reissner energy principle Φ_{RB} is to be rendered stationary with respect to the generalized coordinates, which yields a symmetric system of 4n linear algebraic equations to be solved for the real and imaginary parts of the a_i and b_i.

In order to gain some additional insight as to the significance of rendering Φ_{RB} stationary we consider the variational form,

$$\delta\Phi_{RB} = \frac{1}{2}\int_{C_s} \left\{(\vec{F} - 2\vec{\bar{F}})\bullet \delta\vec{u} \;+\; \vec{u}\bullet\delta\vec{F}\right\} ds$$

$$-\frac{1}{2}\int_{C_d} \left\{(\vec{u} - 2\vec{\bar{u}})\bullet \delta\vec{F} \;+\; \vec{F}\bullet\delta\vec{u}\right\} ds \qquad (5.1.5)$$

Inasmuch as the stresses and displacements corresponding to both the quantities $\vec{F}$ and $\vec{u}$ and the varied quantities $\delta\vec{F}$ and $\delta\vec{u}$ satisfy the equations of elasticity internally we can use the reciprocal theorem to write

$$\int_{C_s+C_d} \vec{u}\bullet\delta\vec{F}\, ds = \int_{C_s+C_d} \vec{F}\bullet\delta\vec{u}\, ds \qquad (5.1.6)$$

Then using (5.1.6) in (5.1.5) we obtain

$$\delta\Phi_{RB} = \int_{C_s} (\vec{F} - \bar{\vec{F}})\cdot\delta\vec{u}\, ds - \int_{C_d} (\vec{u} - \bar{\vec{u}})\cdot\delta\vec{F}\, ds \qquad (5.1.7)$$

This result also follows directly from the variational form of the Reissner energy functional listed in equation (A6-3). In the form (5.1.7) it is clear that the stationary conditions of Φ_{RB}, ie., $\delta\Phi_{RB}=0$, are just the boundary conditions,

$$\begin{aligned} \vec{F} &= \bar{\vec{F}} \quad , \quad \text{on } C_s \\ \vec{u} &= \bar{\vec{u}} \quad , \quad \text{on } C_d \end{aligned} \qquad (5.1.8)$$

It is interesting to compare (5.1.7) with the variational form of the least squares functional. From (5.1.2) we obtain

$$\delta I_{LS} = \int_{C_s} (\vec{F} - \bar{\vec{F}})\cdot\delta\vec{F}\, ds + \int_{C_d} (\vec{u} - \bar{\vec{u}})\cdot\delta\vec{u}\, ds \qquad (5.1.9)$$

It is seen that the form of $\delta\Phi_{RB}$ is similar in character to δI_{LS}.

In the case of least squares the meaning of the functional I_{LS} is easily understood, and the process of extracting the "best" solution from the assumed family of functions by minimizing I_{LS} is intuitively satisfying. We have no similar understanding of Φ_{RB} and the process of rendering Φ_{RB} stationary. This can be remedied in certain special cases (in particular when the boundary conditions are either all stress or all displacement conditions). We do this by adopting a point of view taken by Morley [18]. He introduced the functional,

$$I_M = \frac{1}{2}\int_{C_s+C_d} (\vec{F} - \bar{\vec{F}})\cdot(\vec{u} - \bar{\vec{u}})\, ds \qquad (5.1.10)$$

Up until now we have used $\vec{\bar{F}}$ to denote the prescribed stress vector on C_s and similarly with $\vec{\bar{u}}$ on C_d. In (5.1.10) $\vec{\bar{F}}$ on C_d and $\vec{\bar{u}}$ on C_s are understood to correspond to the exact solution. These quantities are of course unknown a priori. However, as we shall see, knowledge of $\vec{\bar{F}}$ on C_d and $\vec{\bar{u}}$ on C_s is unnecessary in the application of Morley's method when the boundary conditions are either all displacement or all stress conditions. For mixed boundary conditions it does not appear that Morley's method can be used.

A physical meaning can be attached to I_M. This functional is equal to the strain energy in the planar body under the loading $(\vec{F} - \vec{\bar{F}})$. Thus, I_M is non-negative. It is clear that as the approximate solution approaches the exact solution, $I_M \to 0$. In this property and its non-negativeness I_M is similar to the least squares functional I_{LS}. The method of least squares and Morley's method are analogous in that the relevant functional is minimized over the generalized coefficients of the problem in order to obtain approximate solutions. However, unlike I_{LS} (and Φ_{RB}), I_M cannot be used in its unvaried form to obtain approximate solutions. In order to put I_M in a suitable form we first compute

$$\delta I_M = \frac{1}{2}\int_{C_s+C_d} \left\{(\vec{F} - \vec{\bar{F}})\bullet\delta\vec{u} + (\vec{u} - \vec{\bar{u}})\bullet\delta\vec{F}\right\} \, ds \qquad (5.1.11)$$

Next we use the reciprocal theorem to obtain

$$\int_{C_s+C_d} \vec{\bar{F}}\bullet\delta\vec{u} \, ds = \int_{C_s+C_d} \vec{\bar{u}}\bullet\delta\vec{F} \, ds \qquad (5.1.12)$$

If all the boundary conditions are stress conditions then we can use (5.1.12) to rewrite (5.1.11) as

$$\delta I_M = \frac{1}{2}\int_{C_s} \left\{(\vec{F} - 2\vec{\bar{F}})\bullet\delta\vec{u} + \vec{u}\bullet\delta\vec{F}\right\} ds \qquad (5.1.13)$$

In the case when all boundary conditions are displacement conditions (5.1.11) becomes

$$\delta I_M = \frac{1}{2} \int_{C_d} \left\{ (\vec{u} - 2\vec{\bar{u}}) \bullet \delta\vec{F} + \vec{F} \bullet \delta\vec{u} \right\} ds \qquad (5.1.14)$$

The forms of δI_M in (5.1.13) and (5.1.14) are suitable for use in obtaining approximate solutions. Comparison of (5.1.13) and (5.1.14) with (5.1.5) shows that in the case of all stress boundary conditions, $\delta I_M = \delta \Phi_{RB}$. When the boundary conditions are all displacement conditions, $\delta I_M = -\delta \Phi_{RB}$. Thus, in these two special cases the minimum conditions of I_M (ie., $\delta I_M = 0$) are equivalent to the stationary conditions of Φ_{RB}. As mentioned earlier the Morley method cannot be used in the case of mixed boundary conditions. However, since Φ_{RB} can be used to obtain approximate solutions quite generally, and since it yields results equivalent to those of the Morley method whenever the Morley method is applicable, the use of Φ_{RB} to obtain approximate solutions appears to be an appropriate generalization of the Morley method. It should be recognized that for mixed boundary conditions the stationary condition of Φ_{RB} is generally not an extremum.

Before proceeding further it is appropriate to make some observations regarding the relationship between Morley's method and the methods of potential and complementary energy. In Article 3a of Section II it was established that the Reissner energy functional Φ_R is equivalent to the potential energy functional Φ_P if the comparison states satisfy the stress-displacement relations internally and the displacement conditions on the boundary. Now the comparison states for the boundary principles of this section satisfy all of the equations of elasticity internally. Thus, if all boundary conditions are stress conditions Φ_{RB} is equivalent to Φ_P. Similarly when all boundary conditions are displacement conditions Φ_{RB} is equivalent to Φ_C. We then conclude that when the Morley method

is applicable (when the boundary conditions are either all displacement conditions or all stress conditions) it is a special case of either the method of complementary energy or potential energy. It is a special case in that the comparison states are more restrictive than required by the statements of the complementary or potential energy principles.

Before concluding this article it is worthwhile to give a brief discussion of the so-called Trefftz methods. More details about Trefftz methods can be found in the books by Sokolnikoff [11] and Mikhlin [19], and in a paper by Weinstein [20]. One of the important uses of the Trefftz method is to obtain lower (upper) bounds of functionals for which upper (lower) bounds can be obtained by some other method. The distinguishing characteristic of the Trefftz method is that the assumed functions must satisfy certain conditions internally but are not required to satisfy any conditions on the boundaries. Morley introduced his method as a Trefftz method for elasticity problems, which it in fact is. However, as we have seen, whenever the Morley method is applicable it is equivalent to a special case of either the potential energy or complementary energy method. Thus, in these special cases the potential or complementary energy method may be considered a Trefftz method. In Article 3a of Section II it was shown that the methods of potential and complementary energy can be used quite generally to obtain upper and lower bounds respectively to the exact values of the corresponding energy functionals. (See (2.3.10).) However, if displacements (stresses) are prescribed anywhere on the boundary the assumed functions in the method of potential (complementary) energy must satisfy these conditions. Thus, although the potential (complementary) energy method can be used to obtain upper (lower) bounds in these cases it cannot be considered a Trefftz method. On the other hand, although it is not necessary for the assumed functions used in connection with Φ_{RB} to satisfy any boundary conditions, this method does not furnish upper or lower bounds in problems with mixed boundary conditions and so cannot generally be regarded as a Trefftz-method. It should be mentioned that for certain elasticity problems with mixed boundary conditions true Trefftz methods have been developed. See Mikhlin [19], p.375 ff. for a discussion of this. However, this writer is aware of no general Trefftz procedure for arbitrary mixed boundary value problems of elasticity.

2. APPLICATION OF BOUNDARY LEAST SQUARES AND REISSNER ENERGY TO THE FOUR SAMPLE PROBLEMS

In this article we discuss the application of boundary least squares and Reissner energy to specific problems, especially the four sample problems of this study. The question of appropriate choices of trial functions, and how to choose these functions in order to take advantage of problem symmetry is considered. Certain difficulties arising from inappropriate choices of these trial functions are analyzed in some detail. Also, the way in which uniform stresses at infinity are accounted for is discussed. Finally a brief summary of important details of the boundary least squares and Reissner energy formulations from a computational standpoint is given.

2a. Choice of Assumed Functions

In the case of sample problems I (uniform pressure) and III (uniform radial expansion) it is clear that the stresses and displacements must vanish at infinity. In addition we know that the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ must be analytic in the regions of the z_1 and z_2 planes which are the images of the exterior of the unit circle in the physical plane. Each of these regions is the exterior of an ellipse. (See Appendix II for details.) Probably the simplest forms of Φ_1 and Φ_2 which satisfy the conditions just cited are

$$\Phi_1 = \sum_{i=1}^{n} a_i z_1^{-i}$$

$$\Phi_2 = \sum_{i=1}^{n} b_i z_2^{-i} \qquad (5.2.1)$$

As mentioned following (5.1.1) the a_i and b_i are complex constants. Now in problems where there is symmetry in the loading, material, and geometry it may be possible to conclude a priori that the stresses and displacements will also have a

certain symmetry. We discussed this at the end of Article 2b of Section II. There we established that the most general problem can be separated into four different uncoupled problems each with a special type of symmetry (or antisymmetry). They are

i) Symmetry about x and y axes
Even cosine harmonics - Ψ, σ_r, σ_θ, u_r
Even sine harmonics - $\tau_{r\theta}$, u_θ

ii) Symmetry about x-axis, antisymmetry about y-axis
Odd cosine harmonics - Ψ, σ_r, σ_θ, u_r
Odd sine harmonics - $\tau_{r\theta}$, u_θ

iii) Symmetry about y-axis, antisymmetry about x-axis
Odd sine harmonics - Ψ, σ_r, σ_θ, u_r
Odd cosine harmonics - $\tau_{r\theta}$, u_θ

iv) Antisymmetry about x and y axes
Even sine harmonics - Ψ, σ_r, σ_θ, u_r
Even cosine harmonics - $\tau_{r\theta}$, u_θ

Sample Problems I and III correspond to (i), and Sample Problems II and IV correspond to the superposition of (i) and (iv). By suitably restricting the a_i and b_i and the powers of z_1 and z_2 of (5.2.1) we can duplicate the symmetry corresponding to anyone of these four groups. In doing this we must distinguish two different cases; μ_1 and μ_2 purely imaginary, μ_1 and μ_2 complex. In the former we have the correspondence,

i) Odd powers of z_1 and z_2; a_i and b_i real

ii) Even powers of z_1 and z_2; a_i and b_i real

iii) Even powers of z_1 and z_2; a_i and b_i imaginary

iv) Odd powers of z_1 and z_2; a_i and b_i imaginary

For μ_1 and μ_2 complex we have the correspondence,

i) Odd powers of z_1 and z_2; $a_i = \bar{b}_i$

ii) Even powers of z_1 and z_2; $a_i = \bar{b}_i$

iii) Even powers of z_1 and z_2; $a_i = -\bar{b}_i$

iv) Odd powers of z_1 and z_2; $a_i = -\bar{b}_i$

When solving a problem having one of these four types of symmetry it is not necessary to use the corresponding conditions on the a_i and b_i and on the powers of z_1 and z_2 in (5.2.1). Numerical results for problems with special types of symmetry,e.g., Sample Problem I with symmetry about the x and y axes, showed that the correct symmetry was obtained from the general forms of Φ_1 and Φ_2 in (5.2.1) using either the boundary least squares of Reissner energy criteria. This symmetry was obtained to a high degree of accuracy even when the results were otherwise poor.

There are two important reasons why one should appropriately specialize (5.2.1) when the problem has one of the four types of symmetry. The obvious reason is that the order of the matrix system that must be solved is reduced. The second reason is that the time needed to generate an element of the coefficient or right hand side matrix can be shortened considerably. This can be seen by considering,e.g., Sample Problem I with symmetry about the x and y axis. the general

forms of Φ_1 and Φ_2 are used then it is necessary to perform the relevant quadratures over the interval $[0,2\pi]$. However, if the forms of Φ_1 and Φ_2 are appropriately restricted then $[0,\pi/2]$ can be used since integration over $[0,2\pi]$ will give the same results except for a factor of four which appears on both sides of the matrix system and therefore has no effect on the solution. Thus, in this particular case the computing time to generate the coefficient matrix can be reduced by approximately a factor of 64 if the problem symmetry is utilized.

The arguments of the preceding paragraphs also apply to assumed forms of Φ_1 and Φ_2 different than in (5.2.1). For example in finite region problems one may wish to include positive powers of z_1 and z_2 in the forms of Φ_1 and Φ_2. Conclusions analogous to those obtained for Φ_1 and Φ_2 of the form (5.2.1) continue to hold. It may be advantageous to use more general functions than powers of z_1 and z_2 in the assumed forms of Φ_1 and Φ_2. For example one could assume Φ_1 and Φ_2 to be as in (3.2.5). We know from the analysis of Section III that for problems involving an infinite plate with a circular hole the exact solution is contained within this assumed form. In fact the programs for both the boundary least squares and Reissner energy were tested with Φ_1 and Φ_2 which contained both the exact solution and other functions in order to see that the exact solution was extracted. From a practical point of view one would never use the boundary least squares or Reissner energy programs to solve problems involving a circular hole since the analysis of Section III is simpler. However, it is conceivable that one might use a truncated version of the forms of Φ_1 and Φ_2 in (3.2.5) to solve problems involving non-circular holes for which no exact solutions are known. If so it may be possible to capitalize on problem symmetry by suitably restricting the complex coefficients A_k and B_k, and the powers of ζ_1 and ζ_2 in (3.2.5). In order to do this we observe that for $|z_1|>|1+\mu_1^2|^{\frac{1}{2}}$ we can expand ζ_1^{-1} (defined in (3.2.4)) in a convergent series containing only odd negative powers of z_1, and similarly for ζ_2. Consequently odd (even) negative powers of ζ_1 and ζ_2 can be expanded in odd (even) negative powers of z_1 and z_2 respectively. Because of this we can deduce the

appropriate restrictions of the series (3.2.5) in a particular problem from the corresponding restrictions of the series (5.2.1) for the same problem.

Let us now consider a serious problem which may arise when using Φ_1 and Φ_2 of the form (5.2.1). In what follows we refer only to Φ_1, μ_1, and z_1 but identical considerations apply for Φ_2, μ_2, and z_2. In sample problems I and III we know that Φ_1 of the exact solution does in fact have a representation in negative powers of z_1, but only outside the circle in the z_1-plane with center at the origin and with radius $|1 + \mu_1^2|^{\frac{1}{2}}$. In Figure 15 we have illustrated this circle and also the elliptical image of the physical plane circular boundary for two different materials. In Appendix II it was established that the foci of

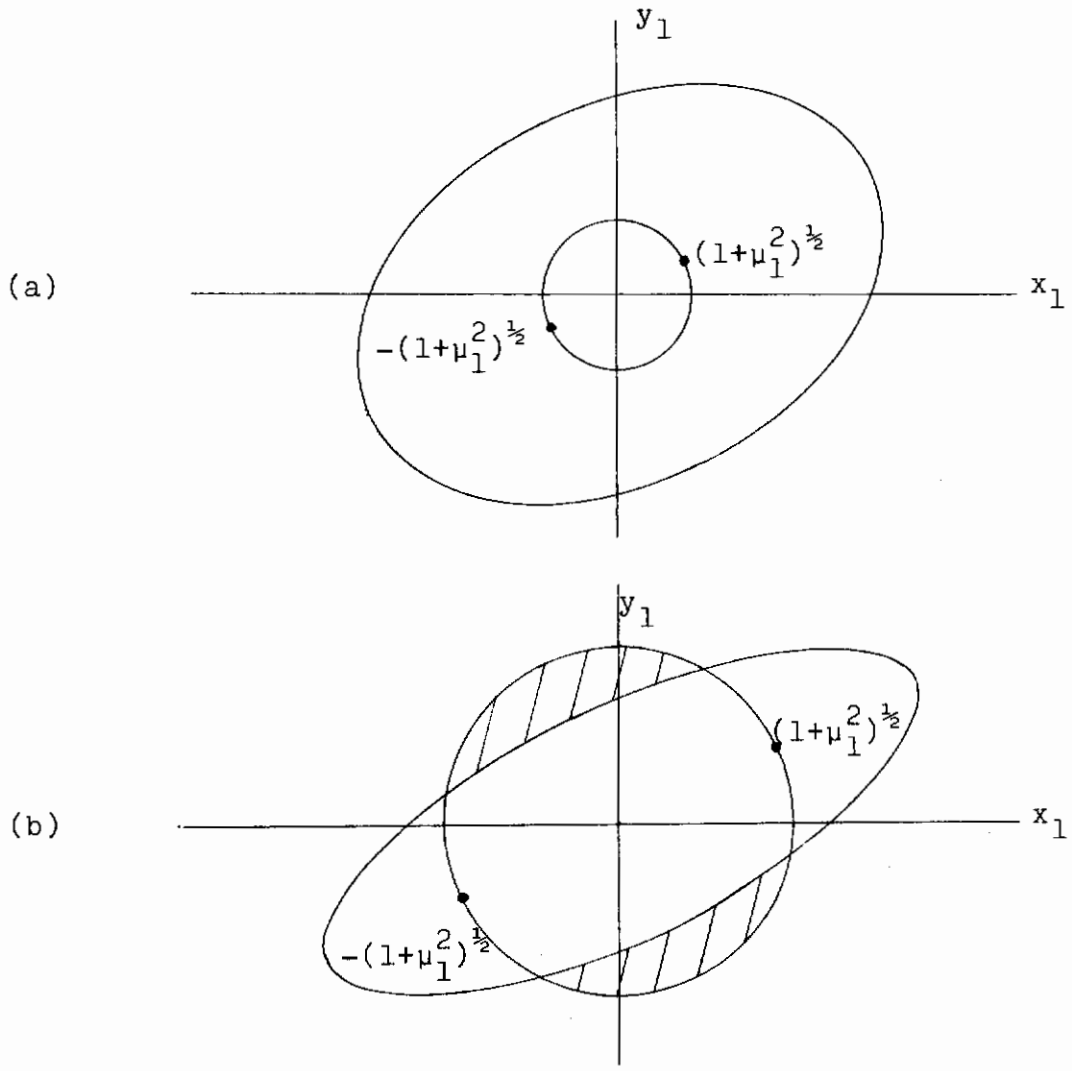


Figure 15. Region of Series Convergence in z_1-plane

the elliptical image are at $z_1 = \pm(1 + \mu_1^2)^{\frac{1}{2}}$. Also in that appendix expressions for the major and minor semiaxes of the ellipse in terms of μ_1 were listed. Now for materials which are only slightly orthotropic, $\mu_1 \approx i$ and the elliptical image is nearly a unit circle as in Figure 15a. This means that the circle passing through the foci lies well within the ellipse, and so Φ_1 of the exact solution has a convergent representation in negative powers of z_1 everywhere in the z_1-plane that corresponds to a region of physical interest, i.e., exterior to the ellipse. For materials such as this both the least squares and Reissner energy programs, when used with the assumed forms (5.2.1), yielded coefficients which were the same as in the expansions of the exact solution. (The leading terms were determined more accurately than the higher negative powers.)

In the case of more strongly orthotropic materials the situation is as illustrated in Figure 15b. The expansion of Φ_1 of the exact solution in negative powers of z_1 does not converge in the shaded region. For such materials one might expect to encounter difficulty in obtaining good approximate solutions when using the assumed form (5.2.1) in conjunction with the boundary least squares or Reissner energy methods. Further discussion of this point is given in Section VII on numerical results. We should point out that it does not take a particularly high degree of orthotropy to result in the situation of Figure 15b. This occurs whenever the distance to the foci ($\equiv c$) exceeds the minor semi-axis ($\equiv b$) of the ellipse. Using the formulas (A2-3) we can write this condition as

$$\left(1 + \mu_{1R}^2 - \mu_{1I}^2\right)^2 + 4\mu_{1R}^2\mu_{1I}^2 > \frac{1}{9}\left(1 + \mu_{1R}^2 + \mu_{1I}^2\right)^2 \qquad (5.2.2)$$

For nearly isotropic materials, $\mu_{1I} \approx 1$, $\mu_{1R} \approx 0$, and inequality (5.2.2) is not satisfied, which corresponds to the situation in Figure 15a. It is clear that it would not take values of μ_{1I} and μ_{1R} greatly different from this to obtain the situation of Figure 15b. For example, $\mu_{1R} = 0$, $\mu_{1I} > \sqrt{2}$ or $\mu_{1I} < \sqrt{2}/2$ satisfy (5.2.2).

It should be pointed out that the problems of series convergence just discussed are not peculiar to orthotropic materials. Similar difficulties arise in isotropic elasticity when analyzing noncircular holes, eg., elliptical holes, using boundary residual methods with series analogous to (5.2.1). This phenomenon was pointed out by Hulbert [21] and Lo, Niedenfuhr, and Leissa [22]. When dealing with noncircular holes in orthotropic materials one could expect the convergence problems to be compounded.

Finally we discuss briefly how Sample Problems II and IV (with uniform stress at infinity) were handled. In Cases I and III we did not have to worry about the line integrals along the outer "boundary" at infinity since the stresses and displacements decayed rapidly enough. However, in Cases II and IV the integrals along the outer "boundary" are unbounded so it would be difficult to solve the problem by a strict application of one of the boundary residual methods. In order to have constant stresses at infinity it is necessary to have terms linear in z_1 and z_2 in the assumed forms of Φ_1 and Φ_2. All other functions of z_1 and z_2 in these assumed forms must grow less rapidly than z_1 and z_2. Thus, these linear functions are the only significant terms at infinity, and it is straightforward to determine the associated complex coefficients by imposing the three stress conditions and one displacement condition (zero rotation). This has already been done in Section III. The formulas for the coefficients are given in (3.2.11). We can then append these known linear terms to assumed forms of Φ_1 and Φ_2, eg., (5.2.1). Since the conditions at infinity are satisfied exactly no matter how the free coefficients in Φ_1 and Φ_2 are chosen it is unnecessary to consider the line integrals at infinity. The solution procedure is then identical to Cases I and III.

2b. Computational Details

The application of both the boundary least squares and Reissner energy is straightforward so we will not give a great deal of detail in this article. Φ_1 and Φ_2 are assumed to have the form

$$\Phi_1 = A_L z_1 + \sum_{k=1}^{n} a_k f_k(z_1)$$

$$\Phi_2 = B_L z_2 + \sum_{k=1}^{n} b_k g_k(z_1), \qquad (5.2.3)$$

where A_L and B_L are known complex constants given by formula (3.2.11). For zero stress at infinity $A_L = B_L = 0$. The other terms are the same as in (5.1.1) and have been discussed following that equation.

In what follows we consider the problem of prescribed stresses on the boundary. The case of prescribed displacements is similar. We also confine our discussion to problems involving a circular hole in an infinite sheet. However, the extension of the method to other geometries is straightforward. We assume that on the boundary $\sigma_r = \bar{\sigma}_r(\theta)$ and $\tau_{r\theta} = \bar{\tau}_{r\theta}(\theta)$, where the barred quantities are known functions. Note that these functions are not the applied boundary stresses, but rather the resulting stresses in the material at the boundary. The next step is to use (5.2.3) in formulas (3.1.18) and (3.1.19) for the stresses and displacements. For example

$$\sigma_x = 2\mathrm{Re}\left[\mu_1^2 \sum_{k=1}^{n} \left(a_{kR} + ia_{kI}\right) f_k'\left(z_1\right) + \mu_2^2 \sum_{k=1}^{n} \left(b_{kR} + ib_{kI}\right) g_k'\left(z_2\right) + \mu_1^2 A_L + \mu_2^2 B_L\right], \qquad (5.2.4)$$

and similarly for σ_y, τ_{xy}, u, and v. Formula (2.1.5) is then used to obtain σ_θ and $\tau_{r\theta}$ in terms of the expressions for σ_x, σ_y, and τ_{xy}. u_r and u_θ are similarly obtained. Next these quantities are used to compute certain line integrals. For least squares the integral is

$$I_{LS} = \int_0^{\theta_U} \left[\left(\bar{\sigma}_r - \sigma_r \right)^2 + \left(\bar{\tau}_{r\theta} - \tau_{r\theta} \right)^2 \right] d\theta \ , \qquad (5.2.5)$$

and for Reissner energy it is

$$\Phi_{RB} = \int_0^{\theta_U} \left[\left(2\bar{\sigma}_r - \sigma_r \right) u_r + \left(2\bar{\tau}_{r\theta} - \tau_{r\theta} \right) u_\theta \right] d\theta \qquad (5.2.6)$$

σ_r, $\tau_{r\theta}$, u_r, and u_θ are functions of z_1 and z_2. By using the relations $z_1 = \cos\theta + \mu_1 \sin\theta$ and $z_2 = \cos\theta + \mu_2 \sin\theta$ on the boundary these stresses and displacements become functions of θ. The integrals can then be computed numerically and the result in either case is a quadratic form in the a_{kR}, a_{kI}, b_{kR}, and b_{kI}. θ_U is equal to 2π in the general case. However, as discussed in Article 2a of this section θ_U may be less than 2π if the loading and assumed forms of Φ_1 and Φ_2 have the appropriate symmetry. The $4n$ coefficients a_{kR}, a_{kI}, b_{kR}, and b_{kI} are determined so that I_{LS} or Φ_{RB} is stationary, i.e., so that $\delta I_{LS} = 0$ or $\delta \Phi_{RB} = 0$. Either of these conditions is equivalent to a linear system of the form,

$$\begin{bmatrix} E_1 & E_2 & E_3 & E_4 \\ E_2^T & E_5 & E_6 & E_7 \\ E_3^T & E_6^T & E_8 & E_9 \\ E_4^T & E_7^T & E_9^T & E_{10} \end{bmatrix} \begin{Bmatrix} A_R \\ A_I \\ B_R \\ B_I \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} , \qquad (5.2.7)$$

where each of the E_i is an n x n matrix. E_1, E_5, E_8, and E_{10} are symmetric so that the $4n$ x $4n$ coefficient is symmetric. The A's, B's, and R's are n x 1 column vectors. A_R, A_I, B_R,

and B_I are the vectors of the a_{kR}, a_{kI}, b_{kR}, and b_{kI} respectively. As an example we list the formula for an element of E_4 in the case of least squares.

$$E_4(j,k) = -8 \int_0^{\theta_U} \mathrm{Re}\left\{\left[\left(\mu_1\cos\theta-\sin\theta\right)^2 - \left(\mu_1^2-1\right)\frac{\sin2\theta}{2} - \mu_1\cos2\theta\right] f_j'(z_1)\right\} \cdot \mathrm{Im}\left\{\left[\left(\mu_2\cos\theta-\sin\theta\right)^2 - \left(\mu_2^2-1\right)\frac{\sin2\theta}{2} - \mu_2\cos2\theta\right] g_k'(z_2)\right\} d\theta \tag{5.2.8}$$

The other E_1 are similar in form. For displacement boundary conditions the integrand in (5.2.8) would contain f_j and g_k instead of f_j' and g_k' and would also have different functions of θ multiplying f_j and g_k. In the Reissner energy formulation the integrand of the formula for any $E_1(j,k)$ contains either f_j and g_k' or f_j' and g_k. The formulas for the elements of the right side vector of (5.2.7) are similar in nature to (5.2.8). It should be noted that the prescribed quantities on the boundary $\bar{\sigma}_r$, $\bar{\tau}_{r\theta}$ or $\bar{u}_r$, $\bar{u}_\theta$ and the constants A_L and B_L appear only on the right side of (5.2.7) as is customary in linear elasticity problems. Once the coefficient matrix and right side vector have been generated, it is simply a matter of solving the matrix system and using the results in (5.2.3) which is used in turn in formulas (3.1.18) and (3.1.19) to obtain stresses and displacements at any point of the region.

SECTION VI

POTENTIAL, COMPLEMENTARY, AND REISSNER ENERGY METHODS

1. INTRODUCTION

In this section we discuss the use of the potential, complementary, and Reissner energy functionals in Ritz fashion to obtain approximate solutions to the sample problems of this study. Inasmuch as the use of the Reissner energy functional in this manner is rather uncommon, and because the method exhibits several unusual characteristics, we give complete algebraic details of the formulation. Since the algebra of the potential and complementary energy formulations is of a similar nature, and because these methods are more conventional, we give a less thorough treatment of them as far as the algebra is concerned.

We note that the use of the Ritz method in connection with the potential, complementary, and Reissner energy functionals is completely equivalent to the Galerkin method as applied to the displacement, stress, and combined stress-displacement formulations of elasticity respectively. This equivalence is established in Appendix IV.

1a. Choice of Assumed Functions

Just as in any weighted residual method, the selection of assumed functions to be used with the energy methods of this section is particularly critical. In this article a thorough discussion of this point is given. Because of the geometry of the sample problems use of polar coordinates suggests itself. Further, it seems reasonable to use trigometric functions to describe the θ-variation in the assumed family of functions. On these hypotheses the assumed functions for use with the energy methods have the following form:

$$\sigma_r = \sigma_{ro}(r,\theta) + \sum_{i=1}^{n} f_{1i}(r) \left\{ \sum_{j=0}^{m} a_k \cos j\theta + \sum_{j=1}^{m} a_\ell \sin j\theta \right\}$$

$$\sigma_\theta = \sigma_{\theta o}(r,\theta) + \sum_{i=1}^{n} f_{2i}(r) \left\{ \sum_{j=0}^{m} b_k \cos j\theta + \sum_{j=1}^{m} b_\ell \sin j\theta \right\}$$

$$\tau_{r\theta} = \tau_{r\theta_o}(r,\theta) + \sum_{i=1}^{n} f_{3i}(r) \left\{ \sum_{j=0}^{m} c_k \cos j\theta + \sum_{j=1}^{m} c_\ell \sin j\theta \right\}$$

$$\Psi = \Psi_o(r,\theta) + \sum_{i=1}^{n} f_{4i}(r) \left\{ \sum_{j=0}^{m} \Psi_k \cos j\theta + \sum_{j=1}^{m} \Psi_\ell \sin j\theta \right\}$$

$$u = u_o(r,\theta) + \sum_{i=1}^{n} f_{5i}(r) \left\{ \sum_{j=0}^{m} u_k \cos j\theta + \sum_{j=1}^{m} u_\ell \sin j\theta \right\}$$

$$v = v_o(r,\theta) + \sum_{i=1}^{n} f_{6i}(r) \left\{ \sum_{j=0}^{m} v_k \cos j\theta + \sum_{j=1}^{m} v_\ell \sin j\theta \right\}, \tag{6.1.1}$$

where

$$k = 2jn + i, \qquad \ell = 2jn - n + i \tag{6.1.2}$$

The a's, b's, c's, Ψ's, u's, and v's are unknown coefficients to be determined in accordance with the particular energy method under consideration. The $f_{ni}(r)$ are known functions of r. There is some flexibility in the choice of these functions. However, they must satisfy certain conditions at infinity, and, in the case of the potential and complementary energy methods, certain conditions on the circular boundary may have to be satisfied. Discussion of the selection of the f_{ni} will be given later in this article. In (6.1.1), the known functions σ_{ro}, $\sigma_{\theta o}$, $\tau_{r\theta o}$,

ψ_o, u_o, and v_o are present when the assumed functions are required to satisfy inhomogeneous boundary conditions. The upper limit of each i(j) summation of (6.1.1) has been taken to be equal to n(m). This is not necessary, and in the computer programs developed for the potential and Reissner energy methods (which use more than one of the assumed functions of (6.1.1) provision has been made for considering a different upper limit for each summation.

When the state of stress and deformation has certain kinds of symmetry (or antisymmetry) terms corresponding to some of the harmonics may be omitted from (6.1.1). We have discussed this point at the end of Article 2b of Section II and also at the beginning of Article 2a of Section V. In Table III we have listed the harmonics, ie., the values taken on by the j-summation index, necessary in each of the assumed forms (6.1.1) for the four basic types of deformation states.

Table III. Values of the j-index for Various Types of Symmetry

Axis	Character of Deformation State about x and y axes			
x / y / Coefficients	Symmetric Symmetric	Symmetric Antisymmetric	Antisymmetric Symmetric	Antisymmetric Antisymmetric
a_k, b_k, ψ_k, u_k	j=0,2,4,..	j=1,3,5,..	---	---
$a_\ell, b_\ell, \psi_\ell, u_\ell$	---	---	j=1,3,5,..	j=2,4,6,..
c_k, v_k	---	---	j=1,3,5,..	j=0,2,4,..
c_ℓ, v_ℓ	j=2,4,6,..	j=1,3,5,..	---	---

Recall that the most general problem can be broken into four uncoupled problems, which have the character indicated in Table III. To obtain the solution to the general problem one can solve each of the four sub-problems separately and then add the individual results. This observation is useful for the approximate methods of this section since the order of the matrix equations that must be solved can be reduced by a factor of two or four. For example, in Sample Problems II and IV with $\phi \neq 0, \pi/2$ one can solve two sub-problems; one symmetric about the x and y axes, the other antisymmetric about the x and y axes; and then add the results to obtain the total solution. (For $\phi = 0, \pi/2$ the problem has symmetry about the x and y axes.)

We now discuss the selection of $f_{ni}(r)$ in (6.1.1). For Sample Problems I and III the stresses and displacements must vanish at infinity. In Cases II and IV there is constant stress at infinity. However, if one were to include constant f_{ni} terms to account for this, the corresponding contribution to the variation of the energy functional would be infinite. One might try the approach used for the boundary residual methods of Section V, ie., determine functions which satisfy the conditions at infinity exactly by other means, and then include these known functions in the σ_{ro}, $\sigma_{\theta o}$, etc., terms of the assumed functions (6.1.1). However, even in this approach individual terms of the variation of the energy functional may be infinite. It might be that a careful consideration of the limiting case ($b\to\infty$) of a finite region problem, one could combine the singular terms in such a way that they would cancel. However, a simpler approach is to use superposition, ie., calculate the stresses at r=1 produced by uniform stress at infinity in a solid sheet, and then solve the problem of an infinite sheet with a circular hole having these stresses prescribed on the circular boundary and zero stress at infinity. The total solution is obtained from the difference of these two solutions. This approach is described in greater detail in Article 3e of this section. Thus, in all four of our sample problems the f_{ni} are to be chosen so that they have no effect on the stresses and displacements at infinity. This suggests choosing the f_{ni} to be negative powers of r. One might select non-integral powers of r. However, there does not appear to be a particularly good reason for doing this. A consistent choice of the f_{ni} in the form of negative intergral powers of r is as follows:

$$
\begin{aligned}
&f_{1i},\ f_{2i},\ f_{3i} = r^{-(i+1)} && , i = 1, 2, \ldots \\
&f_{41} = \log r, \qquad f_{4i} = r^{-(i-1)} && , i = 2, 3, \ldots \qquad (6.1.3) \\
&f_{5i},\ f_{6i} = r^{-i} && , i = 1, 2, \ldots
\end{aligned}
$$

These are consistent in that it is possible to derive the assumed form of the stress state from the assumed form of the stress function or displacement state by means of the usual elasticity relations. We note that when f_{4i} is defined as in (6.1.3) the assumed form of Ψ in (6.1.1) contains all terms of the Michell series (Timoshenko [23], p. 116) consistent with the conditions at infinity and single valued displacements. Since the individual terms of the Michell series are exact solutions to the isotropic problem, the energy methods of this section should be able to solve the four sample problems exactly in the case of isotropic materials when the assumed forms of (6.1.3) are used. This was verified numerically.

In the case of potential energy with displacement boundary conditions or complementary energy with stress boundary conditions, (6.1.3) cannot be used since these conditions are not satisfied for arbitrary choices of the unknown coefficients as required by the statements of these energy principles. In what follows we exclude such problems. However, when we do discuss these problems it will be seen that the conclusions based on the assumed forms of (6.1.3) continue to hold more generally.

We now digress to discuss a method of obtaining exact solutions to orthotropic elasticity problems. This method was developed in a series of papers by Green and Taylor [24] and Green [25]. Although these papers were preceded by the early work of Lekhnitskii (see [8] and [9] for references to this work), they were apparently done independently. In this series of papers the method of solution gradually evolved into a form which can be considered to be equivalent to that of Lekhnitskii, which was discussed in detail in Section III. The approach taken in the first few papers of this series was rather circuitous. However, it does offer us some insight to the question presently under consideration; namely, the selection of assumed functions for use with the Ritz method.

In what follows we adhere as much as possible to the notation of Section III, which differs from that of Green and Taylor. The starting point of their method, just as in Lekhnitskii's, is the observation that solutions of the generalized biharmonic are of the form (3.1.2), which, repeated here, is

$$\Psi = \sum_{j=1}^{4} \Psi_j(x + \mu_j y) \tag{6.1.4}$$

Green and Taylor confine their attention to materials for which the μ_j are purely imaginary. This is not essential but for simplicity we continue to do so here. We also maintain the convention that μ_1 and μ_2 have positive imaginary parts. Also, recall that $\mu_3 = \bar{\mu}_1$ and $\mu_4 = \bar{\mu}_2$. For problems involving an infinite plate with a circular boundary having applied stresses on this boundary which are symmetric about the x and y axes the stress function is assumed to have the form,

$$\Psi = \Psi_P + \mathrm{Re}\left[A_o \log (x + \mu_3 y) + B_o \log (x + \mu_4 y) + \sum_{k=1}^{\infty} \frac{A_{2k}}{(1+\gamma_1)^{2k}(x+\mu_3 y)^{2k}} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(1+\gamma_2)^{2k}(x+\mu_4 y)^{2k}}\right], \tag{6.1.5}$$

where Ψ_P is a known function which satisfies the constant stress conditions at infinity (if any), the A's and B's are real constants, and

$$\gamma_{1,2} = \frac{1 - |\mu_{1,2}|}{1 + |\mu_{1,2}|} \tag{6.1.6}$$

Note that $|\gamma_{1,2}| < 1$. The factors $(1+\gamma_{1,2})^{2k}$ in the denominators of (6.1.5) could be absorbed into the A's and B's. However, subsequent formulas based on (6.1.5) simplify if $(1+\gamma_{1,2})^{2k}$ is retained in the denominators. The fact that Ψ of the form (6.1.5) has the appropriate symmetry can be deduced from the discussion at the beginning of Article 2a of Section V. Also, in that article it was pointed out that for problems involving an infinite sheet with a circular hole and having symmetry about the x and y axes, the exact solution can be expanded in the form (6.1.5). However, as noted there, this representation may not converge everywhere in the region of physical interest. More will be said about this later.

The next step in the solution process is to write the complex variables, $x + \mu_3 y$ and $x + \mu_4 y$, in polar form. The result is

$$x+\mu_{3,4} y = \frac{r(1+\gamma_{1,2} e^{2i\theta})}{(1+\gamma_{1,2})e^{i\theta}} \tag{6.1.7}$$

Using (6.1.7) we can write

$$\log(x+\mu_{3,4}y) = \log r - i\theta - \log(1+\gamma_{1,2}) + \sum_{\ell=1}^{\infty} \frac{(-1)^{(\ell-1)}\gamma_{1,2}^{\ell}}{\ell} e^{2i\ell\theta},$$

$$\frac{1}{(x+\mu_{3,4}y)^{2k}} = \frac{e^{2ik\theta}}{r^{2k}} \left(1+\gamma_{1,2}\right)^{2k} \sum_{\ell=0}^{\infty} \binom{2k+\ell-1}{\ell} \gamma_{1,2}^{\ell}(-1)^{\ell} e^{2i\ell\theta},$$

(6.1.8)

where $\binom{2k+\ell-1}{\ell}$ is the binomial coefficient, $\frac{(2k+\ell-1)!}{\ell!\,(2k-1)!}$. Since $|\gamma_{1,2}e^{2i\theta}|<1$ the summations in (6.1.8) are convergent everywhere in the region of interest. Now we can rewrite (6.1.5) as

$$\Psi = \Psi_P + (A_o+B_o)\log r + \sum_{\ell=1}^{\infty} \left(A_o\gamma_1^{\ell} + B_o\gamma_2^{\ell}\right)\frac{(-1)^{(\ell-1)}}{\ell} \cos 2\ell\theta$$

$$+ \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \binom{2k+\ell-1}{\ell}(-1)^{\ell}\left(A_{2k}\gamma_1^{\ell} + B_{2k}\gamma_2^{\ell}\right)\frac{\cos\, 2(k+\ell)\theta}{r^{2k}}$$

(6.1.9)

We have omitted the constant $\log(1+\gamma_{1,2})$-terms from (6.1.9) since they contribute nothing to the stresses. As we know this series for Ψ may not converge everywhere in the region of physical interest. However, we can proceed formally and obtain expressions for the stresses from (6.1.9) by means of the usual relations, ie., (2.2.14). In order to determine the unknown constants of (6.1.9) we must have the expressions for σ_r and $\tau_{r\theta}$. They are

$$\sigma_r = \frac{1}{r}\frac{\partial \Psi_P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi_P}{\partial \theta^2} + \frac{(A_o+B_o)}{r^2} + \frac{4}{r^2}\sum_{\ell=1}^{\infty}(-1)^{\ell}\ell\left(A_o\gamma_1^{\ell} + B_o\gamma_2^{\ell}\right)\cos 2\ell\theta$$

$$-\sum_{k=1}^{\infty}\sum_{\ell=k}^{\infty}\binom{k+\ell-1}{\ell-k}(-1)^{(\ell-k)}(4\ell^2 + 2k)\left(A_{2k}\gamma_1^{(\ell-k)} + B_{2k}\gamma_2^{(\ell-k)}\right)\frac{\cos 2\ell\theta}{r^{(2k+2)}}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi_P}{\partial \theta}\right) + \frac{2}{r^2}\sum_{\ell=1}^{\infty}(-1)^{\ell}\left(A_o\gamma_1^{\ell} + B_o\gamma_2^{\ell}\right)\sin 2\ell\theta$$

$$-\sum_{k=1}^{\infty}\sum_{\ell=k}^{\infty}\binom{k+\ell-1}{\ell-k}(-1)^{(\ell-k)}(4k\ell + 2\ell)\left(A_{2k}\gamma_1^{(\ell-k)} + B_{2k}\gamma_2^{(\ell-k)}\right)\frac{\sin 2\ell\theta}{r^{(2k+2)}}$$

(6.1.10)

In order to be able to write equations for the determination of the A's and B's it is necessary to reverse the order of the double summations. σ_r and $\tau_{r\theta}$ can then be cast into the form of a cosine series and a sine series respectively. For example, σ_r becomes

$$\sigma_r = \frac{1}{r}\frac{\partial \Psi_P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi_P}{\partial \theta^2} + \frac{(A_o+B_o)}{r^2}$$

$$+\sum_{\ell=1}^{\infty}\left\{\frac{4\ell}{r^2}(-1)^{\ell}\left(A_o\gamma_1^{\ell} + B_o\gamma_2^{\ell}\right)\right.$$

$$\left.-\sum_{k=1}^{\ell}\binom{k+\ell-1}{\ell-k}\frac{(-1)^{(\ell-k)}}{r^{(2k+2)}}(4\ell^2 + 2k)\left(A_{2k}\gamma_1^{(\ell-k)} + B_{2k}\gamma_2^{(\ell-k)}\right)\right\}\cos 2\ell\theta,$$

(6.1.11)

and similarly for $\tau_{r\theta}$. Now if the normal and tangential components of the applied boundary stresses are written as Fourier series one can obtain equations for the A's and B's by requiring that the coefficient of each cosine harmonic of (6.1.11) (evaluated at r=1) be equal to the coefficient of the same harmonic in the Fourier series of the normal component of the applied stress, and likewise for $\tau_{r\theta}$.

As an example consider Sample Problem II (uniform tension at infinity with zero stress on the circular boundary) with the angle of tension, ϕ, equal to zero. For this problem,

$$\Psi_P = \frac{r^2}{4}(1-\cos 2\theta), \tag{6.1.12}$$

and the Fourier coefficients of the applied stresses on the circular boundary are, of course, zero. The formulas for the A's and B's, which were obtained by Green and Taylor [24-III], are as follows:

$$\begin{aligned} A_o &= \frac{(1+\gamma_2)}{2(\gamma_1-\gamma_2)} \\ B_o &= -\frac{(1+\gamma_1)}{2(\gamma_1-\gamma_2)} \\ A_{2k} &= -\frac{\gamma_1^k(1+\gamma_2)}{2(\gamma_1-\gamma_2)}\frac{(2k-1)!}{k!(k+1)!} \\ B_{2k} &= \frac{\gamma_2^k(1+\gamma_1)}{2(\gamma_1-\gamma_2)}\frac{(2k-1)!}{k!(k+1)!} \end{aligned} \tag{6.1.13}$$

Having determined the A's and B's the series (6.1.5) for Ψ and similar ones for σ_x, σ_y, and τ_{xy} can be summed to obtain these quantities for particular values of x and y. These series are equivalent to expansions of the exact solutions of Section III in negative powers of $z_1(\equiv x + \mu_1 y)$ and $z_2(\equiv x + \mu_2 y)$. In Section V the convergence of such series was discussed. If the material orthotropy is pronounced these series do not converge everywhere in the region of physical interest. The regions of convergence in the z_1 and z_2 - planes are illustrated in Figure 15 of Article 2a, Section V. If the series (6.1.5) did not converge for a particular point of interest, one might attempt to sum the series, (6.1.9) and (6.1.10) instead. However, since (6.1.9) is completely equivalent to (6.1.5) the region of convergence for (6.1.9) and (6.1.10) is also as indicated in Figure 13.

In [24-III] Green and Taylor discovered that by rearranging the series (6.1.9) and (6.1.10) into sine and cosine series such as (6.1.11), convergence could be obtained at every point of the physical plane exterior to the circular boundary. Now when the $f_{ni}(r)$ in the assumed functions (6.1.1) are chosen to be negative powers of r as in (6.1.3), series such as (6.1.11) (in truncated form) are contained within the assumed family of functions. It is reassuring to know that it is possible (in principle) to approximate arbitrarily closely the solutions to our sample problems by retaining a sufficient number of terms in the assumed functions.

In our treatment of boundary residual methods in Section V the assumed functions were in the form of negative powers of z_1 and z_2. In Article 2a of that section we discussed how certain powers could be eliminated when the problem had various types of symmetry. However, even if all powers are retained the computer programs for both the boundary least squares and Reissner energy methods yield zero coefficients for the powers of z_1 and z_2 which are unnecessary. For the energy methods of this section it may be possible to omit certain harmonics from the assumed functions when the problem has symmetry. (See Table III.) If these unnecessary harmonics are retained the computer programs for the energy methods eliminate them by yielding zero for the coefficients associated with these harmonics. However, as far as the r-variation of the assumed functions (6.1.1) is concerned, we cannot eliminate certain powers of r by symmetry arguments, even though, eg., only even powers appear in the expansion of the exact solution for σ_r

of the sample problems of this study. (See equation (6.1.11).) Neither do the computer programs eliminate these unnecessary powers of r by yielding zero coefficients.

As we have seen, when the loading (for the sample problems of this study) is symmetric about the x and y axes the exact solution is of the general form,

$$\sigma_r = \sigma_{ro} + \sum_{j=0}^{\infty} g_j(r) \cos 2j\theta, \qquad (6.1.14)$$

and similarly for the other dependent variables. Even if we had known nothing about the exact solution it is likely that we would have chosen the θ-dependence in the assumed form of σ_r to be the same as in (6.1.14). More difficult is the problem of selecting functions of r which can describe adequately the r-variation of the exact solution, ie., functions which can give a good approximation to the $g_j(r)$ of (6.1.14). Using the notation of (6.1.1) this approximation can be written,

$$g_j(r) \approx \sum_{i=1}^{n} a_k f_{1i}(r), \qquad (6.1.15)$$

where $k=2jn+i$. It is not necessary to use the same upper limit on the summation or even the same $f_{1i}(r)$ in the approximation of different $g_j(r)$'s. For example, from (6.1.11) we see that the $g_j(r)$ become more involved as j increases. Thus, if the $f_{1i}(r)$ are taken to be negative powers of r, one could use a smaller upper limit on the summation of (6.1.15) for smaller j. However, it would be difficult to deduce this without knowledge of the exact solution. Also, if the $f_{1i}(r)$ are chosen differently the same conclusion may not be valid.

When the region is infinite one must choose the $f_{1i}(r)$ so that they have no effect at infinity. One could choose functions such as exponentials or Bessel functions. However, the simplest

functions with the required behavior at infinity are negative powers of r, and, as it turns out the $g_j(r)$ can be approximated accurately by these functions. Thus, for these problems intuition leads to appropriate assumed forms, although it is unlikely that one would arrive at the "ideal" assumed form (only the necessary negative powers of r) without doing some careful analysis, and even then it may be impossible without knowledge of the exact solution.

In the case of finite region problems, eg., an annulus, there are no conditions at infinity to satisfy. Consequently, one has far greater flexibility in the selection of the $f_{1i}(r)$. However, it should be noted that in order to avoid singular matrices or non-unique solutions one should not include rigid body terms in the assumed form of the displacements or constant terms in the assumed form of the stress function. The reason for this is that such terms contribute nothing to the energy functionals. Thus, the coefficients corresponding to these terms cannot be determined in the process of obtaining the stationary value of the particular energy functional. Actually, there is never any reason for a constant term in the stress function. However, if there is rigid body motion in a given problem it is necessary to consider it, but it must be done by other means. Once these rigid body components have been determined they can be subtracted out, and the resulting modified problem can then be analyzed by one of the energy methods.

For the potential and complementary energy methods the computer programs have been written so that various problems for an annular geometry can be considered. Among the problems treated are those corresponding to Sample Problems I - IV, ie., the loading on the inner boundary of the annulus is the same as for these sample problems, and the loading on the outer boundary is obtained from the known exact solutions to these problems. Of course, the loading on the outer boundary obtained in this way is rather complicated. However, since exact solutions for orthotropic annuli with simpler boundary conditions, eg., uniform pressure on the inner and outer boundaries, are apparently unknown, there is little alternative to considering complicated conditions on the outer boundary. Since the solutions to these finite region problems are the same as their infinite region counterparts, the assumed functions of r in (6.1.1) or (6.1.15), ie., negative powers of r, can remain the same. However, if one were given such a finite region problem with a moderate outer radius, say r=2, with no idea as to the origin of the boundary conditions it is not likely that he would be led by intuition or common sense to the selection of the $f_{1i}(r)$ in negative powers of r.

Thus, we are confronted with the question of how good can an approximate solution be when the assumed functions are quite different from those of the exact solution? For example, if we assume the $f_{ni}(r)$ of (6.1.1) to be positive powers of r can a good approximate solution be obtained? This question could be rephrased as follows: How well can one approximate negative powers of r over the interval [1,2] with positive powers of r? Some answers to this latter question can be found in approximation theory. However, a thorough consideration of this problem would carry us too far afield. Thus, we content ourselves with the conclusions drawn from the numerical results of Section VII.

Another very important consideration related to the choice of assumed functions is the conditioning of the resulting coefficient matrix. (See Appendix III for a discussion of matrix conditioning in the general case.) It may be that a good approximation to the exact solution is contained within the assumed family of functions, but the matrix conditioning is so poor that this approximation cannot be extracted from the assumed family with the available machine precision. Considerable attention is given to the effect of function choice and method on the conditioning of the associated coefficient matrix in our presentation of numerical results in Section VII.

2. APPLICATION OF THE POTENTIAL AND COMPLEMENTARY ENERGY PRINCIPLES

In this article we present the principal features of the potential and complementary energy formulations. As mentioned earlier full algebraic details will not be given for these formulations since they are similar in nature to the Reissner energy formulation which is treated thoroughly in Article 3 of this section.

2a. The Potential Energy Formulation

We begin by expressing the potential energy functional Φ_p, equation (2.3.5), in terms of the radial and circumferential displacement components, u and v.

$$\Phi_P = \frac{1}{2}\int_0^{\theta_U}\int_1^{b} \Big\{ B_{11}^{r\theta}\left(ru_r^2\right) + 2B_{12}^{r\theta}\left(uu_r + u_r v_\theta\right)$$

$$+ 2B_{16}^{r\theta}\left(u_r u_\theta + ru_r v_r - u_r v\right) + B_{22}^{r\theta}\left(u^2 + 2uv_\theta + v_\theta^2\right)\Big/ r$$

$$+ 2B_{26}^{r\theta}\left(uu_\theta + ruv_r - uv + u_\theta v_\theta + rv_r v_\theta - vv_\theta\right)\Big/ r$$

$$+ B_{66}^{r\theta}\left(u_\theta^2 + 2ru_\theta v_r - 2u_\theta v + r^2 v_r^2 - 2rvv_r + v^2\right)\Big/ r \Big\}\, dr\, d\theta$$

$$-\int_0^{\theta_U}\left[\bar{F}_r u + \bar{F}_\theta v\right]_{r=1} d\theta \; - b\int_0^{\theta_U}\left[\bar{F}_r u + \bar{F}_\theta v\right]_{r=b} d\theta \quad , \tag{6.2.1}$$

where b is the outer radius of the annular region (it may be ∞), and θ_U depends on the symmetry present in the problem. If there is no symmetry $\theta_U = 2\pi$. $\bar{F}_r$ and $\bar{F}_\theta$ are the prescribed stresses on the boundary. If displacements are prescribed then $\bar{F}_r = \bar{F}_\theta = 0$. The $B_{ij}^{r\theta}$ in (6.2.1) are functions of θ and are defined in (2.1.13) and (2.1.14). We note that equations (2.1.13), (2.1.14), and (6.2.1) are valid for general anisotropic materials.

The assumed forms for u and v are special cases of those listed in (6.1.1). They are

$$u = u_o(r) + \sum_{i=1}^{n_u} f_{5i}(r) \sum_{j=1}^{m} u_{ij}\begin{pmatrix}\cos\\ \sin\end{pmatrix} \hat{j}\theta$$

$$v = \sum_{i=1}^{n_v} f_{6i}(r) \sum_{j=1}^{m} v_{ij}\begin{pmatrix}\sin\\ \cos\end{pmatrix} \hat{j}\theta \tag{6.2.2}$$

In (6.2.2) the u_{ij} and v_{ij} are the unknown constants to be determined in the process of minimizing the potential energy. The subscripts i j indicate that the actual subscript depends

on the i and j summation indices, eg., as in (6.1.2). It is convenient to retain this notation for the time being. Later, formulas for the actual subscript in terms of the i and j will be given. The quantity $\hat{j}$ in (6.2.2) is defined as

$$\hat{j} = m_1 + (j - 1)\, m_3, \tag{6.2.3}$$

where m_1 and m_3 are integers specified by the user. Also specified is another integer m_2 which is used in the definition of the upper limit of the j-summations of (6.2.2).

$$m = \frac{m_2 - m_1}{m_3} + 1 \tag{6.2.4}$$

We note that when $m_1 = 0$ the terms corresponding to sin 0θ are unnecessary and should be eliminated. This is to be understood in the discussions to follow. The definitions (6.2.3) and (6.2.4) give the user considerable flexibility in selecting harmonics in the assumed forms of u and v. He selects the initial harmonic by specifying m_1, the increment between successive harmonics by specifying m_3, and the last harmonic by specifying m_2. The difference $(m_2 - m_1)$ must be such that m_3 divides into it evenly.

The notation $\begin{pmatrix}\cos\\ \sin\end{pmatrix}$ and $\begin{pmatrix}\sin\\ \cos\end{pmatrix}$ in (6.2.2) means that when the assumed form for u consists of cosine harmonics the assumed form for v has since harmonics and vice-versa. We know from the discussion of Article 1a, which is summarized in Table III, that the most general problem can be broken into uncoupled subproblems with u = f(cos), v = f(sin) or u = f(sin), v = f(cos) in any one of these subproblems. If a particular problem requires both sine and cosine harmonics in the solutions for u and v, the computer program solves the problem in two steps; in the first step the upper trigonometric functions in (6.2.2) are used, and the lower ones are used in the second step.

When the assumed functions (6.2.2) are substituted into (6.2.1) and the relevant integrations are performed, a quadratic form in the u_{ij} and v_{ij} results. This quadratic form can be written as

$$\frac{1}{2}\left[U \;\vdots\; V\right]\left[\begin{array}{c:c} X_1 & X_2 \\ \hdashline X_2^T & X_3 \end{array}\right]\left\{\begin{array}{c} U \\ \hdashline V \end{array}\right\}-\left[R_1 \;\vdots\; R_2\right]\left\{\begin{array}{c} U \\ \hdashline V \end{array}\right\} \tag{6.2.5}$$

where

$$\left[U \;\vdots\; V\right] = \left[u_1, u_2, \ldots, u_{KD_u}, v_1, v_2, \ldots v_{KD_v}\right] , \tag{6.2.6}$$

where KD_u and KD_v are the number of the unknown constants u_{ij} and v_{ij} respectively. The generation of the coefficient matrix X and the right hand side vector R, although conceptually straightforward, is computationally involved. For example, consider the first term in the integrand of the double integral of (6.2.1).

$$\int_0^{2\pi}\int_1^b B_{11}^{r\theta}\left(ru_r^2\right)\, dr\, d\theta$$

$$= \sum_{i=1}^{n_u}\sum_{k=1}^{n_u}\sum_{j=1}^{m}\sum_{\ell=1}^{m} u_{ij}u_{k\ell}F_{11k}G_{1j\hat{\ell}\hat{\ell}} + \sum_{i=1}^{n_u}\sum_{j=1}^{m} u_{ij}F_{11}^{*}G_{1j}^{*\hat{}} \tag{6.2.7}$$

We have taken the upper limit on the θ-integral to be 2π. However, if a smaller upper limit is desired (because of problem symmetry) then (6.2.7) can be modified simply by multiplying by the appropriate factor. In (6.2.7) the quantities $G_{1j\hat{\ell}\hat{\ell}}$ and F_{lik} (which may be regarded as doubly subscripted arrays) and F^{*}_{li} and G^{*}_{lj} (which may be regarded as singly subscripted arrays) are defined

$$G_{1j\hat{\ell}\hat{\ell}} = \int_0^{2\pi} B_{11}^{r\theta} \begin{pmatrix} \cos \\ \sin \end{pmatrix} \hat{j}\theta \begin{pmatrix} \cos \\ \sin \end{pmatrix} \hat{\ell}\theta \, d\theta$$

$$G^{*}_{1\hat{j}} = \int_0^{2\pi} B_{11}^{r\theta} \begin{pmatrix} \cos \\ \sin \end{pmatrix} \hat{j}\theta \, d\theta \quad , \tag{6.2.8}$$

and

$$F_{lik} = \int_1^{b} f'_{5i}(r) \, f'_{5k}(r) \, r \, dr$$

$$F^{*}_{li} = \int_1^{b} f'_{5i}(r) \, u'_{o}(r) \, r \, dr \tag{6.2.9}$$

In order to evaluate the integrals of (6.2.8) $B_{11}^{r\theta}$ (equation (2.1.13)) must be substituted. Recalling that for orthotropic materials $\tilde{H}_3 = \tilde{H}_4 = 0$, we can write $G_{1j\hat{\ell}\hat{\ell}}$ in the form,

$$G_{1j\hat{\ell}\hat{\ell}} = \tilde{H}_1 I_{1j\hat{\ell}\hat{\ell}} + \tilde{H}_2 I_{2j\hat{\ell}\hat{\ell}} + \tilde{H}_5 I_{5j\hat{\ell}\hat{\ell}} \quad , \tag{6.2.10}$$

where

$$I_{1\hat{j}\hat{\ell}}^{\hat{}\hat{}} = \int_0^{2\pi} \cos 4\theta \begin{pmatrix} \cos \\ \sin \end{pmatrix} \hat{j}\theta \begin{pmatrix} \cos \\ \sin \end{pmatrix} \hat{\ell}\theta \, d\theta \quad , \tag{6.2.11}$$

and similarly for the other integrals of (6.2.10). G_{1j}^{*} is handled in the same way. (6.2.11) and other integrals of this nature can be evaluated analytically. The integrals in (6.2 9) may be evaluated analytically (if possible) or numerically. Even when the $f_{5i}(r)$ in the integrands are simple the analytical evaluation of integrals such as (6.2.9) can be time consuming. For example, the "standard" forms for f_{5i} and f_{6i} that are used in the computer program for the potential energy formulation are

$$\begin{aligned} f_{5i} &= \left(1 - \frac{T_u}{r}\right) r^{\hat{i}_u} \\ f_{6i} &= \left(1 - \frac{T_v}{r}\right) r^{\hat{i}_v} \quad , \end{aligned} \tag{6.2.12}$$

where T_u and T_v are constants supplied by the user, and

$$\begin{aligned} \hat{i}_u &= n_{u1} + (i-1)\, n_{u3} \\ \hat{i}_v &= n_{v1} + (i-1)\, n_{v3} \quad , \end{aligned} \tag{6.2.13}$$

where n_{u1}, n_{u3}, n_{v1}, and n_{v3} are integers supplied by the user, just as in the definition of $\hat{j}$ in (6.2.3). The reasons for choosing f_{5i} and f_{6i} as in (6.2 12) will be given later. Integrals such as (6.2.9) have been computed analytically based on f_{5i} and f_{6i} as defined in (6.2.12). These results are incorporated into the computer program. However, the user has the option of defining f_{5i} and f_{6i} differently by means of user supplied subroutines.

If he exercises this option integrals such as (6.2 9) are evaluated numerically by means of a five point Gauss-Chebyshev quadrature formula. Of course, when these integrals are evaluated numerically the outer radius, b, of the annulus must be finite.

After all of the terms in the surface integral of (6.2.1) have been evaluated in the manner just described, it is necessary to consider the line integrals if there are prescribed stresses. When the region is finite these integrals are evaluated numerically since the prescribed stresses on the outer boundary are not simple functions. The result of these integrations is a linear function of the u_{ij} and v_{ij} similar to the double summation in (6.2.7). When all contributions are added Φ_P can be written

$$\Phi_P = \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} \left\{ \sum_{i=1}^{n_u} \sum_{k=1}^{n_u} X_{1ijk\ell} u_{ij} u_{k\ell} + \sum_{i=1}^{n_u} \sum_{k=1}^{n_v} X_{2ijk\ell} u_{ij} v_{k\ell} + \sum_{i=1}^{n_v} \sum_{k=1}^{n_u} X^{T}_{2ijk\ell} v_{ij} u_{k\ell} + \sum_{i=1}^{n_v} \sum_{k=1}^{n_v} X_{3ijk\ell} v_{ij} v_{k\ell} \right\}$$

$$- \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n_u} R_{1ij} u_{ij} + \sum_{i=1}^{n_v} R_{2ij} v_{ij} \right\} \qquad (6.2.14)$$

The elements of the coefficient matrix and right hand side vector must be placed in the appropriate positions. The row and column addresses of the $X_{1ijk\ell}$, $X_{2ijk\ell}$, and $X_{3ijk\ell}$ in the overall coefficient matrix and the row addresses of the R_{1ij} and R_{2ij} in the overall right hand side vector are computed in terms of i, j, k, and ℓ as follows:

X_1, X_2, R_1: row address $= (j-1)n_u + i$

X_3, R_2: row address $= KD_u + (j-1)n_v + i$

X_1: column address $= (\ell-1)n_u + k$

X_2, X_3: column address $= KD_u + (\ell-1)n_v + k$,

where KD_u is equal to the number of rows (and columns) in X_1, and n_u and n_v are the limits on the i-summations of (6.2.2).

Having established the X matrix and the R vector, the total potential energy is minimized over the u_{ij} and v_{ij}. The result is the system of linear equations,

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} , \qquad (6.2.15)$$

which is then solved for the u_{ij} and v_{ij}. If u and v contain both sine and cosine harmonics it is necessary to repeat this process using the lower trigonometric functions in the assumed forms for u and v in (6.2.2). The two sets of u_{ij} and v_{ij} are then used in (6.2.2) to obtain u and v for particular values of r and θ. One can also obtain approximate answers for the stresses by using these expressions for u and v in the strain-displacement relations (2.2.11), and then using these strains in the stress-strain relationships,

$$\{\sigma_{r\theta}\} = \left[B^{r\theta}\right] \{\varepsilon_{r\theta}\} , \qquad (6.2.16)$$

where the elements of $\left[B^{r\theta}\right]$ are given in (2.1.13). If one is also interested in the stresses in rectangular coordinates it is perhaps simpler to compute $\{\varepsilon_{xy}\}$ in terms of $\{\varepsilon_{r\theta}\}$ using the

transformations of Article 1b of Section II, then compute $\{\sigma_{xy}\}$ using (2.1.18), and finally compute $\{\sigma_{r\theta}\}$ using (2.1.5). The latter procedure is followed in the computer program.

We shall now give some further discussion of our "standard" assumed forms for u and v, ie., using (6.2.12) in (6.2.2). These standard assumed forms are then

$$u = R_u r^{\alpha} + \left(1 - \frac{T_u}{r}\right) \sum_{i=1}^{n_u} r^{\hat{i}_u} \sum_{j=0}^{m} u_{ij} \begin{pmatrix} \cos \\ \sin \end{pmatrix} \hat{j}\theta$$

$$v = \left(1 - \frac{T_v}{r}\right) \sum_{i=1}^{n_v} r^{\hat{i}_v} \sum_{j=0}^{m} v_{ij} \begin{pmatrix} \sin \\ \cos \end{pmatrix} \hat{j}\theta \ , \qquad (6.2.17)$$

where $\hat{i}_u$, $\hat{i}_v$, and $\hat{j}$ are defined in (6.2.3) and (6.2.13). m is defined in (6.2.4) and n_u and n_v are defined analogously.

In Sample Problems III and IV, which have displacements prescribed on the circular boundary (r=1), we must choose $T_u = T_v = 1$. In III (uniform diametral expansion of unit magnitude) we set $R_u = 1$, and in IV (uniform tension at infinity-zero displacement of circular boundary) we set $R_u = 0$. These choices are necessary so that the displacement boundary conditions will be satisfied for arbirtary values of the unknown coefficients u_{ij} and v_{ij}, as required by the potential energy principle. The exponent α in (6.2.17) must be chosen negative for infinite region problems. $\alpha = -1$ seems to be the most reasonable choice, at least for Case III, since the $R_u r^{\alpha}$ term in (6.2.17) is then identical to the isotropic solution. For finite region problems α can be chosen arbitrarily. In Sample Problems I and II stresses are prescribed on the boundary (r=1) so R_u, T_u, and T_v may be set to zero, if desired.

When the potential energy method is used to obtain approximate solutions for finite region problems, the boundary conditions on the outer boundary of the annulus (r=b) are regarded as stress boundary conditions. The reason for this is that, by their very origin (obtained from exact solutions), the stresses and displacements on this outer boundary are not simple functions. Thus, if these complicated displacements were prescribed it would be necessary for the assumed forms of u and v to satisfy these displacement conditions. It would be difficult to do this.

In Sample Problems II and IV there is uniform tension at infinity so the potential energy is infinite. In the paragraph preceding equation (6.1.3) we discussed briefly the use of superposition as a means of circumventing problems associated with infinite energy functionals. This technique is used in the analysis of Cases II and IV by the Reissner energy method. We could use this same approach for the potential and complementary energy methods. However, it was decided to treat Cases II and IV as finite region problems by choosing some suitably large outer radius of the annular region, say b=10, where the effect of the hole is practically negligible, and then to use as prescribed conditions on this outer boundary results from the exact solution. Of course, b should not be too large since the line integral in (6.2.1) around the outer boundary would overshadow the integral around the inner boundary, and also certain elements of the coefficient matrix would be disproportionately large. This would cause loss of significance in the results.

Before concluding this article we note that for certain materials the method of potential energy yields systems of equations which are poorly conditioned. This occurs when the determinant of the compliance matrix in (2.1.15) is small, which gives large values for B_{11}, B_{12}, and B_{22} in (2.1.19). This condition, i.e.,

$$S_{11}S_{22} - S_{12}^2 \approx 0 \ , \qquad (6.2.18)$$

corresponds to the material being nearly "incompressible in the plane". In order to determine whether the material is nearly incompressible in the usual (3-dimensional) sense it is necessary to know the values of the elements, S_{13}, S_{23}, and S_{33}, in the three-dimensional orthotropic compliance matrix ((2.1.1) specialized for orthotropic materials). In [26], Taylor, Pister, and Hermann consider this problem of using the method for potential energy with nearly incompressible materials. They devise a way of modifying the customary potential energy formulation so that nearly incompressible materials can be treated. We refer the reader to the paper cited for details.

2b. The Complementary Energy Formulation

This discussion of the complementary energy formulation follows closely that of the potential energy formulation in the preceding article. Consequently this article is fairly brief since the reader can always refer to Article 2a for further details. According to the statement of the principle of complementary energy (equation (2.3.6)ff.) the assumed form of the stress state must satisfy the equilibrium equations internally. The easiest way to fulfill this requirement is to assume a form for the stress function Ψ and then derive the stresses from Ψ by means of relations (2.2.14). Using these we can write the complementary energy functional (2.3.6) in terms of Ψ as follows:

$$\Phi_C = -\frac{1}{2}\int_0^{\theta_U}\int_1^b \Big\{ S_{22}^{r\theta}\left(r\Psi_{rr}^2\right) + S_{11}^{r\theta}\left(r^2\Psi_r^2 + 2r\Psi_r\Psi_{\theta\theta} + \Psi_{\theta\theta}^2\right)\Big/ r^3$$

$$+ 2S_{12}^{r\theta}\left(r\Psi_{rr}\Psi_r + \Psi_{rr}\Psi_{\theta\theta}\right)\Big/ r + 2S_{26}^{r\theta}\left(-r\Psi_{rr}\Psi_{r\theta} + \Psi_{rr}\Psi_{\theta}\right)\Big/ r$$

$$+ 2S_{16}^{r\theta}\left(-r^2\Psi_r\Psi_{r\theta} + r\Psi_r\Psi_{\theta} - r\Psi_{r\theta}\Psi_{\theta\theta} + \Psi_{\theta\theta}\Psi_{\theta}\right)\Big/ r^3$$

$$+ S_{66}^{r\theta}\left(r^2\Psi_{r\theta}^2 - 2r\Psi_{r\theta}\Psi_{\theta} + \Psi_{\theta}^2\right)\Big/ r^3 \Big\}\, dr\, d\theta$$

$$- \int_0^{\theta_U} \left[\left(\frac{1}{r}\Psi_r + \frac{1}{r^2}\Psi_{\theta\theta}\right)\bar{u} + \left(-\frac{1}{r}\Psi_{r\theta} + \frac{1}{r^2}\Psi_{\theta}\right)\bar{v}\right]_{r=1} d\theta$$

$$+ b\int_0^{\theta_U} \left[\left(\frac{1}{r}\Psi_r + \frac{1}{r^2}\Psi_{\theta\theta}\right)\bar{u} + \left(-\frac{1}{r}\Psi_{r\theta} + \frac{1}{r^2}\Psi_{\theta}\right)\bar{v}\right]_{r=b} d\theta \quad , \tag{6.2.19}$$

where $\bar{u}$ and $\bar{v}$ are the prescribed values of the radial and circumferential displacements. If stresses are prescribed then $\bar{u} = \bar{v} = 0$. Note that the two line integrals in (6.2.19) have opposite signs. This is because $F_r = \sigma_r$ and $F_\theta = \tau_{r\theta}$ on the outer boundary and $F_r = -\sigma_r$ and $F_\theta = -\tau_{r\theta}$ on the inner boundary. (F_r and F_θ are the components of the boundary stresses in the line integral of (2.3.6).) The $S_{ij}^{r\theta}$ are functions of θ and are defined in (2.1.9) and (2.1.10). We note that equations (2.1.9), (2.1.10), and (6.2.19) are valid for general anisotropic materials.

The assumed form for Ψ is a special case of that listed in (6.1.1)

$$\Psi = \Psi_o(r) + \sum_{i=1}^{n} f_{4i}(r) \sum_{j=1}^{m} \Psi_{ij} \begin{pmatrix} \cos \\ \sin \end{pmatrix} \hat{j}\theta \quad , \qquad (6.2.20)$$

where the Ψ_{ij} are unknown coefficients to be determined in the process of maximizing the complementary energy. Note that the form of Ψ in (6.2.20) is the same as that for u in (6.2.2). Almost all of the discussion following (6.2.2) applies to (6.2.20) also.

When (6.2.20) is substituted into (6.2.19) and the relevant integrations are performed, a quadratic form in the Ψ_{ij} results. This quadratic form can be written

$$\frac{1}{2}\{\Psi\}^T [X] \{\Psi\} - \{R\}^T \{\Psi\} \qquad (6.2.21)$$

where

$$\{\Psi\} = [\Psi_1, \Psi_2, \ldots, \Psi_{KD}] \quad , \qquad (6.2.22)$$

where KD is the number of unknown constants Ψ_{ij}. The generation of the coefficient matrix X and the right hand side vector R is quite similar to the corresponding computations of the potential energy formulation. (See equation (6.2.7)ff.) For example, consider the first term in the integrand of the double integral of (6.2.19).

$$\int_0^{2\pi}\int_1^b S_{22}^{r\theta}(r\Psi_{rr}^2)\,dr\,d\theta$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{\ell=1}^{m}\Psi_{ij}\Psi_{k\ell}F_{lik}G_{lj\hat{\ell}\hat{\ell}} + \sum_{i=1}^{n}\sum_{j=1}^{m}\Psi_{ij}F_{li}^{*}G_{lj}^{*}$$

(6.2.23)

(6.2.23) has the same form as (6.2.7). The F's and G's of (6.2.23) are defined (with a few obvious changes) as in (6.2.8) - (6.2.11).

Just as in the potential energy program we have made provisions for user supplied $f_{4i}(r)$ in the assumed form (6.2.20). When this option is exercised the r-integrations are performed numerically, which, of course, requires that the region be finite. If the user does not supply the $f_{4i}(r)$ then the "standard" form,

$$f_{4i}(r) = \left\{ \begin{array}{ll} \left(1-\frac{T}{r}\right)^2 r^{\hat{i}}, & i = 1, 2, \ldots, n-1 \\ \log r - \frac{S}{r} + \frac{S}{r^2}, & i = n \end{array} \right\},$$

(6.2.24)

is used. Here T and S are supplied by the user. When stresses are prescribed on the boundary r=1 (Cases I and II) it is necessary to choose T = S = 1 so that the stress boundary conditions are satisfied for arbitrary values of the Ψ_{ij} in the assumed form (6.2.20). Of course, it is also necessary to choose $\Psi_o(r)$ appropriately. The form we have used for $\Psi_o(r)$ is

$$\Psi_o(r) = R \log r \tag{6.2.25}$$

For Case I R must be chosen to be -1 and in Case II it must be zero. For the other cases it can be chosen in any convenient way. The definition of $\hat{i}$ in (6.2.24) is just as in (6.2.13).

The fact that $f_{4i}(r)$ for i=n is of a different form than $f_{4i}(r)$ for i<n, complicated the analytical integrations of required integrals such as,

$$F_{1ik} = \int_1^b f''_{4i}(r)\, f''_{4k}(r)\, r\, dr, \tag{6.2.26}$$

and also the programming of the results. However, it was felt that inclusion of the log r - term in the assumed form for Ψ is important. That this is true can be seen from the solution form of Green and Taylor (6.1.9).

When treating finite region problems we regard the conditions on the outer boundary (obtained from exact solutions) to be displacement conditions. The line integral in (6.2.19) around the outer boundary can then be evaluated numerically. If these boundary conditions were considered stress conditions, the assumed form of the stress function would have to be chosen to satisfy these conditions. This would be difficult to do since these boundary conditions are not simple. We note that our treatment of Cases II and IV is the same as in the potential energy formulation, ie., we handle them as finite region problems.

After all of the required integrations have been performed Φ_C can be written as the quadratic form (6.2.21). Φ_C is then maximized over the Ψ_{ij} to give the linear system,

$$\left[X\right]\left\{\Psi\right\} = \left\{R\right\} \tag{6.2.27}$$

which is to be solved for the Ψ_{ij}. If the stress function contains both cosine and sine harmonics the procedure is repeated with sine harmonics in the assumed form (6.2.20). The stresses are obtained by differentiating the corresponding approximate solution for Ψ and then using the relations (2.2.14). The recovery of the displacement state corresponding to an approximate stress state is in general not as straightforward as the converse problem, ie., the recovery of the stress state corresponding to an approximate displacement state. The latter problem was discussed in the preceding article (equation (6.2.16) ff.). The former problem is discussed in Appendix V.

3. APPLICATION OF THE REISSNER ENERGY PRINCIPLE

This article describes the solution of the four sample problems by means of Reissner's theorem using the Ritz technique. Explicit equations are presented for each case for any number of assumed functions. The article is organized so that the contributions of Reissner's theorem which are common to all four cases (ie., the volume integral) are presented first, and then subsequently those contributions specific to each case are presented in sequence. Such algebraic details as are necessary for the computation of numerical results are presented in Appendices VII and VIII.

3a. Basic Approach Using Variational Form of the Theorem

The application of Reissner's theorem using the Ritz technique is straightforward and best accomplished via the variational form set out in Appendix VI, equation (A6-3) viz.

$$\delta\Phi_R = \int_A \left[\delta\{\sigma\}^T\left[\{\varepsilon\} - [S]\ \{\sigma\}\right] - \delta\{u\}^T\left[D\ \{\sigma\}\right]\right]dA$$

$$+ \int_{C_s} \delta\{u\}^T\left[\{F\} - \{\bar{F}\}\right]ds - \int_{C_d} \delta\{F\}^T\left[\{u\} - \{\bar{u}\}\right]ds \tag{6.3.1}$$

(Note that the vector notation used in the line integrals in preceding sections has been replaced by matrix notation .) If the stresses and displacements are written in the form

$$\{\sigma\} = [\phi_s]\{X_s\}$$
$$\{u\} = [\phi_d]\{X_d\}, \tag{6.3.2}$$

where $[\phi_s]$, $[\phi_d]$ are the matrices of assumed functions and $\{X_s\}$, $\{X_d\}$ are the column vectors of unknown coefficients. The system of linear equations which results from substitution of equations (6.3.2) into (6.3.1) can be shown to have the form,

$$\begin{bmatrix} [A_1] & [A_2] \\ [A_2]^T & [0] \end{bmatrix} \begin{bmatrix} \{X_s\} \\ \{X_d\} \end{bmatrix} + \begin{bmatrix} \{B_1\} \\ \{B_2\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{0\} \end{bmatrix} \tag{6.3.3}$$

where

$$[A_1] = -\int_A [\phi_s]^T [S] [\phi_s] \, dA$$
$$[A_2] = \int_A [\phi_d]^T [D^T \phi_s] \, dA - \int_{C_d} [\phi_s]^T [L]^T [\phi_d] ds$$
$$\{B_1\} = \int_{C_d} [\phi_s]^T [L]^T \{\bar{u}\} ds \tag{6.3.4}$$
$$\{B_2\} = -\int_{C_s} [\phi_d]^T [L] \{\bar{\sigma}\} ds,$$

where the matrix $[L]$ derives from the boundary equilibrium conditions $\{F\} = [L]\{\sigma\}$. For the matrix of coefficients to be non-singular then the matrix $[A_2]^T [A_1]^{-1} [A_2]$ must also be non-

singular. This will only be true if its order is equal to or less than the order of $[A_1]$. Thus if there are m unknown stress coefficients and n displacement coefficients, $[A_1]$ is of order mxm, $[A_2]$ is of order mxn and $[A_2{}^T][A_1]^{-1}[A_2]$ is of order nxn. Therefore we must have $m \geq n$. This constraint on the freedom of choice of assumed functions is due to the absence in the functional of quadratic terms in the displacements.

3b. Choice of Functions

The choice of functions for the application of the Ritz technique to Reissner's theorem is certainly limited by the considerations of the preceding article. On the other hand, the assumed forms for the stresses and displacements are apparently not required to satisfy any other conditions (other than suitable continuity) either internally or on the boundary. It has already been shown (Article 3 - Section II) that if the assumed stresses satisfy internal equilibrium and the stress boundary conditions, the Reissner theorem is equivalent to the principle of maximum complementary energy. If the stress-displacement relations are satisfied internally and the displacement boundary conditions are satisfied, then the Reissner theorem reduces to the principle of minimum potential energy. There are still other special situations which can arise. For example, consider a problem with stress boundary conditions only. If the stresses are chosen so as to satisfy equilibrium internally the resulting variational form of Reissner's theorem is

$$\delta\Phi_R = \int_A \delta\{\sigma\}^T\Big[\{\varepsilon\} - [S]\{\sigma\}\Big]dA + \int_{C_s} \delta\{u\}^T\Big[\{F\} - \{\bar{F}\}\Big]ds = 0 \qquad (6.3.5)$$

Preliminary work in this study for Sample Problem I indicated that such an approach can lead to a singular coefficient matrix depending on the actual choice of assumed stresses. Therefore, in order to explore the general application of Reissner's theorem and exclude special cases the stresses and displacements were

chosen independently assuming functions which satisfied neither stress-displacement relationships, nor equilibrium, nor the stress and displacement boundary conditions. The assumed forms are as in (6.1.1) with the f_{ni} given by (6.1.3). The corresponding expressions for the strains are

$$\varepsilon_r = \sum_{i=1}^{n} \sum_{j=0}^{m} -ir^{-(i+1)}\left\{u_k \cos j\theta + u_\ell \sin j\theta\right\}$$

$$\varepsilon_\theta = \sum_{i=1}^{n} \sum_{j=0}^{m} r^{-(i+1)}\left\{(u_k + jv_\ell)\cos j\theta + (u_\ell - jv_k)\sin j\theta\right\}$$

$$\gamma_{r\theta} = \sum_{i=1}^{n} \sum_{j=0}^{m} r^{-(i+1)}\left\{\left[ju_\ell - (i+1)v_k\right]\cos j\theta - \left[ju_k + (i+1)v_\ell\right]\sin j\theta\right\} \qquad (6.3.6)$$

There are, of course, no coefficients associated with $j = 0$ for the $\sin j\theta$ terms. The total number of stress coefficients is $3n(2m+1)$ and the displacement coefficients number $2n(2m+1)$. There are therefore more stress than displacement unknowns, which is a necessary condition for a non-singular coefficient matrix.

3c. Sample Problems

This article describes in detail the formulation of the equations for each of the four sample problems described in Article 2b of Section III. The formulation of the equations in polar coordinates requires the following definitions:

$$\{\sigma\} = \{\sigma_r \quad \sigma_\theta \quad \tau_{r\theta}\}$$

$$\{\varepsilon\} = \{\varepsilon_r \quad \varepsilon_\theta \quad \gamma_{r\theta}\}$$

$$\{u\} = \{u \quad v\}$$

$$[D] = \begin{bmatrix} \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) & -\frac{1}{r} & \frac{1}{r}\frac{\partial}{\partial \theta} \\ 0 & \frac{1}{r} & \left(\frac{\partial}{\partial r} + \frac{2}{r}\right) \end{bmatrix} \qquad (6.3.7)$$

$$\{F\} = \{F_r \quad F_\theta\}$$

where $$F_r = -\sigma_r \, r \frac{d\theta}{ds} + \tau_{r\theta} \frac{dr}{ds}$$

$$F_\theta = \sigma_\theta \frac{dr}{ds} - \tau_{r\theta} r \frac{d\theta}{ds}$$

The boundary equilibrium conditions as conventionally stated apply to an external boundary. The present problems all relate to an internal boundary and accordingly the signs in these equations have been reversed. Each of the four problems considered gives rise to the same volume integral (equation (6.3.1)) since the geometry and assumed functions are the same in each case. The differences occur in the boundary integrals only and these will be considered separately.

3d. Volume Integral

The variational terms are $\delta\{\sigma\}$ and $\delta\{u\}$ which in terms of the assumed functions can be written as

$$\delta\sigma_r = \sum_{p=1}^{n} \sum_{q=0}^{m} r^{-(p+1)} \{\delta a_r \cos q\theta + \delta a_s \sin q\theta\}, \qquad (6.3.8)$$

where $$a_r = a_{qn+p} \quad , \quad a_s = a_{2qn-n+p},$$

and similarly for $\delta\sigma_\theta$ and $\delta\tau_{r\theta}$. Likewise,

$$\delta u = \sum_{p=1}^{n} \sum_{q=0}^{m} r^{-p} \{\delta u_r \cos q\theta + \delta u_s \sin q\theta\},$$

and similarly for δv. When computing the matrix $[S]\{\sigma\}$ in the integrand of the volume integral (6.3.1) we must use the compliance matrix in polar coordinates. The elements of this matrix are given in equation (2.1.9). For convenience the matrix $\{\varepsilon\} - [S]\{\sigma\}$ is then written in terms of the expanded expressions

$$\begin{aligned} E_1 &= \varepsilon_r - S_{11}^{r\theta}\sigma_r - S_{12}^{r\theta}\sigma_\theta - S_{16}^{r\theta}\tau_{r\theta} \\ E_2 &= \varepsilon_\theta - S_{12}^{r\theta}\sigma_r - S_{22}^{r\theta}\sigma_\theta - S_{26}^{r\theta}\tau_{r\theta} \\ E_3 &= \gamma_{r\theta} - S_{16}^{r\theta}\sigma_r - S_{26}^{r\theta}\sigma_\theta - S_{66}^{r\theta}\tau_{r\theta} \quad , \end{aligned} \tag{6.3.9}$$

where $S_{ij}^{r\theta}$ is the same as $S_{ij}^{\alpha\beta}$ of (2.1.9). Similarly the matrix $[D\{\sigma\}]$ is written in expanded form as

$$\begin{aligned} E_4 &= \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\sigma_r - \frac{1}{r}\sigma_\theta + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} \\ E_5 &= \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau_{r\theta} \end{aligned} \tag{6.3.10}$$

Premultiplying by the variational terms yields the quantities $\delta\sigma_r E_1$, $\delta\sigma_\theta E_2$, $\delta\tau_{r\theta} E_3$, etc., which are then integrated. The integrations are simple as the variables r and θ are separated. The trigonometric terms involved are of the type

$$\int_0^{2\pi} \begin{pmatrix}\cos \\ \sin\end{pmatrix} q\theta \begin{pmatrix}\cos \\ \sin\end{pmatrix} j\theta \, (\cos\theta)^s (\sin\theta)^t \, d\theta, \text{ and}$$

$$\int_0^{2\pi} \begin{pmatrix}\cos \\ \sin\end{pmatrix} q\theta \, (\cos\theta)^s (\sin\theta)^t \, d\theta,$$

where $s + t = 4$ or $s = t = 0$. The terms $(\cos\theta)^s(\sin\theta)^t$ arise from the $\cos 4\theta$, $\sin 4\theta$, $\cos 2\theta$, and $\sin 2\theta$ terms in the formulas for the $S_{ij}^{r\theta}$ of (2.1.9). One might expect that these integrals for the case, $s + t = 2$, would also be needed. However, it turns out that these are unnecessary. The solutions to the required trigonometric integrals are tabulated in Appendix VII. The r-dependent integrals are all

$$\int_1^\infty r^{-(i+p+2)} \, r \, dr = \frac{1}{i+p} \tag{6.3.11}$$

The complete volume integral can therefore be written explicitly in terms of the unknown coefficients, the orthotropic elastic properties and the trigonometric integrals. Thus,

$$\int_A \left[\delta\{\sigma\}^T\left[\{\varepsilon\} - [S]\{\sigma\}\right] - \delta\{u\}^T\left[D\{\sigma\}\right]\right] dA =$$
$$= \int_1^\infty \int_0^{2\pi} \{\delta\sigma_r E_1 + \delta\sigma_\theta E_2 + \delta\tau_{r\theta} E_3 - \delta u E_4 - \delta v E_5\} r \, dr \, d\theta \tag{6.3.12}$$

The explicit results for the consituent terms are as follows:

$$(i) \quad \int_1^{\infty} \int_0^{2\pi} \delta\sigma_r E_1 r dr d\theta =$$

$$= \sum_{p=1}^{n} \sum_{q=0}^{m} \sum_{i=1}^{n} \sum_{j=0}^{m} \frac{1}{(i+p)} \left\{ \delta a_r \left[-a_k \left[S_{11} I_1 + (2S_{12}+S_{66}) I_3 + S_{22} I_5 \right] \right.\right.$$
$$- b_k \left[S_{12} I_1 + (S_{11}+S_{22}-S_{66}) I_3 + S_{12} I_5 \right]$$
$$- c_\ell \left[(-2S_{11}+2S_{12}+S_{66}) I_7 + (2S_{22}-2S_{12}-S_{66}) I_4 \right]$$
$$\left. -u_k \left[iI_{31} \right] \right]$$
$$+ \delta a_s \left[-a_\ell \left[S_{11} I_{16} + (2S_{12}+S_{66}) I_{18} + S_{22} I_{20} \right] \right.$$
$$-b_\ell \left[S_{12} I_{16} + (S_{11}+S_{22}-S_{66}) I_{18} + S_{12} I_{20} \right]$$
$$-c_k \left[(-2S_{11}+2S_{12}+S_{66}) I_{12} + (2S_{22}-2S_{12}-S_{66}) I_{14} \right]$$
$$\left. \left. -u_\ell \left[iI_{33} \right] \right] \right\} \qquad (6.3.13)$$

where S_{11}, S_{12}, S_{66} are the orthotropic elastic properties (compliances).

(ii) $$\int_1^{\infty}\int_0^{2\pi} \delta\sigma_\theta E_2 r dr d\theta =$$

$$= \sum_{p=1}^{n}\sum_{q=0}^{m}\sum_{i=1}^{n}\sum_{j=0}^{m} \frac{1}{(i+p)} \Big\{ \delta b_r \Big[-a_k \Big[S_{12}I_1 + (S_{11}+S_{22}-S_{66})I_3 + S_{12}I_5 \Big] - b_k \Big[S_{22}I_1 + (2S_{12}+S_{66})I_3 + S_{11}I_5 \Big] - c_\ell \Big[(2S_{22}-2S_{12}-S_{66})I_7 + (-2S_{11}+2S_{12}+S_{66})I_9 \Big] + u_k \Big[I_{31} \Big] + v_\ell \Big[jI_{31} \Big] \Big] + \delta b_s \Big[-a_\ell \Big[S_{12}I_{16} + (S_{11}+S_{22}-S_{66})I_{18} + S_{12}I_{20} \Big] - b_\ell \Big[S_{22}I_{16} + (2S_{12}+S_{66})I_{18} + S_{11}I_{20} \Big] - c_k \Big[(2S_{22}-2S_{12}-S_{66})I_{12} + (-2S_{11}+2S_{12}+S_{66})I_{14} \Big] + u_\ell \Big[I_{33} \Big] - v_k \Big[jI_{33} \Big] \Big] \Big\} \qquad (6.3.14)$$

(iii) $$\int_1^{\infty}\int_0^{2\pi} \delta\tau_{r\theta} E_3 r dr d\theta =$$

$$= \sum_{p=1}^{n}\sum_{q=0}^{m}\sum_{i=1}^{n}\sum_{j=0}^{m} \frac{1}{(i+p)} \Big\{ \delta c_r \Big[-a_\ell \Big[(-2S_{11}+2S_{12}+S_{66})I_7 + (2S_{22}-2S_{12}-S_{66})I_9 \Big] - b_\ell \Big[(2S_{22}-2S_{12}-S_{66})I_7 + (-2S_{11}+2S_{12}+S_{66})I_9 \Big] - c_k \Big[S_{66}I_1 + 2\{2(S_{11}-2S_{12}+S_{22}) - S_{66}\}I_3 + S_{66}I_5 \Big] + u_\ell \Big[jI_{31} \Big] - v_k \Big[(i+1)I_{31} \Big] \Big] + \delta c_s \Big[-a_k \Big[(-2S_{11}+2S_{12}+S_{66})I_{12} + (2S_{22}-2S_{12}-S_{66})I_{14} \Big] - b_k \Big[(2S_{22}-2S_{12}-S_{66})I_{12} + (-2S_{11}+2S_{12}+S_{66})I_{14} \Big] - c_\ell \Big[S_{66}I_{16} + 2\{2(S_{11}-2S_{12}+S_{22}) - S_{66}\}I_{18} + S_{66}I_{20} \Big] - u_k \Big[jI_{33} \Big] + v_\ell \Big[(i+1)I_{33} \Big] \Big] \Big\} \qquad (6.3.15)$$

$$(iv) \quad \int_1^{\infty}\int_0^{2\pi} \delta u E_4 r dr d\theta \quad =$$

$$= \sum_{p=1}^{n}\sum_{q=0}^{m}\sum_{i=1}^{n}\sum_{j=0}^{m} \frac{1}{(i+p)} \left\{ \delta u_r \left[a_k \left[iI_{31} \right] + b_k \left[I_{31} \right] - c_\ell \left[jI_{31} \right] \right] + \delta u_s \left[a_\ell \left[iI_{33} \right] + b_\ell \left[I_{33} \right] + c_k \left[jI_{33} \right] \right] \right\} \tag{6.3.16}$$

$$(v) \quad \int_1^{\infty}\int_0^{2\pi} \delta v E_5 r dr d\theta \quad =$$

$$= \sum_{p=1}^{n}\sum_{q=0}^{m}\sum_{i=1}^{n}\sum_{j=0}^{m} \frac{1}{(i+p)} \left\{ \delta v_r \left[-b_\ell \left[jI_{31} \right] - c_k \left[(i-1)I_{31} \right] \right] + \delta v_s \left[b_k \left[jI_{33} \right] + c_\ell \left[(i-1)I_{33} \right] \right] \right\} \tag{6.3.17}$$

3e. Solutions

Case I:

The absence of displacement boundary conditions reduces the boundary integration to

$$\int_{C_s} \delta\{u\}^T \left[\{F\} - \{\bar{F}\} \right] ds$$

Since the boundary is a circle of radius r = 1 then the components of the boundary stress matrix {F} reduce to

$$F_r = \left[-\sigma_r\right]_{r=1} \quad , \quad F_\theta = \left[-\tau_{r\theta}\right]_{r=1}$$

The prescribed stresses on the boundary $\{\bar{F}_r \ \bar{F}_\theta\}$ are the components due to a uniform radial stress σ_o, ie., $\bar{F}_r = \sigma_o$ and $\bar{F}_\theta = 0$. Thus,

$$\{F\} - \{\bar{F}\} = \left\{ -\left(\left[\sigma_r\right]_{r=1} + \sigma_o\right) - \left[\tau_{r\theta}\right]_{r=1} \right\} \tag{6.3.18}$$

We note that in discussions of the sample problems in earlier sections the loading was taken to be of unit magnitude so the parameter σ_o did not appear. Thus, in all of our numerical work we set $\sigma_o = 1$. The boundary integral can be written explicitly as

$$\int_{C_s} \delta\{u\}^T \left[\{F\} - \{\bar{F}\}\right] ds = \int_0^{2\pi} \left\{ -\left[\delta u\right]_{r=1} \left(\left[\sigma_r\right]_{r=1} + \sigma_o\right) - \left[\delta v\right]_{r=1} \left[\tau_{r\theta}\right]_{r=1} \right\} d\theta, \tag{6.3.19}$$

where

$$\left[\sigma_r\right]_{r=1} = \sum_{i=1}^{n} \sum_{j=0}^{m} \left[a_k \cos j\theta + a_\ell \sin j\theta\right]$$

$$\left[\tau_{r\theta}\right]_{r=1} = \sum_{i=1}^{n} \sum_{j=0}^{m} \left[c_k \cos j\theta + c_\ell \sin j\theta\right] \tag{6.3.20}$$

$$\left[\delta u\right]_{r=1} = \sum_{p=1}^{n} \sum_{q=0}^{m} \left[\delta u_r \cos q\theta + \delta u_s \sin q\theta\right]$$

$$\left[\delta v\right]_{r=1} = \sum_{p=1}^{n} \sum_{q=0}^{m} \left[\delta v_r \cos q\theta + \delta v_s \sin q\theta\right]$$

Therefore,

$$\int_{C_s} \delta\{u\}^T\left[\{F\} - \{\bar{F}\}\right]ds = \sum_{p=1}^{n}\sum_{q=0}^{m} \delta u_r\{-\sigma_o I_{34}\}$$

$$+ \sum_{p=1}^{n}\sum_{q=0}^{m}\sum_{i=1}^{n}\sum_{j=0}^{m} \left\{\delta u_r\left[-a_k I_{31}\right] + \delta u_s\left[-a_\ell I_{33}\right] + \delta v_r\left[-c_k I_{31}\right] + \delta v_s\left[-c_\ell I_{33}\right]\right\}$$

(6.3.21)

Equations (6.3.12) through (6.3.17) combined with (6.3.20) and (6.3.21) comprise the complete Reissner formulation for Case I. Since the variational quantities δa_r, δa_s, δb_r.... δv_s are all arbitrary and independent a set of linear equations, $5n(2m+1)$ in number, can be set up by inspection. A detailed pattern of the coefficient matrix in terms of appropriate submatrices is shown and discussed in Appendix VIII. Also given is the column vector of constant terms.

Case II:

Case II is most simply derived using the principle of superposition. An infinite sheet without a hole subjected to uniaxial tensile stress at infinity will have a constant stress throughout. In what follows, including Case IV we consider only the situation for which the direction of tension is along the x-axis, ie., the angle of tension ϕ is zero. It is therefore elementary to, calculate the stresses in polar coordinates around a circular boundary in the sheet If the problem of an infinite sheet with an equal but opposite stress distribution around a circular hole in the sheet is then solved and superposed on the uniform stress distribution, the solution to Case II is obtained. See Figure 16

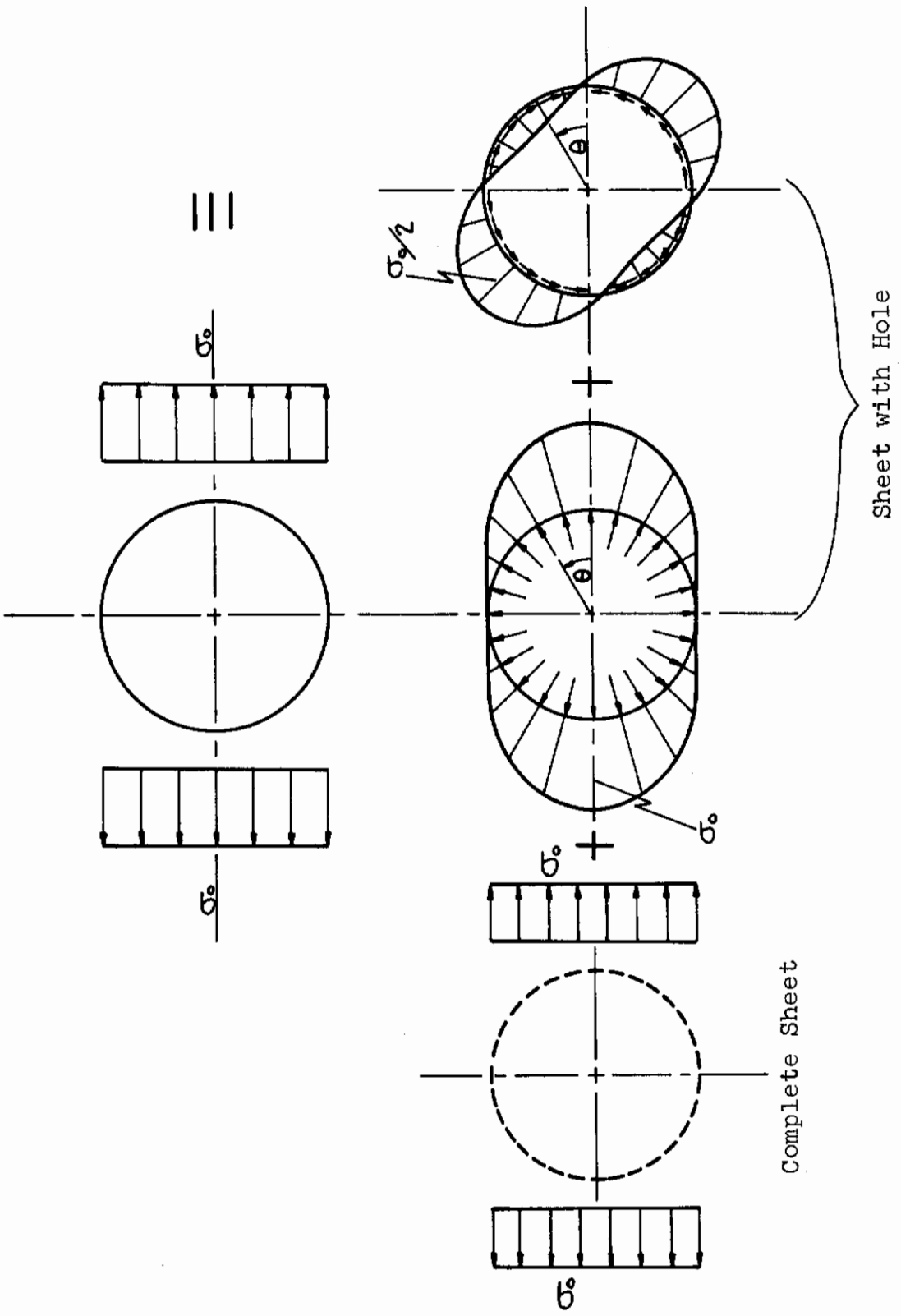


Figure 16. Superposition Model for Case II

The polar components of stress in a uniform stress field $\sigma_x = \sigma_o$ are:

$$\sigma_r = \frac{\sigma_o}{2}(1 + \cos 2\theta)$$

$$\sigma_\theta = \frac{\sigma_o}{2}(1 - \cos 2\theta) \qquad (6.3.22)$$

$$\tau_{r\theta} = \frac{-\sigma_o}{2}\sin 2\theta,$$

where θ is the angle measured with respect to the direction of the stress σ_o. The polar components of strain are likewise

$$\varepsilon_r = \frac{\sigma_o}{2}\left\{(S_{11} + S_{12}) + (S_{11} - S_{12})\cos 2\theta\right\}$$

$$\varepsilon_\theta = \frac{\sigma_o}{2}\left\{(S_{11} + S_{12}) - (S_{11} - S_{12})\cos 2\theta\right\} \qquad (6.3.23)$$

$$\gamma_{r\theta} = -\sigma_o (S_{11} - S_{12}) \sin 2\theta$$

Choosing an origin for displacements such that $[u,v]_{r=0} = 0$, a consistent set of polar displacements is therefore

$$u = \frac{\sigma_o r}{2}\left\{(S_{11} + S_{12}) + (S_{11} - S_{12})\cos 2\theta\right\}$$

$$v = \frac{-\sigma_o r}{2}(S_{11} - S_{12})\sin 2\theta \qquad (6.3.24)$$

The complete set of stresses and displacements for Case II is then found by superposition, ie.,

$$\sigma_r = \frac{\sigma_o}{2}(1 + \cos 2\theta) + \sum_{i=1}^{n}\sum_{j=0}^{m} r^{-(i+1)}\left\{a_k \cos j\theta + a_\ell \sin j\theta\right\}$$

$$\sigma_\theta = \frac{\sigma_o}{2}(1 - \cos 2\theta) + \sum_{i=1}^{n}\sum_{j=0}^{m} r^{-(i+1)}\left\{b_k \cos j\theta + b_\ell \sin j\theta\right\}$$

$$\tau_{r\theta} = \frac{-\sigma_o}{2}\sin 2\theta + \sum_{i=1}^{n}\sum_{j=0}^{m} r^{-(i+1)}\left\{c_k \cos j\theta + c_\ell \sin j\theta\right\}$$

$$u = \frac{\sigma_o r}{2}\left\{(S_{11}+S_{12}) + (S_{11}-S_{12})\cos 2\theta\right\} + \sum_{i=1}^{n}\sum_{j=0}^{m} r^{-i}\left\{u_k \cos j\theta + u_\ell \sin j\theta\right\}$$

$$v = \frac{-\sigma_o r}{2}(S_{11}-S_{12})\sin 2\theta + \sum_{i=1}^{n}\sum_{j=0}^{m} r^{-i}\left\{v_k \cos j\theta + v_\ell \sin j\theta\right\} \qquad (6.3.25)$$

Thus in order to solve the problem the constants $\{a_k\}$ etc. must be determined for a stress distribution applied around the hole of

$$\left[\bar{\sigma}_r\right]_{r=1} = \frac{-\sigma_o}{2}(1 + \cos 2\theta)$$

$$\left[\bar{\sigma}_\theta\right]_{r=1} = \frac{-\sigma_o}{2}(1 - \cos 2\theta) \qquad (6.3.26)$$

$$\left[\bar{\tau}_{r\theta}\right]_{r=1} = \frac{\sigma_o}{2}\sin 2\theta$$

The corresponding boundary stress resultants are

$$\{\bar{F}\} = \{\bar{F}_r \quad \bar{F}_\theta\} \quad , \quad \text{where}$$

$$\bar{F}_r = -\left[\bar{\sigma}_r\right]_{r=1} = \frac{\sigma_o}{2}(1 + \cos 2\theta) \tag{6.3.27}$$

$$\bar{F}_\theta = -\left[\bar{\tau}_{r\theta}\right]_{r=1} = \frac{\sigma_o}{2} \sin 2\theta$$

The boundary integral will be the only part of the Reissner function to differ from Case I and can be written explicitly as

$$\int_{C_s} \delta\{u\}^T \left[\{F\} - \{\bar{F}\}\right] ds =$$

$$= -\int_0^{2\pi} \Bigg\{ (\delta u)_{r=1} \left[(\sigma_r)_{r=1} + \frac{\sigma_o}{2}\left(1 + \cos 2\theta\right)\right] +$$

$$+ (\delta v)_{r=1} \left[(\tau_{r\theta})_{r=1} - \frac{\sigma_o}{2} \sin 2\theta \right] \Bigg\} \, d\theta \quad , \tag{6.3.28}$$

where $(\delta u)_{r=1}$, $(\delta v)_{r=1}$, $(\sigma_r)_{r=1}$ and $(\tau_{r\theta})_{r=1}$ are given by equation (6.3.20). Performing the integrations

$$\int_{C_s} \delta\{u\}^T \left[\{F\} - \{\bar{F}\}\right] \, ds \quad =$$

$$= \sum_{p=1}^{n} \sum_{q=0}^{m} \left[\delta u_r \left\{ - \frac{\sigma_o}{2} \left(I_{34} + I_{35} \right) \right\} + \delta v_s \left\{ \frac{\sigma_o}{2} \, I_{36} \right\} \right] \quad +$$

$$+ \sum_{p=1}^{n} \sum_{q=0}^{m} \sum_{i=1}^{n} \sum_{j=0}^{m} \left\{ \delta u_r \left[-a_k I_{31} \right] + \delta u_s \left[-a_\ell I_{33} \right] + \delta v_r \left[-c_k I_{31} \right] + \delta v_s \left[-c_\ell I_{33} \right] \right\}$$

(6.3.29)

Only the constant terms differ from equation (6.3.21) and therefore the coefficient matrix $\left[A_{ik}\right]$ is unchanged. The matrix of constant {B} for this case is detailed in Appendix VIII.

Case III:

For Case III, which is a uniform radial displacement of the hole boundary, the boundary conditions are solely in terms of displacements and the boundary integral is

$$- \int_{C_d} \delta\{F\}^T \left[\{u\} - \{\bar{u}\}\right] ds$$

The specified boundary displacements are therefore

$$\left[\bar{u}\right]_{r=1} = u_o \qquad , \left[\bar{v}\right]_{r=1} = 0$$

(In our previous discussions of Case III, and also in all of our numerical work, we have normalized the loading by setting u_o to unity.) The displacements at the boundary in terms of the assumed functions are

$$\left[u\right]_{r=1} = \sum_{i=1}^{n}\sum_{j=0}^{m}\{u_k \cos j\theta + u_\ell \sin j\theta\}$$

$$\left[v\right]_{r=1} = \sum_{i=1}^{n}\sum_{j=0}^{m}\{v_k \cos j\theta + v_\ell \sin j\theta\} \qquad (6.3.30)$$

The variational terms are

$$\delta\{F\} = \left\{-(\delta\sigma_r)_{r=1} - (\delta\tau_{r\theta})_{r=1}\right\} ,$$

where

$$\left[\delta\sigma_r\right]_{r=1} = \sum_{p=1}^{n}\sum_{q=0}^{m}\left\{\delta a_r \cos q\theta + \delta a_s \sin q\theta\right\}$$

$$\left[\delta\tau_{r\theta}\right]_{r=1} = \sum_{p=1}^{n}\sum_{q=0}^{m}\left\{\delta c_r \cos q\theta + \delta c_s \sin q\theta\right\} \qquad (6.3.31)$$

Therefore,

$$-\int_{C_d} \delta\{F\}^T\left[\{u\} - \{\bar{u}\}\right]ds = \sum_{p=1}^{n}\sum_{q=0}^{m}\delta a_r\left[-u_o I_{34}\right]$$

$$+ \sum_{p=1}^{n}\sum_{q=0}^{m}\sum_{i=1}^{n}\sum_{j=0}^{m}\left\{\delta a_r\left[u_k I_{31}\right] + \delta a_s\left[u_\ell I_{33}\right] + \delta c_r\left[v_k I_{31}\right] + \delta c_s\left[v_\ell I_{33}\right]\right\} \qquad (6.3.32)$$

The following submatrices in the matrix of coefficients will differ from Cases I and II; $\left[A_{17}\right]$, $\left[A_{28}\right]$, $\left[A_{59}\right]$ and $\left[A_{610}\right]$. These are listed in Appendix VIII together with the constant submatrix $\{B_1\}$.

Case IV:

Case IV is readily solved using the principle of superposition as in Case II. The main difference lies in the fact that the rigid inclusion in the hole gives rise to displacement boundary conditions only. The solution is therefore provided by the superposition of the uniform stress field in a sheet with no hole on the solution obtained by imposing prescribed displacements on a circular hole equal and opposite to those of equation (6.3.24) at a radius of r = 1, ie.,

$$\left[\bar{u}\right]_{r=1} = -\frac{\sigma_o}{2}\left\{\left(S_{11}+S_{12}\right) + \left(S_{11}-S_{12}\right)\cos 2\theta\right\}$$

$$\left[\bar{v}\right]_{r=1} = \frac{\sigma_o}{2}\left(S_{11}-S_{12}\right)\sin 2\theta \qquad (6.3.33)$$

The general expressions for stresses and displacements are identical with equations (6.3.25). The boundary integral will differ from Case III with respect to the constant terms only and after integration is expressed as

$$-\int_{C_d} \delta\{F\}^T\left[\{u\} - \{\bar{u}\}\right] ds =$$

$$= \sum_{p=1}^{n}\sum_{q=0}^{m}\left\{\delta a_r\left[\frac{\sigma_o}{2}\left[\left(S_{11}+S_{12}\right)I_{34}+\left(S_{11}-S_{12}\right)I_{35}\right]\right]-\delta c_s\left[\frac{\sigma_o}{2}\left(S_{11}-S_{12}\right)I_{33}\right]\right\}$$

$$+ \sum_{p=1}^{n}\sum_{q=0}^{m}\sum_{i=1}^{n}\sum_{j=0}^{m}\left\{\delta a_r\left[u_k I_{31}\right] + \delta a_s\left[u_\ell I_{33}\right] + \delta c_r\left[v_k I_{31}\right] + \delta c_s\left[v_\ell I_{33}\right]\right\} \qquad (6.3.34)$$

The appropriate non-zero submatrices of constants are therefore $\{B_1\}$ and $\{B_6\}$ and are tabulated in Appendix VIII.

SECTION VII

NUMERICAL RESULTS

1. INTRODUCTION

In this section we present results obtained from the computer programs for the approximate methods described in the preceding sections. These results are primarily in the form of plots of the deviation (of a particular stress or displacement component) from the exact solution along the circular boundary or along a ray, θ = constant. More of the plots are for $\sigma_{\theta D}$ ($\equiv \sigma_{\theta EXACT} - \sigma_{\theta APPROX}$) since σ_θ is usually the most difficult quantity to approximate (especially near the boundary) in our sample problems. (See the contour plots in Figures 5 and 6.) We note that the maximum deviation has a wide range depending on the particular stress or displacement quantity, the case, the material, the number and type of assumed functions, etc. Consequently it is necessary to use a number of different scales on the ordinates of the plots.

In all of the computer programs for the approximate methods provision has been made for the treatment of at least four different sample problems, which have been designated Cases I - IV in our earlier discussions. In addition the least squares, boundary Reissner energy, potential energy, and complementary energy programs allow the user to define other sample problems. In order to keep computer output within reason we have restricted our numerical comparisons and evaluations to Cases I and III, which are examples of stress and displacement boundary conditions respectively. It is felt that Cases II and IV, with uniform stress at infinity, are not essentially different than Cases I and III respectively, especially when superposition is used to handle the stress at infinity, as it is in all but the potential and complementary energy programs.

In addition to the plots of deviations from the exact solution we present information concerning the conditioning of the system linear equations which must be solved in each of the approximate methods (except perturbation). This information is in the form of condition numbers of the coefficient matrices for these linear systems. (See Appendix III for a discussion of matrix conditioning and condition numbers.) We note that all matrix inversions have been done in double precision even though the coefficient matrices (except for interior Reissner energy) have been formed in single precision. The reason for this is that it was found fairly early in this study that inversion in

single precision often resulted in critical loss of significance. Because of the relatively large coefficient matrices arising in the interior Reissner energy method, it was decided to also form the coefficient matrix and right hand side vector in double precision in order to reduce the effects of round-off in this stage of the solution process.

1a. Perturbation Solutions

At the beginning of this study it was anticipated that perturbation methods could be used successfully only with mildly orthotropic materials, e.g., Material (2) of Table I (Article 3 of Section III). However, it turns out, at least in Sample Problem I, that perturbation solutions have wider applicability. Numerical results (using the third order perturbation series developed in Article 2b of Section IV) for the distinctly orthotropic Material (1) and the extremely orthotropic Material (3) were surprisingly good. In the case of Material (3) the results for σ_θ were substantially in error at r=1, $\theta=90^\circ$. However, inspection of the contour plot of σ_θ (Figure 5) shows that there is a highly localized stress concentration at this point, and one could hardly expect a perturbation solution to be accurate in the neighborhood of this stress concentration. In Figure 17 we have plotted $\sigma_{\theta D}$ and u_{rD} for Material (3) - Case I

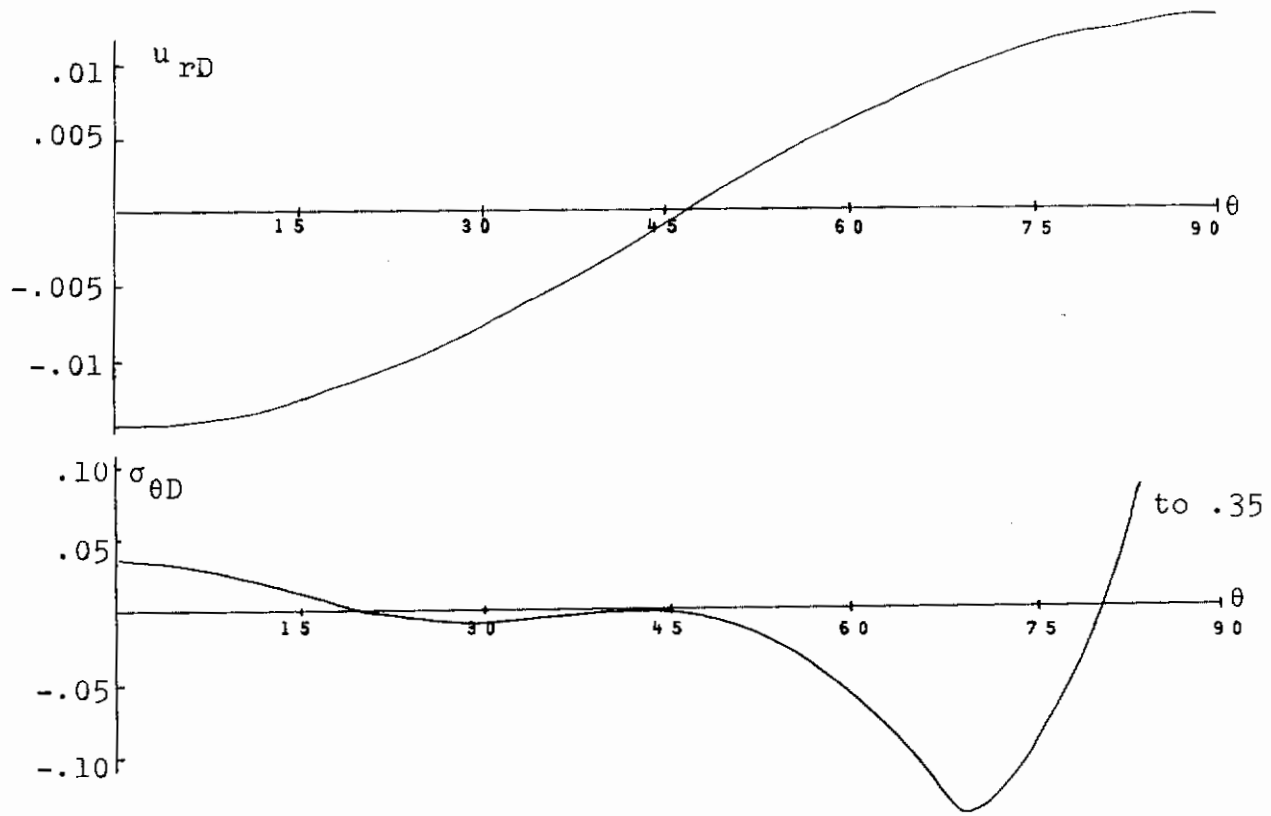


Figure 17. u_{rD} and $\sigma_{\theta D}$ for Perturbation Solution (Case I, Material 3)

along the boundary. The deterioration of σ_θ in the neighborhood of r=1, $\theta=90^\circ$ is apparent. Along the ray $\theta=90^\circ$ $\sigma_{\theta D}$ decreases very rapidly from its maximum of .35. At r=1.1, $\sigma_{\theta D}=-.015$. u_{rD} decreases somewhat less rapidly. σ_{rD} is small everywhere; its maximum is about .01. The quantities $\tau_{r\theta}$ and u_θ are not approximated as accurately, but these quantities are smaller and of less interest than σ_r, σ_θ, and $\tau_{r\theta}$.

In Figure 18 we have plotted $\sigma_{\theta D}$ and u_{rD} along the boundary and σ_{rD} along the ray $\theta=90^\circ$ for Material (1) - Case I. On the first of these graphs there are two curves; one (labeled C_1) corresponds to results obtained when the nearest isotropic compliance matrix $[S^\circ]$ is chosen as in Appendix I, while the other (labeled C_2) corresponds to the definition of $[S^\circ]$ in (4.1.16), as suggested by Sokolnikoff. More will be said about the choice of $[S^\circ]$ later. However, for the time being it is enough to observe that the results for σ_θ and u_r are significantly better for Material (1) than those for Material (3) (plotted in Figure 17). This is to be expected since Material (1) is less strongly orthotropic and does not have an appreciable stress concentration in σ_θ. The results for the slightly orthotropic Material (2) (not plotted here) are even more accurate with $\sigma_{\theta DMAX} \approx .003$, $\sigma_{rDMAX} \approx .0005$, and $u_{rDMAX} \approx .0004$. The results for the highly orthotropic Material (6) were generally poor with $\sigma_{\theta DMAX} \approx 20\%$, $\sigma_{rDMAX} \approx 11\%$, and $u_{rDMAX} \approx 7\%$.

Several runs were made for Case III. The results for all quantities were inferior to those of Case I for the same material (on a percentage basis). For example, for Material (2) $\sigma_{\theta DMAX}$ for Case III was about 6 times that for Case I. The other quantities had errors of similar magnitude. The reasons for poorer results in Case III are not apparent. However, such results are not entirely unexpected since the elastic constants appear in the boundary conditions when displacements are prescribed, and this factor was not considered in the determination of the "nearest" isotropic compliance matrix.

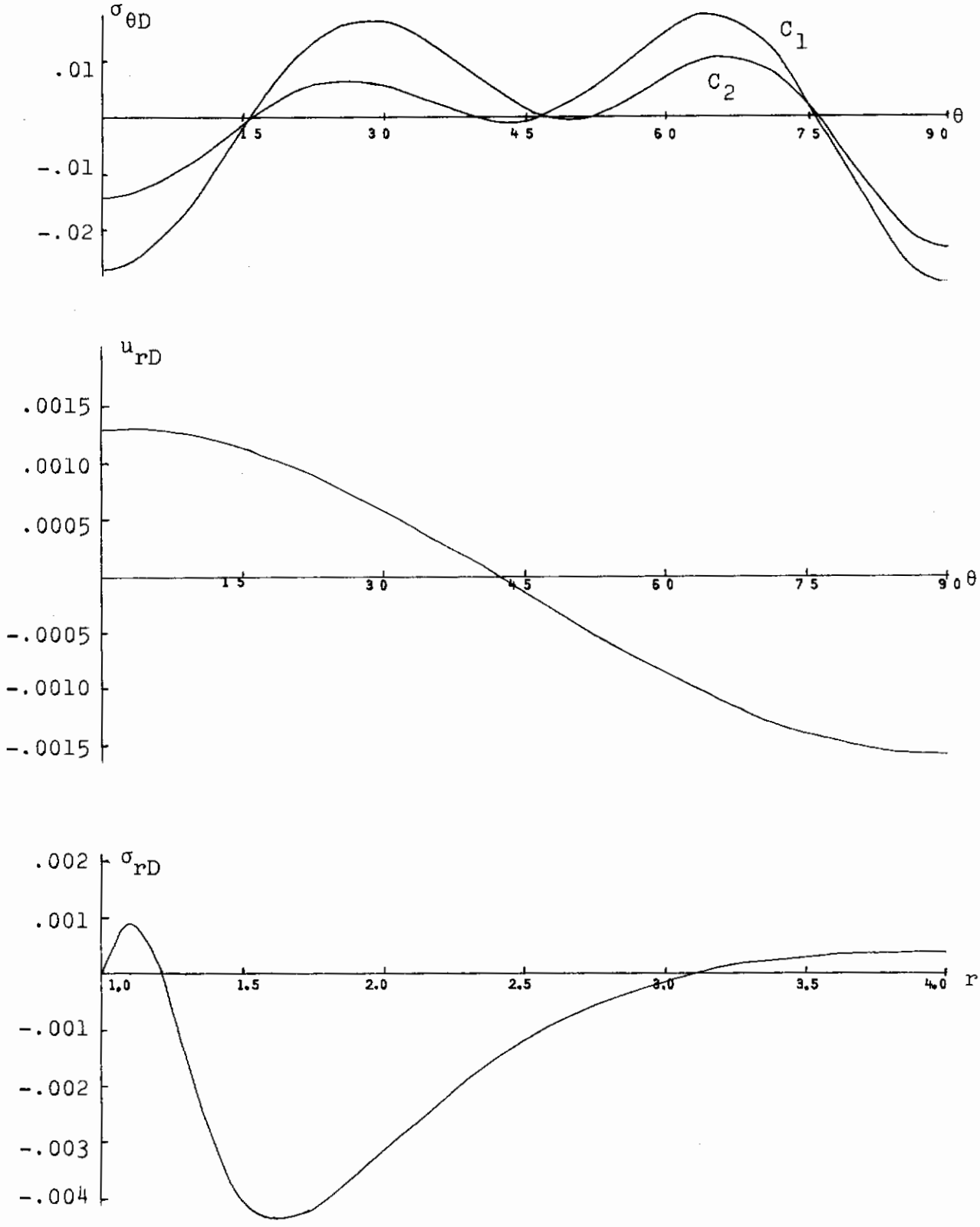


Figure 18. $\sigma_{\theta D}$, σ_{rD}, and u_{rD} for Perturbation Solution (Case I-Material 1)

Let us now return to the question of how the choice of $[S^o]$ affects the accuracy of the results. From the curves of Figure 18 there does not appear to be a significant difference in the results for the two choices of $[S^o]$ used in those plots. The results (not shown in the graphs) of a run for Material (3) with $[S^o]$ defined as in (4.1.16) were similarly very close to those shown in Figure 17. However, a run for Material (1) with $[S^o]$ defined as in (4.1.18) (also suggested by Sokolnikoff) gave results considerably inferior to those of Figure 18. For example, $\sigma_{\theta DMAX}$ was about 10 times as large. Perturbation solutions for a slightly orthotropic material ($E_x = 1.1$, $E_y = 1.$, $\nu_{xy} = .25$, $G_{xy} = .4$) showed that the choice of $[S^o]$ in Appendix I yielded significantly better results than the ad hoc definition of $[S^o]$; $S^o_{11} = S_{11}$, $S^o_{12} = S_{12}$. From these limited numerical results it is clear that the selection of $[S^o]$ can have an important influence on the accuracy of perturbation solutions. On the other hand we cannot conclude, on the basis of these results, that either the definition of $[S^o]$ of Appendix I or the definition of (4.1.16) is superior to the other. However, definition (4.1.16) is inappropriate for certain orthotropic materials, eg., if the quantity $(S_{66}+2S_{12})$ is zero or negative definition (4.1.16) breaks down altogether. Admittedly, this is not a mildly orthotropic material, but as we have seen perturbation techniques can sometimes be used successfully with strongly orthotropic materials. The major advantage of the definition of $[S^o]$ in Appendix I is that S^o_{11} and S^o_{12} (S^o_{22} and S^o_{66} are defined in terms of S^o_{11} and S^o_{12}) depend on all four of the orthotropic elastic compliances, whereas $[S^o]$ defined in (4.1.16) depends on only two, and $[S^o]$ defined in (4.1.18) depends on only one. Thus, a perturbation solution based on the latter two definitions can lead to poor or completely erroneous results in cases where it is possible to obtain satisfactory solutions through the use of $[S^o]$ as defined in Appendix I. There may be materials for which some other definition of $[S^o]$ is distinctly superior to that of Appendix I. However, it is not at all clear what these materials would be.

1b. Least Squares and Reissner Energy on the Boundary

As discussed in Article 2a of Section V the simplest assumed forms for Φ_1 and Φ_2 which satisfy the conditions at infinity for Sample Problems I and III are series in negative powers of $z_1(\equiv x+\mu_1 y)$ and $z_2(\equiv x+\mu_2 y)$ respectively. (See equation (5.2.1).) The condition of symmetry about the x-axis can be enforced by excluding even negative powers of z_1 and z_2 from the assumed forms. This was done in our numerical solutions so it was only necessary to perform the relevant line integrals over the interval $[0,\pi]$. By suitably restricting the unknown complex coefficients, a_i and b_i, in the assumed forms it is possible to obtain symmetry about the y-axis also, in which case the integration interval can be taken to be $[0,\pi/2]$. This point is discussed at the beginning of Article 2a of Section V. In our numerical solutions the unknown coefficients were not specialized in this manner, so symmetry about the y-axis was a consequence of the numerical process. Most of our numerical results are for the assumed forms of Φ_1 and Φ_2 just described. Thus, in future discussions Φ_1 and Φ_2 are understood to be of this form unless noted. We do present a few results for Φ_1 and Φ_2 in the form of odd negative powers of $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$, where $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ are closely related to the exact solution functions ζ_1 and ζ_2 (equation (3.2.4)). The above considerations regarding symmetry and integration interval also apply to these assumed forms. This point is also discussed in Article 2a of Section V.

In Article 2a of Section V we discussed the fact that the exact solutions for Φ_1 and Φ_2 in Cases I and III have expansions in odd negative powers of z_1 and z_2, but that these expansions converge only outside of circles with centers at the origin and with radii equal to $|1+\mu_{1,2}^2|^{\frac{1}{2}}$. (See Figure 15.) In Figure 19 we have drawn to scale in the z_1 and z_2-planes the elliptical image of the unit circle in the x-y plane and the corresponding circles of convergence for Material (1). Material (2) also has the same image in the z_2-plane since μ_2 for Materials (1)

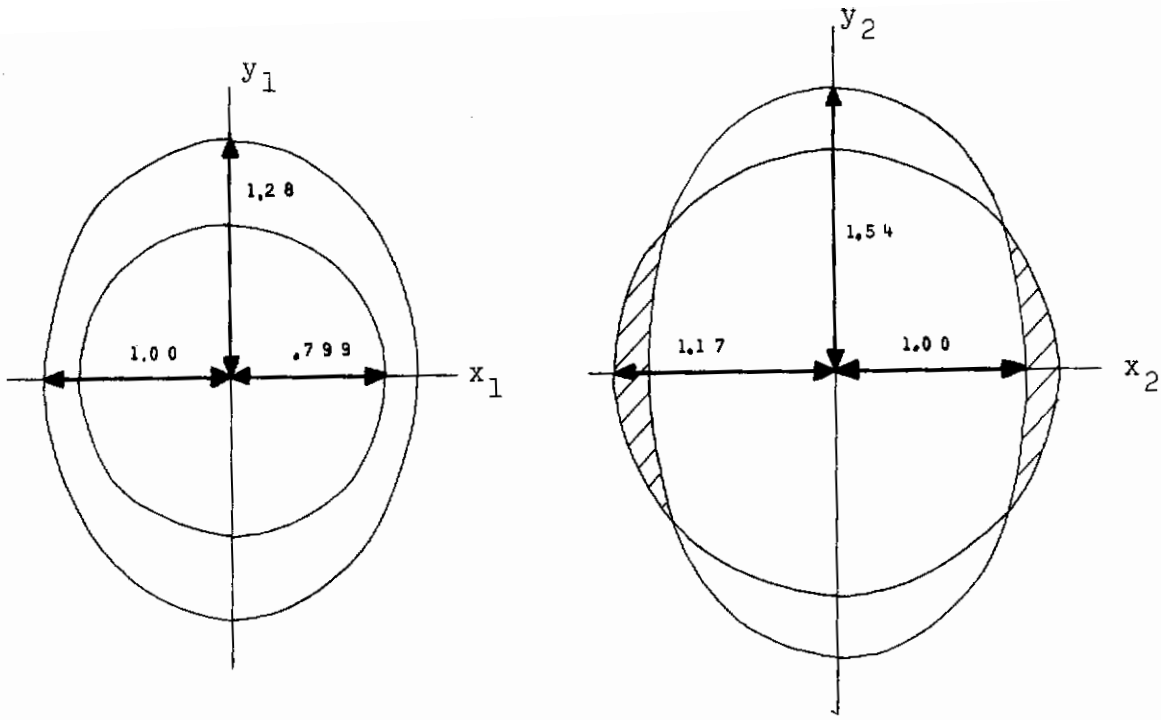


Figure 19. Images in z_1 and z_2-planes for Material (1)

and (2) are equal. The elliptical image in z_1-plane for Material (2) is slightly less eccentric and is rotated by 90°. The images for Material (3) are similar to (2) except that the ellipse in the z_1-plane is much less eccentric and that in the z_2-plane much more eccentric. For Materials (4), (5), and (6) the elliptical image in the z_1-plane has the same eccentricity as that in the z_2-plane. For all three materials the eccentricity is high, and the circles of convergence extend well outside the ellipses. Also, for Materials (4) and (5) the axes of the ellipse are at an angle with respect to the coordinate axes. Thus, for all of our sample materials the series expansions of the exact solutions for Φ_1 and Φ_2 (and consequently for the stresses and displacements) do not converge in parts of the region of physical interest. A small computer program was written to compute the exact solutions for the stresses in Case I from their series expansions. (The coefficients in these series are easily determined from the coefficients in the Green-Taylor series. For Case II these

coefficients are listed in (6.1.13). The coefficients for Case I are similar.) According to Figure 19 these series should converge at $r = 1$, $\theta = 90^{o}$ for Materials (1), (2), and (3). This appeared to be the case, although for Material (3) the series converged so slowly that there was a 10% error in σ_{θ} and a .5% error in σ_{r} after 32 terms (up to $z_{1,2}^{-64}$) were summed. For Materials (1) and (2) σ_{r} and σ_{θ} were accurate to eight significant figures after the same number of terms. Now at $r = 1$, $\theta = 0^{o}$ Figure 19 indicates that these series do not converge. An attempt to sum these series showed that not only did they fail to converge, they began to diverge immediately, and at no point did the partial sums come close to the correct values of the stresses. Thus, if one were to calculate the radial and shear stresses along the boundary from these series (truncated at some point, of course) the squares of the boundary residuals when integrated over the boundary would not be a small number. Further, as more terms are taken in the series the integrated square of the residual would grow without bound. Therefore, we conclude that whenever the circle of convergence of the series form of the exact solution extends outside the elliptical image in the z_{1} or z_{2}-planes it is impossible to reproduce the exact solution series by means of the boundary least squares or Reissner energy principles. This is not to say that a satisfactory approximation cannot be obtained by these methods when Φ_{1} and Φ_{2} are assumed to be odd negative powers of z_{1} and z_{2}. What it does means, however, is that there is some non-zero lower bound to the integrated least squares residual when Φ_{1} and Φ_{2} have the form presently under consideration. Likewise, it means that the approximate value of the Reissner energy functional is bounded away from the exact value. No analytical estimates of these bounds have been made, nor has their presence been indicated numerically. It may be that the existence of such bounds represents no practical limitations in obtaining approximate solutions, since good results have been obtained for Materials (1) and (2).

It is interesting to compare the coefficients in the exact series for Φ_{1} and Φ_{2} with those obtained by one of the approximate methods. Consider, for example, the approximate solution obtained for Case I - Material 1 when 12 terms were taken in each of the assumed forms for Φ_{1} and Φ_{2}. The coefficients in the exact series for Φ_{1} decrease rapidly, and so do the

corresponding approximate coefficients. For five or six terms there is close agreement between the two sets of coefficients. The higher coefficients are different but they are small and have little effect. The first five coefficients of Φ_2 decrease in magnitude, but they start increasing with the sixth and from the behavior of the next 25 it appears that they are unbounded. The first several approximate coefficients of Φ_2 agree to two or three significant figures with the corresponding exact coefficients. However, the approximate coefficients continue to decrease right up to the twelfth one, so there is considerable discrepancy between the exact and approximate coefficients at that point.

Let us now consider briefly the problem of evaluating numerically the line integrals in the integral least squares and boundary Reissner energy methods. For these numerical quadratures we have chosen the five-point Gauss-Chebyshev formula. The reasons for this selection are given in Article 1a of Section V, equation (5.1.3)ff. In order to evaluate the relevant integrals the total interval $[0,\pi]$ is divided into equal subintervals and the quadrature formula is applied in each of these subintervals and the results are then added to obtain the value of the integral. The number of these subintervals is dependent on the required accuracy of the integrals. For the Reissner energy method it is quite important to evaluate the integrals accurately. For example, a run was made of the Reissner energy program with 12 subintervals over $[0,\pi]$ and with eight terms (in odd negative powers of z_1 and z_2) present in the assumed forms of Φ_1 and Φ_2. The results were poor, primarily because integrals involving high negative powers of z_1 and z_2 were considerably in error. When the problem was rerun using 24 subintervals the results were quite good. In contrast the least squares program yielded good results with the same assumed functions and 12 subintervals. In fact even fewer subintervals can be used. Of course the integrals will not be evaluated accurately but this is of little importance since the method is always equivalent to a point least squares method. The only requirement for a point least squares method is that there be as many function evaluations as there are unknowns. In the present application we are actually evaluating two functions, σ_r and $\tau_{r\theta}$, at each point. If, for example, we take five terms in the assumed forms of Φ_1 and Φ_2 there are twenty unknown real constants (10 complex constants) to be determined. In order to obtain twenty function evaluations, ten points (two subintervals) must be chosen over $[0,\pi]$. In Figure 20 we have shown $\sigma_{\theta D}$ and

$_{r\,D}$ (Case I - Material 2) along the boundary for this situation. Also, plotted are the results for four subintervals (twice as many function evaluations as unknowns). The latter results are

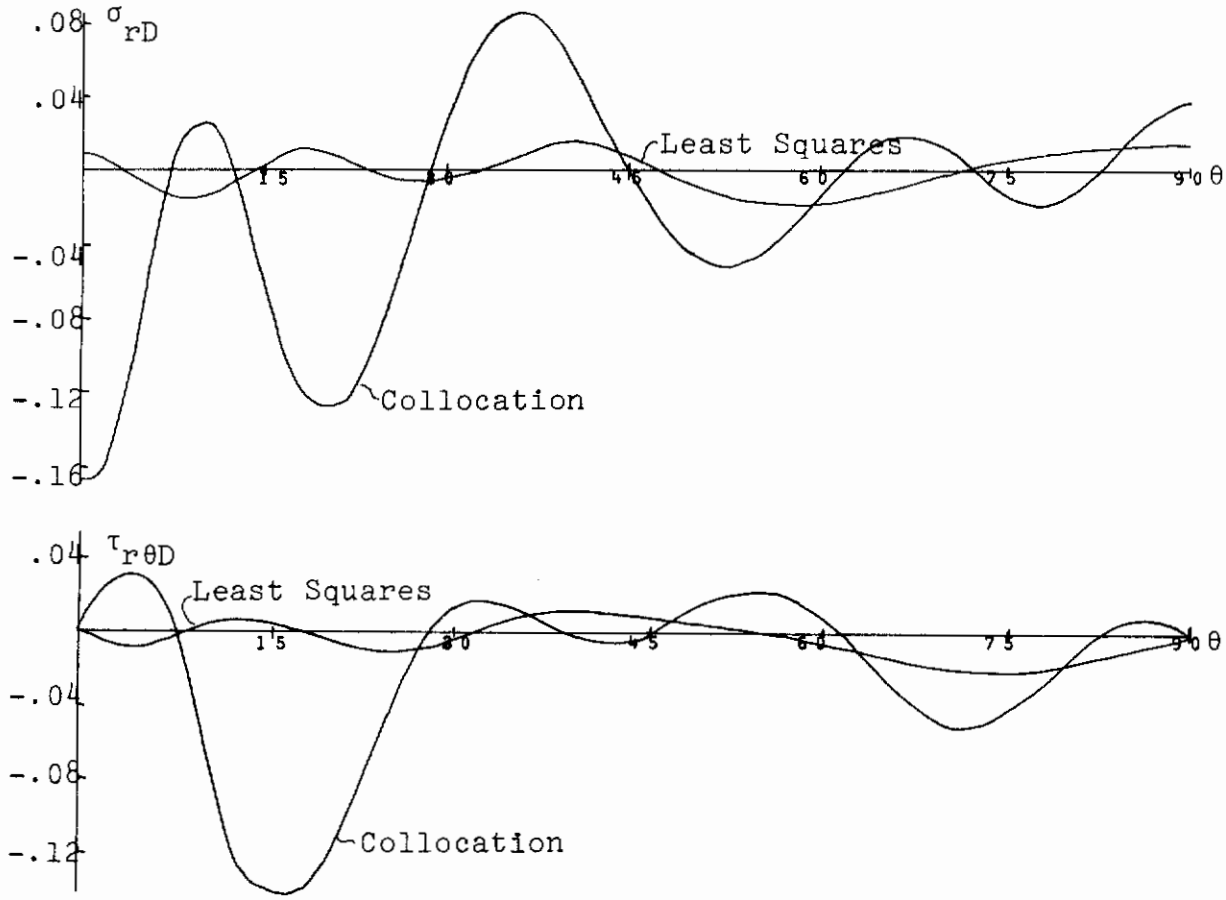


Figure 20. σ_{rD} and $\tau_{r\theta D}$ for Least Squares and Collocation (Case I, Material 2)

are clearly superior. For the two subinterval case (collocation) both σ_{rD} and $\tau_{r\theta D}$ should vanish at the points where the functions are evaluated, ie., $\theta \approx 8^{o}$, 28^{o}, 45^{o}, 62^{o}, and 82^{o}. From Figure 20 it is clear that σ_{rD} and $\tau_{r\theta D}$ do vanish at these points and at a few others also.

Hulbert [21] has studied the question of how the number and position of points (at which the functions are evaluated) affects the accuracy of least squares solutions of harmonic and biharmonic problems. He found that in most cases

it was sufficient to have 1½ to 2 times as many function evaluations as unknowns. Little or no improvement can be obtained with more function evaluations. In this study we have not examined in detail the question of the optimum number of function evaluations. The results of a few numerical experiments are in agreement with the conclusions of Hulbert regarding the number of function evaluations. For most of our numerical work we have performed at least two times as many function evaluations as there were unknowns. When the ratio of function evaluations to unknowns approaches unity the accuracy of the solution deteriorates (clearly indicated in Figure 20) and the conditioning of the linear system becomes poorer. This points strongly against using point least squares with only a few more function evaluations than unknowns. However, when the ratio equals unity the problem can be reformulated as a collocation problem ie., the residuals are not squared, but set directly to zero. The associated linear system generally has better conditioning than that for any least squares formulation using the same assumed functions. Consequently one may be able to compensate for the generally inferior accuracy inherent in collocation solutions by increasing the number of unknowns in his assumed form. We refer the reader to references [15] and [22] for further discussion of the collocation method.

In almost all of our numerical work with the least squares program the points at which functions were evaluated were those of the Chebyshev quadrature formula. However, the program is easily modified to consider other sets of boundary points. In Figure 21 we have plotted $\sigma_{\theta D}$ (Case I, Material 2, 5 terms in Φ_1 and Φ_2) for two different sets of boundary points. For Curve 1 there were 20 points over $[0,\pi]$ determined in accordance with Gauss-Chebyshev formula. For Curve 2 there were 17 equally spaced points starting at $\theta=0$ and ending at $\theta=\pi$. (The functions

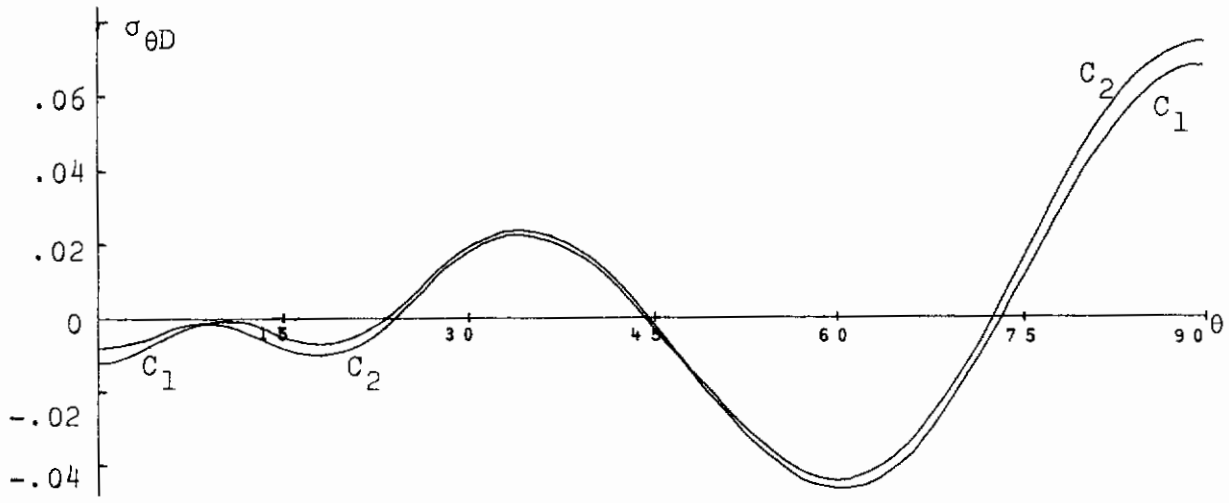


Figure 21. $\sigma_{\theta D}$ for Least Squares (Case I, Material 2)

were evaluated twice at $\theta = \pi/4, \pi/2, 3\pi/4$.) As can be seen from the figure there is very little difference in the two solutions. The results for a third set of boundary points (20 equally spaced points starting at $\theta = 4.5^{\circ}$ and ending at $\theta = 175.5^{\circ}$) were virtually the same as those shown in Figure 21. In Figure 22 we have plotted $\sigma_{\theta D}$ and σ_{rD} (Case I, Material 2, 5 terms in Φ_1 and Φ_2) for two different sets of collocation points. For Curve 1 the points were determined by the Gauss-Chebyshev formula ($\theta \approx 8^{\circ}, 28^{\circ}, 45^{\circ}, 62^{\circ}, 82^{\circ}$ plus images of

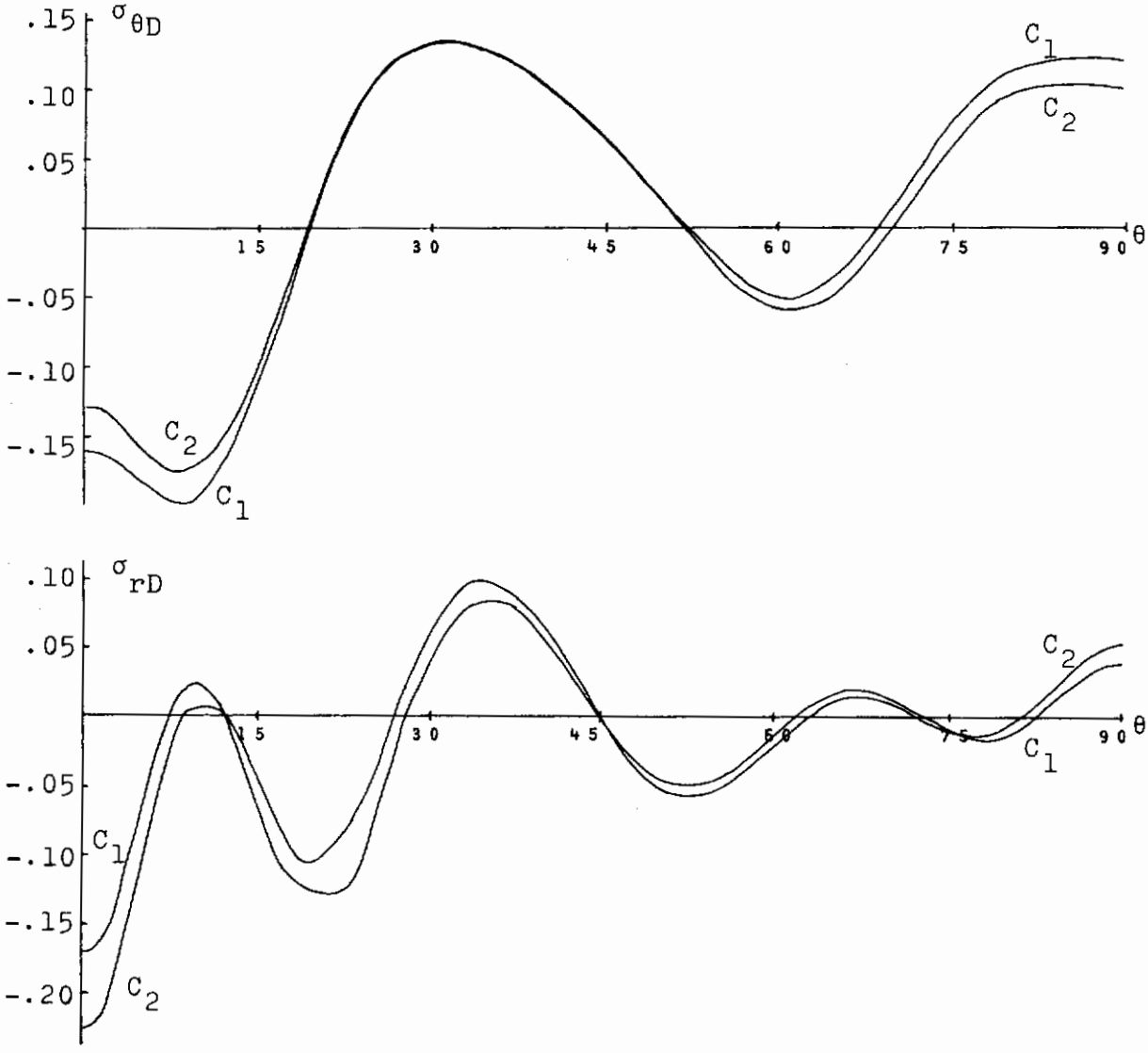


Figure 22. $\sigma_{\theta D}$ and σ_{rD} for Collocation (Case I, Material 2)

these points in the second quadrant). For Curve 2 the points were $\theta = 9^o, 27^o, 45^o, 63^o, 81^o$. The results for these two sets are not greatly different, but the difference is considerably greater than for the least squares test with 20 points over $[0,\pi]$.

In Figure 23 we have plotted σ_{rD}, $\sigma_{\theta D}$, $\tau_{r\theta D}$, u_{rD}, and $u_{\theta D}$ (Case I, Material 1, 8 terms in Φ_1 and Φ_2) for both the least squares and Reissner energy methods. In Figure 24 the same thing has been done for Case III. In Figure 23 it is seen that the least squares results for σ_r and $\tau_{r\theta}$ are superior to the Reissner results, but for σ_θ, u_r, and u_θ the opposite is true. This result is plausible since the least squares method minimizes the integral of $\sigma_{rD}^2 + \tau_{r\theta D}^2$ over the boundary in Case I, and so these quantities should be more accurate. For the same reason one would expect the least squares program to give better results for the displacements in Case III. An examination of Figure 24 shows that while this may be true u_r and u_θ from the Reissner method are almost as good. The Reissner results for σ_r and $\tau_{r\theta}$ are distinctly better and those for σ_θ about the same as the least squares. On the basis of the results plotted in Figures 23 and 24 and also some not presented here it appears that for the class problems being considered in this study the Reissner method is somewhat better than the least squares method for obtaining an accurate solution for all stress and displacement quantities. The least squares method seems to obtain accurate results for certain quantities at the expense of others. As discussed earlier in this article a disadvantage of the Reissner method is that the relevant integrations must be performed accurately. For the solutions of Figures 23 and 24 it was necessary to take 24 subintervals over $[0,\pi]$ for the Reissner method while 12 subintervals were more than enough for least squares. Another advantage of the Reissner method is that the conditioning of the resulting system of linear equations is generally somewhat better than that for least squares. The ratio of the condition number for least squares to that for the Reissner method ranged from approximately .8 to 15. depending on the case, material, and number of unknowns. In Table IV some typical condition numbers for the coefficient matrices in the least squares and Reissner energy methods are listed. The material designated by * is the slightly orthotropic material; $E_x = 1$, $E_y = 1.1$, $\nu_{xy} = .25$, $G_{xy} = .4$. The materials designated by (1') and (2') are Materials (1) and (2) with the axes rotated by 90^o, i.e., for (1'); $E_x=1.$, $E_y=3.86$, $\nu_{xy}=.15492$, $G_{xy}=.743$, and for (2'); $E_x=1.$, $E_y=1.5$, $\nu_{xy}=.375$, $G_{xy}=.4$.

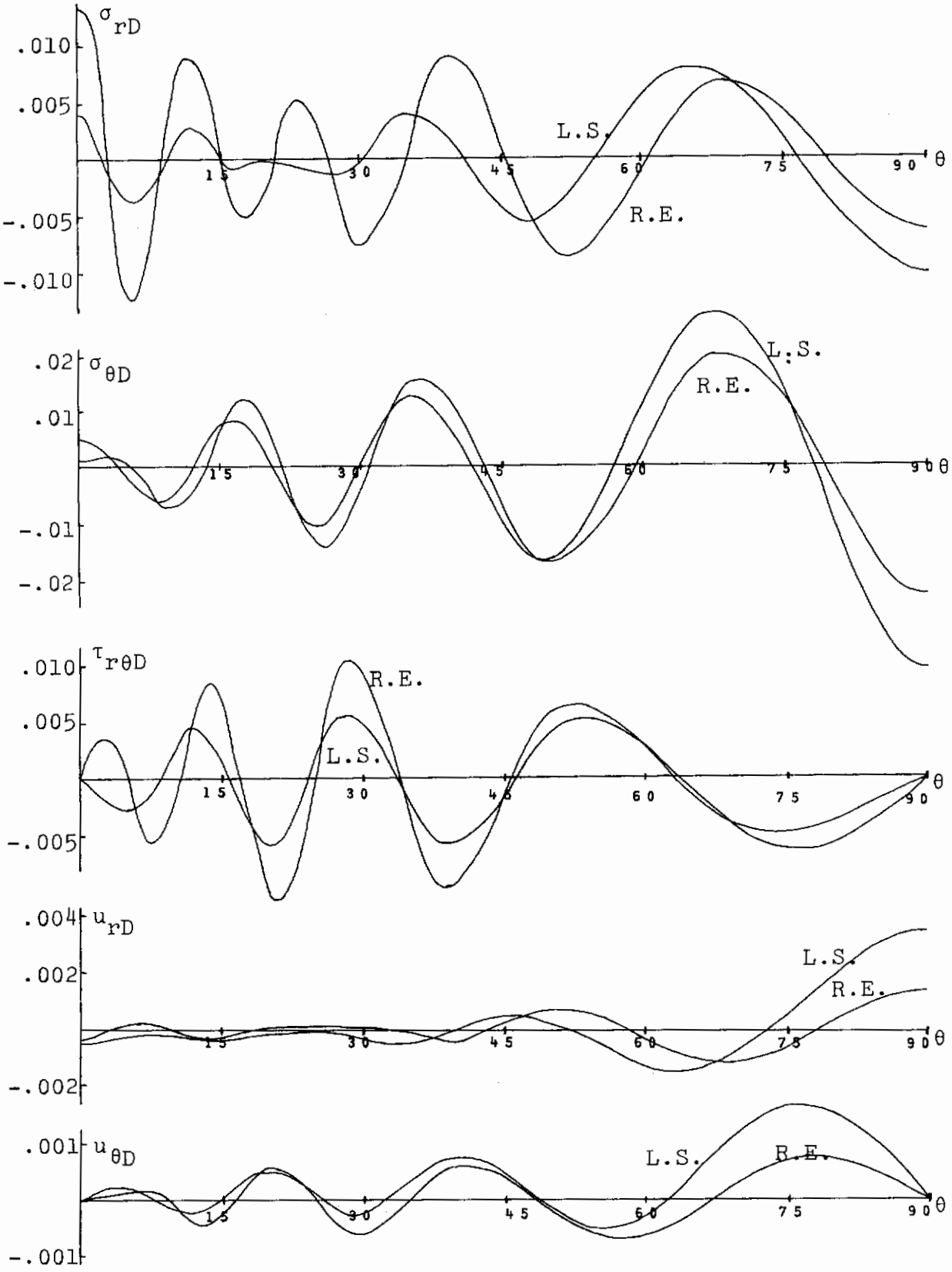


Figure 23. Deviations for Least Squares and Reissner Energy (Case I, Material 1)

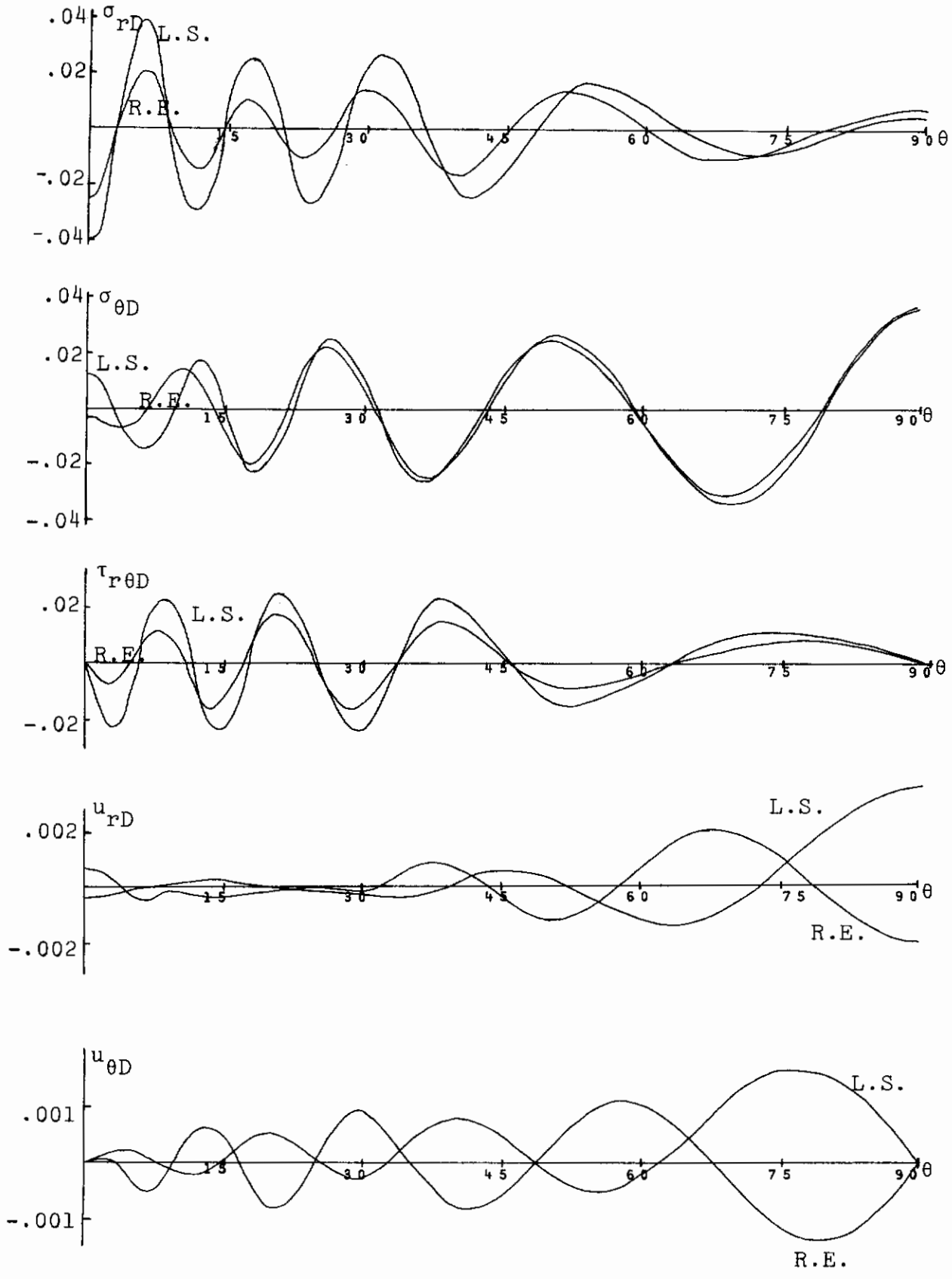


Figure 24. Deviations for Least Squares and Reissner Energy (Case III, Material 1)

Table IV. Condition Numbers of Coefficient Matrices

Case	Material	Unknowns	L.S. Condition No.	R.E. Condition No.
I	2'	16	$2. \times 10^5$	$1. \times 10^4$
I	*	16	$5. \times 10^4$	-
I	5	16	$6. \times 10^7$	$6. \times 10^6$
I	2'	20	$2. \times 10^6$	$9. \times 10^4$
I	2	20	$8. \times 10^4$	$7. \times 10^3$
I	1'	32	$1. \times 10^{10}$	-
I	1	32	$4. \times 10^5$	$9. \times 10^4$
I	2'	32	$9. \times 10^8$	-
III	1	32	$7. \times 10^4$	$9. \times 10^4$
I	1	48	$2. \times 10^7$	-
I	2	48	$9. \times 10^9$	-
I	3	48	$5. \times 10^8$	-

One consistent trend that can be seen in Table IV is that for a given material and case the condition number increases with the number of unknowns. The dependence of the condition number on degree of orthotropy is not clear. For example, from the last three entries of the table we see that the mildly orthotropic Material (2) has a larger condition number than the more strongly orthotropic Materials (1) and (3). It is interesting to note that orienting a material so that $E_y > E_x$, e.g., Materials 1' and 2', has a significant adverse effect on matrix conditioning. We should mention that the bounds for the error of the solution vector (see Appendix III) which depend critically on the condition number were usually quite pessimistic. For example, for Case I, Material (2), and 48 unknowns the condition number is large, 9×10^9, and the bound on the error was such that no significance could be guaranteed in the solution vector. In spite of this the results were good. On the other hand the solution vectors for the same runs with Materials (1) and (3) were less accurate despite the smaller condition numbers. Thus, the condition number of the coefficient matrix may not always be a reliable guide to the accuracy of the solution vector.

In Figure 25 we have plotted $\sigma_{\theta D}$ along the boundary for Case I, Material (2) for 16, 32 and 48 unknowns. The increase in accuracy with increasing number of unknowns is apparent.

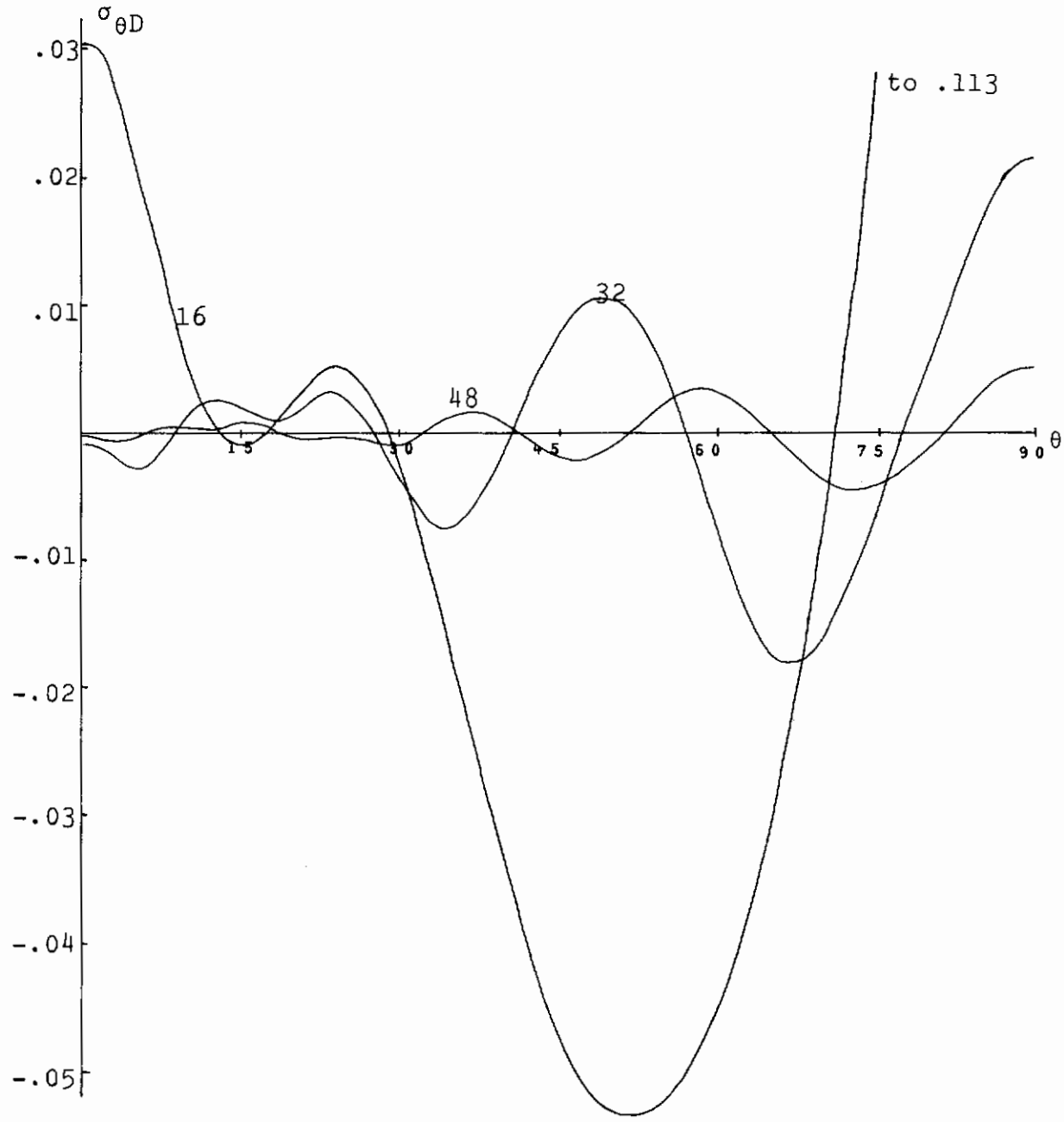


Figure 25. $\sigma_{\theta D}$ for Least Squares with 16, 32, 48 Unknowns (Case I, Material 2)

In Figure 26 we have plotted $\sigma_{\theta D}$ along the boundary for Case I, Materials I and II, with 48 unknowns. It can be seen that the error for Material (1) is slightly greater than for Material (2), and the maximum error for both materials occurs at $\theta = 90^{o}$.

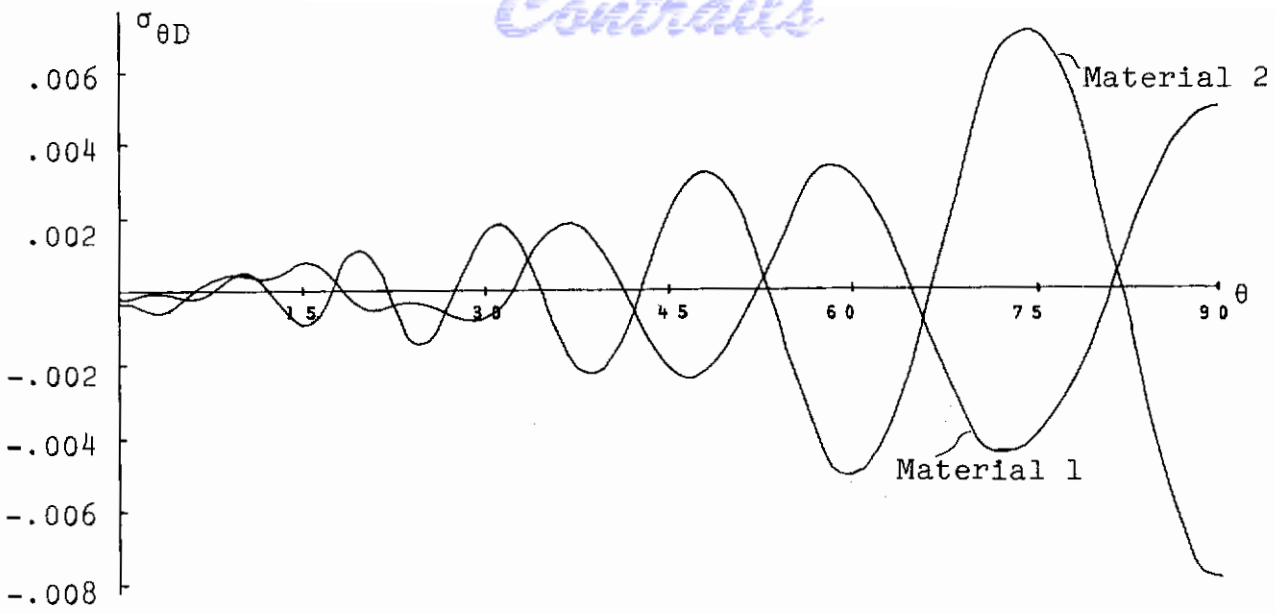


Figure 26. $\sigma_{\theta D}$ for Least Squares (Case I, Materials 1 and 2)

It is interesting to note that the first twelve terms of the series expansion of the exact solution for both Materials (1) and (2) approximate σ_θ at $\theta = 90^\circ$ much more accurately than the least squares solutions. The error is about 1/10 as large as in Figure 25. However, as we know these exact solution series do not converge at all near $r = 1$, $\theta = 0^\circ$.

In this article we have plotted deviations of the approximate solution from the exact solution for Materials (1) and (2) only. Attempts were made to obtain solutions for the other materials using the least squares and Reissner energy programs. However, the results were poor, relative errors being of order unity. In view of the very slow convergence (discussed at the beginning of this article) of the exact solution series for Material (3), it is not surprising that it is difficult (or impossible?) to obtain satisfactory least squares or Reissner energy solutions when Φ_1 and Φ_2 are in the form of negative powers of z_1 and z_2. One direction to proceed in an attempt to overcome these difficulties is the selection of different assumed forms for Φ_1 and Φ_2. If Φ_1 and Φ_2 contain ζ_1^{-1} and ζ_2^{-1} respectively, where ζ_1 and ζ_2 are the exact solution functions defined in (3.2.4), then either the least squares or Reissner energy programs will extract the exact solution from the assumed family. Of course, in problems for which the exact solution is known, there would be no point in using these programs. However, in more complex problems for which no exact solution is available, one might be able to use a form of these exact solution functions in the assumed forms of Φ_1 and Φ_2. For

example, consider a problem involving a non-circular hole in an infinite orthotropic sheet. If the material is strongly orthotropic the images of this hole in the z_1 and/or z_2-planes will be quite elongated. Based on experience with the circular hole problem one is likely to conclude that it would be futile to attempt a least squares or Reissner energy solution with Φ_1 and Φ_2 in the form of negative powers of z_1 and z_2. An approach that might be taken is the following: By inspection (or by other means perhaps) determine ellipses which approximate the elongated images in the z_1 and z_2 planes. Then determine "exact solution" functions, $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$, corresponding to these ellipses, ie.,

$$\tilde{\zeta}_1 = z_1 + \sqrt{z_1^2 - z_{1f}^2} \quad , \tag{7.1.1}$$

where the foci of the ellipse in the z_1-plane are at $\pm z_{1f}$. $\tilde{\zeta}_2$ is defined similarly. (The constant factors in the denominators of the exact solution functions (3.2.4) are not important for the present purposes and have been omitted.) The square root in (7.1.1) is to be defined so that the branch cut lies along the line between the foci of the ellipse. (See Appendix II for a discussion of this point.) Then Φ_1 and Φ_2 are assumed to be in the form of negative powers of $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$. If the problem has symmetry the series may be specialized to account for this symmetry, just as the series in negative powers of z_1 and z_2 can. (See the discussion at the beginning of Article 2a of Section V.) These forms of Φ_1 and Φ_2 are then used with either the least squares or Reissner energy methods, and hopefully a satisfactory solution will be obtained. Since the programs developed during this study were designed for circular boundaries only we could not test this approach on a non-circular hole. However, we did test it in the following manner for circular holes: We defined $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ by taking z_{1f} and z_{2f} to be some fraction of the focal coordinates of the elliptical images of the circular boundary. In Figure 27 we have plotted the least squares solutions for $\sigma_{\theta D}$ (Case I, Material 1, 32 unknowns) for Φ_1 and Φ_2 in the form of odd negative powers of z_1 and z_2 and also in the form of odd negative powers of $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$, where $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ are defined as in (7.1.1) with $z_{1f} = .547i$ and $z_{2f} = .783i$.

(The foci of the elliptical images of the circular boundary are at $z_1 = \pm .790i$ and $z_2 = \pm 1.17i$.) The advantage of the series in negative powers of $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ is apparent from the figure.

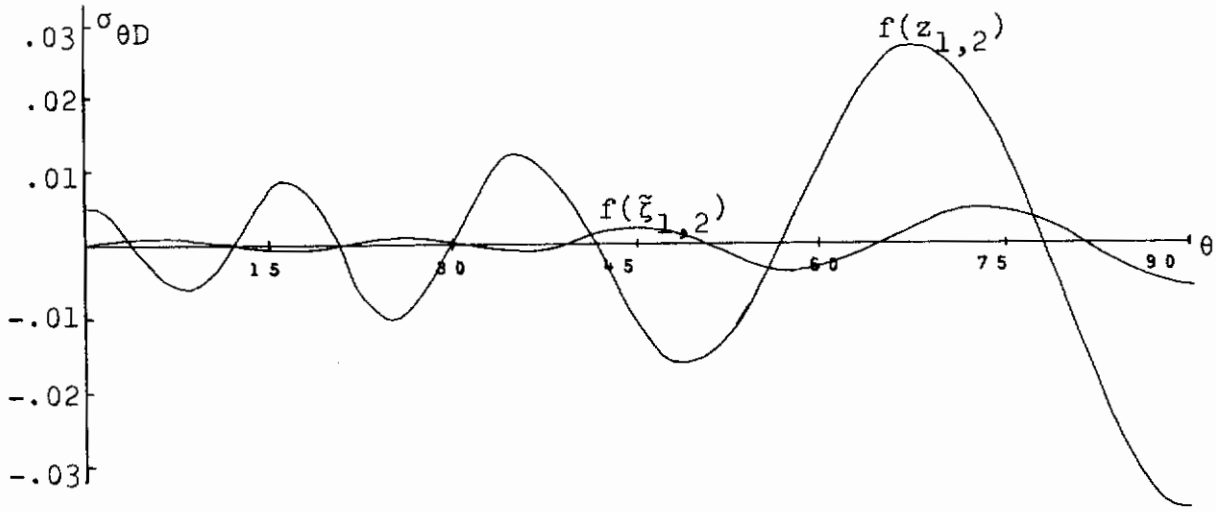


Figure 27. $\sigma_{\theta D}$ for Least Squares Using $\Phi_{1,2}=f(z_{1,2})$ and $\Phi_{1,2}=f(\tilde{\zeta}_{1,2})$ (Case I, Material 1)

As $|z_{1f}|$ and $|z_{2f}|$ are taken smaller and smaller the curve corresponding to Φ_1 and Φ_2 in negative powers of $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ will approach the other curve. If, on the other hand, $z_{1f} \to .790i$ and $z_{2f} \to 1.17i$ the results for $\sigma_{\theta D}$ will decrease until the exact solution is reproduced. This same approach was used to generate a least squares solution for the highly orthotropic Material (6). $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ were defined with $z_{1f} = .9733$ and $z_{2f} = 4.295i$. (The foci of the elliptical images of the circular boundary are at $z_1 = \pm .9740$ and $z_2 = \pm 4.430i$.) The results for Case I with 32 unknowns were fairly good, the maximum error being about 4% in σ_θ. However, another run of the same problem with $z_{1f} = .9354$ and $z_{2f} = 3.873i$ gave very poor results. Thus, it appears that for highly orthotropic materials the choice of z_{1f} and z_{2f} is critical. For noncircular holes in such materials some experimentation may be necessary to locate satisfactory

values of z_{1f} and z_{2f}. The size of the least square residuals is generally an indication of the appropriateness of the choice of z_{1f} and z_{2f}.

Another possible application of the technique just discussed is in problems involving a number of holes in an orthotropic plate. For the purposes of the present discussion let us consider them to be circular holes. Then we could take Φ_1 and Φ_2 to be of the form,

$$\Phi_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \zeta_{1j}^{-i} , \qquad (7.1.2)$$

where a_{ij} are the unknown complex coefficients, m is the number of holes, and ζ_{1j} is defined

$$\zeta_{1j} = (z_1 - z_{1cj}) + \sqrt{(z_1 - z_{1cj})^2 - (z_{1fj} - z_{1cj})^2} , \qquad (7.1.3)$$

where z_{1cj} is the coordinate of the center of the elliptical image of the j^{th} hole, and $\pm z_{1fj}$ are the coordinates of the foci of this elliptical image. Similar definitions hold for Φ_2 and ζ_{2j}. How well this approach would work in practice is open to question. However, a similar technique has been used successfully by Hulbert [21] for the analysis of problems involving multi-holed isotropic plates.

By way of concluding this article on the least squares and boundary Reissner energy methods we summarize the most important points:

i) Both methods can be used to obtain satisfactory solutions to the sample problems of this study if the material is not too highly orthotropic.

ii) The Reissner method generally yields a somewhat better all around solution while the least squares method gives accurate results for certain quantities, apparently at the expense of others.

iii) The Reissner method generally has a somewhat better conditioned coefficient matrix.

iv) The integrations must be performed accurately in the Reissner method which may require many more function evaluations than in the least squares method.

v) Both methods give more accurate results for the displacements in the sample problems of this study.

vi) For problems involving a non-circular hole or more than one hole in a plate exact solution functions may be used in the assumed forms of Φ_1 and Φ_2, which hopefully will be capable of yielding satisfactory solutions.

2. APPROXIMATE SOLUTIONS USING ENERGY METHODS

In this article we present numerical results obtained from the computer programs for the potential, complementary, and Reissner energy methods. However, before these results are given we discuss in detail the nature of the solutions we are attempting to approximate.

2a. Discussion of the Nature of the Exact Solutions

Considerable discussion of the exact solutions from both an analytical and numerical standpoint has already been given (Section III and Article 1 of Section VI). However, we can make a number of additional observations, relating to the use of the energy methods in Ritz fashion, which will enhance our understanding of the approximate numerical results presented later in this article. First let us examine the similarities and differences of Cases I and III. In Case I the applied

stresses are $\sigma_r = -1$, $\tau_{r\theta} = 0$. Numerical results show that the corresponding displacements on the boundary are of the form, $u_r = f(\cos 0\theta, \cos 2\theta)$, $u_\theta = f(\sin 2\theta)$. Away from the boundary other harmonics appear in the stresses and displacements. In Case III the imposed displacements are $u_r = 1$, $u_\theta = 0$. Numerical results show that the resulting stresses on the boundary are of the form $\sigma_r = f(\cos 0\theta, \cos 2\theta)$, $\tau_{r\theta} = f(\sin 2\theta)$. For materials with $E_x = E_y$, eg., Materials (5) and (6), there is symmetry about the rays 45° and 135°. This means that only harmonics which are multiples of four can appear in the solution for the stresses and displacements. It then follows that when $E_x = E_y$ Cases I and III are equivalent except for a scale factor.

It can be seen from the form of the stress function in the solution of Green and Taylor (equation (6.1.9)) that the coefficients of the zeroth harmonic in the Fourier series expansions of σ_r and σ_θ vary as r^{-2}. In Case I, $\sigma_r = -1$ at $r = 1$. This means that the coefficient $(A_o + B_o)$ in (6.1.9) must equal -1. Thus, the coefficient of the zeroth harmonic of the stress function (and the stresses) is just the solution to the isotropic problem. It also turns out in Case I that the coefficients of the second harmonics in the Fourier series expansion of the stresses are zero. Thus, as far as the stresses are concerned in Case I the difference between an isotropic and and orthotropic material is seen only in the fourth and higher harmonics. The same is not true for the displacements. Although the coefficient of the zeroth harmonic in the Fourier series expansion of u_r varies as r^{-1}, just as the isotropic solution, the magnitude of this coefficient depends on the orthotropic material constants. The magnitude of this constant might be determined from the response of some equivalent isotropic material to the loading of Case I. For some orthotropic materials the definition of $[S^o]$ in Appendix I is a good approximation to this equivalent isotropic material. Unlike the stresses the coefficients of the second harmonics in the Fourier series expansions of the displacements are generally non-zero.

In Case III the coefficient of the zeroth harmonic in the Fourier series expansion of u_r is just r^{-1}, the same as for an isotropic materials. The zeroth Fourier coefficients for σ_r and σ_θ vary as r^{-2} but the magnitudes of these coefficients depend on the orthotropic material constants. In general all even harmonics are present in the solutions for the stresses and displacements in Case III.

The considerations of the preceding paragraphs along with the knowledge of the form of the exact solution from the Green-Taylor series furnish guidelines for the selection of assumed forms of the stresses and displacements for use with the approximate energy methods. Of course, in practical problems where the exact solution is unknown, much of this type of information would not be available. However, as we shall see, detailed knowledge regarding the form of the exact solution does not guarantee success. In the preceding articles of this section it was seen that good approximate solutions were difficult to obtain for highly orthotropic materials using the perturbation, least squares, and boundary Reissner energy techniques. One would expect this to continue to be the case with the approximate methods of this article. To illustrate what we are up against we have plotted as a function of r the 4th, 6th, 8th, and 10th Fourier coefficients of σ_θ in Case I for the strongly orthotropic Material (3). (Recall that the zeroth Fourier coefficient is just the isotropic solution ($\sigma_\theta = r^{-2}$) and the second Fourier coefficient vanishes.) From Figure 28 it is clear

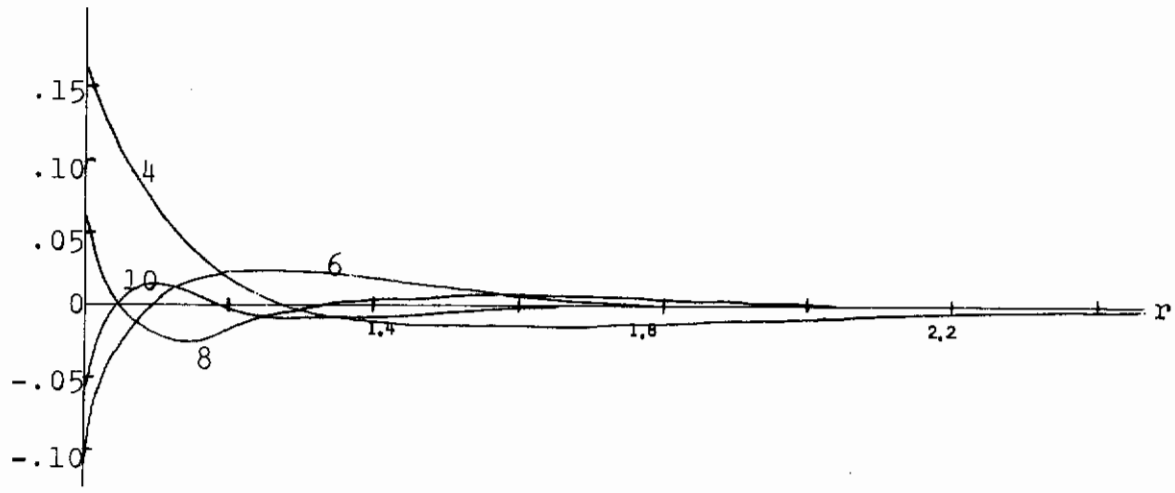


Figure 28. 4th, 6th, 8th, and 10th Fourier Coefficients of σ_θ (Case I, Material 3)

that the higher harmonics have little influence away from the boundary. However, these harmonics do have a significant effect in regions near the boundary which become progressively narrower. Figure 28 shows that the Fourier coefficients for the higher harmonics have more complex behavior. In view of the solution of Green and Taylor this is to be expected since the Fourier coefficients C_F are of the form

$$C_F(2n) = \sum_{i=0}^{n} a_i r^{-2(i+1)} , \qquad (7.2.1)$$

where 2n is the harmonic. As noted earlier the zeroth Fourier coefficient has only one term, r^{-2}. Since the Fourier series for σ_θ in Case I is an even cosine series, one can easily obtain the value of σ_θ at $\theta=0$ by adding algebraically the values of the Fourier coefficients for a given r-coordinate. At $\theta=\pi/2$ the absolute values of the Fourier coefficients are added to obtain σ_θ. Doing this at r=1 using the curves of Figure 28 (recalling $C_F(0)=1$ and $C_F(2)=0$) we obtain the approximate answers $\sigma_\theta(1,0)=1.085$ and $\sigma_\theta(1,\pi/2)=1.408$. The first answer is in error by about 2.5% and the second by about 8%. In order to reduce the error in $\sigma_\theta(1,\pi/2)$ to less than 1% it is necessary to include the next five higher harmonics (12, 14, 16, 18 and 20) in the Green-Taylor series for σ_θ. The total number of terms in a Green-Taylor series which includes harmonics up to and including the 20th is 66. The number of terms in the Green-Taylor series for σ_r, $\tau_{r\theta}$, u_r, and u_θ is comparable. Thus, if the assumed forms for use with the energy methods included the terms of the Green-Taylor series for harmonics up to and including the 20th one would have to deal with approximately 66 unknowns in the complementary energy formulation, 132 in the potential energy formulation, and 310 in the Reissner energy formulation. Systems of these sizes seem rather excessive just to get 1% accuracy in a problem with the simple geometry and loading that we are considering. However, the mere size of these systems is not the whole story. Consider the Green-Taylor series for σ_θ in Case I with Material (3). We can write it in the following form when it is truncated at the twentieth harmonic:

$$\sigma_\theta = \left[\cos 0\theta,\ \cos 2\theta,\ \ldots.,\ \cos 20\theta\right] \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{Bmatrix} r^{-2} \\ r^{-4} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ r^{-22} \end{Bmatrix} \quad (7.2.2)$$

where A has the form,

$$\begin{bmatrix} 1. & 0 & 0 & - & - & - & - & - & 0 \\ 0 & 0 & 0 & & & & & & \\ 0 & X & X & & & & & & \\ 0 & X & X & X & & & & & \\ & & & & & & & & 0 \\ & & & & & & & & 0 \\ 0 & X & X & - & - & - & - & - & X \end{bmatrix} \quad (7.2.3)$$

The non-zero elements are mostly of order unity or greater. They tend to increase in magnitude as the column number increases, reaching magnitudes of 10^4 in the lower right corner. Referring to (7.2.2) it is easily seen that at r=1, θ=0 σ_θ is just the sum of the elements in the A matrix. If the absolute values of these elements were added the result would be of order 10^5. Adding them algebraically gives σ_θ=1.110, i.e., four significant figures are lost in the addition. This means that in order to be able to obtain σ_θ with an error less than 1% the larger elements of the A matrix must be accurate to <u>at least</u> six significant figures, probably more. We note that "A matrices" for σ_r and $\tau_{r\theta}$ also have large elements, which cause similar significance problems. (We have not obtained the "A matrices" for u_r and u_θ, but since they are closely related to the stresses it would not be surprising if these matrices exhibited similar characteristics.) Since the elements

of the solution vectors in the energy methods are related to the elements of the A matrices one can expect serious roundoff problems when using the energy methods with strongly orthotropic materials. (The elements of the A matrices for Material (6) are even larger than those for Material (3).) For the less strongly orthotropic Materials (1) and (3) the elements of the corresponding A matrices do not increase in magnitude (at least up to the 12th harmonic). For these materials the maximum error can be reduced to less than 1% by retaining harmonics up to and including the 8th in the Green-Taylor series, so roundoff is much less of a problem.

2b. Numerical Results of the Potential Energy Method

In the computer program for the potential energy method there are several input variables which are not present in the least squares and boundary Reissner energy programs. One of these is the outer radius, b, of the annular region. For most of the test cases discussed in this article we have taken b=3. or b=∞. In the least squares and boundary Reissner energy programs the assumed functions were in the form of analytic functions of z_1 and z_2. If one used the standard form of these assumed functions (odd negative powers of z_1 and z_2) the only thing left to be chosen is the number of terms in these series. For the potential energy program the standard assumed forms for the are as in (6.2.17), in which there are a number of parameters to be specified. The particular sample problem being treated dictates some of these. For example, in Case III (displacement boundary conditions) $R_u=T_u=T_v=1$. In Case I there seems to be little reason to choose these parameters different than zero. The parameter α may be any negative integer but α=-1 seems to be the most reasonable based on our knowledge of the exact solution. The indexing of the r-functions and trigonometric functions must also be specified. For the sample problems of this study the types of harmonics needed can be determined by symmetry arguments, so the user need only decide on the number of harmonics necessary to obtain a good approximation. Without knowledge of the exact solution it is pretty much up to common sense as to the appropriate powers of r to be used in the assumed functions. From the solution of Green and Taylor we know that only odd negative powers of r are needed. In Case III this cannot be achieved with our standard forms. However, if r is replaced by r^2 in the denominators of the

T_u and T_v terms of (6.2.17) it is possible for the assumed forms to contain only odd negative powers of r. Some numerical experiments were performed to determine the effect of this modification. The results using the modified version were not appreciably different from those obtained using the standard forms. All of the numerical results presented in this article correspond to the standard assumed forms.

In most of our numerical work we have included as many harmonics as was felt necessary to obtain results to a certain accuracy. Generally we have included fewer powers of r than would be consistent with this number of harmonics (according to the solution of Green and Taylor). This is especially true for Materials (3) and (6) which require a relatively large number of harmonics. It was hoped that these powers of r could furnish a good approximation to the higher Fourier coefficients, which as we have seen, are the sum of a relatively large number of negative powers of r. (See Figure 28 for these Fourier coefficients for σ_θ, Case I, Material (3).) In our numerical work it was generally found that increasing the number of unknowns in the assumed forms by the addition of extra powers of r had a more serious effect on matrix conditioning than did the addition of more harmonics. However, it was also found that despite the relatively good conditioning of matrices corresponding to many harmonics and only a few powers of r, the results for displacements and stresses were not completely satisfactory. Thus, for highly orthotropic materials, such as Materials (3) and (6), it is necessary to include most or all of the powers of r of the truncated Green-Taylor series. However, this has such an adverse affect on conditioning that it was impossible to obtain good results with coefficient matrices formed in single precision as they were in the potential (and complementary) energy programs.

Before presenting numerical results we note that the effect of varying the outer radius, b, of the annulus is generally small, at least in the range $2. \leq b \leq \infty$. The condition numbers of the coefficient matrices were slightly smaller for large b, but the effect on the displacements and stresses was insignificant. For $b \approx 1$, the condition number increases substantially with accompanying loss of significance in the results. Further discussion of this point will be given in Article 2c on the complementary energy method. We also note that in finite region problems the assumed functions can contain positive powers of r. Attempts to approximate the solution to Sample Problems I

and III in finite regions using positive powers of r were not particularly successful. Consequently all of the results presented here are for negative powers of r. Also, we do not make any distinction between results for different values of b, since the results given in this article are all for $b \geq 3$, cases for which the magnitude of b has little effect.

As noted earlier when $E_x = E_y$ Sample Problems I and III are equivalent except for a scale factor. For these materials it is easier to determine the effect of using the assumed functions which do and do not satisfy the prescribed boundary conditions. In the potential energy method the assumed functions must satisfy the boundary conditions in Case III but not in Case I. In Figure 29 we have plotted u_{rD} along the rays $\theta=0$, 45° for

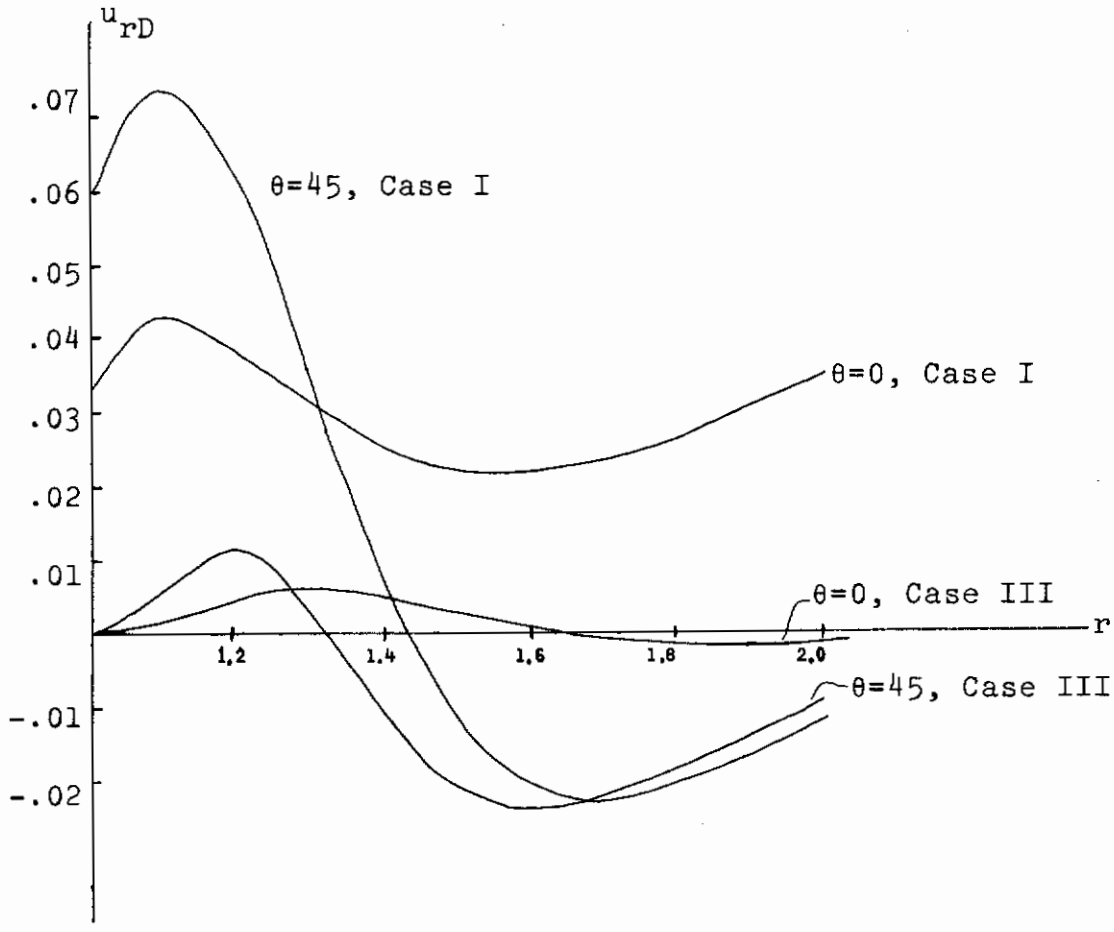


Figure 29. u_{rD} for P.E. Along the Rays $\theta=0$, 45° for Cases I and III (Material 6)

both Cases I and III for Material (6). (The results of Case I have been scaled so that the two cases are equivalent.) There were the same number of unknowns in both problems. In the assumed form for the displacements (6.2.17) the index $\hat{j}$ took on the values 0, 4, 8, 12, and 16. $\hat{i}_u$ and $\hat{i}_v$ took on -5, -3, -1. In Case III $R_u=T_u=T_v=1$, $\alpha=-1$. In Case I these parameters were zero. Now we know from the Green-Taylor solution that only odd negative powers of r are present in the series expansions of the displacements. Thus, from this standpoint the assumed forms for Case I seem to be better. However, the assumed forms for Case III satisfy the prescribed boundary conditions. From Figure 29 it is apparent that u_r for Case III is distinctly better. In Figure 30 we have plotted similar results for $\sigma_{\theta D}$.

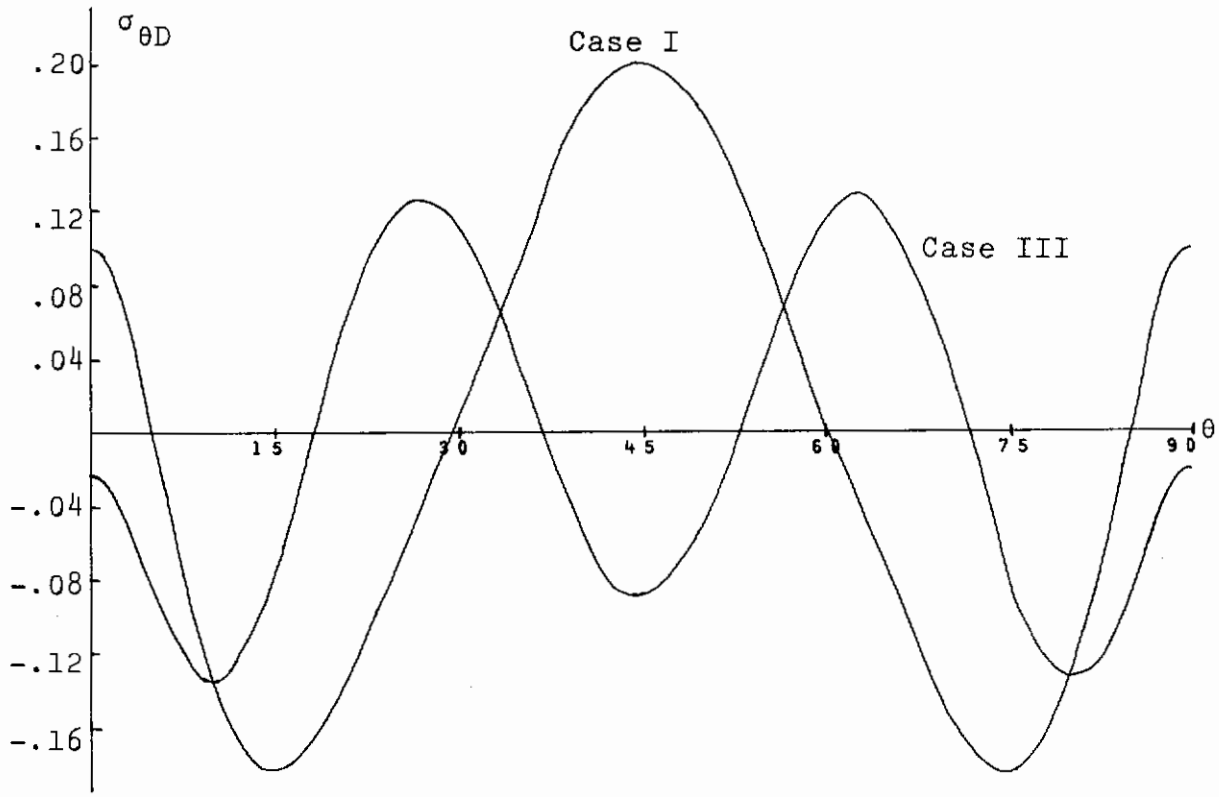


Figure 30. $\sigma_{\theta D}$ for P.E. Along the Boundary for Cases I and III (Material 6)

The results of Case III are also better, but not as decisively. Thus, it appears, on the basis of these results, that satisfaction of prescribed boundary conditions in the assumed form can be quite beneficial. Recall that Material (6) has a very strong stress concentration in σ_θ at $\theta=0, \pi/2$ for both Cases I and III. (See Figures 5, 13 and 14.) Although the errors $(\sigma_{\theta D}/\sigma_{\theta MAX})$ are rather large in both cases (a 14% maximum in Case III and 20% in Case I), one could hardly expect better results in view of the stress concentrations and the fact that the stresses are secondary quantities, which are almost always inferior to the approximate displacements from which they are derived. Surprisingly, the approximations to σ_r, which is better behaved than σ_θ, were inferior to those for σ_θ. The maximum of the ratio $(\sigma_{rD}/\sigma_{rMAX})$ was about 85% in Case I and about 50% in Case III. However, a better measure of the error may be $(\sigma_{rD}/\sigma_{\theta MAX})$ for which the errors in σ_r are comparable to those for σ_θ.

In Figure 31 we have plotted u_{rD} along the ray $\theta=90^\circ$ for Cases I and III for Material (3). (The maximum deviations occurred along this ray.) In both cases the index $\hat{j}$ in the

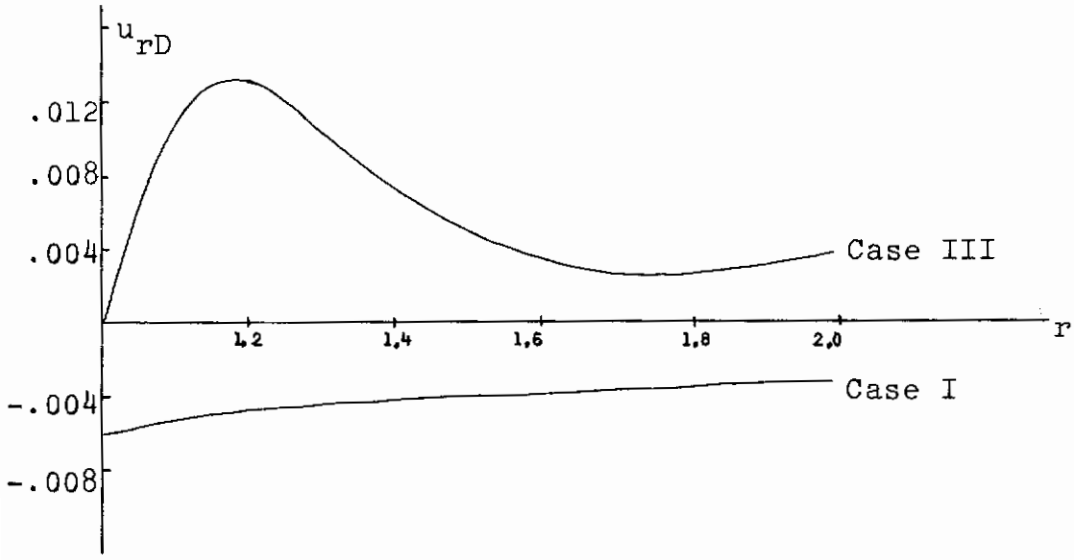


Figure 31. u_{rD} for P.E. Along Ray $\theta=90^\circ$ for Cases I and III (Material 6)

assumed form (6.2.17) took on the values 0, 2, 4, 6, 8, and 10. $\hat{i}_u$ and $\hat{i}_v$ took on the values -5, -3, and -1. For this material Cases I and III are not equivalent as they are for Material 6. For example, in Case I $\sigma_{\theta MAX}$ is 1.53 at $\theta=90^o$. In Case III $\sigma_{\theta MAX}=24.2$. However, along the ray $\theta=90^o$ u_r for the two cases differ in magnitude by only about 10%. It is clear from Figure 31 that the results for Case I are superior, which is the reverse of the situation for Material (6). As far as the circumferential displacement u_θ is concerned the results are significantly better in Case I. However, u_θ is generally much smaller than u_r in both cases and therefore is not of great interest. The stresses in both cases are inaccurate. However, on a percentage basis the stresses in Case III are somewhat better, the maximum of $(\sigma_{\theta D}/\sigma_{\theta MAX})$ being about 11% versus 17% in Case I.

A number of runs for Materials (1) and (2) were made in both Cases I and III. Good results for the displacements, i.e., (u_{rD}/u_{rMAX}) and $(u_{\theta D}/u_{rMAX})$ less than .1%, were obtained when the indices $\hat{i}_u$ and $\hat{i}_v$ in the assumed form (6.2.17) took on the values -5, -3, and -1, and when $\hat{j}$ took on the values 0, 2, and 4 for Material (2) and 0, 2, 4, and 6 for Material (1). Little improvement in the results were obtained when more harmonics were retained. The use of different powers of r in the assumed forms, e.g., $\hat{i}_u$, $\hat{i}_v=-4$, -3, -2, and -1, yielded somewhat inferior results. Since it is possible to obtain quite accurate results for the displacements for Materials (1) and (2), it is perhaps more interesting to consider the stress derived from these displacements. In Figure 32 we have plotted $(\sigma_{\theta D}/\sigma_{\theta MAX})$ along the boundary for Cases I and III and Material (1), where the assumed functions were as indicated above. The results for Case III are distinctly better than for Case I.

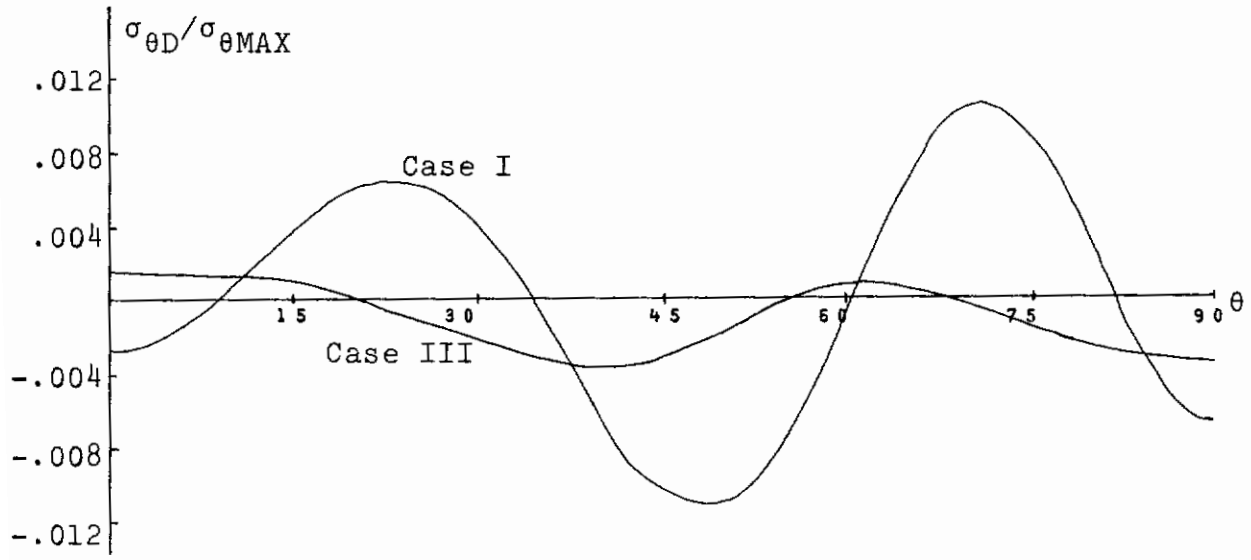


Figure 32. $(\sigma_{\theta D}/\sigma_{\theta DMAX})$ for P.E. for Cases I and III (Material 1)

This is somewhat surprising in view of the strong stress concentration in σ_θ at $\theta=90^o$ in Case III. (See Figure 6.)
In Figure 33 we have plotted $(\sigma_{rD}/\sigma_{rMAX})$ for the same cases.

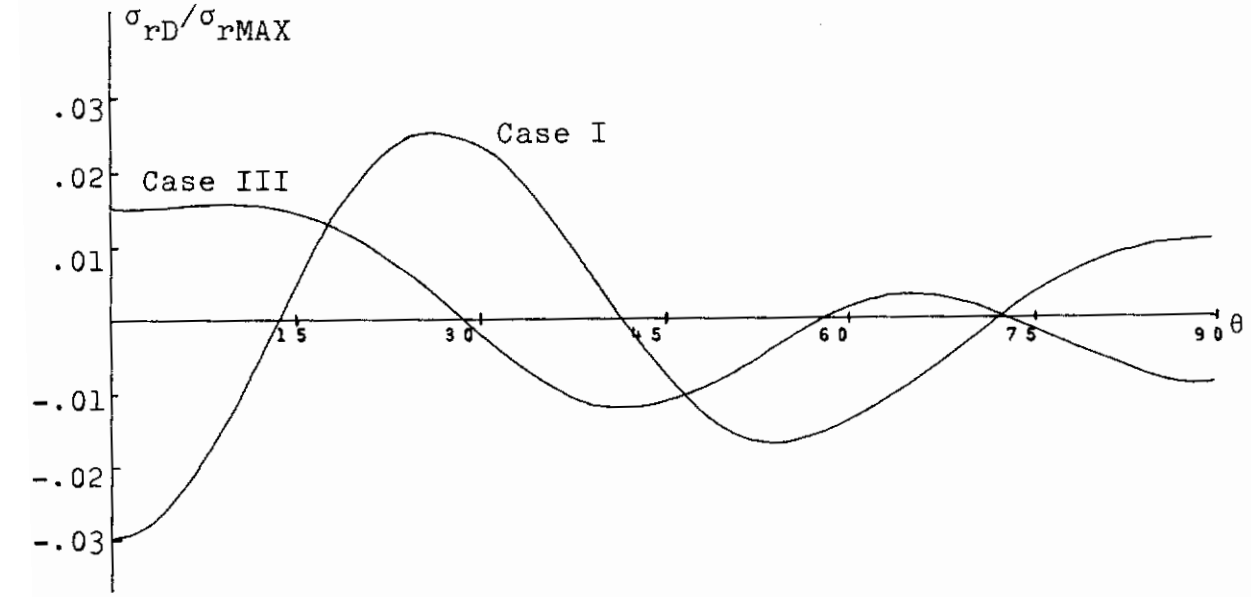


Figure 33. $(\sigma_{rD}/\sigma_{rMAX})$ for P.E. for Cases I and III (Material 1)

Again the results for Case III are better. The solutions shown in Figures 32 and 33 are further evidence of the benefits obtained by choosing the assumed functions so that the boundary conditions are satisfied.

It is interesting to note from Figures 32 and 33 that the results for σ_θ are significantly better than those for σ_r in both cases. Similar results were observed in a number of other runs. This was somewhat unexpected since σ_θ is usually the most ill-behaved stress or displacement component in the sample problems for the various materials.

As far as matrix conditioning is concerned it is somewhat difficult to make comparisons because of the number of different problem parameters involved. However, the following trends were observed:

1) The effect of increasing the outer radius, b, of the annulus is to reduce the condition number. However, except for values of b close to 1 this effect is of little importance.

2) The effect of increasing the number of unknowns is to increase the condition number. However, increasing the number of harmonics in the assumed form has a much less serious effect than increasing the number of powers of r.

3) Incrementing the indices $\hat{i}_u$ and $\hat{i}_v$ in the assumed form (6.2.17) by two gives better conditioning than incrementing by one (for the same number of powers of r, i.e., for the same upper limits n_u and n_v on the summations of (6.2.17)).

4) The more strongly orthotropic materials, e.g., Materials (3) and (6), yield larger condition numbers.

5) For the standard assumed functions Case III appears to have slightly poorer conditioning than Case I.

One of the interesting aspects of the potential energy formulation is that the exact solution corresponds to the minimum of the potential energy functional. This is potentially useful in several ways:

1) Instead of evaluating $\delta\Phi_p$ and setting this quantity to zero to find the minimum point, one can minimize Φ_p directly by a method such as steepest descent. The latter approach is especially useful in nonlinear problems. However, in linear problems there is generally no advantage in doing this.

2) One can determine which of two approximate solutions is better by observing which has the smaller potential energy.

3) By using the method of complementary energy in conjunction with the method of potential energy one can obtain upper and lower bounds to the exact value of these energy functionals. (See equation (2.3.10)f.f.)

Of course, even if one knows how close the potential energy of an approximate solution is to the exact value, he still does not know how accurate the displacements and stresses are. In all of our numerical solutions we have computed the approximate value and the exact value of the total potential energy. In all cases the percentage deviation of the approximate potential energy from the exact value was much smaller than the maximum percentage deviation of the displacements. The ratio of these two percentages was generally in the range 10-100. The fact that this ratio is fairly high indicates that the potential energy surface (for our assumed functions) is relatively "flat" near the minimum point. This is just another way of saying that the system is not particularly well conditioned, which we already know from the size of the matrix condition numbers (generally of the order of 10^4 for the results presented here).

2c. Numerical Results of the Complementary Energy Method

Much of the discussion of the preceding article applies to the complementary energy method also. Consequently, a number of points will be given only very brief mention since they have already been considered. The standard assumed form for use in the complementary energy method is given in (6.2.20), (6.2.24), and (6.2.25). This standard form was used for all of the numerical results presented in this article. The constant R in (6.2.25) was taken to be -1 in Case I and 0 in Case III. The constants T and S were taken to be 1 in Case I and 0 in Case III. Some experimentation was carried out with positive powers of r in the assumed form of the stress function for finite region problems. However, the results were not especially good. Thus, all of the results presented here are for non-positive powers of r. In particular the index $\hat{i}$ in (6.2.24) takes on the value 0 and various other negative values. Further discussion of the indexing of $\hat{i}$ and the total number of powers of r will be given later in this article. We note that in Case III the term $r^0\cos 0\theta$ is usually present in the assumed forms. Such a constant term in the assumed forms would produce a singular coefficient matrix. Thus, the row and column corresponding to this term were eliminated through the use of a row eliminating routine which has been incorporated into the computer program. Also, it appears, from the solution of Green and Taylor, that there is little point in retaining the $f_{4n}(r)$ term of (6.2.24) in Case I since the exact value of the log r term is contained in the Ψ_o term of the assumed form (6.2.20). In Case III it is important to have a (logr cos0θ) term in the assumed form but it is not necessary to have log r multiplying higher harmonics. Thus, for the results of this article these unnecessary terms have been eliminated by means of the row eliminating routine. Also, in Case I it is not necessary to have any cos0θ and cos2θ terms (except for the Ψ_o-term). These harmonics can be eliminated by starting the $\hat{j}$ index at 4.

For most of the numerical results presented here the outer radius, b, of the annulus was either 3 or ∞. It was found that b in this range had little effect on conditioning or results. A few tests were made with values of b close to 1, in particular for b=1.1, 1.05, 1.01. In Case III it was found that the effect of decreasing b had an adverse effect on conditioning, especially for quite small values of b. On the other

hand, in Case I the conditioning for these small values of b was significantly better than for larger values. This is in distinct contrast to the potential energy formulation for which values of b close to 1 produced large condition numbers in both Cases I and III.

In the discussion of the numerical results for the potential energy method it was especially informative to consider Material (6) since Cases I and III are equivalent, and the effect of satisfying or not satisfying the boundary conditions with the assumed functions is readily determined. The same thing is true for the complementary energy method. A number of runs were made with Material (6) for both Cases I and III, all using the following harmonics: 0, 4, 8, 12, 16, and 20. Several different choices of the powers of r in the assumed functions were used. For all of these runs the condition numbers were large, in the range 10^6 - 10^9. These were much larger than the condition numbers (of order 10^4) which were encountered in the treatment of these same problems with the potential energy method for comparable numbers of unknowns. Consequently the complementary energy results for Material (6) were generally poor. The results for σ_θ were no better than those for the potential energy illustrated in Figure 30. The results for σ_r and $\tau_{r\theta}$ were better, the maximum error being about 2% in Case I and 1% in Case III for the best of the runs for Material (6). In Case I the displacements were fair, the maximum error being about 4%. However, for Case III the displacements were quite poor. In view of the large condition numbers of the coefficient matrices, the results just cited are somewhat suspect. It may be that if these same cases were rerun in double precision the answers would be quite different.

For Material (3) a number of runs for both Case I and III were made using harmonics up to either 10 or 14 and using various indexing on the powers of r in the assumed form. Although the condition numbers were generally not as large as for Material (6) they were still large (in the range 10^5 - 10^7). The results for Case III were poor for both displacements and stresses. However, it should be pointed out that there is a very high stress concentration in σ_θ at $\theta=90^\circ$, and difficulty in approximating the stresses could be expected. When Case III, Material (3) was treated using potential energy satisfactory results were obtained for the displacements, which behave much

more regularly than the stresses in this case. (See Figure 31.) For Case I the complementary energy results were much better, but the stress concentration in σ_θ at $\theta=90^o$ was not approximated very well. In order to obtain a good approximation to σ_θ at $\theta=90^o$ several annuli with small outer radii were considered. The effect of doing this is to make the energy in the neighborhood of the stress concentration a greater percentage of the total energy than would be the case for an annulus with a large outer radius. Thus, in the process of minimizing the potential energy, σ_θ near $\theta=90^o$ has a greater influence on the results. In Figure 34 we have plotted $\sigma_{\theta D}$ in Case I for b = 1.01 and b = 3.0.

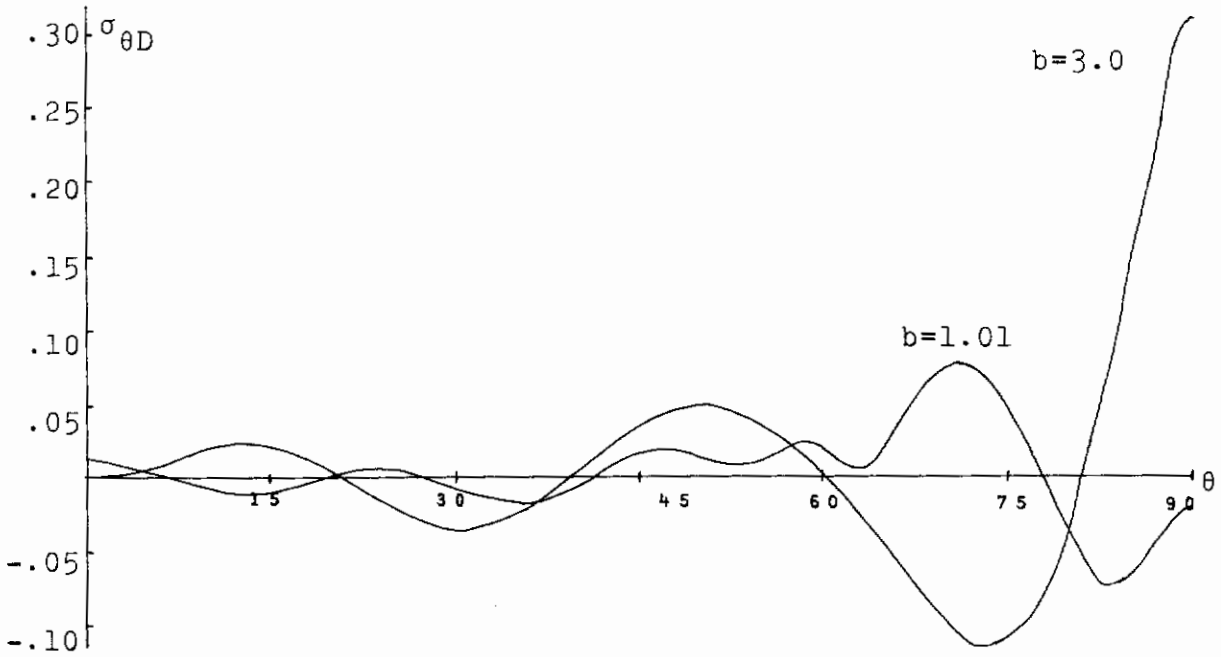


Figure 34. $\sigma_{\theta D}$ for C.E. for Case I and b=3.0, 1.01 (Material 3)

For both of these curves we have used harmonics up to and including the 14th. The index $\hat{i}$ in the assumed form (6.2.24) took on the values -6, -4, -2, and 0. It is clear that the use of a small value of b does produce better results for σ_θ. The results for σ_r and $\tau_{r\theta}$ for both values of b were accurate, the maximum error being less than 1%. It should be noted that the

displacements corresponding to small values of b were quite poor. However, for b = 3 the displacements were good. In Figure 35 we have plotted u_{rD} along the boundary for both the potential and complementary energy. It can be seen that u_r

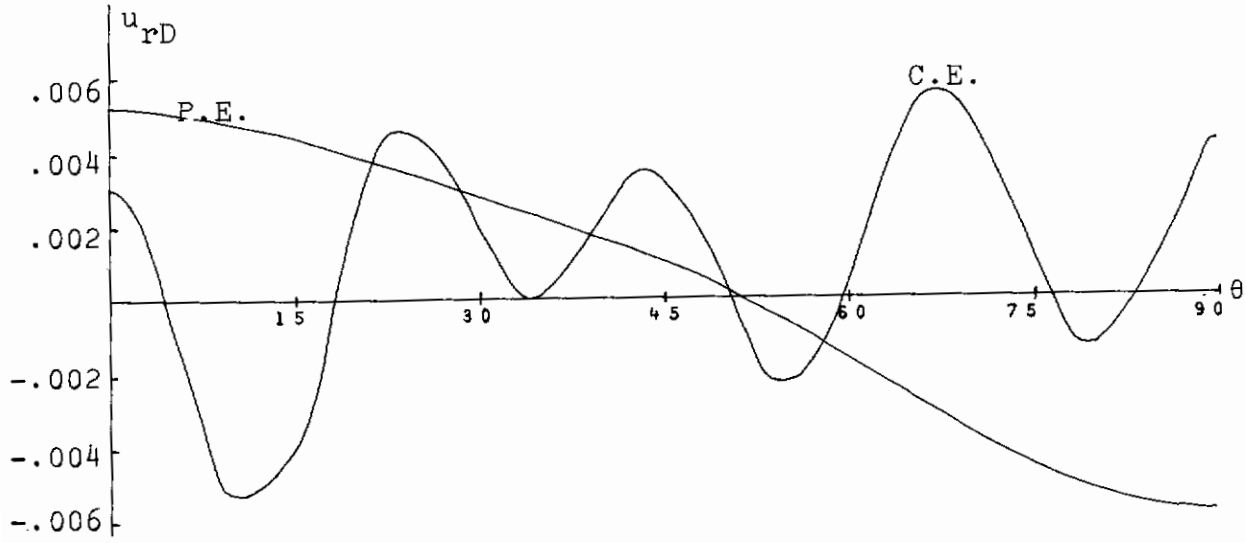


Figure 35. u_{rD} for C.E. and P.E. for Case I (Material 3)

is slightly better for the complementary energy despite the fact that it is a secondary quantity.

A number of runs for Cases I and III with Materials (1) and (2) were made. Good results were obtained. It was found that retention of harmonics up to the 6th for Material (2) and up to the 8th for Material (1), and having the index i in (6.2.24) take on the values -6, -4, -2, and 0 was sufficient to reduce the errors in all quantities to less than 1%. The

condition numbers were generally of the order of 10^4 in these cases, which is somewhat larger than for comparable runs of the potential energy. In Figure 36 we have plotted $(\sigma_{\theta D}/\sigma_{\theta MAX})$ for Material (1) in Cases I and III. The results in Case I are

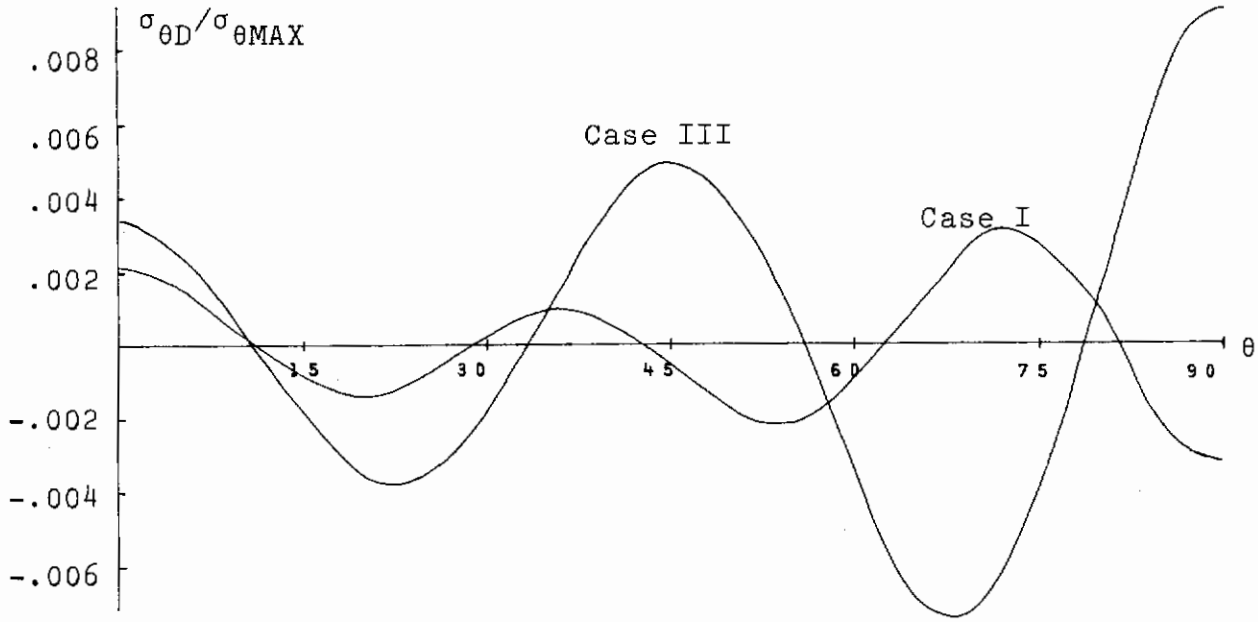


Figure 36. $(\sigma_{\theta D}/\sigma_{\theta MAX})$ for C.E. for Cases I and III (Material 1)

significantly better. It is also interesting to note that the results for σ_θ in Case III for the potential energy (shown in Figure 32) are better than the corresponding results above. This and the fact that better results for u_r in Case I for Material (3) were obtained from the complementary energy method than from the potential energy method are further indications of the advantage of using assumed functions which satisfy the boundary conditions.

Near the end of the preceding article we made five observations regarding the effect of certain variables on matrix conditioning in the potential energy formulation. These continue to be true for the complementary energy with the following modifications: In item (1) the condition number decreases with decreasing b in Case I. Regarding item (5), the effect of the

case on condition number is not clear in the complementary energy formulation. We should also add that the conditioning of the complementary energy formulation was consistently poorer than that of the potential energy formulation for comparable materials, cases, and numbers of unknowns.

At the end of the preceding article we discussed how the percentage deviation of the total potential energy was related to the deviation of the displacements. Similar considerations can be made for the total complementary energy and the stresses. Generally the percentage deviations of σ_r and $\tau_{r\theta}$ were 10-100 times larger than the percentage deviation of Φ_C. However, the percentage deviation of σ_θ was anywhere from 100 to 10,000 times greater than that of Φ_C.

2d. Numerical Results of the Reissner Energy Method

To the authors' knowledge the general Reissner energy formulation has not previously been used in Ritz fashion to obtain approximate solutions to elasticity problems. In Ritz solutions to linear elasticity problems using the potential or complementary energy formulations one can use the positive (or negative) definiteness of the coefficient matrix to prove that a unique solution exists for a given family of assumed functions, and that by increasing the size of this assumed family the approximate energy functional will approach the exact value monotonically. For appropriate assumed functions convergence to the exact solution can also be established. In the Reissner energy formulation the coefficient matrix is generally not positive definite, and similar properties may be difficult or impossible to establish. In fact, numerical results have shown that it is not at all unusual to encounter singular matrices and non-unique solutions with the Reissner energy method. We discuss this point further in the following paragraphs.

It was noted in Article 3b of Section VI that in the initial applications of the Reissner energy formulation the stresses were expressed in terms of a stress function, Ψ. The thinking was that since it is so easy to satisfy equilibrium internally by using a stress function, why not do it? In addition the number of stress unknowns for an approximation to a certain accuracy would be reduced by approximately a factor of 3. The coefficient matrix for this formulation has the form indicated in equation (6.3.3). If the same number of harmonics and r-functions are used in the assumed forms for each of Ψ, u, and v there will be approximately twice as many displacement unknowns as stress unknowns, and the coefficient matrix will be singular because of the size of the null sub-matrix in the lower right corner of the coefficient matrix.

In order to avoid this more harmonics and r-functions were included in the assumed form of Ψ than were included in both the assumed forms for u and v. The coefficient matrix was no longer obviously singular since the size of the null matrix was less than half the size of the total coefficient matrix. However, it turned out that the coefficient matrix was still singular because certain columns in the A_2 submatrix of (6.3.3) were multiples of each other. It may have been that by a judicious elimination of certain of these columns the matrix could have been rendered non-singular and a good approximation would have resulted. However, it was felt that a good approximate technique should not be so critically dependent on the choice of assumed functions. Therefore, investigation of an alternate approach (described in detail in Article 3 of Section VI) to the application of the Reissner energy principle was undertaken. It was found that indiscriminate choices of the assumed functions could still lead to singular matrices. This is discussed in the following paragraphs.

Whenever a singular matrix occurred in the application of the Reissner energy formulation it was found that the reason was because certain columns of the A_2 submatrix were not linearly independent. In some cases certain columns had no non-zero elements. The latter occurred in certain tests of Case I. From the exact solution of Green and Taylor we know that in Case I the only non-zero stress terms corresponding to the zeroth harmonic are $\sigma_r = -r^{-2}$ and $\sigma_\theta = r^{-2}$, and that the second harmonic is completely absent from the stresses. (The second harmonic is present in the displacements.) A run was made for which the rows and columns corresponding to these zero stress terms were eliminated by means of the row eliminating routine mentioned earlier. Columns of the A_2 submatrix corresponding to the second harmonics were zero. The reason for this is clear since all elements of the A_2 submatrix are the result of integrating the product of a stress quantity times a displacement quantity. The integrand contains trigonometric products such as $(\cos j\theta \cos q\theta)$. Only when $j=q$ is the integral non-zero. But this cannot occur for $q=2$ since the terms corresponding to $j=2$ have been eliminated. This example is rather striking since it shows that even with a "perfect" assumed form, i.e., an assumed form containing only the non-zero terms of the exact solution, the Reissner energy principle used in Ritz fashion can fail. Of course, the statement of the Reissner energy principle (following equation (2.3.9)) does not promise that the principle can be used successfully with the Ritz technique. In assuming a form for the stresses and displacements a restriction has been placed on the comparison states, and the Reissner energy principle does not guarantee a unique stationary point for the Reissner energy functional based on these assumed functions.

In an attempt to obtain a non-singular coefficient matrix while still using an assumed form which contained primarily the terms of the Green-Taylor solution, one second harmonic term was added to the assumed forms of each of σ_r, σ_θ, and $\tau_{r\theta}$. There were no longer columns with all zero elements, but the matrix was still singular because certain columns of the A_2 submatrix were linearly dependent. Again the reason why is readily seen. In the A_2 submatrix the columns corresponding to second harmonics can have at most three non-zero elements (in rows corresponding to second harmonics in the assumed form of the stresses). The number of such columns having non-zero elements in the same position is equal to the number of terms in the assumed forms of u and v which contain a second harmonic. This number is equal to the number of different powers of r in these assumed forms. If this number exceeds three the matrix is necessarily singular. More complicated examples of this same thing also occurred. What they all boil down to is this: If, for a particular harmonic, the assumed forms for the displacements have many powers of r and those for the stresses have only a few powers of r, a singular coefficient matrix is likely to occur. This is not the entire story, however, In one of our tests we found that a singular matrix occurred in Case I but not in Case III for the same assumed forms. The only difference between Cases I and III is that the formulas for the elements of the A_2 submatrix are different, but since the positions of the non-zero elements are the same in both cases it is not clear why one case would be singular and the other not. The most obvious case of a singular matrix is when there are more displacement unknowns than stress unknowns. In this case the size of the null submatrix (see (6.3.3)) is such that the coefficient matrix is necessarily singular. The opposite extreme, many more stress unknowns than displacement unknowns, does not yield a singular matrix but does give poor approximations to the stresses and displacements. For the numerical results presented in this article there were the same harmonics and powers of r in the assumed forms of the stresses as there were in the assumed forms of the displacements. No problems with singular coefficient matrices were encountered with these assumed forms.

As noted earlier the coefficient matrix and right hand side vector were formed in double precision in the Reissner energy program. Just as in all programs the matrix inversion and determination of the solution vector was done in double precision. The upper bound on the error of the solution vector was generally less than 10^{-7}. Numerical results for the other approximate methods have indicated that these upper bounds are

quite pessimistic, so the true error is likely to be 10^{-9} or smaller. Because of the large matrices arising in the Reissner energy method and because they were formed in double precision, it was not possible (because of computer storage limitations) to include as many harmonics and powers of r as were desired. Consequently no runs for Material (6) were made, and in the runs for Material (3) only harmonics up to and including the 10th could be retained. In Figure 37 we have plotted $\sigma_{\theta D}$ along the boundary for Material (3) in Case I for both the Reissner and complementary energy methods. In the Reissner

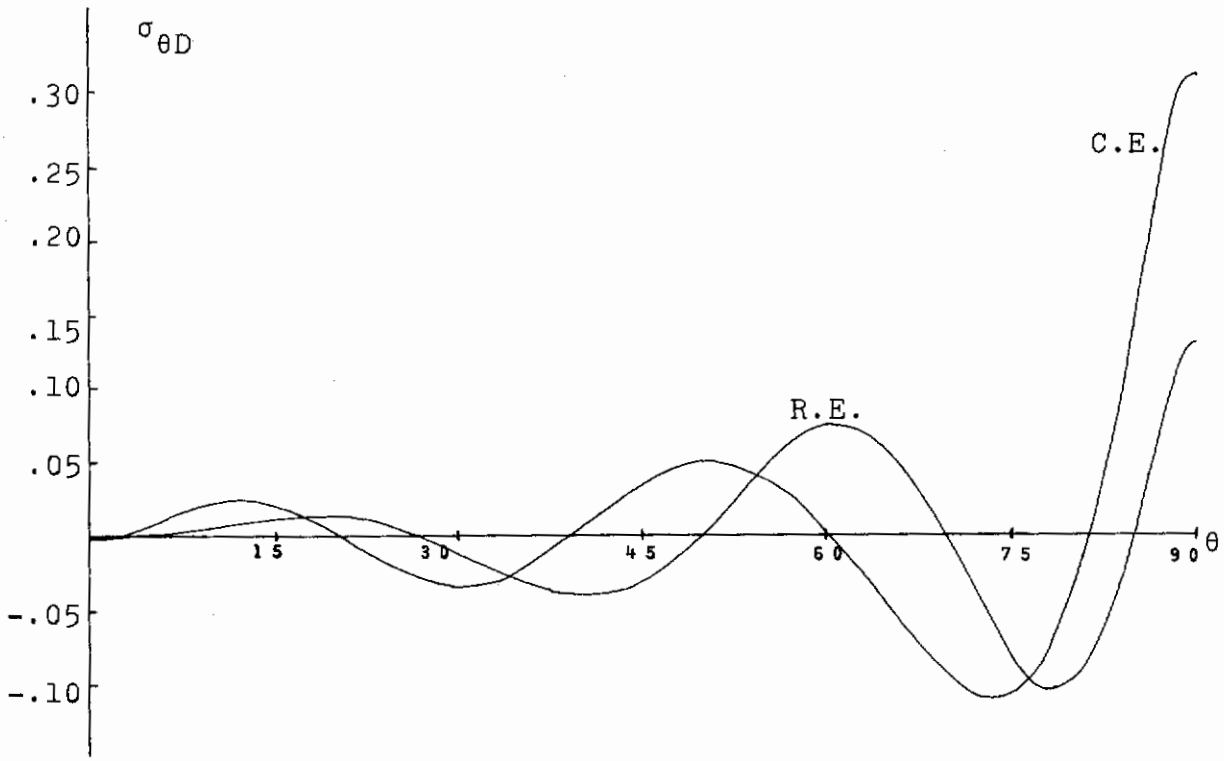


Figure 37. $\sigma_{\theta D}$ for R.E. and C.E. for Case I (Material 3)

energy four powers of r, indexed by 2, and harmonics up to and including the 10th were included in the assumed forms. The powers of r and the harmonics were the same for the complementary energy except that the 12th and 14th harmonics were included. The results for the Reissner energy are clearly superior. However, the difference between the two might be due to loss of significance in the complementary energy, since it is only in single precision. The radial displacement obtained from this run of the Reissner energy had a maximum error less than .25% which is better than the corresponding results from either the potential or complementary energy methods. (See Figure 35.) A run of the Reissner energy for Material (3) in Case III with the same assumed functions as in Case I yielded very poor results. However, as pointed out earlier, there is a very strong stress concentration in σ_θ at r=1, $\theta=90^\circ$, and one would expect difficulty in approximating the stresses. The displacements are better behaved and approximations having a maximum error of about 1% were obtained by the potential energy method. One might hope that results of comparable accuracy for the displacements could be obtained from the Reissner energy method in spite of inaccuracies in the approximations of the stresses. This was not the case, however, since the maximum errors in the displacements were about 10%.

A number of runs were made for Materials (1) and (2) in both Cases I and III. Good results were obtained for all quantities. In Case III the stresses were approximated more accurately than they were in the complementary energy method for comparable assumed functions. In Case I the complementary energy generally gave better results for the stresses, especially σ_r and $\tau_{r\theta}$. In Case I the Reissner energy method gave better results for displacements than did potential energy, but the opposite was true in Case III. This is further evidence of the advantages of using assumed functions which satisfy the boundary conditions.

We now make several observations regarding the conditioning of the coefficient matrix in the Reissner energy formulation.

1) Just as in the potential and complementary energy methods the condition number increases with increasing matrix size. However, increasing the number of harmonics affects the condition number less seriously than does increasing the number of powers of r.

2) Indexing the powers of r by 2 (as in the Green-Taylor solution) gives significantly better conditioning than indexing by 1.

3) Conditioning is poorer for strongly orthotropic materials.

4) For the same assumed functions conditioning was always better in Case III than in Case I for all of the runs that were made during this study.

5) Even though it was necessary to deal with large matrices (up to 112 x 112) in the Reissner energy method the conditioning was not as poor as in the complementary energy for much smaller coefficient matrices. For the 112 x 112 matrices condition numbers were of the order 10^6 in Case III and 10^7 in Case I.

In the Reissner energy method the concept of the total energy is not nearly as useful as it can be in the potential and complementary energy methods. The reason is that the value of the approximate Reissner energy functional can in general be either greater or less than the exact value. We do not even have the assurance that increasing the size of the family of assumed functions will yield a value of the total Reissner energy which is closer to the exact value. Consequently, the total Reissner energy was not obtained in the computer programs.

3. CONCLUSIONS

In this article the important conclusions that have been arrived at during the course of this study are summarized. With the exception of the perturbation method, the methods discussed in this study require that the form of the solution be selected based on intuition, knowledge of the physics of the problem, or on any other information that might be available. Such an approach is often more appealing to the engineer than the use of a method such as finite difference, in which provision are generally not made for the engineer to incorporate any knowledge he might have regarding the problem. From the

classical literature, e.g., Crandall [14] and Timoshenko [23], one is led to believe that the results obtained by these methods, particularly the potential and complementary energy methods, are usually reliable providing that reasonable sense has been used in the selection of the assumed forms. Our experiences in this study show that this is certainly not the case for the sample problems considered. Even during the early phases of the study a considerable amount (in the form of the exact solutions of Lekhnitskii) was known about the solutions we were attempting to approximate. In spite of this satisfactory approximations were generally difficult to obtain, except for mildly orthotropic materials, e.g., Material (2). It was only in the later phases; after the nature of the singularities in the exact solution functions, the convergence of the Laurent expansion of the exact solution, and the exact solution of Green and Taylor had been carefully studied; that enough was known to be able to obtain satisfactory solutions, or at least know why they could not be obtained with a particular assumed family of functions and the available computer precision and storage. It is difficult to imagine how good approximate solutions to these sample problems could have been obtained with only a feel for the physics of the problems to go on.

The knowledge gained during this study might be put to use in at least two different types of practical problems for which exact solutions are unknown. They are problems involving a non-circular hole in an orthotropic sheet and problems involving more than one circular (or noncircular) hole in an orthotropic sheet. The way in which the boundary residual methods might be used to treat such problems is described at the end of Article 1b of this section. The energy methods might be used in a similar manner providing that functions of r (for use in the assumed forms) could be found which are more suitable than the negative powers of r presently being used. Recall that the Fourier coefficients for the stresses in the solution of Green and Taylor are in the form of a summation involving even negative powers of r. (See equation (7.2.1).) For strongly orthotropic materials the coefficients in this summation are large for higher harmonics despite the fact that the Fourier coefficients themselves are rather small. (See Figure 28.) This appears to be the primary reason for the poor conditioning encountered in the energy methods. Further study is necessary to determine families of r-functions which can be used to approximate these higher Fourier coefficients without giving rise to large coefficients as was the case when negative powers of r were used. The use of such families of r-functions in the assumed forms for the energy methods would probably result in

improved conditioning. Assuming the foregoing conjecture is correct, the energy methods might be applied to problems involving noncircular holes and more than one hole. However, there is still the problem of performing the required integrations over the particular region. For problems involving just one non-circular hole it may be impossible to perform the integrations analytically. Conceivably a two-dimensional quadrature (cubature) formula could be developed for the region and satisfactory results could be obtained in this manner. For a problem involving more than one circular hole one might use an assumed form which is the sum of the assumed forms which would be used for each of the individual holes if no other holes were present. The integrations for each term of this sum would be carried out analytically assuming no other holes existed. Then the effect of the other holes could be accounted for by integrating numerically the appropriate part of the assumed function over each of these holes and subtracting these results from the analytical result. The regions over which numerical quadratures are required are just circles, and reliable quadrature formulas are available for such simple regions.

In this study we have considered six different approximate methods, and have treated the same sample problems with all methods. However, in practical applications it is unlikely that one would be able to choose from all six. If the material were strongly orthotropic or if the region were such that simple analytic solutions to the corresponding isotropic problem were unavailable, one would certainly exclude the perturbation method from consideration. If the region were quite irregular one would probably exclude the potential, complementary, and interior Reissner energy methods from consideration. This leaves only the least squares and boundary Reissner energy methods to choose from. The relative merits of these two methods were discussed in Article 1b of this section. If the region were simple enough so that two-dimensional integrations (either analytical or numerical) were feasible one would need to consider the types of boundary conditions and whether stresses or displacements were of greater interest. Complicated displacement (stress) boundary conditions would dictate against the use of the potential (complementary) energy method. Simple displacement (stress) boundary conditions point in favor of the use of the potential (complementary) energy method, since, as we have seen, better solutions are obtained when the assumed functions satisfy the boundary conditions. If displacements (stresses) were of greater interest one would try to use the potential (complementary) energy method. If it appeared that loss of significance would

be a problem, one would probably try to avoid the use of least squares, and would try to use instead one of the energy methods. (This assumes the truth of our conjecture that the poor conditioning encountered with the energy methods is due to poor choices of assumed functions, whereas the poor conditioning in least squares is inherent in the method.)

Another factor to be considered in choosing a particular method is the ease with which the analytical and programming work can be done. In this regard the boundary residual methods are clearly superior. It would not be difficult to design the computer programs for these methods so that a variety of regions could be treated merely by changing the input data. On the other hand, most of the analytical and programming work in the perturbation and interior energy methods is married to the region and, to a lesser extent, the boundary conditions. If computer storage is limited then one might not want to use the interior Reissner energy because of the relatively large coefficient matrices arising in this method.

APPENDIX I

DETERMINATION OF NEAREST ISOTROPIC COMPLIANCE MATRIX

For the analysis of this appendix it is essential that cartesian tensors be used. In tensor notation the stress-strain relationship (2.1.3) is

$$\varepsilon_{ij} = S_{ijk\ell}\sigma_{k\ell} \quad , \quad i,j,k,\ell \rightarrow 1,2 \tag{A1-1}$$

The tensor components of stress and strain are related to their engineering counterparts as follows:

$$\begin{aligned} &\varepsilon_{11} = \varepsilon_x, \ \varepsilon_{22} = \varepsilon_y \quad , \quad \varepsilon_{12} = \varepsilon_{21} = \tfrac{1}{2}\gamma_{xy} \\ &\sigma_{11} = \sigma_x, \ \sigma_{22} = \sigma_y \quad , \quad \sigma_{12} = \sigma_{21} = \tau_{xy} \end{aligned} \tag{A1-2}$$

Note the factor 2 difference between the engineering shear strain γ_{xy} and the tensor shear strain ε_{12}. One can obtain the relationships between the elements of the compliance matrix and compliance tensor by expanding (2.1.3) and (A1-1) and comparing terms. For example

$$\begin{aligned} \varepsilon_{11} &= S_{1111}\sigma_{11} + S_{1112}\sigma_{12} + S_{1121}\sigma_{21} + S_{1122}\sigma_{22} \\ \varepsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{16}\tau_{xy} \end{aligned} \tag{A1-3}$$

We then conclude

$$S_{1111} = S_{11} \ , \ S_{1112} = S_{1121} = \tfrac{1}{2}S_{16} \ , \ S_{1122} = S_{12} \tag{A1-4}$$

Similarly we obtain

$$\begin{aligned} &S_{1211} = \tfrac{1}{2}S_{16}, \ S_{1212} = S_{1221} = \tfrac{1}{4}S_{66}, \ S_{1222} = \tfrac{1}{2}S_{26} \\ &S_{2111} = \tfrac{1}{2}S_{16}, \ S_{2112} = S_{2121} = \tfrac{1}{4}S_{66}, \ S_{2122} = \tfrac{1}{2}S_{26} \\ &S_{2211} = S_{12} \ , \ S_{2212} = S_{2221} = \tfrac{1}{2}S_{26}, \ S_{2222} = S_{22} \end{aligned} \tag{A1-5}$$

We now introduce the following two isotropic compliance matrices:

$$S^{o1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S^{o2} = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{A1-6}$$

These matrices are isotropic in that they are invariant under a coordinate rotation. This is easily checked by substituting the elements of S^{o1} and S^{o2} into formulas (2.1.9) - (2.1.10). The tensor components of these matrices are determined just as in (A1-4) and (A1-5). The results are

$$\begin{aligned} &S^{o1}_{1111} = S^{o1}_{1122} = S^{o1}_{2211} = S^{o1}_{2222} = \frac{1}{2} \\ &S^{o2}_{1111} = S^{o2}_{2222} = S^{o2}_{1212} = S^{o2}_{1221} = S^{o2}_{2112} = S^{o2}_{2121} = 1/\sqrt{8} \\ &S^{o2}_{1122} = S^{o2}_{2211} = -1/\sqrt{8}\ , \end{aligned} \tag{A1-7}$$

and all other components are zero.

Besides being isotropic tensors (or matrices) S^{o1} and S^{o2} form an orthonormal basis in the space of 4th order isotropic tensors (for which the indices take on the values 1 and 2). In other words

$$\begin{aligned} S^{o1}\bullet S^{o1} &= S^{o1}_{ijk\ell}\ S^{o1}_{ijk\ell} = 1 \\ S^{o2}\bullet S^{o2} &= S^{o2}_{ijk\ell}\ S^{o2}_{ijk\ell} = 1 \\ S^{o1}\bullet S^{o2} &= S^{o1}_{ijk\ell}\ S^{o2}_{ijk\ell} = 0 \end{aligned} \tag{A1-8}$$

It should be pointed out that S^{o1} and S^{o2} are not a unique orthonormal basis, and that other choices would be just as good for our purposes.

At this juncture we digress in order to review a familiar result from vector analysis. Suppose we have three vectors $\bar{S}$, $\bar{S}^{o1}$, and $\bar{S}^{o2}$ in three space as shown in the figure.

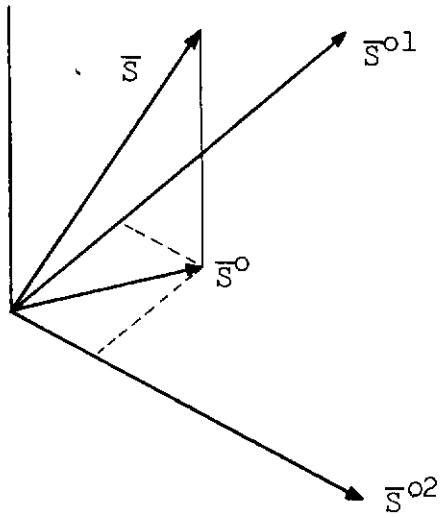


Figure 38. Resolution of Vector into Components

$\bar{S}^{o1}$ and $\bar{S}^{o2}$ are orthonormal and $\bar{S}$ is not in the plane of $\bar{S}^{o1}$ and $\bar{S}^{o2}$. To find the vector $\bar{S}^{o}$ lying in the plane of $\bar{S}^{o1}$ and $\bar{S}^{o2}$ and which is nearest to $\bar{S}$ (in the sense that $|\bar{S}-\bar{S}^{o}|$ is minimized) we use the formula,

$$\bar{S}^{o} = (\bar{S}\bullet\bar{S}^{o1})\bar{S}^{o1} + (\bar{S}\bullet\bar{S}^{o2})\bar{S}^{o2} \qquad \text{(A1-9)}$$

This idea was used by Gazis, Tadjbakhsh, and Toupin [27] to construct an elastic tensor of given symmetry nearest to an anisotropic elastic tensor. We specialize their results to obtain the planar isotropic compliance tensor nearest to a given planar anisotropic compliance tensor. First we compute the tensor inner products

$$\begin{aligned} S\bullet S^{o1} &= \tfrac{1}{2}(S_{11} + S_{22} + 2S_{12}) \\ S\bullet S^{o2} &= \tfrac{1}{\sqrt{8}}(S_{11} + S_{22} - 2S_{12} + S_{66}) \end{aligned} \qquad \text{(A1-10)}$$

We now use these results in the tensor analogue of (A1-9). This gives the tensor components of S^o. More convenient for our purposes are the matrix components of $[S^o]$ obtained from the tensor components using (A1-4) and (A1-5). These are

$$S^o_{11} = S^o_{22} = \frac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66})$$

$$S^o_{12} = \frac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66}) \qquad \text{(A1-11)}$$

$$S^o_{66} = \frac{1}{2}(S_{11} + S_{22} - 2S_{12} + S_{66}) = 2(S^o_{11} - S^o_{12})$$

These have already been obtained, but from a different point of view, in (2.1.10). In the reference just cited it is proved that an isotropic matrix constructed in this manner is positive definite, and therefore an elastic compliance matrix. The matrices $[S^{o1}]$ and $[S^{o2}]$ are positive semidefinite, and therefore do not represent elastic matrices. We now define a residual matrix, which is not an elastic matrix,

$$[S^R] = [S] - [S^o] \qquad \text{(A1-12)}$$

The elements of $[S^R]$ are

$$S^R_{11} = \frac{1}{8}(5S_{11} - 3S_{22} - 2S_{12} - S_{66}) = H_1 + H_2$$

$$S^R_{22} = \frac{1}{8}(5S_{22} - 3S_{11} - 2S_{12} - S_{66}) = H_1 - H_2$$

$$S^R_{12} = \frac{1}{8}(2S_{12} - S_{11} - S_{22} + S_{66}) = -H_1 \qquad \text{(A1-13)}$$

$$S^R_{66} = \frac{1}{2}(S_{66} - S_{11} - S_{22} + 2S_{12}) = -4H_1$$

$$S^R_{16} = S_{16} = 2H_3 + H_4$$

$$S^R_{26} = S_{26} = -2H_3 + H_4 ,$$

where H_1, H_2, H_3, and H_4 are as in (2.1.10). In (A1-13) we have expressed the S^R_{ij} in terms of H_1, H_2, H_3, and H_4. Using $S_{ij} \equiv S^o_{ij} + S^R_{ij}$ in the definitions of the H's in (2.1.10) it is seen that the isotropic parts contribute nothing and H_1, H_2, H_3, and H_4 can be written in terms of the S^R_{ij}. Our choice of S^o is such that the "magnitude" of S^R, given by the inner product,

$$(S^R \cdot S^R)^{\frac{1}{2}} \equiv (S^R_{ijk\ell}\ S^R_{ijk\ell})^{\frac{1}{2}} \qquad \text{(A1-14)}$$

is the minimum possible.

We now define a parameter, ε, which is a measure of material orthotropy

$$\varepsilon = \frac{(S^R \cdot S^R)^{\frac{1}{2}}}{(S^o \cdot S^o)^{\frac{1}{2}}} \qquad \text{(A1-15)}$$

In terms of matrix components

$$\varepsilon = \frac{\left[(S^R_{11})^2 + (S^R_{22})^2 + 2(S^R_{12})^2 + (S^R_{66})^2/4 + (S^R_{16})^2 + (S^R_{26})^2\right]^{\frac{1}{2}}}{\left[(S^o_{11})^2 + (S^o_{22})^2 + 2(S^o_{12})^2 + (S^o_{66})^2/4\right]^{\frac{1}{2}}}$$

$$= \frac{\sqrt{2}\left[4H_1^2 + H_2^2 + 4H_3^2 + H_4^2\right]^{\frac{1}{2}}}{\left[3(S^o_{11})^2 - 2S^o_{11}S^o_{12} + 3(S^o_{12})^2\right]^{\frac{1}{2}}} \qquad \text{(A1-16)}$$

As discussed in Article 1b of Section II, H_1, H_2, H_3, and H_4 each correspond to a particular type of material anisotropy. It is therefore appropriate to break ε into four "components" as follows:

$$\varepsilon_1 = \frac{\sqrt{8}\ H_1}{D}\ ,\quad \varepsilon_2 = \frac{\sqrt{2}\ H_2}{D}\ ,\quad \varepsilon_3 = \frac{\sqrt{8}\ H_3}{D}\ ,\quad \varepsilon_4 = \frac{\sqrt{2}\ H_4}{D}\ , \qquad \text{(A1-17)}$$

where

$$D = \left[3(S^o_{11})^2 - 2\, S^o_{11}\, S^o_{12} + 3(S^o_{12})^2\right]^{\frac{1}{2}} \qquad \text{(A1-18)}$$

As pointed out in that article these ε_i depend on the orientation of the coordinate system. However, in addition to the S^o_{ij}, the quantities,

$$\varepsilon_{13} = \left[\varepsilon_1^2 + \varepsilon_3^2\right]^{\frac{1}{2}}, \quad \varepsilon_{24} = \left[\varepsilon_2^2 + \varepsilon_4^2\right]^{\frac{1}{2}} \qquad \text{(A1-19)}$$

are invariant under coordinate rotations. It can also be shown that the determinant of the compliance matrix $[S^{\alpha\beta}]$, the elements of which are given in (2.1.9), is independent of θ. This determinant can be written

$$\begin{aligned}\mathrm{DET}\left[S^{\alpha\beta}\right] = {} & \left\{2(S^o_{11} + S^o_{12})(S^o_{11} - S^o_{12})^2\right\} - \left\{8(S^o_{11} + S^o_{12})(H_1^2 + H_3^2)\right\} \\ & - \left\{2(S^o_{11} - S^o_{12})(H_2^2 + H_4^2)\right\} + \left\{4(H_1H_2^2 - H_1H_4^2 + 2H_2H_3H_4)\right\}\end{aligned} \qquad \text{(A1-20)}$$

The first brace on the right represents the determinant of $[S^o]$ and is, of course, invariant. The second and third braces are proportional to ε_{13}^2 and ε_{24}^2 respectively and are therefore invariant. Since the determinant of $[S^{\alpha\beta}]$ is invariant it follows that the last brace of (A1-20) is also invariant. We then define

$$\varepsilon_5 = \left|H_1H_2^2 - H_1H_4^2 + 2\, H_2H_3H_4\right|^{1/3} \Big/ D \qquad \text{(A1-21)}$$

ε_5 is a measure of anisotropy which is in general independent of ε_{13} and ε_{24}. However, for orthotropic materials ($H_3 = H_4 = 0$) we have

$$\varepsilon_5^3 = \frac{\sqrt{2}}{8}\, \varepsilon_{13}\, \varepsilon_{24}^2 = \frac{\sqrt{2}}{8}\, |\varepsilon_1| \varepsilon_2^2 , \qquad \text{(A1-22)}$$

i.e., ε_5 is no longer an independent invariant. In summary, we have established for general anisotropic materials five independent material invariants; two of which S^o_{11}, S^o_{12} (S^o_{22} and S^o_{66} can be expressed in terms of S^o_{11} and S^o_{12}) characterize the nearest isotropic material, and the other three ε_{13}, ε_{24}, and ε_5 are measures of the deviation from this isotropic material.

Up to this point our analysis of the nearest isotropic elastic matrix and the deviation from isotropy has been for the compliance matrix. We could have just as well dealt with the stiffness matrix $[B]$. The analysis would have been similar to the foregoing with a few minor changes. In converting the matrix components B_{ij} to tensor components B_{ijkl} the formulas would be as in (A1-5) without the factors ½ and ¼. This would require that the (3,3) element in the second matrix of (A1-6) be changed to 1. The elements of the nearest isotropic stiffness matrix have already been obtained in (2.1.14). The elements of the residual matrix are

$$
\begin{aligned}
B^R_{11} &= \tilde{H}_1 + \tilde{H}_2 \\
B^R_{22} &= \tilde{H}_1 - \tilde{H}_2 \\
B^R_{12} &= -\tilde{H}_1 \\
B^R_{66} &= -\tilde{H}_1 \\
B^R_{16} &= \tilde{H}_3 + \tfrac{1}{2}\tilde{H}_4 \\
B^R_{26} &= -\tilde{H}_3 + \tfrac{1}{2}\tilde{H}_4
\end{aligned}
\qquad \text{(A1-23)}
$$

We can then define another measure of anisotropy

$$
\tilde{\varepsilon} = \frac{(B^R \bullet B^R)^{\frac{1}{2}}}{(B^O \bullet B^O)^{\frac{1}{2}}} \qquad \text{(A1-24)}
$$

$\tilde{\varepsilon}$ is the same as ε in (A1-16) with the H's replaced by $\tilde{H}$'s and the S^O_{ij} replaced by B^O_{ij}. The formulas for ε_1, ε_2, ε_3, ε_4, ε_{13}, ε_{24}, and ε_5, (A1-17), (A1-19) and (A1-21) also continue to hold with tildes above the ε's, D's, and H's and with the S^O_{ij} replaced by B^O_{ij}. The formula for the determinant of $[B^{\alpha\beta}]$ can be obtained from (A1-20) by making the same changes and then dividing by 4.

In this appendix we have established two sets $\{S^o_{11}, S^o_{12}, H_1, H_2, H_3, H_4\}$ and $\{B^o_{11}, B^o_{12}, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4\}$ corresponding to the same material. It is useful to be able to express one set in terms of the other. We list the formulas expressing the second set in terms of the first.

$$
\begin{aligned}
B^o_{11} &= \frac{\left[2S^o_{11}(S^o_{11} - S^o_{12}) - 4(H_1^2 + H_3^2) - \frac{1}{2}(H_2^2 + H_4^2)\right]}{\mathrm{DET}\left[S^{\alpha\beta}\right]} \\
B^o_{12} &= \frac{\left[-2S^o_{12}(S^o_{11} - S^o_{12}) - 4(H_1^2 + H_3^2) + \frac{1}{2}(H_2^2 + H_4^2)\right]}{\mathrm{DET}\left[S^{\alpha\beta}\right]} \\
\tilde{H}_1 &= \frac{\left[-2(S^o_{11} + S^o_{12})H_1 + \frac{1}{2}(H_2^2 - H_4^2)\right]}{\mathrm{DET}\left[S^{\alpha\beta}\right]} \\
\tilde{H}_2 &= \frac{\left[-2(S^o_{11} - S^o_{12})H_2 + 4(H_1H_2 + H_3H_4)\right]}{\mathrm{DET}\left[S^{\alpha\beta}\right]} \\
\tilde{H}_3 &= \frac{\left[-2(S^o_{11} + S^o_{12})H_3 + H_2H_4\right]}{\mathrm{DET}\left[S^{\alpha\beta}\right]} \\
\tilde{H}_4 &= \frac{\left[-2(S^o_{11} - S^o_{12})H_4 - 4(H_1H_4 - H_2H_3)\right]}{\mathrm{DET}\left[S^{\alpha\beta}\right]},
\end{aligned}
\qquad \text{(A1-25)}
$$

where $\mathrm{DET}\left[S^{\alpha\beta}\right]$ is given in (A1-20) in terms of the S^o_{ij} and H_i. The relations inverse to (A1-25), i.e., the expression of the S^o_{ij} and H_i in terms of the B^o_{ij} and $\tilde{H}_i$, are obtained from (A1-25) by interchanging the roles of $\{S^o_{11}, S^o_{12}, H_1, H_2, H_3, H_4\}$ and $\{B^o_{11}, B^o_{12}, \tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4\}$. Using (A1-25) and its inverse it is easy to express the set of invariants $\{B^o_{11}, B^o_{12}, \tilde{\varepsilon}_{13}, \tilde{\varepsilon}_{24}, \tilde{\varepsilon}_5\}$ in terms of $\{S^o_{11}, S^o_{12}, \varepsilon_{13}, \varepsilon_{24}, \varepsilon_5\}$ and vice-versa.

If the material is only mildly anisotropic then H_1, H_2, H_3, and H_4 will be small relative to the S^o_{ij} and we can generally expand the denominators in (A1-25) and obtain

$$B^o_{11} = \left[S^o\right]^{-1}_{11} \left\{1 + \frac{2(H_1^2 + H_3^2)}{S^o_{11}(S^o_{11} + S^o_{12})} + \frac{(3S^o_{11} + S^o_{12})(H_2^2 + H_4^2)}{4S^o_{11}(S^o_{11} + S^o_{12})(S^o_{11} - S^o_{12})}\right.$$

$$\left. - \frac{2(H_1H_2^2 - H_1H_4^2 + 2H_2H_3H_4)}{(S^o_{11} + S^o_{12})(S^o_{11} - S^o_{12})^2} + O\left[\frac{(S^o_{ij})^2 H_i^4}{(S^o_{11} + S^o_{12})^2 (S^o_{11} - S^o_{12})^4}\right]\right\},$$

(A1-26)

where $\left[S^o\right]^{-1}_{11}$ is the 1-1 element of $\left[S^o\right]^{-1}$. The expression for B^o_{12} would be similar but with $\left[S^o\right]^{-1}_{11}$ replaced by $\left[S^o\right]^{-1}_{12}$. The terms of (A1-26) which are quadratic and cubic in the H's are immediately expressible in terms of the material invariants $\tilde{\varepsilon}_{13}$, $\tilde{\varepsilon}_{24}$, and $\tilde{\varepsilon}_5$. We can obtain similar expansions for $\tilde{\varepsilon}_{13}$, $\tilde{\varepsilon}_{24}$, and $\tilde{\varepsilon}_5$. The leading terms are

$$\tilde{\varepsilon}_{13} = X\left[\frac{S^o_{11} + S^o_{12}}{S^o_{11} - S^o_{12}}\right]\left\{\varepsilon_{13} + O(\varepsilon^{3/2})\right\}$$

$$\tilde{\varepsilon}_{24} = X\left\{\varepsilon_{24} + O(\varepsilon^{3/2})\right\} \qquad \text{(A1-27)}$$

$$\tilde{\varepsilon}_5 = X\left[\frac{S^o_{11} + S^o_{12}}{S^o_{11} - S^o_{12}}\right]^{1/3}\left\{\varepsilon_5 + O(\varepsilon^{3/2})\right\},$$

where

$$X = \left[\frac{3(S^o_{11})^2 - 2S^o_{11}S^o_{12} + 3(S^o_{12})^2}{3(S^o_{11})^2 + 2S^o_{11}S^o_{12} + 3(S^o_{12})^2}\right]^{\frac{1}{2}} \qquad \text{(A1-28)}$$

In (2.1.17) several thermodynamic admissibility conditions were listed. We can conclude from these that

$$-1 < \left(\frac{S^{o}_{12}}{S^{o}_{11}}\right) < 1 \qquad \text{(A1-29)}$$

If this ratio is not near either of its extreme values then from (A1-26) we see that the isotropic material corresponding to $[S^{o}]$ is nearly the same as that corresponding to $[B^{o}]$. Likewise, we see that the invariant measures of anisotropy ε_{13}, ε_{24}, and ε_{5} are of the same order of magnitude (but not generally numerically equal to) ε_{13}, ε_{24}, and ε_{5}.

APPENDIX II

DEFINITION OF SQUARE ROOTS NEEDED FOR EXACT SOLUTIONS

In our development of exact solutions in Section III we dealt with three different complex planes and transformations of regions from one plane to another. We started off in the physical plane ($z = x + iy$) with the exterior of the unit circle. We then transformed this region to the $z_1 (= x_1 + iy_1)$ plane by means of the affine transformation,

$$\begin{aligned} x_1 &= x + \mu_{1R} y \\ y_1 &= \mu_{1I} y \end{aligned} \quad , \qquad \text{(A2-1)}$$

where

$$\mu_1 = \mu_{1R} + i\mu_{1I} \qquad \text{(A2-2)}$$

Of course we also transformed the z-plane into the z_2-plane just as in (A2-1) and (A2-2)with the subscript 1 replaced by 2. However, in this appendix we refer only to the z_1-plane since the considerations for the z_2-plane are identical. The exterior of the unit circle under the transformation (A2-1) becomes the exterior of an ellipse in the z_1-plane. (See Figure 39.)

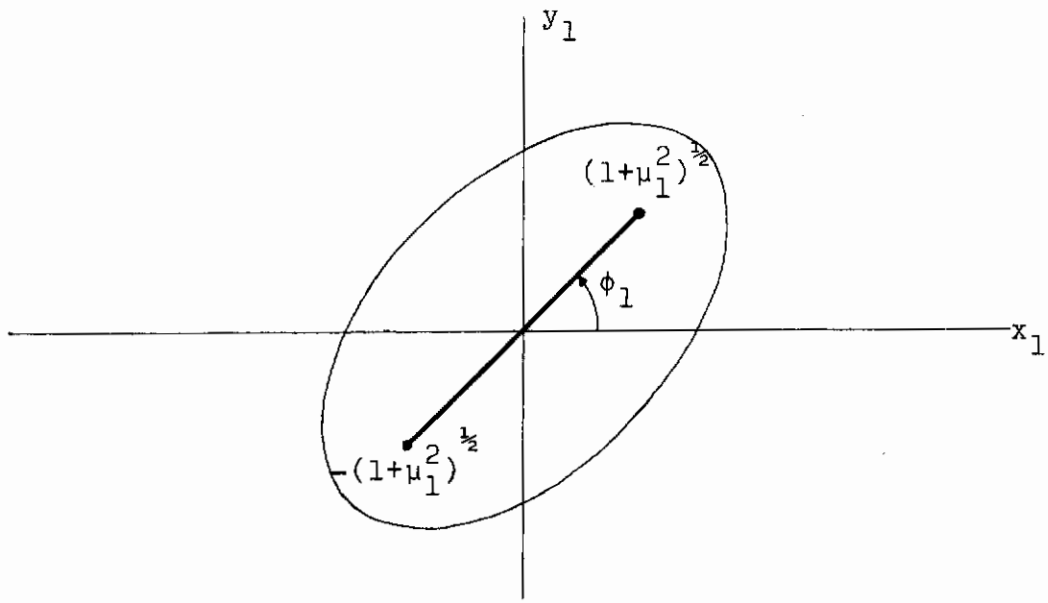


Figure 39. Elliptical Image in z_1-plane

The foci of the ellipse are at $z_1 = \pm (1 + \mu_1^2)^{\frac{1}{2}}$. If we designate the semi-major axis, the semi-minor axis, and the distance from the origin to the foci of the ellipse by a, b, and c respectively, then

$$c = \left[(1 + \mu_{1R}^2 - \mu_{1I}^2)^2 + 4 \mu_{1R}^2 \mu_{1I}^2 \right]^{\frac{1}{4}}$$

$$a = \frac{1}{\sqrt{2}} \left[(1 + \mu_{1R}^2 + \mu_{1I}^2) + c^2 \right]^{\frac{1}{2}} \qquad \text{(A2-3)}$$

$$b = \frac{1}{\sqrt{2}} \left[(1 + \mu_{1R}^2 + \mu_{1I}^2) - c^2 \right]^{\frac{1}{2}}$$

Also, the angle ϕ_1 in Figure 39 is obtained from

$$\tan 2\phi_1 = \frac{2 \mu_{1R} \mu_{1I}}{1 + \mu_{1R}^2 + \mu_{1I}^2} \qquad \text{(A2-4)}$$

It should be noted that if we were to treat a problem involving the exterior of an ellipse in the physical plane (z-plane) the image of this region would be another ellipse in the z_1-plane. Thus, a problem involving the exterior of an ellipse in the physical plane is little more difficult than a problem for the exterior of a circle.

The next step in the solution involves a conformal mapping of the exterior of the ellipse in the z_1-plane onto the exterior of the unit circle in the ζ-plane. Consider the following mapping from the ζ-plane to the z_1-plane:

$$z_1 = \frac{1}{2} \left[(1 - i\mu_1)\zeta + (1 + i\mu_1)\zeta^{-1} \right] \qquad \text{(A2-5)}$$

By substituting $\zeta = e^{i\theta}$ it is seen that the unit circle in the ζ-plane maps onto the boundary of the ellipse in the z_1-plane. The inverse of (A2-5) is

$$\zeta = \frac{z_1 + \sqrt{z_1^2 - (1 + \mu_1^2)}}{(1 - i\mu_1)} \qquad \text{(A2-6)}$$

The square root in (A2-6) has two branches. If the correct branch is chosen (A2-6) will map the exterior of the ellipse in the z_1-plane back onto the exterior of the unit circle in the ζ-plane. By substituting in (A2-6) the value of z_1 on the ellipse, i.e.,

$$z_1 = (\cos\theta + \mu_1 \sin\theta), \tag{A2-7}$$

we obtain

$$\zeta = \frac{(\cos\theta + \mu_1 \sin\theta) \pm i(\sin\theta - \mu_1 \cos\theta)}{(1 - i\mu_1)}, \tag{A2-8}$$

which reduces to $\zeta = e^{i\theta}$ is the upper sign is chosen. Thus, for z_1 on the boundary of the ellipse the determination of the correct value of the square root is straightforward. In general one is also interested in the solution away from the boundary. Thus, it is necessary to determine that branch of the square root which reduces to $+i(\sin\theta - \mu_1 \cos\theta)$ when evaluated on the boundary of the ellipse.

One might define the square root by first writing

$$z_1^2 - (1 + \mu_1^2) = \rho e^{i\xi}, \tag{A2-9}$$

with ξ in the interval $[0, 2\pi]$, say. (See Figure 40.)

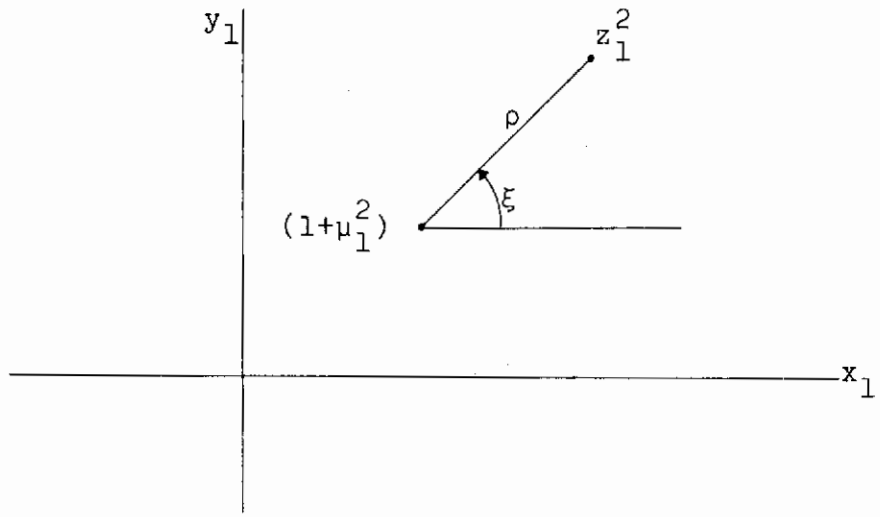


Figure 40. First Definition of Square Root in z_1-plane

Then

$$\sqrt{z_1^2 - (1 + \mu_1^2)} = \pm\sqrt{\rho}\, e^{i\xi/2} \qquad \text{(A2-10)}$$

However, with this definition there is a discontinuity in the square root along the line corresponding to $\xi = 0$, which emanates from $z_1 = (1 + \mu_1^2)$. The image of this line in the physical plane lies at least partially in the region of interest. Thus, the definition of the square root in (A2-10) would yield spurious discontinuities in physical quantities.

This can be avoided by proceeding as follows:

First write

$$\sqrt{z_1^2 - (1 + \mu_1^2)} = \sqrt{z_1 - (1 + \mu_1^2)^{\frac{1}{2}}}\sqrt{z_1 + (1 + \mu_1^2)^{\frac{1}{2}}} \qquad \text{(A2-11)}$$

Next write

$$z_1 - (1 + \mu_1^2)^{\frac{1}{2}} = \rho^- e^{i(\phi_1 + \xi^-)}$$

$$z_1 + (1 + \mu_1^2)^{\frac{1}{2}} = \rho^+ e^{i(\phi_1 + \xi^+)} \quad , \qquad \text{(A2-12)}$$

where ρ^+, ρ^-, ϕ_1, ξ^+, and ξ^- are defined as in the figure with ξ^+ and ξ^- both belonging to the same interval, either $[\,0,\ 2\pi\,]$ or $[\,-\pi, \pi\,]$.

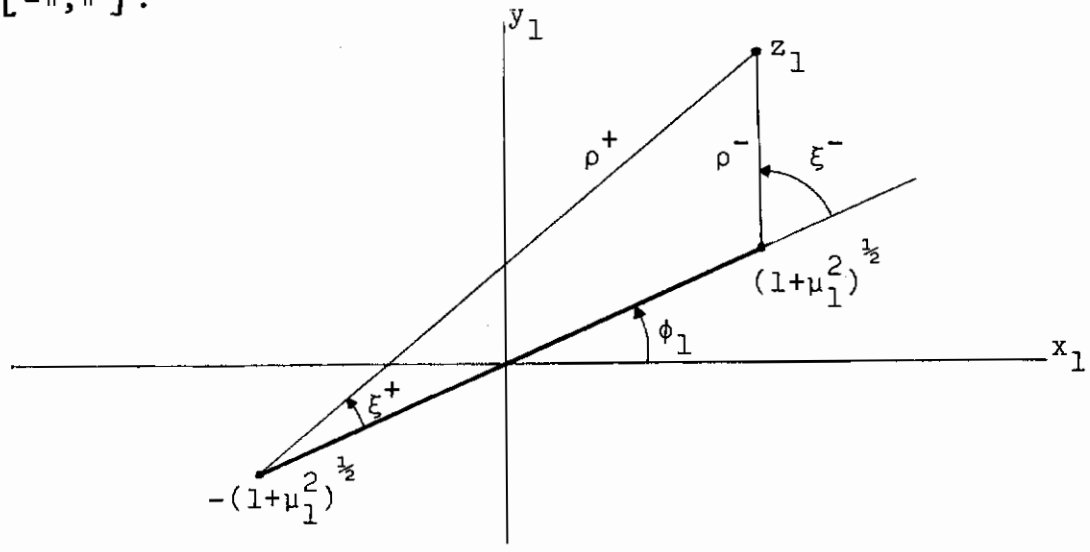


Figure 41 Second Definition of Square Root in z_1-plane

Using (A2-11) and (A2-12) we obtain

$$\sqrt{z_1^2 - (1 + \mu_1^2)} = \pm \sqrt{\rho^- \rho^+}\; e^{i(\phi_1 + \xi^-/2 + \xi^+/2)} \qquad \text{(A2-13)}$$

With definition (A2-13) the square root is discontinuous only across the line extending from $z_1 = -(1 + \mu_1^2)^{\frac{1}{2}}$ to $z_1 = (1 + \mu_1^2)^{\frac{1}{2}}$ i.e., the branch cut of the square root function. Referring to Figure 39 we see that the branch points are just the foci of the ellipse in the z_1-plane. Thus, the image of the branch cut in the physical plane is contained entirely within the circular hole, and so the square root defined as in (A2-13) is analytic throughout the region of interest.

Computationally this procedure is easily carried out by first computing

$$Z \equiv \frac{z_1}{\sqrt{1 + \mu_1^2}} \qquad \text{(A2-14)}$$

and then

$$\sqrt{z_1^2 - (1 + \mu_1^2)} = \pm \sqrt{1 + \mu_1^2}\sqrt{Z - 1}\sqrt{Z + 1}\;, \qquad \text{(A2-15)}$$

where the complex square roots are defined with the branch cut either along the positive or negative x-axis. This is the way complex square roots are defined on most digital computers, e.g., IBM 360 and 7090 series. Thus, by using the simple algorithm of (A2-14) - (A2-15) the correct definition of $\sqrt{z_1^2 - (1 + \mu_1^2)}$ is automatic. The method of defining the square root described in the preceding paragraphs is discussed by Churchill [28] p. 82 ff.

Having appropriately defined the square root it is necessary to select the correct branch, i.e., choose either the + or - sign in (A2-13) or (A2-15). It is sufficient to check for the correct sign at one point on the boundary (where the correct values are known), and then maintain the same sign throughout the calculations.

APPENDIX III

ANALYSIS OF MATRIX CONDITIONING

The treatment of matrix conditioning and its effect on the accuracy of the solution vector presented in this appendix is brief. The analysis follows closely that of Lo, et. al. [22] and Albasiny [29]. The reader is referred to these references, especially the latter, for a more complete analysis. We begin with the matrix equation

$$C_{EX}S_{EX} = R_{EX}, \tag{A3-1}$$

where C_{EX}, S_{EX}, and R_{EX} are the exact values of the coefficient matrix, solution vector, and right hand side vector, respectively. Of course, there will be some roundoff error in the formation of the coefficient matrix and the right hand side vector, and further roundoff errors will occur in obtaining the solution vector. We denote these approximate quantities by C, S, and R. We now rewrite (A3-1) as

$$\left[C + \left(C_{EX}-C\right)\right]\left[S + \left(S_{EX}-S\right)\right] = \left[R + \left(R_{EX}-R\right)\right] \tag{A3-2}$$

From this we obtain

$$\left(S_{EX}-S\right) = C^{-1}\left[\left(R-CS\right) - \left(C_{EX}-C\right)S + \left(R_{EX}-R\right) - \left(C_{EX}-C\right)\left(S_{EX}-S\right)\right] \tag{A3-3}$$

Now the Euclidean norm of a square matrix A of order n is defined

$$||A|| \equiv \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2 \right\}^{1/2} , \qquad \text{(A3-4)}$$

and the norm of a vector is defined similarly. Norms satisfy the familiar conditions,

1) $||AB|| \leq ||A|| \cdot ||B||$

2) $||A+B|| \leq ||A|| + ||B||$

3) $||kA|| = |k| \cdot ||A||$,

where k is a constant and B is either an n x n matrix or a vector with n elements. Taking the norm of both sides of (A3-3) we obtain

$$||S_{EX}-S|| \leq ||C^{-1}|| \left[||R-CS|| + ||C_{EX}-C|| \cdot ||S|| + ||R_{EX}-R|| + ||C_{EX}-C|| \cdot ||S_{EX}-S|| \right] \qquad \text{(A3-5)}$$

To obtain a measure of the relative error of the solution vector we solve for $||S_{EX}-S||$ and divide both sides by $||S||$. This gives

$$\frac{||S_{EX}-S||}{||S||} \leq \frac{||C^{-1}|| \left[||R-CS|| + ||C_{EX}-C|| \cdot ||S|| + ||R_{EX}-R|| \right]}{||S|| \left[1 - ||C^{-1}|| \cdot ||C_{EX}-C|| \right]} , \qquad \text{(A3-6)}$$

which is only valid if the denominator is positive. The three terms in the numerator of the right side of (A3-6) correspond to three different sources of error. The term $||R-CS||$ is a measure of the error incurred in the inversion process. In all of the computer programs developed during this study the inversion and determination of the solution vector were done in double precision whether or not C and R were formed in double precision. Consequently this part of the error term was generally insignificant. In the computer programs the quantity $||R-CS||/||R||$ is printed out under the title RELATIVE ERROR OF INVERSION PROCESS. This quantity was typically $O(10^{-14})$, rarely being greater than 10^{-10}. The quantities $||C_{EX}-C||$ and $||R_{EX}-R||$ correspond to errors in the formation of the coefficient matrix and right hand side vector, respectively. These errors depend on machine precision and the number and type of operations required for the calculation of the elements. If ε is the maximum percentage error in any element we can write

$$||C_{EX}-C|| \leq \varepsilon||C||$$
$$||R_{EX}-C|| \leq \varepsilon||R|| \qquad \text{(A3-7)}$$

We define the condition number of the coefficient matrix as

$$N_C \equiv ||C|| \cdot ||C^{-1}|| \qquad \text{(A3-8)}$$

Using (A3-7) and (A3-8) we can rewrite (A3-6) as

$$\frac{||S-S_{EX}||}{||S||} \leq \frac{||C^{-1}|| \cdot ||R-CS|| + \varepsilon N_C||S|| + \varepsilon||C^{-1}|| \cdot ||R||}{||S||\left[1-\varepsilon N_C\right]} \qquad \text{(A3-9)}$$

On the right hand side of (A3-9) ε and C^{-1} (and N_C, which depends on C^{-1}) are not known. However, ε can be estimated or bounded

by examining the operations involved in obtaining the elements of C and R. We do not know C^{-1} but we do have an approximation to it. If the norm of this approximation is used in place of $||C^{-1}||$ in (A3-9), and if the estimate (A3-9) shows that there is some significance in the solution vector, then one can usually conclude that the use of the norm of the approximation to C^{-1} instead of $||C^{-1}||$ will not cause the inequality to be violated. In the computer programs $||S-S_{EX}||/||S||$ is printed out under the title UPPER BOUND OF RELATIVE ERROR OF SOLUTION VECTOR. In all of the numerical work done in this study the term $\varepsilon N_C||S||$ in the numerator of (A3-9) was dominant because of the large condition number occurring. Our numerical results generally showed the error bound (A3-9) to be quite pessimistic.

APPENDIX IV

DEMONSTRATION OF THE EQUIVALENCE BETWEEN GALERKIN'S METHOD AND THE METHODS OF COMPLEMENTARY, POTENTIAL AND REISSNER ENERGY

There are many variations of the Galerkin method. We refer the reader to Crandall [14], Mikhlin [19], Washizu [30], and Finlayson and Scriven [31] for discussions of this method. The classical Galerkin method is a weighted residual method which can be applied to one ordinary or partial differential equation, and the assumed form of the solution must satisfy all boundary conditions. Generalizations to systems of ordinary or partial differential equations and less restrictive conditions on the assumed forms of the dependent variables suggest themselves. An example of the generalization to systems of equations is seen in our treatment of the displacement and combined stress-displacement formulations later in this Appendix. However, for the time being we consider the stress formulation, ie., the generalized biharmonic, since it fits neatly into the classical Galerkin procedure. We assume the stress function to be of the form,

$$\Psi = \Psi_P + \Psi_H = \Psi_P + \sum_{i=1}^{n} a_i \Psi_i \ , \qquad (A4\text{-}1)$$

where Ψ_P is chosen to satisfy the prescribed stress and displacement boundary conditions, and the Ψ_i satisfy homogeneous boundary conditions. The a_i are unknown coefficients to be determined by the Galerkin procedure. The conditions for their determination are

$$\int_A \left[S_{22}\Psi_{xxxx} + \left(2S_{12} + S_{66} \right) \Psi_{xxyy} + S_{11}\Psi_{yyyy} \right] \Psi_i \, dA = 0, \qquad (A4\text{-}2)$$

where i=1,2,....,n giving a system of n linear algebraic equations for the n unknown coefficients. The part of the integrand in brackets is just the generalized biharmonic

(2.2.9). For simplicity we work in rectangular coordinates but our results are entirely general. One way of interpreting the conditions (A4-2) is that the unknown coefficients are to be chosen so that the residual of the generalized biharmonic is orthogonal to each of the Ψ_i over the region of the problem. The system of equations (A4-2) can be written as

$$\int_A \left[S_{22}\Psi_{xxxx} + \left(2S_{12} + S_{66}\right)\Psi_{xxyy} + S_{11}\Psi_{yyyy} \right] \delta\Psi \; dA = 0, \quad (A4\text{-}3)$$

where the variation of Ψ is understood to be taken with respect to the a_i. In what follows we demonstrate the equivalence between (A4-3) and the complementary energy formulation on the hypothesis that the assumed functions (A4-1) satisfy all boundary conditions. The analysis is just the reverse of the procedure for obtaining the Euler equations and natural boundary conditions of the complementary energy functional. We need the relations,

$$\Psi_{xxxx}\delta\Psi = \delta\left(\tfrac{1}{2}\Psi_{xx}^2\right) + \left(\Psi_{xxx}\delta\Psi - \Psi_{xx}\delta\Psi_x\right)_x,$$

$$\Psi_{yyyy}\delta\Psi = \delta\left(\tfrac{1}{2}\Psi_{yy}^2\right) + \left(\Psi_{yyy}\delta\Psi - \Psi_{yy}\delta\Psi_y\right)_y ,$$

$$\Psi_{xxyy}\delta\Psi = \delta\left(\tfrac{1}{2}\Psi_{xy}^2\right) + \left(\Psi_{xyy}\delta\Psi\right)_x + \left(\Psi_{xxy}\delta\Psi\right)_y - \left(\Psi_{xy}\delta\Psi\right)_{xy}$$

$$= \delta\left(\tfrac{1}{2}\Psi_{xx}\Psi_{yy}\right) + \tfrac{1}{2}\left(\Psi_{xyy}\delta\Psi - \Psi_{yy}\delta\Psi_x\right)_x + \tfrac{1}{2}\left(\Psi_{xxy}\delta\Psi - \Psi_{xx}\delta\Psi_y\right)_y$$

(A4-4)

Using (A4-4) in (A4-3) we obtain

$$\delta\left\{\frac{1}{2}\int_A \left[S_{22}\Psi_{xx}^2 + 2S_{12}\Psi_{xx}\Psi_{yy} + S_{11}\Psi_{yy}^2 + S_{66}\Psi_{xy}^2\right]dA\right\}$$

$$+\int_{C_d}\left\{\left[S_{22}\Psi_{xxx} + \left(S_{12} + \frac{S_{66}}{2}\right)\Psi_{xyy}\right]\delta\Psi dy\right.$$

$$-\left[S_{11}\Psi_{yyy} + \left(S_{12} + \frac{S_{66}}{2}\right)\Psi_{yxx}\right]\delta\Psi dx$$

$$+\left[S_{11}\Psi_{yy} + S_{12}\Psi_{xx}\right]\delta\Psi_y\, dx - \left[S_{22}\Psi_{xx} + S_{12}\Psi_{yy}\right]\delta\Psi_x\, dy$$

$$\left.+\left[\frac{S_{66}}{2}\Psi_{xy}\right]\delta\Psi_x\, dx - \left[\frac{S_{66}}{2}\Psi_{xy}\right]\delta\Psi_y\, dy\right\} = 0, \qquad \text{(A4-5)}$$

where we have made use of Green's lemma in transforming parts of the surface integral of (A4-4) to the line integral of (A4-5). The line integral is only over C_d, the part of the boundary on which displacements are prescribed, since $\delta\Psi$, $\delta\Psi_x$, and $\delta\Psi_y$ vanish on C_s.

Now since Ψ is required to satisfy displacement boundary conditions on C_d it follows that the stress-strain relations (2.1.15) are satisfied on C_d. Using these the line integral of (A4-5) can be written as follows:

$$\int_{C_d}\left\{\varepsilon_x\delta\Psi_y dx - \varepsilon_y\delta\Psi_x dy - \frac{\gamma_{xy}}{2}\delta\Psi_x dx + \frac{\gamma_{xy}}{2}\delta\Psi_y\, dy\right.$$

$$\left.- \frac{d}{dy}(\varepsilon_x)\delta\Psi\, dx + \frac{d}{dx}(\varepsilon_y)\delta\Psi\, dy - \frac{d}{dy}\left(\frac{\gamma_{xy}}{2}\right)\delta\Psi\, dy + \frac{d}{dx}\left(\frac{\gamma_{xy}}{2}\right)\delta\Psi\, dx\right\} \qquad \text{(A4-6)}$$

After some integrations by parts this line integral can be written

$$-\delta \int_{C_d} \left\{ \left(-\Psi_{xy} m + \Psi_{yy} \ell\right) \bar{u} + \left(-\Psi_{xy} \ell + \Psi_{xx} m\right) \bar{v} \right\} ds, \qquad \text{(A4-7)}$$

where $\ell \left(\equiv \frac{dy}{ds}\right)$ and $m \left(\equiv \frac{-dx}{ds}\right)$ are the direction cosines of the external normal, and $\bar{u}$ and $\bar{v}$ are the prescribed displacements. (A4-7) (without the δ) can be rewritten as the line integral of Φ_C (2.3.6) except that it is of oppostie sign. Likewise, the unvaried surface integral of (A4-5) can be identified (except for the sign) with the surface integral of (2.3.6). Thus, it is clear that when the assumed functions satisfy all boundary conditions the Galerkin method as applied to the generalized biharmonic yields the same system of equations for the unknown coefficients a_i as does the method of complementary energy. As pointed out in Article 2a of Section II it is generally difficult to satisfy displacement conditions with assumed stress variables. However, assuming that the stress variables have been suitably chosen, the use of these functions in connection with Φ_C yields results completely equivalent to the Galerkin method. We know that when using the method of complementary energy the assumed functions need satisfy only stress boundary conditions. One might find that the Galerkin procedure used with such functions would yield useful approximations. However, it is unlikely that equivalence between the Galerkin method and the method of complementary energy could be established under these conditions.

Let us now demonstrate the equivalence between the method of potential energy and the Galerkin method as applied to the displacement formulation. The two governing equations for this formulation are listed in equation (2.2.5). Repeated here they are

$$\begin{aligned} B_{11} u_{xx} + B_{66} u_{yy} + (B_{12} + B_{66})\, v_{xy} &= 0 \\ B_{22} v_{yy} + B_{66} v_{xx} + (B_{12} + B_{66})\, u_{xy} &= 0 \end{aligned} \qquad \text{(A4-8)}$$

We assume u and v to be of the following form:

$$u = u_P + u_H = u_P + \sum_{i=1}^{n} a_i u_i$$
$$v = v_P + v_H = v_P + \sum_{i=1}^{n} b_i v_i, \qquad \text{(A4-9)}$$

where u_P and v_P are chosen to satisfy all boundary conditions, and the u_i and v_i satisfy homogeneous boundary conditions. It is not essential that the upper limits on the summations in (A4-9) be the same. We assume them to be the same here merely for convenience. A natural generalization of the Galerkin procedure for one differential equation is the following:

$$\int_A \left[B_{11}u_{xx} + B_{66}u_{yy} + (B_{12} + B_{66})\, v_{xy}\right] u_i \, dA = 0,$$
$$\int_A \left[B_{22}v_{yy} + B_{66}v_{xx} + (B_{12} + B_{66})\, u_{xy}\right] v_i \, dA = 0, \qquad \text{(A4-10)}$$

where i=1,2,....,n, yielding a system of 2n linear equations for the a_i and b_i. As before (A4-10) can be rewritten using variational notation.

$$\int_A \Big\{ \left[B_{11}u_{xx} + B_{66}u_{yy} + (B_{12} + B_{66})\, v_{xy}\right]\delta u + \left[B_{22}v_{yy} + B_{66}v_{xx} + (B_{12} + B_{66})\, u_{xy}\right]\delta v \Big\} \, dA = 0 \qquad \text{(A4-11)}$$

We need the following relations:

$$u_{xx}\delta u = -\delta\left(\frac{1}{2}u_x^2\right) + \left(u_x \delta u\right)_x$$

$$u_{yy}\delta u = -\delta\left(\frac{1}{2}u_y^2\right) + \left(u_y \delta u\right)_y$$

$$v_{xy}\delta u + u_{xy}\delta v = -\delta\left(u_x v_y\right) + \left(v_y \delta u\right)_x + \left(u_x \delta v\right)_y$$

$$= -\delta\left(u_y v_x\right) + \left(v_x \delta u\right)_y + \left(u_y \delta v\right)_x \qquad \text{(A4-12)}$$

Using (A4-12) in (A4-11) we obtain

$$\delta\left\{\frac{1}{2}\int_A \left[B_{11}u_x^2 + B_{22}v_y^2 + 2B_{12}u_x v_y + B_{66}\left(u_y + v_x\right)^2\right] dA \right.$$
$$- \int_{C_s} \left\{ B_{11}\left[u_x \delta u\right]dy - B_{22}\left[v_y \delta v\right]dx - B_{66}\left[u_y \delta u\right]dx + B_{66}\left[v_x \delta v\right]dy \right.$$
$$\left. \left. + B_{12}\left[v_y \delta u\right]dy - B_{12}\left[u_x \delta v\right]dx - B_{66}\left[v_x \delta u\right]dx + B_{66}\left[u_y \delta v\right]dy \right\} \right.$$

(A4-13)

Since the stress-strain relations (2.1.18) are satisfied on C_s we can use them to rewrite the line integral of (A4-13) as

$$\int_{C_s} \left\{ \sigma_x \delta u\, dy - \sigma_y \delta v\, dx + \tau_{xy}\delta v\, dy - \tau_{xy}\delta u\, dx \right\}$$
$$= \int_{C_s} \left\{ \left(\sigma_x \ell + \tau_{xy}\, m\right)\delta u + \left(\sigma_y\, m + \tau_{xy}\ell\right)\delta v \right\} ds$$
$$= \delta \int_{C_s} \vec{\bar{F}} \cdot \bar{u}\, ds \qquad \text{(A4-14)}$$

The final form of the line integral in (A4-14) is identical to the line integral of the potential energy functional Φ_P listed in equation (2.3.5). Since the surface integrals of (A4-13) and (2.3.5) are identical, we have established the equivalence between the method of potential energy and Galerkin's method as applied to the displacement formulation. It should be noted that it is generally somewhat easier to choose assumed displacement functions to satisfy stress boundary conditions than to choose stress variables to satisfy displacement boundary conditions as was necessary when applying the Galerkin method to the stress formulation.

Finally we demonstrate the equivalence between the Reissner energy method and Galerkin's method applied to the combined stress-displacement formulation. We assume the following forms for the stresses and displacements.

$$\sigma_x = \sigma_{xP} + \sigma_{xH} = \sigma_{xP} + \sum_{i=1}^{n} a_i \sigma_{xi}$$

$$\sigma_y = \sigma_{yP} + \sigma_{yH} = \sigma_{yP} + \sum_{i=1}^{n} b_i \sigma_{yi}$$

$$\tau_{xy} = \tau_{xyP} + \tau_{xyH} = \tau_{xyP} + \sum_{i=1}^{n} c_i \tau_{xyi} \qquad \text{(A4-15)}.$$

$$u = u_P + u_H = u_P + \sum_{i=1}^{n} d_i u_i$$

$$v = v_P + v_H = v_P + \sum_{i=1}^{n} e_i v_i \quad ,$$

where the displacements satisfy the prescribed conditions on C_d and the stresses satisfy those on C_s. Again the upper limits on the summations have been taken to be equal only for

convenience. In the combined formulation there are five governing field equations; two equilibrium equations and three stress-displacement relations. Probably the most natural way of defining the Galerkin method is

$$\int_A \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right] u_i \, dA = 0$$

$$\int_A \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right] v_i \, dA = 0$$

$$\int_A \left[\frac{\partial u}{\partial x} - S_{11}\sigma_x - S_{12}\sigma_y\right] \sigma_{xi} \, dA = 0 \qquad \text{(A4-16)}$$

$$\int_A \left[\frac{\partial v}{\partial y} - S_{12}\sigma_x - S_{22}\sigma_y\right] \sigma_{yi} \, dA = 0$$

$$\int_A \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - S_{66}\tau_{xy}\right] \tau_{xyi} \, dA = 0,$$

where i=1,2,....,n giving a system of 5n linear equations. One could define the Galerkin method differently than above. However, the equations (A4-16) appear to be the most natural and symmetric way. If (A4-16) is rewritten using variational notation the resulting integral is identical to the surface integral in the variational form of the Reissner energy theorem in equation (A6-3). Since the assumed functions are required to satisfy the boundary conditions, the line integrals in (A6-3) vanish. Thus, the equivalence between the Reissner energy method and the Galerkin procedure as defined in (A4-16) is immediate.

APPENDIX V

DETERMINATION OF THE DISPLACEMENT STATE CORRESPONDING TO AN APPROXIMATE STRESS STATE

The determination of the stress state corresponding to an approximate displacement state when this displacement state is expressed analytically (as in the potential energy formulation) is completely straightforward. It is just a matter of differentiating the approximate displacements to obtain strains, and then using these strains in the stress-strain relationships. The converse problem, which is encountered in the complementary energy formulation, is generally more difficult. We begin by integrating the strain-displacement equations in polar coordinates (2.2.11). From the first of these equations we obtain

$$u = \int \varepsilon_r dr + z_1(\theta) \ , \tag{A5-1}$$

where $z_1(\theta)$ is an arbitrary function of integration. Using (A5-1) in the second of the stress-strain relations we obtain

$$v = \int r\varepsilon_\theta \, d\theta - \iint \varepsilon_r \, dr \, d\theta - \int z_1(\theta) \, d\theta + z_2(r), \tag{A5-2}$$

where $z_2(r)$ is another arbitrary function of integration. Next we substitute the above expressions for u and v into the third stress-strain relation.

$$\frac{1}{r}\left[\frac{\partial}{\partial\theta}\int \varepsilon_r \, dr + z_1'(\theta)\right] + \frac{\partial}{\partial r}\int r\varepsilon_\theta \, d\theta - \int \varepsilon_r \, d\theta + z_2'(r)$$

$$- \frac{1}{r}\int r\varepsilon_\theta \, d\theta + \frac{1}{r}\iint \varepsilon_r \, dr \, d\theta - \frac{1}{r}\int z_1(\theta) \, d\theta - \frac{1}{r}z_2(r) = \gamma_{r\theta}$$

(A5-3)

Now ε_r, ε_θ, and $\gamma_{r\theta}$ are completely known from the solution for the stresses. Thus, we would like to determine the arbitrary functions $z_1(\theta)$ and $z_2(r)$ so that (A5-3) is satisfied. To see whether this is possible we multiply (A5-3) through by r and then differentiate with respect to r and θ. The result is

$$r\left[\frac{\partial^2}{\partial r^2}(r\varepsilon_\theta) - \frac{\partial \varepsilon_r}{\partial r}\right] + \frac{\partial^2 \varepsilon_r}{\partial \theta^2} = \frac{\partial^2}{\partial r \partial \theta}(r\gamma_{r\theta}), \qquad \text{(A5-4)}$$

which is just the compatibility equation (2.2.13). Recall that the generalized biharmonic is obtained from (A5-4) by expressing the strains in terms of the stress function Ψ. If Ψ satisfies the generalized biharmonic exactly then (A5-3) must be of the form,

$$\frac{1}{r}\left[F(r) + G(\theta) + z_1'(\theta) - \int z_1(\theta)\, d\theta + r z_2'(r) - z_2(r)\right] = 0, \qquad \text{(A5-5)}$$

where F(r) and G(θ) depend on how the indefinite integrals in (A5-1) and (A5-2) are defined. $z_1(\theta)$ and $z_2(r)$ are chosen to satisfy (A5-5) and have the form,

$$\begin{aligned} z_1(\theta) &= A\cos\theta + B\sin\theta + z_{1P}(\theta) \\ z_2(r) &= Cr + z_{2P}(r)\ , \end{aligned} \qquad \text{(A5-6)}$$

where $z_{1P}(\theta)$ and $z_{2P}(r)$ are particular solutions which depend on G(θ) and F(r), respectively. The other terms on the right of (A5-6) correspond to rigid body motion, and the constants A, B, and C are determined by specifying three additional displacement conditions. In our standard sample problems we have excluded rigid body motion so A=B=C=0.

From the foregoing it is apparent that when Ψ is exact the determination of the displacements is straightforward providing analytical expressions for the indefinite integrals are available. However, when Ψ is approximate (A5-4) is not satisfied exactly. Instead of having the form (A5-5), the third stress-strain relation (A5-3) now has the form,

$$\frac{1}{r}\left[H(r,\theta) + z_1'(\theta) - \int z_1(\theta)\, d\theta + rz_2'(r) - z_2(r)\right] = 0, \tag{A5-7}$$

where $H(r,\theta)$ depends on the residual of the generalized biharmonic and the definitions of the indefinite integrals in (A5-1) and (A5-2). Now it is not possible to choose $z_1(\theta)$ and $z_2(r)$ so that (A5-7) will be satisfied exactly. However, if the residual of the generalized biharmonic is small one would expect that the residual of (A5-7) could also be made small by an appropriate choice of $z_1(\theta)$ and $z_2(r)$. If this conjecture is correct then $H(r,\theta)$ must be of the form

$$H(r,\theta) = \hat{H}(r,\theta) + F(r) + G(\theta), \tag{A5-8}$$

where $\hat{H}(r,\theta)$ is related to the residual of the generalized biharmonic, and is presumably small compared to the strain terms of (A5-3). $F(r)$ and $G(\theta)$ are related to the definitions of the indefinite integrals and may be of the same order as the strain terms. $z_{1P}(\theta)$ and $z_{2P}(r)$ are then chosen to cancel the $F(r)$ and $G(\theta)$ terms in (A5-7), leaving $\hat{H}(r,\theta)$ as the residual. The decomposition (A5-8) is not unique. In simple problems one might choose $F(r)$ and $G(\theta)$ to give a small $\hat{H}(r,\theta)$ by inspection. However, in problems such as the ones that are considered in this study this approach is generally not feasible. A possible approach to the resolution of this problem is given in what follows: We assume Ψ to be of the form (6.2.20). For simplicity we consider only cosine harmonics with $j = 0, 1, 2, \ldots, m-1$, and we take $\Psi_o(r) = 0$. The considerations for the general form are similar. The first stress-displacement relation is

$$\frac{\partial u}{\partial r} = S_{11}^{r\theta}(\theta)\left(\frac{1}{r}\Psi_r + \frac{1}{r^2}\Psi_{\theta\theta}\right) + S_{12}^{r\theta}(\theta)\Psi_{rr} - S_{16}^{r\theta}(\theta)\left(\frac{1}{r}\Psi_\theta\right)_r \qquad \text{(A5-9)}$$

Then substituting the assumed form for Ψ and integrating with respect to r we obtain

$$u = S_{11}^{r\theta}(\theta)\left\{\sum_i\sum_j \Psi_{ij}\left[\left(\int\frac{f_{4i}'}{r}dr + K_{1ij}\right) - j^2\left(\int\frac{f_{4i}}{r^2}dr + K_{2ij}\right)\right]\cos j\theta\right\}$$

$$+ S_{12}^{r\theta}(\theta)\left\{\sum_i\sum_j \Psi_{ij}\left[f_{4i}' + K_{3ij}\right]\cos j\theta\right\}$$

$$+ S_{16}^{r\theta}(\theta)\left\{\sum_i\sum_i \Psi_{ij}j\left[\frac{f_{4i}}{r} + K_{4ij}\right]\sin j\theta\right\} \qquad + z_1(\theta)\ , \qquad \text{(A5-10)}$$

where the K_{nij} are constants of integration. If all the θ-functions with factors K_{nij} are collected it is seen that many of the K_{nij} are redundant since they multiply the same θ-functions. In fact, the subscript i is completely unnecessary. Noting the form of the $S_{ij}^{r\theta}(\theta)$ in (2.1.9) and using trigonometric identities such as (2.2.18) it is seen that (A5-10) can be written

$$u = \sum_i\sum_j \Psi_{ij}\left\{\left[\int\frac{f_{4i}'}{r}dr - j^2\int\frac{f_{4i}}{r^2}dr\right] S_{11}^{r\theta}(\theta)\cos j\theta\right.$$

$$\left. + f_{4i}'\, S_{12}^{r\theta}(\theta)\cos j\theta + \frac{f_{4i}}{r}\, S_{16}^{r\theta}(\theta)\, j\sin j\theta\right\}$$

$$+ \sum_{j=0,2,3,\ldots}^{m+3} K_j^*\cos j\theta + A\cos\theta + B\sin\theta\ , \qquad \text{(A5-11)}$$

where K_j^* are constants to be determined. What we have done is to assume the form of $z_1(\theta)$ based on the type of terms arising due to the constants of integration.

Suppose that for the problems being dealt with the region is infinite with the stresses and displacements being zero at infinity. In such problems the f_{4i} would be chosen to give zero stresses at infinity. From (A5-11) it is clear that if the integrals are defined so that they vanish at infinity then one would choose $K_j^*=0$ in order to satisfy the displacement conditions at infinity. For a finite plate without a hole at the center continuity of displacements at r=0 would similarly dictate $K_j^*=0$. However, in problems for an annular region or for an infinite region with non-zero stresses at infinity, it does not appear that the K_j^* can be determined from simple considerations such as these.

To obtain an expression for v we write the second stress-displacement relation.

$$\frac{\partial v}{\partial \theta} = -u+r\left\{S_{12}^{r\theta}(\theta)\left(\frac{1}{r}\Psi_r+\frac{1}{r^2}\Psi_{\theta\theta}\right) + S_{22}^{r\theta}(\theta)\Psi_{rr}-S_{26}^{r\theta}(\theta)\left(\frac{1}{r}\Psi_\theta\right)_r\right\} \qquad \text{(A5-12)}$$

Next we integrate (A5-12) with respect to θ.

$$v = \sum_i \sum_j \Psi_{ij} \Bigg\{ -\left[\int \frac{f'_{4i}}{r} dr - j^2 \int \frac{f_{4i}}{r^2} dr\right]\left[\int S_{11}^{r\theta}(\theta)\, \cos j\theta\, d\theta + L_{1ij}\right]$$

$$-f'_{4i}\left[\int S_{12}^{r\theta}(\theta)\, \cos j\theta\, d\theta + L_{2ij}\right] - j\,\frac{f_{4i}}{r}\left[\int S_{16}^{r\theta}(\theta)\, \sin j\theta\, d\theta + L_{3ij}\right]$$

$$+\left[f'_{4i} - j^2\,\frac{f_{4i}}{r}\right]\left[\int S_{12}^{r\theta}(\theta)\, \cos j\theta\, d\theta + L_{4ij}\right]$$

$$+rf''_{4i}\left[\int S_{22}^{r\theta}(\theta)\, \cos j\theta\, d\theta + L_{5ij}\right]$$

$$+j\left[-\frac{f_{4i}}{r} + f'_{4i}\right]\left[\int S_{26}^{r\theta}(\theta)\, \sin j\theta\, d\theta + L_{6ij}\right]\Bigg\}$$

$$-\int\left[\sum_{j=0,2,3,\ldots}^{m+3} K_j^*\, \cos j\theta + A\cos\theta + B\sin\theta\right] d\theta + z_2(r)$$

(A5-13)

The L_{nij} are constants of integration, and many of them are redundant. The subscript j can be eliminated completely. Depending on the form of the f_{4i} more of the L's might be eliminated. In the most general case the L-terms of (A5-13) can be replaced by

$$\sum_{i=1}^{n}\left\{L_{1i}^* \int \frac{f_{4i}}{r^2} dr + L_{2i}^* \frac{f_{4i}}{r} + L_{3i}^* f'_{4i} + L_{4i}^* rf''_{4i}\right\} \qquad \text{(A5-14)}$$

In other words, we assume the function of integration $z_2(r)$ to be of the form (A5-14) plus the rigid body term Cr. Depending

on the f_{4i} (A5-14) may be simplified further. For example, consider the standard form of the f_{4i}, equation (6.2.24), and suppose $\hat{i}$ takes on the values -4, -3, -2, -1, 0, 1, 2, and 3. Then (A5-14) can be replaced by

$$\sum_{i=1}^{10} L_i^* r^{(i-8)} + L_{11}^* \frac{\log r}{r} + L_{12}^* \log r \qquad \text{(A5-15)}$$

We exclude the term in the summation corresponding to i=9 we already have the rigid body term Cr in $z_2(r)$.

Before proceeding to the question of the determination of the L_i^* and K_j^* we observe that in the expression for v, (A5-13), there are the integrals, $\int S_{11}^{r\theta}(\theta) \cos j\theta \, d\theta$, $\int S_{12}^{r\theta}(\theta) \cos j\theta \, d\theta$, and $\int K_j^* \cos j\theta \, d\theta$. Because $S_{11}^{r\theta}$ and $S_{12}^{r\theta}$ contain a constant term (see (2.1.9)) these integrals yield terms linear in θ for j=0. Such terms do not give single valued displacements, and are therefore unacceptable for the problems we are treating. These θ-terms will be multiplied by different functions of r, depending on the index i in (A5-13). In general these functions of r will not cancel among themselves. However, hopefully the Ψ_{io} (unknown coefficients corresponding to j=0) will be such that the resultant linear-in-θ-term is approximately zero. In a number of runs of the complementary energy program this conjecture was borne out. Since these linear-in-θ-terms should disappear as the approximate solution approaches the exact solution, it was decided to replace all such terms by zero in the computer program.

When the assumed form of the stress function contains only cosine harmonics the stress state is symmetric about the x-axis. Since the material is also symmetric about the x (and y) axis it follows that the circumferential displacement v should be zero along the x-axis (exclusive of rigid body terms). Thus, if the θ-integrals in (A5-13) are defined to reflect this condition all of the L^*'s should be zero. Similarly, if the stress function contains only odd sine harmonics the circumferential

displacement should vanish along the y-axis (exclusive of rigid body terms). This again dictates that the L_i^*'s must be zero. However, if the stress function contains even sine harmonics the symmetry axes of the stress state do not in general coincide with the material axes of symmetry. Thus, it does not appear that a simple determination of the L_i^*'s is possible in this case.

Before proceeding let us summarize the major points. We have established that it is impossible to obtain displacements which satisfy all three stress-displacement equations if the stress state is approximate. We have developed expressions for the displacements containing two arbitrary functions, $z_1(\theta)$ and $z_2(r)$. These expressions satisfy two of the stress-displacement relations. We want to select $z_1(\theta)$ and $z_2(r)$ so that the third stress-displacement relation is approximately satisfied. We have assumed $z_1(\theta)$ and $z_2(r)$ to have certain forms, based on the type of terms arising as a result of the constants of integration in the indefinite integrals. For some problems conditions on the displacements at the origin or at infinity and conditions of symmetry about certain axes may allow $z_1(\theta)$ and/or $z_2(r)$ to be determined by inspection. (Sample Problems I and III fall into this category.) In general, however, these arbitrary functions must be determined from the requirement that the residual of the third stress-displacement relation be small in some sense. We can do this by substituting the expressions for u and v into the third stress-displacement relation, and then using a technique such as least squares, collocation, minimax, etc., to determine the unknown constants K_j^* and L_i^* so that the third stress-displacement relation is approximately satisfied in the region of interest. This technique was not built into the computer program for the complementary energy, so displacements are only obtained in Cases I and III with the standard assumed forms in negative powers of r. It should be noted that it is possible to use the first and third stress-displacement relations to obtain expressions for u and v, and then satisfy the second of these relations in some approximate manner. However, there does not appear to be any particular reason why this would be preferable to the approach we have taken here.

APPENDIX VI

COMPARISON OF THE REISSNER THEOREM IN FUNCTIONAL AND VARIATIONAL FORM

Section II, Article 3 presented the Reissner functional in two equivalent forms (equations (2.3.7) and (2.3.8)). The purpose of this appendix is to derive the alternative variational forms. Equation (2.3.7) states the Reissner functional as

$$\Phi_R = \int_A \left[\{\sigma\}^T \{\varepsilon\} - W\right] dA - \int_{C_s} \{u\}^T \{\bar{F}\} ds - \int_{C_d} \{F\}^T \left[\{u\} - \{\bar{u}\}\right] ds \tag{A6-1}$$

Taking the first variation of equation (A6-1) we obtain

$$\delta\Phi_R = \int_A \left[\delta\{\sigma\}^T \{\varepsilon\} + \delta\{\varepsilon\}^T \{\sigma\} - \delta\{\sigma\}^T [S]\{\sigma\}\right] dA - \int_{C_s} \delta\{u\}^T \{\bar{F}\} ds$$
$$- \int_{C_d} \left[\delta\{F\}^T \left[\{u\} - \{\bar{u}\}\right] + \delta\{u\}^T \{F\}\right] ds \tag{A6-2}$$

Integrating by parts the second term in the volume integral we obtain

$$\delta\Phi_R = \int_A \left[\delta\{\sigma\}^T \left[\{\varepsilon\} - [S]\{\sigma\}\right] - \delta\{u\}^T \left[D\{\sigma\}\right]\right] dA$$
$$+ \int_{C_s} \delta\{u\}^T \left[\{F\} - \{\bar{F}\}\right] ds - \int_{C_d} \delta\{F\}^T \left[\{u\} - \{\bar{u}\}\right] ds \tag{A6-3}$$

This form was given by Reissner [5], [6]. It is perhaps the form of greatest clarity since the statements of internal equilibrium and stress strain conditions together with stress and displacement boundary conditions are contained explicitly. If, in addition, the first term in the volume integral of equation (A6-2) ($\delta\{\sigma\}^T \{\varepsilon\}$) is also integrated by parts then another form of the variational is obtained, i.e.,

$$\delta\Phi_R = \int_A \left[-\delta\{u\}^T \left[D\{\sigma\}\right] - \{u\}^T \left[D\,\delta\{\sigma\}\right] - \delta\{\sigma\}^T \left[S\right]\{\sigma\}\right] dA$$
$$+ \int_{C_s} \delta\{u\}^T \left[\{F\} - \{\bar{F}\}\right] ds + \int_{C_d} \delta\{F\}^T \{\bar{u}\} ds = 0$$

(A6-4)

This, of course, also follows immediately by taking the first variation of equation (2.3.8).

Inspection of equation (A6-2) shows that if stress-displacement relationships are satsified internally together with the displacement boundary conditions then we obtain the potential energy variational, i.e.,

$$\delta\Phi_R = \int_A \delta\{\varepsilon\}^T \left[B\right]\{\varepsilon\} dA - \int_{C_s} \delta\{u\}^T \{\bar{F}\} ds = \delta\Phi_P = 0$$

(A6-5)

Similarly if stresses (and their variations) satisfy equilibrium internally and the boundary conditions then equation (A6-4) is equivalent to the complementary energy variational,

$$\delta\Phi_R = -\int_A \delta\{\sigma\}^T \left[S\right]\{\sigma\} dA + \int_{C_d} \delta\{F\}^T \{\bar{u}\} ds = \delta\Phi_C = 0$$

(A6-6)

The third special case, where both the stress-displacement relationships and equilibrium are satisfied in the interior, can also be obtained by inspection from equation (A6-3). In this case the volume integral vanishes and

$$\delta\Phi_R = \int_{C_s} \delta\{u\}^T \left[\{F\} - \{\bar{F}\}\right] ds - \int_{C_d} \delta\{F\}^T \left[\{u\} - \{\bar{u}\}\right] ds$$

(A6-7)

This is the special form of Reissner's theorem expressed purely as a boundary functional. This form was first given by Reissner [6].

APPENDIX VII

SOLUTIONS TO TRIGONOMETRIC INTEGRALS

For conciseness the usual trigonometric function notation will be abbreviated as follows;

$\cos m\theta = C_m$, $\sin m\theta = S_m$

$(\cos m\theta)^s = C_m^s$ etc.

The solutions to the required integrals are set out in the accompanying table. The numbering of the integrals reflects the total number which will be generated in the analysis but only those which possess non-zero solutions are included in the table.

Table V. Trigometric Integrals

Integral	Symbol	Solution			
		q=0	q=2	q=j=0	q=j≠0
$\int_0^{2\pi} C_q C_j d\theta$	I_{31}	-	-	2π	π
$\int_0^{2\pi} S_q S_j d\theta$	I_{33}	-	-	-	π
$\int_0^{2\pi} C_q d\theta$	I_{34}	2π	-	-	-
$\int_0^{2\pi} C_q C_2 d\theta$	I_{35}	-	π	-	-
$\int_0^{2\pi} S_q S_2 d\theta$	I_{36}	-	π	-	-

Table VI. Trigonometric Integrals

Integral	Symbol	Solution					
		q=j=0	q=j≠0	q=j+2	q=j-2	q=j+4	q=j-4
$\int_0^{2\pi} C_q C_j C_1^4 d\theta$	I_1	3π/4	3π/8	π/4	π/4	π/16	π/16
$\int_0^{2\pi} C_q C_j C_1^2 S_1^2 d\theta$	I_3	π/4	π/8	-	-	-π/16	-π/16
$\int_0^{2\pi} C_q C_j S_1^4 d\theta$	I_5	3π/4	3π/8	-π/4	-π/4	π/16	π/16
$\int_0^{2\pi} C_q S_j C_1^3 S_1 d\theta$	I_7	-	-	-π/8	π/8	-π/16	π/16
$\int_0^{2\pi} C_q S_j C_1 S_1^3 d\theta$	I_9	-	-	-π/8	π/8	π/16	-π/16
$\int_0^{2\pi} S_q C_j C_1^3 S_1 d\theta$	I_{12}	-	-	π/8	-π/8	π/16	-π/16
$\int_0^{2\pi} S_q C_j C_1 S_1^3 d\theta$	I_{14}	-	-	π/8	-π/8	-π/16	π/16
$\int_0^{2\pi} S_q S_j C_1^4 d\theta$	I_{16}	-	3π/8	π/4	π/4	π/16	π/16
$\int_0^{2\pi} S_q S_j C_1^2 S_1^2 d\theta$	I_{18}	-	π/8	-	-	-π/16	-π/16
$\int_0^{2\pi} S_q S_j S_1^4 d\theta$	I_{20}	-	3π/8	-π/4	-π/4	π/16	π/16
		q=0	q=2	q=4			
$\int_0^{2\pi} C_q C_1^4 d\theta$	I_{21}	3π/4	π/2	π/8			
$\int_0^{2\pi} C_q C_1^2 S_1^2 d\theta$	I_{23}	π/4	-	-π/8			
$\int_0^{2\pi} C_q S_1^4 d\theta$	I_{25}	3π/4	-π/2	π/8			
$\int_0^{2\pi} S_q C_1^3 S_1 d\theta$	I_{27}	-	π/4	π/8			
$\int_0^{2\pi} S_q C_1 S_1^3 d\theta$	I_{29}	-	π/4	-π/8			

APPENDIX VIII

EQUATIONS RESULTING FROM THE APPLICATION OF THE REISSNER ENERGY THEOREM

Case I:

The system of equations arising from Case I (uniform pressure on the circular boundary) have the distinct pattern shown in Figure 42. The submatrices $[A_{ik}]$ of the coefficient matrix $[A]$ are presented in tabular form. They may also be obtained directly from the equations by selecting the coefficients of the unknowns. The vector of constants {B} is very sparse for all four cases, and a direct algebraic solution in terms of the submatrices $[A_{ij}]$ can be easily derived. In this, advantage can be taken of the double symmetry (about the x and y axes) of the problem, which excludes all but even cosine harmonics for σ_r, σ_θ, u and even sine harmonics for $\tau_{r\theta}$ and v. (The same is true for Case III, and also for Cases II and IV when the angle of tension ϕ is zero, as is being assumed in the present analysis.) Inspection of Table III (Section VI, Article 1) shows that only the unknowns $\{a_k\}$, $\{b_k\}$, $\{c_\ell\}$, $\{u_k\}$, and $\{v_\ell\}$ will be non-zero .

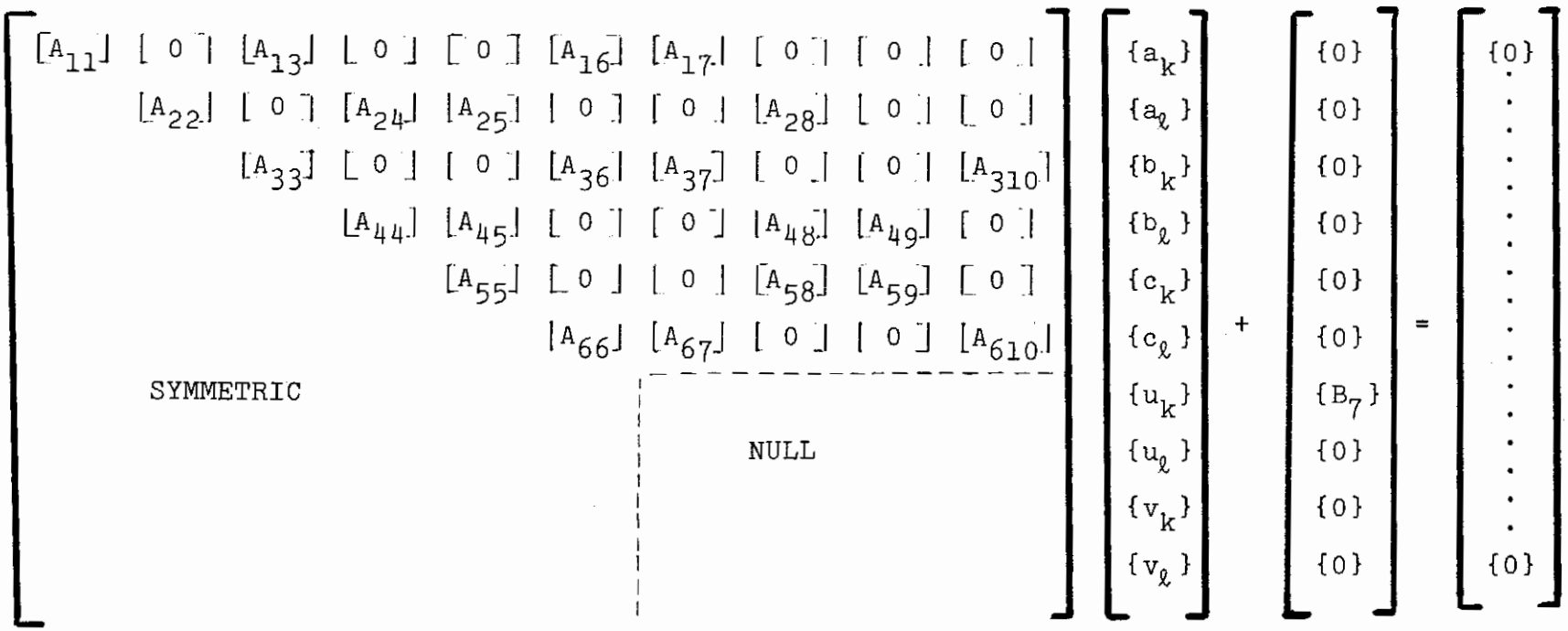


Figure 42. Pattern of Equations for Case I

Table VII. Submatrices for Case I

SUB-MATRIX	GENERAL ELEMENT EXPRESSION	ORDER	ROW AND COLUMN ADDRESSES (i_R AND i_C)	RANGE OF INDICES
$[A_{11}]$	$\frac{-1}{(i+p)}[S_{11}I_1+(2S_{12}+S_{66})I_3+S_{22}I_5]$	$n(m+1)$ x $n(m+1)$	$i_R = (p-1)(m+1)+q+1$ $i_C = (i-1)(m+1)+j+1$	$p=1,n;q=0,m$ $i=1,n;j=0,m$
$[A_{13}]$	$\frac{-1}{(i+p)}[S_{12}I_1+(S_{11}+S_{22}-S_{66})I_3-S_{12}I_5]$	$n(m+1)$ x $n(m+1)$	$i_R = (p-1)(m+1)+q+1$ $i_C = n(2m+1)+(i-1)(m+1)+j+1$	$p=1,n;q=0,m$ $i=1,n;j=0,m$
$[A_{16}]$	$\frac{-1}{(i+p)}[(-2S_{11}+2S_{12}+S_{66})I_7+(2S_{22}-2S_{12}-S_{66})I_9]$	$n(m+1)$ x nm	$i_R = (p-1)(m+1)+q+1$ $i_C = n(5m+3)+(i-1)m+j$	$p=1,n;q=0,m$ $i=1,n;j=1,m$
$[A_{17}]$	$\frac{-1}{(i+p)}[iI_{31}]$	$n(m+1)$ x $n(m+1)$	$i_R = (p-1)(m+1)+q+1$ $i_C = 3n(2m+1)+(i-1)(m+1)+j+1$	$p=1,n;q=0,m$ $i=1,n;j=0,m$
$[A_{22}]$	$\frac{-1}{(i+p)}[S_{11}I_{16}+(2S_{12}+S_{66})I_{18}+S_{22}I_{20}]$	nm x nm	$i_R = n(m+1)+(p-1)m+q$ $i_C = n(m+1)+(i-1)m+j$	$p=1,n;q=1,m$ $i=1,n;j=1,m$
$[A_{24}]$	$\frac{-1}{(i+p)}[S_{12}I_{16}+(S_{11}+S_{22}-S_{66})I_{18}+S_{12}I_{20}]$	nm x nm	$i_R = n(m+1)+(p-1)m+q$ $i_C = n(3m+2)+(i-1)m+j$	$p=1,n;q=1,m$ $i=1,n;j=1,m$
$[A_{25}]$	$\frac{-1}{(i+p)}[(-2S_{11}+2S_{12}+S_{66})I_{12}+(2S_{22}-2S_{12}-S_{66})I_{14}]$	nm x nm	$i_R = n(m+1)+(p-1)m+q$ $i_C = 2n(2m+1)+(i-1)(m+1)+j+1$	$p=1,n;q=1,m$ $i=1,n;j=0,m$

Table VII. Submatrices for Case I (continued)

SUB-MATRIX	GENERAL ELEMENT EXPRESSION	ORDER	ROW AND COLUMN ADDRESSES (i_R AND i_C)	RANGE OF INDICES
$[A_{28}]$	$\frac{-1}{(i+p)}[iI_{33}]$	nm x nm	$i_R = n(m+1)+(p-1)m+q$ $i_C = n(7m+4)+(i-1)m+j$	p=1,n;q=1,m i=1,n;j=1,m
$[A_{33}]$	$\frac{-1}{(i+p)}[S_{22}I_1+(2S_{12}+S_{66})I_3+S_{11}I_5]$	n(m+1) x n(m+1)	$i_R = n(2m+1)+(p-1)(m+1)+q+1$ $i_C = n(2m+1)+(i-1)(m+1)+j+1$	p=1,n;q=0,m i=1,n;j=0,m
$[A_{36}]$	$\frac{-1}{(i+p)}[(2S_{22}-2S_{12}-S_{66})I_7+(-2S_{11}+2S_{12}+S_{66})I_9]$	n(m+1) x nm	$i_R = n(2m+1)+(p-1)(m+1)+q+1$ $i_C = n(5m+3)+(i-1)m+j$	p=1,n;q=0,m i=1,n;j=1,m
$[A_{37}]$	$\frac{1}{(i+p)}[I_{31}]$	n(m+1) x n(m+1)	$i_R = n(2m+1)+(p-1)(m+1)+q+1$ $i_C = 3n(2m+1)+(i-1)(m+1)+j+1$	p=1,n;q=0,m i=1,n;j=0,m
$[A_{310}]$	$\frac{1}{(i+p)}[jI_{31}]$	n(m+1) x nm	$i_R = n(2m+1)+(p-1)(m+1)+q+1$ $i_C = n(9m+5)+(i-1)m+j$	p=1,n;q=0,m i=1,n;j=1,m

Table VIII. Submatrices for Case I

SUB-MATRIX	GENERAL ELEMENT EXPRESSION	ORDER	ROW AND COLUMN ADDRESSES (i_R AND i_C)	RANGE OF INDICES
$[A_{44}]$	$\frac{-1}{(i+p)}\left[S_{22}I_{16}+(2S_{12}+S_{66})I_{18}+S_{11}I_{20}\right]$	nm x nm	$i_R = n(3m+2)+(p-1)m+q$ $i_C = n(3m+2)+(i-1)m+j$	p=1,n;q=1,m i=1,n;j=1,m
$[A_{45}]$	$\frac{-1}{(i+p)}\left[(2S_{22}-2S_{12}-S_{66})I_{12}+(-2S_{11}+2S_{12}+S_{66})I_{14}\right]$	nm x n(m+1)	$i_R = n(3m+2)+(p-1)m+q$ $i_C = 2n(2m+1)+(i-1)(m+1)+j+1$	p=1,n;q=1,m i=1,n;j=0,m
$[A_{48}]$	$\frac{1}{(i+p)}\left[I_{33}\right]$	nm x nm	$i_R = n(3m+2)+(p-1)m+q$ $i_C = n(7m+4)+(i-1)m+j$	p=1,n;q=1,m i=1,n;j=1,m
$[A_{49}]$	$\frac{-1}{(i+p)}\left[jI_{33}\right]$	nm x n(m+1)	$i_R = n(3m+2)+(p-1)m+q$ $i_C = 4n(2m+1)+(i-1)(m+1)+j+1$	p=1,n;q=1,m i=1,n;j=0,m
$[A_{55}]$	$\frac{-1}{(i+p)}\left[S_{66}I_1+2\{2(S_{11}-2S_{12}+S_{22})-S_{66}\}I_3+S_{66}I_5\right]$	n(m+1) x n(m+1)	$i_R = 2n(2m+1)+(p-1)(m+1)+q+1$ $i_C = 2n(2m+1)+(i-1)(m+1)+j+1$	p=1,n;q=0,m i=1,n;j=0,m
$[A_{58}]$	$\frac{1}{(i+p)}\left[jI_{31}\right]$	n(m+1) x nm	$i_R = 2n(2m+1)+(p-1)(m+1)+q+1$ $i_C = n(7m+4)+(i-1)m+j$	p=1,n;q=0,m i=1,n;j=1,m

Table VIII. Submatrices for Case I (continued)

SUB-MATRIX	GENERAL ELEMENT EXPRESSION	ORDER	ROW AND COLUMN ADDRESSES (i_R AND i_C)	RANGE OF INDICES
$\left[A_{59}\right]$	$\frac{-1}{(i+p)}\left[(i+1)I_{31}\right]$	$n(m+1)$ x $n(m+1)$	$i_R = 2n(2m+1)+(p-1)(m+1)+q+1$ $i_C = 4n(2m+1)+(i-1)(m+1)+j+1$	p=1,n;q=0,m i=1,n;j=0,m
$\left[A_{66}\right]$	$\frac{-1}{(i+p)}\left[S_{66}I_{16}+2\{2(S_{11}-2S_{12}+S_{22})-S_{66}\}I_{18}+S_{66}I_{20}\right]$	nm x nm	$i_R = n(5m+3)+(p-1)m+q$ $i_C = n(5m+3)+(i-1)m+j$	p=1,n;q=1,m i=1,n;j=1,m
$\left[A_{67}\right]$	$\frac{-1}{(i+p)}\left[jI_{33}\right]$	nm x $n(m+1)$	$i_R = n(5m+3)+(p-1)m+q$ $i_C = 3n(2m+1)+(i-1)(m+1)+j+1$	p=1,n;q=1,m i=1,n;j=0,m
$\left[A_{610}\right]$	$\frac{-1}{(i+p)}\left[(i+1)I_{33}\right]$	nm x nm	$i_R = n(5m+3)+(p-1)m+q$ $i_C = n(9m+5)+(i-1)m+j$	p=1,n;q=1,m i=1,n;j=1,m
$\{B_7\}$	$-\sigma_o I_{34}$	$n(m+1)$ x 1	$i_R = 3n(2m+1)+(p-1)(m+1)+q+1$	p=1,n;q=0,m

Case II:

As demonstrated in Section VI, Article 3e Case II differs from Case I only with respect to the vector of constants {B}. Inspection of equation (6.3.29) shows that the only non-zero sub-vectors of the vector of constants will be $\{B_7\}$ and $\{B_{10}\}$. These are specified in Table IX

Case III:

The coefficient matrix $[A_{ij}]$ undergoes a few modifications in submatrices $[A_{17}]$, $[A_{28}]$, $[A_{59}]$ and $[A_{610}]$. The modified expressions are tabulated in Table X together with the single non-zero submatrix in the vector of constants {B}.

Case IV:

The coefficient matrix is identical to that for Case III. The vector of constants is shown for non-zero submatrices only in Table XI.

Table IX. Constant Vector Submatrices for Case II

SUB-MATRIX	GENERAL ELEMENT EXPRESSION	ORDER	ROW ADDRESS (i_R)	RANGE OF INDICES
$\{B_7\}$	$\frac{-\sigma_o}{2}\left[I_{34}+I_{35}\right]$	$n(m+1) \times 1$	$i_R = 3n(2m+1)+(p-1)(m+1)+q+1$	$p=1,n;q=0,m$
$\{B_{10}\}$	$\frac{\sigma_o}{2}\left[I_{36}\right]$	$nm \times 1$	$i_R = n(9m+5)+(p-1)m+q$	$p=1,n;q=1,m$

Table X. Submatrices Modified for Case III

SUB-MATRIX	GENERAL ELEMENT EXPRESSION	ORDER	ROW AND COLUMN ADDRESSES (i_R AND i_C)	RANGE OF INDICES
$\left[A_{17}\right]$	$\frac{1}{(i+p)}\left[pI_{31}\right]$	$n(m+1)$ x $n(m+1)$	$i_R = (p-1)(m+1)+q+1$ $i_C = 3n(2m+1)+(i-1)(m+1)+j+1$	$p=1,n;q=0,m$ $i=1,n;j=0,m$
$\left[A_{28}\right]$	$\frac{1}{(i+p)}\left[pI_{33}\right]$	nm x nm	$i_R = n(m+1)+(p-1)m+q$ $i_C = n(7m+4)+(i-1)m+j$	$p=1,n;q=1,m$ $i=1,n;j=1,m$
$\left[A_{59}\right]$	$\frac{1}{(i+p)}\left[(p-1)I_{31}\right]$	$n(n+1)$ x $n(m+1)$	$i_R = 2n(2m+1)+(p-1)(m+1)+q+1$ $i_C = 4n(2m+1)+(i-1)(m+1)+j+1$	$p=1,n;q=0,m$ $i=1,n;j=0,m$
$\left[A_{610}\right]$	$\frac{1}{(i+p)}\left[(p-1)I_{33}\right]$	nm x nm	$i_R = n(5m+3)+(p-1)m+q$ $i_C = n(9m+5)+(i-1)m+j$	$p=1,n;q=1,m$ $i=1,n;j=1,m$
$\{B_1\}$	$-u_o I_{34}$	$n(m+1)$x 1	$i_R = (p-1)(m+1)+q+1$	$p=1,n;q=0,m$

Table XI. Constant Submatrices for Case IV

SUB-MATRIX	GENERAL ELEMENT EXPRESSION	ORDER	ROW ADDRESS i_R	RANGE OF INDICES
B_1	$\frac{\sigma_o}{2}\left[(S_{11}+S_{12})I_{34}+(S_{11}-S_{12})I_{35}\right]$	$n(m+1)$ x 1	$i_R = (p-1)(m+1)+q+1$	$p=1,n;q=0,m$
B_6	$\frac{-\sigma_o}{2}\left[(S_{11}-S_{12})I_{33}\right]$	nm x 1	$i_R = n(5m+3)+(p-1)m+q$	$p=1,n;q=1,m$

APPENDIX IX

USER'S MANUAL FOR COMPUTER PROGRAMS

1. INTRODUCTION

The computer programs discussed in this manual consist of seven FORTRAN IV programs written to run on an IBM 7094 system under IBSYS/IBJOB. These programs are as follows:

(1) exact solution program
(2) perturbation solution program
(3) boundary Reissner energy program
(4) boundary least squares program
(5) potential energy program
(6) complementary energy program
(7) interior Reissner energy program

All of the programs give the user the option of generating a data tape for input to a contour plot program.[5] This plot program, in turn, will generate a plot tape for a CalComp plotter so the user may see the output from these programs in a graphic form. Considerable effort has been devoted to construct the programs in a uniform and consistent manner. This involves using the same subroutines where possible, using the same formats for input, using the same program input controls where applicable, using consistent notation of variable names in the programs, and using the same notation in the output from these programs. For this reason, and to avoid duplication of documentation, a thorough explanation of these program features will be given only once in articles preceding the discussions of the individual programs.

1a. Hardware Requirements

The following pieces of hardware are needed:

i) IBM 7094 computer
ii) two 7-track tape drives
iii) offline CalComp plotter
iv) two devices assigned for normal system input and output.

Item (iii) can be dispensed with if no plotting is to be done.

[5]This program was developed at the Flight Dynamics Laboratory at WPAFB and will be described in a future AFFDL Technical Report.

1b. Deck Set-up for All Programs

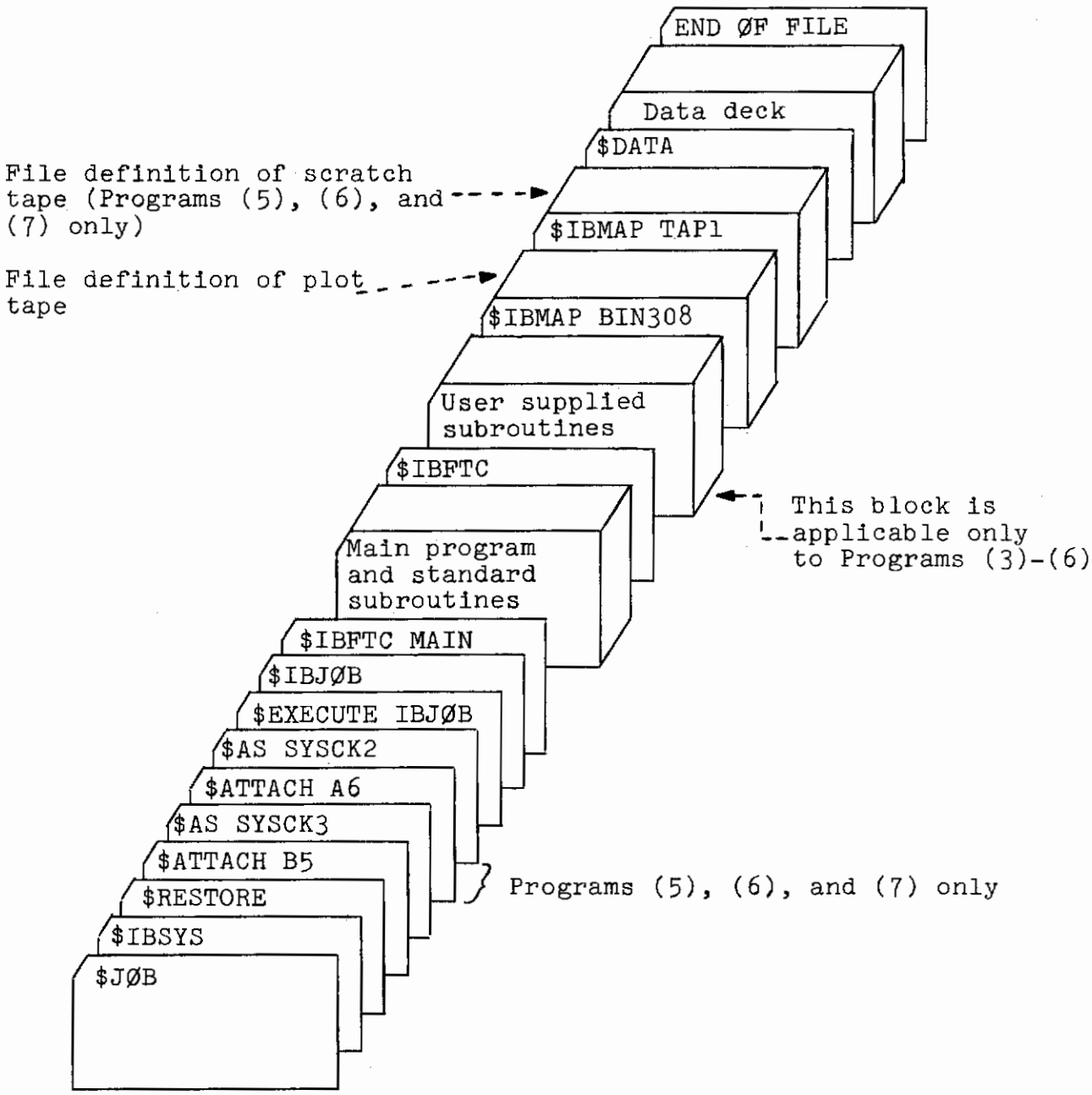


Figure 43. Deck Set-Up for Programs

1c. Special Program Controls

The following inputted program controls apply to all programs unless a program is specifically excluded.

i) IPR

This input code is used to specify whether the points of the mesh will be determined in polar (IPR≤0) or rectangular (IPR>0) coordinates. A discussion of these meshes is given below in Items (iii) and (iv).

ii) IPLOT

This input code determines whether a data tape for input to the plot program will be generated (IPLOT≠0) or will not be generated (IPLOT=0). If IPLOT=2 only the input will be printed but a plot tape will be generated. NOTE: If IPR ≤ 0 (polar coordinates) then no data tape is generated regardless of the value of IPLOT since the plot program requires a rectangular mesh of points.

iii) NR, NTR, NTB, DR, DTHR, and DTB

The purpose of these input variables is to systematically generate a mesh of data points in polar coordinates with minimal input. Using the variables NTB and DTB a pattern of data points is generated on the boundary. The remaining variables described in this paragraph are used to generate a set of rays. NR is the number of points of the mesh along each ray, including the point on the boundary (r=1.0). DR is the distance between successive points along each ray. NTR is the total number of rays (including the ray corresponding to θ=0). DTHR is the angle (in degrees) between successive rays. NTB is the total number of points on the boundary (including the point r=1., θ=0.). DTB is the angle (in degrees) between successive points on the boundary. (See Figure 44.) NOTE: If a point determined on the boundary by NTB and DTB also lies on one of the rays (i.e., if DTHR is an integer multiple of DTB), all computations will be repeated and again printed out.

iv) NPX, NPY, XSR, YST, DX and DY

The purpose of these input variables is to systematically generate a rectangular mesh of data points. NPX and NPY designate the number of points in the rectangular mesh in the

x-direction and y-direction respectively. XST and YST are the coordinates of the initial point in the rectangular mesh (this point must be one of the corners of the mesh, usually the lower left-hand corner). DX and DY are the increments in the x-direction and y-direction, respectively. (See Figure 44.) DX and DY need not be equal and may be positive or negative. However for plotting DX and DY must be positive. It should be noted that for plotting neither NPX nor NPY may exceed 85. It should also be noted that the solutions in programs (2) - (7) are singular at the origin. Ordinarily one would not compute solutions at the origin. However for plotting a rectangular mesh is required, so solutions at points within the hole, although physically meaningless, are usually required. In such situations the origin should be avoided.

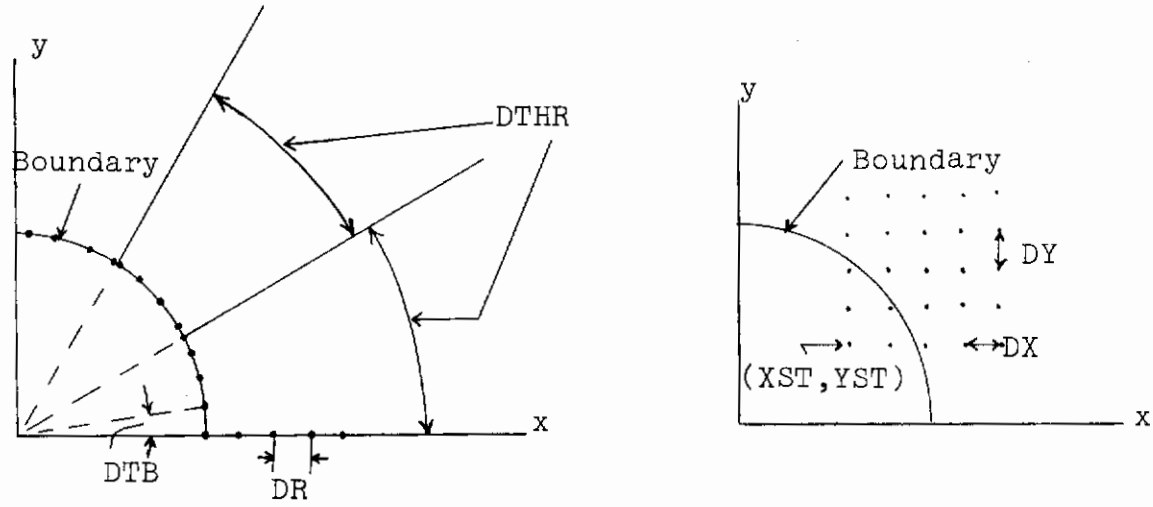


Figure 44. Polar and Rectangular Grids

v) NRUN

This input code allows the user to repeat the running of the entire program starting with a new input data set. This facility for stacking cases is limited in that there may not be more than one case which generates a plot tape, due to the difficulties involved in locating the position on the tape where data from subsequent runs would start.

vi) IDEV

This input code is relevant only to programs (2) - (7). It allows the user to compute deviations of stresses and displacements (obtained by the particular program) from the exact solution. If IDEV≤0 no deviations are computed. If IDEV>0 the deviations are computed. If the deviations are computed and IPLOT>0 then the deviations are written on the data tape as well as the print out.

vii) IRED, NBZXY, IBZXY

These input quantities are relevant only to programs (5), (6), and (7). They allow the user to eliminate rows (and the corresponding columns) from the coefficient matrix and right hand side vector. In effect this gives the user greater flexibility in choosing the assumed forms of the stresses and/or displacements since elimination of a row corresponds to the elimination of a term from the assumed form. If IRED≠0 then rows will be eliminated. NBZXY is the total number of rows to be eliminated, and IBZXY is an integer array containing the numbers of these rows.

1d. Additional Input Quantities

In addition to the special program controls described in the preceding section, the following quantities are input to all of the seven programs:

i) TITLE CARD

All programs have, as their first input quantity, the title card. In these programs this title card is the first item printed out and is also the first item written on the tape (whenever a tape is generated). When the plot program is run it first reads the title card and compares it to the one that is on tape. If they are not identical the plot program terminates ececution. This facility ensures that unwanted information will not be plotted.

ii) ICASE

This input code designates which of the sample problems will be solved. The four standard sample problems are numbered 1 through 4 and are described in Article 2b of Section III. Programs (3) - (6) allow the user to consider a variety of additional cases. (This feature will be discussed in the user's manual for the particular program.) These "non-standard" cases are denoted by integers greater than 4.

iii) EX, EY, VXY, GXY

These quantities are the material constants for a planar orthotropic material. EX and EY are the Young's moduli in the coordinate directions. VXY is Poisson's ratio and GXY is the shear modulus. As discussed at the beginning of Article 1e, EX, EY, and GXY should be scaled so that the maximum values of the stresses and displacements are of order unity.

iv) PHID

This quantity is the angle of tension, ϕ, measured in degrees from the x-axis, and is needed in Sample Problems 2 and 4 for all programs except (7). In program (7), this quantity is assumed to be zero.

1e. Description of Output

In all programs certain input quantities characterizing the problem are printed out. They are:

(i) the title card
(ii) the Case number
(iii) the material constants, EX,EY,VXY and GXY
(iv) the angle of tension, PHI (Cases II and IV only)
(v) information regarding the type (polar or rectangular) and fineness of the grid on which results are obtained
(vi) information as to whether a plot tape was generated.

Most of the programs print out additional input quantities, which, however, vary from program to program and are therefore described in the writeup for the particular program.

In all programs the complex roots of the characteristic equation corresponding to the generalized biharmonic are computed and printed out as MU1 and MU2. If the matrix row (and column) eliminating option is used then detailed information regarding which rows are to be eliminated is printed out.

The programs all have for output the stresses and displacements in both rectangular and polar coordines. In order to improve readability of this output, a decision was made to use F formats. Provision has been made for two significant figures to the left of the decimal point. In order that the output be conformable with these formats it is necessary that the Young's moduli in the coordinate directions (E_x and E_y) and the shear modulus (G_{xy}) be appropriately scaled, preferably so that the expected maximum values of the stresses and displacements are of order unity. For each point of the (polar or rectangular) mesh the quantities listed in Table XII are printed out. (The variable codes in the table indicate the position of the particular variable on the tape record whenever an input tape to the plot program is generated.)

Table XII. Print-Out Notation and Description

Variable Notation	Variable Code	Description of Variable
X	-	x-coordinate of the data point
Y	-	y-coordinate of the data point
SIG X	1	stress in the x-direction (σ_x)
SIG Y	2	stress in the y-direction (σ_y)
T XY	3	shear stress in rectangular coordinates (τ_{xy})
UX	4	displacement in the x-direction (u_x)
UY	5	displacement in the y-direction (u_y)
R	-	radial coordinate of the data point
THETA	-	angular coordinate of the data point
SIG R	6	radial stress (σ_r)
SIG T	7	circumferential stress (σ_θ)
T RT	8	shear stress in polar coordinates ($\tau_{r\theta}$)
UR	9	displacement in the r-direction (u_r)
UT	10	displacement in the θ-direction (u_θ)

Programs (2) - (7), when deviations are computed, all have the following additional output:

Table XIII. Print-Out Notation and Description

Variable Notation	Variable Code	Description of Variable
DSG X	11	deviation of σ_x from the exact solution
DSG Y	12	deviation of σ_y from the exact solution
DTXY	13	deviation of τ_{xy} from the exact solution
DUX	14	deviation of u_x from the exact solution
DUY	15	deviation of u_y from the exact solution
DSG R	16	deviation of σ_r from the exact solution
DSG T	17	deviation of σ_θ from the exact solution
DTRT	18	deviation of $\tau_{r\theta}$ from the exact solution
DUR	19	deviation of u_r from the exact solution
DUT	20	deviation of u_θ from the exact solution

In programs (3) - (7) it is necessary to solve a system of linear equations. In order to obtain the condition number of the associated matrix an inversion routine (as opposed to a direct equation solver) has been employed. The inversion technique is known as "the method of submatrices" or "inversion by partitioning", and is described, eg., by Faddeeva [32] , p.102ff. and Frazer, Duncan, and Collar [33], p.112ff. The inversion is done in double precision even though the coefficient matrices (with the exception of interior Reissner energy) are formed in single precision. There are a number of quantities associated with the matrix system and its conditioning which are printed out. Most of these are listed in Table XIV. (A detailed discussion of the significance of these quantities is given in Appendix III.)

Table XIV. Print-Out Notation and Description

Variable Notation	Description
COEF	coefficient matrix
COEFI	inverse of coefficient matrix
RT	right hand side vector
SOL	solution vector
DRC	product of coefficient matrix and solution vector
RERR	relative error of inversion process
CNMT	condition number of coefficient matrix
UBSV	upper bound of relative error of solution vector
NCOEF	Euclidean norm of COEF
NCOEFI	Euclidean norm of COEFI
NRHS	Euclidean norm of RT
NRCK	Euclidean norm of DRC
NSOL	Euclidean norm of SOL

We note that the upper bound of the relative error of the solution vector is highly dependent on the accuracy of the elements of the coefficient matrix. The maximum percentage error in these elements is designated by ε. (See Appendix III.) ε depends on machine precision. This parameter is defined (as EPS) at the beginning of subroutine SOLVE. Depending on the particular computer being used it may be necessary to appropriately redefine EPS. In the case of some of the variables of Table XIV, the description of the variable rather than the variable name has been printed out. For these cases the computer symbol has been listed and should not be considered print-out notation. The descriptions of the variables RERR, CNMT, and UBSV above are expanded on considerably in Appendix III. There are two messages related to the present discussion which may be printed out. One indicates that an assumption underlying the calculation of UBSV has been violated. The other indicates that the coefficient matrix is singular, or at least appears so to the inversion routine being used.

In addition to the printed output described in the preceding paragraphs there may also be tape output for use as input to the contour plot program. As discussed in Article 1c of this appendix, this will occur if the input code IPLOT equals 1 or 2 and the input code IPR is greater than 0. The first thing written on the tape is the title card. Also written initially are the case number (ICASE), the elastic constants (EX, EY, VXY, GXY), the angle of tension (PHI) in Cases II and IV, the coordinates (NPX, NPY) of the lower left corner of the rectangular mesh, the increments (DX, DY) between successive mesh points in

the coordinate directions, and the number (ICNT) of possible items (stresses, displacements, and deviations) on the tape record corresponding to each mesh point. All quantities in Table XII (except X, Y, R, THETA) are written on the tape at each mesh point. Their position in the record is indicated by the variable code listed in the table. If deviations are computed all items in Table XIII are also written on the tape.

2. EXACT SOLUTION

This program computes the exact solution for the four sample problems being considered in this study. For a detailed discussion of the mathematical model and a description of the four sample problems see Section III. It should be noted that this formulation is not valid for isotropic materials or materials equivalent to isotropic materials, ie., materials for which the roots of the characteristic equation are equal. For a discussion of this point see Article 1a of Section III.

2a. Description of Input

All of the input quantities have been thoroughly described in Article 1c and 1d. Thus, we confine our description of input to the listing of detailed input format sheets in Table XV.

2b. Description of Output

All of the output for the exact solution program has been described in Article 1e of this Appendix.

2c. Sample Case

In Table XVI the output from a run of the exact solution program is shown. All input quantities used for this run can be found on the output. This run illustrates the use of a rectangular mesh over which stresses and displacements are computed. Of course, stresses and displacements for $r < 1$ have no physical meaning. However, for contour plotting purposes these quantities must be defined at every point of the mesh.

Table XV. Input Format Sheet for Exact Solution Programs

Card 1 - Format 12A6

TITLE CARD

Card 2 - Format I5

ICASE

ICASE - 1,2,3, or 4 designating the desired problem to be run

Card 3 - Format 5E10.0

If ICASE is 1 or 3 the following data card is used.

EX | EY | VXY | GXY

If ICASE is 2 or 4 the following data card is used.

EX | EY | VXY | GXY | PHID

EX - Young's modulus in the x-direction
EY - Young's modulus in the y-direction
VXY - Poisson's ratio
GXY - Shear modulus
PHID - Angle of stress W.R.T. the x-axis (in degrees)

Table XV. (Continued)

Card 4 - Format 2I5

IPR	IPLOT

IPR ≤ 0 - Data points will be determined in polar coordinates (no plots)
IPR > 0 - Data points will be determined in rectangular coordinates

IPLOT = 0 - No plot tape will be generated
IPLOT = 1 - Plot tape generated and all data printed
IPLOT = 2 - Plot tape generated and only input printed

If IPR ≤ 0 - (Polar coordinates) use the following 2 cards

Card 5p - Format 3I5

NR	NTR	NTB

NR - Number of points along each ray (including the point on the boundary)
NTR - Number of rays (including first ray where $\theta = 0.$)
NTB - Number of points (first point is $r = 1.$, $\theta = 0.$) on the boundary

Table XV. (Continued)

Card 6p - Format 3E10.0

DR	DTHR	DTB

DR - Distance between points on each ray (first point is r = 1.)
DTHR - Angle between rays (first ray is for θ = 0.) in degrees
DTB - Angle between points on the boundary in degrees

If IPR > 0 (Rectangular coordinates) use the following 2 cards

Card 5_R - Format 2I5

NPX	NPY

NPX - Number of points in the x-direction (including XST)
NPY - Number of points in the y-direction (including YST)

Card 6_R - Format 4E10.0

XST	YST	DX	DY

XST - x-coordinate of the initial point of the mesh
YST - y-coordinate of the initial point of the mesh
DX - distance between successive points in the x-direction
DY - distance between successive points in the y-direction

Table XV. (Continued)

Conclude input with the following card

Card 7 - Format I5

NRUN

NRUN $\leq$ 0 - Call exit - terminate execution

NRUN $>$ 0 - Read in a new case beginning with CARD 1

XVI. SAMPLE CASE OF EXACT SOLUTION PROGRAM

```
                    SAMPLE CASE  EXACT SOLUTION

 THE PROBLEM BEING SOLVED IS CASE  1

 CASE 1  -  FORCE BOUNDARY CONDITIONS, UNIFORM PRESSURE


EX  =  1.50000     EY  =  1.00000     VXY =  0.37500     GXY =  0.40000

MU1  =  0.0            0.79623                    MU2  =  0.0            1.53819

DATA POINTS WILL BE GENERATED IN RECTANGULAR COORDINATES

THE RECTANGULAR MESH WILL BE     6 (POINTS IN THE X-DIRECTION) BY     6 (POINTS IN THE Y-DIRECTION) STARTING AT THE
POINT (   0.0 ,   0.0 ), THEN INCREMENTED IN THE Y-DIRECTION BY 0.200 AND IN THE X-DIRECTION BY 0.400

NO TAPE FOR PLOTTING PURPOSES WILL BE GENERATED
```

X = 0.0	Y = 0.0	SIG X= 1.04929	SIG Y= -3.04929	T XY= 0.0	UX = -1.08989	UY = 3.13330
R = 0.0	THET= 0.0	SIG R= 1.04929	SIG T= -3.04929	T RT= 0.0	UR = -1.08989	UT = 3.13330
X = 0.0	Y = 0.400	SIG X= -0.00194	SIG Y= -1.39114	T XY= 0.48602	UX = -0.97665	UY = 2.21640
R = 0.400	THET= 90.000	SIG R= -1.39114	SIG T= -0.00194	T RT= -0.48602	UR = 2.21640	UT = 0.92665
X = 0.0	Y = 0.800	SIG X= 3.27209	SIG Y= -2.09936	T XY= 0.00000	UX = -0.00000	UY = 1.69733
R = 0.800	THET= 90.000	SIG R= -2.09936	SIG T= 3.27209	T RT= -0.00000	UR = 1.69733	UT = 0.00000
X = 0.0	Y = 1.200	SIG X= 0.70265	SIG Y= -0.70121	T XY= 0.0	UX = 0.0	UY = 1.12888
R = 1.200	THET= 90.000	SIG R= -0.70121	SIG T= 0.70265	T RT= -0.00000	UR = 1.12888	UT = 0.00000
X = 0.0	Y = 1.600	SIG X= 0.38277	SIG Y= -0.41373	T XY= 0.00000	UX = -0.00000	UY = 0.86211
R = 1.600	THET= 90.000	SIG R= -0.41373	SIG T= 0.38277	T RT= -0.00000	UR = 0.86211	UT = 0.00000
X = 0.0	Y = 2.000	SIG X= 0.24499	SIG Y= -0.27342	T XY= 0.0	UX = 0.0	UY = 0.69695
R = 2.000	THET= 90.000	SIG R= -0.27342	SIG T= 0.24499	T RT= -0.00000	UR = 0.69695	UT = 0.00000

X = 0.200	Y = 0.0	SIG X= 1.25294	SIG Y= -3.13536	T XY= -0.99276	UX = 1.39811	UY = 2.95722
R = 0.200	THET= 0.0	SIG R= 1.25294	SIG T= -3.13536	T RT= -0.99276	UR = 1.39811	UT = 2.95722
X = 0.200	Y = 0.400	SIG X= 0.20493	SIG Y= -1.35016	T XY= -1.09357	UX = 1.01348	UY = 2.03446
R = 0.447	THET= 63.435	SIG R= -1.41396	SIG T= 0.76873	T RT= 0.03407	UR = 2.27292	UT = 0.00335
X = 0.200	Y = 0.800	SIG X= 1.19787	SIG Y= -1.15413	T XY= -0.97317	UX = 0.35574	UY = 1.55376
R = 0.825	THET= 75.964	SIG R= -1.47374	SIG T= 1.51748	T RT= 0.30527	UR = 1.58365	UT = 0.03173
X = 0.200	Y = 1.200	SIG X= 0.64905	SIG Y= -0.64976	T XY= -0.20638	UX = 0.12545	UY = 1.09841
R = 1.217	THET= 80.538	SIG R= -0.68160	SIG T= 0.68088	T RT= -0.01539	UR = 1.10409	UT = 0.05694
X = 0.200	Y = 1.600	SIG X= 0.36671	SIG Y= -0.39416	T XY= -0.09234	UX = 0.07067	UY = 0.84719
R = 1.612	THET= 82.875	SIG R= -0.40519	SIG T= 0.37773	T RT= -0.00415	UR = 0.84962	UT = 0.03499
X = 0.200	Y = 2.000	SIG X= 0.23797	SIG Y= -0.26421	T XY= -0.05021	UX = 0.04587	UY = 0.68875
R = 2.010	THET= 84.289	SIG R= -0.26918	SIG T= 0.24294	T RT= -0.00051	UR = 0.68889	UT = 0.02280

X = 0.400	Y = 0.0	SIG X= 1.44025	SIG Y= -3.21453	T XY= -2.49768	UX = 1.73670	UY = 2.35082
R = 0.400	THET= 0.0	SIG R= 1.44025	SIG T= -3.21453	T RT= -2.49768	UR = 1.73670	UT = 2.35082
X = 0.400	Y = 0.400	SIG X= 0.05201	SIG Y= -0.83288	T XY= -1.57014	UX = 1.09229	UY = 1.63546
R = 0.566	THET= 45.000	SIG R= -1.96057	SIG T= 1.17971	T RT= -0.44245	UR = 1.92881	UT = 0.38407
X = 0.400	Y = 0.800	SIG X= 0.58892	SIG Y= -0.70802	T XY= -0.97618	UX = 0.41580	UY = 1.32799

R =	0.894	THET=	63.435	SIG R=	-1.22958	SIG T=	1.11047	T RT=	0.06693	UR =	1.41846	UT =	0.13755
X =	0.400	Y =	1.200	SIG X=	0.51480	SIG Y=	-0.51557	T XY=	-0.35795	UX =	0.23304	UY =	1.01470
R =	1.265	THET=	71.565	SIG R=	-0.62731	SIG T=	0.62653	T RT=	-0.02275	UR =	1.03632	UT =	0.08983
X =	0.400	Y =	1.600	SIG X=	0.32759	SIG Y=	-0.34053	T XY=	-0.16949	UX =	0.13538	UY =	0.80681
R =	1.649	THET=	75.964	SIG R=	-0.38125	SIG T=	0.36332	T RT=	-0.00653	UR =	0.81458	UT =	0.06419
X =	0.400	Y =	2.000	SIG X=	0.21822	SIG Y=	-0.23843	T XY=	-0.09434	UX =	0.08803	UY =	0.66520
R =	2.040	THET=	78.690	SIG R=	-0.25715	SIG T=	0.23694	T RT=	-0.00073	UR =	0.66974	UT =	0.04315
X =	0.600	Y =	0.0	SIG X=	1.60070	SIG Y=	-3.28234	T XY=	-21.90768	UX =	2.10223	UY =	0.40202
R =	0.600	THET=	0.0	SIG R=	1.60070	SIG T=	-3.28234	T RT=	-21.90768	UR =	2.10223	UT =	0.40202
X =	0.600	Y =	0.400	SIG X=	-0.44280	SIG Y=	0.12966	T XY=	-1.56935	UX =	1.08586	UY =	1.11416
R =	0.721	THET=	33.690	SIG R=	-1.71529	SIG T=	1.40715	T RT=	-0.33938	UR =	1.52151	UT =	0.32471
X =	0.600	Y =	0.800	SIG X=	0.22473	SIG Y=	-0.31109	T XY=	-0.81855	UX =	0.59387	UY =	1.07164
R =	1.000	THET=	53.130	SIG R=	-1.00000	SIG T=	0.91364	T RT=	-0.00000	UR =	1.21363	UT =	0.16789
X =	0.600	Y =	1.200	SIG X=	0.35258	SIG Y=	-0.34487	T XY=	-0.43124	UX =	0.31245	UY =	0.89711
R =	1.342	THET=	63.435	SIG R=	-0.55045	SIG T=	0.55806	T RT=	-0.02028	UR =	0.94213	UT =	0.12173
X =	0.600	Y =	1.600	SIG X=	0.26088	SIG Y=	-0.26594	T XY=	-0.22116	UX =	0.18962	UY =	0.74416
R =	1.709	THET=	69.444	SIG R=	-0.34641	SIG T=	0.34135	T RT=	-0.00659	UR =	0.76335	UT =	0.08375
X =	0.600	Y =	2.000	SIG X=	0.18924	SIG Y=	-0.20094	T XY=	-0.12795	UX =	0.12729	UY =	0.62920
R =	2.088	THET=	73.301	SIG R=	-0.23916	SIG T=	0.22745	T RT=	-0.00057	UR =	0.63924	UT =	0.05987
X =	0.800	Y =	0.0	SIG X=	-1.71784	SIG Y=	2.10294	T XY=	-0.00000	UX =	1.23697	UY =	0.00000
R =	0.800	THET=	0.0	SIG R=	-1.71784	SIG T=	2.10294	T RT=	-0.00000	UR =	1.23697	UT =	0.00000
X =	0.800	Y =	0.400	SIG X=	-0.70257	SIG Y=	0.66011	T XY=	-1.01649	UX =	0.98112	UY =	0.69366
R =	0.894	THET=	26.565	SIG R=	-1.24322	SIG T=	1.20076	T RT=	-0.06482	UR =	1.18775	UT =	0.18166
X =	0.800	Y =	0.800	SIG X=	-0.01517	SIG Y=	-0.01194	T XY=	-0.76721	UX =	0.61395	UY =	0.93110
R =	1.131	THET=	45.000	SIG R=	-0.78076	SIG T=	0.75366	T RT=	0.00162	UR =	1.02180	UT =	0.15356
X =	0.800	Y =	1.200	SIG X=	0.20505	SIG Y=	-0.18489	T XY=	-0.43495	UX =	0.36243	UY =	0.76928
R =	1.442	THET=	56.310	SIG R=	-0.46640	SIG T=	0.48656	T RT=	-0.01268	UR =	0.84029	UT =	0.12460
X =	0.800	Y =	1.600	SIG X=	0.19413	SIG Y=	-0.18630	T XY=	-0.24525	UX =	0.23124	UY =	0.67128
R =	1.789	THET=	63.435	SIG R=	-0.30641	SIG T=	0.31424	T RT=	-0.00502	UR =	0.70382	UT =	0.09338
X =	0.800	Y =	2.000	SIG X=	0.15554	SIG Y=	-0.15804	T XY=	-0.14913	UX =	0.15929	UY =	0.58471
R =	2.154	THET=	68.199	SIG R=	-0.21764	SIG T=	0.21514	T RT=	-0.00014	UR =	0.60205	UT =	0.06926

X =	1.000	Y =	0.0	SIG X=	-1.00000	SIG Y=	1.08955	T XY=	-0.00000	UX =	0.88878
R =	1.000	THET=	0.0	SIG R=	-1.00000	SIG T=	1.08955	T RT=	-0.00000	UR =	0.88878
X =	1.000	Y =	0.400	SIG X=	-0.63584	SIG Y=	0.63936	T XY=	-0.58295	UX =	0.85669
R =	1.077	THET=	21.801	SIG R=	-0.86198	SIG T=	0.86551	T RT=	0.01758	UR =	0.86354
X =	1.000	Y =	0.800	SIG X=	-0.13939	SIG Y=	0.14607	T XY=	-0.58714	UX =	0.59863
R =	1.241	THET=	38.660	SIG R=	-0.60082	SIG T=	0.60749	T RT=	0.01036	UR =	0.88477
X =	1.000	Y =	1.200	SIG X=	0.09288	SIG Y=	-0.06337	T XY=	-0.39522	UX =	0.38788
R =	1.562	THET=	50.194	SIG R=	-0.38807	SIG T=	0.41758	T RT=	-0.00557	UR =	0.74461
X =	1.000	Y =	1.600	SIG X=	0.13247	SIG Y=	-0.11447	T XY=	-0.24628	UX =	0.26034
R =	1.887	THET=	57.995	SIG R=	-0.26648	SIG T=	0.28448	T RT=	-0.00306	UR =	0.64300
X =	1.000	Y =	2.000	SIG X=	0.12142	SIG Y=	-0.11557	T XY=	-0.15854	UX =	0.18457
R =	2.236	THET=	63.435	SIG R=	-0.19500	SIG T=	0.20085	T RT=	0.00033	UR =	0.46196

X =	1.200	Y =	0.0	SIG X=	-0.69887	SIG Y=	0.70693	T XY=	0.0	UX =	0.83495
R =	1.200	THET=	0.0	SIG R=	-0.69887	SIG T=	0.70693	T RT=	0.0	UR =	0.83495
X =	1.200	Y =	0.400	SIG X=	-0.51644	SIG Y=	0.51346	T XY=	-0.35784	UX =	0.75111
R =	1.265	THET=	18.435	SIG R=	-0.62816	SIG T=	0.62518	T RT=	0.02270	UR =	0.81326
X =	1.200	Y =	0.800	SIG X=	-0.18606	SIG Y=	0.20219	T XY=	-0.43503	UX =	0.56731
R =	1.442	THET=	33.690	SIG R=	-0.46817	SIG T=	0.48430	T RT=	0.01187	UR =	0.74626
X =	1.200	Y =	1.200	SIG X=	0.01775	SIG Y=	0.01501	T XY=	-0.33844	UX =	0.38502
R =	1.697	THET=	45.000	SIG R=	-0.32206	SIG T=	0.35482	T RT=	-0.00137	UR =	0.66144
X =	1.200	Y =	1.600	SIG X=	0.08143	SIG Y=	-0.05719	T XY=	-0.23208	UX =	0.27873
R =	2.000	THET=	53.130	SIG R=	-0.23008	SIG T=	0.25432	T RT=	-0.00155	UR =	0.59565
X =	1.200	Y =	2.000	SIG X=	0.08999	SIG Y=	-0.07763	T XY=	-0.15850	UX =	0.20342
R =	2.332	THET=	59.036	SIG R=	-0.17312	SIG T=	0.18548	T RT=	0.00064	UR =	0.52160

X =	1.400	Y =	0.0	SIG X=	-0.52626	SIG Y=	0.50617	T XY=	-0.00000	UX =	0.72430
R =	1.400	THET=	0.0	SIG R=	-0.52626	SIG T=	0.50617	T RT=	-0.00000	UR =	0.72430
X =	1.400	Y =	0.400	SIG X=	-0.41728	SIG Y=	0.40500	T XY=	-0.23751	UX =	0.66631
R =	1.456	THET=	15.945	SIG R=	-0.48069	SIG T=	0.46842	T RT=	0.01555	UR =	0.70583
X =	1.400	Y =	0.800	SIG X=	-0.19411	SIG Y=	0.20834	T XY=	-0.32321	UX =	0.53131
R =	1.612	THET=	29.745	SIG R=	-0.37350	SIG T=	0.38774	T RT=	0.00927	UR =	0.65517
X =	1.400	Y =	1.200	SIG X=	-0.02827	SIG Y=	0.05932	T XY=	-0.28146	UX =	0.39208
R =	1.844	THET=	40.601	SIG R=	-0.26932	SIG T=	0.30037	T RT=	0.00024	UR =	0.59241
X =	1.400	Y =	1.600	SIG X=	0.04228	SIG Y=	-0.01553	T XY=	-0.21013	UX =	0.28861
R =	2.126	THET=	48.814	SIG R=	-0.19874	SIG T=	0.22548	T RT=	-0.00075	UR =	0.53416
X =	1.400	Y =	2.000	SIG X=	0.06295	SIG Y=	-0.04627	T XY=	-0.15198	UX =	0.21663

3. PERTURBATION SOLUTION

This program computes approximate solutions to the four sample problems of this study. A detailed discussion of the mathematical model is given in Section IV. The method is based on the assumption that the material being considered is only mildly orthotropic and therefore that the solution to the same problem for a "near" isotropic material furnishes a good first approximately to the orthotropic problem. Thus, one should not attempt to use this program forhighly orthotropic materials.

3a. Description of Input

The input is identical to that for the exact solution program described in Article 2a of this appendix with the following exception: On card 4 of Table XV the quantity IDEV should be added. The modified format for this input is 3I5. If IDEV > 0 then deviations from the exact solution will be computed.

3b. Description of Output

Most of the output for this program has already been described in Article 1e of this appendix. Also printed out are the quantities SZ11, SZ12, EPS1, and EPS2 characterizing the nearest isotropic material and the deviation of this material from the given orthotropic material, respectively.

3c. Sample Case

In Table XVII the output from a run of the perturbation solution is shown. For this run a polar mesh is used. The material for this run is the mildly orthotropic Material (1). (See Table I.) As can be seen the deviations for both stresses and displacements are small.

XVII. SAMPLE CASE OF PERTURBATION SOLUTION PROGRAM

```
                    SAMPLE CASE  PERTURBATION SOLUTION

                THE PROBLEM BEING SOLVED IS CASE  1

EX  =  1.50000    EY  =  1.00000    VXY =  0.37500    GXY =  0.40000

SZ11 (=SZ22) =    0.87499976   SZ12 =  -0.29166675   SZ66 =    2.33333206   EPS1 =    0.023809S7   EPS2 =   -0.0952380S

MU1  =  0.0          0.79623                    MU2  =  0.0          1.53819

DATA POINTS WILL BE GENERATED IN POLAR COORDINATES

ON THE BOUNDARY THERE WILL BE   15 POINTS WITH THETA INCREMENTED BY  5.0 DEGREES STARTING AT 0.  THERE WILL BE    3
RAYS 45.0 DEGREES APART BEGINNING AT THETA = 0. WITH    6 POINTS A DISTANCE  0.20 APART ON EACH RAY.

NO TAPE FOR PLOTTING PURPOSES WILL BE GENERATED
```

XVII. (continued)

X = 1.000	Y = 0.0	SIG X= -1.00000	SIG Y= 1.00843	T XY= 0.0	UX = 0.98611	UY = 0.0	
R = 1.000	THET= 0.0	SIG R= -1.00000	SIG T= 1.00843	T RT= 0.0	UR = 0.98611	UT = 0.0	
		DSG X= -0.00000	DSG Y= 0.00111	DTXY= -0.00000	DUX= 0.00367	DUY= 0.00000	
		DSG R= -0.00000	DSG T= 0.00111	DTRT= -0.00000	DUR= 0.00367	DUT= 0.00000	
X = 0.996	Y = 0.087	SIG X= -0.98417	SIG Y= 1.04755	T XY= -0.18089	UX = 0.98268	UY = 0.11570	
R = 1.000	THET= 5.000	SIG R= -1.00000	SIG T= 1.00337	T RT= 0.00000	UR = 0.98882	UT = 0.02963	
		DSG X= 0.00008	DSG Y= 0.00101	DTXY= -0.00009	DUX= 0.00354	DUY= 0.00105	
		DSG R= -0.00000	DSG T= 0.00102	DTRT= -0.00000	DUR= 0.00361	DUT= 0.00074	
X = 0.985	Y = 0.174	SIG X= -0.93762	SIG Y= 1.00643	T XY= -0.35379	UX = 0.97160	UY = 0.23058	
R = 1.000	THET= 10.000	SIG R= -1.00000	SIG T= 1.00881	T RT= -0.00000	UR = 0.99688	UT = 0.05836	
		DSG X= 0.00002	DSG Y= 0.00074	DTXY= -0.00013	DUX= 0.00314	DUY= 0.00203	
		DSG R= -0.00000	DSG T= 0.00076	DTRT= 0.00000	DUR= 0.00345	DUT= 0.00145	
X = 0.966	Y = 0.259	SIG X= -0.86291	SIG Y= 0.98940	T XY= -0.51162	UX = 0.95394	UY = 0.34383	
R = 1.000	THET= 15.000	SIG R= -1.00000	SIG T= 1.04648	T RT= 0.0	UR = 1.01003	UT = 0.08532	
		DSG X= 0.00002	DSG Y= 0.00034	DTXY= -0.00009	DUX= 0.00252	DUY= 0.00287	
		DSG R= -0.00000	DSG T= 0.00036	DTRT= 0.0	DUR= 0.00318	DUT= 0.00212	
X = 0.940	Y = 0.342	SIG X= -0.76383	SIG Y= 0.78277	T XY= -0.64888	UX = 0.92839	UY = 0.45463	
R = 1.000	THET= 20.000	SIG R= -1.00000	SIG T= 1.01894	T RT= -0.00000	UR = 1.02789	UT = 0.10968	
		DSG X= -0.00001	DSG Y= -0.00008	DTXY= 0.00003	DUX= 0.00170	DUY= 0.00353	
		DSG R= -0.00000	DSG T= -0.00009	DTRT= -0.00000	DUR= 0.00281	DUT= 0.00273	
X = 0.906	Y = 0.423	SIG X= -0.64472	SIG Y= 0.63391	T XY= -0.76190	UX = 0.89629	UY = 0.56217	
R = 1.000	THET= 25.000	SIG R= -1.00000	SIG T= 0.98919	T RT= 0.00000	UR = 1.04990	UT = 0.13071	
		DSG X= -0.00009	DSG Y= -0.00041	DTXY= 0.00019	DUX= 0.00076	DUY= 0.00394	
		DSG R= -0.00000	DSG T= -0.00050	DTRT= 0.00000	DUR= 0.00235	DUT= 0.00325	
X = 0.866	Y = 0.500	SIG X= -0.50992	SIG Y= 0.47024	T XY= -0.84884	UX = 0.85743	UY = 0.66567	
R = 1.000	THET= 30.000	SIG R= -1.00000	SIG T= 0.96032	T RT= -0.00000	UR = 1.07540	UT = 0.14777	
		DSG X= -0.00018	DSG Y= -0.00054	DTXY= 0.00031	DUX= -0.00026	DUY= 0.00410	
		DSG R= -0.00000	DSG T= -0.00072	DTRT= -0.00000	DUR= 0.00182	DUT= 0.00368	
X = 0.819	Y = 0.574	SIG X= -0.36331	SIG Y= 0.29860	T XY= -0.90929	UX = 0.81205	UY = 0.76435	
R = 1.000	THET= 35.000	SIG R= -1.00000	SIG T= 0.93529	T RT= -0.00000	UR = 1.10361	UT = 0.16034	
		DSG X= -0.00022	DSG Y= -0.00046	DTXY= 0.00032	DUX= -0.00127	DUY= 0.00398	
		DSG R= -0.00000	DSG T= -0.00068	DTRT= -0.00000	DUR= 0.00124	DUT= 0.00399	
X = 0.766	Y = 0.643	SIG X= -0.20803	SIG Y= 0.12481	T XY= -0.94383	UX = 0.76043	UY = 0.85744	
R = 1.000	THET= 40.000	SIG R= -1.00000	SIG T= 0.91678	T RT= -0.00000	UR = 1.13367	UT = 0.16804	
		DSG X= -0.00017	DSG Y= -0.00025	DTXY= 0.00021	DUX= -0.00221	DUY= 0.00360	
		DSG R= -0.00000	DSG T= -0.00042	DTRT= 0.00000	DUR= 0.00062	DUT= 0.00418	
X = 0.707	Y = 0.707	SIG X= -0.04649	SIG Y= -0.04648	T XY= -0.95351	UX = 0.70290	UY = 0.94421	
R = 1.000	THET= 45.000	SIG R= -1.00000	SIG T= 0.90703	T RT= -0.00000	UR = 1.16468	UT = 0.17063	
		DSG X= -0.00006	DSG Y= -0.00006	DTXY= 0.00006	DUX= -0.00302	DUY= 0.00299	
		DSG R= -0.00000	DSG T= -0.00012	DTRT= -0.00000	DUR= -0.00002	DUT= 0.00429	
X = 0.643	Y = 0.766	SIG X= 0.11949	SIG Y= -0.21178	T XY= -0.93936	UX = 0.63985	UY = 1.02397	
R = 1.000	THET= 50.000	SIG R= -1.00000	SIG T= 0.90771	T RT= -0.00000	UR = 1.19569	UT = 0.16804	
		DSG X= -0.00003	DSG Y= -0.00002	DTXY= 0.00002	DUX= -0.00363	DUY= 0.00218	
		DSG R= -0.00000	DSG T= -0.00004	DTRT= -0.00000	DUR= -0.00066	DUT= 0.00418	
X = 0.574	Y = 0.819	SIG X= 0.28806	SIG Y= -0.36848	T XY= -0.90191	UX = 0.57172	UY = 1.09605	
R = 1.000	THET= 55.000	SIG R= -1.00000	SIG T= 0.91958	T RT= 0.00000	UR = 1.22576	UT = 0.16034	
		DSG X= -0.00024	DSG Y= -0.00012	DTXY= 0.00017	DUX= -0.00400	DUY= 0.00124	

		DSG R= -0.00000	DSG T= -0.00036	DTRT= 0.00000	DUR= -0.00128	DUT= 0.00399	
X = 0.900	Y = 0.866	SIG X= 0.45663	SIG Y= -0.31445	T XY= -0.84099	UX = 0.49901	UY = 1.15985	
R = 1.000	THET= 60.000	SIG R= -1.00000	SIG T= 0.94218	T RT= 0.00000	UR = 1.25397	UT = 0.14777	
		DSG X= -0.00074	DSG Y= -0.00025	DTXY= 0.00042	DUX= -0.00412	DUY= 0.00023	
		DSG R= -0.00000	DSG T= -0.00098	DTRT= -0.00000	DUR= -0.00196	DUT= 0.00369	
X = 0.423	Y = 0.906	SIG X= 0.62100	SIG Y= -0.64752	T XY= -0.75589	UX = 0.42226	UY = 1.21483	
R = 1.000	THET= 65.000	SIG R= -1.00000	SIG T= 0.97348	T RT= 0.00000	UR = 1.27946	UT = 0.13071	
		DSG X= -0.00127	DSG Y= -0.00028	DTXY= 0.00059	DUX= -0.00396	DUY= -0.00079	
		DSG R= -0.00000	DSG T= -0.00154	DTRT= 0.00000	DUR= -0.00239	DUT= 0.00325	
X = 0.342	Y = 0.940	SIG X= 0.77476	SIG Y= -0.76489	T XY= -0.64596	UX = 0.34206	UY = 1.26050	
R = 1.000	THET= 70.000	SIG R= -1.00000	SIG T= 1.00987	T RT= 0.0	UR = 1.30147	UT = 0.10968	
		DSG X= -0.00135	DSG Y= -0.00018	DTXY= 0.00049	DUX= -0.00354	DUY= -0.00174	
		DSG R= -0.00000	DSG T= -0.00153	DTRT= -0.00000	DUR= -0.00284	DUT= 0.00273	
X = 1.000	Y = 0.0	SIG X= -1.00000	SIG Y= 1.09843	T XY= 0.0	UX = 0.98611	UY = 0.0	
R = 1.000	THET= 0.0	SIG R= -1.00000	SIG T= 1.09843	T RT= 0.0	UR = 0.98611	UT = 0.0	
		DSG X= -0.00000	DSG Y= 0.00111	DTXY= -0.00000	DUX= 0.00367	DUY= 0.00000	
		DSG R= -0.00000	DSG T= 0.00111	DTRT= -0.00000	DUR= 0.00367	DUT= 0.00000	
X = 1.200	Y = 0.0	SIG X= -0.69879	SIG Y= 0.70692	T XY= 0.0	UX = 0.83299	UY = 0.0	
R = 1.200	THET= 0.0	SIG R= -0.69879	SIG T= 0.70692	T RT= 0.0	UR = 0.83299	UT = 0.0	
		DSG X= -0.00008	DSG Y= 0.00001	DTXY= 0.0	DUX= 0.00195	DUY= 0.0	
		DSG R= -0.00008	DSG T= 0.00001	DTRT= 0.0	DUR= 0.00195	DUT= 0.0	
X = 1.400	Y = 0.0	SIG X= -0.52617	SIG Y= 0.50616	T XY= 0.0	UX = 0.72325	UY = 0.0	
R = 1.400	THET= 0.0	SIG R= -0.52617	SIG T= 0.50616	T RT= 0.0	UR = 0.72325	UT = 0.0	
		DSG X= -0.00009	DSG Y= 0.00001	DTXY= -0.00000	DUX= 0.00105	DUY= 0.00000	
		DSG R= -0.00009	DSG T= 0.00001	DTRT= -0.00000	DUR= 0.00105	DUT= 0.00000	
X = 1.600	Y = 0.0	SIG X= -0.41334	SIG Y= 0.38368	T XY= 0.0	UX = 0.63958	UY = 0.0	
R = 1.600	THET= 0.0	SIG R= -0.41334	SIG T= 0.38369	T RT= 0.0	UR = 0.63958	UT = 0.0	
		DSG X= -0.00007	DSG Y= 0.00003	DTXY= -0.00000	DUX= 0.00053	DUY= 0.00000	
		DSG R= -0.00007	DSG T= 0.00003	DTRT= -0.00000	DUR= 0.00053	DUT= 0.00000	
X = 1.800	Y = 0.0	SIG X= -0.33404	SIG Y= 0.30217	T XY= 0.0	UX = 0.57331	UY = 0.0	
R = 1.800	THET= 0.0	SIG R= -0.33404	SIG T= 0.30217	T RT= 0.0	UR = 0.57331	UT = 0.0	
		DSG X= -0.00011	DSG Y= 0.00002	DTXY= -0.00000	DUX= 0.00024	DUY= 0.00000	
		DSG R= -0.00011	DSG T= 0.00002	DTRT= -0.00000	DUR= 0.00024	DUT= 0.00000	
X = 2.000	Y = 0.0	SIG X= -0.27570	SIG Y= 0.24469	T XY= 0.0	UX = 0.51940	UY = 0.0	
R = 2.000	THET= 0.0	SIG R= -0.27570	SIG T= 0.24469	T RT= 0.0	UR = 0.51940	UT = 0.0	
		DSG X= -0.00018	DSG Y= -0.00000	DTXY= -0.00000	DUX= 0.00007	DUY= 0.00000	
		DSG R= -0.00018	DSG T= -0.00000	DTRT= -0.00000	DUR= 0.00007	DUT= 0.00000	
X = 0.707	Y = 0.707	SIG X= -0.04649	SIG Y= -0.04648	T XY= -0.95351	UX = 0.70290	UY = 0.94421	
R = 1.000	THET= 45.000	SIG R= -1.00000	SIG T= 0.90703	T RT= -0.00000	UR = 1.16468	UT = 0.17063	
		DSG X= -0.00006	DSG Y= -0.00006	DTXY= 0.00006	DUX= -0.00302	DUY= 0.00299	
		DSG R= -0.00000	DSG T= -0.00012	DTRT= -0.00000	DUR= -0.00002	DUT= 0.00425	

XVII. (continued)

X = 0.849	Y = 0.849	SIG X= -0.00379	SIG Y= -0.00232	T XY= -0.68621	UX = 0.57776	UY = 0.77873
R = 1.200	THET= 45.000	SIG R= -0.68926	SIG T= 0.68315	T RT= 0.00073	UR = 0.95918	UT = 0.14211
		DSG X= -0.00029	DSG Y= -0.00007	DTXY= 0.00015	DUX= -0.00171	DUY= 0.00178
		DSG R= -0.00002	DSG T= -0.00033	DTRT= 0.00011	DUR= 0.00005	DUT= 0.00246
X = 0.990	Y = 0.990	SIG X= 0.01253	SIG Y= 0.01067	T XY= -0.50433	UX = 0.49823	UY = 0.66091
R = 1.400	THET= 45.000	SIG R= -0.49273	SIG T= 0.51593	T RT= -0.00093	UR = 0.81247	UT = 0.12210
		DSG X= 0.00030	DSG Y= 0.00012	DTXY= 0.00007	DUX= -0.00106	DUY= 0.00116
		DSG R= 0.00028	DSG T= 0.00014	DTRT= -0.00009	DUR= 0.00007	DUT= 0.00157
X = 1.131	Y = 1.131	SIG X= 0.01700	SIG Y= 0.01438	T XY= -0.38276	UX = 0.42241	UY = 0.57364
R = 1.600	THET= 45.000	SIG R= -0.36707	SIG T= 0.39845	T RT= -0.00131	UR = 0.70432	UT = 0.10695
		DSG X= 0.00042	DSG Y= 0.00017	DTXY= 0.00011	DUX= -0.00069	DUY= 0.00081
		DSG R= 0.00041	DSG T= 0.00019	DTRT= -0.00012	DUR= 0.00009	DUT= 0.00106
X = 1.273	Y = 1.273	SIG X= 0.01714	SIG Y= 0.01483	T XY= -0.29940	UX = 0.37228	UY = 0.50671
R = 1.800	THET= 45.000	SIG R= -0.28342	SIG T= 0.31539	T RT= -0.00115	UR = 0.62154	UT = 0.09506
		DSG X= 0.00033	DSG Y= 0.00017	DTXY= 0.00012	DUX= -0.00045	DUY= 0.00060
		DSG R= 0.00037	DSG T= 0.00013	DTRT= -0.00008	DUR= 0.00010	DUT= 0.00074
X = 1.414	Y = 1.414	SIG X= 0.01584	SIG Y= 0.01409	T XY= -0.24031	UX = 0.33290	UY = 0.45382
R = 2.000	THET= 45.000	SIG R= -0.22534	SIG T= 0.25527	T RT= -0.00088	UR = 0.55629	UT = 0.08550
		DSG X= 0.00021	DSG Y= 0.00015	DTXY= 0.00010	DUX= -0.00029	DUY= 0.00046
		DSG R= 0.00028	DSG T= 0.00008	DTRT= -0.00003	DUR= 0.00012	DUT= 0.00053
X = 0.000	Y = 1.000	SIG X= 1.10657	SIG Y= -1.00000	T XY= -0.00000	UX = 0.00000	UY = 1.34325
R = 1.000	THET= 90.000	SIG R= -1.00000	SIG T= 1.10657	T RT= 0.00000	UR = 1.34325	UT = 0.00000
		DSG X= 0.00310	DSG Y= -0.00000	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00371
		DSG R= -0.00000	DSG T= 0.00310	DTRT= -0.00000	DUR= -0.00371	DUT= 0.00000
X = 0.000	Y = 1.200	SIG X= 0.70277	SIG Y= -0.70107	T XY= -0.00000	UX = 0.00000	UY = 1.13095
R = 1.200	THET= 90.000	SIG R= -0.70107	SIG T= 0.70277	T RT= -0.00000	UR = 1.13095	UT = 0.00000
		DSG X= -0.00012	DSG Y= -0.00014	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00209
		DSG R= -0.00014	DSG T= -0.00012	DTRT= -0.00000	DUR= -0.00208	DUT= 0.00000
X = 0.000	Y = 1.400	SIG X= 0.50344	SIG Y= -0.52827	T XY= -0.00000	UX = 0.00000	UY = 0.97883
R = 1.400	THET= 90.000	SIG R= -0.52827	SIG T= 0.50344	T RT= -0.00000	UR = 0.97883	UT = 0.00000
		DSG X= 0.00000	DSG Y= -0.00009	DTXY= -0.00000	DUX= 0.00000	DUY= -0.00129
		DSG R= -0.00009	DSG T= 0.00000	DTRT= 0.00000	DUR= -0.00129	DUT= -0.00000
X = 0.000	Y = 1.600	SIG X= 0.38279	SIG Y= -0.41360	T XY= -0.00000	UX = 0.00000	UY = 0.86296
R = 1.600	THET= 90.000	SIG R= -0.41360	SIG T= 0.38279	T RT= -0.00000	UR = 0.86296	UT = 0.00000
		DSG X= -0.00002	DSG Y= -0.00013	DTXY= -0.00000	DUX= 0.00000	DUY= -0.00085
		DSG R= -0.00013	DSG T= -0.00002	DTRT= 0.00000	DUR= -0.00085	DUT= -0.00000
X = 0.000	Y = 1.800	SIG X= 0.30217	SIG Y= -0.33264	T XY= -0.00000	UX = 0.00000	UY = 0.77146
R = 1.800	THET= 90.000	SIG R= -0.33264	SIG T= 0.30217	T RT= -0.00000	UR = 0.77146	UT = 0.00000
		DSG X= -0.00004	DSG Y= -0.00018	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00058
		DSG R= -0.00018	DSG T= -0.00004	DTRT= -0.00000	DUR= -0.00058	DUT= 0.00000
X = 0.000	Y = 2.000	SIG X= 0.24503	SIG Y= -0.27320	T XY= -0.00000	UX = 0.00000	UY = 0.69735
R = 2.000	THET= 90.000	SIG R= -0.27320	SIG T= 0.24503	T RT= -0.00000	UR = 0.69735	UT = 0.00000
		DSG X= -0.00004	DSG Y= -0.00022	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00040
		DSG R= -0.00022	DSG T= -0.00004	DTRT= -0.00000	DUR= -0.00040	DUT= 0.00000

4. LEAST SQUARES AND BOUNDARY REISSNER ENERGY

These programs compute approximate solutions to the four sample problems. In addition, the user may treat other loading conditions by modifying a certain subroutine, as discussed in Article 4a. A detailed discussion of the mathematical models is contained in Section V of this report. As discussed in that section (equation (5.1.1)ff.) it is necessary (just as in any weighted residual method) to assume the form of the solution. In these programs the assumed forms of Φ_1 and Φ_2 are odd negative powers of z_1 and z_2. (See equation (5.1.1).) However, the user may easily supply different assumed forms by appropriately modifying certain function subprograms as discussed in Article 4a. It should be noted that the formulations upon which these programs are based are not valid for isotropic materials or materials equivalent to isotropic materials, i.e., materials for which the roots of the characteristic equation are equal. (See Article 1a of Section III for a discussion of the roots of the characteristic equation.) It would be relatively easy to design programs (analogous to the present ones) to obtain numerical solutions for problems involving isotropic materials. However, this is not the purpose of the present study.

4a. User Supplied Subroutines

We first describe the subroutine PHIF56, which allows the user to consider problems other than the four standard sample problems. In this subroutine the user supplies the functions (analytic in the region exterior to the circular hole) $\Phi_1(z_1)$, $\Phi_2(z_2)$ and their derivatives $\Phi_1'(z_1)$ and $\Phi_2'(z_2)$. These functions are then used (as described in Section III, equation (3.1.7)ff.) to compute exact values of the stresses and displacements. These exact values are then used as boundary conditions for the least squares or Reissner energy analysis. If Case V is selected (as described in Article 4b) then the prescribed conditions on the boundary are stress conditions. For Case VI they are displacement conditions. In selecting Φ_1 and Φ_2 the user must be careful that the "boundary" conditions at infinity are satisfied exactly since the present programs minimize the residuals only along the inner boundary. (See the end of Article 2a of Section V for related discussion.) In Table XVIII we list the program symbols, formulation symbols, and description of the variables in Subroutine PHIF56.

Table XVIII. Symbols for Subroutine PHIF56

Program Symbol	Formulation Symbol	Description
Z1	z_1	$x + \mu_1 y$
Z2	z_2	$x + \mu_2 y$
I	i	$\sqrt{-1}$
PHI1	Φ_1	analytic function of z_1
PHI2	Φ_2	analytic function of z_2
DPHI1	Φ'	derivative of PHI1
DPHI2	Φ'	derivative of PHI2

The user simply inserts the cards defining PHI1 and DPHI1 in terms of Z1 (and I if needed) into Subroutine PHIF56. Likewise for PHI2 and DPHI2. We note that Subroutine PHIF56 must not be removed from the deck. If Cases V and VI are not used then this subroutine can be a dummy consisting of only the subroutine name card, a RETURN card, and an END card.

Now we describe the function subprograms F, FP, G, and GP and their use. The program symbols, formulation symbols, and their description are given in Table XIX.

Table XIX. Symbols for Function Subprograms F,FP,G, and GP

Program Symbol	Formulation Symbol	Description
Z	z_1	$x + \mu_1 y$ (used in F and FP)
	z_2	$x + \mu_2 y$ (used in G and GP)
K	i	index for summations in equation (5.1.1)
F	f_i	analytic function of z_1
G	g_i	analytic function of z_2
FP	f_i'	derivative of f_i
GP	g_i'	derivative of g_i

As indicated in equation (5.1.1) the f_i and g_i are used to construct the assumed forms of Φ_1 and Φ_2. The programs then determine the complex constants a_i and b_i of (5.1.1) in accordance with the least squares and Reissner energy criteria. The corresponding Φ_1, Φ_2, Φ_1', and Φ_2' are then used to compute approximate stresses and displacements. As mentioned earlier the standard forms (in these programs) for F and G are odd negative powers of Z. In order to use different assumed functions the user simply replaces the standard definitions of F and G, within the corresponding function subprogram, with his own definitions. FP and GP are, of course, also appropriately redefined.

The option of allowing the user to define his own assumed forms of Φ_1 and Φ_2 is very useful in testing the programs. If the assumed forms of Φ_1 and Φ_2 contain the exact solution along with other functions, then either program should extract the exact solution from this assumed form. This is in fact the way in which Cases I - VI were tested out. In these tests it was found that the exact solution was obtained quite accurately when the program was operating correctly. The functions used to test Cases I - IV are described in Article 2a of Section III and in Appendix II. In Cases V and VI the test functions should include those supplied by the user in Subroutine PHIF56.

4b. Description of Input

All input quantities for these programs not discussed below are thoroughly described in Articles 1c and 1d.

i) ICASE

This input code designates which of the four standard sample problems is to be solved. In addition, for these programs, ICASE may assume the values 5 and 6 corresponding to the sample problem 5 and 6 described in Article 4a.

ii) NN

This input is the upper limit of the summation in the assumed forms of Φ_1 and Φ_2 (see equation (5.1.1) of Section V). The order of the coefficient matrix is then 4 x NN.

Before discussing the inputs dealing with the integration we shall first review how the five point Gauss-Chebyshev quadrature formula is used to compute the required integrals along the circular boundary. Consider an interval over which an integration is to be performed. The user may, by an input, divide this interval into subintervals of equal length. The quadrature formula is then used to integrate over each subinterval and these results are then summed to obtain the integral over the entire interval.

iii) NGC

The input NGC is the number of subintervals desired. NGC must be at least 1 (in this case the interval is not subdivided) and may not exceed 32 (if inputted as greater than 32 the program sets it equal to 32).

iv) UPINT

In defining the limits of integration these programs assume zero to be the lower limit and the input UPINT (in degrees) is the upper limit. (UPINT will never exceed 360, not because of any program limitations, but because of the analysis on which these programs are based.)

The data set up and input formats are presented in Table XX.

4c. Description of Output

The initial output consists of the standard information (title card, case number, material constants, etc.) described at the beginning of Article 1e. In addition the following quantities are printed out:

i) NN - upper limit of the summations in the assumed forms of Φ_1 and Φ_2.

ii) UPINT - upper limit of integration around circular boundary.

iii) NGC - number of subintervals in total integration interval.

This is followed by the output described in Tables XIV, XII, IX, and XIII, respectively. In addition, the normalized least squares residual XLSR is printed out in the least squares program. For prescribed stresses this quantity is defined as

$$XLSR = \frac{\int_0^{\theta_U} \left[\left(\sigma_r - \bar{\sigma}_r \right)^2 + \left(\tau_{r\theta} - \bar{\tau}_{r\theta} \right)^2 \right] d\theta}{\int_0^{\theta_U} \left[\bar{\sigma}_r^{\,2} + \bar{\tau}_{r\theta}^{\,2} \right] d\theta},$$

where the barred quantities designate prescribed quantities on the circular boundary. In Case II (uniform tension at infinity - zero stress on the hole boundary) $\bar{\sigma}_r$ and $\bar{\tau}_{r\theta}$ correspond to the stresses produced in a solid sheet with the same loading at infinity. The definition of XLSR in the case of prescribed displacements is the same as above except that σ_r, $\bar{\sigma}_r$, $\tau_{r\theta}$, $\bar{\tau}_{r\theta}$ are replaced by u_r, $\bar{u}_r$, u_θ, $\bar{u}_\theta$ respectively.

4d. Sample Cases

In Tables XXI and XXII the output from runs of the least squares and boundary Reissner energy programs (both having the same input) are shown. The results for σ_r and $\tau_{r\theta}$ in the least squares run are better than those from the Reissner energy run. However, overall the Reissner energy run is somewhat better. This point was discussed in Article 1b of Section VII. Better results for both methods can be obtained by taking more terms in the assumed forms of Φ_1 and Φ_2.

Table XX. Input Format Sheet for Least Squares and Boundary Reissner Energy Programs

Card 1 - Format 12A6

TITLE CARD

Card 2 - Format 3I5

ICASE	NN	NGC

If ICASE is 2 or 4 use the following card:

Card 3_a - Format 6E10.0

UPINT	EX	EY	VXY	GXY	PHID

If ICASE is 1, 3, 5, or 6 use the following card:

Card 3_b - Format 5E10.0

UPINT	EX	EY	VXY	GXY

Card 4 - Format 3I5

IPR	IPLOT	IDEV

Table XX. (Continued)

If IPR < 0 (polar coordinates) use the following 2 cards:

Card 5_P - Format 3I5

NR	NTR	NTB

Card 6_P - Format 3E10.0

DR	DTHR	DTB

If IPR > 0 (rectangular coordinates) use the following 2 cards:

Card 5_R - Format 2I5

NPX	NPY

Card 6_R - Format 4E10.0

XST	YST	DX	DY

Card 7 - Format I5

NRUN

XXI. SAMPLE CASE OF LEAST SQUARES PROGRAM

```
                    SAMPLE CASE  BOUNDARY LEAST SQUARES

                THE CASE BEING SOLVED IS CASE  1

NN  =    5

UPINT=180.00000    EX  =  1.50000    EY  =  1.00000    VXY =  0.37500    GXY =  0.40000

          THE FIVE POINT GAUSS-CHEBYSHEV QUADRATURE FORMULA WILL BE USED TO PERFORM THE NECESSARY INTEGRATIONS
          IN THE FORMATION OF THE COEFFICIENT MATRIX AND THE VECTOR ON THE RIGHT HAND SIDE OF THE MATRIX EQUATION.
          THERE WILL BE 12 SUB-INTERVALS OF EQUAL LENGTH OVER THE INTERVAL 0. TO 180.

MU1  =     0.0              0.796225          MU2  =     0.0              1.538188
```

COEFFICIENT MATRIX - COEF

ROW 1
0.2451E 02

ROW 2
-0.3586E 02 0.4776E 03

ROW 3
0.1428E 02 -0.5287E 03 0.2654E 04

ROW 4
-0.4028E 01 0.2910E 03 -0.3183E 04 0.1096E 05

ROW 5
0.9414E 00 -0.1138E 03 0.2143E 04 -0.1407E 05 0.3959E 05

ROW 6
0.7178E-05 0.1294E-03 0.9664E-04 0.7161E-04 -0.5992E-04 0.2851E 02

ROW 7
-0.9385E-04 0.1303E-04 0.1513E-02 -0.8948E-03 0.1574E-03 -0.3586E 02 0.4776E 03

ROW 8
-0.7680E-04 -0.1126E-02 -0.3152E-03 0.2128E-02 0.2716E-04 0.1428E 02 -0.5287E 03 0.2654E 04

ROW 9
0.3770E-04 0.4550E-03 0.1282E-03 -0.6192E-03 0.2104E-01 -0.4030E 01 0.2911E 03 -0.3184E 04 0.1097E 05

ROW 10
-0.1562E-03 -0.2885E-03 -0.1041E-02 -0.1954E-01 -0.3466E-02 0.9690E 00 -0.1145E 03 0.2150E 04 -0.1411E 05 0.3970E 05

ROW 11
0.1956E 02 0.4743E 02 0.3348E 02 0.1655E 02 0.6876E 01 0.9843E-05 -0.3500E-04 -0.1777E-03 -0.2385E-03 -0.1610E-03
0.3834E 02

ROW 12
-0.1271E 02 0.8504E 02 0.2276E 03 0.2391E 03 0.1703E 03 0.6042E-05 0.2011E-03 0.2225E-03 -0.9507E-04 -0.8908E-04

XXI. (continued)

```
 0.4511E 02  0.2384E 03


    ROW  13
 0.2404E 01 -0.6099E 02  0.6200E 02  0.3705E 03  0.5524E 03  0.1198E-05  0.2605E-03  0.1302E-03  0.1938E-03  0.2137E-03
 0.1771E 02  0.2300E 03  0.8248E 03

    ROW  14
-0.3234E 00  0.1716E 02 -0.9930E 02 -0.5933E 02  0.3344E 03 -0.1558E-04  0.2716E-03  0.2557E-03  0.1798E-03  0.2899E-02
 0.4928E 01  0.1199E 03  0.5065E 03  0.8741E 03

    ROW  15
-0.8133E-03 -0.3346E 01  0.3956E 02 -0.3977E 02 -0.1934E 03  0.8893E-05  0.2501E-03  0.2133E-03  0.1462E-03  0.4394E-04
 0.9961E 00  0.4499E 02  0.3081E 03  0.9507E 03  0.1281E 04

    ROW  16
-0.1034E-04  0.2282E-04  0.1724E-03  0.2140E-03  0.2745E-03  0.1956E 02  0.4743E 02  0.3348E 02  0.1455E 02  0.6876E 01
-0.1198E-04  0.2219E-03  0.2776E-03  0.2960E-03  0.2732E-03  0.3834E 02

    ROW  17
-0.8888E-05 -0.2629E-03 -0.2189E-03  0.1035E-03  0.1374E-03 -0.1271E 02  0.8504E 02  0.2276E 03  0.2391E 03  0.1703E 03
-0.2365E-03 -0.2077E-04  0.1973E-03  0.1853E-03  0.5593E-05  0.4511E 02  0.2384E 03

    ROW  18
 0.3799E-05 -0.2804E-03 -0.1334E-03 -0.1750E-03 -0.5992E-04  0.2404E 01 -0.6099E 02  0.6200E 02  0.3705E 03  0.5524E 03
-0.2045E-03 -0.1190E-03 -0.2485E-03  0.1023E-03  0.3739E-02  0.1771E 02  0.2301E 03  0.5255E 03

    ROW  19
-0.3995E-05 -0.2525E-03 -0.2137E-03 -0.4799E-03 -0.3530E-02 -0.3223E 00  0.1717E 02 -0.9928E 02 -0.5927E 02  0.3345E 03
-0.2197E-03 -0.1917E-04 -0.7570E-04 -0.2173E-03  0.2301E-03  0.4987E 01  0.1205E 03  0.5087E 03  0.8799E 03

    ROW  20
 0.0        -0.2629E-03 -0.8949E-04 -0.1726E-03 -0.3515E-03  0.7384E-02 -0.3303E 01  0.3968E 02 -0.8945E 02 -0.1929E 03
-0.2133E-03 -0.5113E-04 -0.4257E-02 -0.3567E-02  0.7159E-03  0.1255E 01  0.4672E 02  0.3124E 03  0.8567E 03  0.1285E 04
```

XXI. (continued)

```
VECTOR WHICH IS THE RIGHT SIDE OF THE MATRIX EQUATION  -  RT
 -0.1257E 02  0.1917E-04  0.8149E-04 -0.3915E-04  0.3042E-02 -0.9795E-05  0.3244E-04  0.4519E-05  0.3311E-04  0.1113E-03
 -0.1257E 02  0.1055E-03  0.4139E-03  0.4075E-02  0.2787E-01  0.1268E-04  0.6891E-05 -0.3995E-06 -0.8389E-05  0.3196E-05

SOLUTION VECTOR
 -0.3285E 00 -0.2795E-01 -0.6917E-02 -0.1625E-02 -0.2140E-03 -0.5345E-05 -0.2186E-04  0.8844E-07  0.3442E-07  0.2083E-09
 -0.1714E 00  0.5414E-01 -0.2731E-01  0.1214E-01 -0.3243E-02  0.5423E-05 -0.2555E-05  0.1344E-05 -0.5452E-06  0.3848E-07

PRODUCT OF COEFFICIENT MATRIX AND SOLUTION VECTOR
 -0.1257D 02  0.1917D-04  0.8149D-04 -0.3915D-04  0.3042D-02 -0.9795D-05  0.3244D-04  0.4519D-05  0.3311D-04  0.1113D-03
 -0.1257D 02  0.1055D-03  0.4139D-03  0.4075D-02  0.2787D-01  0.1268D-04  0.6891D-05 -0.3995D-06 -0.8389D-05  0.3196D-05

RELATIVE ERROR OF INVERSION PROCESS = 0.23423CE-13

CONDITION NUMBER OF COEFFICIENT MATRIX = 0.7595276 05

UPPER BOUND OF RELATIVE ERROR OF SOLUTION VECTOR = 0.3160756 01

NCOEF= 0.8533736 05    NCOEFI= 0.1162836 01    NRHS= 0.1777156 02    NRCK= 0.4162626-12    NSOL= 0.3767816 00
```

XXI. (continued)

INVERSE OF COEFFICIENT MATRIX - COEFI

ROW 1
0.3860E 00

ROW 2
0.3995E-01 0.2889E-01

ROW 3
0.2762E-02 0.1050E-01 0.6113E-02

ROW 4
-0.1541E-03 0.2577E-02 0.2196E-02 0.1175E-02

ROW 5
-0.2635E-05 0.3433E-03 0.4552E-03 0.3091E-03 0.1149E-03

ROW 6
-0.4475E-06 -0.2157E-05 -0.1214E-05 -0.4154E-06 -0.7333E-07 0.3859E 00

ROW 7
0.1169E-05 -0.9171E-07 -0.3113E-07 0.6189E-08 0.5381E-11 0.3995E-01 0.2894E-01

ROW 8
0.7149E-06 0.1564E-07 -0.6738E-08 0.1568E-09 -0.4457E-09 0.2737E-02 0.1051E-01 0.6109E-02

ROW 9
0.2541E-06 -0.3054E-[illegible] -0.1733E-07 -0.6954E-08 -0.1677E-08 -0.1725E-03 0.2575E-02 0.2192E-02 0.1172E-02

ROW 10
0.4855E-07 0.4102E-[illegible] -0.5213E-05 -0.2511E-08 -0.5841E-09 -0.8150E-05 0.3423E-03 0.4536E-03 0.3082E-03 0.1145E-03

ROW 11
-0.3598E 00 -0.3772E-01 -0.2212E-02 0.2819E-03 0.1933E-04 0.3604E-06 -0.1142E-05 -0.6969E-06 -0.2474E-06 -0.4677E-07
0.3735E 00

ROW 12
0.1132E 00 -0.3351E-01 -0.1896E-01 -0.5131E-02 -0.6699E-03 0.4172E-05 0.7534E-06 0.3301E-06 0.1393E-06 0.2498E-07

```
-0.1175E 00  0.1309E 00


    ROW  13
-0.5276E-01  0.2680E-01  0.8614E-02 -0.1037E-03 -0.6605E-03 -0.1634E-05 -0.7398E-06 -0.2775E-06 -0.6405E-07 -0.5752E-08
 0.5595E-01 -0.9436E-01  0.7840E-01


    ROW  14
 0.2351E-01 -0.1390E-01 -0.1715E-02  0.1858E-02  0.8303E-03 -0.2742E-06  0.7735E-06  0.2459E-06  0.8817E-08 -0.1682E-07
-0.2450E-01  0.4138E-01 -0.4995E-01  0.3928E-01


    ROW  15
-0.4369E-02  0.3978E-02 -0.2042E-03 -0.9606E-03 -0.3432E-03  0.6421E-06 -0.4381E-06 -0.1350E-06  0.4063E-08  0.1214E-07
 0.6699E-02 -0.1166E-01  0.1695E-01 -0.1522E-01  0.7111E-02


    ROW  16
 0.2625E-06  0.2028E-05  0.1169E-05  0.4051E-06  0.7173E-07 -0.3559E 00 -0.3772E-01 -0.2186E-02  0.3014E-03  0.2520E-04
-0.1837E-06 -0.4039E-05  0.1495E-05  0.3784E-06 -0.6925E-06  0.3734E 00


    ROW  17
-0.2630E-05 -0.9219E-06 -0.5775E-06 -0.2363E-06 -0.4033E-07  0.1131E 00 -0.3361E-01 -0.1899E-01 -0.5137E-02 -0.6895E-03
 0.2586E-05  0.5999E-06  0.2367E-06 -0.1807E-05  0.1272E-05 -0.1175E 00  0.1311E 00


    ROW  18
 0.9915E-06  0.6739E-06  0.5216E-06  0.2166E-06  0.3716E-07 -0.5361E-01  0.2692E-01  0.8673E-02 -0.8148E-04 -0.6545E-03
-0.9470E-06 -0.9999E-06 -0.4395E-06  0.1634E-05 -0.1184E-05  0.5580E-01 -0.8454E-01  0.7846E-01


    ROW  19
 0.2666E-06 -0.5340E-06 -0.4018E-06 -0.1500E-06 -0.2293E-07  0.2336E-01 -0.1401E-01 -0.1771E-02  0.1835E-02  0.8240E-03
-0.2951E-06  0.1297E-05 -0.1899E-06 -0.8377E-06  0.7088E-06 -0.2433E-01  0.4152E-01 -0.5002E-01  0.3929E-01


    ROW  20
-0.4493E-06  0.2703E-06  0.1847E-06  0.6073E-07  0.7916E-08 -0.6296E-02  0.4035E-02 -0.1762E-03 -0.9501E-03 -0.3403E-03
 0.4672E-06 -0.8160E-06  0.3306E-06  0.1661E-06 -0.2138E-06  0.6559E-02 -0.1174E-01  0.1698E-01 -0.1524E-01  0.7129E-02

NORMALIZED LEAST SQUARES RESIDUAL =  0.195495E-03

DATA POINTS WILL BE GENERATED IN POLAR COORDINATES

ON THE BOUNDARY THERE WILL BE     7 POINTS WITH THETA INCREMENTED BY 15.0 DEGREES STARTING AT 0.  THERE WILL BE     2
RAYS 90.0 DEGREES APART BEGINNING AT THETA = 0. WITH     7 POINTS A DISTANCE  0.50 APART ON EACH RAY.

NO TAPE FOR PLOTTING PURPOSES WILL BE GENERATED
```

X = 1.000	Y = 0.0	SIG X= -1.00827	SIG Y= 1.10003	T XY= 0.00001	UX = 0.99715	UY = -0.00001
R = 1.000	THET= 0.0	SIG R= -1.00827	SIG T= 1.10003	T RT= 0.00001	UR = 0.99715	UT = -0.00001
		DSG X= 0.00827	DSG Y= -0.01049	DTXY= -0.00001	DUX= 0.00263	DUY= 0.00001
		DSG R= 0.00827	DSG T= -0.01049	DTRT= -0.00001	DUR= 0.00263	DUT= 0.00001
X = 0.966	Y = 0.259	SIG X= -0.96673	SIG Y= 0.91673	T XY= -0.51177	UX = 0.95381	UY = 0.34805
R = 1.000	THET= 15.000	SIG R= -1.00584	SIG T= 1.05295	T RT= 0.00340	UR = 1.01139	UT = 0.09932
		DSG X= 0.00684	DSG Y= -0.00659	DTXY= 0.00006	DUX= 0.00224	DUY= -0.00135
		DSG R= 0.00584	DSG T= -0.00610	DTRT= -0.00340	DUR= 0.00182	DUT= -0.00198
X = 0.866	Y = 0.500	SIG X= -0.50830	SIG Y= 0.45444	T XY= -0.83514	UX = 0.75547	UY = 0.67097
R = 1.000	THET= 30.000	SIG R= -0.99433	SIG T= 0.94047	T RT= -0.00269	UR = 1.07535	UT = 0.15334
		DSG X= -0.00180	DSG Y= 0.01526	DTXY= -0.00939	DUX= 0.00170	DUY= -0.00120
		DSG R= -0.00567	DSG T= 0.01913	DTRT= 0.00269	DUR= 0.00097	DUT= -0.00183
X = 0.707	Y = 0.707	SIG X= -0.05736	SIG Y= -0.03714	T XY= -0.95846	UX = 0.69317	UY = 0.94488
R = 1.000	THET= 45.000	SIG R= -1.00600	SIG T= 0.91092	T RT= 0.01040	UR = 1.16252	UT = 0.17374
		DSG X= 0.01140	DSG Y= -0.00941	DTXY= 0.00500	DUX= 0.00271	DUY= 0.00232
		DSG R= 0.00600	DSG T= -0.00401	DTRT= -0.01040	DUR= 0.00214	DUT= 0.00114
X = 0.500	Y = 0.866	SIG X= 0.50190	SIG Y= -0.49469	T XY= -0.85095	UX = 0.49644	UY = 1.15840
R = 1.000	THET= 60.000	SIG R= -0.99248	SIG T= 0.98970	T RT= -0.00605	UR = 1.24642	UT = 0.15734
		DSG X= -0.04600	DSG Y= -0.02002	DTXY= 0.01039	DUX= 0.00845	DUY= 0.00169
		DSG R= -0.01752	DSG T= -0.04851	DTRT= 0.00605	DUR= 0.00568	DUT= -0.00448
X = 0.259	Y = 0.966	SIG X= 0.90790	SIG Y= -0.87699	T XY= -0.49211	UX = 0.24727	UY = 1.30240
R = 1.000	THET= 75.000	SIG R= -1.00348	SIG T= 1.03439	T RT= -0.02005	UR = 1.32202	UT = 0.09824
		DSG X= 0.00590	DSG Y= 0.01403	DTXY= -0.01936	DUX= 0.00391	DUY= -0.00850
		DSG R= 0.00348	DSG T= 0.01146	DTRT= 0.02005	DUR= -0.00590	DUT= -0.01080
X = 0.000	Y = 1.000	SIG X= 1.04293	SIG Y= -1.01743	T XY= 0.00000	UX = -0.00001	UY = 1.35325
R = 1.000	THET= 90.000	SIG R= -1.01743	SIG T= 1.04293	T RT= -0.00000	UR = 1.35325	UT = 0.00001
		DSG X= 0.06674	DSG Y= 0.01743	DTXY= -0.00000	DUX= 0.00001	DUY= -0.01371
		DSG R= 0.01743	DSG T= 0.06674	DTRT= 0.00000	DUR= -0.01371	DUT= -0.00001
X = 1.000	Y = 0.0	SIG X= -1.00827	SIG Y= 1.10003	T XY= 0.00001	UX = 0.99715	UY = -0.00001
R = 1.000	THET= 0.0	SIG R= -1.00827	SIG T= 1.10003	T RT= 0.00001	UR = 0.99715	UT = -0.00001
		DSG X= 0.00827	DSG Y= -0.01049	DTXY= -0.00001	DUX= 0.00263	DUY= 0.00001
		DSG R= 0.00827	DSG T= -0.01049	DTRT= -0.00001	DUR= 0.00263	DUT= 0.00001
X = 1.500	Y = 0.0	SIG X= -0.46264	SIG Y= 0.43871	T XY= -0.00000	UX = 0.67695	UY = -0.00000
R = 1.500	THET= 0.0	SIG R= -0.46264	SIG T= 0.43871	T RT= -0.00000	UR = 0.67695	UT = -0.00000
		DSG X= -0.00142	DSG Y= -0.00049	DTXY= 0.00000	DUX= 0.00262	DUY= 0.00000
		DSG R= -0.00142	DSG T= -0.00049	DTRT= 0.00000	DUR= 0.00242	DUT= 0.00000
X = 2.000	Y = 0.0	SIG X= -0.27491	SIG Y= 0.24457	T XY= -0.00000	UX = 0.51724	UY = -0.00000
R = 2.000	THET= 0.0	SIG R= -0.27491	SIG T= 0.24457	T RT= -0.00000	UR = 0.51724	UT = -0.00000
		DSG X= -0.00047	DSG Y= -0.00024	DTXY= 0.00000	DUX= 0.00223	DUY= 0.00000
		DSG R= -0.00047	DSG T= -0.00024	DTRT= 0.00000	DUR= 0.00223	DUT= 0.00000
X = 2.500	Y = 0.0	SIG X= -0.19204	SIG Y= 0.15726	T XY= -0.00000	UX = 0.41813	UY = -0.00000
R = 2.500	THET= 0.0	SIG R= -0.19204	SIG T= 0.15726	T RT= -0.00000	UR = 0.41813	UT = -0.00000
		DSG X= -0.00022	DSG Y= -0.00010	DTXY= 0.00000	DUX= 0.00198	DUY= 0.00000

XXI. (continued)

		DSG R= 0.00001	DSG T= -0.00001	DTRT= -0.00000	DUR= 0.00009	DUT= -0.00000
X = 3.000	Y = 0.0	SIG X= -0.12971	SIG Y= 0.10949	T XY= 0.00000	UX = 0.35230	UY = 0.00000
R = 3.000	THET= 0.0	SIG R= -0.12971	SIG T= 0.10949	T RT= 0.00000	UR = 0.35230	UT = 0.00000
		DSG X= 0.00000	DSG Y= -0.00001	DTXY= -0.00000	DUX= 0.00010	DUY= -0.00000
		DSG R= 0.00000	DSG T= -0.00001	DTRT= -0.00000	DUR= 0.00010	DUT= -0.00000
X = 3.500	Y = 0.0	SIG X= -0.09665	SIG Y= 0.08069	T XY= 0.00000	UX = 0.30322	UY = 0.00000
R = 3.500	THET= 0.0	SIG R= -0.09665	SIG T= 0.08069	T RT= 0.00000	UR = 0.30322	UT = 0.00000
		DSG X= -0.00000	DSG Y= -0.00001	DTXY= -0.00000	DUX= 0.00010	DUY= -0.00000
		DSG R= -0.00000	DSG T= -0.00001	DTRT= -0.00000	DUR= 0.00010	DUT= -0.00000
X = 4.000	Y = 0.0	SIG X= -0.07472	SIG Y= 0.06192	T XY= 0.00000	UX = 0.26606	UY = 0.00000
R = 4.000	THET= 0.0	SIG R= -0.07472	SIG T= 0.06192	T RT= 0.00000	UR = 0.26606	UT = 0.00000
		DSG X= -0.00001	DSG Y= -0.00001	DTXY= -0.00000	DUX= 0.00010	DUY= -0.00000
		DSG R= -0.00002	DSG T= -0.00001	DTRT= -0.00000	DUR= 0.00010	DUT= -0.00000
X = 0.000	Y = 1.000	SIG X= 1.06059	SIG Y= -1.00276	T XY= -0.00001	UX = 0.00001	UY = 1.34412
R = 1.000	THET= 90.000	SIG R= -1.00276	SIG T= 1.06059	T RT= 0.00001	UR = 1.34412	UT = -0.00001
		DSG X= 0.04908	DSG Y= 0.00276	DTXY= 0.00001	DUX= -0.00001	DUY= -0.00457
		DSG R= 0.00276	DSG T= 0.04908	DTRT= -0.00001	DUR= -0.00457	DUT= 0.00001
X = 0.000	Y = 1.500	SIG X= 0.43685	SIG Y= -0.46969	T XY= -0.00000	UX = 0.00000	UY = 0.91811
R = 1.500	THET= 90.000	SIG R= -0.46969	SIG T= 0.43685	T RT= -0.00000	UR = 0.91811	UT = -0.00000
		DSG X= -0.00033	DSG Y= 0.00396	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00189
		DSG R= 0.00396	DSG T= -0.00033	DTRT= 0.00000	DUR= -0.00189	DUT= 0.00000
X = 0.000	Y = 2.000	SIG X= 0.24520	SIG Y= -0.27448	T XY= -0.00000	UX = 0.00000	UY = 0.69773
R = 2.000	THET= 90.000	SIG R= -0.27448	SIG T= 0.24520	T RT= -0.00000	UR = 0.69773	UT = -0.00000
		DSG X= -0.00022	DSG Y= 0.00106	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00078
		DSG R= 0.00106	DSG T= -0.00022	DTRT= 0.00000	DUR= -0.00078	DUT= 0.00000
X = 0.000	Y = 2.500	SIG X= 0.15756	SIG Y= -0.17957	T XY= 0.00000	UX = 0.00000	UY = 0.56214
R = 2.500	THET= 90.000	SIG R= -0.17957	SIG T= 0.15756	T RT= -0.00000	UR = 0.56214	UT = -0.00000
		DSG X= -0.00008	DSG Y= 0.00040	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00043
		DSG R= 0.00040	DSG T= -0.00008	DTRT= 0.00000	DUR= -0.00043	DUT= 0.00000
X = 0.000	Y = 3.000	SIG X= 0.10981	SIG Y= -0.12635	T XY= 0.00000	UX = 0.00000	UY = 0.47038
R = 3.000	THET= 90.000	SIG R= -0.12635	SIG T= 0.10981	T RT= -0.00000	UR = 0.47038	UT = -0.00000
		DSG X= -0.00003	DSG Y= 0.00019	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00029
		DSG R= 0.00019	DSG T= -0.00003	DTRT= 0.00000	DUR= -0.00029	DUT= 0.00000
X = 0.000	Y = 3.500	SIG X= 0.08089	SIG Y= -0.09361	T XY= 0.00000	UX = 0.00000	UY = 0.40421
R = 3.500	THET= 90.000	SIG R= -0.09361	SIG T= 0.08089	T RT= -0.00000	UR = 0.40421	UT = -0.00000
		DSG X= -0.00001	DSG Y= 0.00011	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00021
		DSG R= 0.00011	DSG T= -0.00001	DTRT= 0.00000	DUR= -0.00021	DUT= 0.00000
X = 0.000	Y = 4.000	SIG X= 0.06205	SIG Y= -0.07207	T XY= 0.00000	UX = 0.00000	UY = 0.35429
R = 4.000	THET= 90.000	SIG R= -0.07207	SIG T= 0.06205	T RT= -0.00000	UR = 0.35429	UT = -0.00000
		DSG X= -0.00001	DSG Y= 0.00007	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00017
		DSG R= 0.00007	DSG T= -0.00001	DTRT= 0.00000	DUR= -0.00017	DUT= 0.00000

XXII. SAMPLE CASE OF BOUNDARY REISSNER ENERGY PROGRAM

```
                    SAMPLE CASE  BOUNDARY REISSNER ENERGY

                THE CASE BEING SOLVED IS CASE  1

NN  =    5

UPINT=180.00000    EX  =  1.50000    EY  =  1.00000    VXY =  0.37500    GXY =  0.40000

          THE FIVE POINT GAUSS-CHEBYSHEV QUADRATURE FORMULA WILL BE USED TO PERFORM THE NECESSARY INTEGRATIONS
          IN THE FORMATION OF THE COEFFICIENT MATRIX AND THE VECTOR ON THE RIGHT HAND SIDE OF THE MATRIX EQUATION.
          THERE WILL BE 12 SUB-INTERVALS OF EQUAL LENGTH OVER THE INTERVAL 0. TO 180.

MU1  =     0.0              0.796225          MU2  =     0.0              1.528188
```

COEFFICIENT MATRIX - COEF

ROW 1
-0.8059E 01

ROW 2
0.3490E 01 -0.4317E 02

ROW 3
-0.8394E 00 0.3007E 02 -0.1395E 03

ROW 4
0.1694E 00 -0.1200E 02 0.1265E 03 -0.4029E 03

ROW 5
-0.2623E-01 0.3651E 01 -0.6754E 02 0.4267E 03 -0.1114E 04

ROW 6
-0.6012E-05 -0.9628E-05 -0.6573E-05 -0.6136E-05 0.6878E-05 -0.8059E 01

ROW 7
0.1599E-07 -0.1437E-04 -0.7391E-04 0.6950E-04 -0.9482E-04 0.3490E 01 -0.4317E 02

ROW 8
0.9183E-05 0.8755E-04 0.9375E-05 -0.2049E-03 0.8299E-04 -0.8394E 00 0.3007E 02 -0.1395E 03

ROW 9
0.4546E-04 -0.2190E-04 0.4710E-04 -0.9089E-05 -0.1644E-03 0.1698E 00 -0.1201E 02 0.1266E 03 -0.4033E 03

ROW 10
0.1253E-03 -0.1056E-03 0.9889E-04 0.3771E-03 0.6917E-04 -0.3406E-01 0.3728E 01 -0.6803E 02 0.4398E 03 -0.1120E 04

ROW 11
-0.6386E 01 -0.4802E 01 -0.2006E 01 -0.7041E 00 -0.2269E 00 -0.1227E-04 0.2902E-05 0.1157E-04 0.1241E-04 0.1013E-04
-0.8593E 01

ROW 12
0.1287E 01 -0.1084E 02 -0.1486E 02 -0.1067E 02 -0.5807E 01 -0.1483E-04 -0.7989E-05 -0.1306E-04 0.1186E-04 0.1431E-04

-0.3721E 01 -0.1627E 02

ROW 13
-0.1440E 00 0.3964E 01 -0.7295E 01 -0.1877E 02 -0.2010E 02 -0.1318E-04 -0.9398E-05 -0.7767E-05 -0.1248E-04 -0.7590E-05
-0.8945E 00 -0.1058E 02 -0.2061E 02

ROW 14
0.1445E-01 -0.7662E 00 0.5028E 01 -0.9430E 00 -0.1536E 02 -0.1168E-04 -0.1308E-04 -0.1158E-04 -0.9498E-05 -0.1708E-04
-0.1756E 00 -0.4103E 01 -0.1584E 02 -0.2405E 02

ROW 15
0.5582E-02 0.1114E 00 -0.1449E 01 0.4101E 01 0.4100E 01 -0.1748E-04 -0.1569E-04 -0.1378E-04 -0.1598E-04 -0.1279E-04
-0.2009E-02 -0.1195E 01 -0.7951E 01 -0.2000E 02 -0.2713E 02

ROW 16
0.7475E-05 -0.9461E-05 -0.1629E-04 -0.1429E-04 -0.1205E-04 -0.6366E 01 -0.4802E 01 -0.2006E 01 -0.7041E 00 -0.2269E 00
0.1325E-04 -0.8988E-05 -0.1466E-04 -0.1442E-05 0.4182E-06 -0.8593E 01

ROW 17
0.1435E-04 0.1503E-04 0.1260E-04 -0.1240E-04 -0.1912E-04 0.1287E 01 -0.1094E 02 -0.1490E 02 -0.1047E 02 -0.5807E 01
0.4594E-05 0.1049E-04 -0.1673E-04 -0.1394E-04 -0.1099E-04 -0.3721E 01 -0.1627E 02

ROW 18
0.1779E-04 0.1278E-04 0.7590E-05 0.1349E-04 0.3795E-05 -0.1439E 00 0.3965E 01 -0.7295E 01 -0.1877E 02 -0.2010E 02
-0.2796E-05 0.5493E-05 0.1498E-04 -0.1158E-04 -0.5992E-05 -0.8953E 00 -0.1058E 02 -0.2063E 02

ROW 19
0.1259E-04 0.1059E-04 0.1029E-04 0.5992E-05 0.2696E-05 0.1541E-01 -0.7644E 00 0.5032E 01 -0.9367E 00 -0.1535E 02
-0.5293E-05 0.3695E-05 0.1878E-04 0.2097E-04 -0.1598E-04 -0.1827E 00 -0.4132E 01 -0.1592E 02 -0.2419E 02

ROW 20
0.1488E-04 0.1258E-04 0.7590E-05 0.8189E-05 0.5992E-05 0.1186E-01 0.1219E 00 -0.1432E 01 0.4128E 01 0.4143E 01
-0.1598E-05 0.4794E-05 0.1039E-04 0.3194E-05 0.4594E-05 -0.3997E-01 -0.1295E 01 -0.8024E 01 -0.2030E 02 -0.2727E 02

XXII. (continued)

```
VECTOR WHICH IS THE RIGHT SIDE OF THE MATRIX EQUATION  =  RT
  0.3721E 01 -0.4494E-04 -0.2550E-05  0.9924E-06 -0.6573E-04  0.4381E-06 -0.5838E-05 -0.2436E-04 -0.2629E-05 -0.2444E-05
  0.3542E 01 -0.6941E-05 -0.9537E-05 -0.5523E-04 -0.3575E-03 -0.7603E-06 -0.7178E-07 -0.7615E-06 -0.9351E-06 -0.9363E-06

SOLUTION VECTOR
 -0.3260E 00 -0.2920E-01 -0.6356E-02 -0.1037E-02 -0.3149E-04  0.4431E-05  0.5777E-06 -0.4882E-07 -0.2351E-06 -0.1265E-06
 -0.1743E 00  0.5801E-01 -0.3439E-01  0.1895E-01 -0.4858E-02 -0.4449E-05  0.3883E-06  0.5093E-04 -0.4356E-05 -0.4438E-07

PRODUCT OF COEFFICIENT MATRIX AND SOLUTION VECTOR
  0.3721D 01 -0.4494D-04 -0.2550D-05  0.9924D-06 -0.6573D-04  0.4381D-06 -0.5838D-05 -0.2436D-04 -0.2629D-05 -0.2444D-05
  0.3542D 01 -0.6941D-05 -0.9537D-05 -0.5523D-04 -0.3575D-03 -0.7603D-06 -0.7178D-07 -0.7615D-06 -0.9351D-06 -0.9363D-06

RELATIVE ERROR OF INVERSION PROCESS = 0.1954A2E-14

CONDITION NUMBER OF COEFFICIENT MATRIX = 0.7285576E 04

UPPER BOUND OF RELATIVE ERROR OF SOLUTION VECTOR = 0.7680I2E-01

NCOEF= 0.1922578E 04    NCOEFI= 0.3A84ACE 01    NRHS= 0.5137578E 01    NRCK= 0.100432E-13    NSOL= 0.3773235E 00
```

XXII. (continued)

INVERSE OF COEFFICIENT MATRIX - COEFI

ROW 1
-0.6769E 00

ROW 2
-0.6461E-01 -0.1463E 00

ROW 3
-0.7267E-02 -0.6036E-01 -0.5233E-01

ROW 4
-0.7232E-02 -0.9841E-02 -0.2533E-01 -0.1986E-01

ROW 5
-0.8191E-03 -0.4206E-03 -0.6412E-02 -0.6276E-02 -0.3009E-02

ROW 6
-0.6304E-05 -0.8470E-04 -0.7855E-04 -0.4515E-04 -0.1252E-03 -0.6765E 00

ROW 7
0.9065E-04 -0.1706E-05 0.5106E-06 0.6733E-06 0.2355E-06 -0.6481E-01 -0.1471E 00

ROW 8
-0.5263E-07 -0.4180E-06 -0.1661E-06 -0.9923E-08 0.1459E-07 -0.6722E-02 -0.5010E-01 -0.5202E-01

ROW 9
-0.4416E-06 0.9768E-07 -0.2321E-04 -0.1651E-06 -0.4486E-07 -0.1966E-02 -0.9604E-02 -0.2517E-01 -0.1981E-01

ROW 10
-0.2494E-06 0.6889E-07 -0.8694E-07 -0.7241E-07 -0.2159E-07 -0.7186E-03 -0.5598E-03 -0.6566E-02 -0.6262E-02 -0.3000E-02

ROW 11
0.6191E 00 0.5963E-01 0.5842E-02 0.2054E-02 0.8479E-03 0.5890E-05 -0.9275E-06 -0.6475E-07 0.3860E-06 0.2208E-06
-0.6995E 00

ROW 12
-0.2053E 00 0.2331E 00 0.9313E-01 0.1899E-01 0.2468E-03 -0.1043E-05 0.4451E-05 0.9160E-06 -0.4614E-06 -0.2953E-06

```
 0.2320E 00 -0.6473E 00


    ROW  13
 0.1170E 00 -0.1982E 00  0.3153E-01  0.6658E-01  0.2495E-01  0.2817E-05 -0.8494E-05 -0.5368E-06  0.1611E-05  0.7356E-06
-0.1327E 00  0.5144E 00 -0.9177E 00


    ROW  14
-0.6401E-01  0.1181E 00 -0.7227E-01 -0.7270E-01 -0.2272E-01 -0.3157E-05  0.7843E-05 -0.1495E-06 -0.1901E-05 -0.7873E-06
 0.7262E-01 -0.3042E 00  0.7770E 00 -0.8253E 00


    ROW  15
 0.2185E-01 -0.9968E-01  0.3781E-01  0.3094E-01  0.8664E-02  0.2092E-05 -0.3388E-05  0.3141E-06  0.9971E-06  0.3892E-06
-0.2482E-01  0.1025E 00 -0.3186E 00  0.3869E 00 -0.2307E 00


    ROW  16
 0.4756E-05  0.9745E-06  0.8215E-06  0.4570E-06  0.1252E-06  0.6186E 00  0.5982E-01  0.5257E-02  0.1459E-02  0.7460E-03
-0.6389E-05  0.1153E-05 -0.2751E-05  0.3100E-05 -0.2047E-05 -0.6990E 00


    ROW  17
-0.4482E-05  0.9416E-05 -0.1439E-05 -0.1476E-05 -0.5688E-06 -0.2046E 00  0.2348E 00  0.9300E-01  0.1468E-01  0.1646E-03
 0.5347E-05 -0.1070E-04  0.2034E-04 -0.1900E-04  0.8438E-05  0.2313E 00 -0.6505E 00


    ROW  18
 0.4634E-05 -0.4436E-05  0.2702E-05  0.2632E-05  0.8320E-06  0.1138E 00 -0.2018E 00  0.7056E-01  0.4452E-01  0.2432E-01
-0.5075E-05  0.1301E-04 -0.2987E-04  0.3049E-04 -0.1459E-04 -0.1292E 00  0.5204E 00 -0.9243E 00


    ROW  19
-0.3413E-05  0.3313E-05 -0.2549E-05 -0.2227E-05 -0.6941E-04 -0.5541E-01  0.1222E 00 -0.7085E-01 -0.7234E-01 -0.2261E-01
 0.3685E-05 -0.9727E-05  0.2494E-04 -0.2854E-04  0.1307E-04  0.6705E-01 -0.3109E 00  0.7832E 00 -0.8316E 00


    ROW  20
 0.8348E-06 -0.1425E-05  0.1095E-05  0.9862E-06  0.2501E-06  0.1887E-01 -0.4202E-01  0.3677E-01  0.1087E-01  0.8344E-02
-0.2495E-06  0.3819E-05 -0.1044E-04  0.1140E-04 -0.5700E-05 -0.2165E-01  0.1064E 00 -0.1213E 00  0.1909E 00 -0.2322E 00

DATA POINTS WILL BE GENERATED IN POLAR COORDINATES

ON THE BOUNDARY THERE WILL BE     7 POINTS WITH THETA INCREMENTED BY 15.0 DEGREES STARTING AT 0.  THERE WILL BE     2
WAYS 90.0 DEGREES APART BEGINNING AT THETA = 0. WITH     7 POINTS A DISTANCE  0.50 APART ON EACH RAY.

NO TAPE FOR PLOTTING PURPOSES WILL BE GENERATED
```

XXII. (continued)

X = 1.000	Y = 0.0	SIG X= -1.03957	SIG Y= 1.10243	T XY= 0.00001	UX = 0.99219	UY = 0.00001
R = 1.000	THET= 0.0	SIG R= -1.03957	SIG T= 1.10243	T RT= 0.00001	UR = 0.99219	UT = 0.00001
		DSG X= 0.03957	DSG Y= -0.01288	DTXY= -0.00001	DUX= -0.00241	DUY= -0.00001
		DSG R= 0.03957	DSG T= -0.01288	DTRT= -0.00001	DUR= -0.00241	DUT= -0.00001
X = 0.966	Y = 0.259	SIG X= -0.98171	SIG Y= 0.92290	T XY= -0.49295	UX = 0.95656	UY = 0.34543
R = 1.000	THET= 15.000	SIG R= -1.00730	SIG T= 1.04849	T RT= 0.02424	UR = 1.01339	UT = 0.08609
		DSG X= 0.01893	DSG Y= -0.01317	DTXY= -0.01876	DUX= -0.00053	DUY= 0.00127
		DSG R= 0.00730	DSG T= -0.00165	DTRT= -0.02424	DUR= -0.00018	DUT= 0.00136
X = 0.866	Y = 0.500	SIG X= -0.48048	SIG Y= 0.45359	T XY= -0.84276	UX = 0.85457	UY = 0.67102
R = 1.000	THET= 30.000	SIG R= -0.97768	SIG T= 0.95079	T RT= -0.01742	UR = 1.07594	UT = 0.15364
		DSG X= -0.02962	DSG Y= 0.01611	DTXY= -0.00477	DUX= 0.00221	DUY= -0.00125
		DSG R= -0.02232	DSG T= 0.00881	DTRT= 0.01742	DUR= 0.00128	DUT= -0.00219
X = 0.707	Y = 0.707	SIG X= -0.07943	SIG Y= -0.04619	T XY= -0.95736	UX = 0.70391	UY = 0.94651
R = 1.000	THET= 45.000	SIG R= -1.02017	SIG T= 0.89456	T RT= 0.01662	UR = 1.16702	UT = 0.17155
		DSG X= 0.03288	DSG Y= -0.00037	DTXY= 0.00391	DUX= -0.00403	DUY= 0.00069
		DSG R= 0.02014	DSG T= 0.01235	DTRT= -0.01662	DUR= -0.00236	DUT= 0.00333
X = 0.500	Y = 0.866	SIG X= 0.48915	SIG Y= -0.49930	T XY= -0.84667	UX = 0.49201	UY = 1.15636
R = 1.000	THET= 60.000	SIG R= -0.98543	SIG T= 0.97528	T RT= -0.00467	UR = 1.24744	UT = 0.15209
		DSG X= -0.03325	DSG Y= -0.01540	DTXY= 0.00611	DUX= 0.00288	DUY= 0.00372
		DSG R= -0.01457	DSG T= -0.03408	DTRT= 0.00467	DUR= 0.00466	DUT= -0.00064
X = 0.259	Y = 0.966	SIG X= 0.92064	SIG Y= -0.87186	T XY= -0.50127	UX = 0.25970	UY = 1.29587
R = 1.000	THET= 75.000	SIG R= -1.00242	SIG T= 1.05120	T RT= -0.01401	UR = 1.31660	UT = 0.09324
		DSG X= -0.01183	DSG Y= 0.00890	DTXY= -0.01020	DUX= 0.00548	DUY= -0.00197
		DSG R= 0.00241	DSG T= -0.00535	DTRT= 0.01401	DUR= -0.00049	DUT= -0.00580
X = 0.000	Y = 1.000	SIG X= 1.06059	SIG Y= -1.00276	T XY= -0.00001	UX = 0.00301	UY = 1.34412
R = 1.000	THET= 90.000	SIG R= -1.00276	SIG T= 1.06059	T RT= 0.00001	UR = 1.34412	UT = -0.00001
		DSG X= 0.04908	DSG Y= 0.00276	DTXY= 0.00001	DUX= -0.00001	DUY= -0.00457
		DSG R= 0.00276	DSG T= 0.04908	DTRT= -0.00001	DUR= -0.00457	DUT= 0.00001

X = 1.000	Y = 0.0	SIG X= -1.03957	SIG Y= 1.10243	T XY= 0.00001	UX = 0.99219	UY = 0.00001
R = 1.000	THET= 0.0	SIG R= -1.03957	SIG T= 1.10243	T RT= 0.00001	UR = 0.99219	UT = 0.00001
		DSG X= 0.03957	DSG Y= -0.01288	DTXY= -0.00001	DUX= -0.00241	DUY= -0.00001
		DSG R= 0.03957	DSG T= -0.01288	DTRT= -0.00001	DUR= -0.00241	DUT= -0.00001
X = 1.500	Y = 0.0	SIG X= -0.46465	SIG Y= 0.43833	T XY= 0.00000	UX = 0.67949	UY = 0.00000
R = 1.500	THET= 0.0	SIG R= -0.46465	SIG T= 0.43833	T RT= 0.00000	UR = 0.67949	UT = 0.00000
		DSG X= 0.00017	DSG Y= -0.00011	DTXY= -0.00000	DUX= 0.00008	DUY= -0.00000
		DSG R= 0.00017	DSG T= -0.00011	DTRT= -0.00000	DUR= 0.00008	DUT= -0.00000
X = 2.000	Y = 0.0	SIG X= -0.27587	SIG Y= 0.24469	T XY= 0.00000	UX = 0.51938	UY = 0.00000
R = 2.000	THET= 0.0	SIG R= -0.27587	SIG T= 0.24469	T RT= 0.00000	UR = 0.51938	UT = 0.00000
		DSG X= -0.00001	DSG Y= -0.00000	DTXY= -0.00000	DUX= 0.00009	DUY= -0.00000
		DSG R= -0.00001	DSG T= -0.00000	DTRT= -0.00000	DUR= 0.00009	DUT= -0.00000
X = 2.500	Y = 0.0	SIG X= -0.18277	SIG Y= 0.15708	T XY= 0.00000	UX = 0.42002	UY = 0.00000
R = 2.500	THET= 0.0	SIG R= -0.18277	SIG T= 0.15708	T RT= 0.00000	UR = 0.42002	UT = 0.00000
		DSG X= 0.00001	DSG Y= -0.00001	DTXY= -0.00000	DUX= 0.00009	DUY= -0.00000

XXII. (continued)

		DSG R= -0.00072	DSG T= -0.00019	DTRT= 0.00000	DUR= 0.00198	DUT= 0.00000	
X = 3.000	Y = 0.0	SIG X= -0.12911	SIG Y= 0.10960	T XY= -0.00000	UX = 0.35061	UY = -0.00000	
R = 3.000	THET= 0.0	SIG R= -0.12911	SIG T= 0.10960	T RT= -0.00000	UR = 0.35061	UT = -0.00000	
		DSG X= -0.00059	DSG Y= -0.00012	DTXY= 0.00000	DUX= 0.00178	DUY= 0.00000	
		DSG R= -0.00059	DSG T= -0.00012	DTRT= 0.00000	DUR= 0.00178	DUT= 0.00000	
X = 3.500	Y = 0.0	SIG X= -0.09616	SIG Y= 0.08075	T XY= -0.00000	UX = 0.30170	UY = -0.00000	
R = 3.500	THET= 0.0	SIG R= -0.09616	SIG T= 0.08075	T RT= -0.00000	UR = 0.30170	UT = -0.00000	
		DSG X= -0.00050	DSG Y= -0.00007	DTXY= 0.00000	DUX= 0.00161	DUY= 0.00000	
		DSG R= -0.00050	DSG T= -0.00007	DTRT= 0.00000	DUR= 0.00161	DUT= 0.00000	
X = 4.000	Y = 0.0	SIG X= -0.07430	SIG Y= 0.06196	T XY= -0.00000	UX = 0.26468	UY = -0.00000	
R = 4.000	THET= 0.0	SIG R= -0.07430	SIG T= 0.06196	T RT= -0.00000	UR = 0.26468	UT = -0.00000	
		DSG X= -0.00043	DSG Y= -0.00005	DTXY= 0.00000	DUX= 0.00147	DUY= 0.00000	
		DSG R= -0.00043	DSG T= -0.00005	DTRT= 0.00000	DUR= 0.00147	DUT= 0.00000	
X = 0.000	Y = 1.000	SIG X= 1.04293	SIG Y= -1.01743	T XY= 0.00000	UX = -0.00001	UY = 1.35325	
R = 1.000	THET= 90.000	SIG R= -1.01743	SIG T= 1.04293	T RT= -0.00000	UR = 1.35325	UT = 0.00001	
		DSG X= 0.00674	DSG Y= 0.01743	DTXY= -0.00000	DUX= 0.00001	DUY= -0.01371	
		DSG R= 0.01743	DSG T= 0.00674	DTRT= 0.00000	DUR= -0.01371	DUT= -0.00001	
X = 0.000	Y = 1.500	SIG X= 0.43608	SIG Y= -0.47506	T XY= -0.00000	UX = -0.00000	UY = 0.92349	
R = 1.500	THET= 90.000	SIG R= -0.47506	SIG T= 0.43608	T RT= 0.00000	UR = 0.92349	UT = 0.00000	
		DSG X= 0.00043	DSG Y= 0.00936	DTXY= 0.00000	DUX= 0.00000	DUY= -0.00727	
		DSG R= 0.00936	DSG T= 0.00043	DTRT= -0.00000	DUR= -0.00727	DUT= -0.00000	
X = 0.000	Y = 2.000	SIG X= 0.24518	SIG Y= -0.27693	T XY= -0.00000	UX = -0.00000	UY = 0.70131	
R = 2.000	THET= 90.000	SIG R= -0.27693	SIG T= 0.24518	T RT= 0.00000	UR = 0.70131	UT = 0.00000	
		DSG X= -0.00019	DSG Y= 0.00351	DTXY= 0.00000	DUX= 0.00000	DUY= -0.00436	
		DSG R= 0.00351	DSG T= -0.00019	DTRT= -0.00000	DUR= -0.00436	DUT= -0.00000	
X = 0.000	Y = 2.500	SIG X= 0.15757	SIG Y= -0.18091	T XY= -0.00000	UX = -0.00000	UY = 0.56481	
R = 2.500	THET= 90.000	SIG R= -0.18091	SIG T= 0.15757	T RT= 0.00000	UR = 0.56481	UT = 0.00000	
		DSG X= -0.00009	DSG Y= 0.00174	DTXY= 0.00000	DUX= 0.00000	DUY= -0.00310	
		DSG R= 0.00174	DSG T= -0.00009	DTRT= -0.00000	DUR= -0.00310	DUT= -0.00000	
X = 0.000	Y = 3.000	SIG X= 0.10981	SIG Y= -0.12719	T XY= -0.00000	UX = -0.00000	UY = 0.47251	
R = 3.000	THET= 90.000	SIG R= -0.12719	SIG T= 0.10981	T RT= 0.00000	UR = 0.47251	UT = 0.00000	
		DSG X= -0.00004	DSG Y= 0.00103	DTXY= 0.00000	DUX= 0.00000	DUY= -0.00242	
		DSG R= 0.00103	DSG T= -0.00004	DTRT= -0.00000	DUR= -0.00242	DUT= -0.00000	
X = 0.000	Y = 3.500	SIG X= 0.08089	SIG Y= -0.09419	T XY= -0.00000	UX = -0.00000	UY = 0.40600	
R = 3.500	THET= 90.000	SIG R= -0.09419	SIG T= 0.08089	T RT= 0.00000	UR = 0.40600	UT = 0.00000	
		DSG X= -0.00002	DSG Y= 0.00069	DTXY= 0.00000	DUX= 0.00000	DUY= -0.00200	
		DSG R= 0.00069	DSG T= -0.00002	DTRT= -0.00000	DUR= -0.00200	DUT= -0.00000	
X = 0.000	Y = 4.000	SIG X= 0.06204	SIG Y= -0.07249	T XY= -0.00000	UX = -0.00000	UY = 0.35583	
R = 4.000	THET= 90.000	SIG R= -0.07249	SIG T= 0.06204	T RT= 0.00000	UR = 0.35583	UT = 0.00000	
		DSG X= -0.00001	DSG Y= 0.00049	DTXY= 0.00000	DUX= 0.00000	DUY= -0.00171	
		DSG R= 0.00049	DSG T= -0.00001	DTRT= -0.00000	DUR= -0.00171	DUT= -0.00000	

5. POTENTIAL ENERGY

This program computes approximate solutions to the four standard sample problems. In addition the user may consider other loading conditions by means of the user supplied subroutine PHIF56, which has already been discussed in Article 4a of this appendix. A detailed discussion of the mathematical model for this program is given in Article 2a of Section VI. In that article we discussed the standard assumed functions, which consist of functions of r and θ. The θ-dependence of these functions is just sine and cosine harmonics. The user may vary this part of the assumed form only to the extent of selecting the number and type of harmonics (e.g., even cosine (sine) harmonics in the assumed form of u(v) starting with the 4th and ending with the 10th). The user may exert a similar influence on the r-part of the standard assumed forms by deciding which powers of r should be included. However, for finite region problems (where it is possible to perform the required r-integrations numerically) the user may consider different types of r-dependence in the assumed forms by means of the user supplied function subprograms described in Article 5a.

It should be noted that the potential energy program has not been designed for use with isotropic materials. The results will generally be at least partially incorrect for such materials. The reason is that information from the exact solution (which is included in the program as a subroutine) is needed to compute the exact value of the total potential energy, deviations from the exact solution, and, in some cases, prescribed values of the stresses on the boundary. As discussed in Article 1a of Section III, the exact solution breaks down in the isotropic case.

Since the potential energy program has been designed to treat finite region problems it is always necessary to input the outer radius, b, of the annulus. (The inner radius is unity.) In Cases I and III (zero stresses and displacements at infinity) one can use any value of $b>1$. If b exceeds a certain large number (arbitrarily set to 499. in the program) the calculation of integrals along the outer boundary is omitted. (The user may wish to input a much larger value of b to ensure that these integrals on the outer boundary are truly negligible.) In Cases II and IV the stresses approach a constant and the displacements are unbounded as $r\to\infty$, and consequently the potential energy is unbounded. Since the potential energy method is based on minimizing the total potential energy one should not use a large value of b in Cases II and IV. Further discussion of this point is given at the end of Article 2a of Section VI.

5a. User Supplied Subroutines

We have already discussed the user supplied subroutine PHIF56 in Article 4a of this appendix. The use of this subroutine is slightly different than it was in the least squares and boundary Reissner energy programs. In those programs one could choose whether to consider the stresses or the displacements (based on Φ_1 and Φ_2 as defined in PHIF56) as prescribed boundary conditions. In the potential energy method it is necessary that the assumed form of the displacements satisfy displacement boundary conditions. Thus, in order to keep things simple, the boundary conditions for Case V will be regarded as stress conditions, and there will be no Case VI.

We now describe the function subprograms UF, UFD, VF, and VFD. In equation (6.2.2) the general assumed form for the displacements in the potential energy formulation is given. In equation (6.2.12) the standard assumed forms for $f_{5i}(r)$ and $f_{6i}(r)$ are listed. It is straightforward to consider other forms of $f_{5i}(r)$ and $f_{6i}(r)$ by modifying these function subprograms. The relevant program symbols, formulation symbols, and their description are given in Table XXIII. Now the integers IPU and

Table XXIII. Symbols for Function Subprograms UF,VF,UFD, and VFD

Program Symbol	Formulation Symbol	Description
R	r	radial coordinate
IPU	$\hat{i}_u$	integers related to the i-summation indices in the assumed forms of u and v. See (6.2.13)
IPV	$\hat{i}_v$	
JTU	T_u	integer codes in the standard assumed forms of f_{5i} and f_{6i}. See (6.2.12)
JTV	T_v	
UF	f_{5i}	form of r-dependence in assumed form of u
VF	f_{6i}	form of r-dependence in assumed form of v
UFD	f'_{5i}	derivative of f_{5i}
VFD	f'_{6i}	derivative of f_{6i}

IPV were defined especially for use with the standard forms of $f_{5i}(r)$ and $f_{6i}(r)$, and this general definition may not be useful with different forms of $f_{5i}(r)$ and $f_{6i}(r)$. However, by means of other input quantities (to be described in Article 5b) it is possible to make $\hat{i}_u=i$ and $\hat{i}_v=i$. Once IPU and IPV are appropriately defined all that is necessary is to replace UF and VF by the modified definitions, and to change UFD and VFD correspondingly. It is important to note that when using modified versions of these function subprograms all r-integrations must be done numerically. This is accomplished by inputting two integers (described in Article 5b) which indicate that the integration is to be done numerically (using the 5-point Gauss-Chebyshev quadrature formula) and the number of subintervals into which the total integration interval is to be divided. Of course, when integrating numerically one should not use a large outer radius, b, of the annulus since the results of the quadratures will probably be inaccurate.

The user defined subroutines discussed in this article can be used to test the program to see that it is operating correctly. If Φ_1 and Φ_2 in subroutine PHIF56 are defined as positive powers of z_1 and z_2 then u and v can be expressed in closed form as a sum of terms, each of which is the product of a sine or cosine harmonic and a positive power of r. If the assumed forms of u and v contain all of these terms (which is easily accomplished using the standard assumed forms) then the potential energy program should extract the exact solution. When doing this the user should exclude the rigid body terms, $u=A\sin\theta + B\cos\theta$, $v=A\cos\theta - B\sin\theta + Cr$, where A, B, and C are constants which cannot be determined by the potential energy method. It may be possible to define Φ_1 and Φ_2 in other ways so that u and v are expressible as a finite sum of terms, each of which is the product of a sine or cosine harmonic and a function of r. Then if UF, VF, UFD, and VFD were redefined so that these functions of r were contained in the assumed form, the potential energy program should extract the exact solution.

5b. Description of Input

All input quantities for this program not discussed below are thoroughly described in Articles 1c and 1d.

i) ICASE

This input code designates which of the four standard sample problems is to be solved. In addition the user can input ICASE=5 to solve other sample problems as described in Article 5a.

ii) KYKLE

This integer input code determines the type of harmonics that will be included in the assumed forms of u and v:

KYKLE = 0 => u = f(cos), v = f(sin)
KYKLE = 1 => u = f(cos,sin), v = f(cos,sin)
KYKLE = 2 => u = f(sin), v = f(cos)

iii) NU1, NU2, NU3, NV1, NV2, NV3

These integers determine the values taken on by $\hat{i}_u$ and $\hat{i}_v$ in the assumed form (6.2.17). The first value assumed by $\hat{i}_u$ is NU1 (it may be any integer), the final value is NU2 (it must be algebraically greater than NU1), and the increment between successive values is NU3 (it must be positive and must divide into the difference (NU2 - NU1) evenly). NV1, NV2, and NV3 have similar interpretations.

iv) M1, M2, M3

These integers determine the values taken on by $\hat{j}$ in the assumed form (6.2.2) or (6.2.17). The first value assumed by $\hat{j}$ is M1 (it cannot be negative), the final value is M2 (it must be algebraically greater than M1), and the increment between successive values is M3 (it must be positive and must divide into the difference (M2 - M1) evenly).

v) JRU, JTU, JTV, IALPHA

These integers appear in the standard assumed forms of u and v (6.2.17). In Case III it is essential that JRU = JTU = JTV = 1, and the most reasonable choice for IALPHA is -1. In Case IV, JRU = 0, JTU = JTV = 1, and IALPHA is arbitrary since it has no effect when JRU = 0. In Cases I, II, and V all of these quantities may be set to zero.

vi) B

B is the outer radius of the annulus and must be greater than unity. As mentioned in the introduction if B>499 all integrations along the outer boundary will be omitted in Cases I and III.

vii) THETUL

This quantity is the upper limit (in degrees) on the θ-integrals. (The lower limit is always zero.) THETUL should be consistent with problem symmetry. If there is symmetry about the x and y axes then THETUL can be taken to be 90. For symmetry about the x axis only THETUL can be taken to be 180. For any other case THETUL must be 360.

viii) NINVL, INTG, NGC

These integers pertain to the numerical quadratures. If INTG ≠ 0 the r-integrations will be performed numerically. The interval [1,B] will be divided into NINVL subintervals and the 5-point Gauss-Chebyshev quadrature formula will be used in each of these subintervals and the results for each subinterval will be added to obtain the total integral. Since some θ-integrations are always done numerically there is no code analogous to INTG for θ-integrals. NGC is the number of subintervals in the interval [0,THETUL]. Again the 5-point Gauss-Chebyshev formula is used. Neither NGC nor NINVL may exceed 32.

The input format sheets for the potential energy program are listed in Table XXIV.

5c. Description of Output

All of the input described in the preceding article is printed out with the exception of INTG and NINVL when INTG =0. The inputted material constants EX, EY, VXY, and GXY are printed out as are the computed elements of the planar elastic stiffness matrix, B11, B12, B22, and B66. In addition to these quantities and the rest of the standard output described in Article 1e of this appendix the Fourier coefficients of the radial and shear stress on the inner and outer boundaries of the annulus are printed out whenever they are computed. They are computed

whenever there are stresses prescribed on one of the boundaries of the annulus. Also printed out are the exact and approximate values of the total potential energy. These results are scaled by the factor (360./THETUL) so that the potential energy is that for the entire plate and not just for the first or first and second quadrants.

5d. Sample Case

In Table XXV the output from a run of the potential energy program is shown. As can be seen the results for both stresses and displacements are accurate. It should be noted that according to the printout the approximate total potential energy is algebraically less than the exact value. This, of course, is impossible. However, the discrepancy (in the 6th significant figure) is attributable to roundoff.

Table XXIV. Input Format Sheet for Potential Energy Program

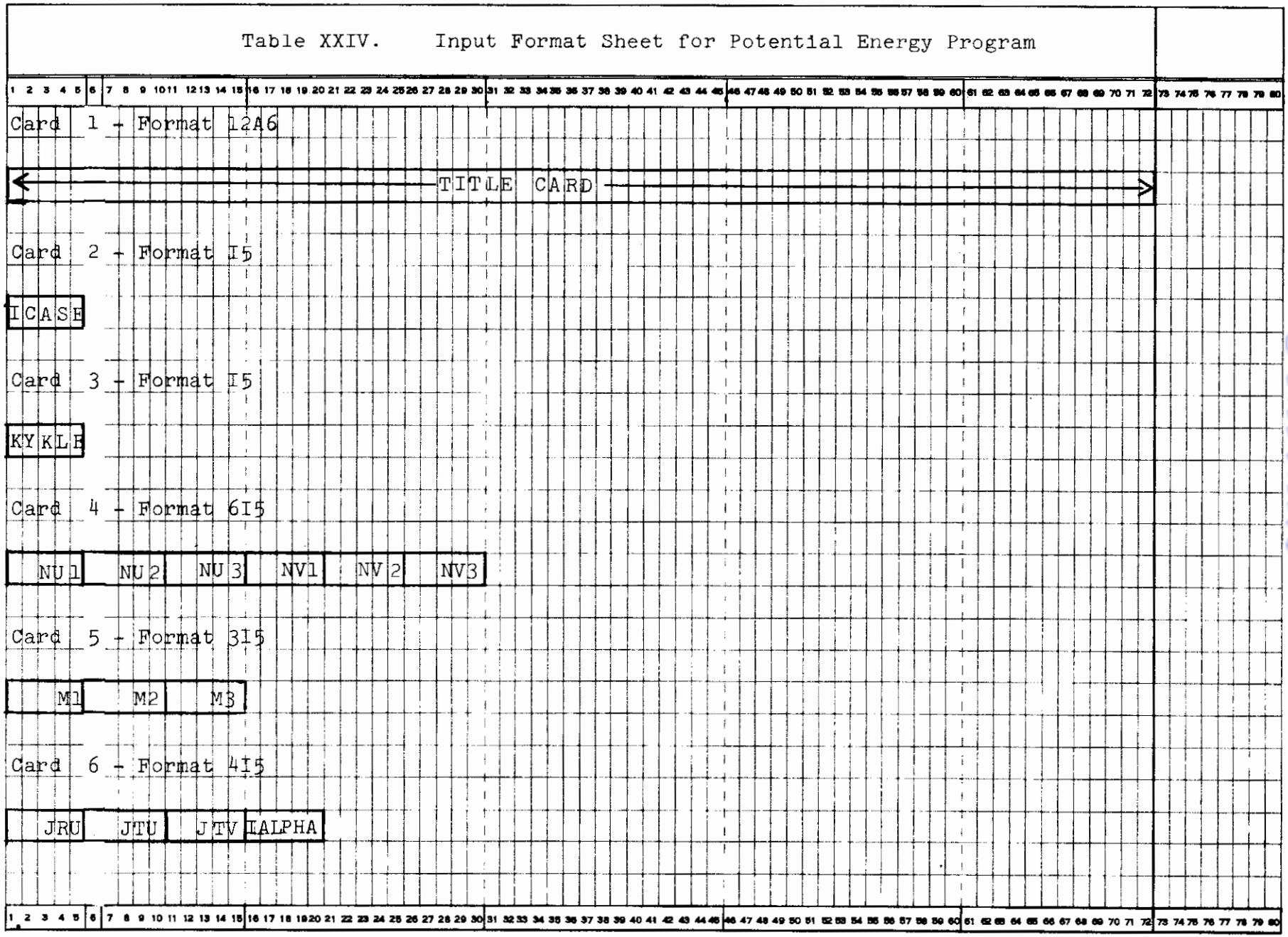
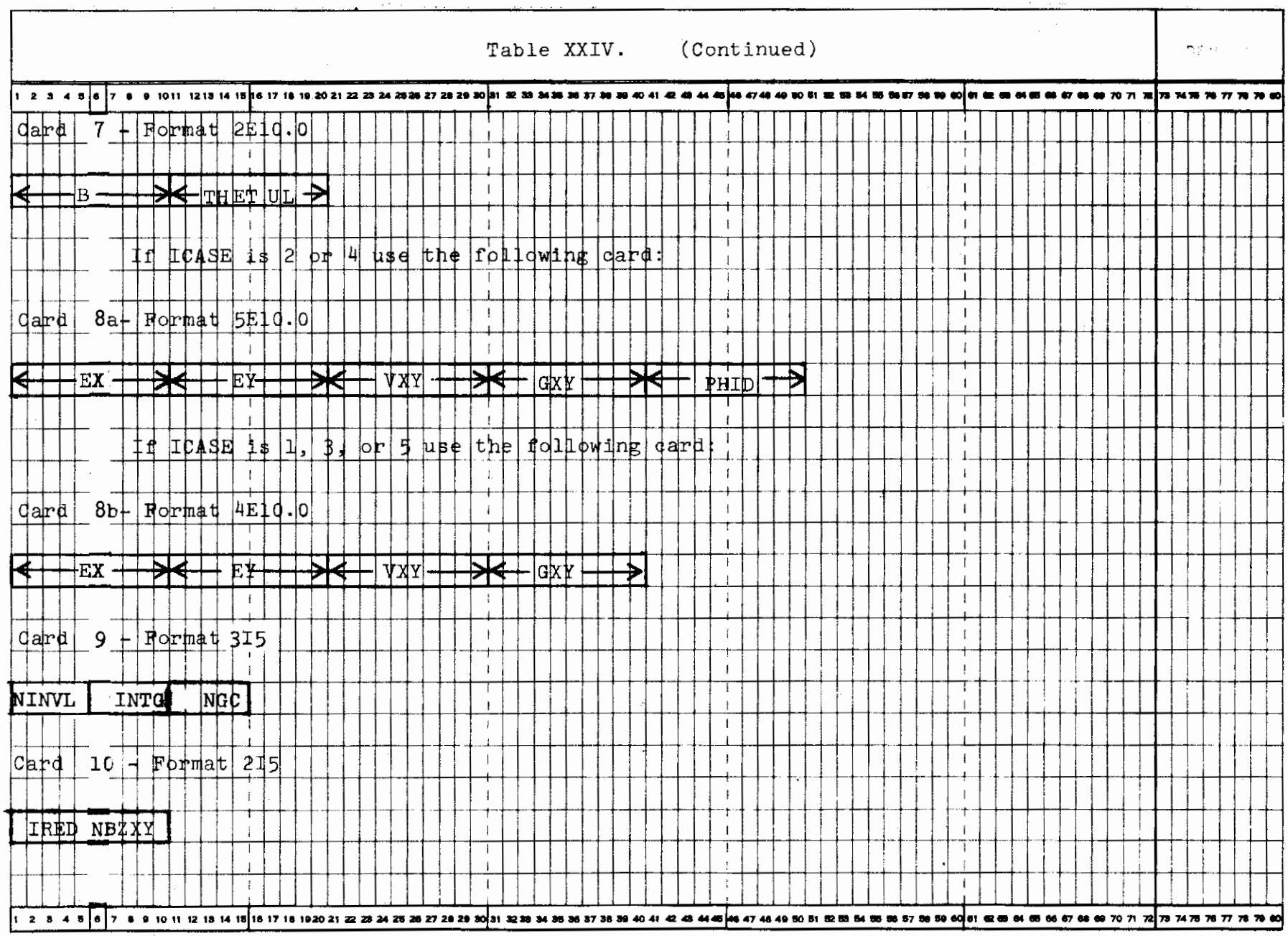


Table XXIV. (Continued)



BAC 0949 REV. 268

☐ 029
☐ 026

COMPUTER INPUT FORM

☐ RUN
☐ HOLD

CUSTOMER _____ EXT. _____ Page _____ of _____

Table XXIV. (Continued)

IDENT NO.

If IRED ≠ 0 read the following card:

Card 10a - Format 14I5 - Repeat this card until I = NBZXY

IBZXY(I)

If KYKLE=1 repeat cards 10 and 10a.

Card 11 - Format 3I5

IPR	IPLOT	IDEV

If IPR < 0 (polar coordinates) use the following 2 cards:

Card 12p - Format 3I5

NR	NTR	NTB

Card 13p - Format 3E10.0

DR	DTHR	DTB

Table XXIV. (Continued)

If IPR > 0 (rectangular coordinates) use the following 2 cards:

Card 12_R - Format 2I5

NPX	NPY

Card 13_R - Format 4E10.0

XST	YST	DX	DY

Conclude input with the following card:

Card 14 - Format I5

NRUN

XXV. SAMPLE CASE OF POTENTIAL ENERGY PROGRAM

```
                    SAMPLE CASE  POTENTIAL ENERGY

                THE CASE BEING SOLVED IS CASE  1

EX  =     1.500000    EY  =     1.000000    VXY =     0.375000    GXY =     0.400000

B11 =     1.655172    B12 =     0.413793    B22 =     1.103448    B66 =     0.400000

STRESS FUNCTION CONTAINS ONLY CCSINE HARMONICS

NU1 =  -5     NU2 =  -1     NU3 =   2     NV1 =  -5     NV2 =  -1     NV3 =   2

M1  =   0     M2  =   6     M3  =   2

Q   =  3.000  JRU =   0     JTU =   0     JTV =   0  IALPHA =  -1

MU1 =     0.0              0.796225          MU2 =     0.0              1.538188

     THE FIVE POINT GAUSS-CHEBYSHEV QUADRATURE FORMULA WILL BE USED TO PERFORM THE NECESSARY INTEGRATIONS WITH RESPECT TO THETA.
     THERE WILL BE 12 SUB-INTERVALS OF EQUAL LENGTH OVER THE INTERVAL 0. TO  90.
```

COEFFICIENT MATRIX

ROW 1
0.9675E 02

ROW 2
0.7109E 02 0.5671E 02

ROW 3
0.2821E 02 0.2790E 02 0.2511E 02

ROW 4
0.8827E 01 0.6620E 01 0.2938E 01 0.5173E 02

ROW 5
0.9516E 01 0.4408E 01 0.2180E 01 0.3939E 02 0.3328E 02

ROW 6
0.1788E-05 0.2861E-05 -0.9537E-06 0.1881E 02 0.2092E 02 0.2511E 02

ROW 7
0.7945E 00 0.6620E 00 0.4408E 00 0.4414E 01 0.3861E 01 0.2938E 01 0.5967E 02

ROW 8
-0.4768E-06 -0.7153E-06 -0.2384E-06 0.2207E 01 0.2204E 01 0.2180E 01 0.4965E 02 0.4716E 02

ROW 9
-0.1322E 01 -0.1308E 01 -0.1177E 01 -0.1469E 01 -0.1090E 01 -0.9537E-06 0.3291E 02 0.4185E 02 0.6277E 02

ROW 10
0.0 0.0 0.0 -0.2980E-06 0.5960E-07 -0.2384E-06 0.4414E 01 0.4413E 01 0.4408E 01 0.7380E 02

ROW 11
0.0 0.0 0.0 -0.6620E 00 -0.6612E 00 -0.6539E 00 0.1695E 01 0.2204E 01 0.3269E 01 0.6730E 02
0.7067E 02

ROW 12
0.0 0.0 0.0 -0.1763E 01 -0.1962E 01 -0.2354E 01 -0.2938E 01 -0.2180E 01 -0.3815E-05 0.5642E 02

XXV. (continued)

```
 0.7672E 02  0.1259E 03


    ROW  13
-0.6179E 01 -0.4413E 01 -0.1469E 01  0.4943E 01  0.1015E 02  0.1881E 02  0.3090E 01  0.2207E 01  0.7346E 00 -0.3576E-06
-0.6620E 00 -0.1763E 01  0.3452E 02


    ROW  14
-0.5516E 01 -0.4408E 01 -0.2180E 01  0.2979E 01  0.9036E 01  0.2092E 02  0.2758E 01  0.2204E 01  0.1090E 01 -0.2980E-07
-0.6612E 00 -0.1962E 01  0.3222E 02  0.3328E 02


    ROW  15
-0.4408E 01 -0.4359E 01 -0.3923E 01 -0.2939E 00  0.6757E 01  0.2511E 02  0.2204E 01  0.2180E 01  0.1962E 01 -0.4768E-06
-0.6539E 00 -0.2354E 01  0.2836E 02  0.3509E 02  0.5061E 02


    ROW  16
-0.7945E 00 -0.6620E 00 -0.4408E 00 -0.4414E 01 -0.3861E 01 -0.2938E 01  0.1095E 02  0.2096E 02  0.3761E 02  0.4414E 01
 0.3861E 01  0.2938E 01 -0.3090E 01 -0.2758E 01 -0.2204E 01  0.5967E 02


    ROW  17
 0.0         0.9537E-06  0.0        -0.4413E 01 -0.4408E 01 -0.4359E 01  0.6620E 01  0.1851E 02  0.4185E 02  0.4413E 01
 0.4408E 01  0.4359E 01 -0.4413E 01 -0.4408E 01 -0.4359E 01  0.6399E 02  0.7581E 02


    ROW  18
 0.1322E 01  0.1308E 01  0.1177E 01 -0.4408E 01 -0.5449E 01 -0.7847E 01 -0.5877E 00  0.1351E 02  0.5022E 02  0.4408E 01
 0.5449E 01  0.7847E 01 -0.6612E 01 -0.7629E 01 -0.9809E 01  0.7111E 02  0.9852E 02  0.1648E 03


    ROW  19
 0.0         0.0         0.0         0.3576E-06  0.0         0.0        -0.6621E 01 -0.6620E 01 -0.6612E 01  0.1642E 02
 0.3144E 02  0.5642E 02 -0.2414E-05 -0.5402E-06  0.2235E-06 -0.6621E 01 -0.6620E 01 -0.6612E 01  0.1025E 03


    ROW  20
 0.0         0.0         0.0         0.6620E 00  0.6612E 00  0.6539E 00 -0.7171E 01 -0.8081E 01 -0.9808E 01  0.9930E 01
 0.2777E 02  0.6277E 02  0.6620E 00  0.6612E 00  0.6539E 00 -0.9378E 01 -0.1028E 02 -0.1199E 02  0.1175E 03  0.1471E 03


    ROW  21
 0.0         0.0         0.0         0.1763E 01  0.1962E 01  0.2354E 01 -0.8081E 01 -0.1090E 02 -0.1766E 02 -0.8816E 00
 0.2027E 02  0.7533E 02  0.1763E 01  0.1962E 01  0.2354E 01 -0.1396E 02 -0.1744E 02 -0.2550E 02  0.1424E 03  0.2042E 03
 0.3551E 03
```

XXV. (continued)

FOURIER COEFFICIENTS OF RADIAL AND SHEAR STRESS AT R = 1 AND R = R

	COSINE COEFFICIENTS		SINE COEFFICIENTS	
HARMONIC	SIGR AT R = R	SIGR AT R = 1	TAURT AT R = R	TAURT AT R = 1
0	-0.222221	-1.000098	0.0	0.0
2	0.000000	0.000003	-0.000000	-0.000000
4	-0.016366	0.000002	-0.002112	-0.000000
6	-0.001741	-0.000003	0.000081	0.000000

RIGHT HAND SIDE VECTOR
0.3196E 02 0.3160E 02 0.2844E 02 -0.4370E-04 -0.4335E-04 -0.4021E-04 -0.3272E-02 -0.2913E-01 -0.2619E 00 -0.2926E-03
-0.3044E-02 -0.2781E-01 0.2444E-05 0.2420E-05 0.2206E-05 -0.4110E-03 -0.3748E-02 -0.3378E-01 0.1251E-04 0.1405E-03
0.1292E-02

SOLUTION VECTOR
0.5789E-05 -0.1059E-04 0.1165E 01 -0.1528E-04 0.4322E-02 -0.1793E 00 0.5500E-01 -0.1377E 00 0.8271E-01 0.2017E-02
-0.3272E-02 0.1077E-02 -0.1356E-04 0.4316E-02 0.1707E 00 0.5485E-01 -0.2719E-01 -0.2771E-01 0.1530E-02 -0.1620E-02
0.2243E-03

PRODUCT OF COEFFICIENT MATRIX AND SOLUTION VECTOR
0.3196D 02 0.3160D 02 0.2844D 02 -0.4370D-04 -0.4335D-04 -0.4021D-04 -0.3272D-02 -0.2913D-01 -0.2619D 00 -0.2926D-03
-0.3044D-02 -0.2781D-01 0.2444D-05 0.2420D-05 0.2206D-05 -0.4110D-03 -0.3748D-02 -0.3378D-01 0.1251D-04 0.1405D-03
0.1292D-02

RELATIVE ERROR OF INVERSION PROCESS = 0.138364E-13

CONDITION NUMBER OF COEFFICIENT MATRIX = 0.428229E 04

UPPER BOUND OF RELATIVE ERROR OF SOLUTION VECTOR = 0.473798E-01

NCOFF= 0.747970E 03 NCOEFI= 0.572521E 01 NRHS= 0.531904E 02 NRCK= 0.735966E-12 NSOL= 0.120462E 01

INVERSE OF COEFFICIENT MATRIX - COEFI

ROW 1
0.2777E 00

ROW 2
-C.4290E 00 0.7031E 00

ROW 3
0.1647E 00 -0.2993E 00 0.1885E 00

ROW 4
-0.6223E-01 0.1012E 00 -0.4265E-C1 0.6452E 00

ROW 5
0.9499E-01 -0.1678E 00 0.7772E-01 -0.1050E 01 0.1843E 01

ROW 6
-0.3751E-01 0.7252E-01 -0.4467E-01 0.4690E 00 -0.9435E 00 0.6923E 00

ROW 7
-0.3500E-02 0.2766E-02 0.4016E-02 -0.3896E-01 0.4982E-01 -0.1212E-03 0.5183E 00

ROW 8
0.4829E-02 -0.3817E-02 -0.8222E-C2 0.5375E-01 -0.7129E-01 -0.3663E-02 -0.7617E 00 0.1187E 01

ROW 9
-0.1332E-02 C.1053E-02 C.4296E-02 -0.1483E-01 0.1947E-01 0.4386E-02 0.2187E 00 -0.3728E 00 0.1577E 00

ROW 10
0.2005E-04 -0.1592E-04 0.1276E-C3 0.2225E-03 -0.7945E-02 0.1461E-01 -0.3209E-01 C.4138E-01 -0.7495E-02 0.4120E 00

ROW 11
-C.3436E-04 0.2727E-04 -0.2187E-C3 -0.3813E-03 0.1155E-01 -0.2302E-01 0.4417E-01 -0.6047E-01 0.1082E-01 -0.5839E 00
0.8799E 00

ROW 12
0.1283E-04 -0.1018E-04 0.8167E-04 0.1423E-03 -0.3404E-02 0.8086E-02 -0.1347E-01 0.1968E-01 -0.3530E-02 0.1603E 00

```
-0.2595E 00  0.9900E-01


    ROW  13
 0.7229E-01 -0.1072E 00  0.3785E-01 -0.5849E-01  0.2533E-01  0.7428E-01 -0.4078E-01  0.5627E-01 -0.1552E-01  0.2335E-03
-0.4002E-03  0.1494E-03  0.1059E 01


    ROW  14
-0.1118E 00  0.1766E 00 -0.7037E-01  0.1811E 00 -0.1884E 00 -0.8048E-01  0.5115E-01 -0.7313E-01  0.1998E-01 -0.7954E-02
 0.1196E-01 -0.3409E-02 -0.1507E 01  0.2290E 01


    ROW  15
 0.4236E-01 -0.7539E-01  0.4209E-01 -0.1785E 00  0.3323E 00 -0.2007E 00 -0.2096E-01  0.3429E-01 -0.1351E-01 -0.1811E-03
 0.3104E-03 -0.1159E-03  0.4112E 00 -0.6772E 00  0.3140E 00


    ROW  16
 0.4310E-02 -0.3406E-02  0.1171E-02  0.4800E-01 -0.7774E-01  0.4615E-01  0.8711E-02 -0.5859E-01  0.2479E-01 -0.2918E-01
 0.9919E-01 -0.1161E-01  0.5023E-01 -0.7936E-01  0.1358E-01  0.6365E 00


    ROW  17
-0.6078E-02  0.4803E-02  0.2298E-03 -0.6769E-01  0.1146E 00 -0.6747E-01  0.9119E-01 -0.8183E-01 -0.2083E-02  0.3526E-01
-0.4901E-01  0.1412E-01 -0.7084E-01  0.1169E 00 -0.2291E-01 -0.7941E 00  0.1085E 01


    ROW  18
 0.1771E-02 -0.1400E-02 -0.1488E-02  0.1972E-01 -0.3485E-01  0.2075E-01 -0.6055E-01  0.8756E-01 -0.2610E-01 -0.1011E-01
 0.1531E-01 -0.9206E-02  0.2064E-01 -0.3551E-01  0.9519E-02  0.1974E 00 -0.2983E 00  0.1000E 00


    ROW  19
 0.2689E-04 -0.2127E-04  0.1716E-03  0.2987E-03  0.4077E-02 -0.4704E-02  0.2676E-01 -0.3702E-01  0.1428E-01  0.3751E-01
-0.9832E-01  0.3055E-01  0.3138E-03  0.4065E-02 -0.2437E-03  0.3067E-01 -0.4320E-01  0.1076E-01  0.4519E 00


    ROW  20
-0.3279E-04  0.2593E-04 -0.2093E-03 -0.3644E-03 -0.6432E-02  0.8940E-02 -0.3428E-01  0.5050E-01 -0.2062E-01  0.2194E-01
 0.1545E-01 -0.1293E-01 -0.3826E-03 -0.6418E-02  0.2973E-03 -0.3905E-01  0.5974E-01 -0.1632E-01 -0.5235E 00  0.6566E 00


    ROW  21
 0.7897E-05 -0.6240E-05  0.5044E-04  0.8781E-04  0.2088E-02 -0.3796E-02  0.8983E-02 -0.1424E-01  0.6611E-02 -0.2730E-01
 0.9384E-01 -0.9361E-02  0.9217E-04  0.2085E-02 -0.7165E-04  0.1013E-01 -0.1705E-01  0.5575E-02  0.1192E 00 -0.1659E 00
 0.5048E-01

DATA POINTS WILL BE GENERATED IN POLAR COORDINATES

ON THE BOUNDARY THERE WILL BE    7 POINTS WITH THETA INCREMENTED BY 15.0 DEGREES STARTING AT 0.  THERE WILL BE    2
RAYS 90.0 DEGREES APART BEGINNING AT THETA = 0. WITH    7 POINTS A DISTANCE  0.50 APART ON EACH RAY.
```

EXACT VALUE OF TOTAL POTENTIAL ENERGY = -3.2506561

APPROXIMATE TOTAL POTENTIAL ENERGY = -3.2506752

X = 1.000	Y = 0.0	SIG X= -1.00825	SIG Y= 1.08869	T XY= 0.0	UX = 0.99956	UY = 0.0	
R = 1.000	THET= 0.0	SIG R= -1.00825	SIG T= 1.08869	T RT= 0.0	UR = 0.99956	UT = 0.0	
		DSG X= 0.00825	DSG Y= 0.00086	DTXY= -0.00000	DUX= 0.00022	DUY= 0.00000	
		DSG R= 0.00825	DSG T= 0.00086	DTRT= -0.00000	DUR= 0.00022	DUT= 0.00000	
X = 0.966	Y = 0.259	SIG X= -0.86028	SIG Y= 0.90890	T XY= -0.51377	UX = 0.95593	UY = 0.34692	
R = 1.000	THET= 15.000	SIG R= -0.99865	SIG T= 1.04728	T RT= -0.00265	UR = 1.01315	UT = 0.08769	
		DSG X= -0.00261	DSG Y= 0.00084	DTXY= 0.00206	DUX= 0.00012	DUY= -0.00023	
		DSG R= -0.00135	DSG T= -0.00043	DTRT= 0.00265	DUR= 0.00006	DUT= -0.00025	
X = 0.866	Y = 0.500	SIG X= -0.90296	SIG Y= 0.47042	T XY= -0.84719	UX = 0.85724	UY = 0.66987	
R = 1.000	THET= 30.000	SIG R= -0.99330	SIG T= 0.96076	T RT= -0.00211	UR = 1.07732	UT = 0.15151	
		DSG X= -0.00714	DSG Y= -0.00072	DTXY= -0.00134	DUX= -0.00006	DUY= -0.00010	
		DSG R= -0.00670	DSG T= -0.00117	DTRT= 0.00211	DUR= -0.00010	DUT= -0.00005	
X = 0.707	Y = 0.707	SIG X= -0.05336	SIG Y= -0.04388	T XY= -0.95389	UX = 0.69993	UY = 0.94708	
R = 1.000	THET= 45.000	SIG R= -1.00251	SIG T= 0.90528	T RT= 0.00474	UR = 1.16461	UT = 0.17476	
		DSG X= 0.00681	DSG Y= -0.00267	DTXY= 0.00044	DUX= -0.00005	DUY= 0.00013	
		DSG R= 0.00251	DSG T= 0.00163	DTRT= -0.00474	DUR= 0.00005	DUT= 0.00013	
X = 0.500	Y = 0.866	SIG X= 0.45286	SIG Y= -0.51611	T XY= -0.84300	UX = 0.49469	UY = 1.16000	
R = 1.000	THET= 60.000	SIG R= -1.00393	SIG T= 0.94068	T RT= 0.00192	UR = 1.25194	UT = 0.15159	
		DSG X= 0.00303	DSG Y= 0.00141	DTXY= 0.00244	DUX= 0.00020	DUY= 0.00007	
		DSG R= 0.00393	DSG T= 0.00051	DTRT= -0.00192	DUR= 0.00016	DUT= -0.00014	
X = 0.259	Y = 0.966	SIG X= 0.91363	SIG Y= -0.86503	T XY= -0.50594	UX = 0.25588	UY = 1.29408	
R = 1.000	THET= 75.000	SIG R= -0.99885	SIG T= 1.04745	T RT= -0.00651	UR = 1.31621	UT = 0.08777	
		DSG X= -0.00482	DSG Y= 0.00207	DTXY= -0.00552	DUX= 0.00029	DUY= -0.00018	
		DSG R= -0.00115	DSG T= -0.00160	DTRT= 0.00650	DUR= -0.00010	DUT= -0.00033	
X = 0.000	Y = 1.000	SIG X= 1.10842	SIG Y= -0.99731	T XY= -0.00000	UX = 0.00000	UY = 1.33985	
R = 1.000	THET= 90.000	SIG R= -0.99731	SIG T= 1.10842	T RT= -0.00000	UR = 1.33985	UT = 0.00000	
		DSG X= 0.00125	DSG Y= -0.00269	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00031	
		DSG R= -0.00269	DSG T= 0.00125	DTRT= 0.00000	DUR= -0.00031	DUT= 0.00000	

X = 1.000	Y = 0.0	SIG X= -1.00825	SIG Y= 1.08869	T XY= 0.0	UX = 0.99956	UY = 0.0	
R = 1.000	THET= 0.0	SIG R= -1.00825	SIG T= 1.08869	T RT= 0.0	UR = 0.99956	UT = 0.0	
		DSG X= 0.00825	DSG Y= 0.00086	DTXY= -0.00000	DUX= 0.00022	DUY= 0.00000	
		DSG R= 0.00825	DSG T= 0.00086	DTRT= -0.00000	DUR= 0.00022	DUT= 0.00000	
X = 1.500	Y = 0.0	SIG X= -0.46441	SIG Y= 0.43858	T XY= 0.0	UX = 0.67981	UY = 0.0	
R = 1.500	THET= 0.0	SIG R= -0.46441	SIG T= 0.43858	T RT= 0.0	UR = 0.67981	UT = 0.0	
		DSG X= -0.00007	DSG Y= -0.00036	DTXY= -0.00000	DUX= -0.00023	DUY= 0.00000	
		DSG R= -0.00007	DSG T= -0.00036	DTRT= -0.00000	DUR= -0.00023	DUT= 0.00000	
X = 2.000	Y = 0.0	SIG X= -0.27732	SIG Y= 0.24500	T XY= 0.0	UX = 0.51929	UY = 0.0	
R = 2.000	THET= 0.0	SIG R= -0.27732	SIG T= 0.24500	T RT= 0.0	UR = 0.51929	UT = 0.0	
		DSG X= 0.00144	DSG Y= -0.00032	DTXY= -0.00000	DUX= 0.00018	DUY= 0.00000	
		DSG R= 0.00144	DSG T= -0.00032	DTRT= -0.00000	DUR= 0.00018	DUT= 0.00000	
X = 2.500	Y = 0.0	SIG X= -0.18370	SIG Y= 0.15711	T XY= 0.0	UX = 0.41951	UY = 0.0	
R = 2.500	THET= 0.0	SIG R= -0.18370	SIG T= 0.15711	T RT= 0.0	UR = 0.41951	UT = 0.0	
		DSG X= 0.00094	DSG Y= -0.00004	DTXY= 0.0	DUX= 0.00060	DUY= 0.0	

XXV. (continued)

		DSG R= 0.00094	DSG T= -0.00004	DTRT= 0.0	DUR= 0.00060	DUT= 0.0	
X = 3.000	Y = 0.0	SIG X= -0.13019	SIG Y= 0.10937	T XY= 0.0	UX = 0.35157	UY = 0.0	
R = 3.000	THET= 0.0	SIG R= -0.13018	SIG T= 0.10937	T RT= 0.0	UR = 0.35157	UT = 0.0	
		DSG X= 0.00048	DSG Y= 0.00011	DTXY= 0.0	DUX= 0.00083	DUY= 0.0	
		DSG R= 0.00048	DSG T= 0.00011	DTRT= 0.0	DUR= 0.00083	DUT= 0.0	
X = 3.500	Y = 0.0	SIG X= -0.09686	SIG Y= 0.08051	T XY= 0.0	UX = 0.30240	UY = 0.0	
R = 3.500	THET= 0.0	SIG R= -0.09686	SIG T= 0.08051	T RT= 0.0	UR = 0.30240	UT = 0.0	
		DSG X= 0.00021	DSG Y= 0.00017	DTXY= 0.00000	DUX= 0.00092	DUY= -0.00000	
		DSG R= 0.00021	DSG T= 0.00017	DTRT= 0.00000	DUR= 0.00092	DUT= -0.00000	
X = 4.000	Y = 0.0	SIG X= -0.07479	SIG Y= 0.06173	T XY= 0.0	UX = 0.26521	UY = 0.0	
R = 4.000	THET= 0.0	SIG R= -0.07479	SIG T= 0.06173	T RT= 0.0	UR = 0.26521	UT = 0.0	
		DSG X= 0.00007	DSG Y= 0.00018	DTXY= 0.0	DUX= 0.00094	DUY= 0.0	
		DSG R= 0.00007	DSG T= 0.00018	DTRT= 0.0	DUR= 0.00094	DUT= 0.0	
X = 0.000	Y = 1.000	SIG X= 1.10842	SIG Y= -0.99731	T XY= -0.00000	UX = 0.00000	UY = 1.33985	
R = 1.000	THET= 90.000	SIG R= -0.99731	SIG T= 1.10842	T RT= -0.00000	UR = 1.33985	UT = 0.00000	
		DSG X= 0.00125	DSG Y= -0.00269	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00031	
		DSG R= -0.00269	DSG T= 0.00125	DTRT= 0.00000	DUR= -0.00031	DUT= 0.00000	
X = 0.000	Y = 1.500	SIG X= 0.43487	SIG Y= -0.46593	T XY= -0.00000	UX = 0.00000	UY = 0.91626	
R = 1.500	THET= 90.000	SIG R= -0.46593	SIG T= 0.43487	T RT= -0.00000	UR = 0.91626	UT = 0.00000	
		DSG X= 0.00165	DSG Y= 0.00021	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00004	
		DSG R= 0.00021	DSG T= 0.00165	DTRT= 0.00000	DUR= -0.00004	DUT= 0.00000	
X = 0.000	Y = 2.000	SIG X= 0.24509	SIG Y= -0.27325	T XY= -0.00000	UX = 0.00000	UY = 0.69710	
R = 2.000	THET= 90.000	SIG R= -0.27325	SIG T= 0.24509	T RT= -0.00000	UR = 0.69710	UT = 0.00000	
		DSG X= -0.00011	DSG Y= -0.00017	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00015	
		DSG R= -0.00017	DSG T= -0.00011	DTRT= 0.00000	DUR= -0.00015	DUT= 0.00000	
X = 0.000	Y = 2.500	SIG X= 0.15833	SIG Y= -0.17894	T XY= -0.00000	UX = 0.00000	UY = 0.56189	
R = 2.500	THET= 90.000	SIG R= -0.17894	SIG T= 0.15833	T RT= -0.00000	UR = 0.56189	UT = 0.00000	
		DSG X= -0.00085	DSG Y= -0.00023	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00019	
		DSG R= -0.00023	DSG T= -0.00085	DTRT= 0.00000	DUR= -0.00019	DUT= 0.00000	
X = 0.000	Y = 3.000	SIG X= 0.11075	SIG Y= -0.12593	T XY= -0.00000	UX = 0.00000	UY = 0.47027	
R = 3.000	THET= 90.000	SIG R= -0.12593	SIG T= 0.11075	T RT= -0.00000	UR = 0.47027	UT = 0.00000	
		DSG X= -0.00098	DSG Y= -0.00023	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00018	
		DSG R= -0.00023	DSG T= -0.00098	DTRT= -0.00000	DUR= -0.00018	DUT= 0.00000	
X = 0.000	Y = 3.500	SIG X= 0.08179	SIG Y= -0.09330	T XY= -0.00000	UX = 0.00000	UY = 0.40417	
R = 3.500	THET= 90.000	SIG R= -0.09330	SIG T= 0.08179	T RT= -0.00000	UR = 0.40417	UT = 0.00000	
		DSG X= -0.00092	DSG Y= -0.00020	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00017	
		DSG R= -0.00020	DSG T= -0.00092	DTRT= 0.00000	DUR= -0.00017	DUT= 0.00000	
X = 0.000	Y = 4.000	SIG X= 0.06284	SIG Y= -0.07183	T XY= -0.00000	UX = 0.00000	UY = 0.35428	
R = 4.000	THET= 90.000	SIG R= -0.07183	SIG T= 0.06284	T RT= -0.00000	UR = 0.35428	UT = 0.00000	
		DSG X= -0.00081	DSG Y= -0.00017	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00016	
		DSG R= -0.00017	DSG T= -0.00081	DTRT= 0.00000	DUR= -0.00016	DUT= 0.00000	

6 COMPLEMENTARY ENERGY

This program computes approximate solutions to the four standard sample problems. Additional sample problems may be considered through the use of the user supplied subroutine PHIF56. A detailed discussion of the mathematical model is given in Article 2b of Section VI. The complementary energy program is quite similar to the potential energy program in input, output, and in the order of the computations within the program. All of the discussion in the introduction to Article 5 applies here (with a few obvious word substitutions), so it will not be repeated here. We proceed directly to the discussion of user supplied subroutines.

6a. User Supplied Subroutines

The use of the subroutine PHIF56 is similar to its use in the potential energy program. In the potential energy program all boundary conditions for Case V were regarded as stress boundary conditions so that the assumed forms of the displacements would not have to satisfy any boundary conditions. In the complementary energy program all boundary conditions are regarded as displacement conditions so that the assumed form of the stress function will not have to satisfy any boundary conditions. Like the potential energy program there is no Case VI for complementary energy.

We now describe the function subprograms FOR, F1R, F2R, and FINT. Equation (6.2.24) shows the standard form for $f_{4i}(r)$, the r-part of the assumed form for the stress function. By means of these user supplied subprograms other forms for $f_{4i}(r)$ may be considered. The relevant program symbols, formulation symbols, and their description are given in Table XXVI.

Table XXVI. Symbols for Function Subprograms FOR,F1R,F2R, and FINT

Program Symbol	Formulation Symbol	Description
R	r	radial coordinate
IP	$\hat{i}$	integer related to the i-summation index in the assumed form of Ψ. See (6.2.13) and (6.2.24).
N2	n_2	maximum value assumed by $\hat{i}$.
XS	S	constants in standard assumed form of $f_{4i}(r)$. See (6.2.24).
XT	T	
FOR	f_{4i}/r^2	related to form of r-dependence in assumed form of Ψ.
F1R	f'_{4i}/r	derivative of f_{4i} divided by r
F2R	f''_{4i}	second derivative of f_{4i}
FINT	$\int^r (f_{4i}/r^2)dr$	indefinite integral of f_{4i}/r^2

The integer IP was defined especially for use with the standard form of $f_{4i}(r)$ and this general definition may not be useful with different forms of $f_{4i}(r)$. However, by means of other input quantities (to be described in Article 6b) it is possible to make $\hat{i} = i$. Once IP is appropriately defined all that is necessary is to replace f_{4i} in the definition of FOR by the new function of r, and to change F1R, F2R, and FINT correspondingly. It may not be possible to determine the integrals in the definition of FINT analytically. However, FINT is needed only to obtain the displacements, which are secondary quantities. Even if the integrals in FINT can be determined analytically there are still the constants of integration to be established. The "correct" way of selecting these constants is not always clear. We refer the reader to Appendix V for a discussion of this point.

Just as in the potential energy program it is necessary, when using non-standard r-functions, to perform numerically the r-integrations required in the formation of the coefficient matrix and right hand side vector (to be distinguished from those required in FINT). This, of course, requires that the

outer radius, b, of the annulus not be too large. Also, much the same as in the potential energy program, the user defined subroutines can be used to test the program for correct operation. See Article 5a of this appendix for details. We caution the user to avoid constant terms in the assumed form of the stress function since such terms produce singular coefficient matrices.

6b. Description of Input

All input quantities for this program not discussed below are thoroughly described in Articles 1c, 1d, or 5b.

i) ICASE (same as for potential energy - Article 5b)

ii) KYKLE

This integer input code determines the type of harmonics that will be included in the assumed forms of the stress function, Ψ.

KYKLE = 0 => Ψ = f(cos)
KYKLE = 1 => Ψ = f(cos,sin)
KYKLE = 2 => Ψ = f(sin)

iii) N1, N2, N3

These integers determine the values taken on by $\hat{i}$ in the assumed form (6.2.24). The first value taken on by $\hat{i}$ is N1 (it may be any integer), the final value is N2 (it must be algebraically greater than N1), and the increment between successive values is N3 (it must be positive and must divide into the difference (N2 - N1) evenly).

iv) M1, M2, M3 (same as for potential energy - Article 5b)

v) JR, JS, JT

These integers appear in the standard assumed form of Ψ (6.2.24). In Case I it is essential that JR = -1, JS = JT = 1. In Case II, JR = 0, JS = JT = 1. In Cases III, IV, and V all of these quantities may be set to zero.

vi) B (same as for potential energy - Article 5b).

vii) THETUL (same as for potential energy - Article 5b).

viii) NINVL, INTG, NGC (same as for potential energy - Article 5b).

We caution the user that in Cases III, IV, and V a constant term will be present in the standard assumed form of the stress function if M1 = 0 and $\hat{i}$ assumes the value 0. In this situation the row eliminating option should be utilized to eliminate the row (and column) corresponding to this term. Otherwise a singular matrix will result. Also, $\Psi = r\cos\theta$ or $\Psi = r\sin\theta$ will give a singular matrix. The input format sheets for the complementary energy program are listed in Table XXVII.

6c. Description of Output

All of the input described in the preceding article is printed out with the exception of INTG and NINVL when INTG=0. The inputted material constants EX, EY, VXY, and GXY are printed out as are the computed elements of the planar elastic compliance matrix S11, S12, S22, and S66, In addition to these quantities and the rest of the standard output described in Article 1e of this appendix the Fourier coefficients of the radial and circumferential displacements on the inner and outer boundaries of the annulus are printed out whenever they are computed. They are computed whenever there are displacements prescribed on one of the boundaries of the annulus. Also printed out are the exact and approximate values of the total complementary energy. These results are scaled by the factor (360./THETUL) so that the complementary energy is that for the entire plate and not just for the first or first and second quadrants.

We have already pointed out in Article 6a that there may be some difficulty in determining the constants of integration in the integrals in the function subprogram FINT when non-standard functions of r are used in the assumed form of the stress function. If these are not determined properly the displacements will be incorrect. Even when the standard assumed forms are used the constants of integration in FINT may be incorrect. This can occur when positive powers of r appear in the assumed form

of Ψ, i.e., when the input quantity N2 is greater than zero. This will be the situation in Cases II and IV for which Ψ behaves as r^2 at infinity. (When only negative powers of r are present the constants of integration in FINT can be established by using the fact that the displacements must be zero at infinity.) In the determination of the displacements from the stresses, θ-integrations are also required. When there is symmetry about the x and y axes (Cases I and III) the constants arising in these integrations are easily determined by requirements of symmetry for the displacements. However, in Cases II and IV when the angle of tension, ϕ, is not equal to 0 or 90 there is not symmetry about the x and y axes and the displacements will generally not be correct. For these reasons, in Cases II, IV, and V a message is printed out to the effect that the displacements may be incorrect. A detailed discussion of the problem of determining the displacement state corresponding to an approximate stress state is given in Appendix V.

6d. Sample Case

In Table XXVIII the output from a run of the complementary energy program is shown. As can be seen the results for both stresses and displacements are accurate. This run illustrates the use of the row eliminating option. The rows (and columns) corresponding to i = n in the assumed form (6.2.24) have been eliminated.

Table XXVII. Input Format Sheet for Complementary Energy Program

Cards 1, 2, and 3 - Same as in the potential energy program. See Table XXIV.

Card 4 - Format 3I5

N1	N2	N3

Card 5 - Format 3I5

M1	M2	M3

Card 6 - Format 3I5

JR	JS	JT

Cards 7, 8, 9, 10, 11, 12, 13, and 14 - Same as in the potential energy program. See Table XXIV.

XXVIII. SAMPLE CASE OF COMPLEMENTARY ENERGY PROGRAM

```
                    SAMPLE CASE  COMPLEMENTARY ENERGY

                THE CASE BEING SOLVED IS CASE  1

EX   =     1.500000     EY   =     1.000000     VXY  =     0.375000     GXY  =     0.600000

S11  =     0.666667     S12  =    -0.250000     S22  =     1.000000     S66  =     2.500000

STRESS FUNCTION CONTAINS ONLY COSINE HARMONICS

N1 =    -6      N2 =     0      N3 =     2      M1 =     4      M2 =    10      M3 =     2

R  =  3.000     JR =    -1      JS =     1      JT =     1

MU1  =     0.0                 0.796225          MU2  =     0.0                 1.539189

     THE FIVE POINT GAUSS-CHEBYSHEV QUADRATURE FORMULA WILL BE USED TO PERFORM THE NECESSARY INTEGRATIONS WITH RESPECT TO THETA.
     THERE WILL BE 12 SUB-INTERVALS OF EQUAL LENGTH OVER THE INTERVAL 0. TO  90.

                THE NUMBER OF ROWS(AND COLUMNS)BEING ELIMINATED =   4

              THE ROWS(AND COLUMNS)THAT WILL BE ELIMINATED ARE AS FOLLOWS

                                          5
                                         10
                                         15
                                         20

          ROW(AND COLUMN)NUMBER     ROW(AND COLUMN)NUMBER
            BEFORE REDUCTION          AFTER REDUCTION

                    1                         1
                    2                         2
                    3                         3
                    4                         4
                    5                         0
                    6                         5
                    7                         6
                    8                         7
                    9                         8
                   10                         9
                   11                         0
                   12                        10
                   13                        11
                   14                        12
                   15                         0
                   16                        13
                   17                        14
                   18                        15
                   19                        16
                   20                         0
```

COEFFICIENT MATRIX

ROW 1
0.1620E 01

ROW 2
0.1967E 01 0.2610E 01

ROW 3
0.2051E 01 0.3509E 01 0.6545E 01

ROW 4
0.1457E 01 0.4014E 01 0.1290E 02 0.5131E 02

ROW 5
0.1110E 00 0.1522E 00 0.2145E 00 0.3076E 00 0.2223E 01

ROW 6
0.6466E-01 0.1029E 00 0.1805E 00 0.3670E 00 0.2932E 01 0.4651E 01

ROW 7
-0.8321E-01 -0.1379E 00 -0.2254E 00 -0.2060E 00 0.3868E 01 0.7921E 01 0.1871E 02

ROW 8
-0.4905E 00 -0.1069E 01 -0.2555E 01 -0.6116E 01 0.4800E 01 0.1453E 02 0.5109E 02 0.2258E 03

ROW 9
-0.4953E-02 -0.7447E-02 -0.1153E-01 -0.2007E-01 0.5853E-01 0.8741E-01 0.1406E 00 0.2473E 00 0.3509E 01

ROW 10
0.4570E-02 0.7066E-02 0.1287E-01 0.2784E-01 -0.7564E-01 -0.1200E 00 -0.1974E 00 -0.3131E 00 0.5228E 01 0.8381E 01

ROW 11
-0.1128E-01 -0.1633E-01 -0.1656E-01 0.6026E-01 -0.4549E 00 -0.8658E 00 -0.1850E 01 -0.4317E 01 0.8285E 01 0.1876E 02
0.4953E 02

ROW 12
-0.2049E 00 -0.4069E 00 -0.8004E 00 -0.1032E 01 -0.1537E 01 -0.3638E 01 -0.1003E 02 -0.3093E 02 0.1409E 02 0.4305E 02

```
 0.1565E 03  C.6904F 03


    ROW  13
 0.0           0.0           0.0           0.0          0.2852E-02  0.4111E-02  0.7383E-02  0.1311E-01 -0.6714E-01 -0.1005E 00
-0.1560E 00 -0.2562E 00  C.5912E 01

    ROW  14
 0.0           0.0           0.0           C.C         -0.6904E-02 -0.1128E-01 -0.1688E-01 -0.1610E-01 -0.3797E 00 -0.6564E 00
-0.1241E 01 -C.2629E 01  C.9656E C1  C.1867E 02

    ROW  15
 0.0           0.0           C.0           C.0         -0.1114E 00 -0.2047E 00 -0.4068E 00 -0.8004E 00 -0.1229E 01 -0.2481E 01
-0.5767E 01 -0.1549E 02  C.1717E 02  C.4069E 02  0.112RE 03

    ROW  16
 0.0           C.0           C.0           0.0         -0.6776E 00 -0.1472E 01 -0.3529E 01 -0.8570E 01 -0.3692E 01 -0.9156E 01
-0.2691E 02 -C.9199E 02  C.3402E C2  0.1034E 03  0.3763E 03  0.1665E 04
```

XXVIII. (continued)

FOURIER COEFFICIENTS OF RADIAL AND CIRCUMFERENTIAL DISPLACEMENTS AT R = 1 AND R = B

	COSINE COEFFICIENTS		SINE COEFFICIENTS	
HARMONIC	UR AT R = B	UR AT R = 1	UT AT R = B	UT AT R = 1
0	C.776435	2.329309	0.0	0.0
4	0.022735	-0.000004	-0.019016	-0.000000
6	0.000681	0.000004	0.000107	-0.000000
B	0.000292	-0.000004	-0.000152	-0.000000
10	0.000015	0.000004	-0.000011	-0.000000

RIGHT HAND SIDE VECTOR
0.88099E-01 C.182BE 00 C.4504E 00 C.1364E 01 -0.1032E-03 -0.8516E-03 -0.6969E-02 -0.5646E-01 -0.3947E-04 -0.4099E-03
-0.4182E-02 -0.4207E-01 -C.2965E-05 -0.3233E-04 -0.3415E-03 -0.3528E-02

SOLUTION VECTOR
0.1123E-01 -C.2791E-01 C.4937E-01 C.1646E-01 0.1924E-02 -0.7809E-02 0.5668E-04 0.1095E-02 0.3550E-02 -0.9615E-03
-0.1372E-02 0.3122E-03 -C.9959E-03 0.1330E-02 -0.4950E-03 0.4538E-04

PRODUCT OF COEFFICIENT MATRIX AND SOLUTION VECTOR
0.88090-01 0.1828D 00 C.4504D 00 0.1364D 01 -0.1032D-03 -0.8516D-03 -0.6969D-02 -0.5646D-01 -0.3947D-04 -0.4099D-03
-0.4182D-02 -0.4207D-01 -C.2965D-05 -0.3233D-04 -0.3415D-03 -0.3528D-02

RELATIVE ERROR OF INVERSION PROCESS = C.259562E-13

CONDITION NUMBER OF COEFFICIENT MATRIX = 0.149177E 06

UPPER BOUND OF RELATIVE ERROR OF SOLUTION VECTOR =-0.307162E 01

CONDITION NUMBER T O O L A R G E - FORMULA FOR THE UPPER BOUND OF RELATIVE ERROR OF SOLUTION VECTOR NOT VALID

NCOEF= 0.192687E 04 NCOEFI= 0.774193E 02 NRHS= 0.145290E 01 NRCK= 0.377119E-13 NSOL= 0.599553E-01

XXVIII. (continued)

INVERSE OF COEFFICIENT MATRIX - CCEFI

ROW 1
0.1987E 02

ROW 2
-0.2569E 02 0.3569E 02

ROW 3
0.9356E C1 -0.1397E 02 C.613CE 01

ROW 4
-0.9207E 00 0.1467E 01 -C.7194E 00 C.1124E 00

ROW 5
-0.6316E 00 0.4224E 00 C.1173E-01 -C.2024E-01 0.1371E 02

ROW 6
0.8808E 00 -0.7396E 00 C.4327E-01 C.2614E-01 -0.1674E 02 0.2211E 02

ROW 7
-0.3660E 00 0.3786E 00 -C.6155E-C1 -C.9256E-02 0.5738E 01 -0.8132E 01 0.3313E 01

ROW 8
0.4515E-01 -0.5636E-01 0.1645E-01 C.5797E-03 -0.5337E 00 0.8034E 00 -0.3607E 00 0.4711E-01

ROW 9
-0.4893E-01 0.1697E 00 -0.7495E-01 C.5392E-02 -0.2902E 00 0.6292E-01 0.7442E-01 -0.1617E-01 0.9679E 01

ROW 10
0.6802E-01 -0.2916E 00 0.1143E C0 -C.8211E-02 0.4092E 00 -0.1908E 00 -0.6673E-01 0.2092E-01 -0.1130E 02 0.1429E 02

ROW 11
-0.2924E-C1 0.9352E-C1 -C.5324E-01 C.3932E-02 -0.1737E 00 0.1302E 00 0.3999E-02 -0.7365E-02 0.3730E 01 -0.5053E 01
0.1956E C1

ROW 12
0.3323E-C2 -0.1C79E-01 C.7C79E-02 -0.5807E-01 0.2164E-01 -0.2207E-01 0.3766E-02 0.6017E-03 -0.3380E 00 0.4844E 00

```
-0.2044E C0  C.2446F-01


    ROW  13
 0.8369E-03  0.1971F-07 -C.5510E-02  0.4446F-03 -0.5721E-01  0.1193F 00 -0.4014F-01  0.2097E-02 -0.1154F 00 -0.6027E-01
 0.7723E-01 -C.1210E-01  C.7013E 01


    ROW  14
-0.1383E-02 -0.1952E-02  0.7578F-02 -0.6797E-03  0.7757E-01 -0.1654F 00  0.6099F-01 -0.3371F-02  0.1764F 00  0.1255E-01
-0.8299E-01  0.1549E-01 -C.7924F 01  C.9655F 01


    ROW  15
 0.774RE-03  0.2627E-03 -C.2971E-02  C.3099E-03 -0.3150F-01  0.6856F-01 -0.2811E-01  0.1692E-02 -0.7923E-01  0.2955E-01
 C.2007E-01 -C.5493E-02  C.2551E C1 -C.3320E 01  0.123RE 01


    ROW  16
-0.1355E-C3  C.7C43E-04  0.3095F-03 -0.4043F-04  0.3662E-02 -0.8044E-02  0.3771E-02 -0.2517E-03  0.1025E-01 -0.7651E-02
-0.1321E-03  0.4923E-03 -C.2278E 00  0.3129F 00 -0.125RE 00  0.1426F-01

DATA POINTS WILL BE GENERATED IN POLAR COORDINATES

ON THE BOUNDARY THERE WILL BE     7 POINTS WITH THETA INCREMENTED BY 15.0 DEGREES STARTING AT 0.  THERE WILL BE     2
RAYS 90.0 DEGREES APART BEGINNING AT THETA = 0. WITH     7 POINTS A DISTANCE  0.50 APART ON EACH RAY.

NC TAPE FOR PLOTTING PURPOSES WILL BE GENERATED
```

XXVIII. (continued)

EXACT VALUE OF TOTAL COMPLEMENTARY ENERGY = -4.0670586

APPROXIMATE TOTAL COMPLEMENTARY ENERGY = -4.0670591

X = 1.000	Y = 0.0	SIG X= -1.00000	SIG Y= 1.09017	T XY= 0.0	UX = 0.99569	UY = 0.0
R = 1.000	THET= 0.0	SIG R= -1.00000	SIG T= 1.09017	T RT= 0.0	UR = 0.99569	UT = 0.0
		DSG X= -0.00000	DSG Y= -0.00062	DTXY= -0.00000	DUX= -0.00591	DUY= 0.00000
		DSG R= -0.00000	DSG T= -0.00062	DTRT= -0.00000	DUR= -0.00591	DUT= 0.00000
X = 0.966	Y = 0.259	SIG X= -0.86799	SIG Y= 0.90971	T XY= -0.51170	UX = 0.95847	UY = 0.34616
R = 1.000	THET= 15.000	SIG R= -1.00000	SIG T= 1.04682	T RT= 0.0	UR = 1.01541	UT = 0.09829
		DSG X= 0.00000	DSG Y= 0.00003	DTXY= -0.00001	DUX= -0.00242	DUY= 0.00094
		DSG R= -0.00000	DSG T= 0.00003	DTRT= 0.0	DUR= -0.00220	DUT= 0.00115
X = 0.866	Y = 0.500	SIG X= -0.51015	SIG Y= 0.46954	T XY= -0.84844	UX = 0.85412	UY = 0.66692
R = 1.000	THET= 30.000	SIG R= -1.00000	SIG T= 0.95939	T RT= 0.0	UR = 1.07315	UT = 0.15051
		DSG X= 0.00005	DSG Y= 0.00016	DTXY= -0.00009	DUX= 0.00306	DUY= 0.00285
		DSG R= -0.00000	DSG T= 0.00021	DTRT= -0.00000	DUR= 0.00407	DUT= 0.00094
X = 0.707	Y = 0.707	SIG X= -0.04663	SIG Y= -0.04663	T XY= -0.95337	UX = 0.69570	UY = 0.94353
R = 1.000	THET= 45.000	SIG R= -1.00000	SIG T= 0.90674	T RT= 0.0	UR = 1.15911	UT = 0.17524
		DSG X= 0.00008	DSG Y= 0.00008	DTXY= -0.00008	DUX= 0.00418	DUY= 0.00367
		DSG R= -0.00000	DSG T= 0.00016	DTRT= -0.00000	DUR= 0.00556	DUT= -0.00036
X = 0.500	Y = 0.866	SIG X= 0.45563	SIG Y= -0.51479	T XY= -0.84041	UX = 0.49215	UY = 1.15024
R = 1.000	THET= 60.000	SIG R= -1.00000	SIG T= 0.96084	T RT= 0.0	UR = 1.24914	UT = 0.15291
		DSG X= 0.00027	DSG Y= 0.00009	DTXY= -0.00016	DUX= 0.00274	DUY= 0.00194
		DSG R= -0.00000	DSG T= 0.00036	DTRT= 0.00000	DUR= 0.00296	DUT= -0.00145
X = 0.259	Y = 0.966	SIG X= 0.90935	SIG Y= -0.86291	T XY= -0.51161	UX = 0.25567	UY = 1.29751
R = 1.000	THET= 75.000	SIG R= -1.00000	SIG T= 1.04644	T RT= 0.0	UR = 1.31947	UT = 0.09896
		DSG X= -0.00054	DSG Y= -0.00004	DTXY= 0.00015	DUX= 0.00051	DUY= -0.00361
		DSG R= -0.00000	DSG T= -0.00058	DTRT= -0.00000	DUR= -0.00336	DUT= -0.00142
X = 0.000	Y = 1.000	SIG X= 1.10938	SIG Y= -1.00000	T XY= -0.00000	UX = 0.00000	UY = 1.34596
R = 1.000	THET= 90.000	SIG R= -1.00000	SIG T= 1.10938	T RT= 0.0	UR = 1.34596	UT = 0.00000
		DSG X= 0.00029	DSG Y= -0.00000	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00642
		DSG R= -0.00000	DSG T= 0.00029	DTRT= -0.00000	DUR= -0.00642	DUT= 0.00000

X = 1.000	Y = 0.0	SIG X= -1.00000	SIG Y= 1.09017	T XY= 0.0	UX = 0.99569	UY = 0.0
R = 1.000	THET= 0.0	SIG R= -1.00000	SIG T= 1.09017	T RT= 0.0	UR = 0.99569	UT = 0.0
		DSG X= -0.00000	DSG Y= -0.00062	DTXY= -0.00000	DUX= -0.00591	DUY= 0.00000
		DSG R= -0.00000	DSG T= -0.00062	DTRT= -0.00000	DUR= -0.00591	DUT= 0.00000
X = 1.500	Y = 0.0	SIG X= -0.46452	SIG Y= 0.43792	T XY= 0.0	UX = 0.68548	UY = 0.0
R = 1.500	THET= 0.0	SIG R= -0.46452	SIG T= 0.43792	T RT= 0.0	UR = 0.68548	UT = 0.0
		DSG X= 0.00004	DSG Y= 0.00030	DTXY= -0.00000	DUX= -0.00591	DUY= 0.00000
		DSG R= 0.00004	DSG T= 0.00030	DTRT= -0.00000	DUR= -0.00591	DUT= 0.00000
X = 2.000	Y = 0.0	SIG X= -0.27585	SIG Y= 0.24482	T XY= 0.0	UX = 0.52538	UY = 0.0
R = 2.000	THET= 0.0	SIG R= -0.27585	SIG T= 0.24482	T RT= 0.0	UR = 0.52538	UT = 0.0
		DSG X= -0.00003	DSG Y= -0.00013	DTXY= -0.00000	DUX= -0.00592	DUY= 0.00000
		DSG R= -0.00003	DSG T= -0.00013	DTRT= -0.00000	DUR= -0.00592	DUT= 0.00000
X = 2.500	Y = 0.0	SIG X= -0.18268	SIG Y= 0.15739	T XY= 0.0	UX = 0.42602	UY = 0.0
R = 2.500	THET= 0.0	SIG R= -0.18268	SIG T= 0.15739	T RT= 0.0	UR = 0.42602	UT = 0.0
		DSG X= -0.00008	DSG Y= -0.00032	DTXY= 0.0	DUX= -0.00591	DUY= 0.0

XXVIII. (continued)

		DSG R= -0.00008	DSG T= -0.00032	DTRT= 0.0	DUR= -0.00591	DUT= 0.0	
X = 3.000	Y = 0.0	SIG X= -0.12973	SIG Y= 0.10972	T XY= 0.0	UX = 0.35827	UY = 0.0	
R = 3.000	THET= 0.0	SIG R= -0.12973	SIG T= 0.10972	T RT= 0.0	UR = 0.35827	UT = 0.0	
		DSG X= 0.00003	DSG Y= -0.00024	DTXY= 0.0	DUX= -0.00588	DUY= 0.0	
		DSG R= 0.00003	DSG T= -0.00024	DTRT= 0.0	DUR= -0.00588	DUT= 0.0	
X = 3.500	Y = 0.0	SIG X= -0.09684	SIG Y= 0.08081	T XY= 0.0	UX = 0.30914	UY = 0.0	
R = 3.500	THET= 0.0	SIG R= -0.09684	SIG T= 0.08081	T RT= 0.0	UR = 0.30914	UT = 0.0	
		DSG X= 0.00018	DSG Y= -0.00013	DTXY= 0.00000	DUX= -0.00582	DUY= -0.00000	
		DSG R= 0.00018	DSG T= -0.00013	DTRT= 0.00000	DUR= -0.00582	DUT= -0.00000	
X = 4.000	Y = 0.0	SIG X= -0.07503	SIG Y= 0.06197	T XY= 0.0	UX = 0.27188	UY = 0.0	
R = 4.000	THET= 0.0	SIG R= -0.07503	SIG T= 0.06197	T RT= 0.0	UR = 0.27188	UT = 0.0	
		DSG X= 0.00031	DSG Y= -0.00006	DTXY= 0.0	DUX= -0.00573	DUY= 0.0	
		DSG R= 0.00031	DSG T= -0.00006	DTRT= 0.0	DUR= -0.00573	DUT= 0.0	
X = 0.000	Y = 1.000	SIG X= 1.10938	SIG Y= -1.00000	T XY= -0.00000	UX = 0.00000	UY = 1.34596	
R = 1.000	THET= 90.000	SIG R= -1.00000	SIG T= 1.10938	T RT= 0.0	UR = 1.34596	UT = 0.00000	
		DSG X= 0.00029	DSG Y= -0.00000	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00642	
		DSG R= -0.00000	DSG T= 0.00029	DTRT= -0.00000	DUR= -0.00642	DUT= 0.00000	
X = 0.000	Y = 1.500	SIG X= 0.43628	SIG Y= -0.46575	T XY= -0.00000	UX = 0.00000	UY = 0.92264	
R = 1.500	THET= 90.000	SIG R= -0.46575	SIG T= 0.43628	T RT= -0.00000	UR = 0.92264	UT = 0.00000	
		DSG X= 0.00024	DSG Y= 0.00007	DTXY= -0.00000	DUX= -0.00000	DUY= -0.00642	
		DSG R= 0.00002	DSG T= 0.00024	DTRT= 0.00000	DUR= -0.00642	DUT= 0.00000	
X = 0.000	Y = 2.000	SIG X= 0.24513	SIG Y= -0.27338	T XY= -0.00000	UX = 0.00000	UY = 0.70338	
R = 2.000	THET= 90.000	SIG R= -0.27338	SIG T= 0.24513	T RT= -0.00000	UR = 0.70338	UT = 0.00000	
		DSG X= -0.00014	DSG Y= -0.00004	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00643	
		DSG R= -0.00004	DSG T= -0.00014	DTRT= -0.00000	DUR= -0.00643	DUT= 0.00000	
X = 0.000	Y = 2.500	SIG X= 0.15775	SIG Y= -0.17914	T XY= -0.00000	UX = 0.00000	UY = 0.56813	
R = 2.500	THET= 90.000	SIG R= -0.17914	SIG T= 0.15775	T RT= 0.00000	UR = 0.56813	UT = 0.00000	
		DSG X= -0.00026	DSG Y= -0.00003	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00642	
		DSG R= -0.00003	DSG T= -0.00026	DTRT= -0.00000	DUR= -0.00642	DUT= 0.00000	
X = 0.000	Y = 3.000	SIG X= 0.10997	SIG Y= -0.12622	T XY= -0.00000	UX = 0.00000	UY = 0.47647	
R = 3.000	THET= 90.000	SIG R= -0.12622	SIG T= 0.10997	T RT= 0.00000	UR = 0.47647	UT = 0.00000	
		DSG X= -0.00019	DSG Y= 0.00006	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00638	
		DSG R= 0.00006	DSG T= -0.00019	DTRT= -0.00000	DUR= -0.00638	DUT= 0.00000	
X = 0.000	Y = 3.500	SIG X= 0.08098	SIG Y= -0.09366	T XY= -0.00000	UX = 0.00000	UY = 0.41031	
R = 3.500	THET= 90.000	SIG R= -0.09366	SIG T= 0.08098	T RT= 0.00000	UR = 0.41031	UT = 0.00000	
		DSG X= -0.00011	DSG Y= 0.00016	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00631	
		DSG R= 0.00016	DSG T= -0.00011	DTRT= -0.00000	DUR= -0.00631	DUT= 0.00000	
X = 0.000	Y = 4.000	SIG X= 0.06208	SIG Y= -0.07224	T XY= -0.00000	UX = 0.00000	UY = 0.36032	
R = 4.000	THET= 90.000	SIG R= -0.07224	SIG T= 0.06208	T RT= 0.00000	UR = 0.36032	UT = 0.00000	
		DSG X= -0.00005	DSG Y= 0.00024	DTXY= 0.00000	DUX= -0.00000	DUY= -0.00620	
		DSG R= 0.00024	DSG T= -0.00005	DTRT= -0.00000	DUR= -0.00620	DUT= 0.00000	

7. INTERIOR REISSNER ENERGY

This program computes approximate solutions to the four standard sample problems in accordance with the general Reissner energy principle. Details of the mathematical model are given in Article 3 of Section VI. Unlike the potential and complementary energy programs, the Reissner energy program cannot handle finite region problems and does not have user supplied subroutines that provide for the treatment of additional sample problems and allow different assumed forms of the stresses and displacements to be used. The user can specify the number of harmonics and powers of r and the increment between successive harmonics and powers of r in the assumed forms of the stresses and displacements (equations (6.1.1) and (6.1.3)). However, the first harmonic is always the zeroth and the highest power of r is r^{-2} in the stresses and r^{-1} in the displacements. The fact that these quantities are fixed does not seem to be a significant loss of generality since there appears to be little reason why one would ever want to choose them differently in an infinite region problem. Cases II and IV, with constant stress at infinity, are treated by means of superposition, and the only part of the solution which is to be obtained by means of the Reissner energy method is that part which vanishes at infinity. The part that is non-zero at infinity is contained the known functions σ_{ro}, $\sigma_{\theta o}$, $\tau_{r\theta o}$, u_o, and v_o in the assumed forms (6.1.1). See Figure 16 for an illustration of this superposition model.

7a. Description of Input

All input quantities for this program not discussed below are thoroughly described in Articles 1c and 1d.

i) N, M1

N is the number of different powers of r in the assumed forms (6.1.1) and (6.1.3) for each of the stress or displacement variables. M1 is the total number of cosine or sine harmonics in the assumed forms for each of these variables. Of course, the zeroth sine harmonic is zero so there are actually (M1-1) non-trivial sine harmonics. The rows and columns of the coefficient matrix and right hand side vector corresponding to these zero sine harmonics are automatically eliminated.

ii) ISKIP, JSKIP

ISKIP is the increment between successive powers of r in the assumed forms and JSKIP is the increment between successive harmonics. Both ISKIP and JSKIP must be positive.

iii) KYKLE

This integer code determines the type of harmonics that will be included in the assumed form of the stresses and displacements:

KYKLE = 0 => σ_r, σ_θ, u = f(cos) ; $\tau_{r\theta}$, v = f(sin)

KYKLE = 1 => σ_r, σ_θ, $\tau_{r\theta}$, u, v = f(cos,sin)

The input format sheet for this program is given in Table XXIX.

7b. Description of Output

Most of the output for this program is the standard output already described in Article 1e. The only additional printout is the planar elastic compliances S_{11}, S_{12}, S_{22}, and S_{66} and the input quantities described in Article 7a.

7c. Sample Cases

In Table XXX the output from a run of the Reissner energy program is shown. As can be seen the results for both stresses and displacements are accurate. Note the significantly greater number of unknowns required in comparison with the complementary and potential energy methods.

Table XXIX. Input Format Sheet for Interior Reissner Energy Program

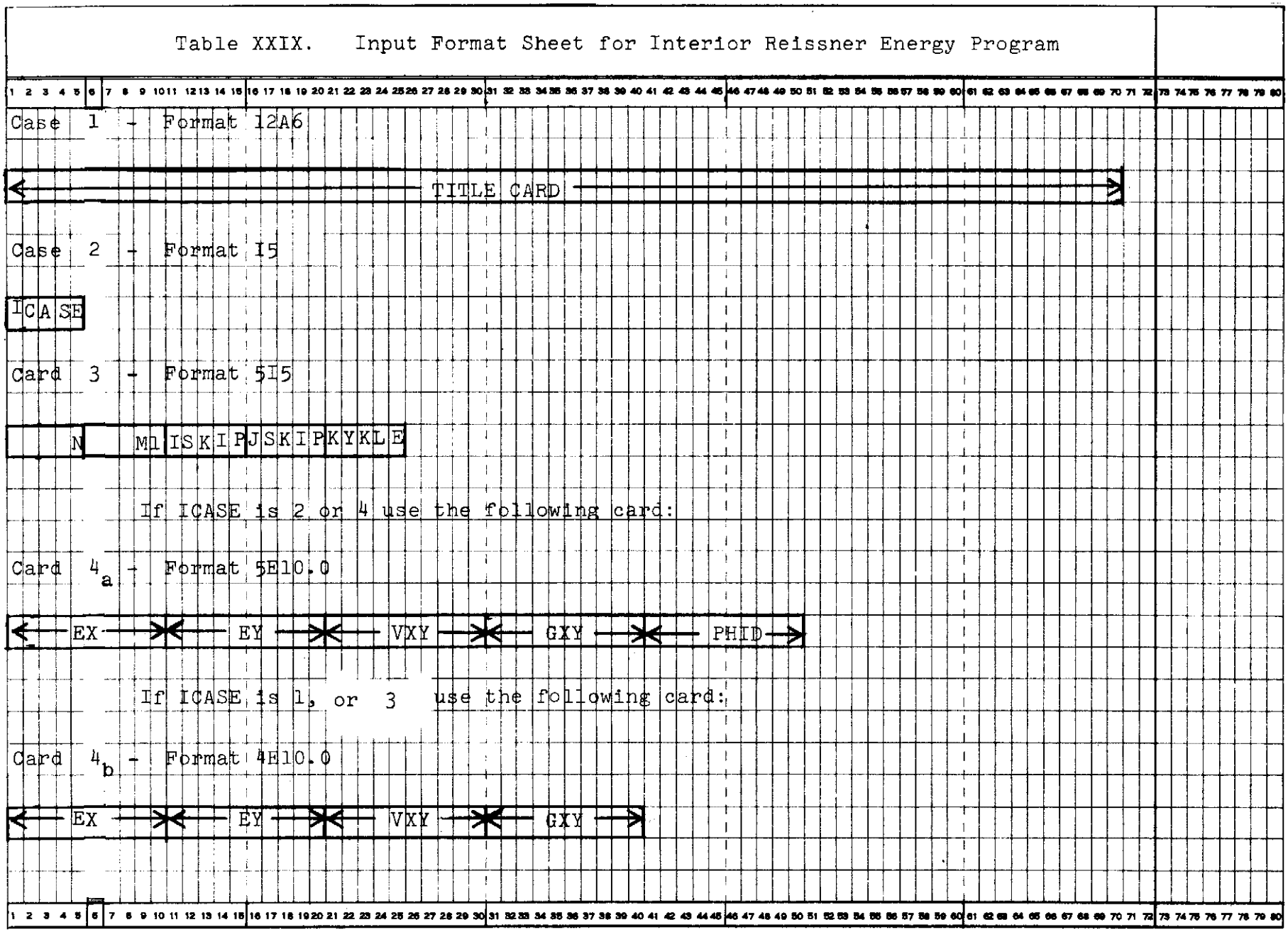


Table XXIX. (Continued)

Card 5 - Format 2I5

IRED	NBZXY

If IRED ≠ 0 read the following card:

Card 5_a - Format 14I5 - Repeat this card until I = NBZXY

IBZXY(I)

Card 6 - Format 3I5

IPR	IPLOT	IDEV

If IPR < 0 (polar coordinates) use the following 2 cards:

Card 7_P - Format 3I5

NR	NTR	NTB

Card 8_P - Format 3E10.0

DF	DTHR	DTB

Table XXIX. (Continued)

If IPR > 0 (rectangular coordinates) use the following 2 cards:

Card 7_R - Format 2I5

NPX	NPY

Card 8_R - Format 4E10.0

Conclude input with following card:

Card 9 - Format I5

NRUN

XXX. SAMPLE CASE OF INTERIOR REISSNER ENERGY PROGRAM

```
                    SAMPLE CASE  INTERIOR REISSNER ENERGY

                THE CASE BEING SOLVED IS CASE  1

EX =     1.500000    EY =     1.000000    VXY =     0.375000    GXY =     0.400000

S11 =    0.666667    S12 =   -0.250000    S22 =     1.000000    S66 =     2.500000

N =   3   M1 =   4   ISKIP =   2   JSKIP =   2   KYKLE =   0

MU1 =     0.0              0.796225          MU2 =     0.0              1.538188
```

XXX. (continued)

COEFFICIENT MATRIX

```
    ROW   1
-0.27490 01

    ROW   2
 0.26180 00 -0.13420 01

    ROW   3
 0.65450-01  0.13090 00 -0.13740 01

    ROW   4
 0.0         0.32720-01  0.13090 00 -0.13740 01

    ROW   5
-0.13740 01  0.13090 00  0.32720-01  0.0         -0.91630 00

    ROW   6
 0.13090 00 -0.67090 00  0.65450-01  0.16360-01  0.87270-01 -0.44720 00

    ROW   7
 0.32720-01  0.65450-01 -0.44720 00  0.65450-01  0.21820-01  0.43630-01 -0.45810 00

    ROW   8
 0.0         0.16360-01  0.65450-01 -0.68720 00  0.0         0.10910-01  0.43630-01 -0.45810 00

    ROW   9
-0.91630 00  0.87270-01  0.21820-01  0.0         -0.68720 00  0.65450-01  0.16360-01  0.0         -0.54980 00

    ROW  10
 0.87270-01 -0.44720 00  0.43630-01  0.10910-01  0.65450-01 -0.33540 00  0.32720-01  0.81810-02  0.52360-01 -0.26830 00

    ROW  11
 0.21820-01  0.43630-01 -0.45810 00  0.43630-01  0.16360-01  0.32720-01 -0.34360 00  0.32720-01  0.13090-01  0.26180-01
-0.27490 00

    ROW  12
 0.0         0.10910-01  0.43630-01 -0.45810 00  0.0         0.81810-02  0.32720-01 -0.34360 00  0.0         0.65450-02
```

```
0.26180D-01 -0.27490 00


   ROW  13
 0.91630 00  0.0        -0.65450-01  0.0         0.45810 00  0.0        -0.3272D-01  0.0         0.30540 00  0.0
-0.2182D-01  0.0        -0.27490 01

   ROW  14
 0.0         0.42540 00  0.0        -0.3272D-01  0.0         0.21270 00  0.0        -0.16360-01  0.0         0.14180 00
 0.0        -0.10910-01 -0.26180 00 -0.13420 01

   ROW  15
-0.6545D-01  0.0         0.45810 00  0.0        -0.3272D-01  0.0         0.22910 00  0.0        -0.2182D-01  0.0
 0.15270 00  0.0         0.6545D-01 -0.13090 00 -0.13740 01

   ROW  16
 0.0        -0.3272D-01  0.0         0.45810 00  0.0        -0.16360-01  0.0         0.22910 00  0.0        -0.10910-01
 0.0         0.15270 00  0.0         0.3272D-01 -0.13090 00 -0.13740 01

   ROW  17
 0.45810 00  0.0        -0.3272D-01  0.0         0.30540 00  0.0        -0.2182D-01  0.0         0.22910 00  0.0
-0.16360-01  0.0        -0.13740 01 -0.13090 00  0.3272D-01  0.0        -0.91630 00

   ROW  18
 0.0         0.21270 00  0.0        -0.16360-01  0.0         0.14180 00  0.0        -0.10910-01  0.0         0.10440 00
 0.0        -0.81810-02 -0.13090 00 -0.67090 00 -0.6545D-01  0.16360-01 -0.8727D-01 -0.44720 00

   ROW  19
-0.3272D-01  0.0         0.22910 00  0.0        -0.2182D-01  0.0         0.15270 00  0.0        -0.16360-01  0.0
 0.11450 00  0.0         0.3272D-01 -0.6545D-01 -0.68720 00 -0.6545D-01  0.2182D-01 -0.43630-01 -0.45810 00

   ROW  20
 0.0        -0.16360-01  0.0         0.22910 00  0.0        -0.10910-01  0.0         0.15270 00  0.0        -0.81810-02
 0.0         0.11450 00  0.0         0.16360-01 -0.6545D-01 -0.68720 00  0.0         0.10910-01 -0.43630-01 -0.45810 00

   ROW  21
 0.30540 00  0.0        -0.2182D-01  0.0         0.22910 00  0.0        -0.16360-01  0.0         0.10330 00  0.0
-0.13090-01  0.0        -0.91630 00 -0.8727D-01  0.2182D-01  0.0        -0.68720 00 -0.6545D-01  0.16360-01  0.0
-0.54980 00

   ROW  22
 0.0         0.14180 00  0.0        -0.10910-01  0.0         0.10440 00  0.0        -0.81810-02  0.0         0.69690-01
 0.0        -0.65450-02 -0.8727D-01 -0.44720 00 -0.43630-01  0.10910-01 -0.6545D-01 -0.33540 00 -0.3272D-01  0.81810-02
```

XXX. (continued)

```
-0.52360-01 -0.26080 00


    ROW  23
-0.21820-01  0.0         0.15270 00  0.0         -0.16360-01  0.0         0.11450 00  0.0         -0.13090-01  0.0
 0.91630-01  0.0         0.21820-01 -0.43630-01  -0.45810 00 -0.43630-01  0.16360-01 -0.32720-01  -0.34360 00 -0.32720-01
 0.13090-01 -0.26180-01  -0.27490 00


    ROW  24
 0.0        -0.10910-01  0.0         0.15270 00  0.0         -0.81810-02  0.0         0.11450 00  0.0         -0.65450-02
 0.0         0.91630-01  0.0         0.10910-01 -0.43630-01  -0.45810 00  0.0         0.81810-02 -0.32720-01  -0.34360 00
 0.0         0.65490-02 -0.26180-01  -0.27490 00


    ROW  25
-0.26180 00 -0.65450-01  0.13090 00  0.65450-01 -0.13090 00 -0.32720-01  0.65450-01  0.32720-01 -0.87270-01 -0.21820-01
 0.43630-01  0.21820-01 -0.26180 00  0.65450-01  0.13090 00 -0.65450-01 -0.13090 00  0.32720-01  0.65450-01 -0.32720-01
-0.87270-01  0.21820-01  0.43630-01 -0.21820-01 -0.35340 01


    ROW  26
-0.13090 00 -0.13090 00  0.0         0.13090 00 -0.65450-01 -0.65450-01  0.0         0.65450-01 -0.43630-01 -0.43630-01
 0.0         0.43630-01  0.13090 00 -0.13090 00  0.0          0.13090 00  0.65450-01 -0.65450-01  0.0          0.65450-01
 0.43630-01 -0.43630-01  0.0         0.43630-01  0.0         -0.36650 01


    ROW  27
 0.0        -0.65450-01 -0.13090 00  0.0         0.0         -0.32720-01 -0.65450-01  0.0         0.0         -0.21820-01
-0.43630-01  0.0         0.0         0.65450-01 -0.13090 00  0.0          0.0         0.32720-01 -0.65450-01  0.0
 0.0         0.21820-01 -0.43630-01  0.0        -0.13090 00  0.0         -0.36650 01


    ROW  28
-0.13090 00 -0.32720-01  0.65450-01  0.32720-01 -0.87270-01 -0.21820-01  0.43630-01  0.21820-01 -0.65450-01 -0.16360-01
 0.32720-01  0.16360-01 -0.13090 00  0.32720-01  0.65450-01 -0.32720-01 -0.87270-01  0.21820-01  0.43630-01 -0.21820-01
-0.65450-01  0.16360-01  0.32720-01 -0.16360-01 -0.17670 01  0.0        -0.65450-01 -0.11780 01


    ROW  29
-0.65450-01 -0.65450-01  0.0         0.65450-01 -0.43630-01 -0.43630-01  0.0         0.43630-01 -0.32720-01 -0.32720-01
 0.0         0.32720-01  0.65450-01 -0.65450-01  0.0         0.65450-01  0.43630-01 -0.43630-01  0.0         0.43630-01
 0.32720-01 -0.32720-01  0.0         0.32720-01  0.0        -0.10330 01  0.0         0.0        -0.12220 01


    ROW  30
 0.0        -0.32720-01 -0.65450-01  0.0         0.0         -0.21820-01 -0.43630-01  0.0         0.0         -0.16360-01
-0.32720-01  0.0         0.0         0.32720-01 -0.65450-01  0.0          0.0         0.21820-01 -0.43630-01  0.0
 0.0         0.16360-01 -0.32720-01  0.0        -0.65450-01  0.0         -0.18330 01 -0.43630-01  0.0        -0.12220 01


    ROW  31
-0.87270-01 -0.21820-01  0.43630-01  0.21820-01 -0.65450-01 -0.16360-01  0.32720-01  0.16360-01 -0.52360-01 -0.13090-01
```

```
 0.26180-01  0.13090-01 -0.87270-01  0.21820-01  0.43630-01 -0.21820-01 -0.65450-01  0.16360-01  0.32720-01 -0.16360-01
-0.52360-01  0.13090-01  0.26180-01 -0.13090-01 -0.11780 01  0.0        -0.43630-01 -0.98360 00  0.0        -0.32720-01
-0.70690 00

    ROW  32
-0.43630-01 -0.43630-01  0.0         0.43630-01 -0.32720-01 -0.32720-01  0.0         0.32720-01 -0.26180-01 -0.26180-01
 0.0         0.26180-01  0.43630-01 -0.43630-01  0.0         0.43630-01  0.32720-01 -0.32720-01  0.0         0.32720-01
 0.26180-01 -0.26180-01  0.0         0.26180-01  0.0        -0.12220 01  0.0         0.0        -0.91630 00  0.0
 0.0        -0.73300 00

    ROW  33
 0.0        -0.21820-01 -0.43630-01  0.0         0.0        -0.16360-01 -0.32720-01  0.0         0.0        -0.13090-01
-0.26180-01  0.0         0.0         0.21820-01 -0.43630-01  0.0         0.0         0.16360-01 -0.32720-01  0.0
 0.0         0.13090-01 -0.26180-01  0.0        -0.43630-01  0.0        -0.12220 01 -0.32720-01  0.0        -0.91630 00
-0.26180-01  0.0        -0.73300 00

    ROW  34
-0.31420 01  0.0         0.0         0.0        -0.15710 01  0.0         0.0         0.0        -0.10470 01  0.0
 0.0         0.0         0.31420 01  0.0         0.0         0.0         0.15710 01  0.0         0.0         0.0
 0.10470 01  0.0         0.0         0.0         0.0         0.0         0.0         0.0         0.0         0.0
 0.0         0.0         0.0         0.0

    ROW  35
 0.0        -0.15710 01  0.0         0.0         0.0        -0.78540 00  0.0         0.0         0.0        -0.52360 00
 0.0         0.0         0.0         0.15710 01  0.0         0.0         0.0         0.78540 00  0.0         0.0
 0.0         0.52360 00  0.0         0.0        -0.31420 01  0.0         0.0        -0.15710 01  0.0         0.0
-0.10470 01  0.0         0.0         0.0         0.0

    ROW  36
 0.0         0.0        -0.15710 01  0.0         0.0         0.0        -0.78540 00  0.0         0.0         0.0
-0.52360 00  0.0         0.0         0.0         0.15710 01  0.0         0.0         0.0         0.78540 00  0.0
 0.0         0.0         0.52360 00  0.0         0.0        -0.62830 01  0.0         0.0        -0.31420 01  0.0
 0.0        -0.20940 01  0.0         0.0         0.0         0.0

    ROW  37
 0.0         0.0         0.0        -0.15710 01  0.0         0.0         0.0        -0.78540 00  0.0         0.0
 0.0        -0.52360 00  0.0         0.0         0.0         0.15710 01  0.0         0.0         0.0         0.78540 00
 0.0         0.0         0.0         0.52360 00  0.0         0.0        -0.94250 01  0.0         0.0        -0.47120 01
 0.0         0.0        -0.31420 01  0.0         0.0         0.0         0.0

    ROW  38
-0.47120 01  0.0         0.0         0.0        -0.31420 01  0.0         0.0         0.0        -0.23560 01  0.0
 0.0         0.0         0.15710 01  0.0         0.0         0.0         0.10470 01  0.0         0.0         0.0
 0.78540 00  0.0         0.0         0.0         0.0         0.0         0.0         0.0         0.0         0.0
 0.0         0.0         0.0         0.0         0.0         0.0         0.0         0.0
```

ROW 39

0.0	-0.2356D 01	0.0	0.0	0.0	-0.1571D 01	0.0	0.0	0.0	-0.1178D 01
0.0	0.0	0.0	0.7854D 00	0.0	0.0	0.0	0.5236D 00	0.0	0.0
0.0	0.3927D 00	0.0	0.0	-0.1571D 01	0.0	0.0	-0.1047D 01	0.0	0.0
-0.7854D 00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	

ROW 40

0.0	0.0	-0.2356D 01	0.0	0.0	0.0	-0.1571D 01	0.0	0.0	0.0
-0.1178D 01	0.0	0.0	0.0	0.7854D 00	0.0	0.0	0.0	0.5236D 00	0.0
0.0	0.0	0.3927D 00	0.0	0.0	-0.3142D 01	0.0	0.0	-0.2094D 01	0.0
0.0	-0.1571D 01	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

ROW 41

0.0	0.0	0.0	-0.2356D 01	0.0	0.0	0.0	-0.1571D 01	0.0	0.0
0.0	-0.1178D 01	0.0	0.0	0.0	0.7854D 00	0.0	0.0	0.0	0.5236D 00
0.0	0.0	0.0	0.3927D 00	0.0	0.0	-0.4712D 01	0.0	0.0	-0.3142D 01
0.0	0.0	-0.2356D 01	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0									

ROW 42

-0.5236D 01	0.0	0.0	0.0	-0.3927D 01	0.0	0.0	0.0	-0.3142D 01	0.0
0.0	0.0	0.1047D 01	0.0	0.0	0.0	0.7854D 00	0.0	0.0	0.0
0.6283D 00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0								

ROW 43

0.0	-0.2618D 01	0.0	0.0	0.0	-0.1963D 01	0.0	0.0	0.0	-0.1571D 01
0.0	0.0	0.0	0.5236D 00	0.0	0.0	0.0	0.3927D 00	0.0	0.0
0.0	0.3142D 00	0.0	0.0	-0.1047D 01	0.0	0.0	-0.7854D 00	0.0	0.0
-0.6283D 00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0							

ROW 44

0.0	0.0	-0.2618D 01	0.0	0.0	0.0	-0.1963D 01	0.0	0.0	0.0
-0.1571D 01	0.0	0.0	0.0	0.5236D 00	0.0	0.0	0.0	0.3927D 00	0.0
0.0	0.0	0.3142D 00	0.0	0.0	-0.2094D 01	0.0	0.0	-0.1571D 01	0.0
0.0	-0.1257D 01	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0						

ROW 45

0.0	0.0	0.0	-0.2618D 01	0.0	0.0	0.0	-0.1963D 01	0.0	0.0
0.0	-0.1571D 01	0.0	0.0	0.0	0.5236D 00	0.0	0.0	0.0	0.3927D 00
0.0	0.0	0.0	0.3142D 00	0.0	0.0	-0.3142D 01	0.0	0.0	-0.2356D 01
0.0	0.0	-0.1885D 01	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0					

ROW 46

```
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.31420 01   0.0          0.0          0.0          0.15710 01   0.0          0.0
 0.0          0.10470 01   0.0          0.0         -0.31420 01   0.0          0.0         -0.15710 01   0.0          0.0
-0.10470 01   0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0

   ROW  47
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.62830 01   0.0          0.0          0.0          0.31420 01   0.0
 0.0          0.0          0.20940 01   0.0          0.0         -0.31420 01   0.0          0.0         -0.15710 01   0.0
 0.0         -0.10470 01   0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0          0.0

   ROW  48
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.94250 01   0.0          0.0          0.0          0.47120 01
 0.0          0.0          0.0          0.31420 01   0.0          0.0         -0.31420 01   0.0          0.0         -0.15710 01
 0.0          0.0         -0.10470 01   0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0          0.0

   ROW  49
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.15710 01   0.0          0.0          0.0          0.10470 01   0.0          0.0
 0.0          0.78540 00   0.0          0.0         -0.31420 01   0.0          0.0         -0.20940 01   0.0          0.0
-0.15710 01   0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0

   ROW  50
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.31420 01   0.0          0.0          0.0          0.20940 01   0.0
 0.0          0.0          0.15710 01   0.0          0.0         -0.31420 01   0.0          0.0         -0.20940 01   0.0
 0.0         -0.15710 01   0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0

   ROW  51
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.47120 01   0.0          0.0          0.0          0.31420 01
 0.0          0.0          0.0          0.23560 01   0.0          0.0         -0.31420 01   0.0          0.0         -0.20940 01
 0.0          0.0         -0.15710 01   0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0

   ROW  52
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.10470 01   0.0          0.0          0.0          0.78540 00   0.0          0.0
 0.0          0.62830 00   0.0          0.0         -0.31420 01   0.0          0.0         -0.23560 01   0.0          0.0
-0.10050 01   0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0
```

XXX. (continued)

```
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0

 ROW  53
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.12570 01   0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0         -0.18850 01   0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0

 ROW  54
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0          0.18850 01   0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
 0.0          0.0         -0.18850 01   0.0          0.0          0.0
 0.0          0.0          0.0          0.0          0.0          0.0
```

XXX. (continued)

```
RIGHT HAND SIDE VECTOR
  0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
  0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
  0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
  0.0          0.0          0.0          0.6283D 01   0.0          0.0          0.0          0.6283D 01   0.0          0.0
  0.0          0.6283D 01   0.0          0.0          0.0          0.0          0.0          0.0          0.0          0.0
  0.0          0.0          0.0          0.0

SOLUTION VECTOR
 -0.1000D 01 -0.7772D-14 -0.1926D 00 -0.2751D-01 -0.5329D-13  0.6661D-13  0.4393D 00  0.1053D 00  0.3197D-13 -0.7039D-13
 -0.2407D 00 -0.8559D-01  0.1000D 01 -0.1009D-13 -0.2889D-13 -0.5759D-02 -0.1165D-12  0.6439D-14 -0.1444D 00  0.4157D-01
  0.1297D-12  0.2376D-13  0.2407D 00 -0.4459D-01  0.9780D-14 -0.4815D-01  0.2139D-03 -0.8982D-13  0.2889D 00  0.1353D-01
  0.1017D-12 -0.2407D 00 -0.1950D-01  0.1165D 01 -0.1792D 00  0.8287D-01  0.2332D-02  0.1532D-13  0.4267D-02 -0.1379D 00
 -0.6623D-02 -0.8882D-14  0.3109D-13  0.5510D-01  0.4003D-02  0.1707D 00 -0.2762D-01  0.7775D-03  0.4267D-02 -0.2758D-01
 -0.3974D-02 -0.3131D-13  0.5510D-01  0.3580D-02

PRODUCT OF COEFFICIENT MATRIX AND SOLUTION VECTOR
  0.2853D-15 -0.1473D-13 -0.7494D-15  0.3542D-15  0.4288D-14 -0.9941D-14  0.4163D-16  0.3114D-15  0.6236D-14 -0.8184D-14
  0.9714D-16  0.2012D-15 -0.2387D-13 -0.1425D-13 -0.1152D-14  0.1015D-15 -0.1491D-13 -0.9686D-14 -0.9853D-15  0.7373D-16
 -0.1331D-13 -0.7878D-14 -0.1388D-16  0.1128D-15  0.1986D-15 -0.3261D-14 -0.2463D-15  0.2825D-14 -0.2554D-14 -0.2862D-16
  0.1568D-14  0.6384D-15 -0.4424D-16  0.6283D 01  0.2271D-14 -0.3456D-14  0.4953D-15  0.6283D 01  0.2089D-15 -0.4718D-15
  0.2272D-15  0.6283D 01  0.1237D-14 -0.1069D-14  0.3036D-15  0.7176D-14  0.2470D-14 -0.7017D-15  0.7219D-14  0.3275D-14
 -0.5378D-15 -0.1346D-14 -0.2776D-15 -0.6072D-17

RELATIVE ERROR OF INVERSION PROCESS = 0.492479E-14

CONDITION NUMBER OF COEFFICIENT MATRIX = 0.235943E 05

UPPER BOUND OF RELATIVE ERROR OF SOLUTION VECTOR = 0.266365E-08

NCOEF= 0.443923E 02    NCOEFI= 0.531496E 03    NRHS= 0.108828E 02    NRCK= 0.535955E-13    NSOL= 0.199493E 01
```

INVERSE OF COEFFICIENT MATRIX - COEFI

ROW 1
-0.33200 01

ROW 2
-0.6461D 00 -0.4342D 01

ROW 3
-0.8939D-01 -0.3457D 00 -0.6073D 01

ROW 4
-0.3419D-02 -0.4537D-01 -0.9529D 00 -0.6344D 01

ROW 5
0.1197D 02 0.2106D 01 0.1986D 00 -0.6432D-14 -0.4561D 02

ROW 6
0.2513D 01 0.1640D 02 0.1193D 01 0.1361D 00 -0.8425D 01 -0.6270D 02

ROW 7
0.4608D 00 0.1743D 01 0.2215D 02 0.1580D 01 -0.8458D 00 -0.6062D 01 -0.9208D 02

ROW 8
0.2091D-01 0.2722D 00 0.2402D 01 0.2402D 02 0.4525D-13 -0.8166D 00 -0.7427D 01 -0.1116D 03

ROW 9
-0.9504D 01 -0.1598D 01 -0.6920D-01 0.3419D-02 0.3707D 02 0.6439D 01 0.3850D 00 -0.2051D-01 -0.3042D 02

ROW 10
-0.2070D 01 -0.1318D 02 -0.8998D 00 -0.9074D-01 0.7021D 01 0.5080D 02 0.4597D 01 0.5444D 00 -0.5390D 01 -0.4136D 02

ROW 11
-0.4230D 00 -0.1577D 01 -0.1758D 02 -0.1080D 01 0.7929D 00 0.5513D 01 0.7748D 02 0.5342D 01 -0.3699D 00 -0.4196D 01
-0.6471D 02

ROW 12
-0.2051D-01 -0.2722D 00 -0.2070D 01 -0.1934D 02 -0.4627D-13 0.8166D 00 0.6680D 01 0.9738D 02 0.2051D-01 -0.5444D 00

XXX. (continued)

```
-0.49270 01 -0.87120 02


    ROW  13
-0.18170 01 -0.15020 00  0.89390-01  0.34130-02  0.85790 01  0.64860 00 -0.46080 00 -0.20510-01 -0.76150 01 -0.56440 00
 0.42300 00  0.20510-01 -0.33200 01


    ROW  14
 0.19800 00 -0.60840 00 -0.15630 00  0.24060-14 -0.10550 01  0.24340 01  0.83370 00 -0.29430-14  0.98860 00 -0.20280 01
-0.78160 00 -0.53690-14  0.59220 00 -0.33740 01


    ROW  15
 0.41520-01  0.42690-01 -0.38030 00 -0.79290-01 -0.15860 00 -0.28440 00  0.13260 01  0.47570 00  0.11710 00  0.29380 00
-0.94840 00 -0.47570 00 -0.41520-01  0.15630 00 -0.17260 01


    ROW  16
 0.34190-02  0.45370-01  0.44160-01 -0.27620 00  0.62270-14 -0.13610 00 -0.43630 00  0.13290 01 -0.34190-02  0.90740-01
 0.44500 00 -0.77950 00 -0.34190-02 -0.12990-13  0.79290-01 -0.11100 01


    ROW  17
 0.51900 01  0.37500-02 -0.47580 00  0.12160-14 -0.27540 02 -0.15000-01  0.25370 01  0.10790-13  0.25770 02  0.12500-01
-0.23790 01 -0.13940-13  0.15360 02 -0.31570 01  0.47580 00 -0.91990-15 -0.81770 02


    ROW  18
-0.41020 00  0.48420 01  0.47400 00 -0.13610 00  0.28180 01 -0.22270 02 -0.28310 01  0.81660 00 -0.29350 01  0.20010 02
 0.28250 01 -0.81660 00 -0.27510 01  0.18810 02 -0.13830 01  0.13610 00  0.16630 02 -0.11930 03


    ROW  19
-0.17360 00  0.75320-01  0.33320 01  0.31910 00  0.84580 00  0.60780 00 -0.18960 02 -0.25990 01 -0.67220 00 -0.96100 00
 0.16510 02  0.28480 01  0.17360 00 -0.83370 00  0.11310 02 -0.70030 00 -0.25370 01  0.82850 01 -0.78840 02


    ROW  20
-0.20510-01 -0.27220 00 -0.15780 00  0.24910 01 -0.35470-13  0.81660 00  0.23770 01 -0.16730 02  0.20510-01 -0.54440 00
-0.25360 01  0.13460 02  0.20510-01  0.84700-13 -0.47570 00  0.75110.01  0.99350-14 -0.81660 00  0.42820 01 -0.52300 02


    ROW  21
-0.38470 01  0.86050-01  0.34600 00 -0.17100-01  0.20120 02 -0.58350 00 -0.19250 01  0.10260 00 -0.19130 02  0.60590 00
 0.14500 01 -0.10260 00 -0.13280 02  0.29600 01 -0.59530 00  0.17100-01  0.76610 02 -0.16940 02  0.33610 01 -0.10260 00
-0.79590 02


    ROW  22
 0.24250 00 -0.38300 01 -0.95240-01  0.22680 00 -0.17650 01  0.20150 02  0.10130 01 -0.13610 01  0.19610 01 -0.19210 02
-0.12340 01  0.13610 01  0.23920 01 -0.17890 02  0.16100 01 -0.22680 00 -0.15780 02  0.12150 03 -0.10100 02  0.13610 01
```

```
 0.16940 02 -0.12810 03


    ROW  23
 0.13580 00 -0.24110 00 -0.34900 01 -0.18820 00 -0.79290 00 -0.58450-01  0.23630 02  0.22700 01  0.65710 00  0.56000 00
-0.21740 02 -0.26850 01 -0.13580 00  0.78160 00 -0.11790 02  0.82360 00  0.23790 01 -0.82790 01  0.84600 02 -0.50760 01
-0.32860 01  0.10320 02 -0.91990 02


    ROW  24
 0.20510-01  0.27220 00  0.11160 00 -0.27710 01  0.34380-13 -0.81660 00 -0.22730 01  0.21010 02 -0.20510-01  0.54440 00
 0.24790 01 -0.17900 02 -0.20510-01 -0.86440-13  0.47570 00 -0.79600 01 -0.12490-13  0.81660 00 -0.43170 01  0.56170 02
 0.10260 00 -0.13610 01  0.51330 01 -0.60680 02


    ROW  25
 0.37120 00  0.89930 00 -0.94690-01 -0.22680-01 -0.15910 01 -0.40810 01  0.45450 00  0.13610 00  0.13410 01  0.36430 01
-0.39770 00 -0.13610 00  0.41910 00 -0.13830 01 -0.56820-01  0.22680-01 -0.15810 01  0.40810 01  0.45450 00 -0.13610 00
 0.11970 01 -0.21920 01 -0.51130 00  0.13610 00 -0.16250 01


    ROW  26
 0.6545D-01  0.19420 00  0.63990 00 -0.78660-01 -0.15860 00 -0.73890 00 -0.54450 01  0.24380 00  0.93130-01  0.59680 00
 0.56050 01 -0.16070 00 -0.65450-01  0.15630 00 -0.67290 00 -0.48430-01  0.47580 00 -0.92850 00  0.23350 01  0.31740 00
-0.46570 00  0.85280 00 -0.13920 01 -0.32890 00  0.18940-01 -0.12080 01


    ROW  27
 0.34190-02  0.45370-01  0.13540 00  0.55530 00  0.74430-14 -0.13610 00 -0.64170 00 -0.53990 01 -0.34190-02  0.90740-01
 0.55910 00  0.57220 01 -0.34190-02 -0.31450-14  0.79290-01 -0.50380 00  0.19770-14  0.13610 00 -0.63190 00  0.24510 01
 0.17100-01 -0.22680 00  0.70990 00 -0.21230 01  0.22680-01 -0.25610-01 -0.87010 00


    ROW  28
-0.11740 01  0.25160-01  0.77660 00  0.13610 00  0.56210 01  0.28010 01 -0.38390 01 -0.81660 00 -0.49740 01 -0.37850 01
 0.34280 01  0.81660 00 -0.19870 01  0.91870 01  0.13250 00 -0.13610 00  0.84250 01 -0.31120 02 -0.16150 01  0.81660 00
-0.67410 01  0.21630 02  0.20260 01 -0.81660 00  0.69830 01 -0.32210 00 -0.13610 00 -0.34370 02


    ROW  29
-0.31720 00 -0.83370 00 -0.57950 00  0.63050 00  0.84580 00  0.33350 01  0.16700 02 -0.24140 01 -0.52860 00 -0.27790 01
-0.19450 02  0.19160 01  0.31720 00 -0.83370 00  0.49930 01  0.13200 00 -0.25370 01  0.55580 01 -0.23320 02 -0.95260 00
 0.26430 01 -0.55580 01  0.19450 02  0.10220 01 -0.15560-13  0.60970 01 -0.49030-02  0.11120 01 -0.33250 02


    ROW  30
-0.20510-01 -0.27220 00 -0.58630 00 -0.84400 00 -0.42990-13  0.81660 00  0.33410 01  0.18310 02  0.20510-01 -0.54440 00
-0.30720 01 -0.21660 02  0.20510-01  0.44820-14 -0.47570 00  0.40990 01 -0.79190-14 -0.81660 00  0.39610 01 -0.23360 02
-0.10260 00  0.13610 01 -0.45400 01  0.22540 02 -0.13610 00  0.21020 00  0.47390 01  0.81660 00 -0.31000 00 -0.28310 02


    ROW  31
 0.86900 00 -0.11270 01 -0.73400 00 -0.11340 00 -0.43930 01  0.20910 01  0.36620 01  0.68050 00  0.39630 01 -0.53330 00
```

XXX. (continued)

-0.32910 01 -0.68050 00 0.17650 01 -0.79290 01 -0.23540-01 0.11340 00 -0.78970 01 0.33310 02 0.88310 00 -0.68050 00
 0.65930D 01 -0.25400 02 -0.12540 01 0.68050 00 -0.58190 01 0.35520 00 0.11340 00 0.30120 02 -0.13900 01 -0.68050 00
-0.26940 02

ROW 32
 0.27940 00 0.66800 00 -0.31360 00 -0.60470 00 -0.79290 00 -0.27860 01 -0.10370 02 0.24880 01 0.51350 00 0.23780 01
 0.13290 02 -0.20720 01 -0.27940 00 0.78160 00 -0.54710 01 -0.30700-01 0.23790 01 -0.55520 01 0.28530 02 0.31810 00
-0.25690 01 0.57790 01 -0.25970 02 -0.37590 00 -0.56820-01 -0.53370 01 0.83380-01 -0.70130 00 0.30480 02 -0.21740 00
 0.10190 01 -0.28790 02

ROW 33
 0.20510-01 0.27220 00 0.49500 00 0.12470-01 0.41760-13 -0.81660 00 -0.31360 01 -0.11580 02 -0.20510-01 0.54440 00
 0.29590 01 0.15160 02 -0.20510-01 -0.25680-14 0.47570 00 -0.47050 01 0.51660-14 0.81660 00 -0.40290 01 0.28420 02
 0.10250 00 -0.13610 01 0.46540 01 -0.28390 02 0.13610 00 -0.23310 00 -0.43720 01 -0.81660 00 0.44650 00 0.27670 02
 0.68050 00 0.10330 00 -0.28000 02

ROW 34
-0.57920 00 0.63360-01 -0.61880-01 -0.44830-02 0.38100 01 -0.24100 00 0.23040 00 0.17390-01 -0.37080 01 0.19470 00
-0.18680 00 -0.13930-01 0.57920 00 0.50970-01 0.15930-01 -0.81260-03 0.30590-01 -0.30270 00 -0.92640-01 0.59910-02
-0.55990 00 0.27790 00 0.99010-01 -0.64730-02 0.61910-02 0.15910-01 0.13810-03 -0.10510 00 -0.69970-01 0.15280-02
 0.11590 00 0.64270-01 -0.24790-02 0.67830 00

ROW 35
 0.18380-01 -0.23790 01 -0.33550 00 -0.14990 00 0.22660-02 0.10330 02 0.11630 01 0.64280 00 -0.20650-01 -0.90170 01
-0.90530 00 -0.54960 00 -0.18380-01 -0.32960 00 0.32930-01 0.69440-02 -0.67970-02 0.82500 00 -0.20730 00 -0.11540-01
 0.10320 00 -0.15010 01 0.23600 00 -0.14670-02 -0.10250 01 0.40960-01 0.32610-01 0.18120 01 -0.11730 00 -0.13200 00
-0.89750 00 0.93270-01 0.10640 00 -0.14370 00 0.19520 01

ROW 36
 0.47010-02 0.10410-03 -0.62800 00 -0.12060 00 0.76300-16 -0.31220-03 0.43340 01 0.52400 00 -0.47010-02 0.20810-03
-0.43040 01 -0.45130 00 -0.47010-02 -0.34250-14 -0.18160 00 0.94110-02 0.97740-14 0.31220-03 0.49260 00 -0.33040-01
 0.23510-01 -0.52030-03 -0.51150 00 0.22920-01 0.52030-04 -0.73700 00 0.29380-01 -0.31220-03 0.19200 01 -0.12680 00
 0.26010-03 -0.13050 01 0.10680 00 0.15930-01 0.58400-02 0.48450 00

ROW 37
-0.11570-15 0.22310-14 0.35640-01 -0.31320 00 -0.10460-15 -0.26640-13 -0.80190-01 0.25190 01 0.19370-15 0.29570-13
 0.44550-01 -0.26040 01 0.29650-15 -0.86990-15 0.42090-14 -0.80420-01 -0.14430-14 0.50000-14 0.26730-01 0.12870-01
 0.97350-15 -0.31800-14 -0.44550-01 0.95830-01 0.10470-14 0.89090-02 -0.62180 00 -0.23540-14 -0.53460-01 0.18910 01
 0.19050-14 0.44550-01 -0.13500 01 0.37120-03 0.23290-01 -0.94670-02 0.25430 00

ROW 38
 0.94470 00 -0.13590 00 0.57770-01 0.13570-14 -0.10160 02 0.54370 00 -0.30810 00 -0.12840-13 0.11120 02 -0.45310 00
 0.28890 00 0.14760-13 -0.94470 00 -0.13590 00 -0.57770-01 0.59260-15 -0.81560-01 0.90620 00 0.30810 00 -0.31540-14
 0.16770 01 -0.90620 00 -0.28890 00 0.29830-14 0.12350-13 -0.57770-01 0.76040-15 0.18120 00 0.30810 00 0.21240-14
-0.22660 00 -0.28990 00 -0.39940-14 -0.11090 01 0.24640 00 0.43420-14 -0.76370-15 0.29570 01

XXX. (continued)

ROW 39

-0.49020-01 0.36030 01 0.35020 00 0.13470 00 -0.60410-02 -0.25690 02 -0.16600 01 -0.73610 00 0.55060-01 0.26200 02
0.14720 01 0.69840 00 0.49020-01 0.87900 00 -0.14320 00 -0.62720-01 0.19120-01 -0.40540 01 0.66270 00 0.34630 00
-0.27930 00 0.49730 01 -0.57260 00 -0.33120 00 0.13620 01 -0.20180 00 -0.85090-01 -0.29780 01 0.96550 00 0.46630 00
0.19090 01 -0.85920 00 -0.44400 00 0.26280 00 -0.28500 01 0.72780-01 -0.96100-02 -0.65700 00 0.69040 01

ROW 40

-0.18800-01 -0.41620-03 0.16510 01 0.26840 00 -0.38800-14 0.12490-02 -0.16350 02 -0.17450 01 0.18800-01 -0.83250-03
0.17730 02 0.16810 01 0.18800-01 0.19330-13 0.72640 00 -0.25340-01 -0.42820-13 -0.12490-02 -0.22980 01 -0.47620-01
-0.94020-01 0.20810-02 0.15320 01 0.15520 00 -0.20810-03 0.17780 01 -0.16340 00 0.12490-02 -0.79260 01 0.79090 00
-0.10410-02 0.57330 01 -0.61290 00 -0.32900-01 -0.43470-03 -0.11980 01 0.43410-01 -0.16640-13 -0.28170 00 0.44760 01

ROW 41

0.79100-15 -0.69200-14 -0.10120 00 0.94050 00 -0.10140-14 0.96030-13 0.22770 00 -0.10060 02 0.49780-15 -0.10950-12
-0.12650 00 0.11360 02 -0.78690-15 0.22920-14 -0.16110-13 0.33050 00 0.33330-14 -0.10460-13 -0.79900-01 -0.14260 00
-0.12270-14 0.30790-14 0.12650 00 -0.72400 00 -0.47840-14 -0.25300-01 0.17910 01 0.16540-13 0.15180 00 -0.79040 01
-0.15060-13 -0.12650 00 0.60440 01 -0.10540-02 -0.68250-01 0.17650-01 -0.73710 00 0.28870-14 0.37440-01 -0.14490 00
0.29320 01

ROW 42

-0.52460 00 0.72580-01 -0.26550-01 0.10410-03 0.63490 01 -0.30270 00 0.14670 00 -0.62430-03 -0.74160 01 0.25840 00
-0.14040 00 0.62430-03 0.52460 00 0.84960-01 0.41840-01 -0.10410-03 0.50970-01 -0.60350 00 -0.23950 00 0.62430-03
-0.11170 01 0.62830 00 0.23220 00 -0.62430-03 -0.61910-02 0.34200-01 -0.10410-03 -0.76130-01 -0.19260 00 0.62430-03
0.21060 00 0.18630 00 -0.62430-03 0.61580 00 -0.13120 00 -0.27390-02 0.53620-15 -0.18480 01 0.39490 00 0.10950-01
-0.19900-14 0.12320 01

ROW 43

0.30640-01 -0.18610 01 -0.14850 00 -0.48210-01 0.37760-02 0.15680 02 0.77940 00 0.28920 00 -0.34410-01 -0.17190 02
-0.72350 00 -0.28920 00 -0.30640-01 -0.54940 00 0.11020 00 0.48210-01 -0.11330-01 0.29110 01 -0.54970 00 -0.28920 00
0.17210 00 -0.34710 01 0.49380 00 0.28920 00 -0.65580 00 0.12940 00 0.48210-01 0.14840 01 -0.66460 00 -0.28920 00
-0.10110 01 0.60870 00 0.28920 00 -0.14700 00 0.14460 01 -0.53080-01 -0.55340-14 0.41060 00 -0.41740 01 0.21230 00
0.20300-13 -0.26370 00 0.27280 01

ROW 44

0.14100-01 0.31220-03 -0.10210 01 -0.14760 00 0.10050-14 -0.93650-03 0.11060 02 0.10330 01 -0.14100-01 0.62430-03
-0.12790 02 -0.10190 01 -0.14100-01 -0.17190-13 -0.54480 00 0.38810-01 0.23630-13 0.93650-03 0.21250 01 -0.12090 00
0.70920-01 -0.15610-02 -0.16500 01 0.59740-01 0.15610-03 -0.10400 01 0.11500 00 -0.93650-03 0.44670 01 -0.56880 00
0.78040-03 -0.37900 01 0.49260 00 0.16490-01 -0.49920-02 0.71380 00 -0.31300-01 0.11470-13 0.19940 00 -0.29740 01
0.11820 00 -0.82140-02 -0.15920 00 0.21150 01

ROW 45

-0.71920-15 0.49180-14 0.61160-01 -0.62790 00 0.12300-14 -0.68490-13 -0.13760 00 0.72460 01 -0.80430-15 0.77460-13
0.76450-01 -0.88470 01 0.13950-15 0.12220-14 0.11140-13 -0.31880 00 0.55740-15 -0.11450-13 0.45870-01 0.87090 00
-0.24070-14 0.17350-13 -0.76450-01 -0.35820 00 0.33370-14 0.15290-01 -0.11120 01 -0.12400-13 -0.91730-01 0.50380 01
0.11790-13 0.76450-01 -0.42440 01 0.63700-03 0.43730-01 -0.13150-01 0.46280 00 -0.20740-14 -0.26970-01 0.10590 00
-0.20350 01 0.14060-14 -0.14200-13 -0.94540-01 0.15300 01

ROW 46

0.72600-02 0.14940 01 0.15900 00 0.49990-03 -0.22660-02 -0.62450 01 -0.64660 00 0.16010-01 -0.49950-02 0.53380 01
0.54660 00 -0.22940-01 -0.72600-02 0.17620 01 -0.61030-01 0.10090-01 0.67970-02 -0.49110 01 0.33480 00 -0.62780-01
0.24970-01 0.35370 01 -0.32680 00 0.63740-01 -0.13390 00 -0.64020-01 0.81900-02 0.22740 01 0.30680 00 -0.53850-01
-0.25080 01 -0.28350 00 0.55750-01 0.13450 00 -0.77420 00 -0.43570-01 -0.74250-03 -0.24640 00 0.12780 01 0.11260 00
0.21090-02 0.13910 00 -0.68660 00 -0.69110-01 -0.12740-02 0.93400 00

ROW 47
-0.47010-02 -0.10410-03 0.51720 00 0.59310-01 -0.73200-15 0.31220-03 -0.24130 01 -0.28930 00 0.47010-02 -0.20810-03
0.21760 01 0.26510 00 0.47010-02 0.73960-14 0.11370 01 -0.22240-01 -0.10750-13 -0.31220-03 -0.36790 01 0.12570 00
-0.23510-01 0.52030-03 0.26390 01 -0.12230 00 -0.52030-04 -0.68950-02 -0.28900-01 0.31220-03 0.63310 00 0.15690 00
-0.26010-03 -0.82340 00 -0.15020 00 -0.71340-02 0.45890-01 -0.11410 00 -0.15860-01 -0.31980-14 -0.89800-01 0.33370 00
0.47170-01 0.27380-02 0.53080-01 -0.21970 00 -0.30990-01 0.25980-01 0.23830 00

ROW 48
0.13390-15 0.93070-16 0.11880-01 0.32000 00 -0.23420-15 0.66810-14 -0.26730-01 -0.13890 01 0.15650-15 -0.83090-14
0.14850-01 0.12540 01 -0.69750-16 -0.33640-15 -0.58900-14 0.82200 00 0.23470-15 0.15850-14 0.49090-02 -0.28600 01
0.83270-16 -0.23840-14 -0.14850-01 0.21540 01 0.29620-15 0.29700-02 0.49370-02 -0.36790-14 -0.17820-01 0.31210 00
0.39370-14 0.14850-01 -0.45000 00 0.12370-03 -0.99220-02 0.15860-01 -0.39010-01 0.40460-15 0.56380-02 -0.38580-01
0.12570 00 -0.27220-15 0.21340-14 0.24940-01 -0.93310-01 -0.24750-03 0.12400-01 0.11080 00

ROW 49
-0.53550-01 -0.19750 01 -0.40520-01 0.13470 00 0.60410-02 0.11900 02 0.46710 00 -0.73610 00 0.47510-01 -0.11490 02
-0.52200 00 0.69840 00 0.53550-01 -0.46990 01 0.25560 00 -0.62720-01 -0.18120-01 0.17850 02 -0.14640 01 0.34630 00
-0.23750 00 -0.13110 02 0.14210 01 -0.33120 00 0.13620 01 0.19690 00 -0.85090-01 -0.10820 02 -0.11610 01 0.46630 00
0.11710 02 0.11350 01 -0.44400 00 -0.23000 00 0.33640 00 0.72780-01 -0.96100-02 0.65700 00 -0.15940 01 -0.28170 00
0.37440-01 -0.42640 00 0.11970 01 0.19940 00 -0.26970-01 -0.19090 01 -0.89800-01 0.56380-02 0.69040 01

ROW 50
0.18800-01 0.41620-03 -0.13140 01 -0.83880-01 0.22080-14 -0.12490-02 0.67590 01 0.47630 00 -0.18800-01 0.83250-03
-0.65670 01 -0.48920 00 -0.18800-01 -0.26400-13 -0.35910 01 0.13250 00 0.34440-13 0.12490-02 0.15680 02 -0.83490 00
0.94020-01 -0.20810-02 -0.12700 02 0.85650 00 0.20910-03 0.45510 00 0.10490 00 -0.12490-02 -0.44610 01 -0.67520 00
0.10410-02 0.54310 01 0.70280 00 0.65630-02 -0.15350 00 0.87960-01 0.40910-01 0.10250-13 0.33060 00 -0.41500 00
-0.15630 00 -0.10950-01 -0.21230 00 0.38780 00 0.11670 00 -0.59980-01 -0.70640 00 -0.39550-01 0.33060 00 0.30100 01

ROW 51
-0.69650-15 0.36870-15 -0.60720-01 -0.96360 00 0.14360-14 -0.26980-13 0.13660 00 0.46570 01 -0.10580-14 0.32370-13
-0.75900-01 -0.46410 01 -0.14270-15 0.65020-14 0.25850-13 -0.28570 01 0.19930-14 -0.37080-13 -0.45540-01 0.13670 02
-0.36140-14 0.39870-13 0.75900-01 -0.11890 02 -0.21920-14 -0.15180-01 0.31080 00 0.22540-13 0.91070-01 -0.29750 01
-0.23960-13 -0.75900-01 0.36260 01 -0.63250-03 0.22710-01 -0.53070-01 0.34010-02 -0.18030-14 -0.19980-01 0.16770 00
-0.23410-01 0.11620-14 -0.79580-14 -0.12010 00 0.11590 00 0.12650-02 -0.35360-01 -0.37020 00 -0.19980-01 0.16090 00
0.17680 01

ROW 52
0.46290-01 0.11170 01 0.80620-02 -0.71640-01 -0.37760-02 -0.62880 01 -0.10520 00 0.42990 00 -0.42510-01 0.61500 01
0.13950 00 -0.42990 00 -0.46290-01 0.29370 01 -0.19450 00 0.71640-01 0.11330-01 -0.12300 02 0.12240 01 -0.42990 00
0.21260 00 0.95770 01 -0.12520 01 0.42990 00 -0.90980 00 -0.10130 00 0.71640-01 0.79090 01 0.66460 00 -0.42990 00
-0.92040 01 -0.69300 00 0.42990 00 0.12330 00 -0.11010 00 -0.54590-01 0.10810-14 -0.41060 00 0.56010 00 0.21840 00
-0.35620-14 0.28730 00 -0.45000 00 -0.16380 00 0.16300-14 0.11580 01 0.54990-01 -0.13880-14 -0.47510 01 -0.21840 00
0.69760-14 0.35930 01

```
     ROW  53
-0.14100-01 -0.31220-03  0.79500 00  0.24360-01  0.22720-14  0.93650-03 -0.33880 01  0.12580-02  0.14100-01 -0.62430-03
 0.34340 01  0.13190-01  0.14100-01  0.17660-13  0.24550 01 -0.13310 00 -0.17090-13 -0.93650-03 -0.12920 02  0.91090 00
-0.70520-01  0.15610-02  0.11010 02 -0.97200 00 -0.15610-03 -0.44860 00 -0.56950-01  0.93650-03  0.44670 01  0.46300 00
-0.78040-03 -0.55650 01 -0.53920 00  0.55600-03  0.10720 00  0.25870-01 -0.31300-01 -0.64790-14 -0.23130 00 -0.22250 00
 0.11820 00  0.82140-02  0.15920 00  0.51410-01 -0.94540-01  0.34030-01  0.46830 00  0.24940-01 -0.23130 00 -0.23640 01
-0.12010 00  0.16380 00  0.21150 01

     ROW  54
 0.49140-15 -0.11030-14  0.54700-01  0.64440 00 -0.97390-15  0.21240-13 -0.12310 00 -0.28700 01  0.72750-15 -0.24150-13
 0.68380-01  0.30790 01  0.38930-15 -0.94620-14 -0.19850-13  0.21270 01 -0.34570-14  0.48350-13  0.41030-01 -0.11790 02
 0.47990-14 -0.49990-13 -0.68380-01  0.11270 02  0.26340-14  0.13680-01 -0.39190 00 -0.24180-13 -0.82050-01  0.34390 01
 0.25480-13  0.68380-01 -0.40940 01  0.56990-03 -0.11140-01  0.38490-01  0.62190-01  0.14660-14  0.13190-01 -0.13510 00
-0.31530 00 -0.92150-15  0.56360-14  0.10360 00  0.1207D 00 -0.11400-02  0.22540-01  0.26830 00  0.13190-01 -0.12290 00
-0.15260 01 -0.60030-14  0.10360 00  0.14710 01

DATA POINTS WILL BE GENERATED IN POLAR COORDINATES

ON THE BOUNDARY THERE WILL BE    7 POINTS WITH THETA INCREMENTED BY 15.0 DEGREES STARTING AT 0.  THERE WILL BE    2
RAYS 90.0 DEGREES APART BEGINNING AT THETA = 0. WITH    7 POINTS A DISTANCE  0.50 APART ON EACH RAY.

NO TAPE FOR PLOTTING PURPOSES WILL BE GENERATED
```

X = 1.000	Y = 0.0	SIG X= -1.00576	SIG Y= 1.08751	T XY= 0.0	UX = 0.99957	UY = 0.0
R = 1.000	THET= 0.0	SIG R= -1.00576	SIG T= 1.08751	T RT= 0.0	UR = 0.99957	UT = 0.0
		DSG X= 0.00576	DSG Y= 0.00204	DTXY= -0.00000	DUX= 0.00021	DUY= 0.00000
		DSG R= 0.00576	DSG T= 0.00204	DTRT= -0.00000	DUR= 0.00021	DUT= 0.00000
X = 0.966	Y = 0.259	SIG X= -0.85992	SIG Y= 0.90807	T XY= -0.51702	UX = 0.95600	UY = 0.34703
R = 1.000	THET= 15.000	SIG R= -1.00000	SIG T= 1.04815	T RT= -0.00576	UR = 1.01324	UT = 0.08777
		DSG X= -0.00297	DSG Y= 0.00167	DTXY= 0.00531	DUX= 0.00006	DUY= -0.00033
		DSG R= -0.00000	DSG T= -0.00130	DTRT= 0.00576	DUR= -0.00003	DUT= -0.00033
X = 0.866	Y = 0.500	SIG X= -0.50552	SIG Y= 0.47192	T XY= -0.84649	UX = 0.85738	UY = 0.66986
R = 1.000	THET= 30.000	SIG R= -0.99424	SIG T= 0.96064	T RT= -0.00000	UR = 1.07745	UT = 0.15142
		DSG X= -0.00458	DSG Y= -0.00222	DTXY= -0.00205	DUX= -0.00021	DUY= -0.00009
		DSG R= -0.00576	DSG T= -0.00104	DTRT= -0.00000	DUR= -0.00023	DUT= 0.00003
X = 0.707	Y = 0.707	SIG X= -0.08391	SIG Y= -0.04239	T XY= -0.95185	UX = 0.70003	UY = 0.94690
R = 1.000	THET= 45.000	SIG R= -1.00000	SIG T= 0.90370	T RT= 0.00576	UR = 1.14456	UT = 0.17456
		DSG X= 0.00736	DSG Y= -0.00416	DTXY= -0.00160	DUX= -0.00015	DUY= 0.00030
		DSG R= -0.00000	DSG T= 0.00320	DTRT= -0.00576	DUR= 0.00019	DUT= 0.00032
X = 0.500	Y = 0.866	SIG X= 0.45586	SIG Y= -0.51855	T XY= -0.94386	UX = 0.49460	UY = 1.15986
R = 1.000	THET= 60.000	SIG R= -1.00576	SIG T= 0.94306	T RT= 0.00000	UR = 1.25177	UT = 0.15159
		DSG X= 0.00004	DSG Y= 0.00385	DTXY= 0.00330	DUX= 0.00029	DUY= 0.00022
		DSG R= 0.00576	DSG T= -0.00187	DTRT= -0.00000	DUR= 0.00033	DUT= -0.00014
X = 0.259	Y = 0.966	SIG X= 0.91382	SIG Y= -0.86568	T XY= -0.50705	UX = 0.25571	UY = 1.29408
R = 1.000	THET= 75.000	SIG R= -1.00000	SIG T= 1.04815	T RT= -0.00576	UR = 1.31517	UT = 0.08794
		DSG X= -0.00502	DSG Y= 0.00272	DTXY= -0.00441	DUX= 0.00047	DUY= -0.00018
		DSG R= -0.00000	DSG T= -0.00229	DTRT= 0.00576	DUR= -0.00006	DUT= -0.00050
X = 0.000	Y = 1.000	SIG X= 1.10508	SIG Y= -0.99424	T XY= -0.00000	UX = 0.00000	UY = 1.33994
R = 1.000	THET= 90.000	SIG R= -0.99424	SIG T= 1.10508	T RT= -0.00000	UR = 1.33994	UT = 0.00000
		DSG X= 0.00459	DSG Y= -0.00576	DTXY= -0.00000	DUX= 0.00000	DUY= -0.00039
		DSG R= -0.00576	DSG T= 0.00459	DTRT= 0.00000	DUR= -0.00039	DUT= -0.00000

X = 1.000	Y = 0.0	SIG X= -1.00576	SIG Y= 1.08751	T XY= 0.0	UX = 0.98957	UY = 0.0
R = 1.000	THET= 0.0	SIG R= -1.00576	SIG T= 1.08751	T RT= 0.0	UR = 0.98957	UT = 0.0
		DSG X= 0.00576	DSG Y= 0.00204	DTXY= -0.00000	DUX= 0.00021	DUY= 0.00000
		DSG R= 0.00576	DSG T= 0.00204	DTRT= -0.00000	DUR= 0.00021	DUT= 0.00000
X = 1.500	Y = 0.0	SIG X= -0.46434	SIG Y= 0.43878	T XY= 0.0	UX = 0.68003	UY = 0.0
R = 1.500	THET= 0.0	SIG R= -0.46434	SIG T= 0.43878	T RT= 0.0	UR = 0.68003	UT = 0.0
		DSG X= -0.00014	DSG Y= -0.00057	DTXY= -0.00000	DUX= -0.00046	DUY= 0.00000
		DSG R= -0.00014	DSG T= -0.00057	DTRT= -0.00000	DUR= -0.00046	DUT= 0.00000
X = 2.000	Y = 0.0	SIG X= -0.27643	SIG Y= 0.24519	T XY= 0.0	UX = 0.51967	UY = 0.0
R = 2.000	THET= 0.0	SIG R= -0.27643	SIG T= 0.24519	T RT= 0.0	UR = 0.51967	UT = 0.0
		DSG X= 0.00055	DSG Y= -0.00051	DTXY= -0.00000	DUX= -0.00020	DUY= 0.00000
		DSG R= 0.00055	DSG T= -0.00051	DTRT= -0.00000	DUR= -0.00020	DUT= 0.00000
X = 2.500	Y = 0.0	SIG X= -0.18275	SIG Y= 0.15725	T XY= 0.0	UX = 0.41991	UY = 0.0
R = 2.500	THET= 0.0	SIG R= -0.18275	SIG T= 0.15725	T RT= 0.0	UR = 0.41991	UT = 0.0
		DSG X= -0.00001	DSG Y= -0.00018	DTXY= 0.0	DUX= 0.00020	DUY= 0.0

		DSG R= -0.00001	DSG T= -0.00018	DTRT= 0.0	DUR= 0.00020	DUT= 0.0	
X = 3.000	Y = 0.0	SIG X= -0.12936	SIG Y= 0.10947	T XY= 0.0	UX = 0.35195	UY = 0.0	
R = 3.000	THET= 0.0	SIG R= -0.12936	SIG T= 0.10947	T RT= 0.0	UR = 0.35195	UT = 0.0	
		DSG X= -0.00034	DSG Y= 0.00001	DTXY= 0.0	DUX= 0.00044	DUY= 0.0	
		DSG R= -0.00034	DSG T= 0.00001	DTRT= 0.0	DUR= 0.00044	DUT= 0.0	
X = 3.500	Y = 0.0	SIG X= -0.09619	SIG Y= 0.08058	T XY= 0.0	UX = 0.30276	UY = 0.0	
R = 3.500	THET= 0.0	SIG R= -0.09619	SIG T= 0.08058	T RT= 0.0	UR = 0.30276	UT = 0.0	
		DSG X= -0.00047	DSG Y= 0.00009	DTXY= 0.00000	DUX= 0.00056	DUY= -0.00000	
		DSG R= -0.00047	DSG T= 0.00009	DTRT= 0.00000	DUR= 0.00056	DUT= -0.00000	
X = 4.000	Y = 0.0	SIG X= -0.07423	SIG Y= 0.06179	T XY= 0.0	UX = 0.26554	UY = 0.0	
R = 4.000	THET= 0.0	SIG R= -0.07423	SIG T= 0.06179	T RT= 0.0	UR = 0.26554	UT = 0.0	
		DSG X= -0.00049	DSG Y= 0.00012	DTXY= 0.0	DUX= 0.00061	DUY= 0.0	
		DSG R= -0.00049	DSG T= 0.00012	DTRT= 0.0	DUR= 0.00041	DUT= 0.0	
X = 0.000	Y = 1.000	SIG X= 1.10508	SIG Y= -0.99424	T XY= -0.00000	UX = 0.00000	UY = 1.33994	
R = 1.000	THET= 90.000	SIG R= -0.99424	SIG T= 1.10508	T RT= -0.00000	UR = 1.33994	UT = 0.00000	
		DSG X= 0.00459	DSG Y= -0.00576	DTXY= -0.00000	DUX= 0.00000	DUY= -0.00039	
		DSG R= -0.00576	DSG T= 0.00459	DTRT= 0.00000	DUR= -0.00039	DUT= -0.00000	
X = 0.000	Y = 1.500	SIG X= 0.43531	SIG Y= -0.46682	T XY= -0.00000	UX = 0.00000	UY = 0.91615	
R = 1.500	THET= 90.000	SIG R= -0.46682	SIG T= 0.43531	T RT= -0.00000	UR = 0.91615	UT = 0.00000	
		DSG X= 0.00120	DSG Y= 0.00110	DTXY= 0.00000	DUX= -0.00000	DUY= 0.00007	
		DSG R= 0.00110	DSG T= 0.00120	DTRT= -0.00000	DUR= 0.00007	DUT= 0.00000	
X = 0.000	Y = 2.000	SIG X= 0.24427	SIG Y= -0.27323	T XY= -0.00000	UX = 0.00000	UY = 0.59684	
R = 2.000	THET= 90.000	SIG R= -0.27323	SIG T= 0.24427	T RT= -0.00000	UR = 0.59684	UT = 0.00000	
		DSG X= 0.00071	DSG Y= -0.00019	DTXY= 0.00000	DUX= -0.00000	DUY= 0.00011	
		DSG R= -0.00019	DSG T= 0.00071	DTRT= -0.00000	DUR= 0.00011	DUT= 0.00000	
X = 0.000	Y = 2.500	SIG X= 0.15733	SIG Y= -0.17866	T XY= -0.00000	UX = 0.00000	UY = 0.56160	
R = 2.500	THET= 90.000	SIG R= -0.17866	SIG T= 0.15733	T RT= 0.00000	UR = 0.56160	UT = 0.00000	
		DSG X= 0.00015	DSG Y= -0.00051	DTXY= -0.00000	DUX= 0.00000	DUY= 0.00011	
		DSG R= -0.00051	DSG T= 0.00015	DTRT= 0.00000	DUR= 0.00011	DUT= -0.00000	
X = 0.000	Y = 3.000	SIG X= 0.10985	SIG Y= -0.12562	T XY= -0.00000	UX = 0.00000	UY = 0.46998	
R = 3.000	THET= 90.000	SIG R= -0.12562	SIG T= 0.10985	T RT= 0.00000	UR = 0.46998	UT = 0.00000	
		DSG X= -0.00007	DSG Y= -0.00054	DTXY= 0.00000	DUX= -0.00000	DUY= 0.00011	
		DSG R= -0.00054	DSG T= -0.00007	DTRT= -0.00000	DUR= 0.00011	DUT= 0.00000	
X = 0.000	Y = 3.500	SIG X= 0.08102	SIG Y= -0.09301	T XY= -0.00000	UX = 0.00000	UY = 0.40390	
R = 3.500	THET= 90.000	SIG R= -0.09301	SIG T= 0.08102	T RT= 0.00000	UR = 0.40390	UT = 0.00000	
		DSG X= -0.00015	DSG Y= -0.00049	DTXY= -0.00000	DUX= 0.00000	DUY= 0.00010	
		DSG R= -0.00049	DSG T= -0.00015	DTRT= 0.00000	DUR= 0.00010	DUT= -0.00000	
X = 0.000	Y = 4.000	SIG X= 0.06220	SIG Y= -0.07157	T XY= -0.00000	UX = 0.00000	UY = 0.35402	
R = 4.000	THET= 90.000	SIG R= -0.07157	SIG T= 0.06220	T RT= 0.00000	UR = 0.35402	UT = 0.00000	
		DSG X= -0.00016	DSG Y= -0.00043	DTXY= 0.00000	DUX= -0.00000	DUY= 0.00010	
		DSG R= -0.00043	DSG T= -0.00016	DTRT= -0.00000	DUR= 0.00010	DUT= 0.00000	

REFERENCES

1. Hearmon, R.F.S., Applied Anisotropic Elasticity, Oxford University Press (1961).

2. Lempriere, B.M., "Poisson's Ratio in Orthotropic Materials," AIAA Journal, Vol. 6, pp. 2226-27 (1968).

3. Hellinger, E., "Die Allgemeinen Ansätze der Mechanik der Kontinua," Enzyclopädie der Mathematischen Wissenschaften, Vol. 4, pp. 654-55 (1914).

4. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. 1, Interscience (1953).

5. Reissner, E., "On Variational Principles in Elasticity," Proceedings of Symposia in Applied Mathematics, Vol. 8, pp. 1-6 (1958).

6. Reissner, E., "On Some Variational Theorems in Elasticity," Problems of Continuum Mechanics, SIAM, pp. 370-81 (1961).

7. Synge, J.L., The Hypercircle in Mathematical Physics, Cambridge University Press (1957).

8. Lekhnitskii, S.G., Theory of Elasticity of an Anisotropic Elastic Body, Holden-Day (1963), (translation of 1950 Russian edition).

9. Lekhnitskii, S.G., Anisotropic Plates, Gordon and Breach Science Publishers (1968), (translation of 1957 Russian edition).

10. Savin, G.N., Stress Concentration Around Holes, Pergamon Press (1961), (translation of amended 1951 Russian edition).

11. Sokolnikoff, I.S., Mathematical Theory of Elasticity, McGraw-Hill (1956).

12. Sokolnikoff, I.S., "Approximate Methods of Solution of Two-Dimensional Problems in Anisotropic Elasticity," Proceedings of Symposia in Applied Mathematics, Vol. 3, pp. 1-11 (1950).

13. Yih-O-Tu, "Perturbation Solution of Plane Stress Problems in Anisotropic Elasticity," SIAM Journal of Applied Mathematics, Vol. 16, pp. 374-86 (1968).

14. Crandall, S.H., Engineering Analysis, McGraw-Hill (1956).

15. Hopper, A.T., Leissa, A.W., Hulbert, L.E., and Clausen, W.E., Numerical Analysis of Equilibrium and Eigenvalue Problems, Technical Report No. AFFDL-TR-67-121 (1967).

16. Ralston, A., A First Course in Numerical Analysis, McGraw-Hill (1965).

17. IBM Corp., System/360 Programmer's Manual for Scientific Subroutine Package (360A-CM-03X), Version III (1968).

18. Morley, L.S.D., "The Approximate Solution of Plate Problems," Proceedings of Ninth International Congress for Applied Mechanics, Vol. 6, pp. 22-9, Brussels (1956).

19. Mikhlin, Variational Methods in Mathematical Physics, MacMillan (1964), (translation of 1957 Russian edition).

20. Weinstein, A., "New Methods for the Estimation of Torsional Rigidity," Proceedings of Symposia in Applied Mathematics, Vol. 3, pp. 1-11 (1950).

21. Hulbert, L.E., The Numerical Solution of Two-Dimensional Problems of the Theory of Elasticity, Bulletin 198 of The Engineering Experiment Station of the Ohio State University (1963).

22. Lo, C.C., Niedenfuhr, F.W., and Leissa, A.W., Further Studies in the Application of the Point Matching Technique to Plate Bending and Other Harmonic and Biharmonic Boundary Value Problems, Technical Report No. AFFDL-TR-65-114 (1966).

23. Timoshenko, S., and Goodier, J.N., Theory of Elasticity, McGraw-Hill (1951).

24. Green, A.E., and Taylor, G.I., "Stress Systems in Aeolotropic Plates. I and III," Proceedings of the Royal Society of London, Vol. 173A, pp. 162-172 (1939), and Vol. 177A, pp. 181-195 (1941).

25. Green, A.E., "Stress Systems in Aeolotropic Plates. II and IV," Proceedings of the Royal Society of London, Vol. 173A, pp. 173-192 (1939), and Vol. 180A, pp. 173-208 (1942).

26. Taylor, R.L., Pister, K.S., and Herrmann, L.R., "On a Variational Theorem for Incompressible and Nearly-Incompressible Orthotropic Elasticity," Int. J. Solids and Structures, Vol. 4, pp. 875-883 (1968).

27. Gazis, D.C., Tadjbakhsh, I., and Toupin, R.A., "The Elastic Tensor of Given Symmetry Nearest to an Anisotropic Elastic Tensor," Acta. Cryst., Vol. 16, pp. 917-21 (1963).

28. Churchill, R.V., Complex Variables and Applications, McGraw-Hill (1960).

29. Albasiny, E.L., "Error in Digital Solution of Linear Problems," Error in Digital Computation, Vol. 1, Wiley, pp. 139-42 (1965).

30. Washizu, K., Variational Methods in Elasticity and Plasticity, Pergamon Press (1968).

31. Finlayson, B.A., and Scriven, L.E., "The Methods of Weighted Residuals - A Review," Applied Mechanics Reviews, Vol. 19, pp. 735-48 (1966).

32. Faddeeva, V.N., Computational Methods of Linear Algebra, Dover (1959).

33. Frazer, R.A., Duncan, W.J., and Collar, A.R., Elementary Matrices, Cambridge University Press (1946).

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Bell Aerosystems Company Buffalo, New York 14240	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP N/A

3. REPORT TITLE
APPROXIMATE SOLUTION OF PLANE ORTHOTROPIC ELASTICITY PROBLEMS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Final Report - April 1968 - October 1969

5. AUTHOR(S) (First name, middle initial, last name)
Joel T. Schimke John Garrison
Kevin Thomas

6. REPORT DATE November 1969	7a. TOTAL NO. OF PAGES 348	7b. NO. OF REFS 33

8a. CONTRACT OR GRANT NO. AF 33(615)-68-C-1560 b. PROJECT NO. 1467 c. Task 146702 d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFFDL-TR-69-84

10. DISTRIBUTION STATEMENT
Distribution of this document is unlimited. It may be released to the Clearinghouse, Department of Commerce, for sale to the general public.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Air Force Flight Dynamics Laboratory (FDTR) Wright-Patterson AFD, Ohio 45433

13. ABSTRACT The purpose of this study is to compare and evaluate a number of approximate methods for the solution of plane orthotropic elasticity problems. The methods under consideration fall into three main categories. The first category is perturbation methods, which are based on the assumption that the behavior of the orthotropic material under a particular loading is not greatly different than the behavior of some nearby isotropic material under the same loading. The second category is the so-called boundary residual methods. In these techniques the assumed families of functions are required to satisfy all the equations of elasticity internally but none of the boundary conditions. The boundary conditions are then satisfied approximately in accordance with the particular boundary residual criterion under consideration. The two criteria that are examined in this report are the integral least squares and boundary Reissner energy methods. The third category of approximate methods is that for which the assumed functions may have to satisfy some, but not all, of the equations of elasticity internally and certain of the boundary conditions. The other equations of elasticity and the other boundary conditions (if any) are then satisfied approximately in accordance with the particular method. The methods of this category which are considered are the potential, complementary, and Reissner energy methods.

DD FORM 1 NOV 65 1473

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Numerical Approximation						
Weighted Residual Methods						
Variational Methods						
Perturbation Methods						
Plane Orthotropic Elasticity						