THE USE OF OPTIMALITY CRITERIA IN AUTOMATED STRUCTURAL DESIGN

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Research in the field of structural optimization in the past decade has led to the development of a number of methods for the automated design of structures of least weight subjected to a multiplicity of loads. These methods which have been principally based upon the use of various algorithmic forms of numerical search have achieved a considerable measure of success for smaller scale problems. Such methods do suffer from the major disadvantage that computational costs increase rapidly with problem size, tending to impose an economic limit on the complexity of the structures which can be optimized. In order to circumvent these economic limitations, the entire problem of structural optimization has been reviewed. A novel approach to the least weight design of indeterminate structures under multiple loading conditions with strength, displacement and fabricational constraints has been developed which overcomes many of the shortcomings of direct numerical search methods. Details of the new approach, which has dramatically reduced the number of cycles compared with mathematical programming formulations are presented along with examples of applications to the weight optimization of representative structures. The incorporation of other types of constraint conditions is also discussed.

I. INTRODUCTION

The last decade has been a period of triumph and tragedy for the technology of structural optimization. At the start of the 60's the initial concepts of performing automatic optimization of complex structures were being defined and translated into working programs. At the heart of this development were the finite element methods of structural analysis and the rapidly expanding electronic computer capabilities. Finite element methods of analysis had been demonstrated to be reliable, accurate, relatively rapid, general in application to all classes of structures and above all, largely automatic in use. Only the minimal quantity of basic data was needed to specify the problem and the analysis methods could generate without further assistance from the engineer all manner of information concerning stresses, displacements and other response phenomena. Although, in some respects, still in their infancy at that time, finite element methods clearly offered major advantages over other analysis procedures while possessing great development potential. Electronic computers were also in a similar state.

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Their basic potentialities had been eminently proven by the socalled first generation mechanics and the second generation computers were being widely used. Third and later generation machines were being promised by the manufacturers which would reduce computation costs and calculation times by orders of magnitudes while providing infinitely large rapid access storages. In this aura of general optimism, it was confidently assumed that the significant computational costs associated with analysis of early finite element problems involving less than a hundred degrees of freedom would be reduced, possibly to inconsequential levels for the majority of analysis problems. Unfortunately, the ambition of the engineer/analyst also increased with passage of time and he was no longer satisfied with the numerical results generated by the small numbers of degrees of freedom. Larger and more complex, problems were undertaken involving the handling of correspondingly larger matrix arrays.

At the same time some, but not all, of the anticipated advances in computer technology have taken place. Computers are large and fast but the net effect to this date has been that the requirements of analytical complexity have consistently exceeded the advances in computational capabilities made available by the successive generations of computers. The analyst has been forced to solve increasingly complex problems with the result that analysis times and costs have tended to increase rather than decrease. It is true that more detailed information has been generated for the more complex models but this has always been at the behest of the design engineer. Although the net result of this has been the opposite of what was generally foreseen, the view in the early 60's was that analysis cost could eventually become negligible.

The development of operations research methods arose through the requirements for determining optimal processes in a wide variety of industrial applications. Through application of various forms of mathematical programming techniques the determination of optimal values of complex functions of several variables was accomplished. From a prime application in process control, the use of mathematical programming quickly spread to a wide variety of other disciplinary activities.

Through use of the one or more of the many standard and nonstandard search algorithms, it is possible to determine the maximum or minimum value of some specified merit function subject to certain constraints, in an iterative manner. These methods of directed search are, in general, far superior to random trial and error procedures and guarantee some form of convergence, albeit to local minima. After each iteration, some form of evaluation of the merit function and constraints is necessary to determine the direction of the next stage of iteration.

In light of the apparent situation with regard to anticipated analysis costs in the late 50's, it was a logical and brilliant step to couple mathematical programming and finite element analysis together to generate an automated method for structural optimization (Reference 1). Mathematical programming methods of optimization required repeated analysis - and finite element methods provided a rapid automatic low cost procedure for such analyses.

Initially, this work met with a great degree of success, particularly with small scale problems. At last it was possible to come to grips with the real problems of design-stiffness as well as strength constraint conditions and fabricational considerations for structures subjected to a multiplicity of loading cases.

Flushed with the successful demonstration of the feasibility of these coupled procedures for constrained weight minimization, development proceeded apace. It appeared that their extension to larger order problems involving up to 100 or more degrees of freedom and/or variables could readily be accomplished. The field of endeavor expanded, large scale problems with practical operational capabilities were developed (References 2 and 3), new formulations (Reference 4) of the structural problem investigated, new algorithms were used (Reference 5), more sophisticated types of problems were tackled (References 6 and 7), and the field seemed limitless. It was recognized with larger and more complex problems that the computational costs would increase. Larger analyses were being performed and some increase in the number of iterations to convergence was only to be expected. It was tacitly assumed by many that the anticipated improvements in computers and advances in analysis methods would keep, the costs of the analyses to economic levels. In addition, the actual increase in the number of iterative analyses would not be excessive.

With further development, a less satisfactory situation began to emerge. As indicated previously, the requirements for more sophisticated analyses had actually tended to increase rather than decrease computational costs. In addition, the number of iterations required and the number of analyses per iterative stage was found to increase more rapidly with problem size than had been first assumed, principally due to the explicit or implicity need to determine the derivatives of the constraint functions with respect to the design variables. Some improvements in this situation appeared with the reformulation of the basic constrained problem as an unconstrained one through the use of penalty functions of various types (References 8 and 9). A greater generality of merit and constraint functions was possible than with the earliest constrained formulations and the use of non-derivative search methods eliminated the need for so many function evaluations - but the gain was more apparent than real. Again, as in the case of the original constrained formulation development, a considerable measure of success was achieved initially. Certainly for the small test problems, the procedure was more efficient and the greater generality of merit function definition was an advantage. On the other hand, the actual improvements were relatively small, and the same difficulty of rapid rise in number of constraint evaluations with problem size was still present*. Although the explicit need for derivatives was eliminated, the implicit requirement had the same effect in the end.

In the 1960's, the mathematical programming approach almost completely dominated the structural optimization scene. The potential payoff in structural optimization aroused considerable general interest resulting in a considerable level of effort being expended by many researchers who approached the problem from a number of slightly different points of view with varying degrees of success. By the use of new search algorithms and new penalty functions or by various schemes for linearization of the problem attempts were made to overcome the critical rise in numbers of analyses with problem size. References 10 and 11 provide effective summaries of the many methods used with indications of their capabilities. Reference 10 documents the variations in approaches to optimization through the use of nonlinear programming methods whereas the papers of Reference 11 tend to highlight some of the barriers encountered in the practical use of nonlinear programming approaches to structural optimization.

In the most general terms, it was beginning to appear that there were remarkably low limits on the size of structure which could be handled economically by optimization programs. Even the most optimistic estimate was only about 150 variables or less. Thus, at the end of the 60's, a general need for structural design and optimization capabilities had been recognized but there appeared little immediate prospect for the development of more efficient nonlinear programming algorithms to overcome the economic barriers to widespread operational usage on real structures. The time was now ripe for a new approach to the problem, if structural optimization was to survive as an economic design potential rather than an interesting research toy.

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* In Reference 10, it has been estimated that the number of function evaluations may go as high as $7N^3$, where the N is the number of variables.

II. NEW APPROACH TO OPTIMIZATION

A. Fully Stressed Design Methods

Since the avenue of nonlinear programming for structural optimization did not appear to offer the hope of large scale design applications, it was appropriate to reconsider some of the basic concepts which are used extensively in structural design.

Numerical search methods determine optimal systems in a purely empirical manner. That is, a set of rules is established which will guarantee a continuous and monotonic decrease in a prescribed merit function, without regard to the nature of that function. No preconditions concerning the optimum are specified apart from the criteria that is impossible or uneconomic to determine a further design which will be an improvement on the present design. Both the strength and weakness of mathematical programming reside in this formulation. The strength is the generality which this independence of problem type imparts; the weakness is that no use is made of any characteristics of the problem which would permit a more efficient solution.

For the structural problem, perhaps the generality and rigor are unnecessary. What is really required for practical design is a simple method (i.e. limited number of function evaluations) which may depend to some extent upon the specialized structural nature of the problem and which will operate with some measure of optimality. That is, a procedure which will rapidly produce designs better than a standard engineering approach but without rigorous guarantees of convergence on the optimum.

In structural design, the classic example of this type of approach is the concept of simultaneous failure. In this, it is assumed, intuitively, that the best design is one for which every possible mode of failure will occur at the same time. Thus, if there are an equal number of possible modes of failure (constraint conditions) and design variables, the result will be a set of simultaneous equations (possibly nonlinear) whose solution will yield the best design. If the numbers of constraints and variables are unequal, then a certain degree of judgement and even trial and error may be necessary. This corresponds to the determination of the vertices of constraint surfaces in the nonlinear programming with the basic difference that the desired vertex is found, not by a direct search, but by direct solution of the equations generated by the conditions assumed to obtain at the optimum. This method has its obvious limitations, but its simpler form of fully stressed design is widely known and used (Reference 12). This time-honored procedure is based upon the intuitively satisfying, although analytically unjustified assumption that in an optimal structure each member will be fully stressed under at least one of the loading conditions.

The basis for the selection of this type of criterion is obvious. Engineering reason clearly indicates that the best structure must be one in which each member is working to its utmost limit. For the case of a statically determinate structure under a single or multiple loading system, it has indeed been demonstrated that the fully stressed design has least weight (Reference 13). It is also fairly certain that this fact was demonstrated well after the concept was used in engineering design. On this basis, the extension of the fully stressed concept to multiple loaded indeterminate structures is a relatively simple step (Reference 14). Unfortunately, this also has proved to be an unjustified assumption (Reference 1). The paradox which exists about fully stressed design is that it may or may not be of minimum weight. True, there is no explicit reference to weight as a merit condition in the usual stress ratio method of determining a fully stressed design, but this alone does not necessarily negate the validity of the assumption. In many cases, a fully stressed design can be determined whose optimality with respect to weight can be verified by application of the empirical criteria used in the numerical search methods (Reference 15). In some cases, a fully stressed design can equally be shown to be heavier than the least weight design (Reference 16) and in still other cases no fully stressed design can be found. That no apparent rules exist to differentiate between these cases, except through a post hoc examination of the fully stressed design has been demonstrated analytically (Reference 17).

In general, operational experience with fully stressed design methods does indicate in the vast majority of problems, not selected for their pathological behavior, that the resultant design - for stress limits only - is indeed either the optimum or close to it. It is also relatively easy to select problems for which the fully stressed design is remote from the optimum, but this is usually achieved by selecting unrealistic and disparate allowable stresses for members which provide parallel load paths in a redundant structure.

Using a simple stress-ratio redesign method, convergence on the fully stressed design will occur in one step for a statically determinate structure and will require a finite number of iterations for an indeterminate structure. The number of iterations required for convergence is another paradoxical problem being apparently controlled by characteristics which are presently poorly understood. One factor which is relatively clear, is that the number of iterations is not, in general, linked closely to the size and number of variables in the problem. It is this last fact which makes the use of fully stressed design so attractive. Each redesign step is simple and in many cases, the convergence is extremely rapid. The fact that a true least weight structure is not generated becomes of lesser importance when compared with the ease with which the improved design is reached (Reference 12).

Clearly, the basis for this type of optimization which is only applicable to strength considerations is the a priori specification of a set of conditions to be satisfied by the optimal design. That these criteria may not be rigorously applicable at the optimum is of little consequence. An a priori approach is in direct contrast to a search algorithm in mathematical programming where the determination of an optimum is predicated on a strictly post hoc basis, e.g. when further improvement in merit is impossible, only then is the search terminated.

From consideration of the known characteristics of the fully stressed design procedure, it might be concluded that the rapid convergence and the relative independence of the number of iterations from problem size was largely attributable to the a priori specification of optimality criteria. The stress ratio method attempts to satisfy these arbitrary criteria and hence converges directly on the prescribed point in the design space without need for exploration of convexity or other use of the constraints.

With this recognition, a new avenue of approach to structural optimization becomes possible, through the use of optimality criteria. In use, this requires the definition of additional criteria for each response phenomena coupled with some procedure which will permit the simultaneous consideration of multiple criteria.

B. Displacement Limited Designs

In developing structural optimization programs, a heirarchy of priorities for the inclusion of different types of constraint conditions has become established. The first consideration has been strength followed closely by fabricational limitations (minimum material sizes) and stiffness constraints (displacement limits). Subsequently, buckling, both overall and local, frequency response and static and dynamic aeroelasticity conditions are considered.

In fully stressed design, the basis for the stress-ratio redesign method is the assumption that at the optimal point, an inequality stress constraint for an element becomes satisfied as an equality condition. The same basic logic may be used for a displacement constraint condition.

The first consideration of this approached was presented by Barnett and Hermann (Reference 18) for the optimal design of determinate trusses in which a value was specified for the displacement of one node point. Berke (Reference 19) re-evaluated this work and proposed its application to an indeterminate structure with a generalized constraint. From this starting point, the optimality criteria have been further generalized for indeterminate structures under multiple loading conditions with a multiplicity of displacement constraints (Reference 20). In deriving these criteria, provision is made for the simultaneous consideration of other redesign algorithms associated with additional criteria appropriate to the other constraint conditions.

The starting point for the development of this approach to minimum weight design is the consideration of a structure under the action of a single loading system.

The structure is represented by an assemblage of discrete elements for which, initially, one stress is sufficient to describe the response behavior of each element. For more complex elements involving multiple stress components, the same overall derivation will apply, but modified slightly to reflect the additional stress components. This will be demonstrated later. Single stress elements may be axial force members or shear panels. For an axial member, the appropriate design variable is selected to be its cross-sectional area, whereas for a plate or panel thickness is the chosen variable.

Application of external loading results in internal forces S defined as the produce of stress and cross-sectional area for axial members and stress and thickness for plates. The internal forces in each member S_i^P arise as the result of the application of an external loading system P which may consist of a single or many load components. Associated with the loading system will be displacements δ_i at all nodes of structure. The value of a generalized displacement, which as a special case is a single nodal displacement, can be determined through use of a generalized virtual load method.

A virtual loading system Q, corresponding to the generalized displacement system in node point and direction, is applied to the structure. The virtual work Δ arising between the virtual system Q and the generalized displacement δ^* is given by

$$\Delta = \sum_{i=1} \frac{S_i^P \; S_i^Q \; L_i}{A_i \; E_i} \tag{1}$$

where S_i^P, S_i^Q are the forces due to the actual and virtual systems respectively

A_i is the (variable) cross-sectional dimension

L_i is the geometric dimension of the element and

E_i is the appropriate elastic modulus

L_i , the geometric dimension is defined by

$$L_i = V_i/A_i \tag{2}$$

where V_i is the volume of the element.

For plate elements L_i is the superficial area and for shear panels the appropriate value for E_i is the shear modulus. For the special case of a single displacement, the corresponding virtual loading system Q consists of a single unit load and hence Δ as calculated by Equation (1) is the value of displacement.

It is assumed now that the variables in the structure may be divided into two groups — active members and passive. Active members are those whose cross-sectional dimension A_i may be varied to achieve an optimized displacement limited design, whereas passive members will remain unchanged. The basis for the allocation of the individual member into the two grouping will be discussed later, and has no influence on the derivation.

With this division Equation (1) can be rewritten as

$$\Delta = \sum_{i=1}^{m} \frac{S_i^Q S_i^P L_i}{A_i E_i} + \sum_{i=m+1} \frac{S_i^P S_i^Q L_i}{A_i E_i} \tag{3}$$

where the first group contains m passive members.

Since in subsequent manipulations, the passive members are unaltered, the first term of Equation (3) is replaced by Δ_o

i.e. $$\Delta = \Delta_o + \sum_{i=m+1} \frac{S_i^P S_i^Q L_i}{A_i E_i} \tag{4}$$

The weight of the structure W is given by

$$W = \sum_{i=1} V_i \rho_i = \sum_{i=1} A_i L_i \rho_i \tag{5}$$

where ρ_i is material density.

Introducing the active and passive member grouping Equation (5) becomes

$$W = W_o + \sum_{i=m+1} A_i L_i \rho_i \tag{6}$$

where W_o is the weight of the passive members.

A constraint on the allowable values of the virtual work of the load system Q can be prescribed as Δ^*, given by

$$\Delta^* = \Sigma\, Q_i \cdot S_i^* \tag{7}$$

where the summation is taken over all terms of the generalized system.

The minimum weight structure for which the generalized displacement will have the specified value Δ^* can now be determined by finding the stationary value of W subject to the equality constraint.

$$\Delta = \Delta^* \tag{8}$$

For optimization problems involving equality constraints, a Lagrange multiplier approach is ideally suited. Using a Lagrange multiplier λ, the problem can be expressed as the minimization of the function

$$F = (W_o + \sum_{i=m+1} A_i L_i \rho_i) + \lambda \left(\sum_{i=m+1} \frac{S_i^P S_i^Q L_i}{A_i E_i} + \Delta_o - \Delta^* \right) \tag{9}$$

For a minimum

$$\frac{\partial F}{\partial A_j} = 0 = L_j \rho_j - \lambda \frac{S_j^P S_j^Q L_j}{A_j^2 E_j} + \lambda \sum_{i=1} \left(\frac{\partial S_i^P}{\partial A_j} S_i^Q + \frac{\partial S_i^Q}{\partial A_j} S_i^P \right) \frac{L_i}{A_i E_i} \tag{10}$$

or

$$0 = L_j \rho_j - \lambda \frac{S_j^P S_j^Q L_j}{A_j^2 E_j} + \lambda \sum_{i=1} C_{ij} \tag{11}$$

It is to be noted that the summation involved in the last term of Equation (11) must be taken over all members active and passive. This term represents the redistribution of the internal forces associated with change in member size. For a statically determinate structure, all Cij are zero. For redundant structures, the terms $\partial S_i / \partial A_j$ will generally be small and will tend to diminish in size rapidly moving away from the j^{th} member.

From Equation (11)

$$A_j = \sqrt{\lambda} \sqrt{\frac{S_j^P S_j^Q L_j}{E_j (L_j \rho_j + \lambda \sum_{i=1} C_{ij})}} \tag{12}$$

Substituting Equation (12) into Equation (14) and noting Equation (8)

$$\Delta^* = \Delta_o + \frac{1}{\sqrt{2}} \sum_{j=m+1} \sqrt{\frac{S_j^P S_j^Q L_j}{E_j} (L_j \rho_j + \sum_{i=1} C_{ij})} \tag{13}$$

Rewriting Equation (13)

$$\sqrt{\lambda} = \left(\frac{1}{\Delta^* - \Delta_o} \right) \left(\sum_{j=m+1} L_j \sqrt{\frac{S_j^P S_j^Q \rho_j}{E_j} (1 + \frac{\lambda}{\rho_j L_j} \sum_{k=1} C_{jk})} \right) \tag{14}$$

Finally substituting Equation (14) into Equation (12)

$$A_i = \left(\frac{1}{\Delta^* - \Delta_o} \right) \left(\sum_{j=m+1} L_j \sqrt{\frac{S_j^P S_j^Q \rho_j}{E_j} (1 + \frac{\lambda}{\rho_j L_j} \sum_{k=1} C_{jk})} \right) \sqrt{\frac{S_i^P S_i^Q}{E_i \rho_i (1 + \frac{\lambda}{\rho_i L_i} \sum_{k=1} C_{ij})}} \tag{15}$$

Equation (15) now represents the criteria which the variables A_i must satisfy at the optimum for the desired displacement constrained minimum weight design.

Use of these relationships would involve the iterative determination of the constant λ and terms C_{ij} which are the local gradients of the stresses with respect to the variables. Their determination by finite differences or other means would be tantamount to a return to the computational situation which exists for nonlinear programming approaches. Since they are generally small, the error introduced by setting all C_{ij} to zero will be negligible. Equation (15) can now be rewritten as

$$A_j = \left(\frac{1}{\Delta^* - \Delta_o}\right) \sqrt{\frac{S_j^P \, S_j^Q}{E_j \, \rho_j}} \sum_{i=m+1} \sqrt{\frac{S_i^P \, S_i^Q \, \rho_i}{E_i}} \tag{16}$$

In finite element analyses, the more usual form of output from a computer is stress rather than element force. Introducing elemental stress Equation (16) becomes

$$A_j = \left(\frac{1}{\Delta^* - \Delta_o}\right) A_j \sqrt{\frac{\sigma_j^P \, \sigma_j^Q}{E_j \, \rho_j}} \sum_{i=m+1} (A_i \, L_i \, \rho_i) \sqrt{\frac{\sigma_i^P \, \sigma_i^Q}{E_i \, \rho_i}} \tag{17}$$

For a statically determinate structure, there is no redistribution of internal forces with variation in member size and all C_{ij} are identically zero. Hence, Equation (17) is exact, and will generate the active member sizes directly for a minimum weight structure with its critical displacement equal to the specified value Δ^* and ignoring any restraints on element sizes or stresses. If any members have been selected as passive, their fixed values will be reflected in the computation of the term Δ_o. This procedure is then analogous to a constrained minimization with one main constraint (the displacement limit) and a number of side constraints.

It has been long known that for a statically determinate structure subject to strength (stress) limitations only, the minimum weight design has all members either fully stressed or at their minimum allowable size. If such a member satisfies the inequality displacement limitation, no redesign is possible or required. If the displacement constraint is violated, certain (active) members must be increased in size. With reference to Equation (3), if the product $(S_i^P \cdot S_i^Q)$ is negative, then an increase in the member size (in the denominator) will decrease the negative component of Δ arising from this member and hence further increase the total value of Δ. On the other hand, reduction of the area is not possible since it will already have its minimum size to satisfy the stress or fabricational requirements. Thus, in effect, no variation in such a member size can be permitted, and this provides one criterion for the selection of a passive member in a statically determinate structure i.e. $(S_i^P \, S_i^Q)$ is negative. As a logical corollary to this definition, it also follows that any member for which an external condition would be governing, i.e. requiring a larger size than that demanded by the displacement constraint condition, must also be treated as a passive member.

For statically indeterminate structures, as indicated previously the relationship Equation (17) is only approximate due to the neglect of the C_{ij} terms.* As such, Equation (17) can be used as a recursion expression for the iterative determination of A_j, exactly analogous to the use of the simple stress-ratio expression for determining a fully stressed design. Equation (17) can then be written

$$A_j^{\nu+1} = \left[\left(\frac{A_j}{\Delta^* - \Delta_o}\right) \sqrt{\frac{\sigma_j^P \, \sigma_j^Q}{E_j \, \rho_j}} \sum_{i=m+1} (A_i \, L_i \, \rho_i) \sqrt{\frac{\sigma_i^P \, \sigma_i^Q}{E_i \, \rho_i}} \right]^{\nu} \tag{18}$$

where the superscripts ν, $\nu+1$ refer to iterations. From some arbitrary starting point, iterative application of Equation (18) will then lead to convergence in a finite number of steps, on the statically indeterminate structure of minimum weight with its

*It has been recently realized that $\Sigma \, C_{ij}$ vanishes identically also for redundant structures.

critical displacement equal to Δ^*. As in the case of the statically determinate structure, certain members may be treated as passive non-participatory in this process based upon external criteria. The rate of convergence on the minimum weight design will be controlled by the magnitude of the omitted terms. If they are truly small and localized in effect, the convergence will be rapid. This is usually the case in large complex structures, but paradoxically in small structures the reverse frequently occurs.

Before a practical use can be made of this recursion expression, for the design of minimum weight structures with displacement constraints, a number of points must be clarified:

First, in using virtual unit load methods, the internal force or stress system corresponding to the virtual external load Q need only be statically equivalent to Q. For a determinate structure only one such system exists, but in an indeterminate structure S_i^Q is not unique and a multiple choice exists for the statically equivalent system. Since the selection of any one particular statically equivalent system means that S_i^Q will be arbitrarily set to zero in some (redundant) members, this is tantamount to an arbitrary selection of active and passive members for redesign. This is also equivalent to introducing side constraints on the design variables which in turn can result only in the increase of the attainable minimum weight. Therefore, when some members are to be excluded from design changes, the correct approach is to use the actual distribution of S_i^Q, which arises from the application of the virtual load Q to the real structure. This permits all members, potentially, to participate in redesign. It also has the virtue of not requiring a separate analysis of a degenerate structure under the virtual load but simply of including Q as an additional loading case in the finite element analysis. Of greater complexity is the division between active and passive members and the associated problems of introducing multiple loads along with displacement and other types of constraints.

The recursion expression, Equation (18) will cause convergence at a vertex in the design space formed by the intersection of the main constraint (displacement condition) and the side constraints (passive member sizes). It is clear that the source and nature of these side constraints is irrelevant to the iteration procedure. This, then, provides the mechanism for the inclusion of additional criteria, whereby other forms of constraint conditions (e.g. multiple displacement constraints, stress limits, fabricational considerations, etc.) can be permitted to specify certain member sizes while other members are sized by the primary constraint. The effect of this type of combination of criteria will be to force the design simultaneously toward all potential individual minima, resulting in convergence in a vertex formed by many active constraints.

It is, therefore, necessary to establish rules or criteria for the selection of active and passive members consistent with the above philosophy.

From the examination of Equation (18) one criterion for the selection of passive member can be observed immediately, similar to the condition used for statically determinate structures. If $(S_i^P \cdot S_i^Q)$ is negative for a member, its inclusion as an active member would require its square root to be found, introducing imaginary numbers which are computationally and physically unacceptable. Thus, one criterion for passive members is

$$(\sigma_i^P \cdot \sigma_i^Q) < 0 \tag{19}$$

Similarly to the corollary for the statically determinate case, if the size of a member is defined dominantly by other than a displacement constraint, then that member must be treated as passive in the redesign to satisfy displacement constraints. That is, if the size required for a member to satisfy a stress or fabricational constraint is greater than the size required to satisfy a displacement constraint then that member should be considered passive in using Equation (18).

The extension of these principles to multiple loads and displacement constraints can be accomplished in a manner similar to that used in determining the appropriate stress ratio for a fully stressed design. In the stress ratio method, the ratio of the actual to the allowable stress is computed for an element for each loading case and the largest ratio selected for the redesign. With displacement constraints the largest area generated using Equation (18) is taken from the combined set resulting from all loads and displacement constraints. Thus if four loading cases are specified and three displacements are constrained, a total of twelve possible areas for one member are generated by Equation (18) and the largest selected as the dominant value. A second iteration of Equation (18) is then performed in which members designed in the first iteration by a specific load/displacement case are treated as passive except when considering that particular case.

The procedure for redesign with displacement constraints which has evolved from an intensive exploration of the numerical aspects of the problem, automatically partitions members into active and passive categories for a given redesign phase. Tests on the sign of $(\sigma^{P}_{i} \cdot \sigma^{Q}_{i})$ are incorporated, as are checks to determine whether stress and fabricational limits dominate for any member and insure that that member is treated as being passive.

An additional feature of note is the method of treating negative displacement limits. Frequently displacement limits are expressed in the form $-\Delta^* < \Delta < +\Delta^*$. For the satisfaction of the negative limit, a negative virtual load system $-Q$ is required. This has the effect of reversing all the signs on the virtual stresses and thereby interchanging the active and passive members selected by the sign of $(\sigma^{P}_{i} \cdot \sigma^{Q}_{i})$. The effect of this is to cause some elements to participate in the satisfaction of the positive displacement while the others (passive in the first case) then participate in the redesign for the satisfaction of the negative displacement. The fact that these two constraint conditions are apparently in direct conflict does not introduce, in practice, any complications. Other conditions of stress limits, minimum sizes and multiple loads and displacements intrude and eliminate one of the contradictory constraints.

In summary, the redesign process is as follows: As a first stage, a stress ratio redesign is effected for each member with due regard to multiple loads and minimum member sizes. For every limited displacement specified, a unit load vector is generated and the corresponding stresses σ^{Q}_{i} are computed for all members along with the actual stresses σ^{P}_{i}. A matrix of the product $(\sigma^{P}_{i} \cdot \sigma^{Q}_{i})$ is computed for all combinations of loads and limited displacements (all P and Q) for each member.

Using Equation (18), new values of A_j are computed for every member. The partitioning into active and passive members at this stage is controlled wholly by the test on the sign of $(\sigma^{P}_{i} \cdot \sigma^{Q}_{i})$. In the presence of both negative and positive limits on the restrained displacements, new areas will be generated for each member.

The largest area is selected for each member from all the loading/displacement combinations and a record is made of the particular combination which controlled the design of each area. The areas so generated are compared with those based upon stresses or minimum sizes and the larger values selected for each member.

Again, using Equation (18), a new evaluation of areas is made for each load/displacement case but in this case the passive members for each case consist not only of those elements for which $(\sigma^{P}_{i} \cdot \sigma^{Q}_{i})$ is negative, but also those which were, at the end of the first iteration, critically designed by stress limits, minimum size or by a load/displacement combination other than that being currently considered. At the end of this redesign, a comparison is again made between the stress-designed and the displacement-designed sizes. This cycle is repeated, up to a maximum of three times, until no transfer occurs between the members of the group designed by stresses and those designed by displacements. The resultant design is then reanalyzed and scaled until critical. This process can be repeated as many times as necessary until a minimum weight design is achieved.

The above derivation has been presented for a structure composed of elements for which only one stress was defined. In the case of membrane plates, three components of stress σ_x, σ_y, τ_{xy} are generated for each element. For the calculation of stress limits, a single reference stress is calculated based on the Von Mises criterion

$$\sigma_{ref} = (\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2)^{1/2} \tag{20}$$

A limiting value is then specified for σ_{ref}. For displacement constraints, Equations (17) and (18) are modified slightly to reflect the additional terms. Wherever the term $\dfrac{(\sigma_i^P\,\sigma_i^Q)}{E_i\,\rho_i}$ appears, it is replaced by the term

$$\frac{1}{\rho_i}\left[\frac{(\sigma_x{}^P - \mu\sigma_y{}^P)\,\sigma_x{}^Q + (\sigma_y{}^P - \mu\sigma_x{}^P)\,\sigma_y}{E} + \frac{\tau_{xy}{}^P\,\tau_{xy}{}^Q}{G}\right]_i \tag{21}$$

A similar modification is introduced in the calculation of Δ. The test on the sign of $(\sigma_i^P \cdot \sigma_i^Q)$ is replaced by a test on Expression (21). Otherwise the logic is unchanged.

The above redesign procedure was coded and used for a number of very successful optimizations, results of which are presented in the following section.

Among the problems considered was a 25-bar transmission tower used as an example on a number of occasions (References 3, 5, 6, and 21). There were basically two loading conditions on the tower, but in order to maintain symmetry of the structure, 4 additional loads were introduced. Of the 25 members in the structure, the conditions of symmetry partitioned them into 5 groups containing 4 identical members, 2 groups of two members plus a single bar. Application of the program to this problem was only moderately successful and a great deal of operator intervention was necessary to maintain the expected symmetry of design.

The logic of the program was reviewed and it was finally observed that such lack of symmetry was inevitable as the program stood. The difficulty was caused by the presence of the single and the two groups of two members. In the redesign, the single member is equally designed by any of the sets of loads introduced for symmetry. As the logic exists it will be selected as being designed by the first of these cases, and in a subsequent iteration will be allocated to that load case as an active member and be treated as passive by the other three loading cases of the symmetrizing set. A similar effect occurs for the twin members and the result is a nonsymmetric design. To overcome this difficulty, a symmetry option was introduced which constrains members of any specified groups to be identical. This in turn, provides an additional criterion for the selection of active/passive members in the redesign process since only one member of each symmetric group is now independently designed. With this modification, the need for additional symmetrizing loads was eliminated in the tower problem, and excellent rapidly converging results were obtained.

III. RESULTS

In order to demonstrate the practicality of the optimality criteria approach to the design of least weight structures, a computer program was developed based upon a finite element analysis procedure.

At the heart of a finite element program is the element library. At present four simple elements are provided in the optimization program. These are the axial force member, the triangular membrane plate, the quadrilateral shear panel and the special half web shear panel used for symmetric wing analysis (Figure 1). The triangle is a constant stress element and generates the three components of stress at its centroid, σ_x, σ_y, and τ_{xy}. The use of the half web element is necessary when performing the analysis and design of thin symmetric wing-type structures. In these structures inclusion of all degrees of freedom in both upper and lower halves usually leads to severe conditioning problems associated with the numerical disparities of out-of-plane and in-plane stiffness characteristics. The solution to this difficulty lies in considering symmetric (in-plane) and anti-symmetric (out-of-plane) behavior separately and only treating half the structure. For elements lying completely in either half, this is no problem, but a special half-web element is required to cross the centerline. This trapezoidal shear web is defined by only two node points.

The program uses a simple stress-ratio method to account for strength constraints and the recursion relationship (Equation (18)) for displacement constraints. Fabricational constraints are included and the program performs the partitioning between active and passive members automatically, according to the criteria defined previously. The results presented here illustrate the characteristics of the iterative process and emphasize some points about the behavior which still remain to be resolved.

A large scale version of the program has been developed. This program is currently being applied to the optimization of large scale wing structures for practical aircraft in their design stage, where stiffness and strength considerations are both of critical importance.

A. Four Bar Pyramid (Figure 2)

The four bar pyramid has been used extensively (Reference 22) as a test case with a variety of loading cases and constraints.

The structure is subjected to the three non-simultaneous loads at the free node

Px = 5,000 lb.

Py = 5,000 lb.

Pz = 7,500 lb.

The constraint conditions are stresses of ±25,000 psi in all members with a minimum area of 0.1 in^2 and displacement limits of ±0.3 ins. in the y-direction and ±0.4 in. in the z-direction.

Using a uniform structure as starting point, the minimum weight design of 14.283 lb. was reached in four iterations. Table Ia lists the design history and final design. For information purposes, the program was allowed to run past the fourth iteration for another eleven cycles. Redesign continued and the weight varied but eventually returned to 14.283 lb. with slightly different values (Figure 3).

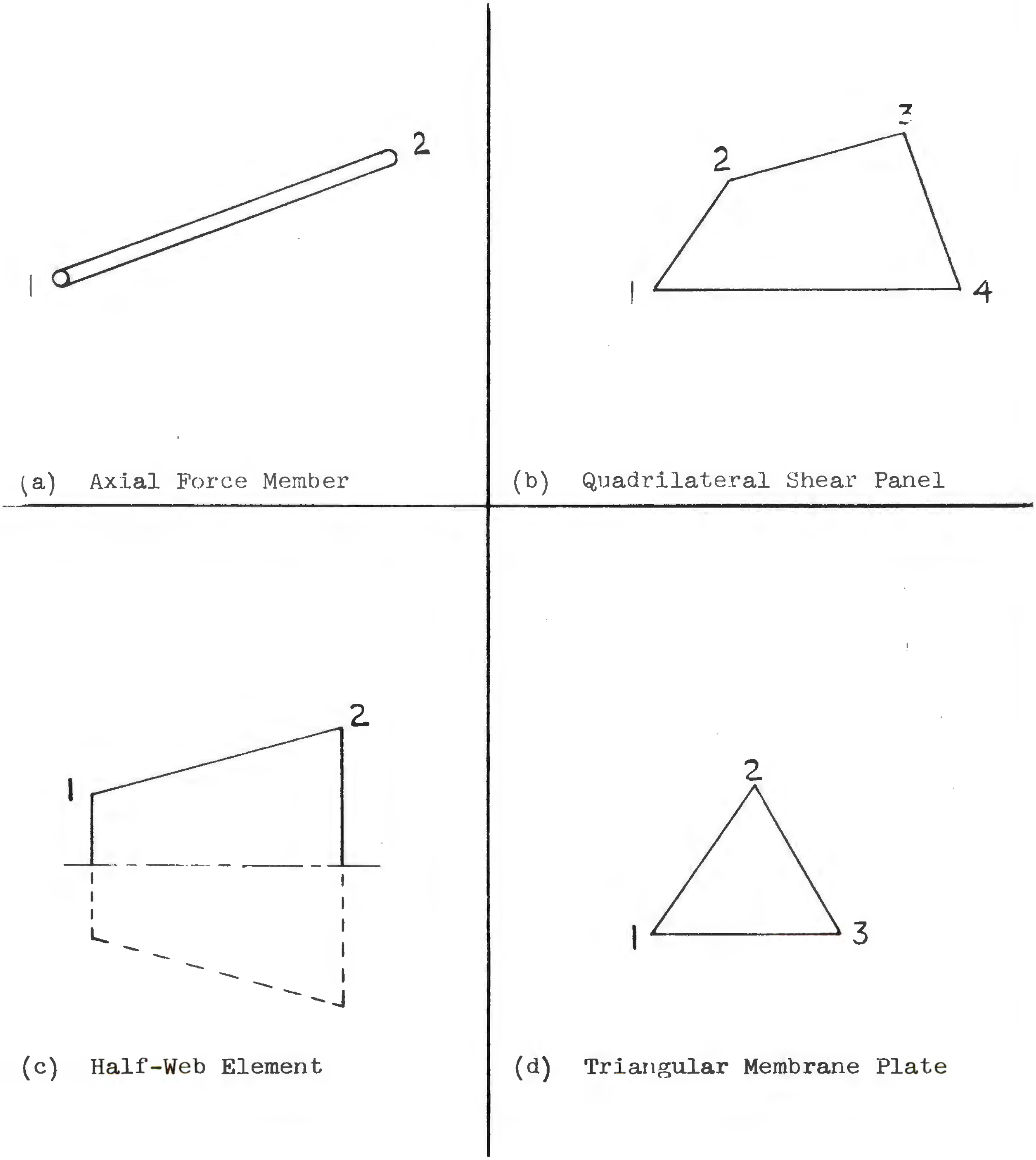


Figure 1. Finite Elements

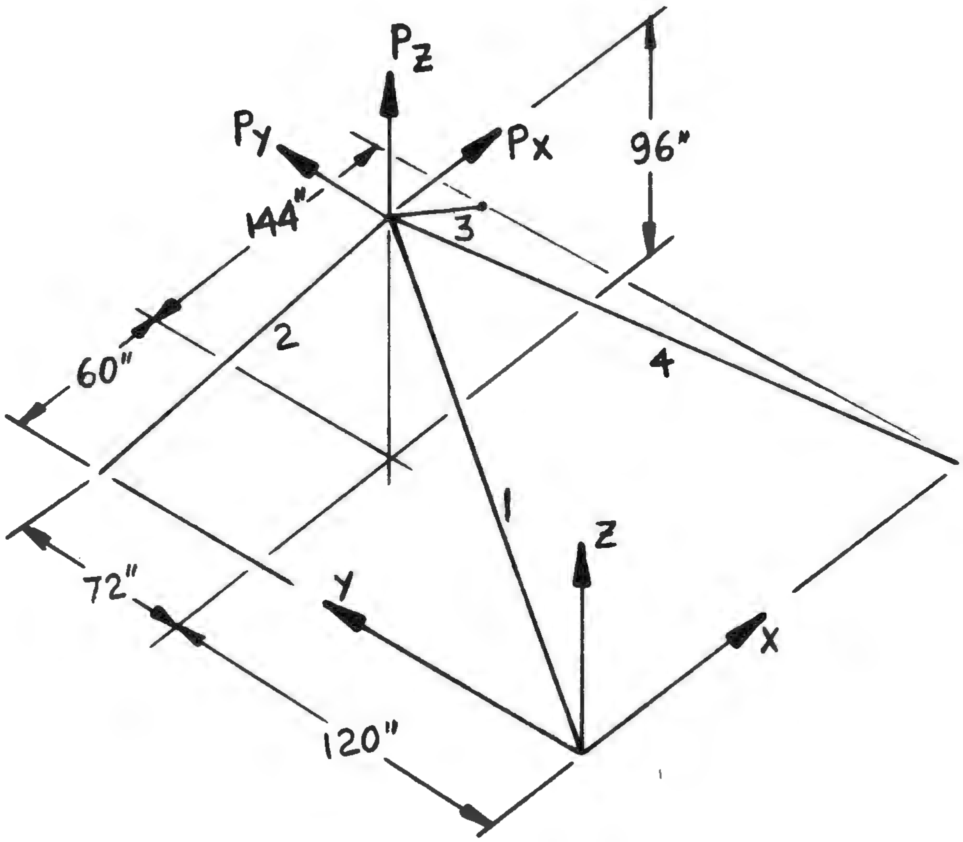


Figure 2. Four-Bar Pyramid

TABLE I

FOUR-BAR PYRAMID WITH THREE LOADS

Iteration	Weight	A_1	A_2	A_3	A_4
1	15.8407	0.2272	0.2272	0.2272	0.2272
2	14.5198	0.2714	0.2960	0.1552	0.1502
3	14.2949	0.2362	0.3226	0.1691	0.1377
4	14.2833	0.2241	0.3277	0.1756	0.1376
5	14.2861	0.2194	0.3303	0.1764	0.1391
6	14.2871	0.2172	0.3319	0.1751	0.1410
7	14.2868	0.2160	0.3329	0.1733	0.1428
8	14.2860	0.2152	0.3337	0.1714	0.1445
9	14.2853	0.2147	0.3344	0.1697	0.1460
10	14.2847	0.2143	0.3349	0.1682	0.1473
11	14.2842	0.2140	0.3353	0.1669	0.1484
12	14.2839	0.2138	0.3357	0.1658	0.1494
13	14.2836	0.2136	0.3360	0.1648	0.1502
14	14.2834	0.2134	0.3363	0.1640	0.1508

(a) Iteration History, Minimum Area = 0.1 in^2

Iteration	Weight	A_1	A_2	A_3	A_4
1	15.3821	0.3176	0.3065	0.2033	0.1054
2	15.2915	0.2181	0.3011	0.2108	0.0970
3	15.2126	0.3196	0.2964	0.2174	0.0897
4	14.3204	0.3264	0.2571	0.2571	0.0365
5	14.2986	0.3192	0.2592	0.2566	0.0351

(b) Iteration History, Minimum Area = 0.01 in^2

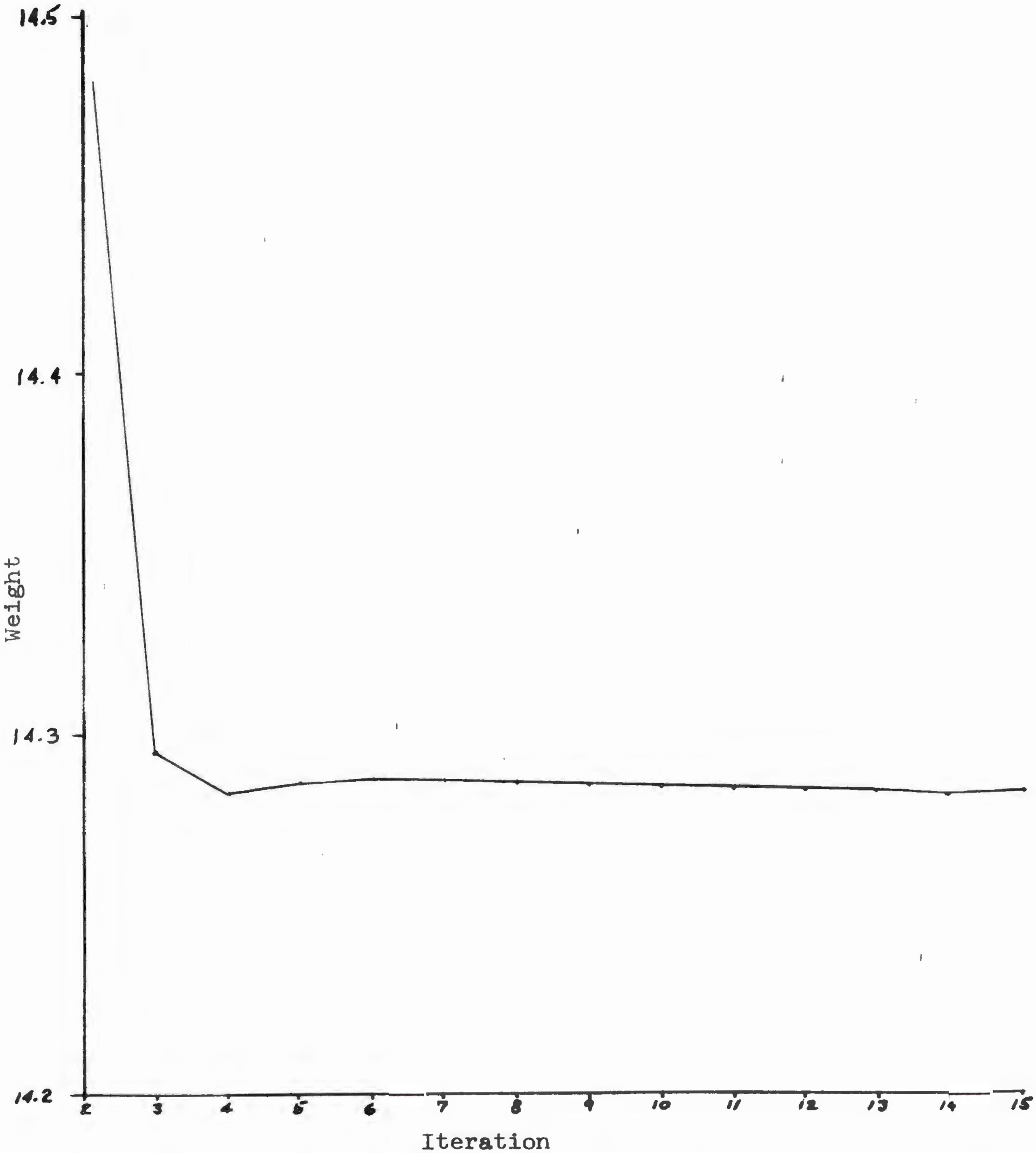


Figure 3. Iteration History for Four Bar Pyramid with Three Loads

The final designs, essentially the same as that of Reference 22, were all limited by the two displacement constraints. Neither stress nor lower side constraints (minimum member sizes) were active in the final designs. In an early development stage of the program, some difficulty was experienced in handling the side constraints effectively. Since the known least weight design was not constrained by the minimum member sizes, these were altered from 0.1 in^2 to 0.01 in^2. With this alteration a minimum weight design of 14.298 lb. was determined and is given in Table Ib. This design is considerably different from the previous design although of essentially the same weight. The indirect influence of the inactive side constraints on the final design is of interest.

B. 25-Bar Transmission Tower (Figure 4)

This structure has been used as a demonstration problem on a number of previous occasions (References 3, 5, 6, and 21). The tower, representative of a structure carrying transmission lines, has 25 axial force members. There are two loading cases (Table 3a). In spite of the directional nature of these loads, the structure is required to be doubly symmetric about the X and Y axes. The maximum displacements of the upper nodes 1 and 2 are ±0.35 in. in the X and Y directions. The bar members are all designed for a 35,000 psi tensile stress and a compressive stress based upon buckling allowables for thin wall circular tubes (Table 2b). Minimum member size is 0.01 in^2.

In some optimizations, symmetry is achieved through the use of additional loading cases but this is not necessary with the present program. Among the previous studies, slight differences are found between the final designs generated. These differences are principally associated with variations in tolerances and specified allowable stresses. An average value is 550 lb. and using a numerical search method over 100 analyses were required.

Using the current program, convergence occurred in seven iterations on a weight of 545.4 lb. Details of the history and final design are given in Tables 2c and 2d and Figure 5.

C. 72-Bar Four Level Tower (Figure 6)

The doubly symmetric tower of Figure 6 presented by Venkayya (Reference 21) was optimized using the present computer programs. Two loading cases were applied (Table 3a) and as in the case of the 25-bar transmission tower, symmetry was achieved by use of the input option. The stress limits were ±25,000 psi on all members and the displacements of the uppermost node points were limited to ±0.25 in. in the X and Y directions. Minimum member size is 0.1 in^2.

In spite of the greater size of the problem, convergence was very rapid and only eight iterations were required. Two subsequent iterations were performed to check convergence and the final design weight was 396 lb. Details of the design are given in Tables 3b, 3c, and Figure 7.

D. Cantilever Frames (Figure 8)

The 10-bar cantilever plane frame is subjected to a single loading case as indicated in Figure 8. The stress limits on all members is ±25,000 psi and limits of ±2.0 in. are also specified on the vertical displacement of the nodes at the free end. The minimum member size is 0.1 in^2, $E = 10 \times 10^6$ psi and $\rho = 0.1$ lb/in^3.

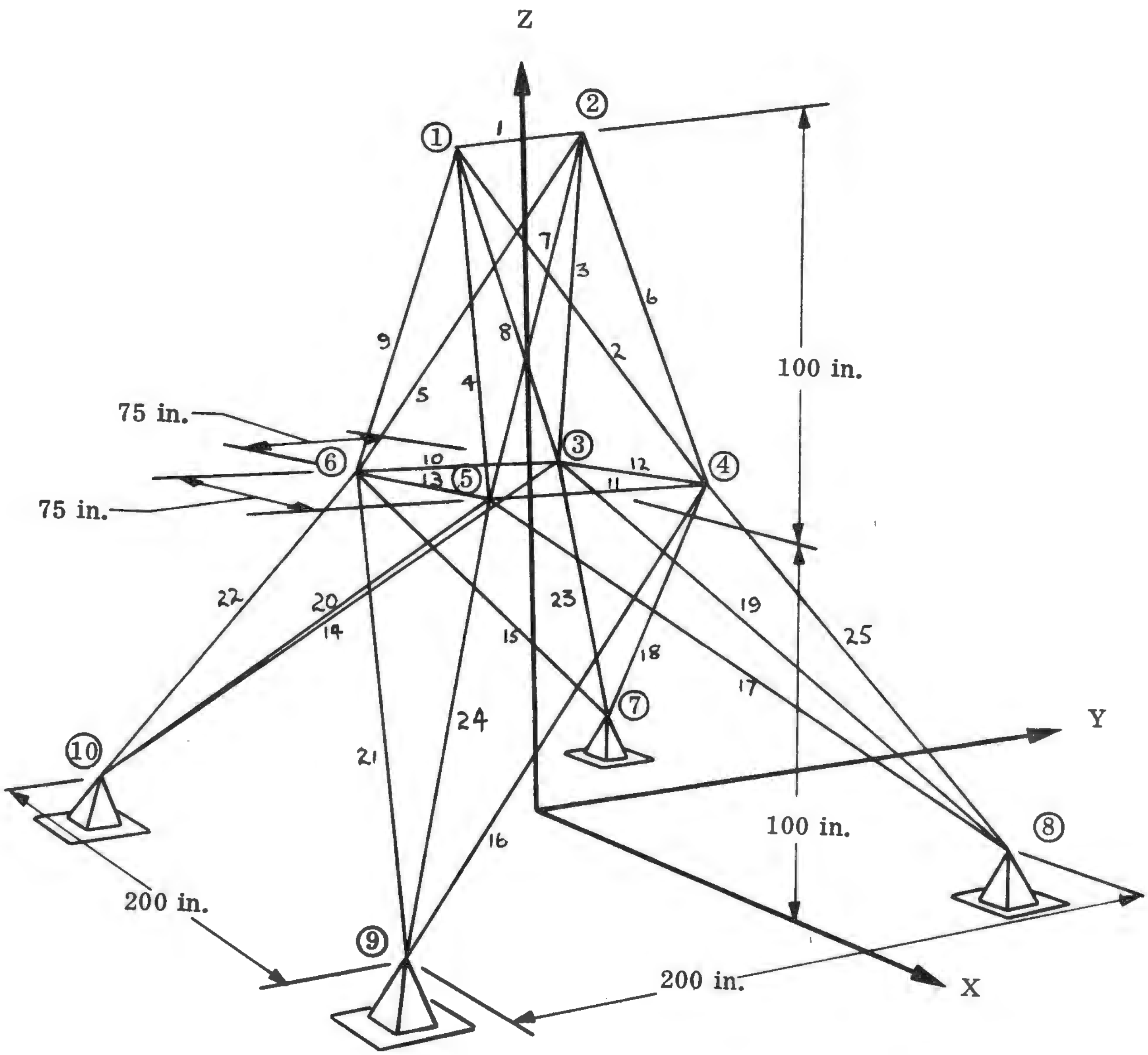


Figure 4. Twenty-five Bar Transmission Tower

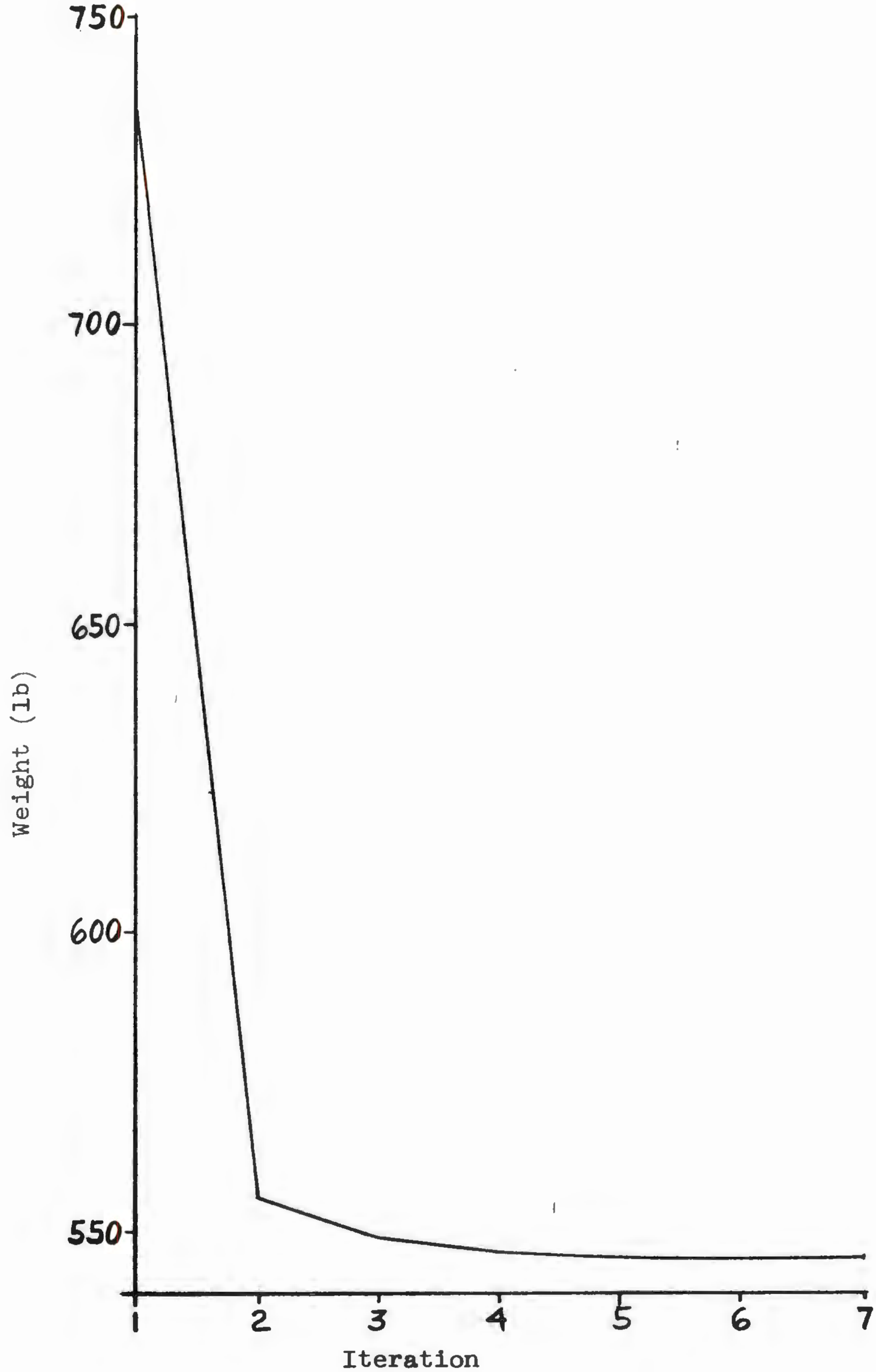


Figure 5. Iteration History for Twenty-five Bar Transmission Tower

TABLE II

25-BAR TRANSMISSION TOWER

Load Condition	Node	Direction X	Direction Y	Direction Z
1	1	1,000	10,000	-5,000
	2	0	10,000	-5,000
	3	500	0	0
	6	500	0	0
2	5	0	20,000	-5,000
	6	0	-20,000	-5,000

a) Applied Loading Systems

Member	Allowable Stress	Member	Allowable Stress
1	-35092.	12, 13	-35092.
2, 3, 4, 5	-11590.	14, 15, 16, 17	-6759.
6, 7, 8, 9	-17305.	18, 19, 20, 21	-6959.
10, 11	-35092.	22, 23, 24, 25	-11082.

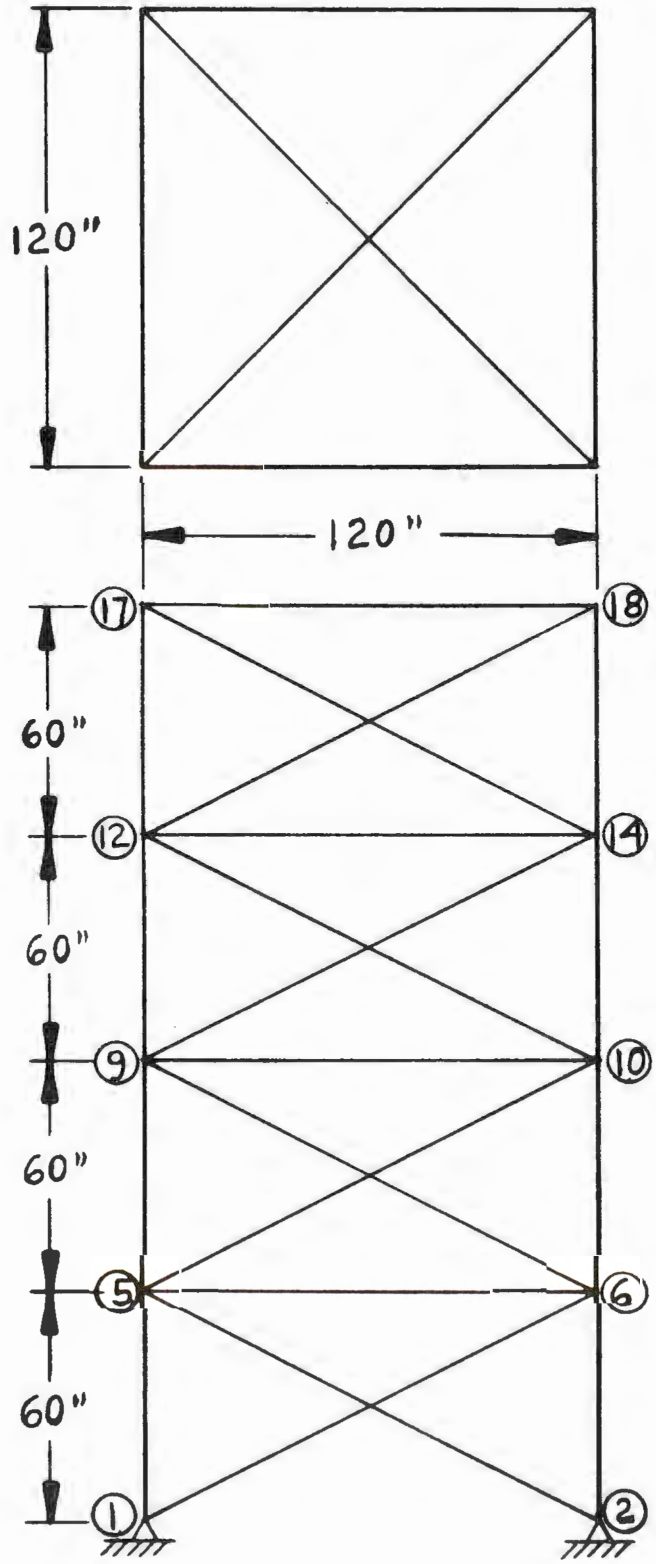
b) Allowable Compressive Stresses

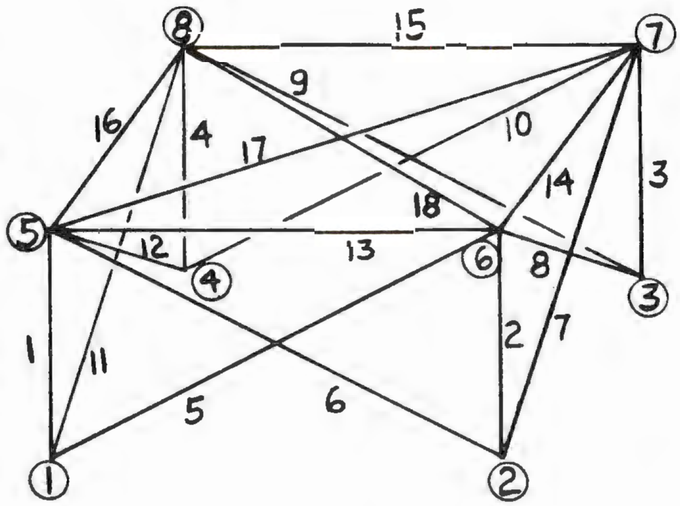
Iteration	1	2	3	4	5	6	7
Weight	734.38	555.72	549.08	546.54	545.92	545.45	545.36

c) Iteration History

Member	Area	Member	Area
1	0.01	12, 13	0.01
2, 3, 4, 5	2.0069	14, 15, 16, 17	0.6876
6, 7, 8, 9	2.9631	18, 19, 20, 21	1.6784
10, 11	0.01	22, 23, 24, 25	2.6638

d) Final Design





Typical Element Numbering
for First Level

Figure 6. 72-Bar Four Level Tower

TABLE III

72-BAR TOWER

Load Condition	Node	Direction		
		X	Y	Z
1	17	5,000	5,000	-5,000
2	17	0	0	-5,000
	18	0	0	-5,000
	19	0	0	-5,000
	20	0	0	-5,000

a) Applied Loading Systems

Iteration	1	2	3	4	5	6	7	8
Weight	656.77	416.07	406.21	399.06	396.82	396.25	396.02	395.97

b) Iteration History

Member	Area	Member	Area
1, 2, 3, 4	1.4636	37, 38, 39, 40	0.5521
5, 6, 7, 8, 9, 10, 11, 12	0.5207	41, 42, 43, 44, 45, 46, 47, 48	0.6084
13, 14, 15, 16	0.1	49, 50, 51, 52	0.1
17, 18	0.1	53, 54	0.1
19, 20, 21, 22	1.0235	55, 56, 57, 58	0.1492
23, 24, 25, 26, 27, 28, 29, 30	0.5421	59, 60, 61, 62, 63, 64 65, 66	0.7733
31, 32, 33, 34	0.1	67, 68, 69, 70	0.4534
35, 36	0.1	71, 72	0.3417

c) Final Design

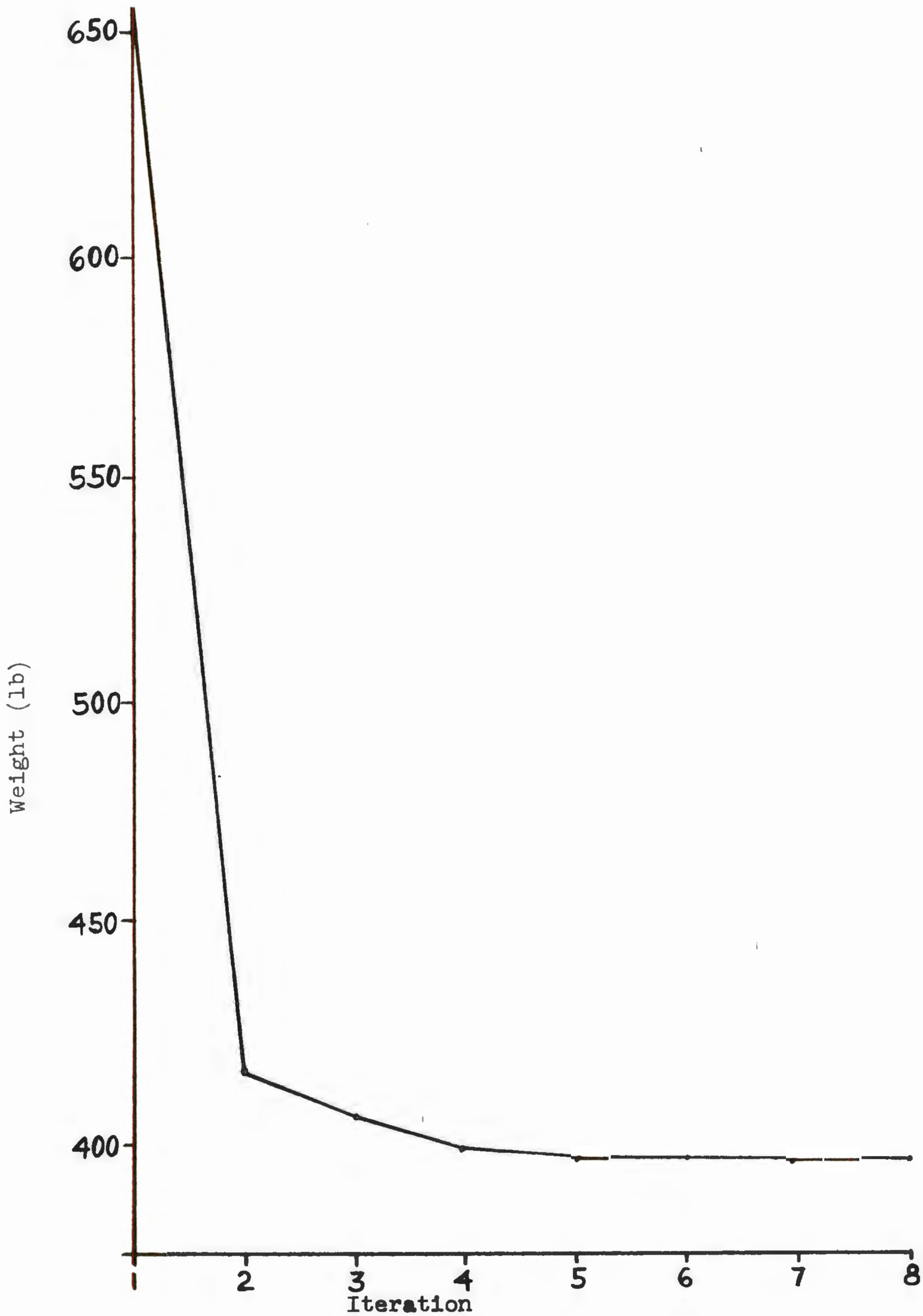


Figure 7. Iteration History for Seventy-two Bar Tower

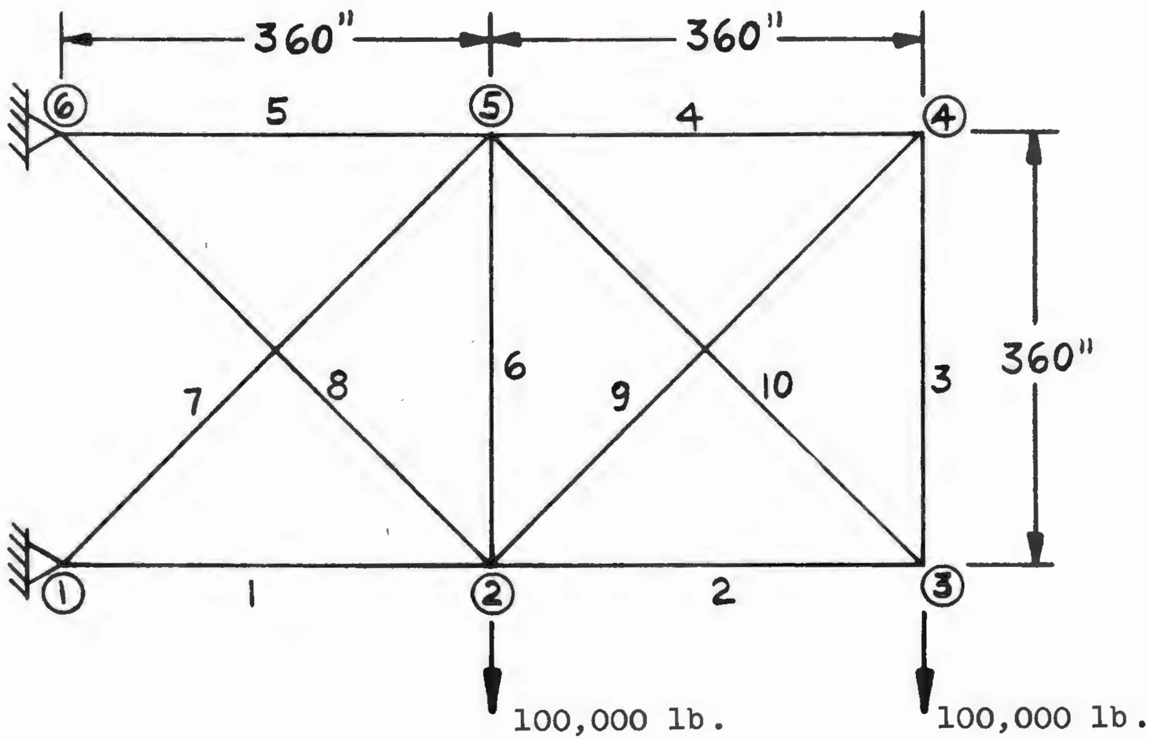


Figure 8. Cantilever Frame

This problem proved to be a particularly interesting one requiring a number of attempts for solution. In the first attempt, since program execution time is a function of the number of constraints, a limit of 2.0 in. was imposed on the vertical displacement of point 3 for which the largest displacement was anticipated. After a considerable number of iterations (using an early edition of the program) a design weight of 5233 lb was achieved, but unfortunately the vertical displacement of point 4 was 2.037 in. In view of this result, the problem was then rerun with the limit of 2.0 in. imposed on node point 4 instead of 3. After three iterations, a design weight of 4625 lb was generated, but at iteration 4, the weight jumped to 4833 lb. The program was allowed to run on and eventually converged after 29 iterations on a design of 4714 lb. Unfortunately, this design was also unacceptable having a displacement of 2.9 in. at point 3. The least weight design at 4625 lb also exceeded the acceptable displacement at point 3 by 0.833 ins.

A third run was carried out with both points 3 and 4 limited to 2.0 in. vertical displacement. The program was run for 47 iterations and the weights varied as indicated in Table 4 and Figure 9. The best design of 5112 lb. occurred at iteration 18 but subsequently the weight did go as high as 9029 lb. The process did not converge as in the previous run but was apparently settling down towards the end. Also of interest in this run is the reversal at iteration 14. The slight weight increase between iterations 13 and 14 was only of minor effect, but its presence did make the selection of an automatic termination criterion difficult.

The termination mode finally selected is a threefold test on (i) number of iterations (ii) convergence of successive iterations or (iii) if a non-convergent process generates a design which exceeds the least weight system found by a specified percentage. A typical value for this last test may be about 5% to allow the perturbation at Iterations 13 and 14, but to cut off at Iteration 19.

The dramatic rise in weight at Iteration 19 appears to have been caused by the sudden emergence of a constraint condition hitherto dormant which became associated with a major redistribution of internal forces in the frame.

This problem has been studied analytically (Reference 23) with a view to comparing the minimum weight achievable for the given configuration with an equivalent Michell structure. If members 3, 4, 6, and 9 are omitted, the theoretical minimum weight for the degenerate structure, satisfying the displacement constraint, is approximately 4970 lb. The corresponding Michell structure lying within the bounds of the present system has a weight of 4760 lb. The imposition of minimum sizes for the above omitted members necessitates some redistribution of internal strains, and prevents the ideal minimum weight being achieved by the small margin of 2.5%.

E. 18 Element Wing Box (Figure 10)

To evaluate the use of plate elements, the wing structure of Figure 10 was selected. This structure is essentially similar to that used in Reference 5, but with a small modification in the idealization. In Reference 5, one-half of the symmetric structure was idealized using five axial members, eight half-web shear panels, two quadrilateral and one triangular membrane elements. Since membrane quadrilaterals are not currently available in the new optimization program, the two quadrilaterals were replaced by four triangles as indicated in Figure 10. The change in dealization introduces differences between the two optimizations and hence results are not strictly comparable. In order to approximate the original model as closely as possible, thicknesses of the pairs of triangles forming the quadrilaterals are coupled using the program symmetry option.

TABLE IV

CANTILEVER FRAME

Iter. No.	Weight	Iter. No.	Weight	Iter. No.	Weight	Iter. No.	Weight
1	8266	13	5195	25	7508	37	5432
2	6356	14	5206	26	7239	38	5407
3	5980	15	5191	27	7050	39	5393
4	5779	16	5169	28	5818	40	5387
5	5625	17	5147	29	5793	41	5352
6	5547	18	5112	30	5788	42	5324
7	5470	19	5897	31	5767	43	5290
8	5392	20	7640	32	5707	44	5274
9	5323	21	8708	33	5677	45	5263
10	5266	22	9029	34	5595	46	5255
11	5225	23	8371	35	5564	47	5250
12	5200	24	7862	36	5488		

a) Iteration History

Member	Area	Member	Area	Member	Area
1	20.03	5	31.35	9	0.10
2	15.60	6	0.14	10	22.06
3	0.24	7	22.21		
4	0.10	8	8.35		

b) Least Weight Design (5112 lb.)

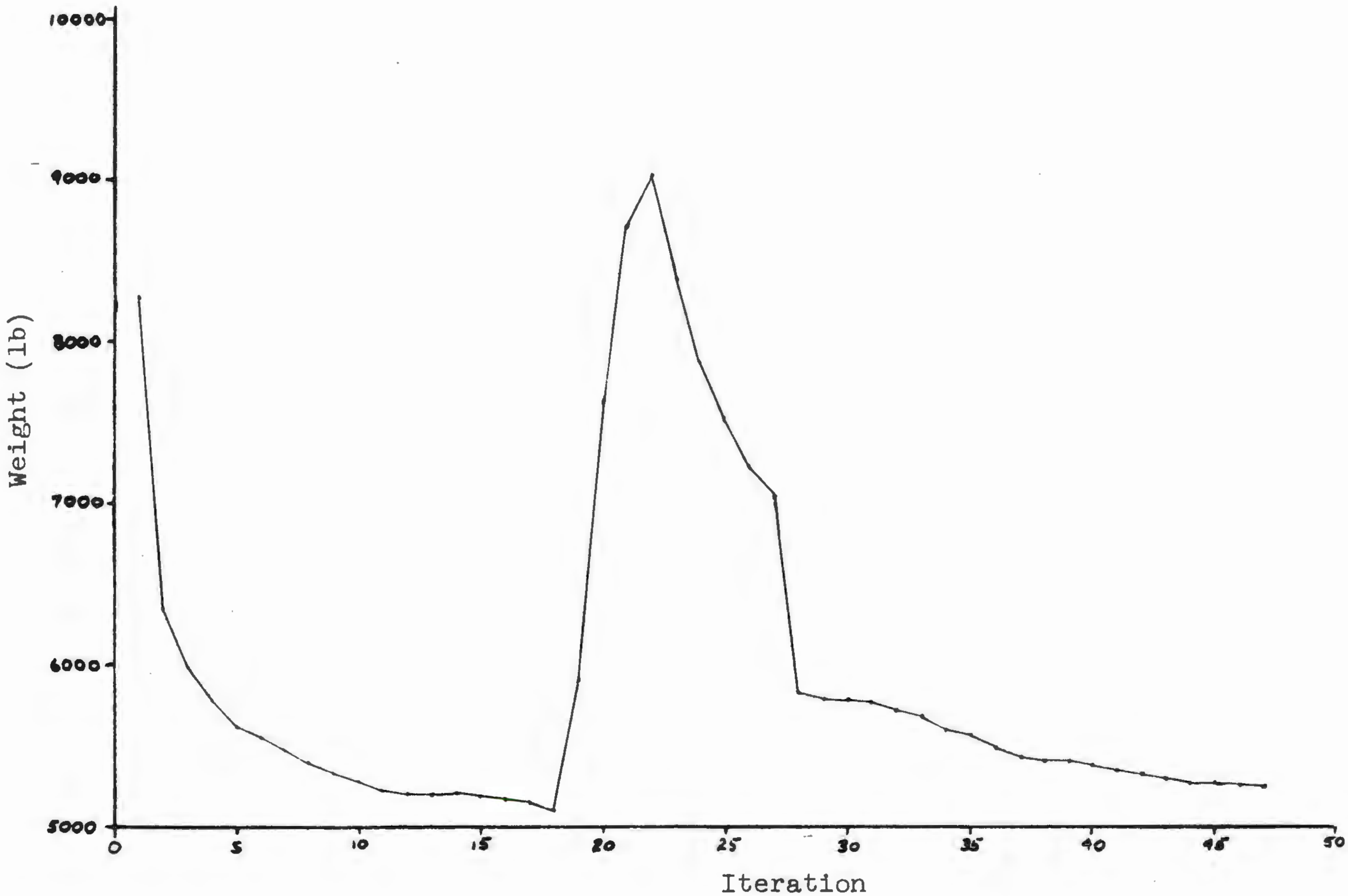
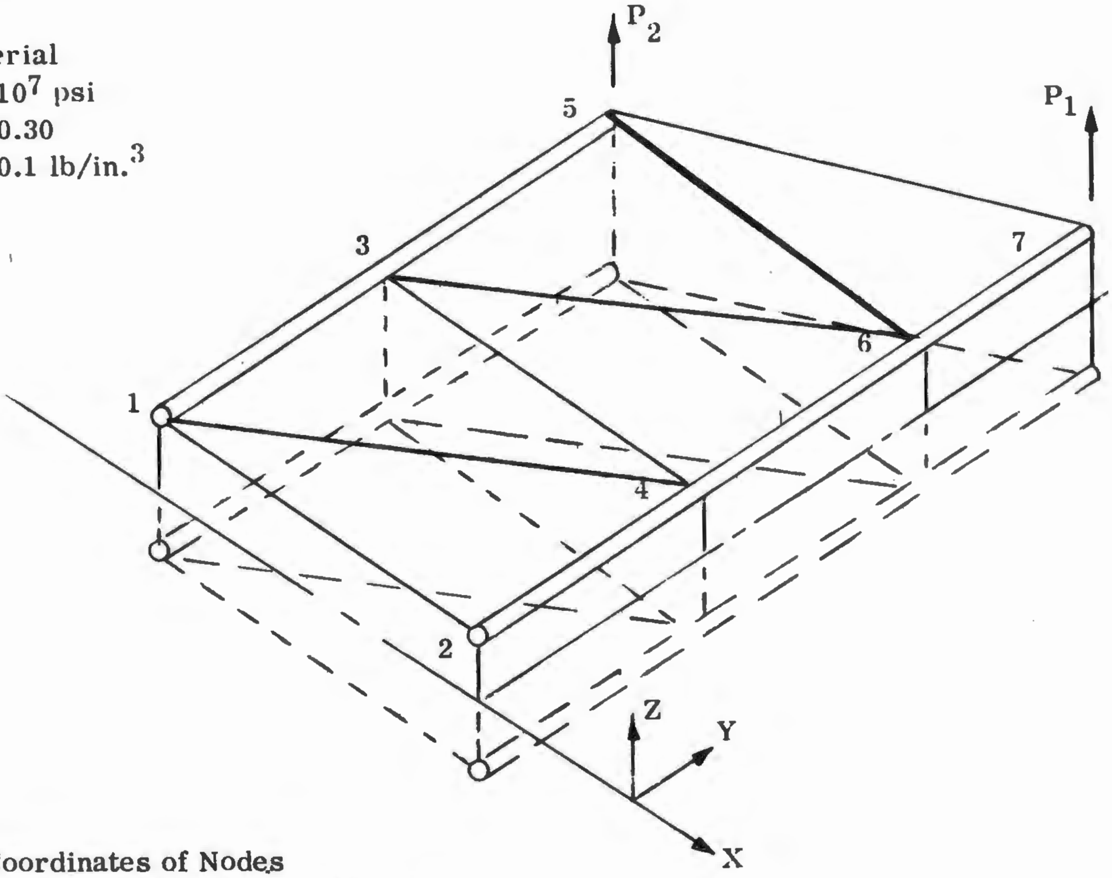


Figure 9. Iteration History for Cantilever Frame

Material
E = 10^7 psi
μ = 0.30
ρ = 0.1 lb/in.^{3}



Coordinates of Nodes

	X	Y	Z
1	0.0	0.0	10.0
2	100.0	0.0	8.0
3	0.0	70.0	10.0
4	100.0	70.0	8.0
5	0.0	140.0	10.0
6	100.0	140.0	8.0
7	100.0	190.0	8.0

Boundary Conditions

Points 1 and 2 Completely Restrained

Two Loading Conditions

(1) $P_1 = 10^4$ lb

(2) $P_2 = 2 \times 10^4$ lb

All Stress Limits = $\pm 10^4$ psi

All Deflection Limits = ± 2.0 in.

Lower Limits on Axial Members = 0.10 in.^{2}

Lower Limits on Plate Members = 0.02 in.

Figure 10. Eighteen Element Wing Box

Two vertical loads are applied to the tip of the wing and the maximum allowable displacements are 2.0 in. at the tip. The maximum stresses are ±10,000 psi in all members, and the minimum sizes are 0.1 in^2 for axial members and 0.02 in. for plates.

Using the present program, a least weight design of 387.6 lb was generated in 4 iterations. The process was relatively stable and as indicated in Table 5a and Figure 11, the weight increased slightly but steadily on subsequent iterations. By comparison, the numerical search approach generated a design (with quadrilateral cover plates) weighing 381.2 lb. The search procedure was relatively lengthy and required a total of 193 analyses.

This final design was used as an input to the current optimization program, which immediately indicated that the stresses in the triangular elements were in violation. Upon scaling to eliminate this violation, which arises from the different idealizations, a weight of 389.8 lb was produced. The redesign process was continued from this point and produced a design of 378.8 lb at the second iteration. These latter designs differed considerably from the former although of similar weight as indicated in Table 5b. A third starting point was selected which again generated after 22 iterations a design weighing 378.8 lb. although considerably different in member sizes from the previous design.

In the idealization of the wing both direct stress-carrying covers and spar caps are provided. The results generated tend to indicate that the wind bending stresses can be carried equally efficiently by either covers and spar caps. This implies that a large number of designs of similar weight probably exist with various distributions of materials between spar caps and covers.

Also of relevance is the difference in weights between the wing designed using two idealizations - triangles and quadrilateral membrane cover panels, indicating a considerable degree of sensitivity of the optimal design to the precise nature of the analytical model.

IV. CONCLUSIONS

The approach to the weight minimization of fixed geometry structures with displacement and stress constraints based upon the use of optimality criteria appears to offer considerable advantages over mathematical programming based methods. For comparable problems the present method reaches a similar or better design in considerably less iterations than most numerical search methods and at considerable reduced computational costs.

The most efficient optimization method apparently published so far is attributable to Venkayya (References 21 and 22). For identical problems, the numbers of iterations required to determine the minimum weight designs are essentially the same for the mathematical programming and optimality criteria approaches. A difference does reside in the computational effort since presence of active displacement constraints requires the determination of gradients of constraint functions which, as noted previously, will increase computational time considerably for large numbers of variables. With the use of optimality criteria, these calculations are avoided and the computational effort needed at each redesign stage is relatively small and approximately directly proportional to problem size.

TABLE V

WING BOX

Iteration	Weight	Iteration	Weight	Iteration	Weight
1	593.44	6	387.91	11	388.15
2	407.09	7	387.68	12	388.14
3	388.95	8	387.85	13	388.23
* 4	387.67	9	387.97	14	388.26
5	387.90	10	388.07	15	388.28

* Least Weight Design

a) Iteration History

Member		Area/Thickness	
Type	Nodes	Present Method	Ref. 14 (scaled)
Axial	13	0.6505	1.0431
	35	0.1001	0.1036
	24	0.2366	0.3508
	46	0.2352	0.3315
	67	0.1001	0.1035
Half Web	13	0.0876	0.0876
	35	0.0889	0.0895
	24	0.0808	0.0664
	46	0.0768	0.0553
	67	0.0815	0.0537
	34	0.0200	0.0219
	56	0.0200	0.0215
	57	0.0337	0.0256
Triangle	124	0.1328	0.1441
	431	0.1328	0.1441
	346	0.0702	0.0599
	653	0.0702	0.0599
	567	0.0449	0.0435

b) Final Designs

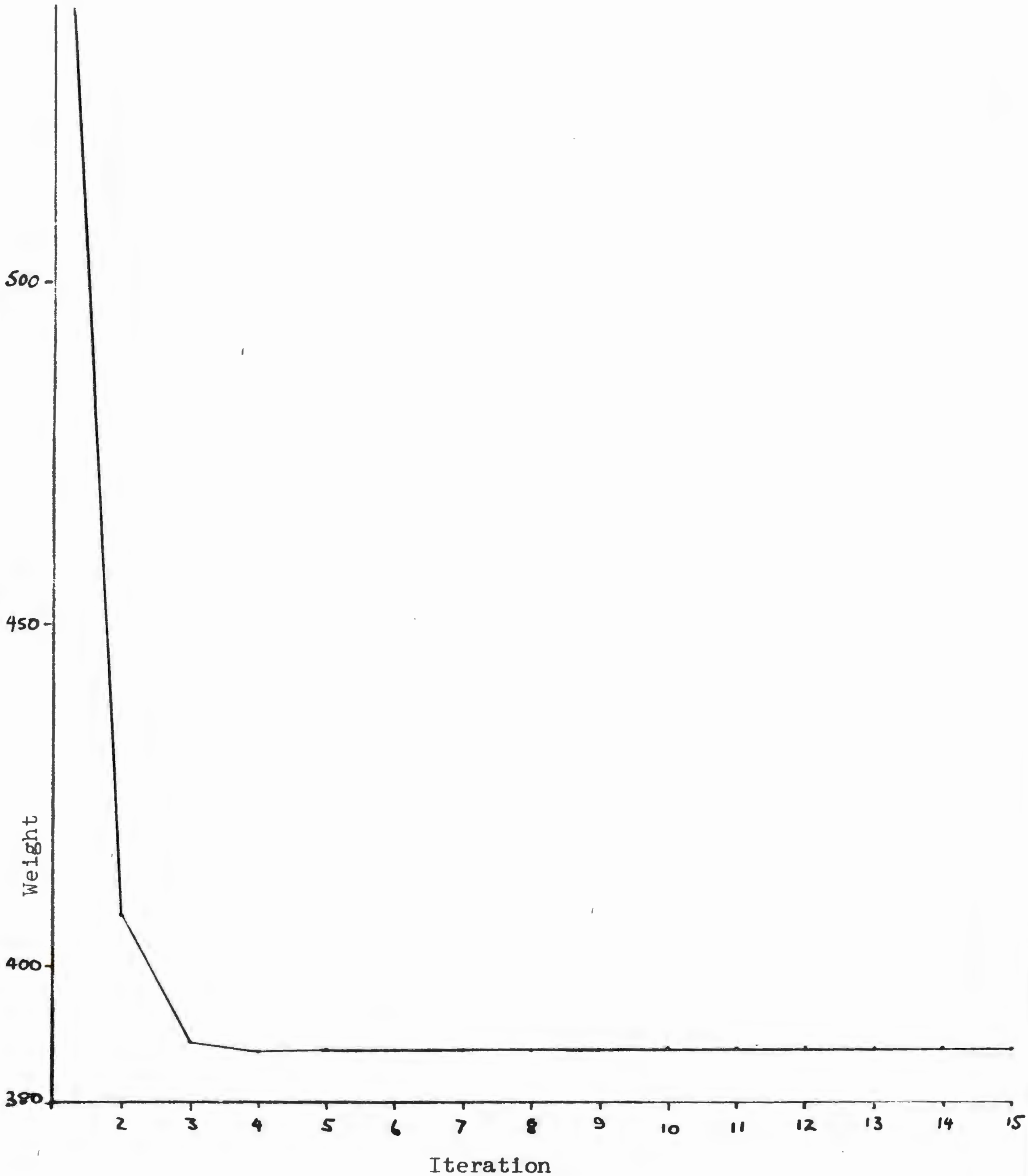


Figure 11. Iteration History for Wing Box

The results presented for the specific examples are very encouraging and indicate that some, if not all, of the difficulties encountered in large scale optimization problems can be eliminated through this type of approach. It is recognized that certain problems still remain to be resolved, particularly with regard to convergence characteristics. While the use of the omitted terms in Equation (15) might possibly affect some of the convergent behavior, their inclusion would certainly tend to negate the computational advantages indicated by the present approach.

The extension of the procedures presented above beyond stress and displacement constraint conditions can be accomplished within the general framework discussed in this paper. The criteria for the automatic partitioning between active and passive members is generally valid for other types of response phenomena. It is then only necessary to derive the criteria which must be satisfied (exactly or approximately) at the optimal design in a form analogous to Equation (15), containing provision for both variant and invariant member sizes. In such an extension, it must be recognized while the convergence can be expected to be more rapid than a nonlinear-programming approach, that there will still be a considerable level of computational complexity due to the fundamental nature of the analysis processes associated with the relevant response phenomena.

The approach presented herein represents a major step in a new direction of exploration and also provides a basis for many more developments in structural optimization.

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