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FRICTIONAL DAMPING AND RESONANT VIBRATION CHARACTERISTICS OF AN AXIAL SLIP LAP JOINT

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FOREWORD

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ABSTRACT

A brief summary of the various methods of damping analysis is presented. Data are procured on the frictional properties of mild steel in reciprocating sliding motion. The variation of the kinetic coefficient of friction as a function of normal load, lubrication, and number of cycles of motion is studied. The friction testing apparatus is considered as a vibrating system with Coulomb damping and its frequency response and damping energy are analysed.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

M. E. Sorte for

M. E. SORTE
Colonel, USAF
Chief, Materials Laboratory
Directorate of Research

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SECTION I. INTRODUCTION

Near-resonant vibration is generally considered to be a common cause for fatigue and other types of service failures in many fields of engineering. Even if actual failure can be avoided, the rough and noisy operation associated with the near-resonant condition frequently necessitates correction. Current trends, especially in the aircraft industry, have increased the importance of resonant vibrations as a factor in design.

The dangers of resonance can usually be minimized through one of three approaches:

(a) decreasing the exciting force,

(b) changing the natural frequency of the system, or

(c) increasing the damping in the system.

It is frequently difficult in many applications such as turbines or compressors to decrease the exciting force beyond the point which causes serious vibrations. Also, it is not always feasible to change the natural frequency of the system; design oftentimes will not permit changes, or operation must be over a wide frequency range. For these reasons designers are often forced to consider the various methods of increasing damping as a means of reducing vibration amplitudes near resonance.

The damping of a vibrating system may be increased in one of the following ways:

(a) by use of tuned mechanical absorbers which are attached to the vibrating structure,

(b) by more effective use of hysteresis or internal damping in the material, or

(c) by use of built-up members or similar structural units which dissipate energy through sliding friction.

Method (a) has been rather carefully studied, but comparatively little is known about (b) and (c).

The problems of energy dissipation in engineering materials and structures are most generally solved only in approximate or simplified form since the exact mechanism of damping in any specific case is usually not understood, (1,2,3,4,5)[1/]. If the entire resonant region is of interest, practical considerations dictate the use of dissipation functions

1/ Numbers in parentheses refer to references in the bibliography.

or damping terms so chosen that the equation of motion can be solved. It is also usually assumed, with justification in most instances, that the solution obtained in this manner offers a "good" approximation to physical reality.

However, if a reasonable choice of damping terms leads to an unmanageable equation or where such a choice does not exist, one must rely solely on measurements of energy dissipated per cycle (4) as a basis for the prediction of system behavior, and in many such cases the solution of the response exactly at resonance can be determined.

The study of energy losses in built-up structures has been largely neglected until recently (6,7,8). In order to reduce aircraft flutter or to increase shock and impact resistance, it is often desirable to incorporate spliced joints or similar members which will dissipate significant energy. In many cases, of course, a combination of material damping and frictional damping exists in varying proportions; some deliberately included and some accidentally. The contribution of each to the total can usually only be estimated. It is evident from the above that an increased knowledge of structural (connection) damping is highly desirable.

In the past the terms "structural" or "connection" damping have been used to describe energy dissipation in built-up assemblies. These terms include all sources of energy loss, both internal and external, but are usually understood to mean only frictional losses. This paper will use the term "slip" damping to describe those losses in an assembly which arise through friction caused by the slip of component parts on one another.

In the interest of a better understanding of damping due primarily to frictional phenomena, this report deals first with a brief investigation of the frictional properties of a material under reciprocating sliding motion. Secondly, the test setup for determining the above mentioned friction properties was treated as an axial motion lap joint and its damping and response characteristics were investigated.

SECTION II. CLASSIFICATION OF TYPES OF DAMPING AND REVIEW OF PRIOR WORK ON DAMPING ANALYSIS

Damping in any given system can usually be classified as one of the following, (see also Table I):

(a) Viscous. — Damping force varies as velocity. Characteristic of low velocity motion in a high viscosity fluid: dashpots, shock absorbers.

(b) Coulomb.	Damping force is kinetic frictional force. Characteristic of dry sliding surfaces: brakes, clutches, Lanchester Damper.
(c) Air.	Damping force varies as the square of the velocity. Characteristic of high velocity motion in low viscosity fluid: body in turbulent air at less than 0.6 Mach.
(d) Hysteresis.	Damping energy varies as function of maximum stress, usually a power function. Energy loss results from internal friction in solid bodies.
(e) Complex.	Damping force is proportional to displacement and in phase with velocity. Included as complex stiffness coefficient. An artifice to cover all cases of small damping.
(f) Negative.	Damping force feeds energy into the system. Always results in system instability: oscillator with positive feedback.
(g) Relaxation.	Damping force with both positive and negative characteristics. Results in the existence of a limit cycle: Prony brake, Froude pendulum, multivibrators.
(h) Magnetic.	Hysteresis energy loss varies as the product of frequency and maximum flux density to 1.6 power. Eddy current energy loss varies as the product of frequency squared and the maximum flux density squared. Used for damping of servo motors and electrical measuring instruments.

(i) Slip. Essentially Coulomb damping except that slip distribution gives an energy dissipation which does not, in general, vary as the first power of the amplitude of motion: built-up structures, joints.

In all practical vibrating systems a combination of two or more of the above types of damping will be present, usually in unknown proportions. The usual case is that in which one type is greatly predominant and analysis proceeds on that basis. Frequency and phase angle response characteristics for a single degree of freedom system with either viscous or complex damping are easily obtained and well known (9). The response characteristics of a system with Coulomb or combined Coulomb and viscous damping are more difficult to obtain and rather awkward to use (10). Systems with damping other than viscous, Coulomb, or complex are extremely difficult to handle even if an exact expression for the damping function is known (11,12).

In a good many mathematical treatments therefore, damping is introduced as a force proportional to the velocity with the constant of proportionality regarded as a measure of the equivalent viscous damping (3). Another common technique is the method of complex damping (13). Here the spring constant or stiffness coefficient is replaced by a complex constant, as $k(1 + ig)$ in the ordinary simple spring-mass equation of motion. The damping force, as introduced in this way, is a force proportional to the displacement and in phase with the velocity. A major drawback in using this type of damping is that the damped natural frequency increases with increasing damping, a phenomena contrary to experience.

Reference 14 presents complex damping in a slightly different and somewhat more logical manner. Here the damping is also introduced in the form of a complex spring constant $\bar{k} = k\, e^{2bi}$, where $2b$ is called the complex damping factor. For all practical cases of small damping either form of complex damping leads to the same result. However, the method of Myklestad is based on a consideration of hysteresis loop of elliptical shape and is thus satisfactory from both a mathematical and a physical standpoint. The method of complex damping is most commonly used in flutter analysis and similar fields.

Damping in vibrating built-up structures is generally a complex mixture of:

(a) Coulomb damping (dry friction),

(b) hysteresis damping (internal friction), and

(c) aerodynamic damping (air friction).

Energy dissipation in a connection which is not too tight is due primarily to sliding friction, and secondarily to hysteresis damping. In riveted connections the situation may be reversed. Thus, almost any combination of internal and external (slip) damping may occur in practice. With a few important exceptions, connections are designed for maximum rigidity and the damping is small enough so that a linear analysis is possible. For connections where a certain amount of looseness is permitted, the ever-present dangers of mechanical wear and seizure exclude this means of energy dissipation except in rare instances. The development of excellent dry and silicone lubricants has lessened some of this difficulty, but unknown resonance response characteristics still block most of the development in this field.

If, for any reason, the damping in a structure can be reduced to that due to Coulomb friction, or to a combination of Coulomb and equivalent viscous friction; the amplitude and phase angle response characteristics of a linear, single degree of freedom system with this type of damping is given in Reference 10. However, the analytical expressions for the response curves are quite involved, see Table I, and for small damping the results do not differ markedly from those obtained by using either an equivalent viscous damping term or a complex spring constant in the equations of motion.

In view of the situation indicated above, T. H. H. Pian and F. C. Hallowell, Jr. have developed an expression for the energy dissipation per cycle for simple built-up beams (6). This expression is given in terms of the beam dimensions, elastic constants and loading involved, and agrees well with their experimental results. Reference 6 appears to provide the best approach thus far available to a basic understanding of the phenomena of structural damping in a simple jointed structure at low frequencies. Reference 8 describes work on damping of turbine and compressor blades by friction at the connection of blade to wheel and illustrates some of the difficulties in separating the various types of damping.

The problem of expressing analytically the exact damping properties of any specific material or structure under all conditions of stress and temperature and all manner of loadings is to date almost completely unsolved (4). It is well known that in practical cases the damping must be small and the motion nearly sinusoidal in order to make use of the approximations necessary to solve the equations of motion. In other words, the damping and motion must be such that an equivalent viscous damping does not differ appreciably from the actual damping. The behavior of the approximating viscous damped system will then oftentimes follow closely the behavior of the actual system. That this does not always hold true is evidenced by the infinite amplitudes at resonance in a simple spring-mass system with only pure Coulomb damping.

Frequently, however, the damping is not small and one of the following complications may be encountered.

(a) The nonlinearity may be such that the linear approximation to the damping is largely invalid.

(b) The hysteresis loop may not resemble an ellipse.

(c) The damping may vary as an unknown function of frequency, amplitude, and velocity.

Of these possibilities we may separate out those cases where the type of damping is known and an exact expression, or close approximation, for the damping can be used in the equations of motion. These differential equations are nonlinear and can be solved approximately by means of a phase-plane method, numerical iteration procedure, and sometimes even analytically by classical methods. Reference 15 provides a powerful practical method for dealing with nonautonomous systems of differential equations, both of linear and nonlinear type.

If an approximate expression for the damping cannot be determined for inclusions into the equations of motion, or if the equations obtained cannot be solved satisfactorily, then we are forced to fall back on empirically derived expressions for energy loss per cycle; each expression possibly being confined to a given system, stress level, temperature or similar particular set of conditions. Since in all practical cases damping is one of the more elusive variables that the designer has to deal with, it is imperative that the extent of approximation be thoroughly investigated for each system in which energy dissipation is to be considered. A brief summary of some of the methods for expressing damping will be found in Table I.

It is perhaps necessary at this point to clarify the concept of "resonance," or at least to define what is meant in this paper by the terms resonance and resonant response characteristics. When speaking of the forced vibration of a linear, single degree of freedom system without damping, resonance is always clearly defined as that case in which the forcing frequency is the same as the natural frequency of the system. With undamped multi-degree of freedom systems, resonance is considered only as it affects the various modes of vibration and is again defined for any particular mode as the case for which the forcing frequency is equal to the natural frequency of that mode.

For damped systems the situation becomes slightly more involved. The damped natural frequency is lower than the undamped natural frequency and decreases with increasing damping, except for the artifice of complex damping. In addition, the frequency of maximum amplitude is different from either of the above, see Figure 1, and is sometimes called the "resonant frequency". For small damping all three of the above frequencies are very close to one another and no real distinction need be made. For this reason the frequency at which a 90 degree phase angle exists between the vectors representing input force and the displacement is here utilized as a definition of resonance with very little error in most cases. This frequency coincides with that determined as the undamped natural frequency.

For damping that can be approximated as viscous damping, the 90 degree phase angle definition is convenient from the viewpoint of energy relations. At this point the work done per cycle by the external force must be equal to the energy dissipated per cycle by damping. Hence, by equating these two quantities a very good approximation to the maximum amplitude can be obtained for any system with small damping. This scheme will oftentimes work surprisingly well for systems with rather large damping.

It is sometimes convenient to describe the severity of a resonant condition by using the resonance amplification factor as a measure of these conditions. The resonance amplification factor, A_r, 1/ or Q of an electrical circuit, has been defined as:

(a) the ratio of the maximum amplitude to the statical deflection or amplitude as the frequency of the exciting force approaches zero,

(b) 2 π times the ratio of the elastic energy stored at maximum stress to the energy dissipated per cycle by damping, and

(c) the ratio of the amplitude of the system when the forcing frequency equals the undamped natural frequency to the amplitude when the forcing frequency equals zero.

There is negligible difference between the above definitions providing the system is linear and the damping is small. For systems in which the damping is not small, definition (a) will give larger numerical results than either definition (b) or (c). In nonlinear systems none of the above definitions are valid (11). In nonlinear oscillations the natural frequency of oscillation is dependent on the amplitude and hence the frequency of the impressed force must be changed in order to continue to build up the amplitude of oscillation for a resonant condition. In addition, phase angle relationships are greatly changed for a nonlinear system and may even be double valued at some frequencies.

The relations involving A_r in Table I are all derived in accordance with the definition of resonance as that condition where the impressed frequency is the same as that of the undamped natural frequency of the system. The exciting force is 180 degrees out of phase with the velocity of motion. Appendix II and Figure 11 indicate the extent of approximation involved in using the above definition of resonance.

1/ Refer to Appendix I for definition of symbols and terms.

SECTION III. STATEMENT OF THE PROBLEM

Two types of problems concerning resonance are encountered in engineering practice. In the more comprehensive type, the response of a system over a large frequency range is desired. However, as discussed previously the dynamic equations for the entire resonant region can become extremely involved. Therefore, in this paper the second type of problem is considered: the determination of the vibration amplitude exactly at resonance. This second type of problem, although less general, is the primary concern in most engineering applications in which resonant vibrations are critical from a noise and fatigue viewpoint.

As indicated previously, damping is a critical factor in defining A_r. This report is therefore concerned with that important facet of structural damping which involves frictional or slip effects.

In the interest of a better understanding of damping due primarily to frictional phenomena, this report deals first with a brief investigation of the frictional properties of a material under reciprocating sliding motion. Secondly, the test setup and its damping and resonant response were investigated. Mild steel was used in this first investigation because of the great abundance of published data on this material.

Since, as indicated previously, the systems involved may be greatly nonlinear, the reliability of a simple linear system equation was partially determined.

SECTION IV. TEST EQUIPMENT

A measure of the amount of damping in any given system is usually arrived at experimentally by one of the following five methods:

(a) rate of decay of a structure in free vibration (16),

(b) measurements of the shape of a resonance curve (1),

(c) measurements of the area of a hysteresis loop (16,17),

(d) measurements of the energy input by an oscillator at or near resonance (18,19), or

(e) measurements of the lateral deflection of a rotating beam (20, 21).

Experimental results are expressable as either energy loss per cycle or in terms of a calculated equivalent viscous damping coefficient. It is often more convenient to use the energy loss per cycle as a measure of damping and its affect on A_r. Unfortunately, an expression of this nature cannot be included in the dynamic equations of motion (12).

It is generally accepted that a reliable measure of the damping in a structure can be obtained from the shape of the resonance curve and the use of a one degree of freedom analysis providing:

(a) the damping is small,

(b) points of force application and amplitude measurements are appropriate from the standpoint of normal mode considerations, and

(c) only amplitudes close to the resonant peak are used to determine the damping parameters.

To obtain values of damping energy and resonance amplification factor in the present investigation, a resonant vibration excitor and controller was used. This machine will automatically maintain vibration amplitude at any predetermined value at or near resonance regardless of changes in specimen stiffness or damping properties. Such changes can be evaluated from instrument readings while the test is in operation. Control is acquired by maintaining the phase angle between the vector representing force and the vector representing displacement at the desired value (90 degrees at resonance). A complete description of the machine operation and its capabilities can be found in reference 19.

The test setup for the work of Sections V and VI is shown in Figures 2 and 3. The sliding block is cylindrical with the contact area of 0.50 square inch. The lower block is rectangular and fixed in a grip by means of two set screws. For each test the specimen surfaces were ground and cleaned with acetone before lubricants were added or assembly was made. The grip holding the lower block has two flexible sections which permits the frictional force developed at the contact surfaces to be transmitted to the strain gage pickup. The output of the gage is fed to an Ellis Bridge, Model BA-11, and then to an oscilloscope.

The alternating exciting force is provided by the revolving eccentric and transmitted to the upper assembly by the connecting rod. The vertical bar with two flexible sections is used to prevent rocking or fishtailing of the upper specimen block during its oscillation. Normal loading is obtained by two calibrated extension springs. A velocity pickup is attached to the upper assembly to provide a velocity signal and, after integration, an amplitude signal. The amplitude signal is used primarily to observe the wave form representing the displacement. The measurements of the amplitude of motion are taken with a micrometer microscope. The signals representing friction force and velocity were applied to the appropriate deflecting plates of a DuMont 304-H oscilloscope and the resulting trace photographed with a Polaroid-Land camera. Figure 4 illustrates the results of this technique.

SECTION V. FRICTIONAL CHARACTERISTICS OF A RECIPROCATING SLIDING SURFACE UNDER VARIOUS CONDITIONS OF PRESSURE AND LUBRICATION

The problems of slip damping are very intimately related to those of the frictional properties of sliding surfaces. Thus, in connection with this investigation on axial joints and future work on slip damping in simple built-up beams, data are required on the frictional characteristics of mild steel under various conditions of pressure and lubrication.

The frictional damping is a function of the contact pressures, the type and amount of lubrication, the sliding velocity, and the properties of the materials in contact. In most engineering applications it is desirable to reduce or eliminate friction, but for the purposes of slip damping it is necessary that a given amount of friction be included and maintained.

No general relationship exists for specifying the optimum frictional properties for maximum damping in joint or connection. Indeed it seems that each friction problem must be solved individually (22). Although a given coefficient of friction may be obtained it can be held constant only under very carefully controlled conditions.

Using the setup of Figures 2 and 3, described previously, test data were procured on mild steel sliding surfaces under various operating conditions. The resulting test data are shown plotted in Figures 5 and 6. The test data of the unlubricated dry surfaces, initially ground, have a maximum scatter of approximately $\pm$ 15 per cent. This scatter was caused at least partially by seizure characteristics. During the first ten thousand cycles of the test there is seizure, oxidation, and tearing out of material which radically changes the surface conditions and makes the acquisition of reproducible data impossible (22). After an initial period of violently unstable motion there appears a gradual change to a state where the amount of fretted material generated between the surfaces is constant and the coefficient appears to stabilize.

It was found in the cases of dry surfaces and those lubricated with MoS_2, that deviations from the steady state amplitude of motion resulted in deviations in the measured coefficient of friction. Larger amplitudes gave lower values of the coefficient because of the opportunity for the wear material to be ejected from between the sliding surfaces and possibly because of motion over the comparatively virgin surfaces at the ends of travel. Small amplitudes were impossible to sustain for a long period of time and usually terminated in seizure (23).

The test data of the ground surfaces lubricated with MoS_2 dusted on indicate that the effect of this lubrication is only temporary under these test conditions. The number of cycles necessary for appreciable mechanical wear to begin, in general, decrease with increasing normal load. The coefficient of friction increases gradually to a value which corresponds to the surfaces operating without any lubrication. No work was done using other methods of applying the MoS_2 (25) in such a way as to make a more permanent lubricating surface.

If the surfaces are lubricated with SAE 30 oil, the coefficient of friction exhibits very little change with number of cycles. The coefficient decreases during a wear-in period and thereafter appears to stabilize or continue to slowly decrease. During the test there was a liberal supply of oil surrounding the test surface at all times. The edges of the slider were beveled slightly to eliminate any scraping action which would prevent oil from entering between the surfaces.

The maximum sliding velocity in all of the above tests was of the order of 25 feet per minute. Within the range of variables investigated and within the accuracy of measurement made, the coefficient of friction was found to be independent of velocity and pressure for the dry and MoS_2 surfaces.

Figures 7 and 8 are included to indicate the appearance of the sliding surfaces after testing. The photographs are negative reproductions of Faxfilm (Brush Development Company) impressions.

Of particular interest is the instantaneous variation between frictional force and velocity during the vibration cycle. Figure 4 shows Polaroid-Land photographs of the oscilloscope trace, with the exception of the displacement versus time curve which was drawn in for completeness, which were obtained by the means described in Section IV. These photographs show the existence of a somewhat unsteady pause at the amplitude extremes which indicate that the friction is high enough to exhibit motion with one stop per half cycle. It can also be seen from the force versus velocity trace that there is a slight peaking at zero velocity, maximum amplitude, where the static coefficient of friction is overcome.

The frictional force remains essentially constant throughout each half cycle of motion in close approximation to an assumption of Coulomb friction. This behavior except for an initial run-in period of a few thousand cycles was noted in most cases for the range of pressure and sliding velocity which was investigated.

SECTION VI. DAMPING ENERGY AND RESPONSE CHARACTERISTICS OF AN AXIAL MOTION JOINT

In order to provide a stepwise approach to the understanding of resonant amplification effects, tests and analyses were made on the axial slip joint setup shown in Figures 2 and 3. This background, it was felt, would aid in the analysis of more complicated joints.

The axial slip joint shown in Figures 2 and 3 can be analysed as a simple, single degree of freedom system with Coulomb damping. A complete analysis of this type of system is given in reference 10. Several attempts were made to check the theoretical response curves shown in Figure 9 for this system. However, due to the high degree of instability, experimental curves could not be procured. Referring to Figure 9, at the high F/P_o ratio necessary to obtain a finitely peaked response curve, it was found to be impossible to maintain this ratio over the required frequency range because of the 15 per cent variation in the coefficient of friction mentioned in Section V. The low F/P_o ratios lead to extremely large and unstable slope. Thus, the experimental determination of the curves near resonance could not be made.

In view of the above instability, either the above or below resonance region of operation had to be used. The above resonance region was selected. Most of the tests with this setup were run with $\omega/\omega_n = 1.5$ or in that neighborhood. The high F/P_o and ω/ω_n ratios place operation in or near that region of the response curve diagram in which motion with one stop per half cycle occurs. This is the region below the dashed line in Figure 9. Illustration of this type of motion is shown in Figure 4.

An attempt was made to determine the magnitude of the energy dissipated by joint friction. Two methods were used: (1) energy was calculated using the frictional properties of the surfaces as discussed in Section V, and (2) energy input was determined from machine readings during a vibration test. In all of the computations it was assumed that the motion was sinusoidal. The results and equations used are shown below. It will be observed that for some of the given F/P_o and ω/ω_n ratios there is no phase angle given in Figure 10. The phase angles in the tabulated results were obtained experimentally by "stopping" the motion with a General Radio Strobotac so that the displacement was zero, and measuring the angle made by the weighted eccentric (direction of input force) and the direction of motion. Hence, even though the motion was not sinusoidal, the phase angle determined in this manner does provide a reasonable value for use in the below formulae

$$W_i = \pi \; P_o \, X_o \sin \phi \qquad (1)$$

$$D_o = 4\,F\,X_o = 4\,\mu\,F_n\,X_o \tag{2}$$

Lubrication F_n	$\frac{\omega}{\omega_n}$	ϕ Measured	From Fig. 10	F/P_o	μ	W_i	D_o
Dry 21	1.41	90		0.78	0.60	1.42	1.38
Dry 77.5	1.65	90		0.71	0.61	3.12	3.97
MoS_2 42	1.41	71		0.74	0.61	3.04	3.05
MoS_2 77.5	1.58	66		0.71	0.59	5.97	5.82
Oil 61	0.90	35	38	0.51	0.033	0.46	0.52
Oil 77.5	1.75	168	167	0.16	0.040	0.55	0.56

It can be seen that though the motion is not sinusoidal and the phase angle was determined as previously described, a fairly good check exists between the energy input and the energy dissipated using equations 1 and 2. Therefore, an energy input equation can be provided even though the motion is not strictly sinusoidal. If a phase angle can be determined in the manner described above, then the energy input can be determined for use in damping calculations.

SECTION VII. SUMMARY AND CONCLUSIONS

A brief summary of various methods of damping analysis is presented which points out the difficulty of analytically handling most forms of energy dissipation in a vibrating system. These difficulties are accentuated with the dissipation of large quantities of energy as are encountered in connection damping. The use of an equivalent viscous damping coefficient is most often used in design considerations of a vibrating system. This artifice works surprisingly well even in cases where small non-linearities in damping and elasticity are present. A complex stiffness coefficient has also been used with good results in cases where the damping is small. In general, however, the phenomena of energy dissipation in materials and structures is not clearly understood.

Since the energy dissipation in slip damping takes place, for the most part, through sliding friction, an investigation of the properties of mild steel in reciprocating sliding motion was undertaken. For the range of normal pressures, sliding velocities, and lubrication conditions investigated it was found that:

(a) After an initial wear-in period the frictional force versus velocity characteristic was very similar to that usually assumed for dry friction or Coulomb damping.

(b) Regardless of whether oil (SAE No. 30), MoS_2, or no lubrication was used there exists an initial stage of rapid change in the coefficient of friction under reciprocating sliding motion.

(c) After the wear-in period the coefficient of friction and amount of mechanical wear, if any, appears to stabilize or change very slowly, if at all.

Since the frictional characteristics of the test setup does approximate those of Coulomb damping, the apparatus was considered from this viewpoint. Theory predicts that for the large damping energy experienced, motion with one stop per half cycle should exist. Experimental observation of this condition was made. A_r was not computed for this slip joint since high damping and small variations in coefficient of friction did not provide stable motion nor reliable data. It was also found that a check of energy balance was possible and reasonably accurate despite the fact that nonsinusoidal motion occurred in some cases.

Future work in slip damping should take into account possible changes in the frictional characteristics of the mating parts. A nonconstant slip distribution could easily lead to a varying frictional characteristic along the length of slip. It is thus possible that a joint in bending vibrations could have damping characteristics which change with number of cycles in a manner similar to that experienced in hysteresis damping. Proper lubrication would perhaps diminish some of these difficulties. However, it is easily seen that connection damping in jointed members is a problem relatively untouched as yet.

BIBLIOGRAPHY

1. Coleman, R. P., "Damping Formulas and Experimental Values of Damping in Flutter Models," NACA-TN-751, 1940.

2. Theodorsen, T. and Garrick, I. E., "Mechanism of Flutter - A Theoretical and Experimental Investigation of the Flutter Problem," NACA-TN-685, 1940.

3. Soroka, W. W., "Note on the Relations Between Viscous and Structural Damping Coefficients," J. Aeronautical Science, Vol. 16, 7 (1949).

4. Lazan, B. J., "Effect of Damping Constants and Stress Distribution on the Resonance Response of Members," J. Applied Mechanics, Vol. 19, 8 (1952).

5. Fearnow, D. O., "Investigation of the Structural Damping of a Full-Scale Airplane Wing," NACA-TN-2594, 1952.

6. Pian, T. H. H. and Hallowell, F. C., Jr., "Investigation of Structural Damping in Simple Built-Up Beams," Technical Report, ONR Contract N5 ori-07833, Massachusetts Inst. of Technology 1949; also, Proc. First U. S. Nat'l. Congress of Applied Mechanics, pp. 97-102, 1952.

7. Cooper, D. H. D., "A Suggested Method of Increasing the Damping of Aircraft Structures," (British) Memoranda No. 2398, 1946.

8. Hanson, M. P., Meyer, A. J., and Manson, S. S., "A Method of Evaluating Loose Blade Mounting as a Means of Suppressing Turbine and Compressor Blade Vibration," Proc. Soc. Exptl. Stress Anal. 10, 2, 103-116, (1953).

9. Den Hartog, J. P., "Mechanical Vibrations," McGraw-Hill Book Company, New York, 1940.

10. Den Hartog, J. P., "Forced Vibrations with Combined Coulomb and Viscous Friction," Am. Soc. Mech. Eng., Vol. 53, 9 (1931), also, Phil. Mag., Vol. 9, No. 59, (1930).

11. Stoker, J. J., "Nonlinear Vibrations," Interscience Publishers, Inc., New York, New York, 1950.

12. Pian, T. H. H. and Siddall, J. N., "The Prediction of Stresses in a Structure Under an Arbitrary Dynamic Loading," Technical Report, ONR Contract N5 ori-07833, Massachusetts Inst. of Technology, 1949.

13. Scanlon, R. H. and Rosenbaum, "Aircraft Vibration and Flutter," Macmillan Company, 1951.

14. Myklestad, N. O., "The Concept of Complex Damping," J. Applied Mechanics, Vol. 19, No. 3 (1952).

15. Klotter, Karl, "Nonlinear Vibration Problems Treated by the Averaging Method of W. Ritz," Technical Report No. 17, Parts I and II, ONR Contract N6 onr-251 Stanford University; also, Proc. First Nat'l. Congress of Applied Mechanics, pp. 125-132, 1952.

16. von Heydekampf, G. S., "Damping Capacity of Materials," Proc. Am. Soc. Test. Mat., Vol. 31, Part II, pp 157-171 (1931).

17. Catalog of "Cheenard Testing Apparatus," distributed by Buehler Company and Amsler Testing Machine Company. Alternating Torsion Machine Type M1/TA, "Michromachine".

18. Lazan, B. J., "Some Mechanical Properties of Plastics and Metals Under Sustained Vibrations," Trans. Am. Soc. Mech. Eng., Vol. 65, pp. 87-102 (1943).

19. Lazan, B. J., Brown, J., Gannett, A., Kirmser, P., and Klumpp, J., "Dynamic Testing of Materials and Structures with a New Resonance Vibration Exciter and Controller," Proc. Am. Soc. Test. Mat., Vol. 52 (1952).

20. Kimball, A. L., "Vibration Prevention in Engineering," Wiley, 1932.

21. Lazan, B. J., "A Study with New Equipment of the Effects of Fatigue Stress on the Damping Capacity and Elasticity of Mild Steel." Trans. Am. Soc. Metals, Vol. 142, pp. 499-558 (1950).

22. Pomey, J., "Friction and Wear," NACA Technical Memorandum No. 1318, 1952.

23. Feng, I-Ming, and Rightmire, B. G., "The Mechanism of Fretting," J. American Society of Lubrication Engineers, Vol. 9, No. 3 (1953).

24. Myklestad, N. O., "Vibration Analysis," McGraw-Hill Book Company, New York, New York, 1944.

25. Godfrey, D. and Bisson, E., "Bonding of Molybdenum Disulfide to Various Materials to Form a Solid Lubricating Film," NACA-TN-2628, 1952.

26. Lazan, B. J., "Fatigue Failure Under Resonant Vibration Conditions," presented as part of the ASM Lecture Course on Fatigue, 1953.

27. Cochardt, A. W., "The Origin of Damping in High-Strength Ferro-magnetic Alloys," J. Applied Mechanics, Vol. 20, No. 2, pp. 196-200 (1953).

APPENDIX I

DEFINITION OF SYMBOLS AND TERMS

A_r = resonance amplification factor.

= A_v when $f_r = f_v$.

A_v = vibration amplification factor at frequency f_v.

ratio of amplitude of vibration near $f_v = 0$ to amplitude of vibration at frequency f_v under the same exciting force.

c = damping coefficient, lb-sec/in.

c_c = critical viscous damping coefficient = $2\sqrt{km}$, lb-sec/in.

C = discord = frequency ratio = f_v/f_r (= 1 at resonance).

= ω/ω_n if damping is relatively small.

D_o = total damping energy absorbed by a test volume or joint; in-lbs/cycle.

E = static modulus of elasticity, psi.

f_r = resonant frequency, usually where $\phi = 90^\circ$ (Appendix II).

F = frictional force, lbs (References 9 and 10).

F_t = difference between upper and lower limits of applied load for bending joints, lbs (ΔF in Reference 6).

F_d = damping force, lbs.

F_N = normal force, lbs.

F_o = exciting force, $\pm$ lbs (usually P_o in Table I).

g = fraction of L that locates end of spar cap from free end of cantilever beam with reinforcing spar caps.

h = depth of beam, in.

I = moment of inertia of beam, in^4.

J, n = constants in equation $D_o = J\ S^n$ (Reference 4).

k = spring constant, lbs/in.

f_v = frequency of exciting force = frequency of vibration in normal mode, cps.

K_c = cross sectional shape factor (Reference 4).

K_s = longitudinal stress distribution factor (Reference 4).

K_m = material factor = $\pi/E_d \; J \; S^{n-2}$ (Reference 4).

K_v = volume-stress factor (Reference 26).

L = length of beam.

m = mass, lb-sec^2/in.

p = ratio of half length of cap (cover plate) to total length of beam (Reference 6).

P_o = exciting force, lbs.

q_M = limiting shear force (per unit length) between the cap and the beam, lbs/in (Reference 6).

r = $2 I/Ah^2$, nondimensional constant, (λ , in Reference 6).

s = portion of total beam length measured from free end of beam to splice (Reference 6).

W_i = energy input to a system or joint, in-lbs/cycle.

W_E = elastic energy stored in system at maximum stress, in-lbs.

x = displacement or amplitude of vibration, $\pm$ inches (Table I).

x_o = maximum of amplitude during a cycle, inches.

$\dot{x}$ = velocity, in/sec^2.

$\ddot{x}$ = acceleration, in/sec^2.

x_{st} = amplitude as ω approaches zero = P_o/k, inches.

z = $\dfrac{F_t}{2q_M h}$, nondimensional notation convenience (Reference 6).

δ = logarithmic decrement.

δ = $\ln(\frac{x_{n-1}}{x_n}) \cong \Delta x/x_n$, where x_{n-1} and x_n are the amplitudes of successive decay vibrations and $\Delta x = x_{n-1} - x_n$.

μ = coefficient of sliding friction.

ϕ = phase angle between the rotating vector representing sinusoidal exciting force and the rotating vector representing sinusoidal displacement.

ψ = specific damping capacity $2\pi/A_r = +2\delta$ (Reference 18).

ω = frequency of impressed force (Table I), rad/sec.

ω_d = damped natural frequency, rad/sec.

ω_n = $\sqrt{k/m}$ = undamped natural frequency, rad/sec.

APPENDIX II

SOME RESONANCE RELATIONSHIPS FOR DAMPED VIBRATIONS

A. Viscous Damping

Forced vibration of a linear spring-mass system with viscous damping can be described by the equation:

$$m\ddot{x} + c\dot{x} + kx = P_o \sin \omega t \qquad (1)$$

The solution of this equation of motion can be written as:

$$\frac{x}{x_{st}} = \frac{\sin(\omega t - \phi)}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\frac{c}{c_c}\frac{\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}} \qquad (2)$$

where;

$$\phi = \tan^{-1}\left[\frac{2\frac{c}{c_c}\ \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \qquad (3)$$

From these relations we find:

1. If $\omega = \omega_n$;

$$\left|\frac{x}{x_{st}}\right| = \frac{1}{2\frac{c}{c_c}} \qquad \text{and} \qquad \phi = 90 \text{ degrees} \qquad (4)$$

2. If $\omega = \omega_d = \omega_n \sqrt{1 - \left(\frac{c}{c_c}\right)^2}$

$$\left|\frac{x}{x_{st}}\right| = \frac{1}{2\frac{c}{c_c}\left[1.25 - \left(\frac{c}{c_c}\right)^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi = \tan^{-1}\left[\frac{2\sqrt{1-\left(\frac{c}{c_c}\right)^2}}{\frac{c}{c_c}}\right] \qquad (5)$$

3. If $\omega = \omega_n \sqrt{1-2\left(\frac{c}{c_c}\right)^2}$ $\left|\frac{x}{x_{st}}\right| =$ a maximum

$$\left|\frac{x}{x_{st}}\right|_{max} = \frac{1}{2\frac{c}{c_c}\left[1-\left(\frac{c}{c_c}\right)^2\right]^{\frac{1}{2}}} \text{ and } \phi = \tan^{-1}\left[\frac{\sqrt{1-2\left(\frac{c}{c_c}\right)^2}}{\frac{c}{c_c}}\right] \tag{6}$$

From an engineering viewpoint Equation 6 is the most significant since the maximum amplitudes of a vibrating system are of primary interest. For systems representable by a single degree of freedom with viscous or equivalent viscous damping, Equation 6 gives the maximum amplitude obtainable. Fortunately, in most cases the damping is small and Equation 4, 5, and 6 give results which are very close to each other. Figure 11 shows the variation of phase angle and amplitude ratio as a function of the amount of damping. Thus, of many purposes the resonance amplification factor, A_r, may be obtained from Equation 4 as:

$$A_r = \frac{1}{2\frac{c}{c_c}} = \frac{m\omega_n}{c} \; ; \qquad \phi = 90 \text{ degrees.} \tag{7}$$

The damping ratio can be that of equivalent viscous damping for most engineering applications without undue error. Figure 11 shows that for a damping ratio, actual or equivalent, less than 0.2, A_r calculated on the basis of ω_n and A_r calculated on the basis of $\omega_{x\ max.}$ differ by very little: 4 per cent at $c/c_c = 0.2$. The difference decreases, of course, with decreasing damping.

The logarithmic decrement δ can also be associated with A_r for a viscously damped system. It can be shown that:

$$A_r = \frac{\pi}{\delta}\left(1 + \frac{\delta^2}{8\pi^2}\right) \tag{8}$$

This equation is subject to the assumption that the points of tangency between the decay curve and its envelope are the same or the points of maximum amplitude. It is also assumed that $\omega_n = \omega_d$ and that the resonant amplitude is obtainable from Equation 7.

B. Unspecified Type of Damping, Such as Hysteresis or Slip

In most engineering applications the damping is not purely viscous in nature. It is therefore desirable to investigate the applicability and limitations in the equations of Section A, above. Since the primary interest in this paper is the special case of resonance and damping terms required for defining resonance amplification factors and related terms, the equations defining the shape of the resonance curve will not be discussed further in this Appendix.

As mentioned previously, for systems which can be described approximately by Equation 1 with an equivalent viscous damping, the values given by Equation 7 and the actual maximum resonant amplification factor will not differ by more than 4 per cent if the damping factor is less than 0.2.

For a system with unspecified type of damping, sufficiently linear force-deflection relationship, and motion close enough to sinusoidal so that,

$$W_i = \pi P_o x_o \sin \phi$$

Then,

$$A_r = 2\pi \frac{W_E}{D_o} \qquad (9)$$

Thus, A_r from Equation 9 for ϕ = 90 degrees and the assumed viscous or equivalent viscous damping of Section A, above:

$$A_r = \frac{2\pi \frac{1}{2} k x_o^2}{\pi c \omega x_o^2} = \frac{1}{2 \frac{c}{c_c}}$$

reduces to Equation 7 as could be expected.

However, for all cases covered to date and in general (subject to the restrictions mentioned above) A_r for maximum amplitude may be assumed to be equal to A_r for ϕ = 90 degrees.

The logarithmic decrement for a system with unspecified damping can also be related to A_r through the specific damping capacity ψ of the system. By definition of the logarithmic decrement:

$$\delta = \log_e \quad (x_{n-1}/x_n)$$

Also,

$$\psi = \frac{D_o}{W_E} = \frac{k\,(x_{n-1}^2 - x_n^2)}{k\,x_{n-1}^2}$$

Therefore:

$$\frac{x_{n-1}}{x_n} = \left[1 - \psi\right]^{-\frac{1}{2}}$$

and

$$\delta = -\frac{1}{2}\log_e \;(1-\psi)$$

$$\psi = 1 - e^{-2\delta}$$

$$\psi = 2\left[\delta - \delta^2 + \frac{4\delta^3}{3} \quad ---- \quad \right] \tag{10}$$

Using Equations 9 and 10 for a system with unspecified damping:

$$A_r = \frac{\pi}{\delta - \delta^2} \tag{11}$$

This equation applies with fair accuracy to the case of viscous damping and is, of course, more general than Equation 8.

Since, however, A_r is a function of stress distribution, stress history, and specimen shape (4, 26), the equations of this Appendix should be applied with care unless the exact form of damping is known.

TABLE I

DAMPING ANALYSIS AND A_r PRESENTATION

Type of Damping Equation of Motion	Amplitude and Phase Angle Response Characteristics	D_o = Energy Dissipated Per Cycle δ = Logarithmic Decrement ψ = Specific Damping Capacity	A_r = Resonance Amplification Factor
Viscous or Velocity Damping $F_d = c\dot{x}$ $m\ddot{x} + c\dot{x} + kx = P_o \sin \omega t$ Ref. 9	$A_v = \frac{x}{x_{st}} = \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\frac{c}{c_c}\ \frac{\omega}{\omega_n}\right)^2\right]^{-\frac{1}{2}}$ $\phi = \tan^{-1}\left[\frac{2\frac{c}{c_c}\ \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$	$D_o = \pi c \omega x_o^2$ $\delta = \frac{2\pi \frac{c}{c_c}}{\left[1 - \left(\frac{c}{c_c}\right)^2\right]^{\frac{1}{2}}} =$ $\cong 2\pi \frac{c}{c_c}$ if $\frac{c}{c_c} << 1$ $\psi = \frac{2\pi c \omega}{k}$	$A_r = \frac{\left[\left(\frac{A_r}{A_v}\right)^2 - C^2\right]^{\frac{1}{2}}}{2(1-C)}$ $= -\frac{1}{2}\left[\frac{d}{dC}(\cot \phi)\right]_{C=1}$ $\cong \frac{\pi}{\delta}$ $A_r = \frac{2\pi}{\psi} = \frac{k}{c\omega}$
Coulomb or Dry Friction Damping $F_d = F$ (sign opp. of $\dot{x}$) $m\ddot{x} + kx \pm F = P_o \cos(\omega t + \phi)$ Ref. 9, 10	Non-Stop Motion Only. Ref. 10 for Stop Motion $\frac{x}{x_{st}} = \left[V^2 - \left(\frac{F}{P_o}\right)^2 U^2\right]^{\frac{1}{2}}$ $V = \frac{\left(\frac{\omega_n}{\omega}\right)^2}{\left(\frac{\omega_n}{\omega}\right)^2 - 1}$ $U = \frac{\frac{\omega_n}{\omega} \sin \frac{\omega_n}{\omega}\pi}{1 + \cos \frac{\omega_n}{\omega}\pi}$ Discontinuous at Resonance $\phi = \sin^{-1}\left(-\frac{F}{P_o}\ \frac{U}{V}\right)$	$D_o = 4 F x_o$ Decay $= \frac{4F}{k}$ per Cycle From : $\frac{D_o}{2W_E} = \frac{4F}{kx_o}$	Infinite Amplitudes at Resonance if $\frac{F}{P_o} < \frac{\pi}{4}$ Based on Equivalent Viscous Damping Coeff. $A_v = \pm \frac{\left[1 - \left(\frac{4}{\pi}\ \frac{F}{P_o}\right)^2\right]^{\frac{1}{2}}}{1 - C^2}$
Air or Velocity Squared Damping $F_d = c\dot{x}\lvert\dot{x}\rvert$ $m\ddot{x} + c\dot{x}\lvert\dot{x}\rvert + kx = P_o \sin \omega t$ c = constant, lb.-sec.^{2}/in.^{2} Ref. 9, 24	Based on Equiv. Viscous Damping Coefficient $\left(\frac{x}{x_{st}}\right)^2 = \frac{3\pi}{8D}\left[-\frac{3\pi}{16D}\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \sqrt{\frac{9\pi^2}{256D^2}\left(1 - \frac{\omega^2}{\omega_n^2}\right)^4 + 1}\right]$ Where $D = \frac{c\omega^2 P_o}{k^2}$	$D_o = \frac{8}{3} c\, \omega^2\, x_o^3$	Based on Equiv. Visc. Damping Coefficient $A_r = \frac{A_v \left(\frac{A_v}{A_r}\right)^2}{1 - A_v^2\ (1 - C^2)^2}$
Hysteresis or Material Damping Ref. 4, 24, 26, 27		If $D_o = f(S)$ → If $D_o = J S^n$ →	Then $A_r = K_m \cdot K_v$ Where: $K_m = \frac{\pi S_m^2}{E D}$ $K_v = \frac{\int \left(\frac{S}{S_m}\right)^2 \frac{dV}{dS} dS}{\int \left(\frac{D}{D_m}\right) \frac{dV}{dS} dS}$ Then $K_v = K_c \cdot K_s$ Equiv Viscous Coeff. $A_r = \frac{\left[\left(\frac{A_r}{A_v}\right)^2 - 1\right]^{\frac{1}{2}}}{2(1-C)}$
Connection or Structural Damping Ref. 6	Nonlinear Characteristics, Coupled Stiffness and Damping	Slow Vibrations Only Spliced Joint $D_o = \frac{F_t^3\ L^3}{12EIq_M h}\left[\frac{(s+p)^3 + s^3 r^3}{(1+r+z)^2} + \frac{(s-p)^3 + r^3 s^3}{(1+r-z)^2}\right]$ Reinforcing Spar Caps $D_o = \frac{F_t^3\ L^3 g^3}{12EIq_M h(1+r-z)^2}$	Exact Expression Unknown

TABLE I (cont.)

DAMPING ANALYSIS AND A_r PRESENTATION

Type of Damping Equation of Motion	Amplitude and Phase Angle Response Characteristics	D_o = Energy Dissipated Per Cycle δ = Logarithmic Decrement ψ = Specific Damping Capacity	A_r = Resonance Amplification Factor
Combined Viscous and Coulomb Damping $F_d = c\dot{x} \pm F$ $m\ddot{x} + c\dot{x} \pm F + kx = P_o \cos(\omega t + \phi)$ Ref. 10	$\frac{x}{x_{st}} = -G\left(\frac{F}{P_o}\right) + \sqrt{\frac{1}{q^2} - H^2\left(\frac{F}{P_o}\right)^2}$ $q = \sqrt{\frac{1}{V^2} + \left(2\frac{c}{c_c}\frac{\omega}{\omega_n}\right)^2}$ $V = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$ $G = \frac{\sinh\left(\pi\frac{\omega_n}{\omega}\cdot\frac{c}{c_c}\right) - \frac{\frac{c}{c_c}}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}}\sin\frac{\omega_n}{\omega}\pi\sqrt{1-\left(\frac{c}{c_c}\right)^2}}{\cosh\left(\pi\frac{\omega_n}{\omega}\frac{c}{c_c}\right) + \cos\frac{\omega_n}{\omega}\pi\sqrt{1-\left(\frac{c}{c_c}\right)^2}}$ $H = \frac{\frac{\omega_n}{\omega}}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}} \cdot \frac{\sin\pi\sqrt{1-\left(\frac{c}{c_c}\right)^2}}{\cosh\left(\frac{\omega_n}{\omega}\pi\frac{c}{c_c}\right) + \cos\frac{\omega_n}{\omega}\pi\sqrt{1-\left(\frac{c}{c_c}\right)^2}}$	$D_o = \pi c \omega x_o^2 + 4 F x_o$	If $\frac{F}{P_o} < \frac{\pi}{4}$ Based on Equiv. Viscous Damping Coefficient A_v from: $A_v^2 + A_v\left\{\frac{\frac{16}{\pi}\frac{F}{P_o}\frac{c}{c_c}C}{(1-C^2)^2 + 4C\left(\frac{c}{c_c}\right)^2}\right\}$ $+\frac{\left(\frac{4}{\pi}\frac{F}{P_o}\right)^2 - 1}{\left[(1-C^2)^2 + 4C^2\left(\frac{c}{c_c}\right)^2\right]} = 0$
Damping Not Specified, Assumed Prop. to Displ. Complex Notation $m\ddot{x} + k(1+ig)x = P_o e^{i\omega t}$ g = Coeff. of Complex Damping Ref. 13, 24	$\frac{x}{x_{st}} = \left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + g^2\right]^{-\frac{1}{2}}$ $\phi = \tan^{-1}\left[\frac{g}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$	$D_o = \pi k g x_o^2$, small g. $\delta = \frac{2\pi g}{1+\sqrt{1+g^2}} \cong \pi g$	$A_r = \frac{\left[\left(\frac{A_r}{A_v}\right)^2 - 1\right]^{\frac{1}{2}}}{2(1-C)}$ $\cong \frac{1}{g}$ for small g.
Damping Not Specified, Assumed Prop. to Displ. Complex Notation $m\ddot{x} + ke^{2bi}x = P_o e^{i\omega t}$ 2b = Complex Damping Factor Ref. 14	$\frac{x}{x_{st}} = \frac{e^{-i\alpha_c}}{\sqrt{1-2\left(\frac{\omega}{\omega_n}\right)^2\cos 2b + \left(\frac{\omega}{\omega_n}\right)^4}}$ $\alpha_c = \tan^{-1}\left[\frac{\sin 2b}{\cos 2b - \left(\frac{\omega}{\omega_n}\right)^2}\right]$	$D_o = \pi k \sin 2b\, x_o^2$ $= \pi k\, 2b\, x_o^2$, small damping $\delta = 2\pi\frac{\sin b}{\cos b}$ $\cong 2\pi b$, small damping	$A_r = \left[2(1-\cos 2b)\right]^{-\frac{1}{2}}$ $A_r = \frac{\left[\left(\frac{A_r}{A_v}\right)^2 - C^2\right]^{\frac{1}{2}}}{2(1-C)}$
Relaxation Damping $F_d = -(c_1 - c_2 x_2)\dot{x}$ or $F_d = -\alpha\dot{x} + \gamma\dot{x}^3$ $\ddot{x} - \left(\frac{c_1}{m} - \frac{c_2}{m}x^2\right)\dot{x} + \omega_n^2 x = 0$ $\ddot{x} - \alpha\dot{x} + \gamma\dot{x}^3 + \omega_n^2 x = F\cos\omega t$ Ref. 9, 11	General Closed Solution Not Known Limit Cycle Exists.	Non Sinusoidal Motion	

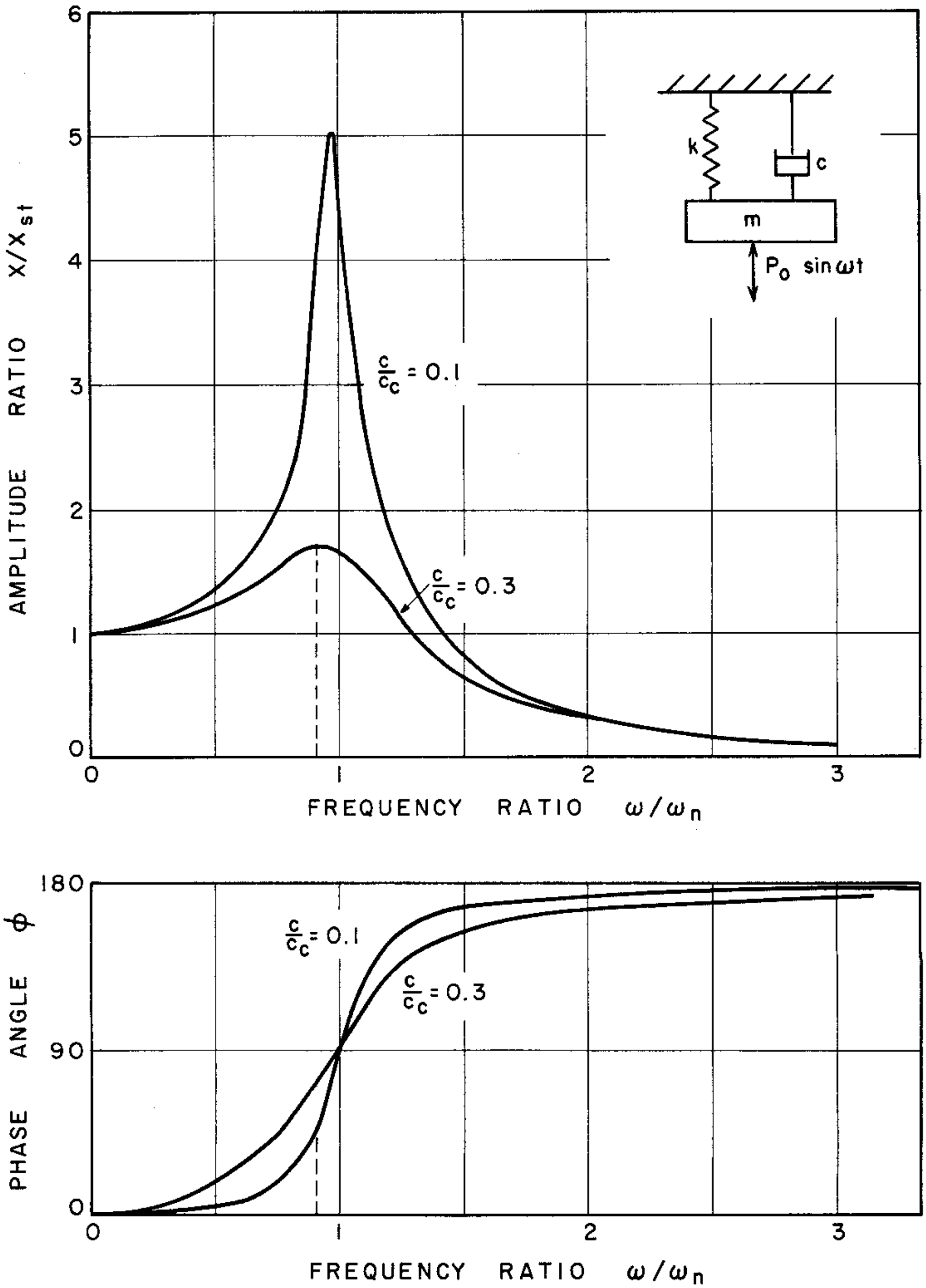

FIG. 1 - RESPONSE CURVES FOR VISCOUS DAMPED, SINGLE DEGREE OF FREEDOM SYSTEM.

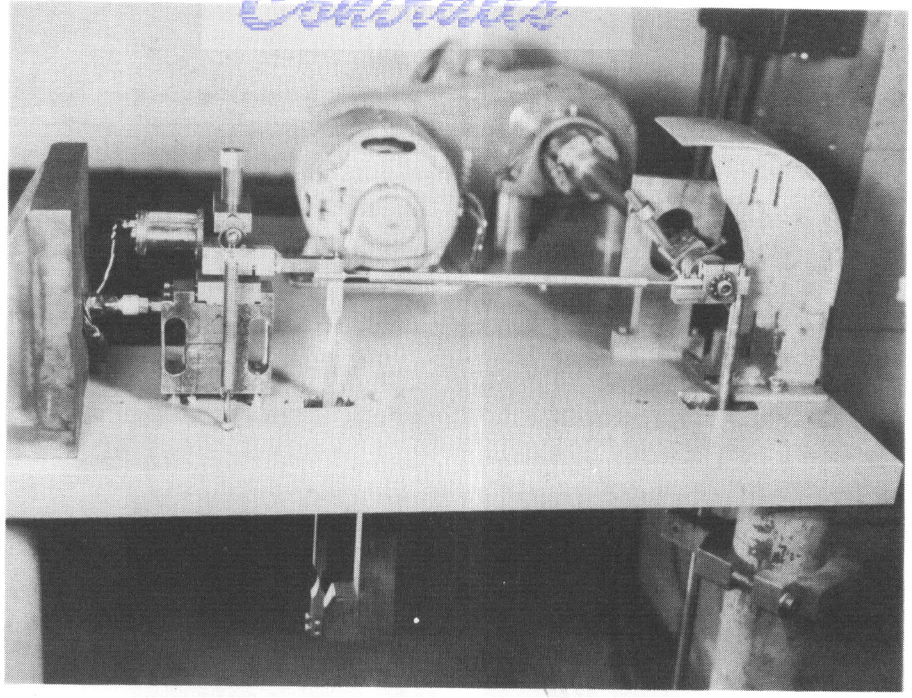

FIG. 2 - PHOTOGRAPH OF EQUIPMENT FOR FRICTION MEASUREMENTS.

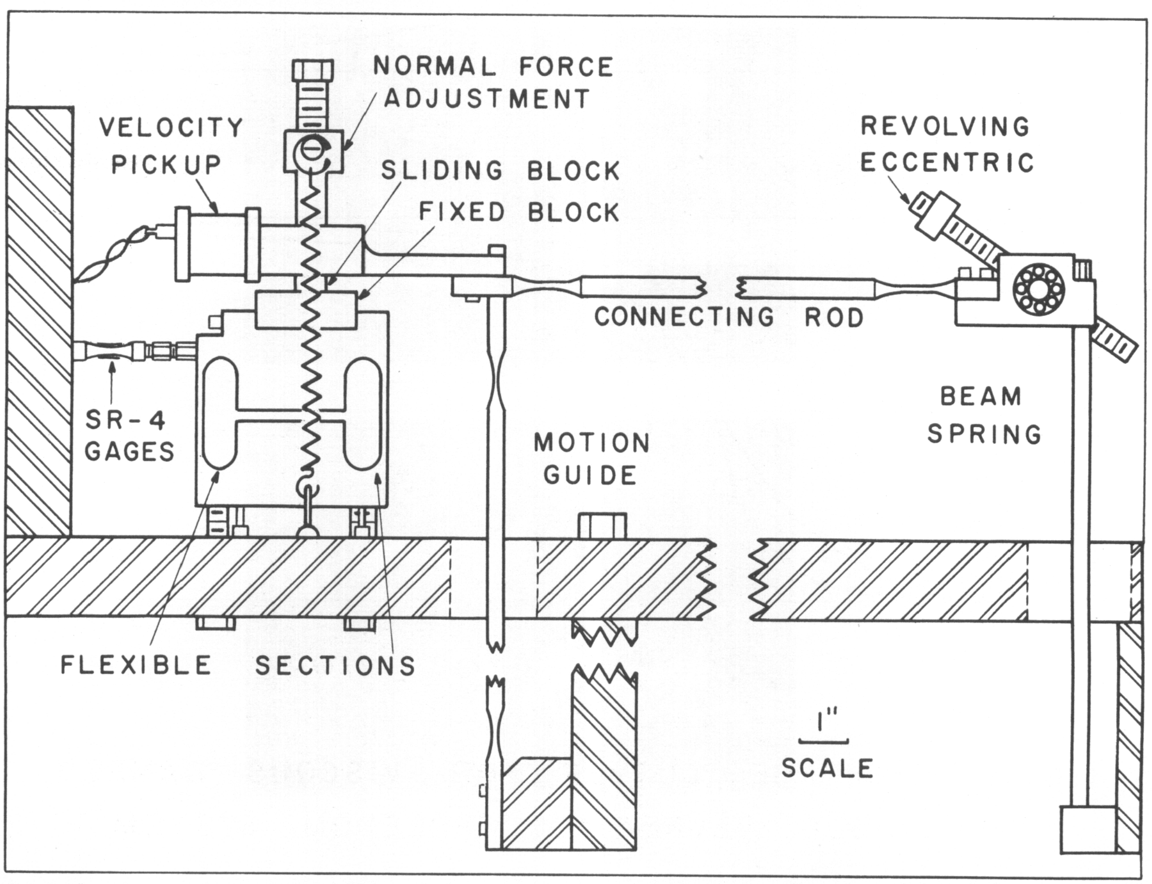

FIG. 3 - SCHEMATIC OF FRICTION TEST SET-UP.

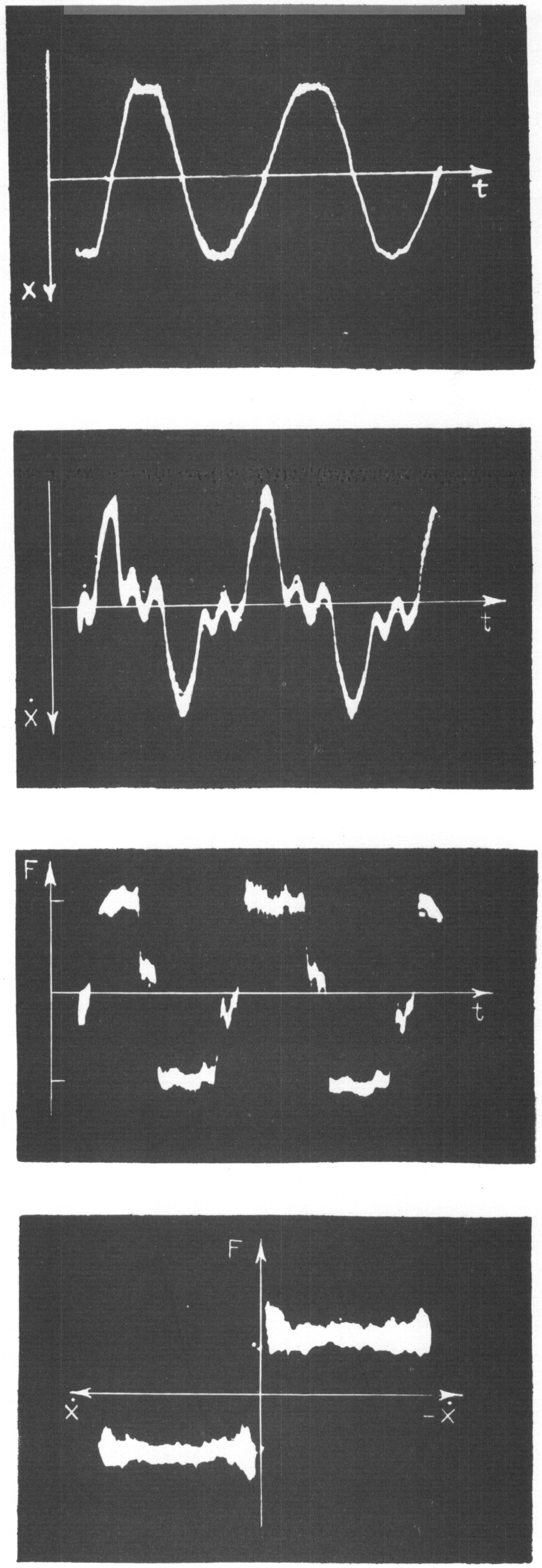

FIG. 4 - FORCE, VELOCITY, TIME RELATIONSHIPS DURING A VIBRATION CYCLE.

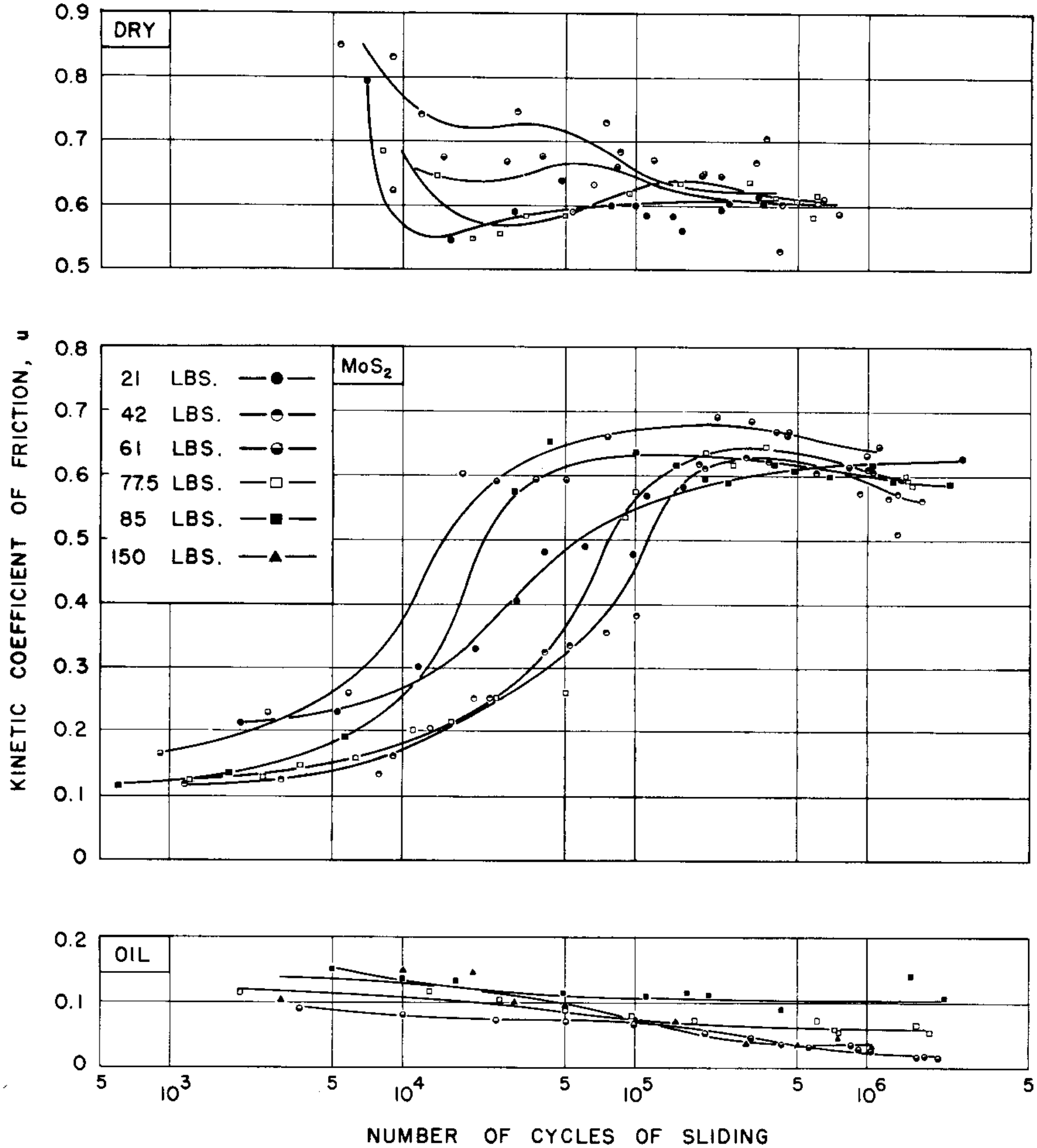

FIG. 5. COEFFICIENT OF FRICTION VS. NUMBER OF CYCLES WITH VARIOUS LOADS AND LUBRICANTS.

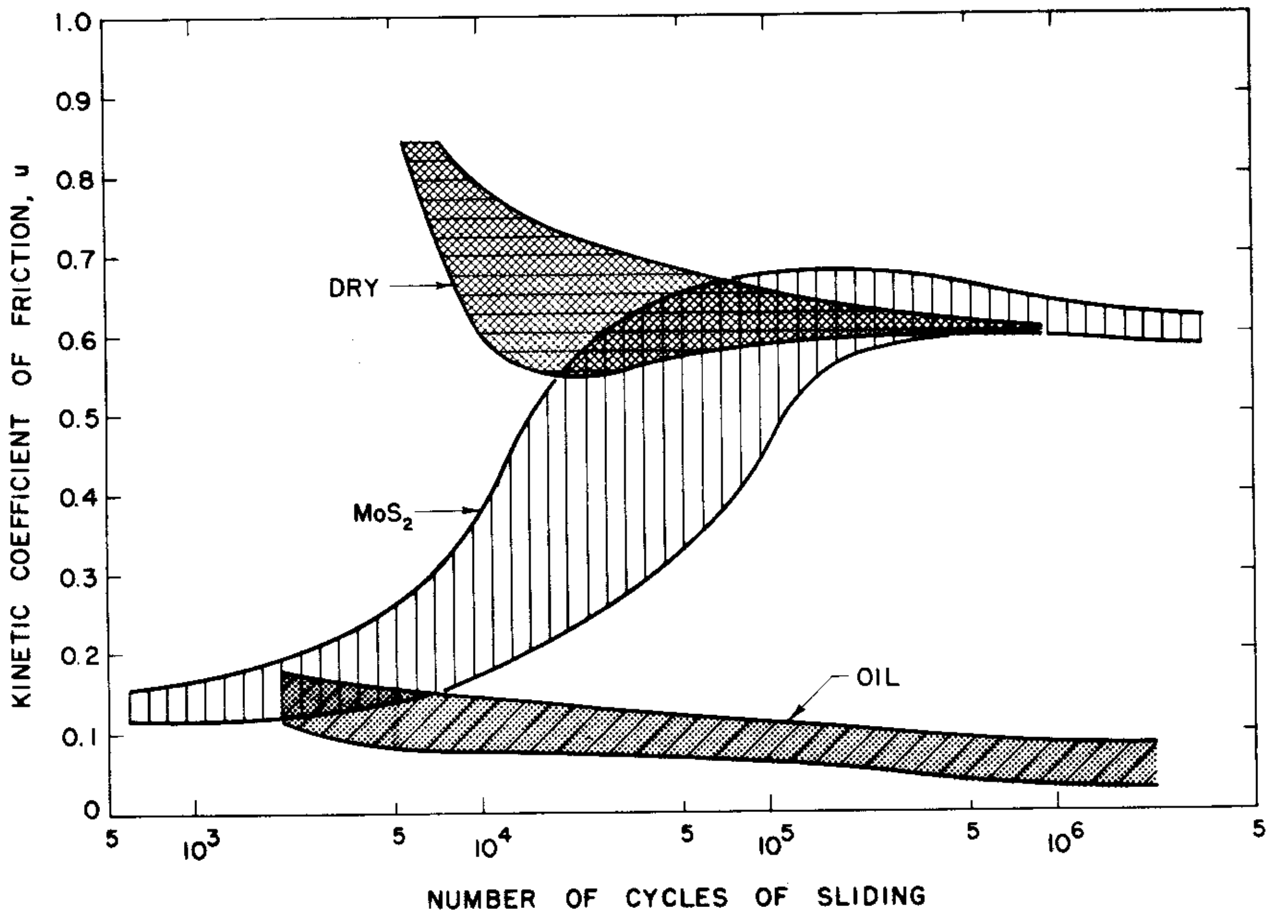

FIG. 6. TRENDS IN COEFFICIENT OF FRICTION AS FUNCTION OF NUMBER OF CYCLES FOR THREE LUBRICANTS. NOMINAL NORMAL PRESSURE 42 TO 170 PSI.

FIG. 7 – SURFACE AFTER 1,603,000 CYCLES AT 77.5 LB. LOAD. MoS_2 DUSTED ON. (X7)

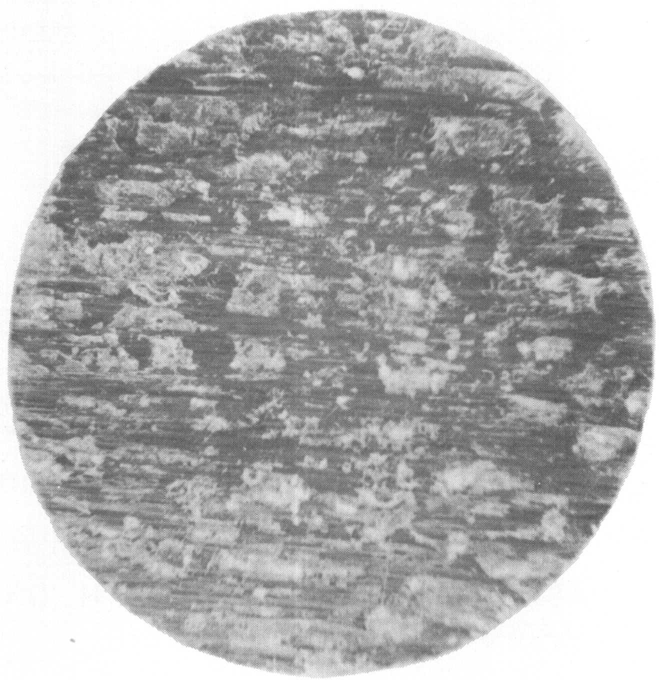

FIG. 8 – SURFACE AFTER 637,000 CYCLES AT 85 LB. LOAD. NO LUBRICATION. (X7)

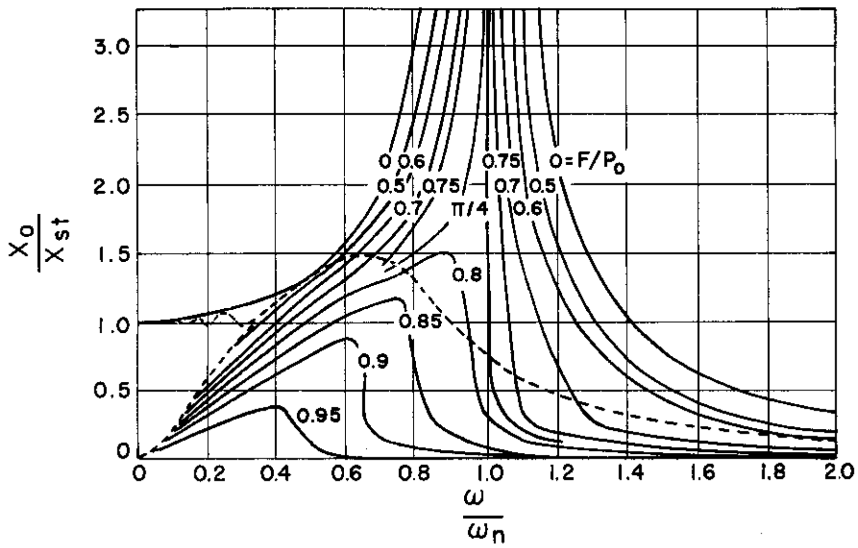

FIG. 9 – RESONANCE DIAGRAM FOR A SYSTEM WITH DRY FRICTION DAMPING. DEN HARTOG (9)

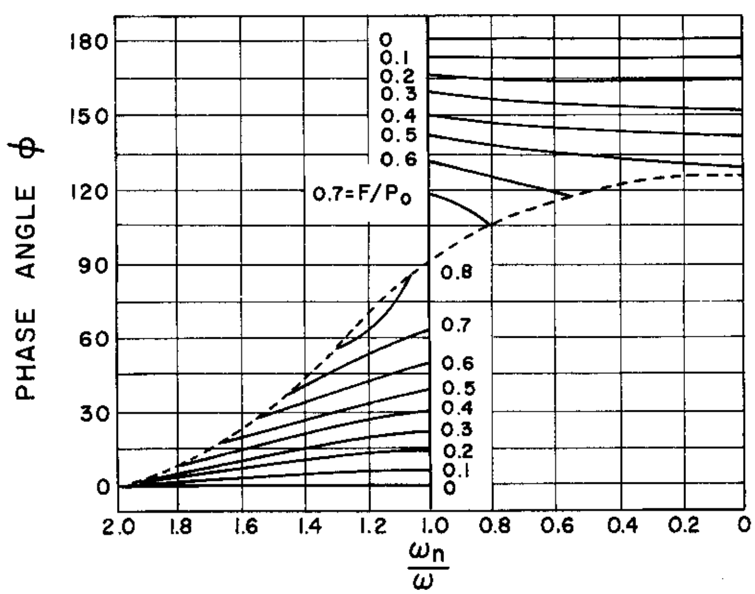

FIG. 10 – PHASE-ANGLE DIAGRAM WITH DRY FRICTION DAMPING. DEN HARTOG (9)

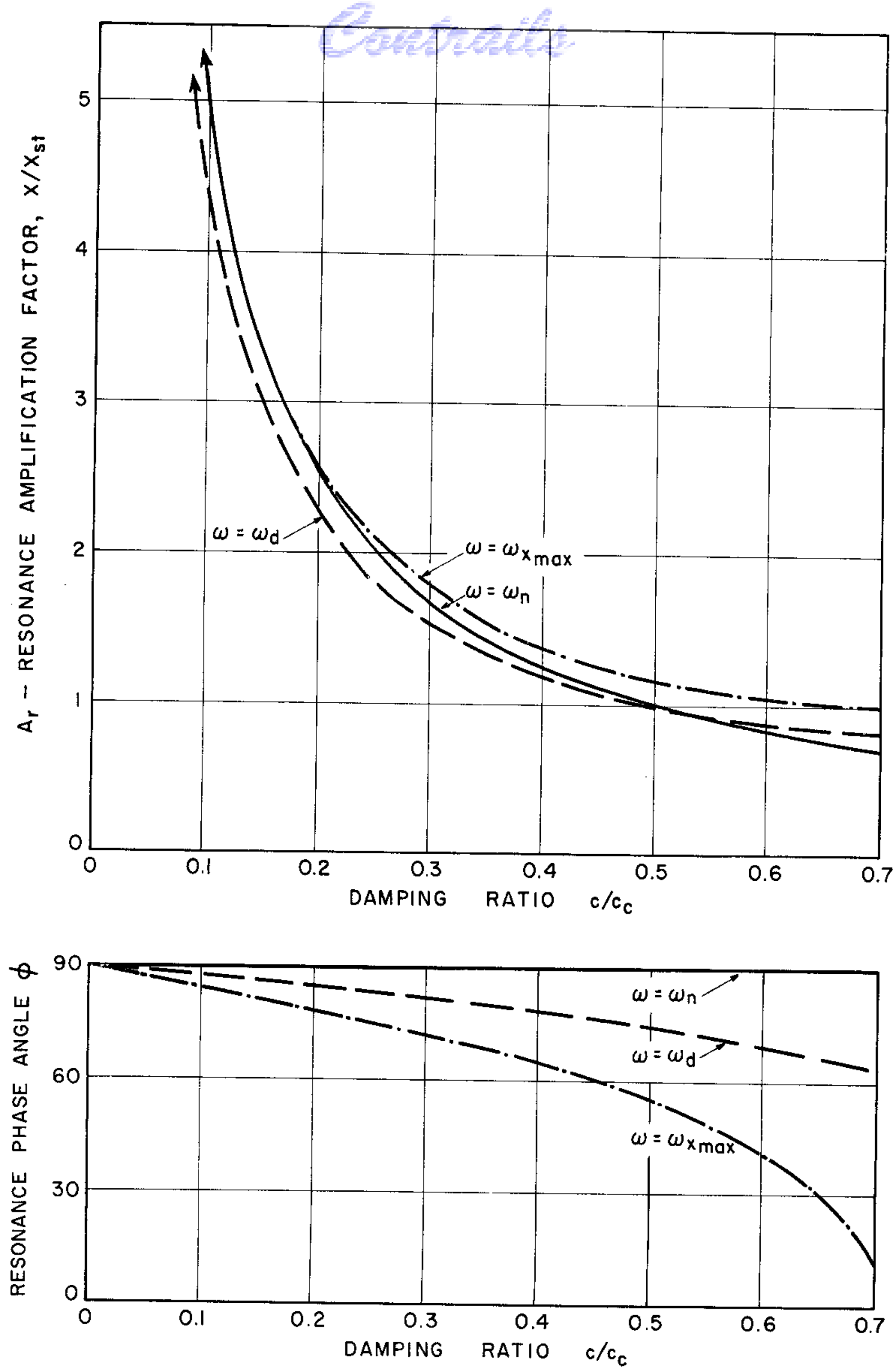

FIG. 11 – VARIATION OF AMPLITUDE AND PHASE WITH DAMPING DEPENDING ON THE DEFINITION OF RESONANCE.