Dynamic Analysis of Finite, Three Dimensional, Linear, Elastic Solids With Kelvin Viscoelastic Inclusions: Theory with Applications to Asymmetrically Damped Circular Plates

I. Y. Shen †
and
C. D. Mote, Jr.

Department of Mechanical Engineering
University of California, Berkeley
Berkeley, CA 94720
U. S. A.

ABSTRACT

Eigensolutions and Green's functions of finite, three dimensional, linear, elastic solids with Kelvin viscoelastic inclusions are analyzed. The eigensolutions satisfy a set of integral equations expressing the reciprocal theorem of viscoelasticity. Successive approximations to these integral equations lead to asymptotic solutions and an iteration scheme for the eigensolutions. The Green's function is also determined through the integral equation approach. Finally, the vibration of Kirchhoff circular plates with evenly spaced, radial, viscoelastic inclusions, which cause some of the repeated vibration modes to split into distinct ones, is analyzed both analytically and numerically for the eigensolutions and the Green's function.

† (415) 642-6371

1. Introduction

Tuned dampers and surface damping treatments are commonly used damping designs [1-3]. The tuned damper can be a one degree of freedom system consisting of a mass and a viscoelastic element attached to the structures to be damped [4,5]. They can also be viscoelastic links connecting complex structures. When the structures vibrate, the tuned dampers dissipate energy. The surface damping treatments include thin layers of viscoelastic materials bonded onto surfaces of the structures [6,7]. The vibration energy is dissipated via cyclic bending or shearing of the viscoelastic layers. A review of surface damping treatment is given by Torvik [7].

An alternative damping design is to replace part of an elastic structure by a viscoelastic component. For instance, slots and holes filled with viscoelastic material can reduce vibration of circular saws. The holes and slots can be arranged so that the viscoelastic material is significantly strained when the structure vibrates in particular modes. The damping design procedure, however, is one of trial and error; dynamic analysis of such designs has not been presented.

The purpose of this paper is to provide a dynamic analysis of damping designs through viscoelastic inclusions. The damped structure is modeled as a finite, three dimensional, linear, elastic solid containing Kelvin viscoelastic inclusions. Eigensolutions and Green's functions of the damped structure are determined analytically and numerically. Special attention is given to degenerate systems, like axisymmetric circular plates, that occur when the corresponding, homogeneous, linear, elastic solid (without inclusions) possesses repeated eigenvalues.

According to the reciprocal theorem of viscoelasticity (Section 7.3 of [8]), the eigensolutions satisfy a set of regular, homogeneous, Fredholm integral equations of the second kind. Successive approximations to the integral equations yield perturbation formulas and a numerical iteration scheme for the eigensolutions. The real and imaginary parts of each eigenvalue represents the modal damping coefficient and the damped natural frequency, respectively. The eigenfunctions may or may not be complex depending on the geometry and the viscosity of the inclusions. The Green's function is also determined through the integral equation approach.

The analysis is illustrated on the transverse vibration of a classical circular plate with evenly spaced, radial, viscoelastic inclusions. The perfect plate is axisymmetric and spectrally degenerate. The perturbation theory for degenerate systems predicts that some of the repeated vibration modes split into two distinct ones when the inclusions are introduced. Perturbed eigensolutions are also derived explicitly. In addition, eigensolutions and Green's functions at two different excitation frequencies are predicted numerically. The numerical results show that vibration modes with higher natural frequencies possess greater damping and the nodal curves may be time dependent.

2. Formulation

Consider a homogeneous, isotropic, linear, elastic solid containing Kelvin viscoelastic inclusions shown as System 1 in Fig. 1(a). The elastic solid occupies a finite, three dimensional region $\tau^{(1)}$ with Lamé constants λ_0, μ_0 and density ρ_0. The perfectly bonded viscoelastic inclusions occupy a not necessarily small region τ_c with Lamé constants λ'_0, μ'_0, density ρ'_0, and Kelvin damping coefficients λ_0^*, μ_0^*.

The response of System 1 is equivalent to that of System 2 in Fig. 1(b) which consists of an inhomogeneous Kelvin viscoelastic solid occupying a region $\tau\,(\equiv \tau^{(1)} \cup \tau_c)$ with stiffness $\lambda(\mathbf{r})$, $\mu(\mathbf{r})$, density $\rho(\mathbf{r})$, and damping $\lambda^*(\mathbf{r})$ and $\mu^*(\mathbf{r})$ [9]

$$\lambda(\mathbf{r}) = \lambda_0 - \lambda_1 J(\mathbf{r})\,,\ \ \mu(\mathbf{r}) = \mu_0 - \mu_1 J(\mathbf{r})\,,\ \ \rho(\mathbf{r}) = \rho_0 - \rho_1 J(\mathbf{r}) \tag{1a}$$

$$\lambda^*(\mathbf{r}) = \lambda_0^* J(\mathbf{r})\,,\ \ \mu^*(\mathbf{r}) = \mu_0^* J(\mathbf{r}) \tag{1b}$$

where

$$J(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} \in \tau_c \\ 0 & \mathbf{r} \in \tau^{(1)} \end{cases} \tag{1c}$$

and

$$\lambda_1 = \lambda_0 - \lambda'_0\,,\ \ \mu_1 = \mu_0 - \mu'_0\,,\ \ \rho_1 = \rho_0 - \rho'_0 \tag{1d}$$

The constitutive equation of System 2 is then

$$\tau_{ij}(\mathbf{w};\lambda,\mu,\lambda^*,\mu^*) = \lambda\delta_{ij}\varepsilon_{kk}(\mathbf{w}) + 2\mu\varepsilon_{ij}(\mathbf{w}) + \lambda^*\delta_{ij}\dot{\varepsilon}_{kk}(\mathbf{w}) + 2\mu^*\dot{\varepsilon}_{ij}(\mathbf{w}) \tag{2a}$$

where the infinitesimal strain $\varepsilon_{ij}(\mathbf{w})$ and the infinitesimal strain rate $\dot{\varepsilon}_{ij}(\mathbf{w})$ associated with the displacement field $\mathbf{w}(\mathbf{r},t)$ are

$$\varepsilon_{ij}(\mathbf{w}) = \frac{1}{2}(w_{i,j} + w_{j,i})\,,\ \ \dot{\varepsilon}_{ij}(\mathbf{w}) = \frac{1}{2}(\dot{w}_{i,j} + \dot{w}_{j,i})\,.$$

When the displacement and stress fields are both harmonic, i.e., $\mathbf{w}(\mathbf{r},t) = \mathbf{u}(\mathbf{r})e^{\nu t}$ and $\tau_{ij}(\mathbf{w};\lambda,\mu,\lambda^*,\mu^*) = \sigma_{ij}(\mathbf{u},\nu;\lambda,\mu,\lambda^*,\mu^*)e^{\nu t}$, (2a) implies

$$\sigma_{ij}(\mathbf{u},\nu;\lambda,\mu,\lambda^*,\mu^*) = \lambda\delta_{ij}\varepsilon_{kk}(\mathbf{u}) + 2\mu\varepsilon_{ij}(\mathbf{u}) + \nu\,[\lambda^*\delta_{ij}\varepsilon_{kk}(\mathbf{u}) + 2\mu^*\varepsilon_{ij}(\mathbf{u})] \tag{2b}$$

In addition, the reciprocal theorem is (Section 7.3 of [8])

$$\begin{aligned}
&\int_{\sigma_2} \sigma_{ij}(\mathbf{u},\nu;\lambda,\mu,\lambda^*,\mu^*) n_j u'_i \, d^2\mathbf{r} - \int_\tau \frac{d}{dx_j}[\sigma_{ij}(\mathbf{u},\nu;\lambda,\mu,\lambda^*,\mu^*)] u'_i \, d^3\mathbf{r} \\
&\quad = \int_{\sigma_2} \sigma_{ij}(\mathbf{u}',\nu;\lambda,\mu,\lambda^*,\mu^*) n_j u_i \, d^2\mathbf{r} - \int_\tau \frac{d}{dx_j}[\sigma_{ij}(\mathbf{u}',\nu;\lambda,\mu,\lambda^*,\mu^*)] u_i \, d^3\mathbf{r} \\
&\quad = \int_\tau \sigma_{ij}(\mathbf{u},\nu;\lambda,\mu,\lambda^*,\mu^*)\varepsilon_{ij}(\mathbf{u}') \, d^3\mathbf{r} = \int_\tau \sigma_{ij}(\mathbf{u}',\nu;\lambda,\mu,\lambda^*,\mu^*)\varepsilon_{ij}(\mathbf{u}) \, d^3\mathbf{r} \\
&\quad = \int_\tau I(\mathbf{u},\mathbf{u}';\lambda,\mu) \, d^3\mathbf{r} + \nu \int_\tau I(\mathbf{u},\mathbf{u}';\lambda^*,\mu^*) \, d^3\mathbf{r}
\end{aligned} \tag{3}$$

with

$$I(\mathbf{u},\mathbf{u}';\lambda,\mu) = \int_\tau [\lambda\varepsilon_{kk}(\mathbf{u})\varepsilon_{kk}(\mathbf{u}') + 2\mu\varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{u}')] \, d^3\mathbf{r}$$

where $\mathbf{u}(\mathbf{r})e^{\nu t}$ and $\mathbf{u}'(\mathbf{r})e^{\nu t}$ are two harmonic displacement fields satisfying zero displacements on σ_1.

3. Eigensolutions

Exact Solutions. The complex-valued eigenfunction $\psi(\mathbf{r}) \equiv [\psi^{(1)}(\mathbf{r}), \psi^{(2)}(\mathbf{r}), \psi^{(3)}(\mathbf{r})]^T$ and the corresponding complex eigenvalue ν of system 2 under zero body force and vanishing surface tractions satisfy

$$\frac{d}{dx_j}[\sigma_{ij}(\psi,\nu;\lambda,\mu,\lambda^*,\mu^*)] = \nu^2\rho(\mathbf{r})\psi^{(i)}(\mathbf{r}), \quad i=1,2,3 \tag{4}$$

with boundary conditions

$$\psi(\mathbf{r}) = 0, \quad \text{on } \sigma_1 \tag{5a}$$

$$\sigma_{ij}(\psi,\nu;\lambda,\mu,\lambda^*,\mu^*)n_j = 0, \quad \text{on } \sigma_2, \quad i=1,2,3 \tag{5b}$$

in the unprimed system. The complex-valued Green's function $\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0)$ of System 3†, shown in Fig. 1(c) under an interior concentrated force $\delta(\mathbf{r}-\mathbf{r}_0)e^{\nu t}$ acting in direction x_k $(k=1,2,3)$, is represented in the primed system. Therefore, $\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0) \equiv [\,G_1^k(\mathbf{r}\,|\,\mathbf{r}_0),\ G_2^k(\mathbf{r}\,|\,\mathbf{r}_0),\ G_3^k(\mathbf{r}\,|\,\mathbf{r}_0)]^T$ satisfies

$$\frac{d}{dx_j}[\sigma_{ij}(\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0),\nu;\lambda_0,\mu_0,0,0)] - \rho_0\nu^2 G_i^k(\mathbf{r}\,|\,\mathbf{r}_0) = -\delta_{ik}\delta(\mathbf{r}-\mathbf{r}_0), \quad i,k=1,2,3 \tag{6}$$

with boundary conditions

$$\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0) = 0, \quad \text{on } \sigma_1, \quad k=1,2,3 \tag{7a}$$

$$\sigma_{ij}(\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0),\nu;\lambda_0,\mu_0,0,0)n_j = 0, \quad \text{on } \sigma_2, \quad i,k=1,2,3 \tag{7b}$$

With the unprimed solution specified by (4), (5a,b), and the primed by (6), and (7a,b), the equalities in (3) give the following integral equations governing the free vibration of System 2 [9]:

$$\begin{aligned}\psi^k(\mathbf{r}_0) &= H[\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0),\psi(\mathbf{r})]\\ &\equiv \nu^2\int_\tau \rho_1 J(\mathbf{r})\psi(\mathbf{r})\cdot\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0)\,d^3\mathbf{r} - \nu\int_\tau I(\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0),\psi(\mathbf{r});\lambda_0^* J(\mathbf{r}),\mu_0^* J(\mathbf{r}))\,d^3\mathbf{r}\\ &\quad + \int_\tau I(\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0),\psi(\mathbf{r});\lambda_1 J(\mathbf{r}),\mu_1 J(\mathbf{r}))\,d^3\mathbf{r}, \quad k=1,2,3\end{aligned} \tag{8}$$

$\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0)$ is seldom known for numerical or perturbation evaluation of (8). An orthonormal eigenfunction expansion of $\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0)$ is

$$\mathbf{G}^k(\mathbf{r}\,|\,\mathbf{r}_0) = \sum_{n=1}^{\infty}\frac{\phi_n^k(\mathbf{r}_0)}{\nu^2+\omega_n^2}\phi_n(\mathbf{r}), \quad k=1,2,3 \tag{9}$$

where ω_n and $\phi_n(\mathbf{r}) \equiv [\,\phi_n^1(\mathbf{r}), \phi_n^2(\mathbf{r}), \phi_n^3(\mathbf{r})\,]^T$ are the n-th eigensolution‡ of System 3 with orthonormality

$$\int_\tau \rho_0\phi_n(\mathbf{r})\cdot\phi_m(\mathbf{r})\,d^3\mathbf{r} = \delta_{nm} \tag{10a}$$

† System 3 is defined by a perfect, homogeneous, linear, elastic solid *without* viscoelastic inclusions occupying the same domain τ and satisfying boundary conditions (5a,b).

‡ ω_n and $\phi_n(\mathbf{r})$ are real, because the eigenvalue problem associated with System 3 is self adjoint.

$$\int_\tau I(\phi_n(\mathbf{r}),\phi_m(\mathbf{r});\lambda_0,\mu_0)\,d^3\mathbf{r} = \omega_n^2\delta_{nm} \tag{10b}$$

Substitution of (9) into (8), recalling the definition of $J(\mathbf{r})$ in (1c), and discarding index k give

$$\psi(\mathbf{r}_0) = \sum_{n=1}^{\infty} \frac{\phi_n(\mathbf{r}_0)}{\nu^2+\omega_n^2} U(\psi(\mathbf{r}),\phi_n(\mathbf{r});\nu) \tag{11a}$$

where

$$U(\psi(\mathbf{r}),\phi_n(\mathbf{r});\nu) = \nu^2<\psi\,|\,\phi_n>_{\rho_1} - \nu<\psi\,|\,\phi_n>_{I^*} + <\psi\,|\,\phi_n>_I \tag{11b}$$

with

$$<\psi\,|\,\phi_n>_{\rho_1} \equiv \int_{\tau_c} \rho_1\psi(\mathbf{r})\cdot\phi_n(\mathbf{r})\,d^3\mathbf{r} \tag{12a}$$

$$<\psi\,|\,\phi_n>_{I^*} \equiv \int_{\tau_c} I(\psi(\mathbf{r}),\phi_n(\mathbf{r});\lambda_0^*,\mu_0^*)\,d^3\mathbf{r} \tag{12b}$$

$$<\psi\,|\,\phi_n>_I \equiv \int_{\tau_c} I(\psi(\mathbf{r}),\phi_n(\mathbf{r});\lambda_1,\mu_1)\,d^3\mathbf{r} \tag{12c}$$

In addition, (11a) is homogeneous allowing normalization of $\psi(\mathbf{r})$ such that

$$\psi(\mathbf{r}) = T_1[\psi,\nu] = \phi_n(\mathbf{r}) + \sum_m{}' \frac{U(\psi,\phi_m;\nu)}{\nu^2+\omega_m^2}\phi_m(\mathbf{r}), \quad \sum_m{}' \equiv \sum_{\substack{m=1\\ m\neq n}}^{\infty} \tag{13a}$$

when

$$\nu^2 = T_2[\psi,\nu] = -\omega_n^2 + U(\psi,\phi_n;\nu) \tag{13b}$$

The $\psi(\mathbf{r})$ and ν of (13a,b) are the eigensolutions of System 2 (and hence System 1). In Appendix A, T_1 and T_2 in (13a,b) are shown to be contraction mappings for sufficiently small τ_c and their contraction constants α_1 and α_2 are also estimated. The convergence of the infinite series in (13a) is also shown in the Appendix A.

As $\tau_c \to 0$, $U(\psi,\phi_m;\nu)$ vanishes if the solution is regular. Otherwise, it is singular. Singular solutions are not discussed here.

Perturbation Solutions. For regular and nondegenerate solutions, first order perturbation is obtained by replacing $\psi(\mathbf{r})$ and ν on the right hand side of (13a,b) by $\phi_n(\mathbf{r})$ and $i\omega_n$. The n-th eigenfunction $\psi_n(\mathbf{r})$ is

$$\psi_n(\mathbf{r}) = \phi_n(\mathbf{r}) + \sum_m{}' \frac{d_{nm}^*}{\omega_m^2-\omega_n^2}\phi_m(\mathbf{r}) + O(\tau_c^2) \tag{14a}$$

where

$$d_{nm}^* = U(\phi_n,\phi_m;i\omega_n) = -\omega_n^2<\phi_n\,|\,\phi_m>_{\rho_1} - i\omega_n<\phi_n\,|\,\phi_m>_{I^*} + <\phi_n\,|\,\phi_m>_I$$

The n-th eigenvalue ν_n is

$$\nu_n = \sqrt{-\omega_n^2 + d_{nn}^* + O(\tau_c^2)}$$

$$= -\frac{1}{2}<\phi_n\,|\,\phi_n>_{I^*} + i\left\{\omega_n + \frac{1}{2\omega_n}\left[\omega_n^2<\phi_n\,|\,\phi_n>_{\rho_1} - <\phi_n\,|\,\phi_n>_I\right]\right\} + O(\tau_c^2) \tag{14b}$$

where the branch selected for the square root satisfies $\lim_{\tau_c \to 0} \nu_n = i\omega_n$ for a regular solution. I the inclusion is dissipative $<\phi_n | \phi_n>_{I^*} > 0$, and if it is elastic $<\phi_n | \phi_n>_{I^*} = 0$.

Degeneracy occurs when any ω_n is repeated. The contraction mappings in (13a,b) are valid if the initial trials $[\nu]^{(0)}$ and $[\psi(\mathbf{r})]^{(0)}$ for the iteration of ν_n and $\psi_n(\mathbf{r})$ satisfy $[\nu^2]^{(0)} \neq -\omega_n^2$. Otherwise, the denominators of the terms containing the repeated eigenvalues in (13a) vanish and the iteration fails. A perturbation theory for degenerate systems is presented in Appendix B.

4. Green's Functions

Exact Solutions. The Green's function of System 1, excited by a concentrated force $\delta(\mathbf{r}-\mathbf{r}_1)e^{\nu t}$ acting in the direction x_l $(l=1,2,3)$, is $\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1)$ in the unprimed system. $\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1)$ and ν satisfy

$$\frac{d}{dx_j}[\sigma_{ij}(\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1),\nu;\lambda,\mu,\lambda^*,\mu^*)] - \nu^2\rho(\mathbf{r})R_i^l(\mathbf{r}|\mathbf{r}_1) = -\delta_{il}\delta(\mathbf{r}-\mathbf{r}_1), \quad i,l = 1,2,3 \tag{15}$$

and boundary conditions

$$\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1) = 0, \quad \text{on } \sigma_1, \quad l = 1,2,3 \tag{16a}$$

$$\sigma_{ij}(\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1),\nu;\lambda,\mu,\lambda^*,\mu^*)n_j = 0, \quad \text{on } \sigma_2, \quad i,l = 1,2,3 \tag{16b}$$

The Green's function $\mathbf{G}^k(\mathbf{r}|\mathbf{r}_0)e^{\nu t}$ of System 3 is in the primed system. Then $\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1)$ satisfies the integral equation

$$R_k^l(\mathbf{r}_0|\mathbf{r}_1) = G_l^k(\mathbf{r}_1|\mathbf{r}_0) + H[\mathbf{G}^k(\mathbf{r}|\mathbf{r}_0),\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1)], \quad k,l = 1,2,3 \tag{17}$$

where $R_k^l(\mathbf{r}_0|\mathbf{r}_1)$ is the k-th element of the Green's function $\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1)$ $(k=1,2,3)$, and $H[\cdot,\cdot]$ is the integral operator defined in (8).

The eigenfunction expansion in (9) converts (17) into

$$\mathbf{R}^k(\mathbf{r}_0|\mathbf{r}_1) = \mathbf{G}^k(\mathbf{r}_0|\mathbf{r}_1) + T_3[\mathbf{R}^k(\mathbf{r}|\mathbf{r}_1),\nu]$$

$$\equiv \mathbf{G}^k(\mathbf{r}_0|\mathbf{r}_1) + \sum_{m=1}^{\infty} \frac{U(\mathbf{R}^k(\mathbf{r}|\mathbf{r}_1),\phi_m(\mathbf{r});\nu)}{\nu^2+\omega_m^2}\phi_m(\mathbf{r}_0) \tag{18}$$

in which the symmetry $G_l^k(\mathbf{r}_1|\mathbf{r}_0) = G_k^l(\mathbf{r}_0|\mathbf{r}_1)$ has been used. T_3 is a contraction mapping for sufficiently small τ_c (see Appendix A); therefore, iteration of (18) converges to $\mathbf{R}^l(\mathbf{r}|\mathbf{r}_1)$.

Perturbation Solution. Use of $\mathbf{G}^k(\mathbf{r}|\mathbf{r}_1)$ for $\mathbf{R}^k(\mathbf{r}|\mathbf{r}_1)$ in $T_3[\cdot,\nu]$ yields a first order perturbation

$$\underline{\mathbf{R}}(\mathbf{r}_0|\mathbf{r}_1) = \underline{\mathbf{G}}(\mathbf{r}_0|\mathbf{r}_1) + \sum_{m,n=1}^{\infty} \frac{U(\phi_n,\phi_m;\nu)}{(\nu^2+\omega_m^2)(\nu^2+\omega_n^2)}\phi_m(\mathbf{r}_0)\phi_n^T(\mathbf{r}_1) + O(\tau_c^2) \tag{19}$$

where $\underline{\mathbf{R}} \equiv [\mathbf{R}^1,\mathbf{R}^2,\mathbf{R}^3]$ and $\underline{\mathbf{G}} \equiv [\mathbf{G}^1,\mathbf{G}^2,\mathbf{G}^3]$ are Green's matrices and the superscript T denotes the transpose. The perturbation solution (19) is valid only when ν is far from $\pm i\omega_k$ (and therefore ν_k) avoiding the small divisors in (19) and resonance. The perturbation formulas at resonance can be obtained through an approach similar to that shown in Appendix B; they are not discussed here.

5. Circular Plates with Evenly Spaced, Radial, Viscoelastic Inclusions

Consider the transverse vibration of a Kirchhoff circular plate of uniform thickness h with k evenly spaced, radial, Kelvin viscoelastic inclusions each spanning a small angle ε and located at $\overline{\theta}_i = \frac{2\pi}{k} i$, $i = 1, 2, \cdots, k$ from $r = r_0$ to $r = b$. The inner and outer rim at $r = a$ and $r = b$ are free, clamped, or simply supported. The eigensolutions and the Green's function of the asymmetrically damped circular plate are evaluated by the methods derived previously.

Axisymmetric Circular Plates. Let $\omega_{m,\pm n}$ and $\Phi_{m,\pm n}(\mathrm{r})$ be the eigensolutions of an axisymmetric circular plate with n nodal diameters and m nodal circles. When $n = 0$, the eigenfunctions are axisymmetric; i.e.,

$$\Phi_{m0}(\mathrm{r}) = R_{m0}(r) \tag{20a}$$

When $n > 0$, the eigenfunctions

$$\Phi_{mn}(\mathrm{r}) = R_{mn}(r)\cos(n\theta+\alpha_{mn}) \tag{20b}$$

$$\Phi_{m,-n}(\mathrm{r}) = R_{mn}(r)\sin(n\theta+\alpha_{mn}) \tag{20c}$$

correspond to repeated eigenvalues $\omega_{mn} = \omega_{m,-n}$. In (20a,b,c), α_{mn} is an arbitrary constant and $R_{mn}(r)$ is a linear combination of Bessel's functions satisfying boundary conditions at both rims and the orthonormality conditions

$$\int_A \rho_0 h\, \Phi_{mn}(\mathrm{r})\Phi_{pq}(\mathrm{r})\, dA = \delta_{mp}\delta_{nq},\quad m,p = 0,1,2,\ldots,\infty,\quad n,q = 0,\pm1,\pm2,\ldots,\pm\infty$$

$$\int_A I(\Phi_{mn}(\mathrm{r}),\Phi_{pq}(\mathrm{r});D_0,\sigma_0)\, dA = \omega_{mn}^2\delta_{mp}\delta_{nq}, m,p = 0,1,2,\ldots,\infty, n,q = 0,\pm1,\pm2,\ldots,\pm\infty$$

with the bilinear operator

$$I(u,v;D_0,\sigma_0) = D_0\left[\nabla^2 u\, \nabla^2 v + 2(1-\sigma_0)\left\{\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)\left(\frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial v}{\partial\theta}\right)\right.\right.$$
$$\left.\left. - \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right)\frac{\partial^2 v}{\partial r^2} + \left(\frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}\right)\frac{\partial^2 u}{\partial r^2}\right]\right\}\right]$$

h, D_0, ρ_0, and σ_0 are the thickness, flexural rigidity, density, and Poisson ratio of the axisymmetric plate.

Perturbation Solutions. According to the perturbation theory for degenerate problems in Appendix B, the k viscoelastic inclusions affect the plate eigenfunctions $\Phi_{m,\pm n}(\mathrm{r})$ corresponding to eigenvalue $\omega_{mn} = \omega_{m,-n}$ through

$$<\Psi_{mn}\,|\,\Phi_{pq}>_{\rho_1} = \sum_{j=1}^{k}\int_{\overline{\theta}_j-\frac{\varepsilon}{2}}^{\overline{\theta}_j+\frac{\varepsilon}{2}}\int_{r_0}^{b}\rho_1 h\,\Psi_{mn}(\mathrm{r})\Phi_{pq}(\mathrm{r})\, r dr d\theta$$

$$<\Psi_{mn}\,|\,\Phi_{pq}>_{I^*} = \sum_{j=1}^{k}\int_{\overline{\theta}_j-\frac{\varepsilon}{2}}^{\overline{\theta}_j+\frac{\varepsilon}{2}}\int_{r_0}^{b} I(\Psi_{mn},\Phi_{pq};D_0^*,\sigma_0^*)\, r dr d\theta$$

and

$$<\Psi_{mn} \mid \Phi_{pq}>_I = \sum_{j=1}^{k} \int_{\bar\theta_j-\frac{\varepsilon}{2}}^{\bar\theta_j+\frac{\varepsilon}{2}} \int_{r_0}^{b} I(\Psi_{mn},\Phi_{pq};D_1,\sigma_1)]\, r dr d\theta$$

with D_0^*, σ_0^* derived from λ_0^*, μ_0^* in (1b) and D_1, σ_1 derived from λ_1, μ_1 in (1d), respectively. For example, to transform $<\Psi_{mn} \mid \Phi_{pq}>_{\rho_1}$ above to the asymptotic form (B-1a), define

$$\Pi(\theta) = \int_{r_0}^{b} \rho_1 h\, \Psi_{mn}(r)\Phi_{pq}(r)\, r dr \tag{21a}$$

and

$$\Gamma(\theta) = \int_{0}^{\theta} \Pi(\phi) d\phi \tag{21b}$$

Therefore,

$$<\Psi_{mn} \mid \Phi_{pq}>_{\rho_1} = \sum_{j=1}^{k} [\Gamma(\bar\theta_j+\frac{\varepsilon}{2}) - \Gamma(\bar\theta_j-\frac{\varepsilon}{2})]$$

Use of the Taylor expansion around $\theta=\bar\theta_j$ gives

$$<\Psi_{mn} \mid \Phi_{pq}>_{\rho_1} = \varepsilon \sum_{j=1}^{k} \left[\Pi(\bar\theta_j) + \frac{\varepsilon^2}{24}\frac{d^2\Pi(\bar\theta_j)}{d\theta^2} + \cdots \right] \tag{22}$$

Compare (22) with (B-1a) to obtain

$$<\Psi_{mn} \mid \Phi_{pq}>_{\rho_1}^{(0)} = \sum_{j=1}^{k} \int_{r_0}^{b} [\rho_1 h\, \Psi_{mn}(r)\Phi_{pq}(r)]_{\theta=\bar\theta_j}\, r dr\ , \quad <\Psi_{mn} \mid \Phi_{pq}>_{\rho_1}^{(1)} = 0\ , \cdots \tag{23a}$$

Similarly,

$$<\Psi_{mn} \mid \Phi_{pq}>_{I}^{(0)} = \sum_{j=1}^{k} \int_{r_0}^{b} [I(\Psi_{mn},\Phi_{pq};D_0^*,\sigma_0^*)]_{\theta=\bar\theta_j}\, r dr\ , \quad <\Psi_{mn} \mid \Phi_{pq}>_{I}^{(1)} = 0\ , \cdots \tag{23b}$$

and

$$<\Psi_{mn} \mid \Phi_{pq}>_{I}^{(0)} = \sum_{j=1}^{k} \int_{r_0}^{b} [I(\Psi_{mn},\Phi_{pq};D_1,\sigma_1)]_{\theta=\bar\theta_j}\, r dr\ , \quad <\Psi_{mn} \mid \Phi_{pq}>_{I}^{(1)} = 0\ , \cdots \tag{23c}$$

With $<\Psi_{mn} \mid \Phi_{pq}>_{\rho_1}^{(i)}$, $<\Psi_{mn} \mid \Phi_{pq}>_{I}^{(i)}$, and $<\Psi_{mn} \mid \Phi_{pq}>_{I}^{(i)}$ $(i=0,1,\cdots)$ in (23a,b,c), the results derived in Appendix B can be applied directly. From (B-5) in Appendix B the following coefficients are needed for the explicit expression of $\Psi_{mn}(r)$.

$$d_{rs}^{pq(0)} = \sum_{j=1}^{k} \int_{r_0}^{b} [-\rho_1 h\, \omega_{mn}^2 \Phi_{rs}(r)\Phi_{pq}(r) - i\,\omega_{mn}\, I(\Phi_{rs},\Phi_{pq};D_0^*,\sigma_0^*) + I(\Phi_{rs},\Phi_{pq};D_1,\sigma_1)]_{\theta=\bar\theta_j}\, r dr$$

$$e_{rs}^{pq(0)} = \sum_{j=1}^{k} \int_{r_0}^{b} [2i\,\omega_{mn}\mu_{mn}\rho_1 h\, \Phi_{rs}(r)\Phi_{pq}(r) - \mu_{mn} I(\Phi_{rs},\Phi_{pq};D_0^*,\sigma_0^*)]_{\theta=\bar\theta_j}\, r dr$$

$$d_{rs}^{pq(1)} = e_{rs}^{pq(1)} = 0\ , \cdots$$

For each pair of repeated eigensolutions (m,n) and $(m,-n)$, the unperturbed eigenfunctions are specified via diagonalization of the matrix

$$\mathbf{D}^{(0)} \equiv \begin{bmatrix} d_{mn}^{mn(0)} & d_{mn}^{m,-n(0)} \\ d_{m,-n}^{mn\;\;(0)} & d_{m,-n}^{m,-n(0)} \end{bmatrix} = \begin{bmatrix} A_{mn}^{mn}\theta_{11} - B_{mn}^{mn}\theta_{22} & (A_{mn}^{mn} + B_{mn}^{mn})\theta_{12} \\ (A_{mn}^{mn} + B_{mn}^{mn})\theta_{12} & A_{mn}^{mn}\theta_{22} - B_{mn}^{mn}\theta_{11} \end{bmatrix}$$

where A_{mn}^{pq} and B_{mn}^{pq} are complex coefficients given by

$$A_{mn}^{pq} = -\rho_1 h\,\omega_{mn}^2 \int_{r_0}^{b} R_{mn}(r)R_{pq}(r)rdr$$

$$+ (D_1 - i\,\omega_{mn} D_0^*) \int_{r_0}^{b} (\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2})R_{mn}(r)(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{q^2}{r^2})R_{pq}(r)\,rdr$$

$$- [D_1(1-\sigma_1) - i\,\omega_{mn} D_0^*(1-\sigma_0^*)] \int_{r_0}^{b} \left[\frac{d^2R_{mn}(r)}{dr^2}(\frac{1}{r}\frac{d}{dr} - \frac{q^2}{r^2})R_{pq}(r) + \frac{d^2R_{pq}(r)}{dr^2}(\frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2})R_{mn}(r) \right] rdr$$

$$B_{mn}^{pq} = -2[D_1(1-\sigma_1) - i\,\omega_{mn} D_0^*(1-\sigma_0^*)] \int_{r_0}^{b} [(\frac{n}{r}\frac{d}{dr} - \frac{n}{r^2})R_{mn}(r)]\,[(\frac{q}{r}\frac{d}{dr} - \frac{q}{r^2})R_{pq}(r)]\,rdr$$

and

$$\theta_{11} = \sum_{j=1}^{k} \cos^2 n(\bar{\theta}_j + \alpha_{mn}) = \begin{cases} \frac{k}{2}, & 2n \neq \mathbf{M}(k) \\ \frac{k}{2}(1+\cos 2\alpha_{mn}), & 2n = \mathbf{M}(k) \end{cases}$$

$$\theta_{22} = \sum_{j=1}^{k} \sin^2 n(\bar{\theta}_j + \alpha_{mn}) = \begin{cases} \frac{k}{2}, & 2n \neq \mathbf{M}(k) \\ \frac{k}{2}(1-\cos 2\alpha_{mn}), & 2n = \mathbf{M}(k) \end{cases}$$

$$\theta_{12} = \sum_{j=1}^{k} \sin n(\bar{\theta}_j + \alpha_{mn})\cos n(\bar{\theta}_j + \alpha_{mn}) = \begin{cases} 0, & 2n \neq \mathbf{M}(k) \\ \frac{k}{2}\sin 2\alpha_{mn}, & 2n = \mathbf{M}(k) \end{cases}$$

in which $2n = \mathbf{M}(k)$ means $2n$ is an integer multiple of k. Following Appendix B, the unperturbed eigenfunctions $\Phi_{mn}(\mathbf{r})$ and $\Phi_{m,-n}(\mathbf{r})$ require $\mathbf{D}^{(0)}$ to be diagonal; i.e., $\theta_{12}=0$, or equivalently,

$$\alpha_{mn} = \begin{cases} 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots, & \text{if } 2n{=}\mathbf{M}(k) \\ \text{arbitrary}, & \text{if } 2n{\neq}\mathbf{M}(k) \end{cases}$$

In the sequel, $\alpha_{mn}=0$ when $2n=\mathbf{M}(k)$. If $2n \neq \mathbf{M}(k)$, α_{mn} can be shown to be arbitrary at least up to the second order. Therefore

$$
\mathbf{D}^{(0)} = \begin{cases} \begin{bmatrix} \frac{k}{2}(A_{mn}^{mn} - B_{mn}^{mn}) & 0 \\ 0 & \frac{k}{2}(A_{mn}^{mn} - B_{mn}^{mn}) \end{bmatrix}, & 2n \neq \mathrm{M}(k) \\ \begin{bmatrix} kA_{mn}^{mn} & 0 \\ 0 & -kB_{mn}^{mn} \end{bmatrix}, & 2n = \mathrm{M}(k) \end{cases}
$$

For the α_{mn} classified above, the repeated $\Phi_{m,\pm n}(\mathbf{r})$ evolve into two distinct groups depending on the number of nodal lines n and the number of inclusions k that are spaced equally on the circular plate. Application of the perturbation formulas in Appendix B gives the following results.

The eigenfunctions $\Psi_{m,\pm n}(\mathbf{r})$ of the asymmetric plate remain arbitrarily positioned and the eigenvalues remain repeated (to the first order) if $2n \neq \mathrm{M}(k)$, $n \neq 0$:

$$
\nu_{mn} = \nu_{m,-n} = i\,\omega_{mn} - i\,\frac{\varepsilon k}{4\omega_{mn}}(A_{mn}^{mn} - B_{mn}^{mn}) + O(\varepsilon^2) \tag{24}
$$

Note that $\nu_{m,\pm n}$ are not imaginary because A_{mn}^{mn} and B_{mn}^{mn} are complex. The complex eigenfunctions, perturbed to first order, are

$$
\Psi_{mn}(\mathbf{r}) = R_{mn}(r)\cos(n\theta + \alpha_{mn})
$$

$$
-\frac{\varepsilon k}{2}\left[\bar{\sum_{q=\mathrm{M}(k)-n}} \frac{A_{mn}^{pq} + B_{mn}^{pq}}{\omega_{mn}^2 - \omega_{pq}^2}\Phi_{pq}(\mathbf{r}) + \bar{\sum_{q=\mathrm{M}(k)+n}} \frac{A_{mn}^{pq} - B_{mn}^{pq}}{\omega_{mn}^2 - \omega_{pq}^2}\Phi_{pq}(\mathbf{r})\right] + O(\varepsilon^2) \tag{25a}
$$

$$
\Psi_{m,-n}(\mathbf{r}) = R_{mn}(r)\sin(n\theta + \alpha_{mn})
$$

$$
+\frac{\varepsilon k}{2}\left[\bar{\sum_{q=\mathrm{M}(k)-n}} \frac{A_{mn}^{pq} + B_{mn}^{pq}}{\omega_{mn}^2 - \omega_{pq}^2}\Phi_{p,-q}(\mathbf{r}) - \bar{\sum_{q=\mathrm{M}(k)+n}} \frac{A_{mn}^{pq} - B_{mn}^{pq}}{\omega_{mn}^2 - \omega_{pq}^2}\Phi_{p,-q}(\mathbf{r})\right] + O(\varepsilon^2) \tag{25b}
$$

with α_{mn} arbitrary up to the first order perturbation, and $\bar{\sum}_{q=\mathrm{M}(k)-n}$ denotes double summation on integers $p \geq 0$ and $q > 0$ with $q = \mathrm{M}(k) - n$ and $(p,q) \neq (m,n)$.

The eigenfunctions $\Psi_{m,\pm n}(\mathbf{r})$ are termed split modes if $2n = \mathrm{M}(k)$, $n \neq 0$, because the complex eigenvalues are distinct:

$$
\nu_{mn} = i\,\omega_{mn} - i\,\frac{\varepsilon k}{2\omega_{mn}} A_{mn}^{mn} + O(\varepsilon^2) \tag{26a}
$$

$$
\nu_{m,-n} = i\,\omega_{mn} + i\,\frac{\varepsilon k}{2\omega_{mn}} B_{mn}^{mn} + O(\varepsilon^2) \tag{26b}
$$

The corresponding complex eigenfunctions, perturbed to the first order, are

$$\Psi_{mn}(\mathrm{r}) = R_{mn}(r)\cos n\theta$$

$$- \left\{ \begin{array}{ll} \epsilon k \displaystyle\sum_{q=\mathrm{M}(k)+\frac{k}{2}}^{-} \frac{A_{mn}^{pq}\,\Phi_{pq}(\mathrm{r})}{\omega_{mn}^2 - \omega_{pq}^2}, & n \neq \mathrm{M}(k) \\ \epsilon k \left[\displaystyle\sum_{p=0}^{\infty} \frac{A_{mn}^{p0}\,\Phi_{p0}(\mathrm{r})}{\omega_{mn}^2 - \omega_{p0}^2} + \sum_{q=\mathrm{M}(k)}^{-} \frac{A_{mn}^{pq}\,\Phi_{pq}(\mathrm{r})}{\omega_{mn}^2 - \omega_{pq}^2} \right], & n = \mathrm{M}(k) \end{array} \right\} + O(\epsilon^2) \tag{27a}$$

$$\Psi_{m,-n}(\mathrm{r}) = R_{mn}(r)\sin n\theta + \epsilon k \sum_{q=\mathrm{M}(k)+n}^{-} \frac{B_{mn}^{pq}\,\Phi_{p,-q}(\mathrm{r})}{\omega_{mn}^2 - \omega_{pq}^2} + O(\epsilon^2) \tag{27b}$$

The eigenfunctions $\Psi_{m0}(\mathrm{r})$ are not axisymmetric to first order perturbation:

$$\Psi_{m0}(\mathrm{r}) = R_{m0}(r) - \epsilon k \left[\sum_{\substack{p=0 \\ p\neq m}}^{\infty} \frac{A_{m0}^{p0}\,\Phi_{p0}(\mathrm{r})}{\omega_{m0}^2 - \omega_{p0}^2} + \sum_{p=0}^{\infty} \sum_{\substack{q=1 \\ q=\mathrm{M}(k)}}^{\infty} \frac{A_{m0}^{pq}\,\Phi_{pq}(\mathrm{r})}{\omega_{m0}^2 - \omega_{pq}^2} \right] + O(\epsilon^2) \tag{28}$$

with eigenvalue

$$\nu_{m0} = i\,\omega_{m0} - i\,\frac{\epsilon k}{2\omega_{mn}} A_{m0}^{m0} + O(\epsilon^2) \tag{29}$$

where $\sum_{\substack{q=1 \\ q=\mathrm{M}(k)}}^{\infty}$ denotes the summation on the positive integer q with $q = \mathrm{M}(k)$.

Numerical Solutions. The eigensolutions and Green's functions of a circular plate with three equally spaced, radial, Kelvin viscoelastic inclusions are computed numerically by the perturbation iteration method (13a,b) and (18). The inclusions are thin sector bars extending from $r/b = 0.75$ to $r/b = 1$. The angle ϵ spanned by each inclusion is 0.035 rad ($\approx 2.0°$). The material properties satisfy

$$\frac{\rho_0'}{\rho_0} = \frac{E_0'}{E_0} = 0.5;\ \sigma_0' = 0.3;\ \xi = \frac{E_0^*}{E_0}\sqrt{\frac{E_0 h^2}{4\rho_0 b^4}} = 0.05;\ \sigma_0^* = 0.3$$

Eigenfunctions with 0 to 20 nodal diameters and 0 and 1 nodal circles are used in the series in (13a) and (18). In calculating the eigensolutions of the asymmetrically damped plate, the iteration converges if the differences in $|\hat{\nu}_{mn}| \equiv \left| \nu_{mn}\sqrt{\frac{4\rho_0 b^4}{E_0 h^2}} \right|$ and in $||\Psi_{mn}(\mathrm{r})||^2 \equiv ||\sqrt{\rho_0 h b^2}\,\Psi_{mn}(\mathrm{r})||^2$ between successive iterations are less than 10^{-6} and 10^{-10}, respectively.

The upper bounded estimates of the contraction constants α_1 and α_2 in Appendix A are calculated in advance to guarantee convergence. For a 3-inclusion plate, calculation shows that $\alpha_1, \alpha_2 < 1$ for modes up to 7 nodal diameters and zero nodal circles. For these modes, the perturbation iteration is guaranteed contractive and convergent. For modes (0, 11), (0, 12) and (0,13) $\alpha_1 = 24.3$, 14.5, and 17.7, but the iteration converges; the results are shown in

Table 1.

The normalized complex eigenvalues $\tilde{\nu}_{mn}$ of the 3-inclusion plate are listed in Table 1. The exact eigenvalues of the axisymmetric plates are also listed for reference. The results in Table 1 show the split in $\Phi_{mn}(\mathbf{r})$ when $2n = M(k)$. In addition, Table 1 shows that damping coefficient $-\mathrm{Re}[\tilde{\nu}_{mn}]$ and damping ratio $\zeta_{mn} = -\dfrac{\mathrm{Re}[\tilde{\nu}_{mn}]}{\mathrm{Im}[\tilde{\nu}_{mn}]}$ both increase as $\mathrm{Im}[\tilde{\nu}_{mn}]$ increases. For example, $\zeta_{0,13} = 4.54\%$ while $\zeta_{01} = 0.05\%$. This suggests that the damped circular plate possesses large stability margins for high frequency modes.

The eigenfunctions of the damped circular plate are characterized by nodal lines that periodically shift their positions at twice the characteristic frequency. The evolution of the nodal lines of (0, 12) cosine mode is shown in Fig. 2 for one-half of a period.

The loci of eigenvalues $\tilde{\nu}_{mn}$ with respect to ξ on the complex $\tilde{\nu}$ plane from three to six nodal diameter modes are shown in Fig. 3. Bifurcations occur for split modes 3^c, 3^s, 6^c, and 6^s as predicted by (26a,b).

Green's Functions. The Green's function $R(\mathbf{r}\,|\,\mathbf{r}_0)$ of the asymmetrically damped circular plate under harmonic excitation is also found by the perturbation iteration method. Two Green's function displacement contours are shown in Fig. 4 and 5. In Fig. 4 the load is applied on an antinodal line of the unperturbed modes with circumferential distribution $\sin 3\theta$, $\sin 9\theta, \ldots,$ and $\cos 6\theta$, $\cos 12\theta, \ldots$. The perturbation iteration terminates when difference in $\left|\left|\dfrac{Eh^3}{4b^2} R(\mathbf{r}\,|\,\mathbf{r}_0)\right|\right|^2$ between consecutive iterations is less than 10^{-7}. In Fig. 5 the excitation frequency is near the 3 nodal diameter cosine mode resonance (cf. Table 1).

6. Conclusions

1. Explicit perturbation formulas and a numerical iteration scheme are developed to determine eigensolutions and Green's functions for finite, three dimensional, linear, elastic solids containing Kelvin viscoelastic inclusions under the condition that the solutions are regular at the inclusions. The perturbed eigensolutions are represented in a convergent series of orthonormal eigenfunctions of the perfect elastic solid. The perturbation iteration generates results to the precision required provided the perfect solid solutions are known.

2. Perturbation analyses show that all vibration modes are damped by a viscoelastic inclusion. The damping of the n-th vibration mode is determined by $\langle \phi_n \,|\, \phi_n \rangle_I$, which cannot vanish. The viscosity of the inclusions affects natural frequencies through second order perturbation.

3. Circular plates with k evenly spaced, sector, Kelvin viscoelastic inclusions are studied by this technique for eigensolutions and Green's functions. The repeated eigensolutions $\Phi_{mn}(\mathbf{r})$ with m nodal circles and n nodal diameters of the corresponding, axisymmetric, circular plate split into two distinct eigensolutions when $2n$ is an integer multiple of k. Otherwise, $\Phi_{mn}(\mathbf{r})$ remain repeated. Numerical results show that vibration modes with higher natural frequencies possess relatively greater damping ratios. The location of the nodal curves of the perturbed eigenfunctions on the plate changes periodically at twice the eigenfrequency.

Appendix A

This appendix shows that T_1 and T_2 in (13a,b) and T_3 in (18) are contraction mappings for sufficiently small τ_c, and determines an upper bound to the contraction constant of each mapping. The convergence of the series in T_1 and T_3 is also discussed.

Mapping T_1. Substitute (13b) into (13a) and recall $U(\psi,\phi_m;\nu) \approx O(\tau_c)$ to give

$$\psi(\mathbf{r}) = \phi_n(\mathbf{r}) + \sum_m{}' \frac{U(\psi,\phi_m;\nu)}{\omega_m^2 - \omega_n^2 + U(\psi,\phi_n;\nu)} \phi_m(\mathbf{r}) \tag{A-1a}$$

$$= \phi_n(\mathbf{r}) + \sum_m{}' \frac{U(\psi,\phi_m;i\omega_n)}{\omega_m^2 - \omega_n^2} \phi_m(\mathbf{r}) + O(\tau_c^2) \tag{A-1b}$$

Equation (A-1b) is shown to be a contraction mapping up to $O(\tau_c)$ under the strain energy norm

$$||\mathbf{x}||_{SE} = \sqrt{\int_\tau I(\mathbf{x},\bar{\mathbf{x}};\lambda_0,\mu_0)\, d^3\mathbf{r}} \tag{A-2}$$

where $\bar{\mathbf{x}}$ is the complex conjugate of $\mathbf{x}$. Consider the first order mapping T_1^* defined by

$$T_1^*\psi = \phi_n(\mathbf{r}) + \sum_m{}' \frac{\omega_n^2 <\psi|\phi_m>_{\rho_1} + i\omega_n <\psi|\phi_m>_{I^*} - <\psi|\phi_m>_I}{\omega_n^2 - \omega_m^2} \phi_m(\mathbf{r}) \tag{A-3}$$

then

$$||T_1^*\psi_1 - T_1^*\psi_2||_{SE}^2 =$$

$$\sum_m{}' \left| \frac{\omega_n^2 <\psi_1-\psi_2|\phi_m>_{\rho_1} + i\omega_n <\psi_1-\psi_2|\phi_m>_{I^*} - <\psi_1-\psi_2|\phi_m>_I}{\omega_n^2 - \omega_m^2} \right|^2 \omega_m^2 \tag{A-4}$$

Because every ψ is normalized according to (13a), the eigenfunction representation

$$\psi_1 - \psi_2 = \sum_k{}' a_k \phi_k(\mathbf{r}) \tag{A-5}$$

implies that $||\psi_1 - \psi_2||_{SE}^2 = \sum_k{}' \omega_k^2 |a_k|^2$ and

$$||T_1^*\psi_1 - T_1^*\psi_2||_{SE}^2 = \sum_m{}' \left| \frac{\omega_m}{\omega_n^2 - \omega_m^2} \sum_k{}' a_k(\omega_n^2 d_{km} + i\omega_n f_{km} - e_{km}) \right|^2 \tag{A-6}$$

where $d_{km} = <\phi_k|\phi_m>_{\rho_1}$, $f_{km} = <\phi_k|\phi_m>_{I^*}$, and $e_{km} = <\phi_k|\phi_m>_I$. With the Schwartz inequality

$$\left| \sum_k{}' a_k(\omega_n^2 d_{km} + i\omega_n f_{km} - e_{km}) \right|^2 \le \sum_k{}' \left| \frac{\omega_n^2 d_{km} + i\omega_n f_{km} - e_{km}}{\omega_k} \right|^2 ||\psi_1 - \psi_2||_{SE}^2 \tag{A-7}$$

(A-6) is reduced to

$$||T_1^*\psi_1 - T_1^*\psi_2||_{SE}^2 \le \alpha_1 ||\psi_1 - \psi_2||_{SE}^2 \tag{A-8a}$$

with

$$\alpha_1 = \sum_m{}' \sum_k{}' \left| \frac{\omega_n^2 d_{km} + i\omega_n f_{km} - e_{km}}{[1 - (\omega_n/\omega_m)^2]\omega_k \omega_m} \right|^2 \tag{A-8b}$$

$\alpha_1 < 1$ is a sufficient condition for the contraction mapping (A-1b) to first order of τ_c.

Mapping T_2. Substitute (13a) into (13b) and retain terms up to $O(\tau_c)$

$$\nu^2 = -\omega_n^2 + U(\phi_n(r), \phi_n(r); \nu) + O(\tau_c^2)$$

Then

$$\nu_2^2 - \nu_1^2 = (\nu_2^2 - \nu_1^2)\left[<\phi_n|\phi_n>_{\rho_1} - \frac{1}{\nu_1+\nu_2}<\phi_n|\phi_n>_{I^*}\right] + O(\tau_c^2)$$

$$= (\nu_2^2 - \nu_1^2)\left[d_{nn} + i\frac{f_{nn}}{2\omega_n}\right] + O(\tau_c^2)$$

where $\nu_1, \nu_2 = i\omega_n + O(\tau_c)$ has been used. Therefore, T_2 is a contraction mapping up to $O(\tau_c)$ if

$$\alpha_2 = \left| d_{nn} + i\frac{f_{nn}}{2\omega_n} \right| < 1 \qquad \text{(A-9)}$$

Mapping T_3. The proof that T_3 is contractive follows that given for T_1^* because T_3 is the same form as T_1^* with $\sum_m'$ and $i\omega_n$ replaced by $\sum_{m=1}^{\infty}$ and ν.

$$\alpha_3 = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left|\frac{\nu^2 d_{km} - \nu f_{km} + e_{km}}{[1+(\nu/\omega_m)^2]\omega_k\omega_m}\right|^2 \qquad \text{(A-10)}$$

$\alpha_3 < 1$ is a sufficiency condition that T_3 is contractive.

Convergence. The convergence of the orthonormal eigenfunction series in (13a) and (18), according to the Riesz-Fisher theorem, is determined by the series

$$S \equiv \sum_m{}' \left|\frac{\nu^2<\psi|\phi_m>_{\rho_1} - \nu<\psi|\phi_m>_{I^*} + <\psi|\phi_m>_I}{\nu^2+\omega_m^2}\right|^2 \qquad \text{(A-11)}$$

Apply the parallelogram law twice to (A-11)

$$S \le 2\sum_m{}' \left|\frac{\nu^2<\psi|\phi_m>_{\rho_1}}{\nu^2+\omega_m^2}\right|^2 + 4\sum_m{}' \left|\frac{\nu<\psi|\phi_m>_{I^*}}{\nu^2+\omega_m^2}\right|^2 + 4\sum_m{}' \left|\frac{<\psi|\phi_m>_I}{\nu^2+\omega_m^2}\right|^2 \qquad \text{(A-12)}$$

Furthermore, the Schwartz inequalities associated with the inner products $<\cdot|\cdot>_{\rho_1}$, $<\cdot|\cdot>_{I^*}$, and $<\cdot|\cdot>_I$ give

$$S \le 2\sum_m{}' \left[\frac{\nu^2||\phi_m||_{\rho_1}}{\nu^2+\omega_m^2}\right]^2 ||\psi||_{\rho_1}^2$$

$$+4\sum_m{}' \left[\frac{\nu||\phi_m||_{I^*}}{\nu^2+\omega_m^2}\right]^2 ||\psi||_{I^*}^2 + 4\sum_m{}' \left[\frac{||\phi_m||_I}{\nu^2+\omega_m^2}\right]^2 ||\psi||_I^2 \qquad \text{(A-13)}$$

where $||\cdot||_{\rho_1}$, $||\cdot||_{I^*}$, and $||\cdot||_I$ are the natural norms

$$||\phi_m||^2_{\rho_1} \le \frac{|\rho_1|}{\rho_0}\int_\tau \rho_0 \phi_m \cdot \phi_m \, d^3r = \frac{|\rho_1|}{\rho_0}$$

$$||\phi_m||^2_{I^*} \le c_1 \int_\tau I(\phi_m,\phi_m;\lambda_0,\mu_0)\, d^3r = c_1 \omega_m^2$$

and

$$||\phi_m||^2_{I} \le c_2 \int_\tau I(\phi_m,\phi_m;\lambda_0,\mu_0)\, d^3r = c_2 \omega_m^2$$

where $c_1 = Max \left[\frac{|\lambda_0^*|}{\lambda_0}, \frac{|\mu_0^*|}{\mu_0}\right]$, and $c_2 = Max \left[\frac{|\lambda_1|}{\lambda_0}, \frac{|\mu_1|}{\mu_0}\right]$. Therefore,

$$S \le 2\frac{|\rho_1|}{\rho_0} \sum_m{}' \left[\frac{\nu^2}{\nu^2+\omega_m^2}\right]^2 ||\psi||^2_{\rho_1}$$

$$+4c_1 \sum_m{}' \left[\frac{\nu\omega_m}{\nu^2+\omega_m^2}\right]^2 ||\psi||^2_{I^*} + 4c_2 \sum_m{}' \left[\frac{\omega_m}{\nu^2+\omega_m^2}\right]^2 ||\psi||^2_{I} \qquad \text{(A-14)}$$

The three infinite series in (A-14) converge if the Green's function representation (9) exists. If ψ is regular, then $||\psi||_{\rho_1}$, $||\psi||_{I^*}$, and $||\psi||_{I}$ are finite. Therefore, the series in (A-11) converges for it is monotonically increasing and bounded above by the RHS of (A-14). The convergence of the orthonormal eigenfunction series in (13a) and (18) is guaranteed by the Riesz-Fisher theorem.

Appendix B

Assume that the first β eigenvalues of a finite, elastic solid without inclusions (System 3 in Fig. 1(c)) are repeated and the remainder are distinct. The perturbation formulas (14a,b) are not valid for the eigenfunctions $\boldsymbol{\psi}_n(\mathbf{r})$ $(n \le \beta)$ because small divisors appear for the repeated eigensolutions in (14a).

Let ε be a dimensionless, small parameter measure of the inclusions (e.g., volume ratio of the inclusions to the solid). Being defined on the small inclusion domain τ_c, $<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{\rho_1}$, $<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{I^*}$, and $<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_I$ can be expanded in asymptotic series of ε; i.e.,

$$<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{\rho_1} = \int_{\tau_c} \rho_1 \boldsymbol{\psi}_n(\mathbf{r}) \cdot \boldsymbol{\phi}_m(\mathbf{r})\, d^3\mathbf{r} = \varepsilon \sum_{i=0}^{\infty} \varepsilon^i <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{\rho_1}^{(i)} \tag{B-1a}$$

$$<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{I^*} = \int_{\tau_c} I(\boldsymbol{\psi}_n, \boldsymbol{\phi}_m; \lambda_0^*, \mu_0^*)\, d^3\mathbf{r} = \varepsilon \sum_{i=0}^{\infty} \varepsilon^i <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{I^*}^{(i)} \tag{B-1b}$$

and

$$<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_I = \int_{\tau_c} I(\boldsymbol{\psi}_n, \boldsymbol{\phi}_m; \lambda_1, \mu_1)\, d^3\mathbf{r} = \varepsilon \sum_{i=0}^{\infty} \varepsilon^i <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_I^{(i)} \tag{B-1c}$$

where $<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{\rho_1}^{(i)}$, $<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{I^*}^{(i)}$, and $<\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_I^{(i)}$ are coefficients of the asymptotic series. These coefficients can be determined explicitly by expanding the integrals $\int_{\tau_c} (\cdots)\, d^3\mathbf{r}$ into series of ε and comparing these series with those in (B-1a,b,c). This process is illustrated in Section 5 when obtaining (23a). Therefore,

$$U(\boldsymbol{\psi}_n, \boldsymbol{\phi}_m; \nu_n) = \varepsilon \sum_{i=0}^{\infty} \varepsilon^i \left[\nu_n^2 <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{\rho_1}^{(i)} - \nu_n <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{I^*}^{(i)} + <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_I^{(i)}\right]$$

Assume also that the eigenfunction $\boldsymbol{\psi}_n(\mathbf{r})$ and the eigenvalue ν_n take the asymptotic series representations

$$\boldsymbol{\psi}_n(\mathbf{r}) = \boldsymbol{\phi}_n(\mathbf{r}) + \varepsilon \sum_j{}' a_{nj} \boldsymbol{\phi}_j(\mathbf{r}) + \varepsilon^2 \sum_j{}' b_{nj} \boldsymbol{\phi}_j(\mathbf{r}) + \cdots \tag{B-2a}$$

$$\nu_n = i\omega_n + \varepsilon\mu_n + \varepsilon^2 \lambda_n + \cdots \tag{B-2b}$$

or

$$\nu_n^2 = -\omega_n^2 + \varepsilon(2i\omega_n \mu_n) + \varepsilon^2(2i\omega_n \lambda_n + \mu_n^2) + \cdots \tag{B-2c}$$

Equation (13a) for the exact mode shape $\boldsymbol{\psi}_n(\mathbf{r})$ $(n \le \beta)$ is then rearranged as

$$\boldsymbol{\psi}_n(\mathbf{r}) = \boldsymbol{\phi}_n(\mathbf{r}) + \sum_{\substack{m=1 \\ m \ne n}}^{\beta} \frac{\sum_{i=0}^{\infty} \varepsilon^i \left[\nu_n^2 <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{\rho_1}^{(i)} - \nu_n <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_{I^*}^{(i)} + <\boldsymbol{\psi}_n | \boldsymbol{\phi}_m>_I^{(i)}\right]}{2i\omega_n \mu_n + \varepsilon(2i\omega_n \lambda_n + \mu_n^2) + \cdots} \boldsymbol{\phi}_m(\mathbf{r})$$

$$+ \sum_{m=\beta+1}^{\infty} \frac{U(\boldsymbol{\psi}_n, \boldsymbol{\phi}_m; \nu_n)}{\nu_n^2 + \omega_m^2} \boldsymbol{\phi}_m(\mathbf{r}), \quad n \le \beta \tag{B-3}$$

where the troublesome small divisors in (13a) are addressed by the second term on the right of (B-3), which contains an order ε^0 term $d_{nm}^{(0)} \equiv -\omega_n^2 <\boldsymbol{\phi}_n | \boldsymbol{\phi}_m>_{\rho_1}^{(0)} - i\omega_n <\boldsymbol{\phi}_n | \boldsymbol{\phi}_m>_{I^*}^{(0)} + <\boldsymbol{\phi}_n | \boldsymbol{\phi}_m>_I^{(0)}$ $(n, m \le \beta)$. In order that the perturbed mode shape $\boldsymbol{\psi}_n(\mathbf{r})$ remains nearby the unperturbed one,

$\phi_n(\mathbf{r}), n=1,2,\ldots,\beta$ must be specified such that $\mathbf{D}^{(0)} \equiv [d_{nm}^{(0)}]$ $(n,m \le \beta)$ is a diagonal matrix. The specification of $\phi_n(\mathbf{r})$, $n=1,2,\ldots,\beta$ is unique only when all the diagonal elements of $\mathbf{D}^{(0)}$ are distinct. Otherwise, those $\phi_n(\mathbf{r})$ corresponding to the repeated diagonal $d_{nn}^{(0)}$ can be arbitrarily chosen within any orthogonal transformation.

With diagonal $\mathbf{D}^{(0)}$, (B-1), and (B-2a,b), equations (13a,b) for the exact $\psi_n(\mathbf{r})$ and ν_n $(n \le \beta)$ become

$$\psi_n(\mathbf{r}) = \phi_n(\mathbf{r}) + \sum_{\substack{m=1 \\ m\neq n}}^{\beta} \frac{\phi_m(\mathbf{r})}{2i\omega_n\mu_n} \Bigg\{ \varepsilon [\sum_{j=\beta+1}^{\infty} a_{nj} d_{jm}^{(0)} + a_{nm} d_{mm}^{(0)} + e_{nm}^{(0)} + d_{nm}^{(1)}] +$$

$$\varepsilon^2 \left[\sum_j{}' a_{nj} (d_{jm}^{(1)} + e_{jm}^{(0)}) + e_{nm}^{(1)} + d_{nm}^{(2)} - (\frac{\lambda_n}{\mu_n} + \frac{\mu_n}{2i\omega_n}) d_{nm}^{(1)} + \right.$$

$$\left. [b_{nm} - (\frac{\lambda_n}{\mu_n} + \frac{\mu_n}{2i\omega_n}) a_{nm}] d_{mm}^{(0)} + \sum_{j=\beta+1}^{\infty} [b_{nj} - (\frac{\lambda_n}{\mu_n} + \frac{\mu_n}{2i\omega_n}) a_{nj}] d_{jm}^{(0)} \right] + O(\varepsilon^3) \Bigg\}$$

$$+ \sum_{m=\beta+1}^{\infty} \frac{\phi_m(\mathbf{r})}{\omega_m^2 - \omega_n^2} \left\{ \varepsilon d_{nm}^{(0)} + \varepsilon^2 \left[\sum_j{}' a_{nj} d_{jm}^{(0)} + d_{nm}^{(1)} + e_{nm}^{(0)} - \frac{2i\omega_n\mu_n}{\omega_m^2 - \omega_n^2} d_{nm}^{(0)} \right] + O(\varepsilon^3) \right\} \qquad \text{(B-4a)}$$

and

$$\nu_n^2 = -\omega_n^2 + \varepsilon d_{nn}^{(0)} + \varepsilon^2 [\sum_{j=\beta+1}^{\infty} a_{nj} d_{jn}^{(0)} + d_{nn}^{(1)} + e_{nn}^{(0)}] + O(\varepsilon^3) \qquad \text{(B-4b)}$$

where

$$\left. \begin{aligned} d_{pq}^{(i)} &= -\omega_n^2 \langle\phi_p | \phi_q\rangle_{\rho_1}^{(i)} - i\omega_n \langle\phi_p | \phi_q\rangle_{l}^{(i)} + \langle\phi_p | \phi_q\rangle_{f}^{(i)} \\ e_{pq}^{(i)} &= 2i\omega_n\mu_n \langle\phi_p | \phi_q\rangle_{\rho_1}^{(i)} - \mu_n \langle\phi_p | \phi_q\rangle_{l}^{(i)} \end{aligned} \right\} n \le \beta,\ i = 0,1,2,3,\ldots,\infty \qquad \text{(B-5)}$$

First Order Perturbation. Comparison of (B-4a,b) with (B-2a,c) to the first order of ε implies

$$\mu_n = \frac{d_{nn}^{(0)}}{2i\omega_n} = -\frac{1}{2} \langle\phi_n | \phi_n\rangle_{l}^{(0)} + \frac{i}{2\omega_n} \left[\omega_n^2 \langle\phi_n | \phi_n\rangle_{\rho_1}^{(0)} - \langle\phi_n | \phi_n\rangle_{f}^{(0)} \right] \qquad \text{(B-6a)}$$

and

$$a_{nm} = \begin{cases} \dfrac{d_{nm}^{(0)}}{\omega_m^2 - \omega_n^2}, & \text{for } n \le \beta, m > \beta \\[2ex] \dfrac{\sum_{j=\beta+1}^{\infty} a_{nj} d_{jm}^{(0)} + e_{nm}^{(0)} + d_{nm}^{(1)}}{d_{nn}^{(0)} - d_{mm}^{(0)}}, & \text{for } n,m \le \beta \end{cases} \qquad \text{(B-6b)}$$

If $\mu_n \neq \mu_m$, the perturbation a_{nm} $(n,m \le \beta)$ is well defined by (B-6b). If $\mu_n = \mu_m$, a_{nm}, $\phi_n(\mathbf{r})$, and $\phi_m(\mathbf{r})$, $n,m \le \beta$, must be determined by higher order perturbations.

Second Order Perturbation. Assume that μ_n are repeated for all $n = 1, 2, \ldots, \beta$. The procedure shown below can also be applied to the case when some of the μ_n, $n = 1, 2, \ldots, \beta$, are repeated. Select the unperturbed orthogonal eigenfunctions $\phi_n(\mathbf{r})$, $n = 1, 2, \ldots, \beta$, such that the matrix $\mathbf{P}^{(1)} \equiv [p_{rs}]$ $(r, s \leq \beta)$ with

$$p_{rs} = \sum_{j=\beta+1}^{\infty} \frac{d_{rj}^{(0)} d_{j\beta}^{(0)}}{\omega_j^2 - \omega_n^2} + e_{rs}^{(0)} + d_{rs}^{(1)}, \quad (n \leq \beta) \tag{B-7}$$

is diagonal. Comparison of (B-4a,b) with (B-2a,c) to the second order perturbation ε^2 gives

$$\lambda_n = \frac{p_{nn} - \mu_n^2}{2i\,\omega_n} = \frac{1}{2i\,\omega_n}\left[\sum_{j=\beta+1}^{\infty} a_{nj} d_{jn}^{(0)} + e_{nn}^{(0)} + d_{nn}^{(1)} + \left[\frac{d_{nn}^{(0)}}{2\omega_n}\right]^2\right] \tag{B-8a}$$

as well as the following relations between a_{nm} and b_{nm}

$$b_{nm} = \frac{1}{\omega_m^2 - \omega_n^2}\left[\sum_{\substack{j=1 \\ j\neq n}}^{\beta} a_{nj} d_{jm}^{(0)} + p_{nm} - d_{nn}^{(0)} a_{nm}\right] \tag{B-8b}$$

for $n \leq \beta, m > \beta$, and

$$\sum_j{}' a_{nj}(e_{jm}^{(0)} + d_{jm}^{(1)}) + \sum_{j=\beta+1}^{\infty} b_{nj} d_{jm}^{(0)} - p_{nn} a_{nm} + f_{nm}^{(0)} + e_{nm}^{(1)} + d_{nm}^{(2)} = 0 \tag{B-8c}$$

for $n, m \leq \beta$, where $f_{pq}^{(i)} = (2i\,\omega_n \lambda_n + \mu_n^2)\langle\phi_p \mid \phi_q\rangle_{\rho_1}^{(i)} - \lambda_n \langle\phi_p \mid \phi_q\rangle_{l}^{(i)}$. Notice that a_{nm} for $n, m \leq \beta$ is unknown in both (B-8b,c). Substitute (B-8b) into (B-8c) to eliminate b_{nj} and recall that $[p_{nm}]$ is diagonal for $n, m \leq \beta$. These reduce (B-8b,c) to

$$a_{nm} = \frac{1}{p_{nn} - p_{mm}}\left[\sum_{j=\beta+1}^{\infty} \frac{(e_{nj}^{(0)} + d_{nj}^{(1)} - d_{nn}^{(0)} a_{nj}) d_{jm}^{(0)}}{\omega_j^2 - \omega_n^2} + \sum_{j=\beta+1}^{\infty} a_{nj} p_{jm} + f_{nm}^{(0)} + e_{nm}^{(1)} + d_{nm}^{(2)}\right] \tag{B-9}$$

for $n, m \leq \beta$. If $p_{nn} = p_{mm}$ (i.e. $\lambda_n = \lambda_m$), then a_{nm} is not specified uniquely. Higher order perturbations similar to the one described above can be developed.

Table 1 - Splitting of Eigenvalues in a Circular Plate by Three Viscoelastic Inclusions

Complex eigenvalue $\tilde{\nu}_{mn} \equiv \nu_{mn} \sqrt{\dfrac{4\rho_0 b^4}{E_0 h^2}}$

Plate: clamping ratio 0.5; $\sigma=0.3$; fixed at inner rim, free at outer rim.

Inclusions extend from $r=0.75b$ to $r=b$; $\varepsilon=0.035$.

$\dfrac{\rho_0'}{\rho_0}=\dfrac{E_0'}{E_0}=0.5$; $\sigma_0'=0.3$; $\xi=\dfrac{E_0^*}{E_0}\sqrt{\dfrac{E_0 h^2}{4\rho_0 b^4}}=0.05$; $\sigma_0^*=0.3$.

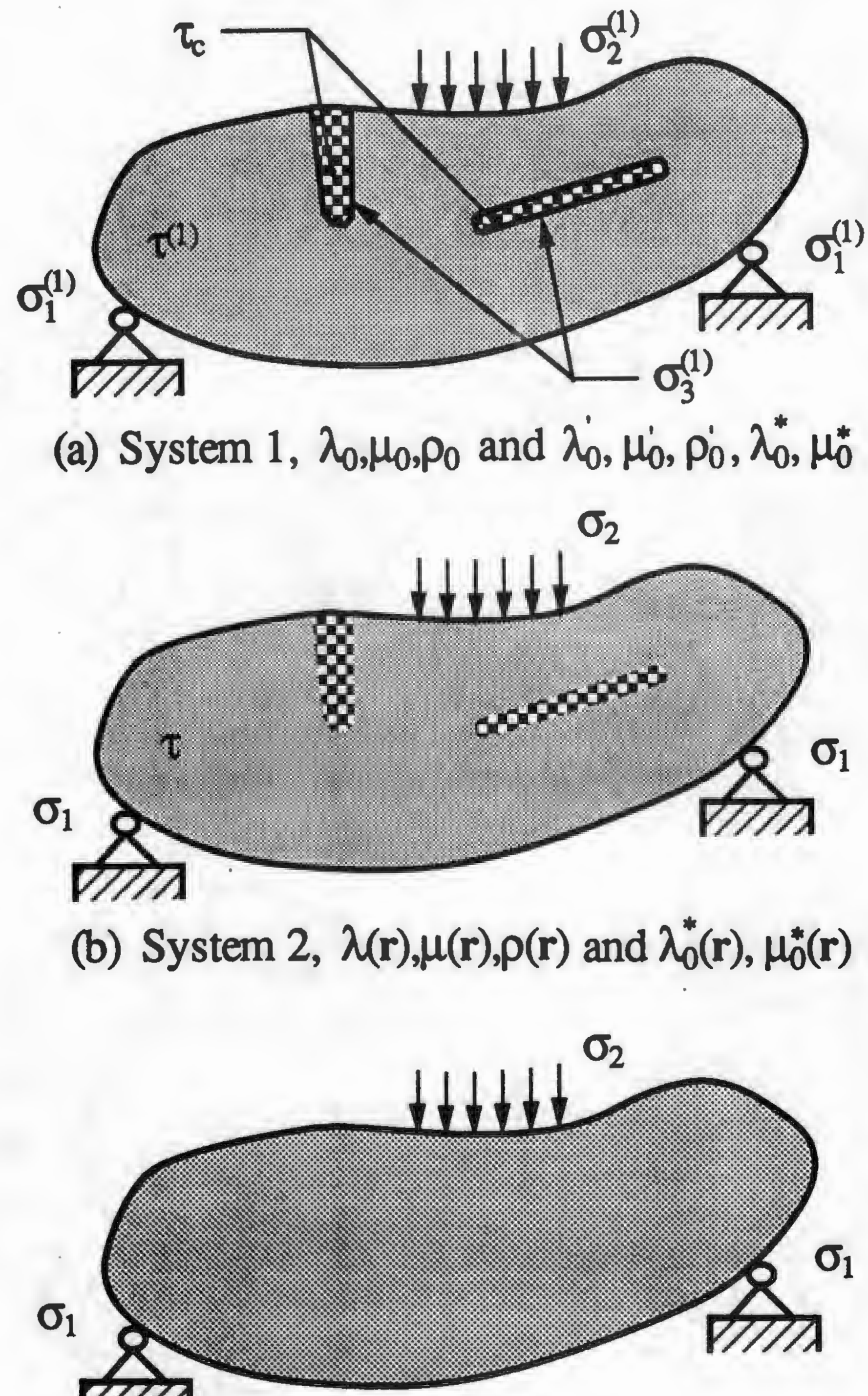
(m,n)	Mode	Eigenvalues Without Inclusions‡	With Three Inclusions†	
			$\mathrm{Re}[\tilde{\nu}_{mn}]$	$\mathrm{Im}[\tilde{\nu}_{mn}]$
(0,0)	axisym.	(0, 7.88264)	-3.5974×10^{-3}	7.90959
(0,1)	cos	(0, 8.04334)	-4.0253×10^{-3}	8.06966
	sin	(0, 8.04334)	-4.0253×10^{-3}	8.06966
(0,2)	cos	(0, 8.89915)	-7.2558×10^{-3}	8.92844
	sin	(0, 8.89915)	-7.2558×10^{-3}	8.92844
(0,3)	cos	(0, 11.23423)	-1.4344×10^{-2}	11.31474
	sin	(0, 11.23423)	-3.1725×10^{-2}	11.20828
(0,4)	cos	(0, 15.49129)	-6.4819×10^{-2}	15.51938
	sin	(0, 15.49129)	-6.4819×10^{-2}	15.51938
(0,5)	cos	(0, 21.62483)	- 0.15262	21.66391
	sin	(0, 21.62483)	- 0.15262	21.66391
(0,6)	cos	(0, 29.45800)	- 0.43977	29.63642
	sin	(0, 29.45800)	- 0.18127	29.41778
(0,11)	cos	(0, 90.41296)	- 3.2630	91.58853
	sin	(0, 90.41296)	- 3.2630	91.58853
(0,12)	cos	(0, 106.61302)	- 7.1748	109.84900
	sin	(0, 106.61302)	- 1.8536	106.82114
(0,13)	cos	(0, 124.10038)	- 5.6289	126.72937
	sin	(0, 124.10038)	- 5.6289	126.72937

† Perturbation iteration with error of $|\tilde{\nu}_{mn}| < 10^{-6}$, and error of norm square < 10^{-10}.

‡ The eigenvalues are represented by complex numbers (a,b), where a and b are real and imaginary parts of the eigenvalues.

References

1. Nashif, Ahid D., Jones, David I. G., and Henderson, John P., *Vibration Damping*, John Wiley & Sons, Inc., New York, NY., 1985.
2. Rogers, Lynn, "Vibration Damping as a Design Parameter," *Mechanical Engineering*, vol. 108, pp. 66-69, Jan. 1986.
3. Drake, Michael L., "Passive Damping: Has Its Time Finally Come," *Mechanical Engineering*, vol. 107, pp. 20-24, Dec. 1985.
4. Yamaguchi, H. and Saito, H., "Vibrations of Beams with an Absorber Consisting of a Viscoelastic Solid and a Beam," *Earthquake Engineering and Structural Dynamics*, vol. 12, pp. 467-479, 1984.
5. Snowdon, J. C., "Platelike Dynamic Vibration Absorbers," *Journal of Engineering for Industry*, vol. 97(1), pp. 88-93, Feb. 1975.
6. El-Raheb, M. and Wagner, P., "Damped Response of Shells by a Constrained Viscoelastic Layer," *Journal of Applied Mechanics*, vol. 53, pp. 902-908, Dec. 1986.
7. Torvik, P. J., "The analysis and Design of Constrained Layer Damping Treatments," in *Damping Applications for Vibration Control*, ed. Peter J. Torvik, pp. 85-112, ASME, New York, NY., 1980.
8. Christensen, R. M., *Theory of Viscoelasticity*, Academic Press, Inc., New York, N.Y., 1982.
9. Shen, I. Y. and Mote, C. D. Jr., "Dynamic Analysis of Three Dimensional, Finite, Linear Elastic Solids Containing Small Imperfections: Theory with Application to Asymmetric Circular Plates," *submitted to Journal of Sound and Vibration*.



(a) System 1, λ_0, μ_0, ρ_0 and $\lambda_0', \mu_0', \rho_0', \lambda_0^*, \mu_0^*$

(b) System 2, $\lambda(\mathbf{r}), \mu(\mathbf{r}), \rho(\mathbf{r})$ and $\lambda_0^*(\mathbf{r}), \mu_0^*(\mathbf{r})$

(c) System 3, λ_0, μ_0, ρ_0

Fig. 1 (a) An Elastic Solid Containing Viscoelastic Inclusions
(b) The Equivalent Inhomogeneous Viscoelastic Solid
(c) The Homogeneous Elastic Solid without Inclusions

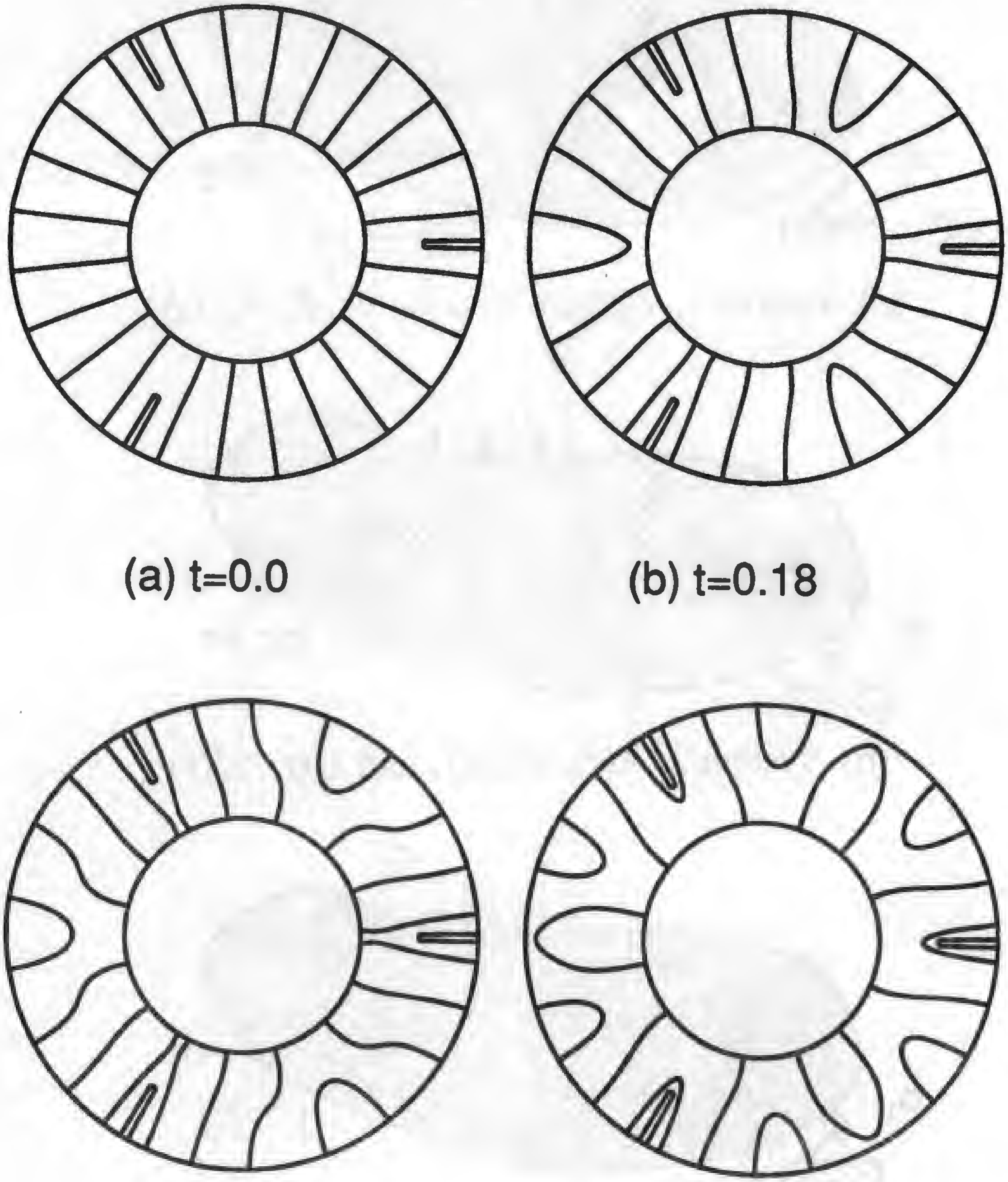
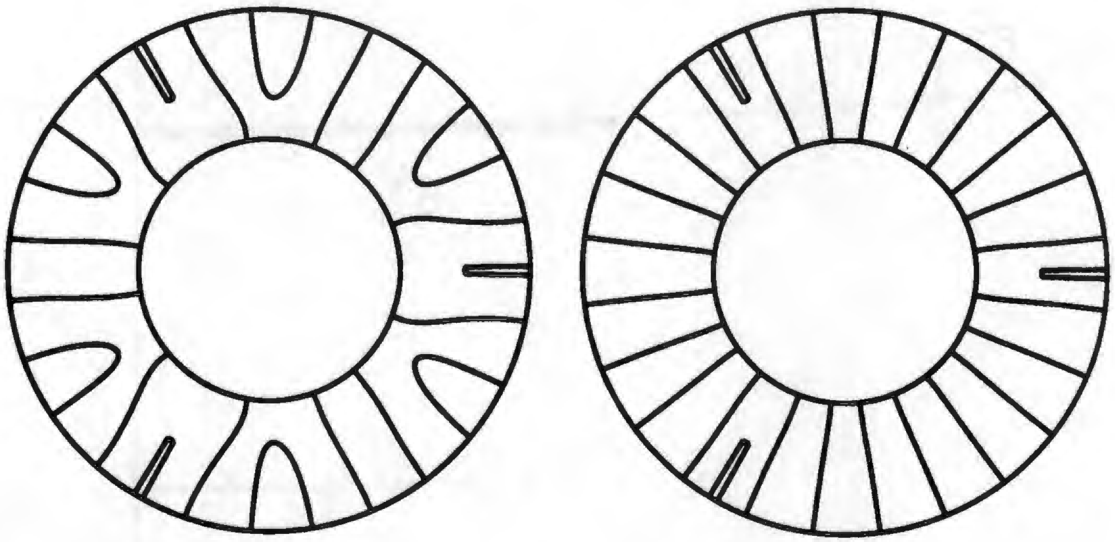


Fig. 2 Evolution of Nodal Curves of (0, 12^c)



(e) t=0.30 (f) t=0.50

Fig. 2 (Continued)

Fig. 3 Loci of Eigenvalues $\tilde{v}_{mn}$

$\xi = 0 - 0.2$ for 5^c, 5^s, 6^c, and 6^s

$\xi = 0 - 0.4$ for 3^c, 3^s, 4^c, and 4^s

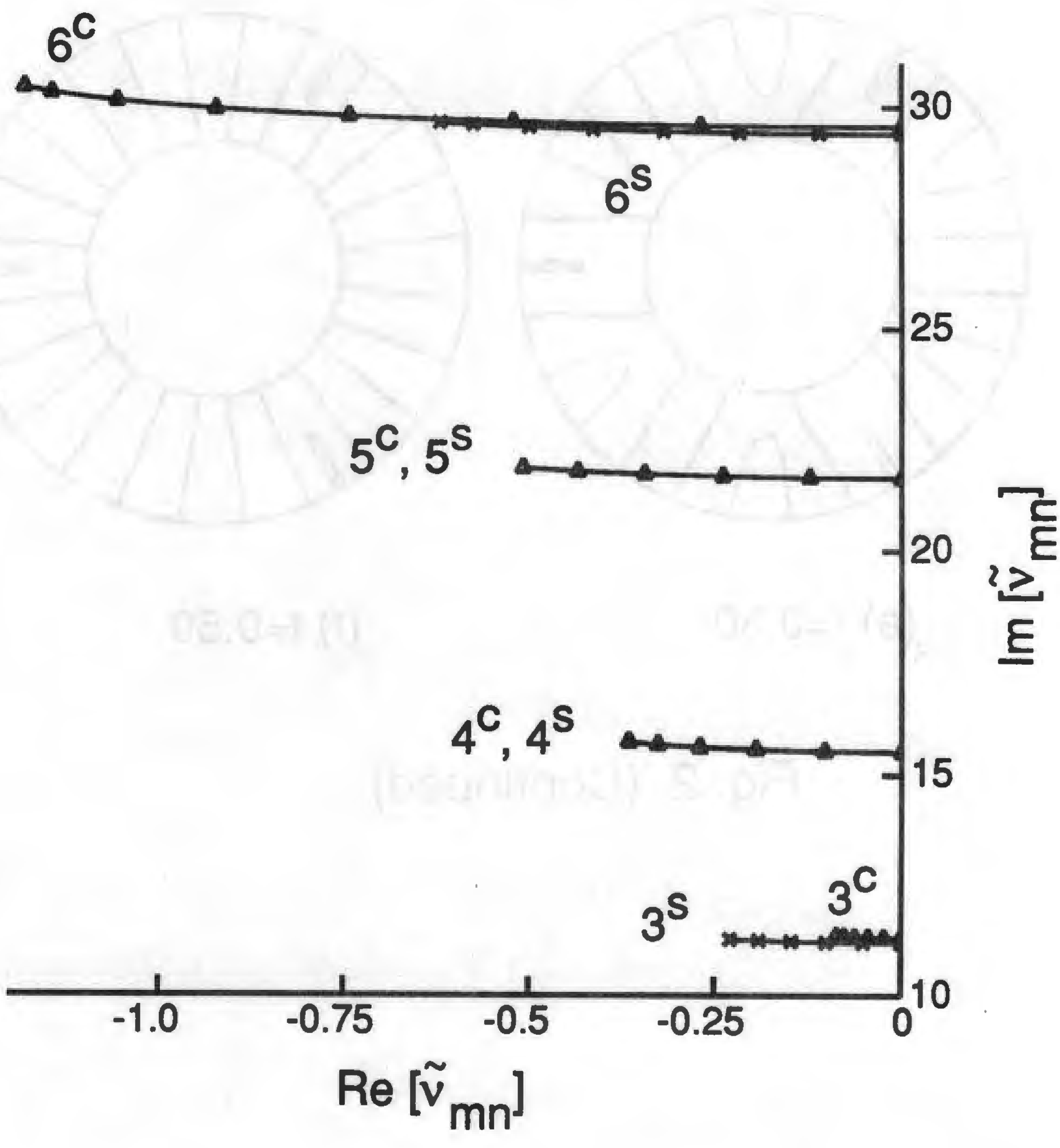


Fig. 4 Displacement Contours of Green's Function with

$t=0$, $r_0=(b,30^o)$, $\tilde{\nu}=6.00$, $\Delta w=0.05$

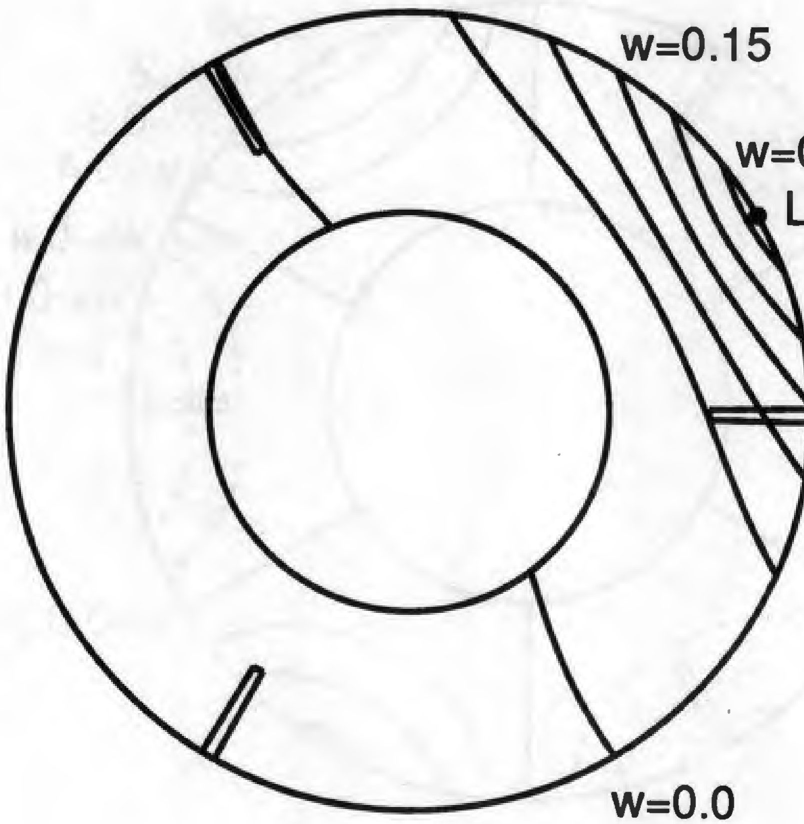


Fig. 5 Displacement Contours of Green's Function with

$t=0,\ r_0=(b,60^o),\ \tilde{\nu}=11.40,\ \Delta w=0.4$