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PART I

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THE FLUTTER CHARACTERISTICS OF LOW ASPECT RATIO WINGS IN INCOMPRESSIBLE FLOW

PART I. DEVELOPMENT OF THE BASIC THEORY OF THE METHOD

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FOREWORD

This report was prepared by the Cornell Aeronautical Laboratory, Inc. Buffalo, New York, for the Aircraft Laboratory, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio. The research and development work was accomplished under Air Force Contract No. AF 33(616)-317, Project No. 1370, 'Aeroelasticity, Vibration and Noise,' and Task No. 13471, 'Subsonic and Transonic Theoretical Flutter'. Mr. Walter J. Mykytow of the Dynamics Branch, Aircraft Laboratory, is task engineer. Research started in 1954 and is continuing. This is Part I of this report which will be published in two separate parts. Part II will contain applications of the Lawrence-Gerber method of flutter analysis and will contain comparisons of theoretical and experimental results.

ABSTRACT

The method for determining the aerodynamic characteristics of low aspect ratio wings oscillating in an incompressible flow which has recently been developed at the Cornell Aeronautical Laboratory (Refs. 1, 2) is combined with the equations of motion of a vibrating wing to produce a relatively simple flutter analysis procedure. The newly-developed method has been worked out for both symmetric and antisymmetric wing mode shapes. The aerodynamic pressure on the fluttering wing is introduced into the final flutter equation in the form of a nondimensional influence function which represents the virtual work performed by the aerodynamic pressure due to one mode shape acting on the displacements of the wing due to another mode shape. The method is applicable to wings with straight trailing edges of aspect ratio less than four.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

JAMES O. COBB
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LIST OF SYMBOLS

AR	=	wing aspect ratio, 4 B_0^2/S
b_f	=	total flap span, ft.
B_0	=	maximum semi-span, $B(c)$, ft.
$B(X)$	=	wing local semi-span, ft.
c	=	wing root semi-chord, ft.
$\bar{c}$	=	mean aerodynamic chord, $\frac{2}{S}\int_0^{B(c)} c^2(Y)\,dY$, ft.
$\bar{c}_f$	=	root mean square flap chord, ft.
C_{h_δ}	=	hinge moment $/0.5\rho U^2 b_f \bar{c}_f^2 \delta$
C_{ℓ_δ}	=	rolling moment$/\rho U^2 S B_0 \delta$
C_{ℓ_p}	=	rolling moment$/\rho U S B_0^2 p$
C_{L_α}	=	lift $/0.5\rho U^2 S\alpha$
$C_{L_{\dot\alpha}}$	=	lift $/0.25\rho U S\bar{c}\dot\alpha$
C_{L_δ}	=	lift$/0.5\rho U^2 S\delta$
C_{M_α}	=	moment (about wing apex)$/\rho U^2 S c$
$C_{m_{\dot\alpha}}$	=	moment (about wing apex)$/0.25\rho U S\bar{c}^2\dot\alpha$
C_{m_q}	=	moment (about wing apex)$/0.25\rho U S\bar{c}^2 q$
$e(X)$		see Eq. (36)
$E(X)$		see Eq. (43)
$f(X)$		see Eq. (8)
$F(X)$		see Eq. (23)
$g(X)$		see Eq. (9)

$g_s(X)$		see Eq. (17)
G		see Eqs. (19 through 21)
h_{mn}	=	influence function, see Eq. (45)
$H(\theta), \frac{B}{c}$		see Eq. (22)
k	=	reduced frequency, $\omega c/U$
K_j	=	numerical integration coefficients, see Eq. (52)
L	=	lift force, lbs.
L_α	=	$-iA\!\!R \times$ (Lift due to rotary oscillation about quarter root chord)$/\rho U^2 c^2 \pi k$
L_h	=	$-iA\!\!R \times$ (Lift due to vertical translation oscillation)$/\rho U^2 c^2 \pi k$
$m(X)$		see Eq. (37)
$m_s(X)$		see Eq. (42)
M_α	=	$-iA\!\!R \times$ (Quarter-root-chord moment due to rotary oscillation about quarter root chord)$/\rho U^2 c^3 \pi k$
M_h	=	$-iA\!\!R \times$ (Quarter-root-chord moment due to vertical translatory oscillation)$/\rho U^2 c^3 \pi k$
$M\left(\frac{X}{c}, \frac{B_o}{c}, k\right)$		see Eq. (30)
$N\left(k, \frac{B_o}{c}\right)$		see Eq. (29)
p	=	rolling angular velocity, radians/sec.
$P\left(\frac{X}{c}, \frac{B_o}{c}, k\right)$		see Eq. (28)

P_{mn}		see Eq. (64)
Q_r		see Eq. (26)
Q_{mn}		see Eq. (65)
$R\left(\frac{X}{c}, \frac{B}{c}\right)$		see Eq. (27)
S	=	wing area, sq. ft.
$S\,[Z]$	=	structural load per unit area, lbs/sq. ft.
T	=	time in seconds
U	=	free-stream velocity, ft/sec.
w	=	angle of attack of oscillating wing, radians, $Z' + i\,\frac{k}{c}\,Z$
$\bar{w}$	=	complex conjugate of w, $Z' - i\,\frac{k}{c}\,Z$
$w^{(i)}$	=	induced angle of attack, radians
x, y, z	=	X/c, Y/c, Z/c, see Fig. 1
X, Y, Z	=	Cartesian coordinate distances, ft., see Fig. 1
α	=	wing angle of attack, radians
$\dot{\alpha}$	=	$d\alpha/dT$
$\gamma(X, Y)$	=	mass per unit area of wing, slugs/sq. ft.
Γ_{pq}	=	elements of inverted G matrix, see Eq. (19)
δ	=	deflection angle of flaps and ailerons, radians
δ_{mn}	=	Kronecker delta, unity for $m = n$ and zero otherwise
$\Delta(\ )_r$	=	difference operator, $(\ )_{r+1} - (\ )_{r-1}$
η	=	spanwise variable, ft., see Fig. 1

θ	=	$\cos^{-1}(X/c)$
$\mu(k)$		see Eq. (18)
ρ	=	air mass density, slugs/cu. ft.
τ	=	$\cos^{-1}(Y/B(X))$
$\phi(X,Y)$	=	Perturbation potential, see Eq. (6)
ω	=	angular frequency of oscillation, radians/sec.
Ci, Si	=	Cosine and sine integrals, see Ref. 5
$(\ )'$	=	$\partial(\ )/\partial X$
$(\ )_X (\ )_Y$	=	$\partial(\ )/\partial X, \partial(\ )/\partial Y$
$[\ ]$	=	square matrix or function involving derivatives and integrals
$\{\ \}$	=	column matrix

INTRODUCTION

Estimation of the flutter characteristics of low aspect ratio wings in incompressible flow is contingent on the solution of two problems: calculation of the lifting pressures on an oscillating wing in an arbitrary mode shape, and evaluation of the dynamic properties of the wing from its structural characteristics and mass distribution. For an oscillating wing of arbitrary planform and mode, the most general method for computing the surface pressures is the direct solution of the two variable integral equation of lifting surface theory (Ref. 3). Although this procedure would be expected to produce good results, the labor necessary to evaluate the surface pressures for a range of frequencies and mode shapes is prohibitive with the computing equipment available at present.

A number of approximate methods for calculating surface pressures by reducing the two variable integral equation to an integral equation in one variable have been proposed both for the steady and unsteady case. This technique reduces the labor to the point where numerical results can be obtained rapidly with desk calculators.

The above technique has been applied in References 1 and 2 to the calculation of surface pressures on a low aspect ratio wing with a straight trailing edge oscillating in an incompressible flow. Excellent agreement between calculations based on this theory and experimental data for the rigid body modes of triangular and rectangular wings and control surface motion have been reported (References 4, 5). Although the lack of experimental data eliminates the possibility of making direct comparisons between surface pressures predicted by the theory and experimental results for nonrigid modes, the evidence cited above appears sufficient to warrant the tentative use of the theory in flutter calculations.

The dynamical properties of the wing will be considered as known and expressed in the form of uncoupled normal modes and frequencies. However, even when the normal modes and natural frequencies are known, the analysis

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for low aspect ratio wings is considerably more complex than for high aspect ratio configurations. In the high aspect ratio case, the normal modes can usually be represented as combinations of bending and twisting deformations. Thus, the virtual work performed by the aerodynamic forces arising from one mode on the displacement in another mode may be expressed as a line integral. In the low aspect ratio case, such a decomposition of deformations is not known, so that the virtual work performed by the aerodynamic forces must be expressed as a surface integral. Much of the complexity in the numerical calculation of the flutter properties of low aspect ratio wings may be traced to this source.

The method of estimating surface pressures on a low aspect ratio wing which was first presented in References 1 and 2 is derived in a simplified manner. The results of this analysis are then combined with the dynamic properites of the wing as expressed in normal mode form to yield the flutter equations. The flutter equations contain the aerodynamic characteristics of the wing in the form of aerodynamic influence coefficients which represent the virtual work performed by the aerodynamic surface pressures generated in one mode acting through the displacements defining another mode. In order to illustrate the physical meaning of the aerodynamic influence coefficients, a number of stability derivatives are expressed in this form.

DEVELOPMENT OF THE THEORY

THE LIFT PER UNIT AREA ON A SYMMETRICALLY OSCILLATING WING

An approximate method for computing the aerodynamic forces on a straight trailing edge low aspect ratio wing oscillating in an incompressible flow have recently been developed at the Cornell Aeronautical Laboratory (Reference 1). A modification of this work presented in Reference 2 will be used in this section to evaluate the lift per unit area on a straight trailing edge low aspect ratio wing oscillating in an incompressible flow.

The analysis will be carried out using the Cartesian coordinates X, Y, Z shown on Figure 1. The nondimensional coordinates x, y, z used commonly in Reference 1 are related to the displacements X, Y, Z by the equations

$$x = X/c, \quad y = Y/c, \quad z = Z/c \tag{1}$$

where c is the root semi-chord (see Figure 1).

If a wing is oscillating with an angular frequency of ω radians per second in a free stream of velocity U feet per second, the commonly used reduced frequency, k, is defined as $\omega c/U$. Defining the time variable in seconds as T, the vertical displacement of the wing is found at any instant by the real part of the expression $Z(X,Y)e^{i\omega T} = Z(X,Y)e^{ikUT/c}$, where $Z(X,Y)$ is the maximum displacement measured positive downward.

From Reference 1, the angle of attack of the oscillating wing is expressed as

$$w(X,Y) = Z'(X,Y) + i\,\frac{k}{c}\,Z(X,Y) \tag{2}$$

where the prime denotes differentiation by the streamwise variable X.

With the change of spanwise variable $Y = B(X)\cos\tau$ where $B(X)$ is the wing semi-span at the chordwise station X, the deflection shape $Z(X,Y)$ may be expanded in a Fourier series of the form

$$Z(X,\tau) = \frac{1}{2} Z_0(X) + \sum_{r=2,4,\cdots}^{\nu} Z_r(X) \cos r\tau \qquad (3)$$

The expansion coefficients $Z_r(X)$ are given by the formula

$$Z_r(X) = \frac{2}{\pi} \int_0^{\pi} Z(X,\tau) \cos r\tau \, d\tau \qquad r = 0, 2, \cdots, \nu \qquad (4)$$

Thus, with the notation $w_r(X) = Z'_r(X) + i \frac{k}{c} Z_r(X)$, Eq. (2) may be rewritten as

$$w(X,\tau) = \frac{1}{2} w_0(X) + \sum_{r=2,4,\cdots}^{\nu} w_r(X) \cos r\tau \ . \qquad (5)$$

One now seeks an expression for the angle of attack, $w(X,Y)$, from the oscillating airfoil theory in Reference 1. It is desirable, in an effort to attach a maximum physical significance to the following derivations, to formulate the theoretical expression for the attack angle in the manner suggested in Reference 2. Accordingly, one commences with the integral equation (Reference 7) valid for a Jones (Reference 6) wing

$$w^*(X,Y) = \frac{c}{\pi} \int_{-B(X)}^{B(X)} \frac{\phi_\eta(X,\eta)}{Y-\eta} \, d\eta \qquad (5a)$$

where $w^*(x,y)$ is the angle of attack of a Jones wing.

For a finite-span wing, the value $w^*(X,Y)$ as given by integration from Eq. (5a) is less than the true wing attack angle, $w(X,Y)$. If this difference is denoted by $w^{(i)}(X,Y)$, that is, if

$$w^*(X,Y) = w(X,Y) - w^{(i)}(X,Y) \qquad (5b)$$

then $w^{(i)}(X,Y)$ is the induced angle of attack and is representative of the additional aerodynamic contribution due to the lifting surface elements. Consequently, as the wing span and surface area shrink to zero, $w^{(i)}(X,Y)$ approaches zero, and, by (5b), the actual attack angle becomes the attack angle of a Jones wing.

By inverting the integral equation (5a) and making use of the definition (5b),

$$\phi_Y(X,Y) = \frac{-1}{c\pi\sqrt{B^2(X)-Y^2}} \int_{-B(X)}^{B(X)} \frac{\sqrt{B^2(X)-\eta^2}}{Y-\eta}\left[w(X,\eta)-w^{(i)}(X,\eta)\right] d\eta. \quad (6)$$

Eq. (6), when integrated over the span, gives the perturbation potential valid over the wing surface and wake.

One now assumes that for the symmetrically oscillating wing $w^{(i)}(X,Y)$ is constant along the span. Then, substituting relation (5b) in Eq. (5a), multiplying both sides of the resulting equation by $\sqrt{B^2(X)-Y^2}$ and integrating over the span, one obtains

$$w^{(i)}(X,Y) = w^{(i)}(X) = \frac{2c^2}{\pi B^2(X)}\left[f(X)-g(X)\right] \quad (7)$$

where (Reference 2)

$$f(X) = \frac{1}{c^2}\int_{-B(X)}^{B(X)} w(X,Y)\sqrt{B^2(X)-Y^2}\, dY \quad (8)$$

$$g(X) = \frac{1}{c}\int_{-B(X)}^{B(X)} \phi(X,Y)\, dY. \quad (9)$$

From the simplified Bernoulli relationship (Reference 3) that $p(X,Y) = 2\rho U^2 c\, \phi_X(X,Y)$ one deduces from Eq. (9) that the lift per unit chord of the wing in steady flow is given as $2\rho U^2 c^2 g'(X)$. In the limiting case of zero aspect ratio, the value of $w^{(i)}(X,Y)$ approaches zero, hence, by Eq. (7), $f(X)$ and $g(X)$ become equal. Thus, $2\rho U^2 c^2 f'(X)$ is the lift per unit chord of a Jones (Reference 6) wing in steady flow.

If the values given by Eqs. (5) and (7) are inserted into Eq. (6) and the indicated integration carried out, there results

$$\phi_Y(X,\tau) = -\frac{1}{c\pi B(X)\sin\tau}\left\{\frac{\pi}{2}B(X)w_0(X)\cos\tau\right.$$

$$-\frac{\pi}{2}B(X)\sum_{r=2,4,\cdots}^{\nu} w_r(X)\left[\cos(r-1)\tau-\cos(r+1)\tau\right] \tag{10}$$

$$\left.-\frac{2c^2}{B(X)}\cos\tau\left[f(X)-g(X)\right]\right\}.$$

Since $$\phi(X,Y) = \int_{-B(X)}^{Y}\phi_\eta(X,\eta)\,d\eta \tag{11}$$

where, $\phi(X,-B(X)) = 0$, Eq. (10) may be immediately integrated to give

$$\phi(X,\tau) = \frac{B(X)}{2c}w_0(X)\sin\tau - \frac{2c}{\pi B(X)}\left[f(X)-g(X)\right]\sin\tau \tag{12}$$

$$+\frac{B(X)}{2c}\sum_{r=2,4,\cdots}^{\nu} w_r(X)\left[\frac{\sin(r+1)\tau}{r+1}-\frac{\sin(r-1)\tau}{r-1}\right].$$

The value of $f(X)$ may be found by performing the indicated integration in Eq. (8) with the aid of Eq. (5). There results

$$f(X) = \frac{\pi B^2(X)}{4c^2}\left[w_0(X)-w_2(X)\right] \tag{13}$$

which, when substituted into Eq. (12), gives

$$\phi(X,\tau) = \frac{2c}{\pi B(X)}g(X)\sin\tau - \frac{B(X)}{2c}\sum_{r=3,5,\cdots}^{\nu+1}\Delta w_r(X)\frac{\sin r\tau}{r} \tag{14}$$

where $\Delta w_r(X) = w_{r+1}(X) - w_{r-1}(X)$ and $w_{\nu+1}(X) = w_{\nu+2}(X) \equiv 0$.

From References 1 and 2, the lift per unit area on the oscillating wing is expressed as

$$\frac{dL}{dS}(x,y) = 2\rho U^2 c\left[\phi_X(X,Y) + i\frac{k}{c}\phi(X,Y)\right] \tag{15}$$

The value of Eq. (15) may be determined at any point using Eq. (14), when $g(X)$ is known.

The function $g(X)$ represents the solution of the integral equation (Reference 1)

$$f(X) = \frac{1}{2} g(X) + \frac{1}{4}\int_{-c}^{c} g'(\xi)\left[1 + \frac{\sqrt{(X-\xi)^2 + B^2(X)}}{X-\xi}\right] d\xi$$
$$- \frac{ik}{4c} g(c) \int_{c}^{\infty} e^{-ik\left(\frac{\xi}{c}-1\right)}\left[1 + \frac{\sqrt{(X-\xi)^2 + B_0^2}}{X-\xi}\right] d\xi \qquad (15a)$$

for a low aspect ratio wing oscillating in an incompressible flow.

A solution for $g(X)$ satisfying the integral equation (15a) may be written as (Reference 1)

$$g(X) = g_s(X) - \mu(k) g_s(c)\left(1 + \frac{X}{c}\right) \qquad (16)$$

where $g_s(X)$ is a point function defined as the series (Reference 8)

$$g_s(\theta) = (\pi - \theta)(A_0 + A_1) + \sum_{r=1}^{N} (A_{r-1} - A_{r+1}) \frac{\sin r\theta}{r} \qquad (17)$$
$$A_{N+1} \equiv A_N \equiv 0.$$

The angle θ is introduced by the substitution $X = c\cos\theta$. The function $\mu(k)$ is defined as

$$\mu(k) = \frac{ik}{2ik+1} = \frac{2k^2 + ik}{4k^2+1} \quad . \qquad (18)$$

It will be observed that the series expansion for $g(X)$ given by Eqs. (16) and (17) satisfies the conditions that $g(X)$ vanish at the leading edge $(X = -c)$ and that an integrated Kutta-Joukowsky condition be satisfied at the trailing edge, i.e.,

$$g'(c) + i\frac{k}{c} g(c) = 0 . \qquad (18a)$$

It is clear that a solution of Eq. (17) for the coefficients A_r will allow the determination of $g(X)$ at all X and, hence, will enable evaluation of Eq. (15) by means of Eq. (14). The determination of the coefficients A_r amounts only to the solution of that equation resulting when Eqs. (13), (16), (17), and (18) are substituted into the integral equation (15a) and the resulting equation rearranged and solved by collocation at N points. Such a collocation procedure satisfies any imposed conditions at the collocation points $\theta_j = j\pi/N$ only and results in N simultaneous algebraic equations, involving certain tabulated definite integrals, which are solved herein by matrix algebra methods. The details of the solution for the coefficients A_r may be found in Reference 1, but the results are stated in the following discussion in a concise form so that the numerical work involved can be carried out in a straightforward fashion.

The expression for the coefficients A_r to be presented involves the use of a square matrix, $\left[\Gamma_{pq}\right]$, defined as

$$\left[\Gamma_{pq}\right] = \begin{bmatrix} G_0(0) & G_1(0) & \cdots & G_{N-1}(0) \\ G_0\left(\frac{\pi}{N}\right) & G_1\left(\frac{\pi}{N}\right) & \cdots & G_{N-1}\left(\frac{\pi}{N}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ G_0\left(\frac{N-1}{N}\pi\right) & G_1\left(\frac{N-1}{N}\pi\right) & \cdots & G_{N-1}\left(\frac{N-1}{N}\pi\right) \end{bmatrix}^{-1} \tag{19}$$

where

$$G_0(\theta_j) = \frac{2}{\pi}(\sin\theta_j - \theta_j) + \frac{B(\theta_j)}{c} + 3 + \frac{1}{\pi}\int_0^{\pi}(\cos\tau - 1)H\left(\theta_j, \frac{B\theta_j}{c}, \tau\right)d\tau \tag{20}$$

$$G_r(\theta_j) = \frac{2}{\pi}\left[\frac{\sin(r+1)\theta_j}{r+1} - \frac{\sin(r-1)\theta_j}{r-1}\right] + 2\,\frac{B(\theta_j)}{c}\cos r\theta_j$$
$$- \frac{2}{\pi}\int_0^{\pi} \sin r\tau \sin\tau\, H\!\left(\theta_j, \frac{B(\theta_j)}{c}, \tau\right) d\tau \qquad r > 0 \tag{21}$$

and, $$H\!\left(\theta_j, \frac{B(\theta_j)}{c}, \tau\right) = \frac{\sqrt{(\cos\tau - \cos\theta_j)^2 + B^2(\theta_j)/c^2} - B(\theta_j)/c}{\cos\tau - \cos\theta_j}\,. \tag{22}$$

Numerical values of the matrix elements of $\left[\Gamma_{pq}\right]$ are given in Reference 8 when N = 6 and the wing is of the type depicted in Figure 1 (i.e., completely defined by a stated value of aspect ratio and taper ratio and with a straight trailing edge). The matrix elements of $\left[\Gamma_{pq}\right]$ are given directly if the wing has an aspect ratio 0, 0.25, 0.5, 1, 2, or 4 and a taper ratio 0, 0.25, 0.5, 0.75, or 1. For other values of aspect ratio and taper ratio, one can linearly interpolate between the given values. Reference 8 also contains curves from which one can read off the matrix elements of the inverse matrix, $\left[\Gamma_{pq}\right]^{-1}$, for any given wing shape, provided it has a straight trailing edge. The elements of this latter inverse matrix may then be inverted, a task of only three hours for the average computer.

If one now defines the column matrix $\left\{F(\theta_j)\right\}$ as

$$\left\{F(\theta_j)\right\} = \frac{4}{\pi}\left\{f(\theta_j)\right\} = \left\{\frac{B^2(\theta_j)}{c^2}\left[w_0(\theta_j) - w_2(\theta_j)\right]\right\} \tag{23}$$

then, the results in Reference 1 for the determination of the coefficients A_r may be rearranged to give

$$\left\{A_r\right\} = \left[\left[\Gamma_{pq}\right] - \frac{1}{1 + Q_0 + Q_1}\left\{Q_r\right\}\left[\Gamma_{0q} + \Gamma_{1q}\right]\right]\left\{F(\theta_j)\right\} \tag{24}$$

where the row matrix $\left[\Gamma_{0q} + \Gamma_{1q}\right]$ is defined as

$$\left[\Gamma_{0q} + \Gamma_{1q}\right] = \left[\Gamma_{00} + \Gamma_{10}, \Gamma_{01} + \Gamma_{11}, ----, \Gamma_{0,N-1} + \Gamma_{1,N-1}\right]. \tag{25}$$

The column matrix $\left\{Q_r\right\}$ is given by the equation

$$\left\{Q_r\right\} = -\mu(k)\left[\Gamma_{pq}\right]\left\{R\left(\frac{X_j}{c}, \frac{B(X_j)}{c}\right) + P\left(\frac{X_j}{c}, \frac{B_0}{c}, k\right)\right\} \tag{26}$$

The functions $R\left(\frac{X}{c}, \frac{B(X)}{c}\right)$ and $P\left(\frac{X}{c}, \frac{B_0}{c}, k\right)$ are given as, (Reference 1):

$$\begin{aligned} R\left(\frac{X}{c}, \frac{B(X)}{c}\right) = 4 + 2\frac{X}{c} - \sqrt{\left(\frac{X}{c} - 1\right)^2 + \frac{B^2(X)}{c^2}} + \sqrt{\left(\frac{X}{c} + 1\right)^2 + \frac{B^2(X)}{c^2}} \\ + \frac{B(X)}{c}\sinh^{-1}\left|\frac{B(X)}{X-c}\right| - \frac{B(X)}{c}\sinh^{-1}\left|\frac{B(X)}{X+c}\right| \end{aligned} \tag{27}$$

$$\begin{aligned} P\left(\frac{X}{c}, \frac{B_0}{c}, k\right) = \frac{B_0}{c} e^{ik(1-\frac{X}{c})}\left[N\left(k, \frac{B_0}{c}\right) - M\left(\frac{X}{c}, \frac{B_0}{c}, k\right)\right. \\ \left. + C_i\left[k\left(1 - \frac{X}{c}\right)\right] + i\left\{\frac{\pi}{2} - Si\left[k\left(1 - \frac{X}{c}\right)\right]\right\}\right] \end{aligned} \tag{28}$$

and

$$N\left(k, \frac{B_0}{c}\right) = \int_0^\infty e^{-it}\left(\frac{c}{kB_0} + \frac{1}{t} - \frac{\sqrt{t^2 + k^2 B_0^2/c^2}}{k B_0 t/c}\right) dt \tag{29}$$

$$M\left(\frac{X}{c}, \frac{B_0}{c}, k\right) = \int_0^{k(1-\frac{X}{c})} e^{-it}\left(\frac{c}{kB_0} + \frac{1}{t} - \frac{\sqrt{t^2 + k^2 B_0^2/c^2}}{k B_0 t/c}\right) dt \tag{30}$$

where the Ci and Si are the sine and cosine integrals of Reference 5. Short tables of the functions R , P , and M may be found in Reference 1. Rather complete tables of both the complete and incomplete Cicala functions,

$N\left(k, \frac{B_0}{c}\right)$ and $M\left(\frac{X}{c}, \frac{B_0}{c}, k\right)$ have been compiled by the Harvard Computation Laboratory and will be published shortly.

It will be observed that as x/c approaches unity the functions R and P respectively approach positive and negative infinity; however, the sum, $R + P$, needed in Eq. (26), is finite and is given as (Reference 1)

$$\lim_{x \to 1}\left[R\left(1, \frac{B_0}{c}\right) + P\left(1, \frac{B_0}{c}, k\right)\right] = 6 + \sqrt{4 + \frac{B_0^2}{c^2}} - \frac{B_0}{c}\left[1 + \sinh^{-1}\left|\frac{B_0}{2c}\right| - N\left(k, \frac{B_0}{c}\right) - \log 2\gamma k \frac{B_0}{c} - i\frac{\pi}{2}\right] \tag{31}$$

where $\gamma = 1.781072 - - -$.

The solution of Eq. (15) for the lift per unit area on the oscillating wing has now been fully indicated, although a more compact form for practical analysis can be constructed. Eq. (15) is solved using Eq. (14) where the values of $g(X)$ are determined from Eq. (16) et seq.

THE LIFT PER UNIT AREA ON AN ANTISYMMETRICALLY OSCILLATING WING

In a manner analogous to the symmetrically oscillating wing discussed above, the deflection shape $Z(X, Y)$ may be expanded in a Fourier series of the form

$$Z(X, \tau) = \sum_{r=1,3,5---}^{\nu} Z_r(X) \cos r\tau \tag{32}$$

where, as before, $Y = B(X)\cos\tau$. Eq. (32) differs from Eq. (3) by the use of an odd series in r necessary to describe an antisymmetric deflection shape. The expansion cœfficients $Z_r(X)$ are given by the formula

$$Z_r(X) = \frac{2}{\pi}\int_0^{\pi} Z(X, \tau)\cos r\tau \, d\tau \qquad r = 1, 3, \cdots, \nu \tag{33}$$

The antisymmetric angle of attack, similar to Eq. (5), is now

$$w(X,\tau) = \sum_{r=1,3,5,\cdots}^{\nu} w_r(X) \cos r\tau \tag{34}$$

where,

$$w_r(X) = Z_r'(X) + i\frac{k}{c} Z_r(X) .$$

In order to integrate Eq. (6) for the perturbation potential, one requires a value of the antisymmetric induced angle of attack, $w^{(i)}(X,Y)$. It will be assumed that the antisymmetric induced attack angle varies linearly in Y according to the equation

$$w^{(i)}(X,Y) = \frac{Y}{B(X)} w^{(i,a)}(X) \tag{34a}$$

Thus, substituting relation (5b) into Eq. (5a), multiplying both sides of the resulting equation by $Y\sqrt{B^2(X)-Y^2}$ and integrating over the span, one obtains

$$w^{(i)}(X,Y) = \frac{16\,Yc^3}{\pi B^4(X)}\Big[e(X) - m(X)\Big] \tag{35}$$

where (Reference 7),

$$e(X) = \frac{1}{2c^3}\int_{-B(X)}^{B(X)} w(X,Y)\, Y\sqrt{B^2(X)-Y^2}\; dY \tag{36}$$

$$m(X) = \frac{1}{c^2}\int_{-B(X)}^{B(X)} Y\phi(X,Y)\, dY . \tag{37}$$

By comparing Eqs. (36) and (37) with Eqs. (8) and (9) respectively, it can be seen that $2\rho U^2 c^3 m'(X)$ is the rolling moment per unit chord of a low aspect ratio wing in steady flow. Further, as the aspect ratio approaches zero, the induced attack angle approaches zero and $e(X) = m(X)$ Thus, $2\rho U^2 c^3 e'(X)$ is the rolling moment per unit chord of a Jones (Reference 3) wing in steady flow.

In a manner directly analogous to the evaluation of Eq. (12), the antisymmetric perturbation potential may be obtained from Eqs. (6), (34), and (35) as

$$\phi(X,\tau) = \frac{B(X)}{4c} w_1(X)\sin 2\tau - \frac{4c^2}{\pi B^2(X)}\Big[e(X) - m(X)\Big]\sin 2\tau$$
$$+ \frac{B(X)}{2c}\sum_{r=3,5,\cdots}^{\nu} w_r(X)\left[\frac{\sin(r+1)\tau}{r+1} - \frac{\sin(r-1)\tau}{r-1}\right]. \qquad (38)$$

The value of $e(X)$ is easily determined by performing the indicated integration in Eq. (36) with the aid of Eq. (34). There results

$$e(X) = \frac{\pi B^3(X)}{16c^3}\Big[w_1(X) - w_3(X)\Big] \qquad (39)$$

which, substituted into Eq. (38), gives

$$\phi(X,\tau) = \frac{4c^2}{\pi B^2(X)} m(X)\sin 2\tau - \frac{B(X)}{2c}\sum_{r=4,6,\cdots}^{\nu+1} \Delta w_r(X)\frac{\sin r\tau}{r} \qquad (40)$$

where

$$w_{\nu+1}(X) = w_{\nu+2}(X) \equiv 0 \text{ and } \Delta w_r(X) = w_{r+1}(X) - w_{r-1}(X).$$

The lift per unit area of the wing is found, as in the preceding section, by evaluating Eq. (15) with the aid of Eq. (40) provided values of $m(X)$ are available. It can be shown (Reference 1) that the solutions of the symmetric and antisymmetric wing oscillation equations are identical in form so that one need only substitute $e(X)$ for $f(X)$ and $m(X)$ for $g(X)$ to get the antisymmetric solutions when the symmetric solutions are available. Thus, analogous to Eq. (16), one writes

$$m(\theta) = m_s(\theta) - \mu(k) m_s(0)(1 + \cos\theta) \tag{41}$$

where $m_s(\theta)$ is a point function defined as the series

$$m_s(\theta) = (\pi - \theta)(A_0 + A_1) + \sum_{r=1}^{N-1} (A_{r-1} - A_{r+1}) \frac{\sin r\theta}{r} \tag{42}$$

$$A_N \equiv 0$$

The column matrix $\left\{E(\theta_j)\right\}$ is now defined as

$$\left\{E(\theta_j)\right\} = \frac{4}{\pi}\left\{e(\theta_j)\right\} = \left\{\frac{B^3(X)}{4c^3}\left[w_1(\theta_j) - w_3(\theta_j)\right]\right\} \tag{43}$$

which equation is used in the matrix equation below, analogous to Eq. (24),

$$\left\{A_r\right\} = \left(\left[\Gamma_{pq}\right] - \frac{1}{1 + Q_0 + Q_1}\left\{Q_r\right\}\left[\Gamma_{0q} + \Gamma_{1q}\right]\right)\left\{E(\theta_j)\right\} \; . \tag{44}$$

The functions needed to solve Eq. (44) are defined by Eqs. (19) through (31) which remain the same for both the symmetric and antisymmetric cases.

In summary, the lift per unit area on the antisymmetrically oscillating wing is found from Eqs. (15) and (40), where the function $m(x)$ is evaluated by Eqs. (41) through (44) and Eqs. (19) through (31).

AN INFLUENCE COEFFICIENT FOR THE OSCILLATING WING

The aerodynamic influence coefficients h_{mn} which represent the virtual work performed by the aerodynamic forces arising from the n^{th} mode acting through the displacements specified by the m^{th} mode are defined as follows

$$h_{mn} = \frac{1}{2\rho U^2 c^3}\iint_S Z_m(X,Y)\frac{dL}{dS}(Z_n)\,dX\,dY \; . \tag{45}$$

Using the value of dL/dS from Eq. (15) in Eq. (45), one obtains

$$h_{mn} = \frac{1}{c^2} \iint_{-B(X),-c}^{B(X),c} Z_m(X,Y)\left[\phi_X(X,Y) + i\frac{k}{c}\phi(X,Y)\right] dX\,dY$$
$$= \frac{1}{c^2}\int_{-B(X)}^{B(X)} \left[Z_m(c,Y)\phi(c,Y) - \int_{-c}^{c}\phi(X,Y)\left[Z_m'(X,Y) - i\frac{k}{c}Z_m(X,Y)\right]dX\right]dY \tag{46}$$

since $\phi(-c,Y) = 0$. It is implied that $\phi(X,Y)$ is evaluated corresponding to the deflection shape $Z_n(X,Y)$. The influence coefficient h_{mn} will be used subsequently for the determination of flutter characteristics.

THE INFLUENCE COEFFICIENT FOR A SYMMETRICALLY OSCILLATING WING

When the wing is oscillating symmetrically, the deflection shape $Z_m(X,Y)$ may be expressed in the form of Eq. (3). If $\bar{w}_{mr}(X)$ is defined as $\bar{w}_{mr}(X) = Z'_{mr}(X) - i\frac{k}{c}Z_{mr}(X)$, then $\bar{w}_{mr}(X)$ is the complex conjugate of $w_{mr}(X)$ and, analogous to Eq. (5),

$$\bar{w}_m(X,\tau) = 0.5\,\bar{w}_{mo}(X) + \sum_{r=2,4,\cdots}^{\nu} \bar{w}_{mr}(X)\cos r\tau \; . \tag{47}$$

Now substitute Eqs. (3), (14), and (47) into Eq. (46) to get

$$h_{mn} = \frac{1}{c}\int_0^{\pi}\left[0.5\,Z_{mo}(c) + \sum_{s=2,4,\cdots}^{\nu} Z_{ms}(c)\cos s\tau\right]\left[\frac{2}{\pi}g_n(c)\sin\tau - \frac{B_o^2}{2c^2}\sum_{r=3,5,\cdots}^{\nu+1}\Delta w_{nr}(c)\frac{\sin r\tau}{r}\right]\left[\sin\tau\,d\tau\right]$$
$$-\frac{1}{c}\iint_{0,-c}^{\pi,c}\left[0.5\,\bar{w}_{mo}(X) + \sum_{s=2,4,\cdots}^{\nu}\bar{w}_{ms}(X)\cos s\tau\right]\left[\frac{2}{\pi}g_n(X)\sin\tau - \frac{B^2(X)}{2c^2}\sum_{r=3,5\cdots}^{\nu+1}\Delta w_{nr}(X)\right.$$
$$\left.\frac{\sin r\tau}{r}\right]\sin\tau\,d X\,d\tau. \tag{48}$$

The integration of Eq. (48) with respect to τ can be written at once by use of the formula

$$
\begin{aligned}
\int_0^{\pi} \cos S\tau \sin r\tau \sin\tau \, d\tau &= \frac{\pi}{2} \quad \text{when} \quad r = 1,\ S = 0 \\
&= \frac{\pi}{4} \quad \text{when} \quad r - S = 1,\ S \neq 0 \\
&= -\frac{\pi}{4} \quad \text{when} \quad S - r = 1,\ r \neq 0 \\
&= 0 \quad \text{otherwise.}
\end{aligned}
\tag{49}
$$

Thus,

$$
\begin{aligned}
h_{mn} = &-\frac{1}{2c} g_n(c)\Delta Z_{m1}(c) + \frac{\pi B_o^2}{8c^3} \sum_{r=3,5\cdots}^{\nu+1} \frac{\Delta w_{nr}(c)\Delta Z_{mr}(c)}{r} \\
&+ \frac{1}{c}\int_{-c}^{c} \left\{ 0.5 g_n(X)\Delta \bar{w}_{m1}(X) - \frac{\pi B^2(X)}{8c^2} \sum_{r=3,5,\cdots}^{\nu+1} \frac{\Delta w_{nr}(X)\Delta \bar{w}_{mr}(X)}{r} \right\} dX
\end{aligned}
\tag{50}
$$

where

$$
Z_{m,\nu+1} = Z_{m,\nu+2} = Z'_{m,\nu+1} = Z'_{m,\nu+2} \equiv 0
$$

Now, the values of the function $g_n(X)$ are known by the collocation procedure, Eqs. (16) through (31), at the specific values of x given by $x = c\cos\theta_j$, for $\theta_j = j\pi/N$. Hence, the numerical integration indicated in Eq. (50) should be carried out by a method utilizing these selected values of the function. Such a method is the Newton-Cotes formula (Reference 12) which, when used in Eq. (50) for the values of x, yields

$$
\begin{aligned}
h_{mn} = &-\frac{1}{2c} g_n(0)\Delta Z_{m1}(0) + \frac{\pi B_o^2}{8c^3} \sum_{r=3,5\cdots}^{\nu+1} \frac{\Delta w_{nr}(0)\Delta Z_{mr}(0)}{r} \\
&+ \sum_{j=0}^{N-1} K_j \left\{ 0.5 g_n(\theta_j)\Delta \bar{w}_{m1}(\theta_j) - \frac{\pi B^2(\theta_j)}{8c^2} \sum_{r=3,5,\cdots}^{\nu+1} \frac{\Delta w_{nr}(\theta_j)\Delta \bar{w}_{mr}(\theta_j)}{r} \right\} .
\end{aligned}
\tag{51}
$$

The values of the coefficients K_j may be computed using Reference 12. For the particular case where $N = 6$, i.e., a six-point collocation procedure used, the coefficients are

$$\begin{array}{llllllll} \theta_j & = & 0° & 30° & 60° & 90° & 120° & 150° \\ K_j & = & 1/35 & 16/63 & 16/35 & 164/315 & 16/35 & 16/63 \end{array} \tag{52}$$

Eq. (51) yields various values of the influence coefficient, h_{mn} when the deflection shapes $Z_n(X,Y)$ and $Z_m(X,Y)$ are known along with the geometry of the wing and the reduced frequency. The values of $g_n(X)$ are solved from Eqs. (16) through (31).

THE INFLUENCE COEFFICIENT FOR AN ANTISYMMETRICALLY OSCILLATING WING

For the wing oscillating antisymmetrically, the deflection shape $Z_m(X,Y)$ may be expressed in the form of Eq. (32). The value of $\bar{w}_m(X,\tau)$ is, analogous to Eq. (34),

$$\bar{w}_m(X,\tau) = \sum_{r=1,3,5,\cdots}^{\nu} \bar{w}_{mr}(X) \cos r\tau \tag{53}$$

Now, substituting Eqs. (32), (40), and (53) into Eq. (46),

$$\begin{aligned} h_{mn} = {} & \frac{1}{c} \sum_{s=1,3,5,\cdots}^{\nu} Z_{ms}(c) \int_0^{\pi} \cos s\tau \left[\frac{4c}{\pi B_o} m_n(c) \sin 2\tau - \frac{B_o^2}{2c^2} \sum_{r=4,6,\cdots}^{\nu+1} \Delta w_{nr}(c) \frac{\sin r\tau}{r} \right] \sin \tau \, d\tau \\ & - \frac{1}{c} \iint_{0;-c}^{\pi,c} \sum_{s=1,3,5\cdots}^{\nu} \bar{w}_{ms}(X) \cos s\tau \left[\frac{4c}{\pi B(X)} m_n(X) \sin 2\tau - \frac{B^2(X)}{2c^2} \sum_{r=4,6,\cdots}^{\nu+1} \Delta w_{nr}(X) \frac{\sin r\tau}{r} \right] \sin \tau \, dX \, d\tau \end{aligned} \tag{54}$$

which, by using Eq. (49), yields

$$h_{mn} = -\frac{m_n(c)}{B_o}\Delta Z_{m2}(c) + \frac{\pi B_o^2}{8c^3}\sum_{r=4,6,\cdots}^{\nu+1}\frac{\Delta w_{nr}(c)\Delta Z_{mr}(c)}{r} \tag{55}$$

$$+\frac{1}{c}\int_{-c}^{c}\left\{\frac{m_n(X)c}{B(X)}\Delta\bar{w}_{m2}(X) - \frac{\pi B^2(X)}{8c^2}\sum_{r=4,6,\cdots}^{\nu+1}\frac{\Delta w_{nr}(X)\Delta\bar{w}_{mr}(X)}{r}\right\}dX$$

where $Z_{m,\nu+1} = Z_{m,\nu+2} = Z'_{m,\nu+1} = Z'_{m,\nu+2} \equiv 0$

The values of $m_n(X)$ are found by the collocation procedure, Eqs. (41) through (44) and Eqs. (19) through (31), at the specific value of x given by $x = c\cos\theta_j$, for $\theta_j = j\pi/N$. If the integration in Eq. (55) is carried out by a method such as the Newton-Cotes formula (Reference 12) for the selected values of x, there results

$$h_{mn} = -\frac{m_n(0)}{B_o}\Delta Z_{m2}(0) + \frac{\pi B_o^2}{8c^3}\sum_{r=4,6,\cdots}^{\nu+1}\frac{\Delta w_{nr}(0)\Delta Z_{mr}(0)}{r} \tag{56}$$

$$+\sum_{j=0}^{N-1}K_j\left\{\frac{m_n(\theta_j)c}{B(\theta_j)}\Delta\bar{w}_{m2}(\theta_j) - \frac{\pi B^2\theta_j}{8c^2}\sum_{r=4,6,\cdots}^{\nu+1}\frac{\Delta w_{nr}(\theta_j)\Delta\bar{w}_{mr}(\theta_j)}{r}\right\}.$$

As in the previous section, the values of the coefficients K_j may be computed using Reference 12. For the particular case where $N = 6$, the coefficients are given by Eq. (52).

Eq. (56) gives values of the influence coefficient, h_{mn}, when the deflection shapes $Z_n(X, Y)$ and $Z_m(X, Y)$ are known along with the geometry of the wing and the reduced frequency. Values of $m_n(X)$ are obtained from Eqs. (41) through (44) and Eqs. (19) through (31).

CALCULATION OF THE WING FLUTTER SPEEDS AND FREQUENCIES

Consider a low aspect ratio wing to be in a state of undamped flutter at a flight speed and frequency at which such a motion exists. The maximum deflection of the wing is then $Z(X,Y)$ which is either symmetric or antisymmetric (or a combination of both) depending upon whether the lift per unit area developed on the wing is symmetric or antisymmetric (or a combination of both). The structural load per unit area, $S[Z]$ which results from the wing being deformed into the shape $Z(X,Y)$ and the inertial force per unit area, $-\gamma(X,Y)\,\omega^2 Z(X,Y)$ are related to the aerodynamic lift per unit area by the equation of motion of the wing

$$S\left[Z\right]+\gamma(X,Y)\omega^2 Z(X,Y)=\frac{dL}{dS}(X,Y) \tag{57}$$

where $\gamma(X,Y)$ is the mass per unit area of the wing and ω is the flutter frequency. The notation $S\left[Z\right]$ indicates that the load function may contain derivatives and integrals of Z_n as well as powers.

Introducing the reduced frequency $k=\omega c/U$, Eq. (57) becomes

$$S\left[Z\right]+\frac{k^2U^2}{c^2}\gamma(X,Y)\,Z(X,Y)=\frac{dL}{dS}(X,Y)\,. \tag{58}$$

Assume, now, that the deflection shape $Z(X,Y)$ can be expanded in a series of the form

$$Z(X,Y)=\sum_{n=1}^{R} B_n Z_n(X,Y) \tag{59}$$

where the functions $Z_n(X,Y)$ are often taken as normal oscillation modes for simplicity.

It will be assumed that the structural force function $S\left[Z\right]$ is a linear function of the deflection $Z(X,Y)$ and its derivatives or integrals, if any. Then,

$$S\left[Z\right]=\sum_{n=1}^{R} B_n S\left[Z_n\right]\,. \tag{60}$$

The aerodynamic forces are known to be linear. Hence, with the notation

$$\left[\frac{dL}{dS}(X,Y)\right]_n = \left[\frac{dL}{dS}(X,Y)\right]_{z=z_n} ,$$

$$\frac{dL}{dS}(X,Y) = \sum_{n=1}^{R} B_n\left[\frac{dL}{dS}(X,Y)\right]_n . \tag{61}$$

Upon combining Eqs. (58) through (61), there results

$$\sum_{n=1}^{R} B_n\left\{S\left[Z_n\right] + \frac{k^2U^2}{c^2}\gamma(X,Y)Z_n(X,Y)\right\} = \sum_{n=1}^{R} B_n\left[\frac{dL}{dS}(X,Y)\right]_n \tag{62}$$

Now, multiply both sides of Eq. (62) by $Z_m(X,Y)$ and integrate over the wing surface, which, using the definition given by Eq. (45), yields

$$\sum_{n=1}^{R} B_n\iint_S Z_m(X,Y)\left\{S\left[Z_n\right] + \frac{k^2U^2}{c^2}\gamma(X,Y)Z_n(X,Y)\right\}dX\,dY = 2\rho U^2c^3\sum_{n=1}^{R} B_n h_{mn} . \tag{63}$$

If, $$P_{mn} = \iint_S Z_m(X,Y)S\left[Z_n\right]dX\,dY \tag{64}$$

$$Q_{mn} = \iint_S \gamma(X,Y)Z_m(X,Y)Z_n(X,Y)\,dX\,dY \tag{65}$$

then, Eq. (63) may also be written as

$$\sum_{n=1}^{R} B_n\left(P_{mn} + \frac{k^2U^2}{c^2}Q_{mn}\right) = 2\rho U^2c^3\sum_{n=1}^{R} B_n h_{mn} \tag{66}$$

If the functions $Z_n(X,Y)$ are coupled normal modes considerable simplification is possible. For example, the function Q_{mn} is now subject to orthogonality principles (Reference 17) and can be written as

$$Q_{mn} = Q_{mn}\,\delta_{mn} \tag{67}$$

where, δ_{mn} the Kronecker delta, is zero when $m \neq n$ and unity when $n = m$.

Furthermore, when $Z_n(X,Y)$ are normal modes, Eq. (58) will become

$$S\left[Z_n\right] + \left(\frac{k^2U^2}{c^2}\right)_n \gamma(X,Y)\,Z_n(X,Y) = 0 \tag{68}$$

for, in a normal mode, the wing will oscillate with no external force applied at one of its natural frequencies,

If Eq. (68) is multiplied by $Z_m(X,Y)$ and integrated over the wing surface, there results

$$\begin{aligned} P_{mn} &= -\left(\frac{k^2U^2}{c^2}\right)_n \iint_S \gamma(X,Y)\,Z_m(X,Y)\,Z_n(X,Y)\,dX\,dY = \\ &-\left(\frac{k^2U^2}{c^2}\right)_n Q_{mn} = -\left(\frac{k^2U^2}{c^2}\right)_n Q_{mn}\,\delta_{mn}\,. \end{aligned} \tag{69}$$

Thus, Eqs. (67) through (69) can be used to simplify Eq. (66) to the form

$$B_m\,Q_{mn}(k^2 - k_m^2) = 2\rho c^5 \sum_{n=1}^{R} B_n\,h_{mn} \tag{70}$$

when the functions $Z_n(X,Y)$ are normal modes.

Eq. (66), representing the general case of arbitrary $Z_n(X,Y)$ functions, can conveniently be put into matrix form as

$$\frac{1}{2\rho U^2 c^3}\left[\left[P_{mn}\right] + \frac{k^2U^2}{c^2}\left[Q_{mn}\right]\right]\left\{B_n\right\} = \left[h_{mn}\right]\left\{B_n\right\} \tag{71}$$

or

$$\left| P_{mn} + \frac{k^2U^2}{c^2} Q_{mn} - 2\rho U^2 c^3 h_{mn} \right| = 0 \tag{72}$$

In Eqs. (71) and (72), the functions Q_{mn} and P_{mn} are known completely once the physical data for the wing and the functions $Z_n(X,Y)$ are specified. The function h_{mn} is known, for a specific wing case, as a function of the unknown reduced frequencies k . Thus, for a given wing case at a selected reduced frequency, Eqs. (71) and (72) may be solved for the various critical flight speeds. At each of these results, the oscillation frequency, ω , is specified because of the preselected reduced frequency.

The method of solution of Eqs. (71) or (72) is left to the discretion of the reader. Such methods as found in References 13 and 14 are frequently employed.

COMPUTATION OF VARIOUS AERODYNAMIC COEFFICIENTS USING THE h_{mn} FUNCTIONS

In order to illustrate the physical meaning of the influence functions it will be noted that they may be used to evaluate aerodynamic coefficients (References 8, 15, 16). An outline is given on the opposite page for finding various aerodynamic coefficients from the h_{mn} values computed by Eqs. (51) and (56) for the respective cases of symmetrically and antisymmetrically oscillating low aspect ratio wings. No theoretical derivation is given, since it is felt that the underlying principles are easily deduced.

Several other coefficients not tabulated may be evaluated by similar procedures.

1. Low aspect ratio wing in steady symmetric flight.

Coefficient	k	$Z_n(X,Y)$	$Z_m(X,Y)$	Equation (h_{mn} from Eq. (51)
C_{L_α}	0	X	c	$(c^2 A\!\!R/B_0^2)h_{mn} = C_{L_\alpha}$
C_{M_α}	0	X	$c+X$	$(c^2 A\!\!R/2B_0^2)h_{mn} = C_{M_\alpha}$
$(C_{L_\delta})_{\alpha=0}$	0	0 for $X \leq X_f$ X for $X \geq X_f$	c	$(c^2 A\!\!R/B_0^2)h_{mn} = (C_{L_\delta})_{\alpha=0}$
$(C_{M_\delta})_{\alpha=0}$	0	0 for $X \leq X_f$ X for $X \geq X_f$	$c+X$	$(c^2 A\!\!R/2B_0^2)h_{mn} = (C_{M_\delta})_{\alpha=0}$
$(C_{h_\delta})_{\alpha=0}$	0	0 for $X \leq X_f$ X for $X \geq X_f$	$X-X_f$ for $X \geq X_f$ 0 for $X \leq X_f$	$(4c^3/b_f \bar{c}_f^2)h_{mn} = (C_{h_\delta})_{\alpha=0}$
$(C_{h_\alpha})_{\delta=0}$	0	X	$X-X_f$ for $X \geq X_f$ 0 for $X \leq X_f$	$(4c^3/b_f \bar{c}_f^2)h_{mn} = (C_{h_\alpha})_{\delta=0}$
C_{m_q}	0	$\frac{X}{c}\left(\frac{X}{2}+c\right)$	$c+X$	$-2A\!\!R(c/\bar{c})^2(c/B_0)^2 h_{mn} = C_{m_q}$

2. Low aspect ratio wing in steady antisymmetric flight.

Coefficient	k	$Z_n(X,Y)$	$Z_m(X,Y)$	Equation (h_{mn} from Eq. (56)
C_{ℓ_δ}	0	0 for $X \leq X_f$ $XY/\lvert Y \rvert$ for $X \geq X_f$	Y	$(c^3 A\!\!R/2B_0^3)h_{mn} = C_{\ell_\delta}$
C_{ℓ_p}	0	$\frac{1}{c}\int_{-c}^{c} Y\,dX$	Y	$-(c^4 A\!\!R/2B_0^4)h_{mn} = C_{\ell_p}$

3. Low aspect ratio wing in unsteady symmetric flight.

Coefficient	k	$Z_n(X,Y)$	$Z_m(X,Y)$	Equation (h_{mn} from Eq. (51)
L_h	k	$c^3/2k^2B_0^2$	c	$-(2iA\!\!R/\pi k)h_{mn} = L_h$
M_h	k	$c^3/2k^2B_0^2$	$X+0.5c$	$-(2iA\!\!R/\pi k)h_{mn} = M_h$

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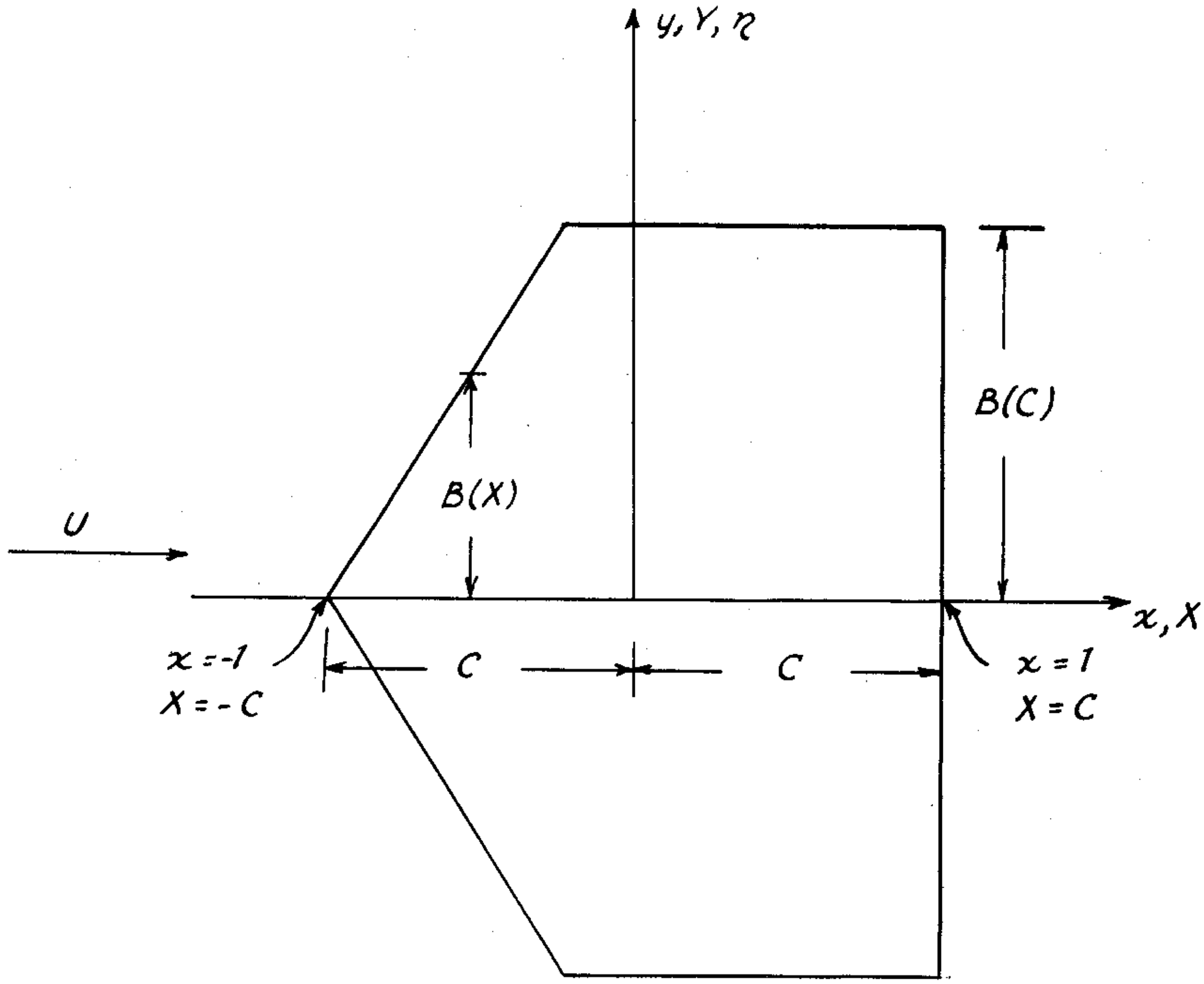

Z is positive downward

Fig. 1

COORDINATE SYSTEM FOR OSCILLATING WING