AFFDL-TR-68-150

A FINITE ELEMENT APPLICATION OF THE HELLINGER–REISSNER VARIATIONAL THEOREM*

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The Hellinger-Reissner Variational Theorem of linear elastostatics is used to construct a Finite Element Method of solution for boundary value problems in generalized plane stress. A "stiffness" matrix is derived for the case where stress and displacement fields are linear at the element level. Two example problems are included.

* This paper was partially supported by the Army Research Office, Durham, North Carolina and the Naval Ordnance Test Station, China Lake, California.

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NOMENCLATURE

a, $\{a\}$	generalized coordinates for displacements
α_{ij}	coefficient of thermal expansion tensor
$\{\alpha T\}$	thermal expansion vector
b, $\{b\}$	generalized coordinates for stress
C_{ijkl}, $\{C\}$	strain-stress constitutive tensor
∇	gradient operator
δ	variation symbol
f_i, $\{f\}$	body force per unit volume
$\bar{p}_i$, $\{\bar{p}\}$	prescribed traction on S
R	region occupied by the solid body
$[S]$	stiffness matrix
S_τ	surface of prescribed stresses
S_u	surface of prescribed displacements
T	temperature above a reference temperature
τ_{ij}, $\{\tau\}$	stress tensor
$[\Phi_\tau]$	coordinate functions for stress
u_i, $\{u\}$	displacement vector
$\bar{u}_i$, $\{\bar{u}\}$	prescribed displacement on S_u
$[\Phi_u]$	coordinate functions for displacement
V	Hellinger-Reissner functional
W	complementary energy density

SECTION I

INTRODUCTION

A method often used for the solution of initial-boundary value problems in structural mechanics has been the Finite Element Method (References 1, and 2). Many applications of the Finite Element Method in linear elastostatics have been derived from the Theorem of Minimum Potential Energy or its equivalent, the Principle of Virtual Work (Displacements). Consequently, the set of primary field variables is the displacement field; strains and stresses are then expressed as functions of the displacements through the field equations.

It has been assumed that the Finite Element Method yields displacements and stresses which converge as the mesh size is decreased (References 3, 4, and 5). Smooth convergence of the displacement field has been widely reported for many applications (Reference 2 a), some violating certain of the compatibility conditions (Reference 1 a). On the other hand, smooth convergence of the stress field has not always been observed; poor convergence and spatial oscillations are found (Reference 6). Several authors have given stress "averaging" techniques (References 7 and 8). For many engineering purposes it must be recognized that a knowledge of the stress distribution even in localized regions, is the desired result and the displacements are of secondary interest.

Some of these difficulties can be circumvented by mesh resolution and adoption of averaging techniques mentioned above in localized regions with high gradients. However, for some problems this is impossible or impracticable and other alternatives must be sought. One alternative is to employ a higher order displacement expansion. Investigators have pursued this approach with encouraging results (Reference 9 and 10); however, the stress vector remains discontinuous over contiguous element interfaces, giving a stress distribution with a histograph appearance.

Another approach is to obtain the stress directly as a primary variable through an application of the Theorem of Minimum Complementary Energy. Investigators have had limited success with the method, one difficulty being to establish a stress distribution identically satisfying the equilibrium equations and expressible in terms of a convenient set of global generalized coordinates. At present the method has not been applied with the same degree of generality as the displacement method (References 2b, 11, 12 and 13). It is surprising that applications of mixed variational theorems have not been explored in this regard since they offer the generality of the displacement method and retain the stress field as a primary variable. To this end the present paper introduces a Finite Element application of the Hellinger-Reissner Variational Theorem.

It is shown that expansions for the displacement and stress fields can be made independently in Finite Element analyses and that continuity of the stress vector can be maintained across contiguous element interfaces. Stresses from a mixed variational theorem are shown to be improved over those of comparable displacement models. The limitation principle (Reference 2b) is discussed in this connection and is shown to limit the scope of mixed variational theorems.

A stiffness matrix for plane and axisymmetric solids is discussed and several examples are presented with comparisons made to a frequently used displacement model (Reference 17).

SECTION II

HELLINGER-REISSNER THEOREM

The functional in the theorem is expressed in terms of the displacement and stress field* (Reference 14)

$$V(u,\tau) = \int_R \left[\tau_{ij}\, u_{(i,j)} - W(\tau) - f_i\, u_i\right] dv - \int_{S_\tau} \overline{p}_i\, u_i\, da - \int_{S_u} n_j\, \tau_{ji}\,(u_i - \overline{u}_i)\, da \qquad (1)$$

where

$$W(\tau) = \tau_{ij}\left(\frac{1}{2} C_{ijkl}\, \tau_{kl} + \alpha_{ij}\, T\right) \qquad (2)$$

The theorem asserts that of all admissible** displacement and stress states, those states that satisfy the stress equations of equilibrium, the strain stress equations and the boundary conditions render the functional stationary

$$\delta V(u,\tau) = 0 \qquad (3)$$

Unlike the minimum principles the stationary value cannot be shown to be an extremuum value of the functional (Reference 14).

The form of the functional is such that separate expansions are possible without the restrictions necessary in the minimum principles. The stress field need not satisfy the equilibrium or stress-strain-displacement equations in R, nor need either the displacement or stress fields satisfy the prescribed boundary conditions. The Euler Equations are the stress equations of equilibrium, the strain-stress law and the boundary conditions. Since the stationary value of the functional contains the constitutive equation as an Euler Equation, the stress field obtained through the stress-strain-displacement relations may not agree with the stress field given by the stationary value.

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* Standard tensor notation is used and a cartesian reference frame is assumed; comma denotes differentiation, repeated indices are summed, and parentheses denote the symmetric part of a tensor. Interpretation of the symbols is given in the Appendix.

** Admissibility and convergence questions are beyond the scope of the present work (Reference 15).

If independent expansions are used for both fields, where components of the displacement and stress fields are adjoined across contiguous element boundaries using a system of common displacement and stress nodal points, then a direct solution for the primary variables should differ from corresponding applications of the minimum principles. However, the limitation principle requires that the stress variables be left free (Reference 2b). The notion of connectivity in the stress variables is central to this concept. To illustrate this, consider the following*:

$$u_i = \phi_i^n (x_p) a_n \tag{4}$$

$$\tau_{ij} = \psi_{ij}^n (x_p) b_n \tag{5}$$

where the a's and b's are the generalized coordinates for displacement and stress, respectively. Suppose there exists a set of scalars β_n^m such that

$$\psi_{ij}^n (x_p) \beta_n^m = C_{ijkl} \phi_{k,l}^m (x_p) \tag{6}$$

Then if there is no interelement connectivity in the stress variables, Euler Equation of the functional is identically satisfied. The stationary value yields, in part,

$$\frac{\partial V}{\partial \tau_{ij}} = 0 \Rightarrow \frac{\partial V}{\partial b_n} = 0 \tag{7}$$

using Equation 4 in Equation 1, and assuming

$$\begin{aligned} u_i &= \overline{u}_i \quad \text{on } S_u \\ f_i &= 0 \quad \text{in } R \\ T &= 0 \end{aligned} \tag{8}$$

Equation 7 becomes

$$\int_R \left[\phi_{i,j}^n a_n - C_{ijkl} \psi_{kl}^n b_n\right] \psi_{ij}^m \delta b_m \, dv = 0 \tag{9}$$

from Equation 6, if

$$b_n = \beta_n^m a_m$$

then

$$(\phi_{i,j}^m - C_{ijkl} \psi_{kl}^n \beta_n^m) a_m \equiv 0 \tag{10}$$

* The superscripts refer to generalized coordinates and are treated as ordinary indices.

Equations 6 and 10 imply that the stress is computed from the stress-strain-displacement relations and

$$\tau_{ij} = C^{-1}_{ijkl} \; u_{k,l} \tag{11}$$

thus

$$\tau_{ij} \, u_{(i,j)} - W(\tau) = \frac{1}{2} u_{i,j} \, C^{-1}_{ijkl} \; u_{k,l} \tag{12}$$

in which case Equation 12 and Equation 8 imply the functional in Equation 1 can be reduced to

$$V(u) = \int_R \frac{1}{2} u_{i,j} \, C^{-1}_{ijkl} \, u_{k,l} \, dv - \int_{S_\tau} \bar{p}_i \, u_i \, da. \tag{13}$$

Equation 13 is the functional in the Theorem of Minimum Potential Energy.

The above is a demonstration of the limitation principle which is seen to hold in the restrictive case where the stress variables are not connected from element to element.

SECTION III

A FINITE ELEMENT MODEL

Following the approach in (Reference 16) a linear expansion is taken for both the stress and displacement, and a system of generalized coordinates is established at the vertices of a triangular element for a plane stress or axisymmetric solid. Matrix notation is employed for convenience. Displacement boundary conditions are to be satisfied in the usual fashion. Derivation of an element stiffness matrix for a two-dimensional plane stress solid is given for a linear triangular element. A more complete exposition for both plane and axisymmetric solids is given in (Reference 16).

For the stated application, the functional in Equation 1 can be expressed

$$V(u,\tau) = \int_R \left(\langle\tau\rangle \{\nabla u\} - W(\tau) - \langle u\rangle \{f\} \right) dv - \int_{S_\tau} \langle u\rangle \{\bar{p}\}\, da \tag{14}$$

where

$$W(\tau) = \langle\tau\rangle \left(\frac{1}{2} [C] \{\tau\} + \{\alpha T\} \right) \tag{15}$$

and the brackets $\langle\ \ \rangle$ and $\{\ \ \}$ are used to represent row and column matrices, respectively. The coordinate functions expansions are

$$\{u\} = [\Phi_u]\{a\}$$
$$\{\tau\} = [\Phi_\tau]\{b\} \tag{16}$$

where

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \langle\phi\rangle & \langle 0\rangle \\ \langle 0\rangle & \langle\phi\rangle \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \langle\phi\rangle & \langle 0\rangle & \langle 0\rangle \\ \langle 0\rangle & \langle\phi\rangle & \langle 0\rangle \\ \langle 0\rangle & \langle 0\rangle & \langle\phi\rangle \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_9 \end{bmatrix} \tag{18}$$

$$\langle\phi\rangle = \langle 1,\ x,\ y\rangle \tag{19}$$

and the displacement gradients are

$$\{\nabla u\} = [\Phi'_u]\{a\} \tag{20}$$

The generalized coordinates are expressed in terms of the values of the physical components of the displacement and stress fields at the element vertices, (Figure 1)

$$\{u_o\} = [\Phi_{u_o}]\{a\}$$
$$\{\tau_o\} = [\Phi_{\tau_o}]\{b\} \tag{21}$$

where

$$\langle u_o \rangle = \langle u_i, u_j, u_k, v_i, v_j, v_k \rangle$$
$$\langle \tau_o \rangle = \langle \tau_{xx_i}, \tau_{xx_j}, \tau_{xx_k}, \tau_{yy_i}, \cdots \tau_{xy_k} \rangle \tag{22}$$

$$[\Phi_{u_o}] = \begin{bmatrix} [\phi_o] & 0 \\ 0 & [\phi_o] \end{bmatrix} \tag{23}$$

$$[\Phi_{\tau_o}] = \begin{bmatrix} [\phi_o] & 0 & 0 \\ 0 & [\phi_o] & 0 \\ 0 & 0 & [\phi_o] \end{bmatrix} \tag{24}$$

$$[\phi_o] = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \tag{25}$$

Thus

$$\{a\} = [\Phi_{u_o}^{-1}]\{u_o\}$$
$$\{b\} \quad [\Phi_{\tau_o}^{-1}]\{\tau_o\} \tag{26}$$

and

$$\{u\} = [\Phi_u][\Phi_{u_o}^{-1}]\{u_o\}$$
$$\{\tau\} = [\Phi_\tau][\Phi_{\tau_o}^{-1}]\{\tau_o\} \tag{27}$$
$$\{\nabla u\} = [\Phi'_u][\Phi_{u_o}^{-1}]\{u_o\}$$

The functional becomes

$$V = \frac{1}{2} < \tau_o > [S_{\tau\tau}] \{\tau_o\} + < \tau_o > [S_{\tau u}] \{u_o\} - < \tau_o > \{F_\tau\} - < u_o > \{F_u\} \tag{28}$$

where

$$[S_{\tau\tau}] = -[\Phi^{-1}_{\tau_o}]^T \left(\int_R [\Phi_\tau]^T [c][\Phi_\tau]\, dv \right) [\Phi^{-1}_{\tau_c}]$$

$$[S_{\tau u}] = [\Phi^{-1}_{\tau_o}]^T \left(\int_R [\Phi_\tau]\, [\Phi'_u]\, dv \right) [\Phi^{-1}_{u_o}]$$

$$\{F_\tau\} = [\Phi^{-1}_{\tau_o}]^T \left(\int_R [\Phi_\tau]^T \{\alpha T\}\, dv \right) \tag{29}$$

$$\{F_u\} = [\Phi^{-1}_{u_o}]^T \left(\int_R [\Phi_u]^T \{f\}\, dv + \int_{S_\tau} [\Phi_u]^T \{\bar{p}\}\, da \right)$$

The stationary value yields the governing equations

$$\begin{bmatrix} S_{\tau\tau} & S_{\tau u} \\ S^T_{\tau u} & 0 \end{bmatrix} \begin{bmatrix} \tau_o \\ u_o \end{bmatrix} = \begin{bmatrix} F_\tau \\ F_u \end{bmatrix} \tag{30}$$

The linear triangular element is used to form the quadrilateral element shown in Figure 1. Condensation is then performed on the stiffness matrix in order to reduce the bandwidth of the governing Equations 9 and 17. However, in the case of the present mixed model this cannot be effected directly due to a singularity found during the Gaussian elimination. What is done

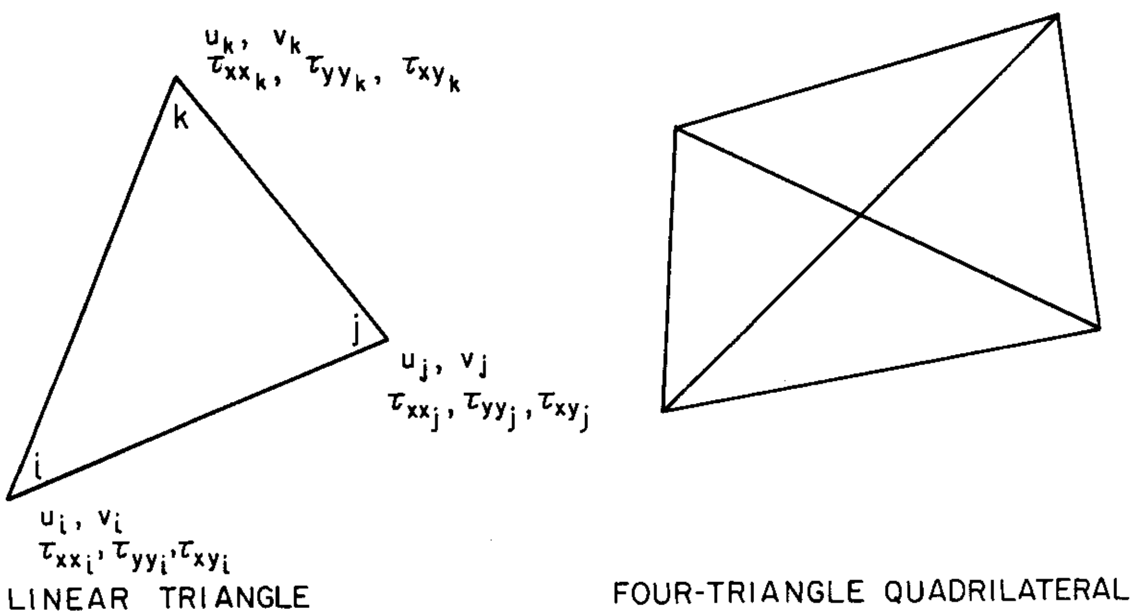

Figure 1. Four-Triangle Quadrilateral

is to constrain the displacements at the center node to be the average of the four exterior nodes

$$u_5 = \frac{1}{4}(u_1 + u_2 + u_3 + u_4)$$
$$v_5 = \frac{1}{4}(v_1 + v_2 + v_3 + v_4) \tag{31}$$

or

$$\{u_0\} = [\Psi]\{u_0\}_C \tag{32}$$

where

$$[\Psi] = \begin{bmatrix} [\Psi] & [0] \\ [0] & [\Psi] \end{bmatrix} \tag{33}$$

and

$$[\Psi] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \tag{34}$$

The constrained stiffness matrix becomes

$$[S_{\tau u}]_C = [S_{\tau u}][\Psi] \tag{35}$$

Standard techniques are used to solve the resulting system of linear algebraic Equation 30. Although Equation 30 is not a positive definite system, no difficulty is encountered in Gaussian elimination if nodal point stress are eliminated before the displacements.

SECTION IV

EXAMPLES

To illustrate the method, two example problems are given with direct solutions obtained using the mixed model and a comparable displacement model Reference 17.

A boundary value problem with an applied linear stress distribution is shown in Figure 2. The problem is equivalent to combined extension and flexure in elementary beam theory. The two-dimensional plane stress solution is

$$\begin{aligned} &\tau_{xx} = y, \quad \tau_{xy} = \tau_{yy} = \tau_{zz} = 0 \\ &\epsilon_{xx} = \frac{y}{E}, \quad \epsilon_{yy} = -\frac{\nu y}{E}, \quad \epsilon_{xy} = 0 \\ &u(x,y) = \frac{xy}{E}, \quad v(x,y) = -\frac{1}{2E}(x^2 + \nu y^2) \end{aligned} \tag{36}$$

The linear stress distribution is contained within the expansion for the stress field in Equation 16, but the displacement field is parabolic.

Figure 2 shows a plot of the stress field along the top fiber of the beam and through the depth at the edge of applied traction. The mixed model gives accurate values for the stress field including those points on the boundary. The displacement model gives values for the stress only at the element centroid. The mixed model also predicts the deformed profile with considerable accuracy.

Figure 3 illustrates a finite length hollow cylinder, fixed at the outer radius and acted upon by an internal pressure. The geometry and physical properties are

$$\begin{aligned} &a = 4, \quad b = 6, \quad \ell = 5.5 \\ &\frac{p}{E} = 1, \quad \nu = 0.4 \end{aligned} \tag{37}$$

A plot of the normal stress distribution along the fixed boundary is given in Figure 3 for the two Finite Element models. The stress singularity is seen to be more severe in the mixed model, the displacement model suppressing much of the gradient in the vicinity of the singularity.

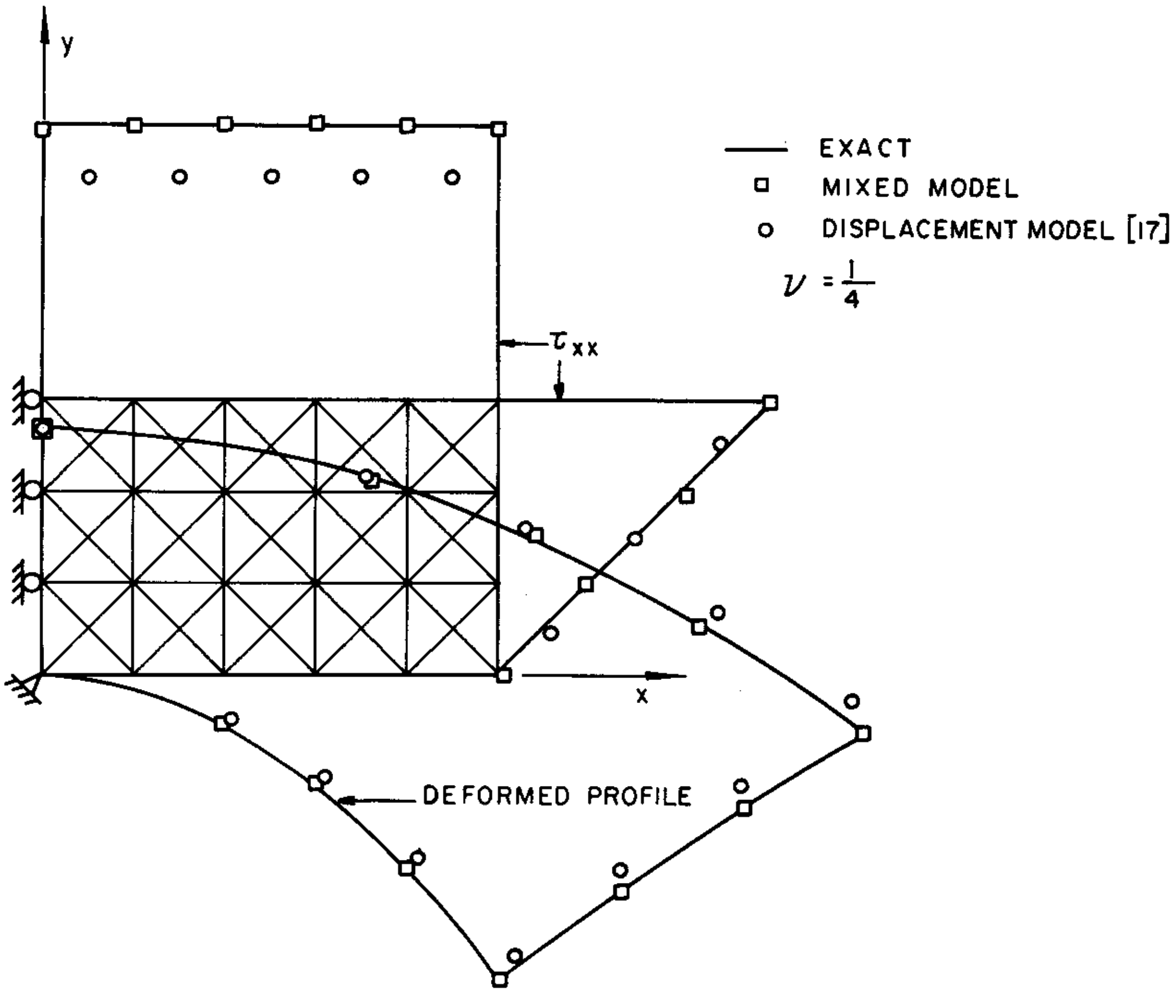

Figure 2. A Linear Stress Problem

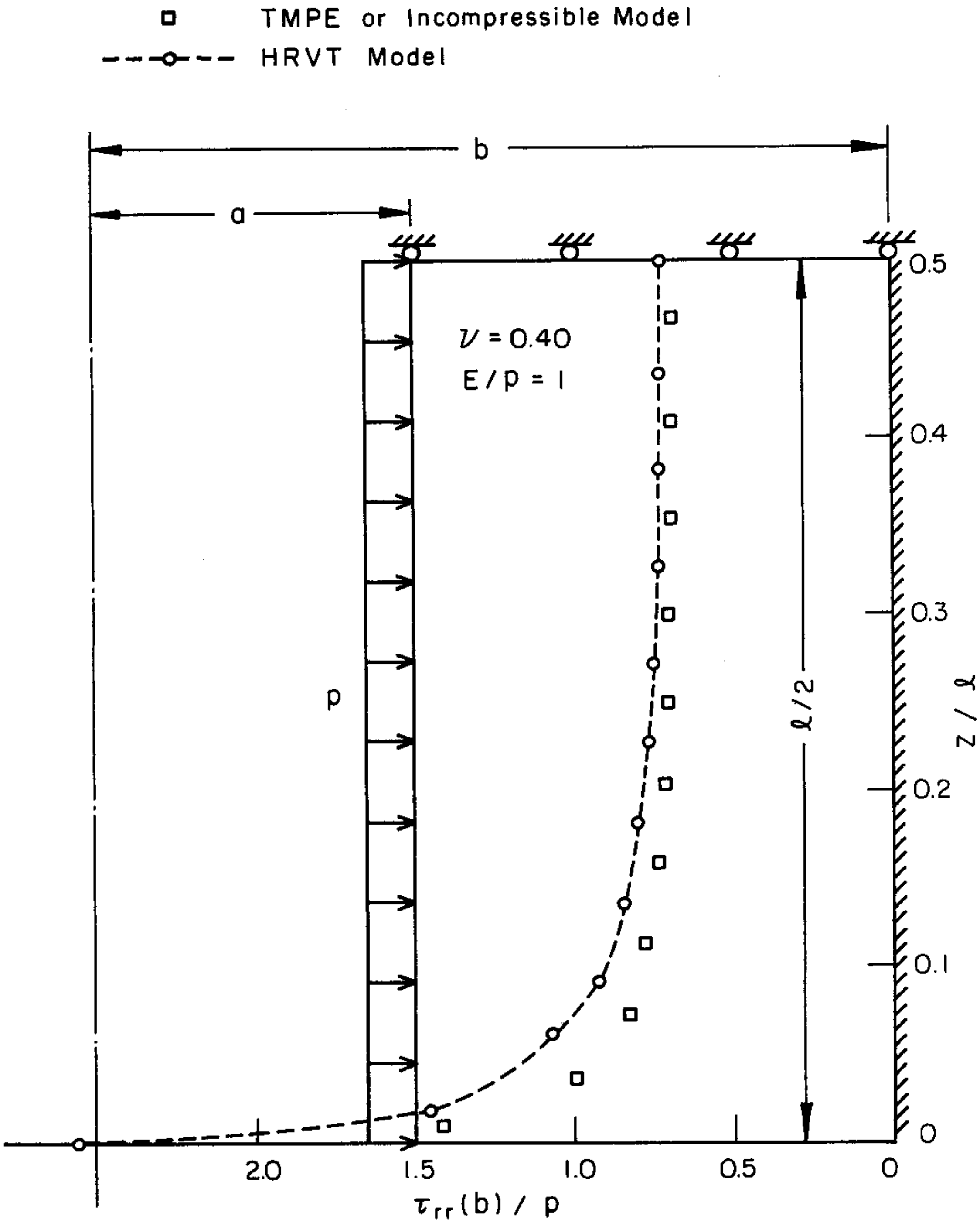

Figure 3. Pressurized Finite Cylinder

SECTION V

CONCLUSIONS

For the examples considered the present application of the Hellinger-Reissner Theorem gives a more accurate field description of both displacement and stress than do existing applications of displacement models. The mixed model yields solutions in which all components of the stress tensor are continuous from element to element, eliminating the histograph distribution often found in displacement models. The displacement vector obtained from the mixed model is considerably more accurate than that given by comparable displacement models. Results for the stress field are superior, but tend to be dependent on material properties and mesh configuration. In part, this can be attributed to identically satisfying the displacement boundary conditions. Since the stress distribution is the desired result, it would be advantageous to bias the model in this direction. An alternate form of the functional which accomplishes this is currently being studied.*

The mixed model is shown to be highly effective in capturing steep stress or displacement gradients that can occur near singularities in boundary value problems. These are often not predicted by displacement models. The mixed model is most effectively used for this class of problems since an excess of generalized coordinates can require considerable computational effort for a general problem.

The present effort should be regarded as introductory, and considerable additional research is indicated. Alternate forms should be considered. The number of generalized coordinates should be reduced by employing other coordinate functions and element configurations. Herrmann has shown that this can be done for plate bending (Reference 1b), and Prager (Reference 18), has extended and systematized the relevant principles. What has been clearly demonstrated here is that Finite Element Methods can be successfully used with mixed variational theorems for direct solutions to boundary value problems in mechanics.

*This form has been suggested by S. Pawsey in private communications.

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