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APPROXIMATE STRUCTURAL ANALYSIS TECHNIQUES  
EMPLOYING UNDETERMINED PARAMETERS

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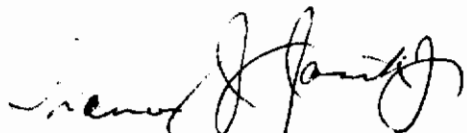
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FOREWORD

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## ABSTRACT

A unified approach for the approximate solution of boundary value problems by undetermined parameters is presented. It is demonstrated how the general technique reduces to the well known methods of Ritz, Galerkin, least-squares, collocations, Mikhlín, and point-matching, as well as some new methods. Two examples from plate-bending theory and one from steady-state heat conduction are treated numerically by the various methods to demonstrate their application. Based on detailed results, comparisons are made among the methods as to accuracy, ease of application, and over-all applicability. The feasibility of automating several of the methods is demonstrated. Suggestions are made concerning selection of trial functions and methods to be used in connection with digital computer applications.

## CONTENTS

<u>SECTION</u>		<u>PAGE</u>
I	INTRODUCTION	1
II	GENERAL METHOD OF UNDETERMINED PARAMETERS	2
	A. Introduction	2
	B. Mathematical Preliminaries	2
	C. Statement of Problem and General Solution Method	11
III	RELATIONSHIP BETWEEN THE GENERAL TECHNIQUE AND SPECIFIC METHODS	15
	A. Discussion of the Well-Known Methods	15
	B. Reduction of General Technique to Specific Method	24
	C. New Methods	28
	D. Summary of General Method	33
IV	APPLICATIONS OF THE METHODS	38
	A. Introduction	38
	B. Example 1 - Clamped Square Plate with a Uniform Load	40
	C. Example 2 - Heat Conduction in Annular Wedge	83
	D. Example 3 - Clamped Rectangular Plate with Central Hole	114
V	DISCUSSION	139
	A. Summary of Advantages and Disadvantages	139
	B. Accuracy and Ease of Application	140
	C. Digital Computer Automation	140
	D. Conclusions	141

CONTENTS (Cont'd)

	<u>PAGE</u>
APPENDIX I      ALGEBRAIC LEAST-SQUARES	142
APPENDIX II      EQUIVALENCE OF THE RITZ AND GALERKIN METHODS FOR THE HARMONIC AND BIHARMONIC OPERATORS	145
ACKNOWLEDGEMENTS	148
REFERENCES	149

## ILLUSTRATIONS

<u>FIGURE</u>		<u>PAGE</u>
1	One-Dimensional Heaviside Step Function	7
2	Combination of Heaviside Functions to Give a Uniform Pulse	8
3	Two-Dimensional Subdomain	9
4	Domain D with Boundaries B and Pertinent Variables	11
5	Rectangular Domain with Subdomain $D_k$	27
6	"Sub-Boundaries" for Modified Biezeno-Koch Method	31
7	Numerical Examples Considered	39
8	Subdomains used for a Four Parameter Biezeno-Koch Solution of Example 1	46
9	One Parameter Ritz-Galerkin Boundary and Interior Errors for Example 1	54
10	One Parameter Least-Squares Boundary and Interior Errors for Example 1	55
11	Biezeno-Koch Boundary and Interior Errors for Example 1	56
12	Four Term Collocation Boundary and Interior Errors for Example 1	57
13	Least-Squares Collocation Boundary and Interior Errors for Example 1	58
14	Mikhlin Boundary and Interior Errors for Example 1	59
15	4x4 Point-Matching Interior and Boundary Errors for Example 1	60
16	Equal Spaced Point-Matching Errors for Example 1	61
17	Comparison of Deflection Errors for Example 1	63
18	Comparison for Moment Errors for Example 1	65
19	Comparison of Slope Errors for Example 1	67
20	Comparison of Residuals for Example 1	69
21	Biezeno-Koch Residuals for Example 1	71
22	Deflection Along Diagonal for Example 1	72

## ILLUSTRATIONS (Cont'd)

<u>FIGURE</u>		<u>PAGE</u>
23	Approximate Moments along X-Axis for Example 1	73
24	Comparison of Particular Solutions for Example 1	82
25	Example 2, Range of Interior Temperature and Residual Errors for One-Term Ritz-Galerkin, Least-Squares, Collocation, and Biezeno-Koch Solutions	92
26	Example 2, Range of Residual Errors on the Boundaries for One-Term Ritz-Galerkin, Least-Squares, Collocations and Biezeno-Koch Solutions	94
27	Example 2, Comparison of Temperature Errors for Four-Term Point-Matching, Mikhlin, and Least-Squares Point-Matching Solutions	96
28	Example 2, Maximum Residuals and Temperature Errors vs Number of Terms for Ritz-Galerkin and Collocation Solutions	107
29	Ritz-Galerkin and Least-Squares Collocation Residuals and Temperature Errors for Example 2	108
30	Ritz-Galerkin and Collocations Residuals and Temperature Errors for Example 2	111
31	Ritz-Galerkin, Collocation and Least-Squares Collocations Residuals and Temperature Errors for Example 2	113
32	Loading Selected for Example 3	115
33	Example 3: Comparison of Deflections for One-Term Ritz-Galerkin, Biezeno-Koch and Least-Squares Collocations	118
34	Example 3: Comparison of Moments for One-Term Ritz-Galerkin, Biezeno-Koch and Least-Squares Collocations	119
35	Example 3: Comparison of Residual Errors for One-Term Ritz-Galerkin, Biezeno-Koch and Least-Squares Collocations	120
36	Collocation Points used for Example 3	121
37	Example 3: Comparison of Exact Solution with 5x5 Collocation and 15x5 Least-Squares Collocation Solutions	122
38	Exact Solution for Example 1	123

# Contrails

## TABLES

<u>TABLE</u>		<u>PAGE</u>
I	Tabulation of Known Approximate Methods by Undetermined Parameters	35
II	Point-Matching Solutions to Example 1	75
III	Collocations Solutions to Example 1	76
IV	Least-Squares Collocation Solutions to Example 1	77
V	Biezeno-Koch Clamped Solutions to Example 1	79
VI	Effect on Solution Accuracy of Number of Parameters and Particular Solution for Example 1	81
VII	Point Matching Results for Example 2	100
VIII	Least-Squares Point Matching Results for Example 2	101
IX	Collocation Results for Example 2	102
X	Ritz-Galerkin Results for Example 2	105
XI	"Exact" Solution for Example 1	130
XII	One Term Ritz-Galerkin Solution for Example 1	131
XIII	One Term Least-Squares Solution for Example 1	132
XIV	Four Term Biezeno-Koch Solution for Example 1	133
XV	Four Term Collocation Solution for Example 1	134
XVI	30x4 Least-Square Collocation Solution for Example 1	135
XVII	Four Term Mikhlin Solution for Example 1	136
XVIII	Four Term Point Matching Solution for Example 1	137
XIX	Four Term Equal Spaced Point Matching Solution for Example 1	138



## SYMBOLS

$\underline{a}, a_k$	Undetermined parameters ( $k = 1, 2, \dots, n$ )
$a, b, R, \theta_0$	Plate dimensions
$\{a\}$	Column matrix with elements $a_1, a_2, \dots, a_m$
$\{b\}$	Column matrix with elements $b_1, b_2, \dots, b_n$
B	Boundary
$B_i$	Differential Operator
B-K	Biezeno-Koch
$[c]$	$m \times n$ matrix with elements $c_{11}, c_{12}, \dots, c_{21}, c_{22}, \dots, c_{mn}$
COL	Collocation
D	Domain
$D_k$	Subdomain
ESPM	Equal spaced point matching
$f, F, g$	Functions
h	Homogeneous solution (subscript)
$H(x-\bar{x})$	Heaviside step function
$H_{S.D.}$	Two dimensional Heaviside step function
I	Function
L, M, $M_i$	Differential operators
LS	Least Squares
LS-COL	Least Squares Collocation
LS-PM	Least Squares Point Matching
m	Number of undetermined parameters
MIK	Mikhlin
$M_x, M_y$	Bending moments

# Contrails

$n$	Number of equations in a set
$p$	Particular solution (subscript)
$P$	Point within domain
PM	Point matching
$q/\bar{D}$	Loading to stiffness ratio for a plate
$Q$	Point on boundary
$R(P)$	Domain residual
$R_i(Q)$	Boundary residual
R-G	Ritz-Galerkin
T	Temperature
$u$	Function
$u_n, w_n$	Trial solutions
$w$	Plate deflection
$x, y$ or $r, \theta$	Coordinate axes
$\alpha_k, \beta_k$	Functions
$\delta$	Variational symbol
$\delta I$	Variation in I
$\delta(x - \bar{x})$	Dirac delta function
$\nabla^2$	Two-dimensional Lapacian or harmonic operator
$\nabla^4$	Biharmonic or "del-fourth" operator
$\epsilon_w$	Deflection error
$\epsilon_{SL}$	Slope error
$\epsilon_M$	Moment error
$\epsilon_R$	Residual error
$\epsilon_T$	Temperature
$\{\epsilon\}$	Error column matrix

# Contrails

$\eta$	Outward drawn normal to B
$\nu$	Poisson's Ratio
$\rho$	Weighting function
$\phi_k, \Phi, \psi_k, \chi_k, \Omega$	Functions
$[ ]^T$	Transpose of a matrix, $[ ]$

# *Contrails*

## SECTION I INTRODUCTION

Numerous approximate analytical procedures for obtaining solutions to boundary value problems have evolved and exist as either unrelated methods or special cases of certain more general approaches. The results of a preliminary study program that critically examines these techniques from a unified view-point, (as related to problems in structural mechanics and heat transfer) is presented. The advantages of casting many methods as special cases of a more general procedure are obvious because it permits a systematic comparison among them as to solution accuracy, set-up and computational effort, and over-all applicability (not to mention the possibility of indicating new methods that could prove more accurate or convenient than existing ones).

Among the more important general approaches for solving boundary value problems that give rise to analytically continuous approximate solutions are the methods of least-squares, weighted equation-residuals, and admissible functions applied to extremum problems. Included in these are the methods of Ritz, Galerkin, Biezeno-Koch, point-matching, and collocation (References 1, 2, and 3). Methods that replace the continuous variables by discrete variables that change stepwise, such as finite-differences, are not included in this report.

The general procedure considered in this study unifies the previous methods through an extension of the method of undetermined parameters (References 4, and 5). It is demonstrated that all techniques mentioned, as well as some new ones, can be considered as special cases of this extended method. Three specific problems are treated, each by several of these methods, and detailed numerical comparisons are made.

For simplicity, attention is restricted to linear equilibrium problems in finite continuum domains. Although the development included is applicable to arbitrary linear operators, the actual investigation, including numerical studies, is concerned with problems associated with Laplacian and biharmonic operators in conjunction with suitable linear boundary conditions. Selection of these operators is based upon their comparatively simple analytical tractability and their importance to many physical problems included in such disciplines as structural analysis, heat-flow, and fluid-mechanics.

SECTION II  
GENERAL METHOD OF UNDETERMINED PARAMETERS

A. INTRODUCTION

A presentation of approximate analytical methods to be considered is greatly facilitated if a number of important mathematical terms, which will be freely used throughout the report, are understood. Therefore, before embarking upon the solution techniques treated, some of these terms will be defined and, where appropriate, illustrated with examples. It is understood that most of these terms will be familiar to the engineer with a strong mathematical background. However, they are intended to accommodate the reader without this advantage. Also, because ambiguities could arise from different common usages of certain of the terms defined, it is also intended that the following be taken as a glossary for their present intended meaning.

B. MATHEMATICAL PRELIMINARIES

- 1) Function or functional --  $u$  is called a functional of one variable,  $x$ , if a scalar is defined for every  $x$ . Because the value of  $u$  is governed by the value of  $x$  (which itself assumes more than one value),  $x$  is called the independent variable and  $u$  the dependent variable. Often,  $u$  is written  $u(x)$  and, then,  $u$  is called a function of  $x$ .

A functional is said to be linear if

$$u(ax + by) = au(x) + bu(y)$$

where:  $a$  and  $b$  are scalar quantities or constants.

- 2) Operator -- Abstractly, an operator,  $L$ , is short-hand notation for something that converts a quantity, e.g.,  $u$ , into some other quantity, e.g.,  $f$ . Such a conversion is called a mapping of  $u$  into  $f$ . Mathematically, this is written as

$$Lu = f$$

where the  $L$ , which achieves the mapping, can be a matrix operator (in which case  $u$  and  $f$  could be vectors) or an integral operator, e.g.,

$$Lu = \int_{\alpha}^{\beta} u \, dx = f \quad .$$

as well as many other type operators.

- 3) Linear differential operator --  $L$  is a differential operator if  $Lu = f$  is a differential equation. For example, if

$$L = \frac{d^2(\quad)}{dx^2} + x^2(\quad) ,$$

# Contrails

$Lu = f$  implies the differential equation

$$\frac{d^2 u}{dx^2} + x^2 u = f .$$

An operator,  $L$ , is defined as linear when:

$$L(au_1 + bu_2) = af_1 + bf_2$$

where  $a$  and  $b$  are scalars and if

$$Lu_1 = f_1 \text{ and } Lu_2 = f_2 .$$

Similarly, a linear partial-differential operator is a differential operator in more than one independent variable, which also satisfies the condition of linearity. Examples of such  $L$ 's are:

$$a) \quad L( ) = \frac{\partial^2( )}{\partial x^2} + \frac{\partial^2( )}{\partial y^2}$$

which is called the two-dimensional Laplacian (or harmonic operator) in cartesian coordinates. Often, it is abbreviated as:

$$\nabla^2 = \bar{\nabla} \cdot \bar{\nabla}$$

where  $\bar{\nabla}$  is the (vector) gradient operator. It can also be written in polar coordinates as:

$$\nabla^2( ) = \frac{\partial^2( )}{\partial r^2} + \frac{1}{r} \frac{\partial( )}{\partial r} + \frac{1}{r^2} \frac{\partial^2( )}{\partial \theta^2}$$

$$b) \quad L( ) = \nabla^2(\nabla^2( )) = \nabla^4( )$$

which is known as the biharmonic or "del-fourth", operator. In two-dimensional cartesian coordinates, it is written as:

$$L( ) = \frac{\partial^4( )}{\partial x^4} + 2 \frac{\partial^4( )}{\partial x^2 \partial y^2} + \frac{\partial^4( )}{\partial y^4}$$

It should be noted that the "order" of the biharmonic is four (i.e., the highest derivative appearing is the fourth), and that the order for the Laplacian is two.

If all the terms of a linear differential operator,  $L$ , contain derivatives of the same order, then  $L$  is said to be homogeneous (Reference 31).

# Contrails

- 4) Space, domain, and subdomain -- Space and domain, which are used interchangeably, are the infinite collection of points included by all permissible values that the independent variables (used to describe the points) can assume. A subdomain, can be any portion or zone of a domain, including (as a special case) the entire domain itself.
- 5) Boundary -- The curve surrounding the domain is called its boundary. For three-dimensional problems (i. e., three independent variables), the boundary becomes a two-dimensional surface. For a one-dimensional domain, a boundary is specified by a point. A space can surround holes, in which case, it possesses more than one boundary. Furthermore, the outer boundary will be closed upon itself only if the domain is finite.
- 6) Differential equation and boundary conditions -- Here, differential equation is used to mean an equation that applies to a dependent functional within the boundaries of a domain. Boundary conditions, which are also equations applied to a functional, are only applicable to its values on the boundary. Furthermore, these boundary equations can involve differential operators.
- 7) Boundary-value problem -- The determination of a functional satisfying a differential equation within a domain and lower-order conditions on all surrounding boundaries is termed a boundary-value problem. This definition assumes that the independent variable does not appear in the differential equation. A problem is either well- or ill-posed, depending on whether it uniquely defines solutions or not.
- 8) Residuals -- The extent to which an equation is not satisfied by some functional is defined as the residual. Accordingly, the exact solution of a boundary value problem has a zero residual for the differential equation within the domain and for the equations on the boundary. Approximate solutions will not satisfy all these conditions entirely. Thus, the residual represents a measure of the inaccuracy of a solution.
- 9) Orthogonality -- Two functions,  $f_1(x)$  and  $f_2(x)$ , are said to be orthogonal over an interval  $(x_1, x_2)$  if the integral of the product  $f_1 \cdot f_2$  vanishes over that interval, i. e., when:

$$\int_{x_1}^{x_2} f_1(x) \cdot f_2(x) dx = 0$$

- 10) Weighting function -- if

$$\int_{x_1}^{x_2} f_1(x) f_2(x) \rho(x) dx = 0$$

then,  $f_1$  and  $f_2$  can be said to be orthogonal with respect to the weighting function  $\rho(x)$  over the interval  $(x_1, x_2)$ . An important use of weighting functions is to accent or attenuate functions where it is believed they are either most or least important. For example, suppose a parameter,  $a$ , is to be determined such that

$$f(x, a) = 0, \quad x_1 \leq x \leq x_2$$



# Contrails

If  $f$  is such that it is not possible to satisfy this equation exactly, this condition can be approximated by setting its average value to zero and then solving for  $\underline{a}$  from:

$$\int_{x_1}^{x_2} f(x, \underline{a}) dx = F(\underline{a}) = 0$$

An alternative is to set

$$\int_{x_1}^{x_2} \rho(x) f(x, \underline{a}) dx = 0$$

where  $\rho(x)$  acts to influence the condition  $f \approx 0$ . For example, if  $\rho(x) = x$  and  $x_2 > x_1 > 0$ , it favors  $f = 0$  more heavily near  $x_2$  than  $x_1$ . Thus,  $\rho(x)$  can be termed a weighting function.

There are other ways in which to weight quantities. One method found useful with systems of algebraic equations is presented in the following definition.

- 11) Least-squares -- As with the previous example, suppose it is desired to have  $f(x, \underline{a}) = 0$  throughout some interval  $(x_1, x_2)$ . However, it is known that this cannot be done for many functions, regardless of what value is assigned to the parameter  $\underline{a}$ . Therefore, as an approximation, the average of  $f$ , within  $(x_1, x_2)$ , can be set equal to zero

$$\int_{x_1}^{x_2} f(x, \underline{a}) dx = 0$$

However, because this does not guarantee that the magnitude of

$$|f(x, \underline{a})|, \quad x_1 \leq x \leq x_2$$

remains small (because large positive and negative values of  $f$  could cancel one another),  $f(x)$  is sometimes squared to guarantee a positive quantity (assuming  $f(x)$  is real and not imaginary or complex) and then integrated to obtain  $S$ :

$$S(\underline{a}) = \int_{x_1}^{x_2} f^2(x, \underline{a}) dx$$

Then, one can proceed to minimize  $S(\underline{a})$  by satisfying:

$$\frac{dS}{d\underline{a}} = 0, \quad \frac{d^2S}{d\underline{a}^2} \geq 0$$

Having ensured these last two expressions,  $f(x, \underline{a})$  becomes a minimum within the interval  $(x_1, x_2)$  in a least-squares sense.

# Contrails

An algebraic least-squares form arises as follows. Suppose it is desired to satisfy the following system of equations:

$$c_{11} a_1 + c_{12} a_2 + \dots + c_{1n} a_n - b_1 = 0$$

$$c_{21} a_1 + c_{22} a_2 + \dots + c_{2n} a_n - b_2 = 0$$

.....

$$c_{m1} a_1 + c_{m2} a_2 + \dots + c_{mn} a_n - b_m = 0$$

Through simultaneous evaluation of the a's (unknowns), given the c's and b's. These m equations can be expressed in matrix form as:

$$[c] \{a\} - \{b\} = 0$$

where [c] is the m x n matrix of the coefficients  $c_{11}, c_{12}, \dots, c_{nm}$ ; {a} is the column matrix  $a_1, a_2, \dots, a_n$ ; and {b} is the column matrix  $b_1, b_2, \dots, b_m$ . The system of equations is said to be overdetermined if  $m > n$  (i. e., m-n more conditions or equations have been imposed than there are unknown y's available to accommodate this). It is possible that the additional m-n equations are redundant, and so the system can be satisfied exactly. However, this is not the case for many problems. In such cases, an approximate solution is sought to minimize the error column matrix { $\epsilon$ } as follows:

$$\{\epsilon\} = [c] \{a\} - \{b\} ,$$

where { $\epsilon$ } has elements  $\epsilon_1, \epsilon_2, \dots, \epsilon_m$ . One method of effecting this is to minimize the sum of their squares.

$$S = \{\epsilon\}^T \{\epsilon\} = \sum_{i=1}^m \epsilon_i^2$$

thus justifying the name least-squares. To achieve this, evaluate the y's from the n simultaneous equations:

$$[c]^T ([c] \{a\} - \{b\}) = 0$$

The proof of this last statement is given in Appendix I.

If it is desired that certain of the equation-errors, or residuals, be smaller than others, each equation can be weighted accordingly:

# Contrails

$$\rho_1 (c_{11} a_1 + c_{12} a_2 + \dots + c_{1n} a_n - b_1) = \rho_1 \epsilon_1$$

$$\rho_2 (c_{21} a_1 + \dots) = \rho_2 \epsilon_2$$

. . . . .

$$\rho_m (c_{m1} a_1 + \dots + c_{mn} a_n - b_m) = \rho_m \epsilon_m$$

or if  $\begin{bmatrix} \rho \\ \vdots \end{bmatrix}$  is the  $m$  term diagonal matrix composed of  $\rho_1, \rho_2, \dots, \rho_m$  (with zeros for all off-diagonal terms), solve the following equation for  $\{y\}$ :

$$[c]^T \begin{bmatrix} \rho \\ \vdots \end{bmatrix} ([c] \{a\} - \{b\}) = 0$$

- 12) Adjoint and self-adjoint -- A linear differential operator  $\bar{L}$  and boundary conditions operator  $\bar{B}$  will be called the adjoint of another set of linear differential operators  $L$  and  $B$ , if:

$$\int_{x_1}^{x_2} u_1 L u_2 dx = \int_{x_1}^{x_2} u_2 \bar{L} u_1 dx$$

where the conditions  $B$  and  $\bar{B}$  were used in transforming the left-hand side into the right-hand side, through integrating by parts, by application of the divergence theorem, or by some other suitable method. Thus, it follows that  $L$  is also the adjoint of  $\bar{L}$ .

If  $\bar{L} = L$ , then  $L$  is said to be formally self-adjoint. Also, if  $B = \bar{B}$ , then  $L$  is self-adjoint.

- 13) Heaviside step-function -- This function is represented by the symbol  $H$ . It possesses the following properties (Reference 31):

$$H(x - \bar{x}) \begin{cases} 0 & \text{for } x < \bar{x} \\ 1 & \text{for } x > \bar{x} \\ \text{is not defined} & \text{for } x = \bar{x} \end{cases}$$

Graphically, it appears as shown in Figure 1.

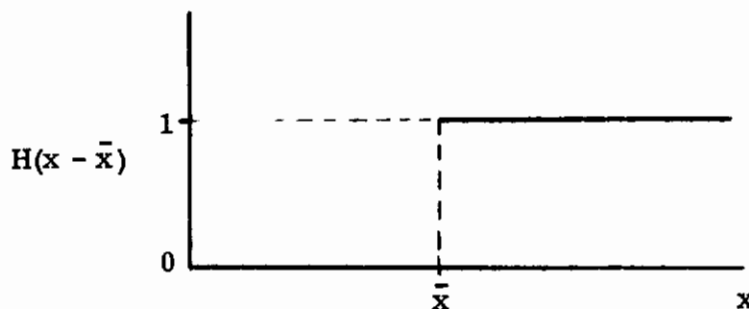


Figure 1. One-Dimensional Heaviside Step Function

Two Heaviside functions can be combined to yield a uniform pulse (see Figures 2a, 2b, and 2c).

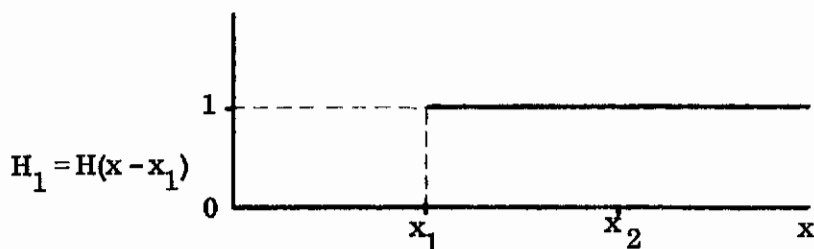


Figure 2a

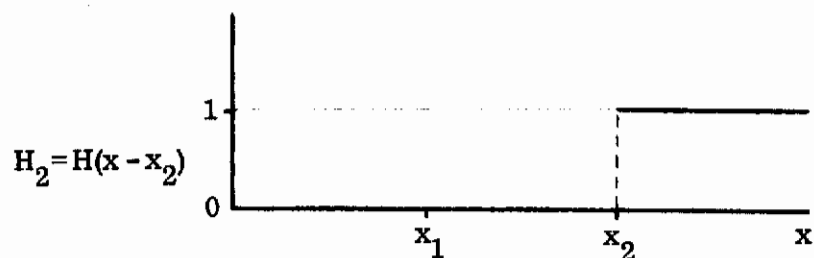


Figure 2b

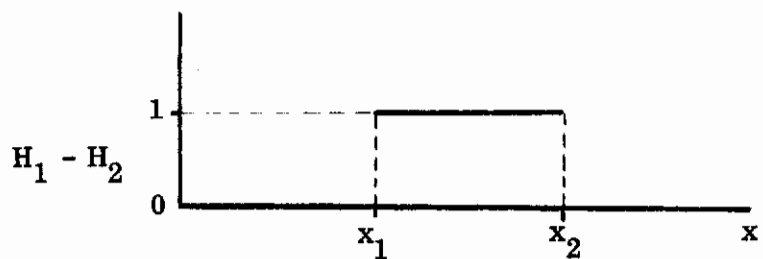


Figure 2c

Figure 2. Combination of Heaviside Functions to Give a Uniform Pulse

In two dimensions, it is defined as:

$$H = H(x - \bar{x}) \cdot H(y - \bar{y})$$

where each component of the product behaves as its one-dimensional counterpart. Thus

$$H_{S.D.} = H(x - x_1) \cdot H(y - y_1) - H(x - x_2) \cdot H(y - y_2)$$

has the value unity within the rectangular subdomain (S. D.) bounded by  $x_1$ ,  $x_2$ , and  $y_1$ ,  $y_2$  (see Figure 3). Shading denotes area within which

$$H_{S. D.} = H(x - x_1) \cdot H(y - y_1) - H(x - x_2) \cdot H(y - y_2) = 1$$

$$H_{S. D.} = 0 \text{ outside the shaded area}$$

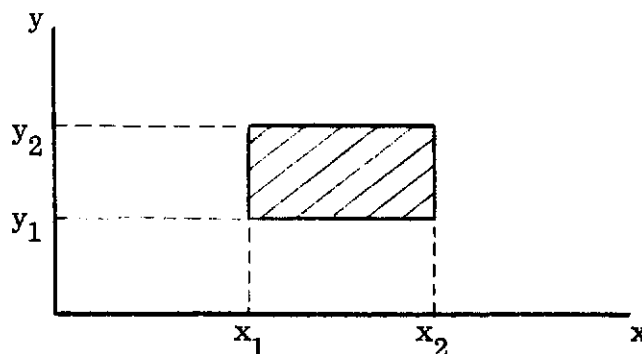


Figure 3. Two-Dimensional Subdomain

- 14) Dirac delta function -- Denoting this function by the symbol  $\delta$ , its properties are defined as follows (Reference 2):

$$\delta(x - \bar{x}) \begin{cases} 0 & \text{for } x \neq \bar{x} \\ \text{is not defined at } x = \bar{x} \end{cases}$$

However,

$$\int_{x=-\infty}^{x=+\infty} f(x) \delta(x - \bar{x}) dx = f(\bar{x})$$

Thus, it follows that the delta function, when used as a weighting function, picks-out the value of  $f(x)$  at  $x = \bar{x}$  in the interval containing  $\bar{x}$ . Its two-dimensional counterpart is:

$$\delta = \delta(x - \bar{x}) \delta(y - \bar{y}) \begin{cases} = 0 & \text{for } x \neq \bar{x} \text{ or } y \neq \bar{y} \\ \text{is not defined for } x = \bar{x} \text{ and } y = \bar{y} \end{cases}$$

However,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x - \bar{x}) \delta(y - \bar{y}) dx dy = f(\bar{x}, \bar{y})$$

- 15) Collocate -- If  $f(x, a) \neq 0$  throughout some interval  $(x_1, x_2)$ , but can be made zero at some point  $\bar{x}$  within this interval, through suitable adjustment of the parameter  $\underline{a}$ , i. e.,

$$f(\bar{x}, a) = 0 \quad x_1 < \bar{x} < x_2 ,$$

# Contrails

then,  $f = 0$  has been collocated at  $x = \bar{x}$ . In other words, the value of the function at some point has been fixed. This tying down could be achieved through use of the delta function as follows:

$$0 = \int_{x_1}^{x_2} f(x, a) \delta(x - \bar{x}) dx = f(\bar{x}, a)$$

- 16) Rank -- The rank of a matrix is defined as the order of the largest square array in that matrix whose determinant is not zero.
- 17) Convergence and completeness -- If the function  $u$  and the sequence  $u_n$  ( $n = 1, 2, 3, \dots$ ) are square-integrable (i. e., their squares can be integrated) real functions,  $u_n$  is said to converge (within domain  $D$ ) to  $u$  in a mean square sense with respect to weighting function  $\rho$  when:

$$\lim_{n \rightarrow \infty} \int_D \rho [u - u_n]^2 dA = 0$$

where  $dA$  is the differential element in  $D$  and  $\rho$ ,  $u$ , and  $u_n$  are functions of the coordinates of  $D$  (Reference 33). If, in addition:

$$u_n = \sum_{k=1}^n a_k \phi_k, \quad n=1, 2, \dots$$

where the  $a_k$  are scalars and the  $\phi_k$  are functions of the coordinates of  $D$ , the  $\phi_k$  are called a complete set of functions. Sets or sequences are complete over an interval if they have the capability to exactly represent a function  $u$  within the interval in the limit as  $n \rightarrow N$ , where  $N \rightarrow \infty$  in some instances.

Examples of complete sets of functions within finite one-dimensional domains are the trigonometrics, Bessel functions, the polynomials of Legendre and Chebyshev (and others), and the power series ( $x, x^2, x^3, \dots$ ).

Furthermore, if  $\alpha_1(x), \alpha_2(x), \dots, \alpha_i(x), \dots$  form a complete system of functions within the interval  $a \leq x \leq b$ , and  $\beta_{1i}(y), \beta_{2i}(y), \dots, \beta_{ji}(y), \dots$  is a similar system in the interval  $c \leq y \leq d$ , then the products:

$$\phi_{ij}(x, y) = \alpha_i(x) \beta_{ij}(y)$$

form a complete system in the rectangle  $a \leq x \leq b; c \leq y \leq d$  (Reference 34). Thus, in this manner, a complete system of functions can be constructed in two or more dimensions.

**C. STATEMENT OF PROBLEM AND GENERAL SOLUTION METHOD**

This discussion is restricted to finite, two-dimensional, continuum domains:  $D(x, y)$  in cartesian coordinates or  $D(r, \theta)$  for polar coordinates. The independent space variable  $P = P(x, y)$ , or alternately  $P(r, \theta)$ , will represent any point in  $D$ , while  $Q$  refers to points on its boundary  $B$  with outward drawn normal  $\eta$  (see Figure 4).

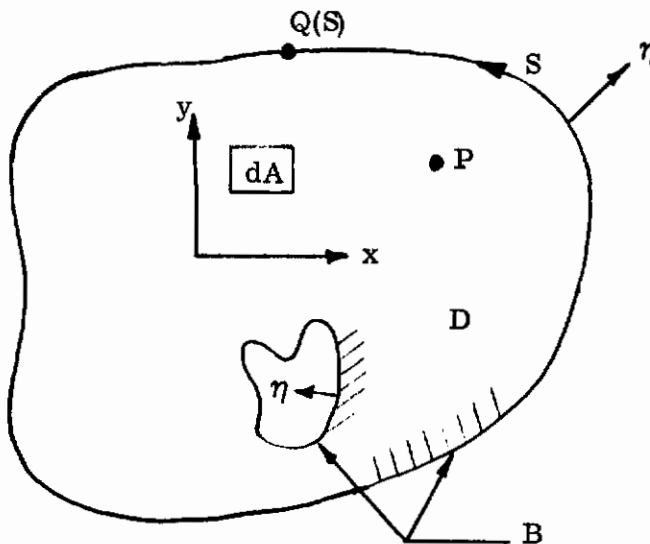


Figure 4. Domain  $D$  with Boundaries  $B$  and Pertinent Variables

To begin the survey of approximate procedures for equilibrium problems in continuous systems, the problem is defined through the system of equations:

$$Lu(P) = f(P) \quad \text{in } D \quad (1)$$

where  $L$  is a linear differential operator of order  $2m$  ( $m$  is a positive integer) and  $P$  is a typical point in the space domain,  $D$ , and

$$B_i u(Q) = g_i(Q) \quad \text{on } B \quad (i = 1, 2, \dots, m) \quad (2)$$

where  $B_i$  is a linear-differential operator of order  $2m-1$ . For example, if it is desired to solve the differential equation:

$$\nabla^2 u = f \quad \text{in } D \quad (3)$$

subject to appropriate general linear conditions on the boundary:

$$\alpha \frac{\partial u}{\partial \eta} + \beta u = g \quad (4)$$

# Contrails

Then,  $m = 1$ ,  $(P) = (x, y)$ ,  $L = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  and  $B(\ ) = \alpha \frac{\partial(\ )}{\partial \eta} + \beta(\ )$  in Equations (1) and (2). Thus, a concise statement of the general problem for arbitrary even-ordered, linear operators is effected.

For approximations to Equations (1) and (2) it is convenient to define residuals:

$$R(P) = L u_n - f \quad \text{in } D \quad (5)$$

and:

$$R_i(Q) = B_i u_n - g_i, \quad i = 1, 2, \dots, m \quad \text{on } B \quad (6)$$

where:  $u_n$  is an approximation to  $u$ , and is cast in the form:

$$u_n = \varphi_0(P) + \sum_{k=1}^n a_k \varphi_k(P) \quad (7)$$

The  $\varphi_k$  are linearly independent known functions in  $D$ ,  $\varphi_0$  is selected in some convenient manner, and the  $a_k$  are, as yet, undetermined constants independent of  $P$ , but nevertheless variable parameters of the solution.

To evaluate the  $a_k$ , a total of  $n$  conditions is imposed that in some way cause the residuals of Equations (5) and (6) to be (hopefully) as close as possible to zero throughout  $D$  and on  $B$ . Therefore, the conditions will represent  $n$  weighted averages, or orthogonalizations, out of  $(m + 1) \times n$  possible ones available on the  $R$ 's throughout  $D$  and  $B$ : i. e.,

$$n \text{ possible conditions: } \int_D R M \varphi_k dA = 0, \quad k = 1, 2, \dots, n \quad (8)$$

and:

$$m \times n \text{ possible conditions: } \int_B R_i M_i \varphi_k dS = 0, \quad \begin{matrix} i = 1, 2, \dots, m \\ k = 1, 2, \dots, n \end{matrix} \quad (9)$$

where  $M$  and the  $M_i$  are linear (but not necessarily homogeneous) differential operators. Upon performing the integrations involved in Equations (8) and (9), and defining two sets of elements,  $c_{jk}$  and  $b_k$ :

$$c_{jk} = \int_D L \varphi_j M \varphi_k dA, \quad \text{if } R_i = 0 \quad \text{but } R \neq 0 \quad (10. a)$$

or

$$c_{jk} = \int_B B_i \varphi_j M_i \varphi_k dS, \quad \text{if } R = 0 \quad \text{but } R_i \neq 0 \quad (10. b)$$

or

$$\text{and } \left. \begin{matrix} c_{jk} = \int_D L \varphi_j M \varphi_k dA \\ \int_B B_i \varphi_j M_i \varphi_k dS \end{matrix} \right\} \text{ if both } R \neq 0 \quad \text{and } R_i \neq 0 \quad (10. c)$$



# Contrails

$$b_k = \int_D (f - L\varphi_0) M\varphi_k dA \quad \text{if } R_1 = 0, R \neq 0 \quad (11. a)$$

or:

$$b_k = \int_B (g_i - B_i\varphi_0) M_i\varphi_k dS \quad \text{if } R = 0, R_1 \neq 0 \quad (11. b)$$

or:

$$\left. \begin{aligned} b_k &= \int_D (f - L\varphi_0) M\varphi_k dA \\ b_k &= \int_B (g_i - B_i\varphi_0) M_i\varphi_k dS \end{aligned} \right\} \text{if both } R_1 \neq 0 \text{ and } R \neq 0 \quad (11. c)$$

the following system of linear algebraic equations results:

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & \dots \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ c_{m1} & \dots & \dots & c_{mn} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{Bmatrix} \quad (12. a)$$

or written in matrix notation:

$$[c_{jk}]_{m \times n} \{a_k\}_{n \times 1} = \{b_j\}_{m \times 1} \quad (12. b)$$

Assume the matrix  $[c_{jk}]$  is of rank  $n$ . If  $m > n$ , Equations (12) can be over-determined, in which case a set of  $a$ 's satisfying all  $m$  conditions is not possible. However, it is possible to solve for the  $a$ 's in a least-squares sense. If  $m = n$ , Equations (12) can be inverted directly. Operationally, these cases are treated as follows:

$$m > n; [c_{jk}]^T [c_{jk}] \{a_k\} = [c_{jk}]^T \{b_j\} \quad (13)$$

$$\{a_k\} = [c_{jk}]^T [c_{jk}]^{-1} [c_{jk}]^T \{b_j\}$$

$m = n$

$$\begin{aligned} [c_{jk}] \{a_k\} &= \{b_j\} \\ \{a_k\} &= [c_{jk}]^{-1} \{b_j\} \end{aligned} \quad (14)$$

# Contrails

If  $[c_{jk}]$  is not of rank  $n$ , (i. e., there are fewer than  $n$  independent equations) its inverse (for an  $n \times n$  system) or the inverse of:

$$[c_{jk}]^T [c_{jk}]$$

(for an  $m \times n$  system) cannot be evaluated. This implies that the  $\{a_k\}$  are not uniquely defined. In such cases, it becomes necessary to generate more linearly independent equations of types similar to Equations (10) and/or (11) or the number of parameters  $a_k$  must be reduced until  $[c_{jk}]$  can be made of rank equal to the number of undetermined parameters.

Solution accuracy will obviously be governed by the  $\varphi_0$ ,  $\varphi_k$ ,  $M$ ,  $M_i$ , and  $n$  selected. In subsection III-B specific combinations of the general parameters are indicated that lead to the following well known methods:

- 1) Ritz
- 2) Galerkin
- 3) Least Squares
- 4) Collocations
- 5) Biezeno-Koch or Subdomain
- 6) Point Matching
- 7) Mikhlin

These methods divide naturally into two general categories:

- 1) Approximate solutions that satisfy the boundary conditions (2) but not the differential Equation (1) (i. e., the first five methods listed)
- 2) Approximate solutions that satisfy the differential Equation (1) but not the boundary conditions (2) (i. e., the last two methods listed)

A third grouping is possible if solutions that satisfy neither the differential nor boundary equations are considered.

SECTION III

RELATIONSHIP BETWEEN THE GENERAL TECHNIQUE AND SPECIFIC METHODS

A. DISCUSSION OF THE WELL-KNOWN METHODS

1. Introduction

The following is a presentation of the approximate solution methods listed in Section IIC, with regard to mathematical properties, application techniques, and historical development.

2. The Ritz Method

The underlying idea of the Ritz Method (Reference 6), which developed from considerations other than the general approach outlined in Section II, is that boundary value problems (i. e., differential equation and boundary conditions) can be replaced by an equivalent problem of making an appropriate functional stationary with respect to admissible functions. For example, the boundary value (or equilibrium) problem:

$$\nabla^2 u = f(x, y) \tag{15}$$

in a planar region D subject to:

$$u = \phi(s) \text{ on } B \tag{16}$$

is equivalent to determining the function u (x,y) which makes the functional:

$$I(u) = \int_D [\bar{\nabla}u \cdot \bar{\nabla}u + 2fu] dA \tag{17}$$

stationary. The functions f (x,y) and  $\phi(s)$  are given functions of position in the domain D and boundary B, respectively. The admissible functions u (x,y) must possess continuous second derivatives in D and must satisfy the boundary condition of Equation (16). The differential Equation (15) is the Euler equation (References 7 and 8) determined by the requirement that Equation (17) be stationary.

More generally, suppose the functional:

$$I(u) = \int_D F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) dA \tag{18}$$

is to be "extremized" subject to the condition of Equation (16). Let  $u = u^*(x, y)$  be the function which actually † minimizes Equation (18) (existence of a minimum is assumed).

† In actuality, it could be a maximum, minimum, or neither. However in many significant applications, a minimum is actually required, such as when the principle of minimum potential energy applies to linear elastic systems.

Introduce a sequence of admissible functions:

$$u_n(x, y) = \phi(x, y, a_1, a_2, \dots, a_n) \quad n = 1, 2, \dots \quad (19)$$

with parameters  $a_i$ , and suppose that each family  $u_k(x, y)$  includes in it families with subscripts less than  $k$ . Then, the Ritz method requires the formation of  $I(u_n)$ , where:

$$I(u_n) = \int_D F\left(x, y, u_n, \frac{\partial u_n}{\partial x}, \frac{\partial u_n}{\partial y}\right) dA \quad (20)$$

= Function of  $a_1, a_2, \dots, a_n$

Thus, the condition  $\delta I = 0$  implies that:

$$\frac{\partial I}{\partial a_j} = 0 \quad j = 1, 2, \dots, n \quad (21)$$

must hold. The solution of the resulting system of algebraic equations that minimizes Equation (20) represents an approximation to the actual or required result. This method has been used with great success in the solution of problems as they occurred in equilibrium (Reference 9), vibration, and instability of structures (Reference 10). The primary attraction of the method is:

- a) It is simple to apply if functions that satisfy the boundary conditions can be obtained
- b) Even if  $u_n$  does not approximate  $u^*$  well, the Ritz method can yield very satisfactory results in the case of eigenvalue problems, such as determining the buckling load in columns, plates, as well as natural frequencies (Reference 11)
- c) It is a well established procedure, and there is a rather extensive literature wherein this method is exploited

There is a vast difference between using the Ritz Method in a formal manner and satisfactorily answering the questions of convergence of this technique to the required solution. In eigenvalue problems, for example, it is known that the Ritz method yields an upper bound to the exact solution. It is also true that, with increase in  $n$ , the computed values will form a monotonic nondecreasing sequence that is bounded from below by the exact solution (according to the assumed mathematical model). The method converges, in a practical sense, when the results change negligibly with increasing  $n$ . This procedure can be very time consuming because a large value of  $n$  could be required.

# Contrails

In order to obviate the necessity of using large values of  $n$ , the Ritz procedure is often used in conjunction with the Trefftz Method (Reference 12), which yields a lower bound for an appropriate functional. These two methods serve to bracket the solution. In some instances, the Trefftz Method has been abused, resulting in sequences that did not furnish lower bounds. The problem of convergence has been satisfactorily treated for a restricted class of functionals. A fundamental notion used to resolve this is that of relative completeness. The set  $u_n = \phi_n(x, y, a_1, a_2, \dots, a_n)$  ( $n = 1, 2, \dots$ ) is said to be relatively complete in  $D$  if for every admissible  $u(x, y)$  there exists a function  $\bar{u}_n(x, y, \bar{a}_1, \dots, \bar{a}_n)$  such that, for any positive  $\epsilon$ , the inequalities:

$$|\bar{u}_n - u| < \epsilon$$

$$\left| \frac{\partial \bar{u}_n}{\partial x} - \frac{\partial u}{\partial x} \right| < \epsilon,$$

and  $\left| \frac{\partial \bar{u}_n}{\partial y} - \frac{\partial u}{\partial y} \right| < \epsilon$

hold simultaneously for all  $x, y$  in  $D$ . For the case of the general functional, Equation (20), there is no assurance that  $\bar{u}_n$  converges to  $u^*$ . In fact, the sequence  $\{\bar{u}_n\}$  need not converge at all, although  $I\{\bar{u}_n\}$  does converge to  $I\{u_n\}$  (Reference 13). This same conclusion holds for the special functional Equation (17); i.e., convergence is not guaranteed.

However, suppose that:

$$\nabla^4 u = f(x, y) \tag{22}$$

is considered subject to the boundary conditions:

$$u = 0, \frac{\partial u}{\partial \nu} = 0 \text{ on } B \tag{23}$$

This system, Equations (22) and (23) appear in the study of a transversely loaded clamped elastic plate. The associated functional that is to be extremized is:

$$I(u) = \iint_D \left[ (\nabla^2 u)^2 - 2fu \right] dx dy \tag{24}$$

Again, it can be shown that, if a relatively complete set is used, that  $I\{\bar{u}_n\}$  converges to  $I\{u^*\}$  where  $u^*$  minimizes Equation (24) and even more surprisingly that  $\bar{u}_n \rightarrow u^*$ . However, this is in accord with an observance by Courant (Reference 14) that the convergence of the minimizing sequence  $\{u_n\}$  and of its derivatives improves when the order of derivatives in the functional is increased.

### 3. The Galerkin Method

In 1915, B.G. Galerkin (Reference 15) proposed a method of approximate solution of the boundary value problems in mathematical physics that is of much wider scope than the Ritz procedure. In fact, the Galerkin method reduces to the Ritz development when applied to variational problems with quadratic functionals.

One of the primary applications of the Galerkin procedure is to the approximate solution of differential equations. For example, suppose that it is required to solve the boundary value problem:

$$L u = f, \text{ in } D \text{ (two-dimensional domain)} \quad (25)$$

subject to linear homogeneous boundary conditions on B.

An approximate solution of the problem is sought in the form of Equation (7) without a  $\varphi_0$  (unless  $\varphi_0$  also satisfies boundary conditions on B):

$$u_n(x, y) = \sum_{k=1}^n a_k \phi_k(x, y) \quad (26)$$

where  $a_1, \dots, a_n$  are constants (to be determined) and  $\phi_1, \phi_2, \dots, \phi_n$  are coordinate functions that are chosen so as to satisfy the boundary conditions.

In general,  $u_n$  does not satisfy Equation (25) so that:

$$L u_n - f = R_n(x, y) \text{ in } D \quad (27)$$

where the residual  $R_n$  can be viewed as an error function. Suppose it is possible to represent  $u(x, y)$  by an infinite series:

$$u(x, y) = \sum_{k=1}^{\infty} a_k \phi_k(x, y) \quad (28)$$

where the  $\phi_k(x, y)$  are known linearly independent functions that satisfy the boundary conditions, and the  $a_k$  are constants to be determined. Let:

$$u_n(x, y) = \sum_{k=1}^n a_k \phi_k(x, y) \quad n = 1, 2, \dots \quad (29)$$

represent a sequence of partial sums. Then, it can be shown that the orthogonality condition:

$$\int_D (L u_n - f) \phi_k(x, y) \, dA \quad k = 1, 2, \dots, \infty \quad (30)$$

implies the differential equation:

$$L u = f \quad (31)$$

Galerkin incorporated this principle by imposing that the error function be orthogonal to each of the functions  $\phi_k$  namely:

$$\int_D (L u_n - f) \phi_k(x, y) dA = 0 \quad (k = 1, 2, \dots, n) \quad (32)$$

and substitution of Equation (29) into Equation (32) yields the conditions:

$$\int_D \left[ L \left( \sum_{j=1}^n a_j \phi_j \right) - f \right] \phi_k dA \quad (33)$$

This results in a system of  $n$  linear algebraic equations in  $n$  unknowns,

$$a_1, a_2, \dots, a_n.$$

It is important to note that, when the differential equation and the boundary conditions are self-adjoint and the corresponding functional  $I(u)$  in the problem  $I(u) = \text{minimum}$ , is positive definite, then Equation (33) is equivalent to the equations of the Ritz system:

$$\frac{\partial I(u_n)}{\partial a_k} = 0 \quad (34)$$

However, the Galerkin method is directly applicable to boundary value problems regardless of whether an energy-type functional can be found. In many instances finding an appropriate functional is very difficult, if not impossible and, in this sense, the Galerkin method is more useful than the more restrictive Ritz procedure.

#### 4. The Method of Least Squares

In the Galerkin method, the error function in an approximate solution:

$$u_n = \sum_{k=1}^n a_k \phi_k(x, y) \quad (35)$$

of equation  $L u = f$  was defined as:

$$R(x, y) = L u - f \quad (36)$$

A technique based on the construction of approximating sequences and in accordance with the criterion:

$$E \equiv \int \int_D R^2(x, y) dx dy = \text{Minimum}$$

is known as the method of least squares. The equations for the  $a_k$  are obtained by forming  $\frac{\partial E}{\partial a_k} = 0$ . This leads to the system:

$$\int \int_D R \frac{\partial R}{\partial a_k} dx dy = 0 \tag{37}$$

Investigations to ensure that the resulting systems have a sequence of solutions that converge to the exact one has been the subject of a number of studies (References 1 and 2).

It can be shown that the system of equations possesses a unique solution, and that the resulting sequence  $\{u_n(x, y)\}$  converges to the solution  $u(x, y)$  if restrictions are imposed on the operator  $L$  and on the choice of the coordinate functions  $\phi_k$ . In general, a complete set of functions  $\phi_1, \phi_2, \dots$  are sought, each of which satisfy the boundary conditions.

To demonstrate the method of least squares for  $L u_n = \nabla^4 u_n = f$ , set  $M \phi_k = \nabla^4 \phi_k$  and  $B_i u_n = g_i$ , for  $i = 1$  and  $2$ . Thus, Equation (8) becomes:

$$\int_D (\nabla^4 u_n - f) \nabla^4 \phi_k dA = 0, k = 1, 2, \dots, n$$

### 5. Collocation Method

This method is another procedure for reducing the error or residual function  $R_n(x, y)$ . The essence of this technique is that the error function is set equal to zero at  $n$  specified points in the domain  $D$ . Thus the error is collocated or assigned to be zero at  $n$  points, so that  $n$  algebraic equations are obtained for the coefficients  $a_k$  in  $u_n = \sum_{k=1}^n a_k \phi_k(x, y)$  without requiring integration over the domain. It is desirable to

distribute these  $n$  points fairly uniformly over the region  $B$ . However, very few investigations have been made into the suitable choice of collocation points.

### 6. Point Matching

#### a. Original Form

A new and interesting technique, known as point-matching, has received a good deal of attention in recent years. The name of the method derives from



the solution procedure, which matches boundary conditions pointwise, thus permitting its convenient application to irregularly shaped bodies. The solution scheme begins with an exact particular solution and a homogeneous series solution to a given linear partial-differential equation. Introduction of linear boundary conditions at discrete boundary points leads to a system of algebraic equations involving the series constants. Judicious selection of a coordinate system in which to express the solution will sometimes permit simple periodicity or regularity requirements to establish the eigenvalues. The series is then truncated and the coefficients evaluated, assuming that the simultaneous equations are linearly independent. Coefficients can also be obtained by using an approach that matches many more boundary points than there are undetermined solution coefficients. This leads to an over-determined system, which is reduced to a linearly independent set of equations through minimization of boundary errors (in a least-square sense). This latter technique is developed in some detail.

The procedure of matching boundary conditions at points is similar to the previously discussed method of collocations, in which solutions do not satisfy differential equations exactly but satisfy them pointwise.

Conway (References 15, 16, and 17), Leissa (References 16, and 18), Niedenfuhr (Reference 18), and Deverall (Reference 19), in recent efforts have applied the point-matching technique to many examples in the steady state bending, buckling, and vibration of elastic plates. Variations of the method are described in earlier works on static plate theory by Throne (Reference 20), Nash (Reference 21), and Sekiya (Reference 22). Furthermore, Conway (Reference 15) and Shuleshko (Reference 23) have applied point-matching to solve Saint-Venant torsion problems and, more currently, Conway and Leissa (Reference 24) have solved some shallow spherical shell problems by this method. While results of these investigators appear promising for the cases treated, many uncertainties exist regarding solution accuracy when the method is applied to new problems. Ojalvo and Linzer (Reference 25) have proposed improvements and extensions to the methods and have developed error-bounds. A computer program to handle a wide variety of Laplacian and biharmonic problems by this method has been developed by Niedenfuhr, Leissa, and Lo (Reference 26). The large number of problems solved by this program has greatly increased confidence in the accuracy of point-matched results.

The method proceeds as follows. For example, consider the two-dimensional solution to Poisson's equation:

$$\nabla^2 T + f(r, \theta) = 0 \quad (38)$$

The solution in polar coordinates is:

$$T = T_p + A_1 \theta + A_2 \theta \ln r + A_3 \ln r + \sum_n \left\{ (B_n r^n + C_n r^{-n}) \cos n \theta + (D_n r^n + E_n r^{-n}) \sin n \theta \right\} \quad (39)$$

It corresponds physically to a steady state temperature distribution,  $T$ , in a bounded body with distributed heat source and constant conductivity.  $T_p$ , assumed to be known,

is the particular solution to Equation (38). The coefficients A, B, C, D, and E are constants that are evaluated based upon known boundary conditions.

It is advantageous to locate the origin of coordinates within the body or inside a hole surrounded by the body because  $\theta$  can then be taken to be unbounded. Consequently, the single valuedness of T requires that  $A_1 = A_2 = 0$  and  $\cos n \theta$  and  $\sin n \theta$  be periodic in  $2\pi$ . Hence, n, the summation index used in Equation (39) must be an integer.

General, linear, boundary conditions for a continuum domain D, in which Equation (38) is operative, with boundary B are:

$$\alpha \frac{\partial T}{\partial \eta} + \beta T = F \text{ on B} \tag{40}$$

in which  $\eta$  is the outward drawn normal to B, and  $\alpha$ ,  $\beta$ , and F depend on the boundary points chosen.

To solve Equation (38) and (40) by point-matching, the right-hand side of Equation (39) is substituted into Equation (40) to yield:

$$\begin{aligned} F = \alpha & \left[ \frac{\partial T}{\partial r} \cos \varphi + \frac{\partial T_p}{\partial \theta} \frac{\sin \varphi}{r} + \frac{A_3 \cos \varphi}{r} \right. \\ & + \sum_n \left\{ r^{n-1} \left( B_n \cos (n\theta + \varphi) + D_n \sin (n\theta + \varphi) \right) \right. \\ & \left. \left. - r^{-(n+1)} \left( C_n \cos (n\theta - \varphi) + E_n \sin (n\theta - \varphi) \right) \right\} \right] \\ & + \beta \left[ T_p(r, \theta) + A_0 + A_3 \ln r + \sum_n \left\{ \left( B_n \cos n\theta + D_n \sin n\theta \right) \right. \right. \\ & \left. \left. + r^{-n} \left( C_n \cos n\theta + E_n \sin n\theta \right) \right\} \right] \end{aligned} \tag{41}$$

where  $\varphi$  is the angle the radius vector makes with  $\eta$ . Substitution of  $\alpha$ ,  $\beta$ , and F at discrete points  $(r_i, \theta_i, \varphi_i)$  on B into Equation (41) and appropriate truncation of the resulting series in n leaves a system of algebraic equations which is solved for the constants.

### b. Least-Squares Point-Matching

Many evenly spaced points, enumerated 1, 2, ..., j, ..., m are selected on the boundary. The error:

$$\epsilon_B = T_B - T'_B,$$

is expressed in terms of the undetermined coefficients  $a_k$ ,  $k = 1, 2, \dots, n$ , where  $n \leq m$ . The summed error-squared  $S'$ ,

$$S' (a_k) \equiv \sum_{j=1}^m \epsilon_{B_j}^2$$

is then minimized through the  $n$  relationships:

$$\frac{\partial S'}{\partial a_k} = 0, \quad k = 1, 2, \dots, n$$

and the resulting system of algebraic equations is solved for the  $a_k$ . This approach is similar to obtaining the best  $n$ -term approximation to  $T$ , in the least squared integral sense because  $S$ , where

$$S \equiv \int_B (T_B - T'_B)^2 ds$$

written in approximate finite summation form corresponds to  $\eta'$  for equal boundary length increments,  $\Delta s_m = 1$ .

### 7. Subdomain Method (Biezeno-Koch)

This method was introduced by Biezeno and Koch (Reference 28) and is often referred to by their names. In essence, the technique is to divide the domain  $D$  into sub-domains  $D_i$  preferably according to a simple pattern. Then the integral of the error function (or residual) over each subdomain is set equal to zero yielding  $n$  equations for the  $n$  parameters  $a_1, a_2, \dots, a_n$ , i.e.,

$$\iint_{D_i} \epsilon_n dx dy = 0 \quad i = 1, 2, \dots, n \quad (42)$$

This method has been used effectively for the cases of thin plates, elastically supported beams, and the more general case of elastic equilibrium.

### 8. Mikhlin Method

This method is also a least-squares procedure. It differs from the usual least-squares method in that it requires that the coordinate functions satisfy the differential equation  $Lu = f$  in the domain  $D$ . The coefficients  $a_k$  of the coordinate functions  $\phi_k$  are to be selected so that the integral of the square of the error function, taken over the boundary, be minimized. Depending upon the problem, this approach often yields more rapidly convergent sequences  $\{u_n\}$  than the usual least-squares procedure, and

usually requires simpler integration because only a boundary, rather than a higher dimensional domain, is involved.

**B. REDUCTION OF GENERAL TECHNIQUE TO SPECIFIC METHOD**

**1. Introduction**

Reduction of Equations (5) and (6) to the well-known approximate methods listed at the end of Section II is achieved through specialization of  $\phi_0$ ,  $\phi_k$ ,  $M$ , and the  $M_i$ . A discussion of the means by which this can be effected follows.

**2. Ritz**

In this approach,  $u_n$  must satisfy the boundary conditions, or Equations (6) exactly.  $M\phi_k$  is taken as  $\phi_k$  when  $L$ , together with the  $B_i$ , is self-adjoint\*; which, in turn, makes the  $c_{jk}$  matrix symmetric in Equation (12). This condition is fulfilled when a given problem can be associated with the extremization of a quadratic functional. If  $L$  together with  $B_i$  are not self-adjoint, then  $M$  arises from the result of making  $I$  stationary, where:

$$I = \int_D \phi(u, P) dA$$

and  $\phi(u, P)$  is an appropriate function in  $D$ .

Mathematically, this is written as follows:

$$\delta I = \int_D \delta \phi dA = \int_D (L u - f) M (\delta u) dA = 0$$

Therefore, the extremization of  $I$  implies Equation (1) when  $u$  satisfies the boundary conditions, or Equation (2).

If for example:

If  $\phi = \bar{\nabla} u_n \cdot \bar{\nabla} u_n + 2f u_n$  and  $u_n$  satisfies the boundary conditions, then:

$\delta I = 0$  implies

$$\int_D (\nabla^2 u_n - f) \delta u_n dA = 0$$

Now in  $D$

$$\delta u_n = \delta \left( \phi_0 + \sum_{k=1}^n a_k \phi_k \right) = \sum_{k=1}^n \phi_k \delta a_k$$

\* This is, incidentally, the case for the harmonic and biharmonic problems, which are treated numerically.

Therefore,

$$\sum_{k=1}^n \delta a_k \int_D (\nabla^2 u_n - f) \varphi_k dA = 0$$

Since the  $\delta a_k$  are arbitrary variations, the following is obtained:

$$\int (\nabla^2 u_n - f) \varphi_k dA = 0, k = 1, 2, \dots, n$$

This corresponds to Equation (8) where:  $M\varphi_k = \varphi_k$  = some element of the set of trial functions.

### 3. Galerkin

In this procedure, which also requires that  $u_n$  satisfy the boundary conditions of Equation (2) set  $M\varphi_k = \varphi_k$ . This is identical with the Ritz Method for the class of self-adjoint operators that, as noted previously, include both the harmonic and biharmonic operators. However, it should be noted that the Galerkin technique is not necessarily associated with an extremum problem, and so the self-adjointness property on the operators is no longer necessary for choosing  $M\varphi_k = \varphi_k$ . Consequently, the  $c_{jk}$  matrix of Equation (12) will not always be symmetrical for the Galerkin method.

### 4. Least Squares

The criterion of minimizing

$$E = \int_D (L u_n - f)^2 dA$$

in accordance with a  $u_n$ , given by Equation (7) and that satisfies boundary conditions of Equation (2), leads to the method of least squares. The equivalence of this procedure with Equation (8) dictates that  $u_n$  satisfy Equation (2) exactly and that:

$$M\varphi_k = \frac{\partial (L u_n)}{\partial a_k} = L \varphi_k$$

### 5. Collocations

As with the preceding methods, this approach begins with a value for  $u_n$  that satisfies Equation (2). To establish  $n$  undetermined parameters,  $a_k$ , Equation (1) is satisfied at  $n$  points  $P_k$ . This is equivalent to selecting:

$$M\varphi_k = \delta (P - P_k)$$

where  $\delta (P - P_k)$  is the Dirac delta function. Thus:

$$\int_D \varphi_j (P) \delta (P - P_k) dA = \varphi_j (P_k)$$

Thus, Equation (8) becomes:

$$L_{2m} u_n (P_k) = f (P_k), \quad k = 1, 2, \dots, n$$

### 6. Point Matching

While similar to collocations, this technique is markedly distinct from its predecessor because it requires that  $u_n$  satisfy differential Equation (1) exactly, instead of boundary conditions of Equation (2). In fact, in this respect, point-matching is quite unique from all the methods previously mentioned.

Formally, in point-matching, it is assumed that:

$$M_i \varphi_k (Q) = \delta (Q - Q_k) \quad \text{for } m = 1$$

If  $m > 1$ , no clearly established procedure exists by which a combination of  $B_i$  and  $Q_k$  should be selected, and, thus, the method is somewhat arbitrary here.

Substituting Equation (6) into Equation (9) and performing the integrations indicated yields:

$$(B_i u_n (Q_k) - g_i (Q_k)) = 0$$

for various  $i, k$  combinations totaling  $n$  conditions.

### 7. Biezeno-Koch

This procedure, also referred to as the subdomain method, requires that  $u_n$  satisfy the boundary conditions, or Equation (2). The domain weighting,  $M\varphi_k$ , is taken as:

$$M\varphi_k = H (P - P_{k-1}) - H (P - P_k)$$

where  $H$  is the Heaviside step function.

Hence, within the subdomains  $D_k$  ( $k = 1, 2, \dots, m$ ),  $M\varphi_k = 1$ , while outside  $D_k$ ,  $M\varphi_k = 0$ .

To illustrate this method, suppose  $D$  is a rectangular domain and  $P = P (x, y)$

where:  $0 \leq x \leq a$  and  $0 \leq y \leq b$ , then Equation (8) becomes:

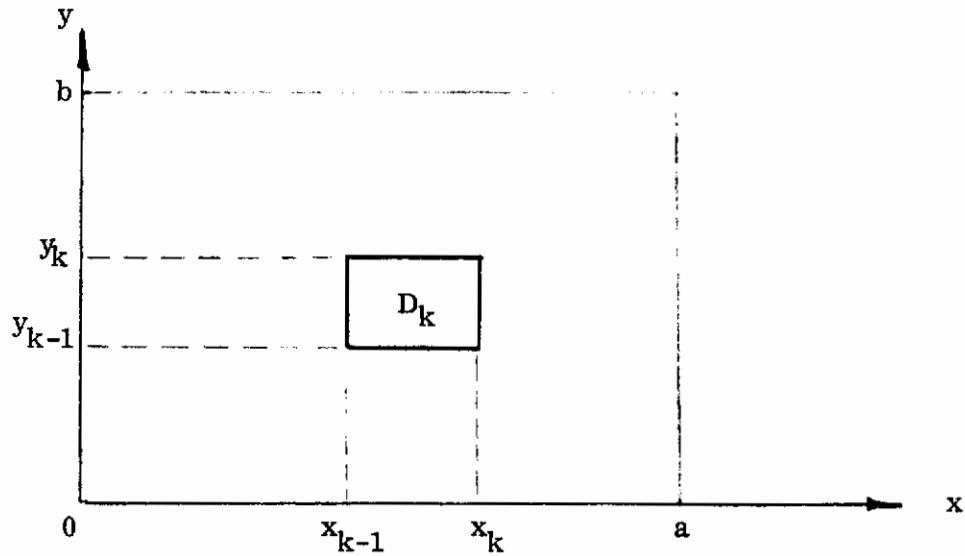


Figure 5. Rectangular Domain with SubDomain  $D_k$

$$\int_{y_{k-1}}^y \int_{x_{k-1}}^{x_k} (L u_n - f) dx dy = 0, \quad k = 1, 2, \dots, \bar{n}$$

where:

$$\iint_{D_k} dx dy = (x_k - x_{k-1}) (y_k - y_{k-1})$$

(See Figure 5) and  $\bar{n}$  is the number of subdomains into which  $D$  is subdivided. Thus, equilibrium (or Equation 8) is satisfied on an average basis for each subdomain  $D_k$ .

8. Mikhlin

To obtain Mikhlin's method,  $u_n$  was chosen so that it satisfies Equation (1) in  $D$  exactly, and then, a least-square integral boundary technique is used to satisfy Equation (2) approximately, i.e. let:

$$M_i \phi_k(Q) = \frac{\partial (B_i u_n - g_i)}{\partial a_k} = B_i \phi_k(Q)$$

Therefore, Mikhlin's method can be thought of as a cross between Point-Matching, in which  $u_n$  satisfies Equation (2), and least-squares, where:

$$M \phi_k = \frac{\partial (B_i u_n - g_i)}{\partial a_k}$$

and error-squared integrals are satisfied within  $D$

C. NEW METHODS

1. Introduction

Many other approximate methods are possible within the framework of equation systems obtainable from Equations (8) and (9), because each different method of selecting operators  $M$  or  $M_i$  leads to a new method. However, the determination of convenient operators that yield accurate results is quite another matter. Although the numerical investigation of these new methods are not included, several promising solution techniques are presented. Therefore, these can only be discussed qualitatively.

2. Generalized Ritz

As previously mentioned, the Ritz method is associated with the minimization of some function,  $I$ , in  $D$ . The following treatment is of a different type extremum problem, in that a combined boundary and domain functional will be minimized. To demonstrate how this leads to an approximate solution scheme similar to those included in Equations (8) and (9), consider the two functionals:

$$I_1 \equiv \int_D (\nabla^2 u \cdot \bar{\nabla} u + 2 fu) dA \tag{43}$$

$$I_2 \equiv \int_B (\beta u^2 - 2 F u) dS \tag{44}$$

Setting the variation of Equations (43) and (44) to zero and employing Green's Theorem yields:

$$-1/2 \delta I_1 = \int_D (\nabla^2 u - f) \delta u (P) dA - \int_B \frac{\partial u}{\partial \eta} \delta u (P) dS = 0 \tag{45}$$

$$1/2 \delta I_2 = \int_B (\beta u - F) \delta u (Q) dS = 0 \tag{46}$$

For arbitrary variations in  $u$ , both in  $D$  and on  $B$ , Equations (45) and (46) are equivalent to the boundary value problem:

$$\nabla^2 u = f \quad \text{in } D$$

$$\alpha \frac{\partial u}{\partial \eta} + \beta u = F \quad \text{on } B (\alpha = \text{constant}) \tag{47}$$

where  $\eta$  is the outward drawn normal to  $B$ . To demonstrate this, substitute Equation (47) into Equation (46) to get:

$$1/2 \delta I_2 = -\alpha \int_B \frac{\partial u}{\partial \eta} \delta u (Q) dS = 0 \tag{48}$$

Thus, this reduces Equation (45) to:

$$\int_D (\nabla^2 u - f) \delta u (P) dA = 0 \tag{49}$$



Selecting trial solution of the form given by Equation (7) and taking variations to mean variations in the undetermined parameters only, the following is obtained:

$$\delta u_n = \sum_{k=1}^n \delta (a_k \varphi_k) = \sum_{k=1}^n \varphi_k \delta a_k,$$

and by Equation (45) and (46):

$$\sum_{k=1}^n \delta a_k \int_D (\nabla^2 u_n - f) \varphi_k (P) dA = \sum_{k=1}^n \delta a_k \int_B \frac{\partial u_n}{\partial \eta} \varphi_k (Q) dS \quad (50)$$

and 
$$\sum_{k=1}^n \delta a_k \int_B (\beta u_n - F) \varphi_k dS = 0. \quad (51)$$

Because the  $a_k$  variations,  $\delta a_k$ , are arbitrary, their coefficients must cancel termwise. Thus, from Equations (50) and (51), there results the system of equations:

$$\int_D (\nabla^2 u_n - f) \varphi_k dA - \int_B \frac{\partial u_n}{\partial \eta} \varphi_k dS = 0, \quad k = 1, 2, \dots, n$$

$$\int_B (\beta u_n - F) \varphi_k dS = 0, \quad k = 1, 2, \dots, n$$

These 2 n algebraic equations can then be solved in a least squares sense.

### 3. Modified Galerkin

For this technique, the following is selected:

$$M \varphi_k (P) = \varphi_k (P)$$

and

$$M_i \varphi_k (P) = \gamma_i \varphi_k (Q)$$

in Equation (10), where the  $\gamma_i$  are parameters that contribute additional weighting to the integrals involved.

### 4. Modified Least-Squares

Substituting

$$M \varphi_k = L \varphi_k$$

and

$$M_i \varphi_k = B_i \varphi_k$$

into Equations (8) and (9) is equivalent to the minimization of the errors  $E_1$ , and  $E_{2i}$

$$E_1 \equiv \int_D (L u_n - f)^2 dA$$

$$E_{2i} \equiv \int_B (B_i u_n - g_i)^2 dS, \quad i = 1, 2, \dots, m$$

### 5. Combined Collocations and Point-Matching

Selecting

$$M \varphi_k = \delta (P - P_k)$$

and

$$M_i \varphi_k = \gamma_i \delta (Q - Q_k), \quad i = 1, 2, \dots, m$$

in Equations (8) and (9) leads to a system of  $n$  equations in the  $a_k$ . Satisfaction of Equation (1) by  $u_n$  reduces this to the Point-Matching method; while a  $u_n$ , which satisfies Equation (2), reduces the method to the collocation procedure.

### 6. Modified Biezeno-Koch

$$M \varphi_k (P) = H (P - P_{k-1}) - H (P - P_k),$$

and

$$M_i \varphi_i (Q) = \gamma_i [H (Q_{k_2} - Q) - H (Q_{k_1} - Q)], \quad Q_{k_2} > Q_{k_1} \quad i = 1, 2, \dots, m$$

If  $B_i u_n = g_i$ , this reduces to the Biezeno-Koch method. However, when  $L u_n - f = 0$ , the technique averages boundary conditions  $B_i u_n = g_i$  over small boundary regions, one at a time, until these sub-boundaries surround the domain (Reference Figure 6)

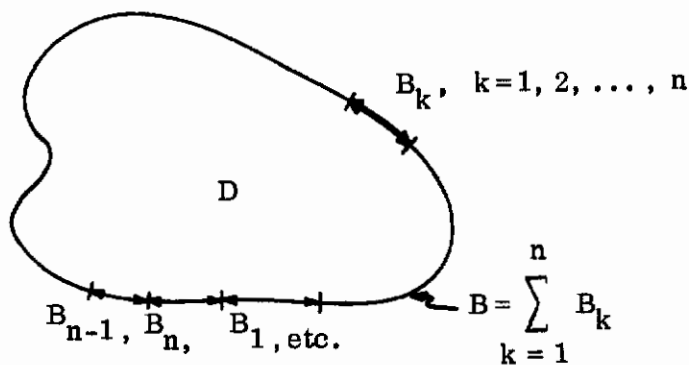


Figure 6. "Sub-Boundaries" for Modified Biezeno-Koch Method

**7. Multiple Particular Solutions**

This technique is really a variation upon any of the methods that use a trial family that satisfies the domain equation exactly. It requires that more than one particular solution ( $\varphi_0$ ) be known and proceeds as follows:

Suppose both

$$L \varphi_{01} = f$$

and

$$L \varphi_{02} = f \text{ in } D$$

Then, assume an approximate solution:

$$u_n = \lambda \varphi_{01} + (1-\lambda) \varphi_{02} + \sum_{k=1}^n a_k \varphi_k$$

where  $\lambda$  is an undetermined parameter similar to the  $a_k$ 's, and the  $\varphi_k$  satisfy

$$L \varphi_k = 0 \text{ in } D.$$

However, the  $B_i u_n \neq g_i$  on  $B$ .

Thus,  $u_n$  is only an approximation to  $u$  where  $u$  satisfies Equations (1) and (2) of Section II.

Proceed to determine  $\lambda$  and the  $a_k$  via one of the methods already listed which has zero residual in D (e.g., Point-matching).

Note that a  $\lambda$  of 0 selects  $\varphi_{02}$  for the particular solution, whereas a  $\lambda$  of 1 selects  $\varphi_{01}$ . Values between 0 and 1 for  $\lambda$  select various component combinations of each particular solution. However, there is no reason at all why  $\lambda$  must lie between 0 and 1 because  $\lambda \varphi_{01} + (1 - \lambda) \varphi_{02}$  always satisfies Equation (1). Moreover, the technique offers even more flexibility to several of the methods already considered if many particular solutions are known. In general, let  $s$  particular solutions,  $\varphi_{op}$  ( $p = 1, 2, \dots, s$ ) satisfy Equation (1).

Select  $u'$  of the form:

$$u' = \sum_{p=1}^{s-1} \lambda_p \varphi_{op} + \left(1 - \sum_{p=1}^{s-1} \lambda_p\right) \varphi_{os} + \sum_{k=1}^n a_k \varphi_k$$

from which it follows that the sum of the coefficients of the  $\varphi_{op}$  ( $p = 1, 2, \dots, s$ ) totals unity, and so the  $\varphi_{op}$ , taken collectively, satisfy Equation (1).

Obviously, the introduction of the  $\lambda_p$  leads to  $s - 1$  additional unknowns requiring at least  $n + s - 1$  equations for evaluation of them and the  $a_k$ 's. Furthermore, no additional algebraic difficulties are introduced by the methods under investigation because the equations remain linear in the unknowns, including the  $\lambda_p$ .

## 8. Ritz-Collocation

If an expression can be found that, when squared and integrated, results in a functional, the stationary value of which implies a boundary-value problem, it can be combined with a collocation technique to obtain approximate solutions. Furthermore, if the trial solution used satisfies the problem's boundary conditions and a sufficient number of points are collocated, the method approaches the Ritz method.

To demonstrate this, consider the heat-conduction problem:

$$\nabla^2 T = 0 \quad \text{in } D$$

$$T = F \quad \text{on } B$$

Assume an approximate solution  $T_n$ ,

$$T_n = \varphi_0 + \sum_{k=1}^n a_k \varphi_k$$

where

$$\varphi_0 = F \quad \text{on } B$$

and

$$\varphi_k = 0 \quad \text{on } B$$

Collocate the gradient of  $T_n$  at a sufficient number of points ( $m$ ) to generate an overdetermined system of algebraic vector (since the del operator is a vector operator) equations. That is:

$$\bar{\nabla} T_n (P_j) = \bar{\nabla} \phi_0 (P_j) + \sum_{k=1}^n a_k \bar{\nabla} \phi_k (P_j) = 0, \quad j = 1, 2, \dots, m \quad (52)$$

If  $D$  is an  $N$ -dimensional domain, Equation (52) results in  $N \times m$  equations. These equations are then solved in a least-squares sense, i.e., first premultiply Equation (52) by the transpose of  $\bar{\nabla} \phi_k (P_j)$ , to obtain:

$$\left[ \bar{\nabla} \phi_k (P_j) \right]^T \cdot \left[ \bar{\nabla} \phi_k (P_j) \right] \{a_k\} = - \left[ \bar{\nabla} \phi_0 (P_j) \right]^T \{ \nabla \phi_0 (P_j) \}, \quad (53)$$

and proceed to solve Equation (53) for  $\{a_k\}$ .

If a sufficient number of equally spaced points  $P_j$  are selected, this method approaches the Ritz method. This can be shown by multiplying Equation (52) by equal volume increments,  $A/M$ , where  $A$  is the total volume of  $D$ , and letting  $M \rightarrow \infty$ . Thus:

$$V \equiv \lim_{M \rightarrow \infty} \sum_{j=1}^M \left( \bar{\nabla} T_n (P_j) \right) \cdot \left( \bar{\nabla} T_u (P_j) \right) \frac{A}{M} = \int_D \bar{\nabla} T_n \cdot \bar{\nabla} T_n \cdot dA$$

and setting

$$\delta V = 0$$

is equivalent to solving the problem by the Ritz method.

## D. SUMMARY OF GENERAL METHOD

### 1. Introduction

Many new approximate methods are suggested within the framework outlined in Equation (8) and (9). The generality of this scheme was arrived at through a logical extension of existing techniques. While solution accuracy and applicability remains, for the present, questionable, there are several heuristic arguments that recommend the investigation of certain specific forms of Equation (8) and (9).

It is important to note that proper selection of  $\phi_0$  and the  $\phi_k$  remains the most important step in applying Equations (8) and (9). Good results cannot be obtained if good approximations are not included within the trial family  $\phi_k$ . While it is desirable

to determine the mathematical conditions under which  $u_n \rightarrow u$ , the principal attraction of the methods discussed lies in the possibility of obtaining good approximations with a limited number of terms  $a_k$  ( $k = 1, 2, \dots, n$ ). In selecting the  $\phi_0$  and  $\phi_k$ , sufficient consideration should be given to any special characteristics of the solution which are known, such as symmetry and solution components that satisfy either Equations (1) or (2) exactly. Examples of methods for selecting the  $\phi_0$  and  $\phi_k$ , will be shown in the following section.

Several additional methods that do not fall into the general scheme of the methods of undetermined parameters should be mentioned.

## 2. Method of Finite Differences

When the boundaries of the domain are not easily represented analytically and the differential equation within the domain is also difficult to satisfy exactly, it is then obligatory to turn to some numerical or graphical method to obtain an approximate solution to the problem. The most widely used of these techniques is the method of finite differences. In this method, the differential equation is replaced by an approximating difference equation and the continuous region  $D$  by a set of discrete points. This reduces the problem to the solution of systems of algebraic equations. For large systems, some iterative technique could be required together with the use of high-speed electronic computers. One of the important techniques is known as the relaxation method due originally to Southwell (Reference 29).

## 3. Method of Reduction of Partial Differential Equations to Ordinary Differential Equations

This is a generalization of the Ritz Method, and it was first proposed by L.V. Kantorovich in 1932 (Reference 13). In brief, it reduces the problem of solving a partial differential equation to systems of ordinary differential equations and, in a number of instances, has been shown to be more accurate than the Ritz procedure (using the same number of coordinate functions).

TABLE 1: TABULATION OF KNOWN APPROXIMATE METHODS BY UNDETERMINED PARAMETERS


KNOWN METHODS	Trial function satisfies differential equation	Trial function satisfies boundary conditions	Weighting Functions (Referring to notation of Section II-B-10)	SYMBOLS:  $a_k$ = Undetermined parameters $u_n$ = trail function $D$ = domain $P$ = point within domain $B$ = boundary $Q$ = point on boundary $DE$ = differential equation $BC$ = boundary conditions	COMMENTS
RITZ	NO	YES	$M_D \phi_k(P)$	Procedure used to generate a set of linear simultaneous equations in the undetermined parameters  $\text{set } \frac{\partial I}{\partial a_k} = 0$ $k = 1, 2, \dots, n$	Minimization of an appropriate functional, $I$ , is equivalent to solving a particular boundary value problem. (See Section III-2)
GALERKIN	NO	YES	$\phi_k(P)$ ( $= \frac{\partial u_n}{\partial a_k}$ )	$\text{set } \int_D R_{DE} \phi_k dA = 0$ where $u_n = \sum_{k=1}^n a_k \phi_k$ $k = 1, 2, \dots, n$	The differential equation residual is made orthogonal to each of the $n$ elements of the trial function, $u_n$ (i. e., $\phi_k$ , $k = 1, 2, \dots, n$ ) over the entire domain. If $L$ and $B_i$ are self-adjoint, Ritz method and Galerkin method are equivalent.
LEAST SQUARES	NO	YES	$L \phi_k(P)$ ( $= \frac{\partial R_{DE}}{\partial a_k}$ )	$\int_D (R_{DE})^2 dA = \text{Minimum}$ i. e., $\text{set } \int_D R_{DE} \frac{\partial R_{DE}}{\partial a_k} dA = 0$ $k = 1, 2, \dots, n$	Minimization of the integral of the differential equation residual squared leads to a set of $n$ simultaneous equations in the undetermined parameters

TABLE 1 (Cont'd)

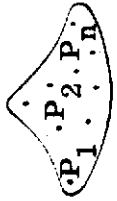
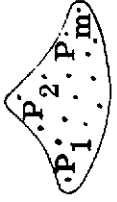


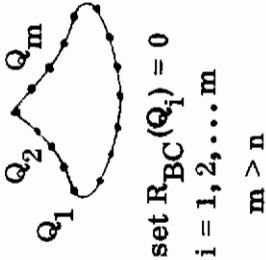
COLLOCATIONS	NO	YES	$\partial (P-P_k)$ $k = 1, 2, \dots, n$	 <p>set <math>R_{DE}(P_i) = 0</math> <math>i = 1, 2, \dots, n</math></p>	The differential equation residual is made to vanish at the same number of points within the domain as there are undetermined parameters.
LEAST-SQUARES COLLOCATIONS	NO	YES	$\delta (P-P_k)$ $k = 1, 2, \dots, m$ $m > n$	 <p>set <math>R_{DE}(P_i) = 0</math> <math>i = 1, 2, 3, \dots, m, m &gt; n</math></p>	Same as collocations except that the differential equation residual is made to vanish at more interior points than there are undetermined parameters. The resulting overdetermined system of equations is then solved in a least squares sense. (See Sect. II-B-11 and Appendix I)
BIEZENO-KOCH or SUB-DOMAIN	NO	YES	$H (P-P_{k-1})$ $-H (P-P_k)$ $k = 1, 2, \dots, n$	 <p>set <math>\int_{D_k} R_{DE} dA = 0</math> <math>k = 1, 2, \dots, n</math></p>	The domain is divided into smaller subdomains and the differential equation residual is made to vanish on an average basis over each of the subdomains. The number of subdomains equals the number of undetermined parameters.
POINT-MATCHING	YES	NO	$\delta (Q-Q_k)$ $k = 1, 2, \dots, n$	 <p>set <math>R_{BC}(Q_i) = 0</math> <math>i = 1, 2, \dots, n</math></p>	The boundary conditions are enforced at the same number of points on the boundary as there are undetermined parameters in the trial solution.



TABLE 1 (Cont'd)

<p>LEAST SQUARES POINT- MATCHING</p>	<p>YES</p>	<p>NO</p>	<p><math>\delta (Q-Q_k)</math> <math>k = 1, 2, \dots, m</math></p>	 <p>set <math>R_{BC}(Q_i) = 0</math> <math>i = 1, 2, \dots, m</math> <math>m &gt; n</math></p>	<p>Same as point matching except that the boundary conditions are enforced at more points than there are undetermined parameters and the resulting over-determined system solved in a least squares sense (see Sect. II-B-11)</p>
<p>MIKHLIN</p>	<p>YES</p>	<p>NO</p>	<p><math>B_i \phi_k</math> <math>k = 1, 2, \dots, n</math></p>	<p><math>\int_B (R_{BC})^2 dS = \text{Minimum}</math> i. e., set <math>\int_B R_{BC} \frac{\partial R_{BC}}{\partial a_k} dS = 0</math> <math>k = 1, 2, \dots, n</math></p>	<p>Similar to least squares but here the integral of the square of the boundary condition residual is minimized.</p>

## SECTION IV APPLICATIONS OF THE METHODS

### A. INTRODUCTION

Unlike very specialized methods for obtaining exact solutions to boundary-value problems, all procedures previously described possess the merit of being applicable to a wide range of problems. However, because the techniques are approximate, the question as to solution accuracy always remains.

In limited cases, convergence to the exact solution can be proved theoretically (e.g., Ritz's method when used in conjunction with a complete set of trial functions). This usually involves consideration of an infinite series. However, in reality, only a finite number of terms of a sequence can be treated. Thus, realistic error-bounds could be more applicable than a statement of eventual convergence. Unfortunately, the quantity of literature dealing with error bounds and of the sort desired seems rather scarce (examples can, however, be given; see References 25 and 29). Therefore, the method used to demonstrate the power of the preceding solution techniques will be the application of them to examples, the exact solutions of which are either known or for which accurate approximate solutions are known to a high confidence level.

Comparisons of results, as well as the steps leading to numerical answers, are presented for three problems, thus affording a detailed demonstration of the various methods. The presentation procedure followed is to state the problem and the manner of selecting a trial solution. Detailed application of the various methods are then followed with the exact (or highly accurate) solution. Finally, comparisons between the methods with regard to over-all suitability, are made.

The specific examples (Figures 7a, b, and c) selected to serve as expository vehicles for demonstrating application and accuracy of the methods are as follows:

- 1) A uniformly loaded clamped square plate in bending; this problem was selected because of its intractability via conventional exact solution methods, as well as the ease with which it lends itself to the known methods discussed. Over the years, the exact analytical solution for the clamped rectangular plate has remained an enigma despite numerous attempts to obtain it. Although it retains the status of a classic for this reason, general agreement exists as to what its correct numerical solution. This is based on the mutual concurrence of many investigators, who have obtained very similar results using a variety of different approximate methods.
- 2) Two-dimensional, steady state heat conduction in a  $\frac{2\pi}{3}$  radian sector of an annulus (ratio of outer to inner radii is 1.6), zero reference temperature along three sides, and a symmetrical cosine temperature distribution over the outer edge;

Originally, the problem involved three insulated surfaces with the same outer surface cosine temperature distribution. However, it was difficult to find suitable trial functions for the methods which required exact

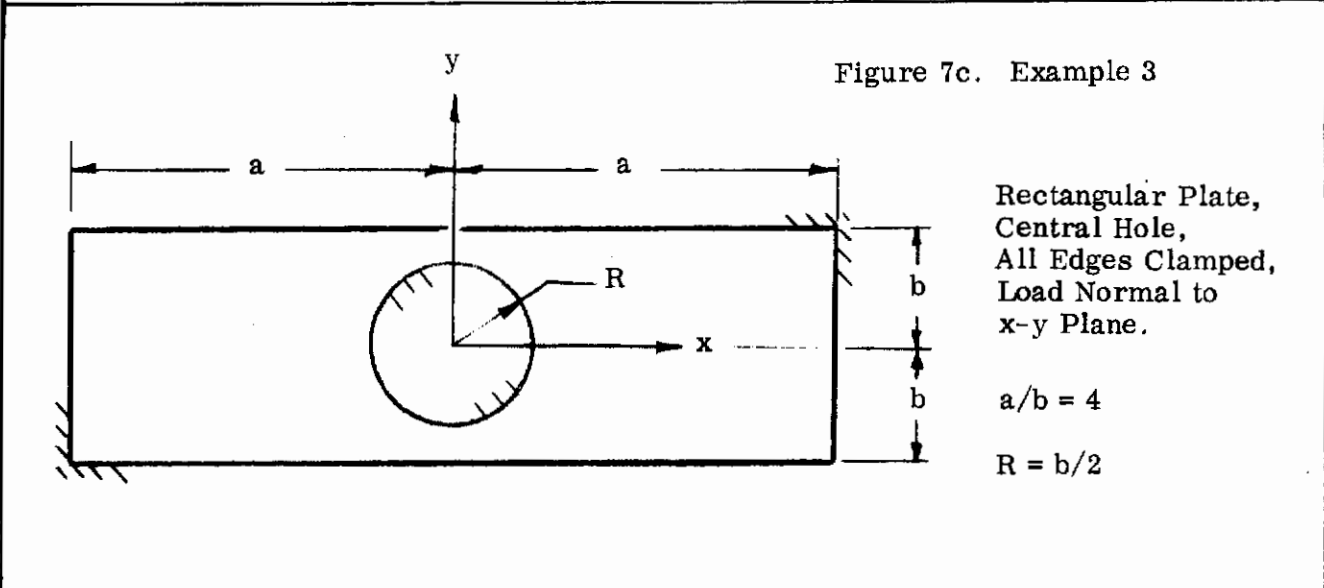
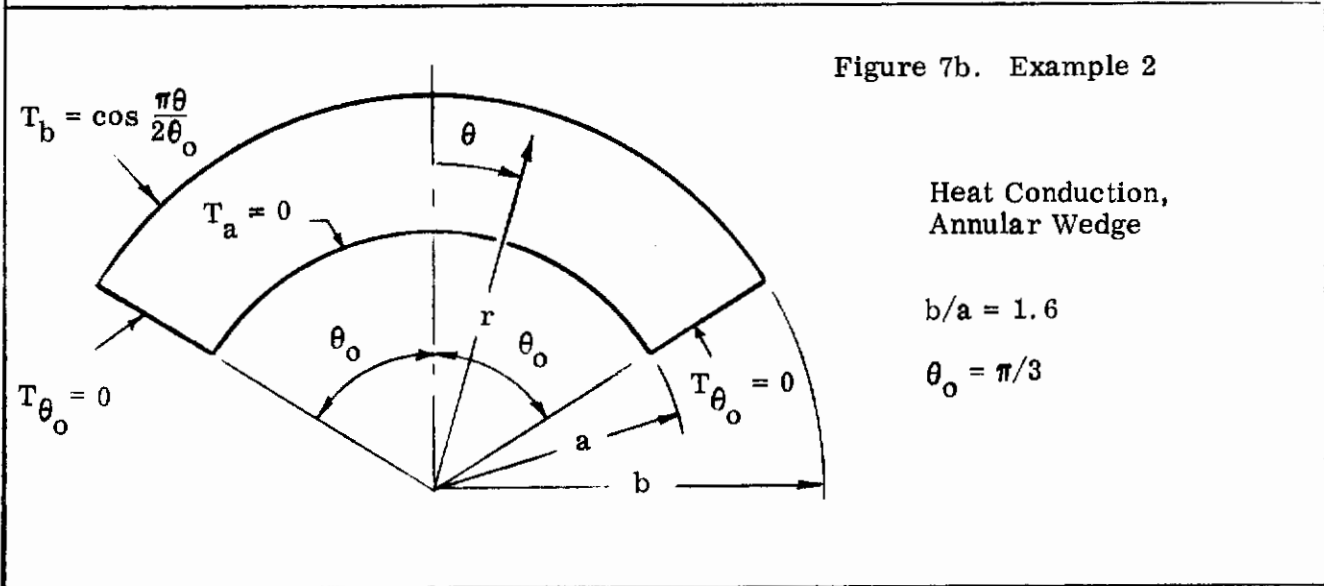
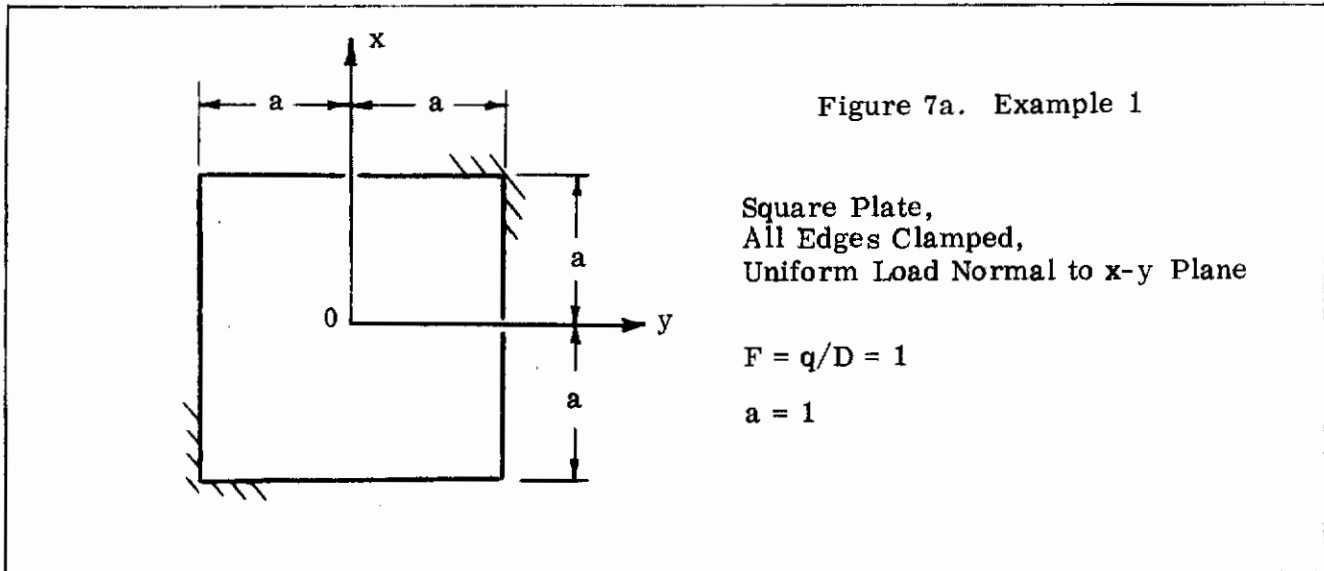


Figure 7. Numerical Examples Considered

satisfaction of the boundary conditions (e.g., Galerkin's method). The opening angle was carefully selected to avoid coincidence with any that would cause the trial functions used to include the exact solution. This would favor the methods using such functions, and thus give them an unfair advantage. The exact solution to the problem, as stated, is easily obtainable through application of the separation-of-variables solution approach and results in a closed solution form.

- 3) Bending of rectangular plate, 4 to 1 aspect ratio, with a symmetrically located internal circular hole having a diameter equal to one-half the shorter side, and clamped on all boundaries. The transverse loading is determined by an inverse method; i.e., a deflection function compatible with the boundary conditions is selected, and the loading required to produce this deflection is determined. This, then, becomes the plate loading considered.

The trial functions used to obtain approximate solutions by the methods that satisfy boundary conditions exactly do not include the exact solution. This is intentional because inclusion of the exact solution in the trial family would result in exact results by all the approximate methods tried.

Selection of this problem was based upon its complexity, in order to highlight some of the difficulties one can expect to encounter in many real problems. Furthermore, selection of clamped boundaries was based upon anticipated difficulties that could be expected in selecting trial functions for methods that require exact satisfaction of boundary conditions.

## B. EXAMPLE 1 - CLAMPED SQUARE PLATE WITH A UNIFORM LOAD

### 1. Problem Statement and Selection of Trial Functions

Equations governing this example are:

$$\nabla^4 w = 1 \quad \text{in } D \quad (54)$$

$$w = 0 \quad \text{on } B \quad (55)$$

and

$$\frac{\partial w}{\partial \eta} = 0 \quad \text{on } B \quad (56)$$

where  $D$  is the two-dimensional domain surrounded by  $B$ , defined by  $x = \pm 1$  and  $y = \pm 1$ ;  $\eta$  is the outward normal coordinate to  $B$  (see Figure 7a).

Quantities of interest for this problem are:

Deflection,  $w(x, y)$

Slope,  $\frac{\partial w}{\partial x}$  or  $\frac{\partial w}{\partial y}$

Bending moment,  $M_x = \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}$ ,  $M_y = \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}$

# Contrails

where, for numerical comparisons, Poisson's ratio,  $\nu$ , is taken as 0.3. The problem, as just defined, has been nondimensionalized through selection of boundaries  $x = \pm 1$ ,  $y = \pm 1$ , and loading to bending stiffness ratio of 1. If the sides of the plate are  $2a \times 2a$  with loading  $q$  and stiffness  $\bar{D}$ , then the preceding quantities must be changed as follows:

$$\text{Deflection; } w = ( ) \frac{q a^4}{\bar{D}}$$

$$\text{Slope; } \frac{\partial w}{\partial x} = ( ) \frac{q a^3}{\bar{D}}, \quad \frac{\partial w}{\partial y} = ( ) \frac{q a^3}{\bar{D}}$$

$$\text{Moment; } M = ( ) q a^2$$

The trial solution used to obtain approximations,  $w_n$ , that satisfy the boundary conditions is:

$$w_n = \varphi_0 + \sum_{k=1}^n a_k \varphi_k \quad (57)$$

One way in which  $w_n$  could satisfy Equation (55) is to construct each term with the factor  $(x^2 - 1)(y^2 - 1)$ . However, this factor must be squared as well if each term is to also satisfy Equation (56). Thus, Equation (57) becomes:

$$w_n = (x^2 - 1)^2 (y^2 - 1)^2 \left( \psi_0 + \sum_{k=1}^n a_k \psi_k \right)$$

where the  $\psi$ 's have yet to be defined.

At this point, the selection of the  $\psi$ 's must be based upon experience, with the understanding that some degree of uncertainty of a suitable choice exists. For simplicity, select:

$$\psi_0 = 0$$

$$\psi_k = r^{2(k-1)} = (x^2 + y^2)^{k-1}, \quad k = 1, 2, \dots, n$$

Note that these  $\psi$ 's are convenient to work with, e.g., derivatives can be taken with a minimum of algebraic manipulation and possess the symmetry required by the physical problem (i.e.,  $w$  is symmetric about the  $x$  and  $y$  axes, as well as the lines  $x = \pm y$  and, therefore, the  $\varphi$ 's incorporate this symmetry). Thus, the following trial family is obtained:

$$w_n = (x^2 - 1)^2 (y^2 - 1)^2 \sum_{k=1}^n a_k (x^2 + y^2)^{k-1} \quad (58)$$

for methods that must satisfy Equations (55) and (56) exactly.

To demonstrate the degree of freedom permitted in selection of the  $\psi$ 's, the following several convenient alternatives are presented:

$$\psi_k = (x y)^{2k}, \quad k = 1, 2, \dots, n$$

or

$$\psi_k = e^{k(x^2 + y^2)}, \quad k = 1, 2, \dots, n$$

The selection of  $\psi_0$  as zero was simply a matter of convenience because it did not appear necessary to retain a quantity without an undetermined multiplier. In many cases, however, this term is called upon to perform a special function in the over-all approximate solution, e. g., to accommodate special load or support conditions in the approximate solution.

The trial family used to obtain approximations,  $w_n$ , that satisfy Equation (54) is:

$$w_n = w_p + w_h$$

where

$$\nabla^4 w_p = 1, \quad \nabla^4 w_h = 0$$

and (see Reference 17)

$$\begin{aligned} w_h = & A_0 + B_0 \ln r + C_0 r^2 + D_0 r^2 \ln r + E_0 r^2 \theta + F_0 \theta \\ & + [A_1 r + B_1 r^{-1} + C_1 r^3 + D_1 r \ln r + E_1 r \theta] \cos \theta \\ & + [\bar{A}_1 r + \bar{B}_1 r^{-1} + \bar{C}_1 r^3 + \bar{D}_1 r \ln r + \bar{E}_1 r \theta] \sin \theta \\ & + \sum_{p=2}^m [A_p r^p + B_p r^{-p} + C_p r^{p+2} + D_p r^{-p+2}] \cos p \theta \\ & + \sum_{p=2}^m [\bar{A}_p r^p + \bar{B}_p r^{-p} + \bar{C}_p r^{p+2} + \bar{D}_p r^{-p+2}] \sin p \theta \end{aligned}$$

This was selected because it possesses sufficient flexibility to suit the needs of many plate bending problems and is well known.

Specifically, if the origin is placed at the plate's center, the required single valuedness of solutions in  $\theta$  (if the  $\theta$  coordinate is permitted to continue circling the origin more than once) requires that

$$E_0 = F_0 = 0$$

Furthermore, because  $w_n$  and its derivatives should be finite at the origin for this problem, the following constants are set to zero:

$$B_0, B_1, \bar{B}_1, B_p, \bar{B}_p$$

$$D_0, D_1, \bar{D}_1, D_p, \bar{D}_p$$

Measuring  $\theta$  from the  $x$  axis, symmetry considerations dictate that all terms involving  $\sin p\theta$  be dropped, and only those cosine terms for which  $p$  is a multiple of four be retained. Thus,  $w_n$  reduces to:

$$w_n = w_p + A_0 + C_0 r^2 + \sum_{p=4,8,\dots} (A_p r^p + C_p r^{p+2}) \cos p\theta \quad (59)$$

Two suitable values for  $w_p$  are  $r^4/64$  and  $x^2 y^2/8$  because both retain the required symmetry and satisfy Equation (54). Many other particular solutions can be found. However, the first of these is arbitrarily selected. Thus,

$$\phi_0 = \frac{r^4}{64}$$

and

$$\left. \begin{aligned} \phi_k &= r^{4(k-1)} \cos 4(k-1)\theta \\ \phi_k &= r^{4(k-1)+2} \cos 4(k-1)\theta \end{aligned} \right\} k = 1, 2, \dots$$

## 2. Ritz-Galerkin Applied to Example 1

As shown in Appendix II, the Ritz and Galerkin methods are identical for the  $\nabla^4$  operator. Therefore, for the present, this method has been termed Ritz-Galerkin (R-G). As previously developed, the procedure starts with the solution form of Equation (58), which satisfies the boundary condition of Equations (55) and (56), and the following  $n$  conditions are enforced:

$$\int_{-1}^1 \int_{-1}^1 (x^2 - 1)^2 (y^2 - 1)^2 (x^2 + y^2)^{k-1} (\nabla^4 w_n - 1) dx dy = 0$$

$$k = 1, 2, \dots, n$$

This yields a system of algebraic equations in the  $a_k$  which can be solved simultaneously. To demonstrate this for  $n=1$ , start with

$$w_1 = a_1 (x^2 - 1)^2 (y^2 - 1)^2$$

then

$$\nabla^4 w_1 - 1 = a_1 \left[ 24(y^2 - 1)^2 + 24(x^2 - 1)^2 + 32(3x^2 - 1)(3y^2 - 1) \right] - 1$$

and

$$4 \int_0^1 \int_0^1 (x^2 - 1)^2 (y^2 - 1)^2 \left\{ a_1 \left[ 24(y^2 - 1)^2 + 24(x^2 - 1)^2 + 32(3x^2 - 1)(3y^2 - 1) \right] - 1 \right\} dx dy = 0$$

The preceding condition is integrated to yield:

$$a_1 = \frac{.28444}{13.375} = .021266$$

Thus,

$$w_1 = .02127 (x^2 - 1)^2 (y^2 - 1)^2$$

To demonstrate the augmented labor and opportunity to develop errors by retention of only one more term in the trial family, the beginnings of a two-term solution are presented:

$$w_2 = a_1 (x^2 - 1)^2 (y^2 - 1)^2 + a_2 (x^2 - 1)^2 (y^2 - 1)^2 (x^2 + y^2)$$

$$\begin{aligned} \nabla^4 w_2 - 1 = a_1 & \left[ 24(y^2 - 1)^2 + 24(x^2 - 1)^2 + 32(3x^2 - 1)(3y^2 - 1) \right] \\ & + a_2 \left[ 24(y^2 - 1)^2 (x^2 + y^2) + 192 x^2 (y^2 - 1)^2 + 48(3x^2 - 1)(y^2 - 1)^2 \right. \\ & + 24(x^2 - 1)^2 (x^2 + y^2) + 192 y^2 (x^2 - 1)^2 + 48(3y^2 - 1)(x^2 - 1)^2 \\ & + 32(3x^2 - 1)(3y^2 - 1)(x^2 + y^2) + 128 y^2 (3x^2 - 1)(y^2 - 1) + 63 (3x^2 - 1)(y^2 - 1)^2 \\ & \left. + 128 x^2 (3y^2 - 1)(x^2 - 1) + 16 (3y^2 - 1)(x^2 - 1)^2 \right] - 1 \end{aligned}$$

Set:

$$4 \int_0^1 \int_0^1 (x^2 - 1)^2 (y^2 - 1)^2 \left\{ \nabla^4 w_2 - 1 \right\} dx dy = 0$$

and

$$4 \int_0^1 \int_0^1 (x^2 - 1)^2 (y^2 - 1)^2 (x^2 + y^2) \left\{ \nabla^4 w_2 - 1 \right\} dx dy = 0$$

Upon detailed evaluation of these two integrals, two algebraic equations in  $a_1$  and  $a_2$  result. These are then solved simultaneously for a two-term Ritz-Galerkin solution.



3. Least-Squares Applied to Example 1

Starting with the solution form of Equation (58) (which satisfies the boundary conditions), set:

$$\int_0^1 \int_0^1 \frac{\partial(\nabla^4 w_n - 1)}{\partial a_k} (\nabla^4 w_n - 1) dx dy = 0, \quad k = 1, 2, \dots, n$$

After performing the integrations indicated, a system of  $n$  equations in the  $n a_k$  results that can be solved simultaneously.

Regardless of the number of terms retained in the solution form, least-squares requires several times the effort required by the Ritz-Galerkin method for a similar term solution. Thus, for comparison purposes, only the  $n=1$  case is solved here, if,

$$w = a_1 \phi_1 = a_1 (x^2 - 1)^2 (y^2 - 1)^2$$

$$\frac{\partial(\nabla^4 w_1 - 1)}{\partial a_1} = \nabla^4 \phi_1 = 24(y^2 - 1)^2 + 24(x^2 - 1)^2 + 32(3x^2 - 1)(3y^2 - 1)$$

$$\nabla^4 w_1 - 1 = a_1 \nabla^4 \phi_1 - 1$$

$$4 \int_0^1 \int_0^1 \nabla^4 \phi_1 (\nabla^4 w_1 - 1) dx dy = 0$$

$$a_1 = \frac{4480}{253,950} = .017641$$

Hence,

$$w_1 = .01764 (x^2 - 1)^2 (y^2 - 1)^2$$

4. Biezeno-Koch Applied to Example 1

Selecting a trial family in the form of Equation (58), set:

$$\int_{D_k} (\nabla^4 w_n - 1) dA_k = 0, \quad k = 1, 2, \dots, n$$

where  $dA_k$  is a differential element of the subdomain  $D_k$ , which is itself contained within the square domain  $D$ .

For illustrative purposes, a one parameter solution with only one sub-domain that coincides with the entire domain is selected first. Hence:

$$w_1 = a_1 (x^2 - 1)^2 (y^2 - 1)^2$$

$$\nabla^4 w_1 - 1 = a_1 [24(y^2 - 1)^2 + 24(x^2 - 1)^2 + 32(3x^2 - 1)(3y^2 - 1)] - 1$$

$$4 \int_0^1 \int_0^1 (\nabla^4 w_1 - 1) dx dy = 0$$

This is integrated to obtain:

$$a_1 = \frac{5}{128} = .03906$$

Thus:

$$w_1 = .03906 (x^2 - 1)^2 (y^2 - 1)^2$$

This analysis is now extended to a four parameter, and therefore, a four-subdomain solution, where the subdomains for one quadrant are shown in Figure 8.

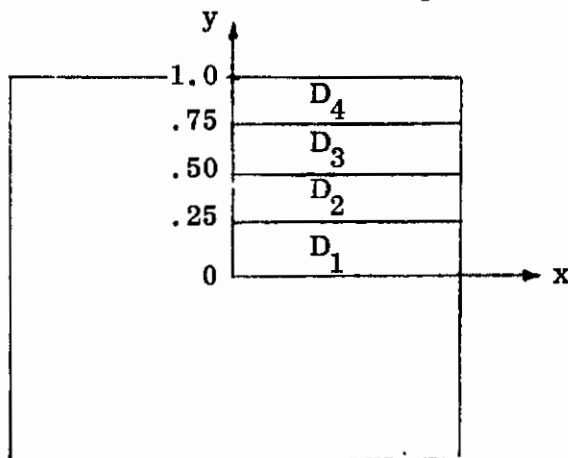


Figure 8. Subdomains used for a Four Parameter Biezeno-Koch Solution of Example 1

Actually, because the trial functions are symmetrical about the  $x = y$  line, the subdomains over which equilibrium  $(\nabla^4 w_n - 1)$  are averaged overlap one another across this line, as well as across the  $x$  and  $y$  axes.

Thus, from Equation (58):

$$w_4 = (x^2 - 1)^2 (y^2 - 1)^2 [a_1 + a_2(x^2 + y^2) + a_3(x^2 + y^2)^2 + a_4(x^2 + y^2)^3]$$

# Contrails

$$\begin{aligned}
 \nabla^4 w_4 - 1 = & a_1 \left[ 24(y^2 - 1)^2 + 24(x^2 - 1)^2 + 32(3x^2 - 1)(3y^2 - 1) \right] \\
 & + a_2 \left[ 24(y^2 - 1)^2(x^2 + y^2) + 192x^2(y^2 - 1)^2 + 48(3x^2 - 1)(y^2 - 1)^2 \right. \\
 & \quad + 24(x^2 - 1)^2(x^2 + y^2) + 192y^2(x^2 - 1)^2 + 48(3y^2 - 1)(x^2 - 1)^2 \\
 & \quad + 32(3x^2 - 1)(3y^2 - 1)(x^2 + y^2) + 128y^2(3x^2 - 1)(y^2 - 1) + 16(3x^2 - 1)(y^2 - 1)^2 \\
 & \quad \left. + 128x^2(3y^2 - 1)(x^2 - 1) + 16(3y^2 - 1)(x^2 - 1)^2 \right] \\
 & + a_3 \left[ 24(y^2 - 1)^2(x^2 + y^2)^2 + 384x^2(y^2 - 1)^2(x^2 + y^2) + 96(3x^2 - 1)(y^2 - 1)^2(3x^2 + y^2) \right. \\
 & \quad + 16 \times 24x^2(x^2 - 1)(y^2 - 1)^2 + 24(x^2 - 1)^2(y^2 - 1)^2 \\
 & \quad + 24(x^2 - 1)^2(x^2 + y^2)^2 + 384y^2(x^2 - 1)^2(x^2 + y^2) + 96(3y^2 - 1)(x^2 - 1)^2(3y^2 + x^2) \\
 & \quad + 16 \times 24y^2(y^2 - 1)(x^2 - 1)^2 + 24(x^2 - 1)^2(y^2 - 1)^2 \\
 & \quad + 32(3x^2 - 1)(3y^2 - 1)(x^2 + y^2)^2 + 256y^2(3x^2 - 1)(y^2 - 1)(x^2 + y^2) \\
 & \quad + 32(3x^2 - 1)(y^2 - 1)^2(x^2 + 3y^2) + 256x^2(3y^2 - 1)(x^2 - 1)(x^2 + y^2) \\
 & \quad + 8 \times 128x^2y^2(x^2 - 1)(y^2 - 1) + 128x^2(x^2 - 1)(y^2 - 1)^2 + 32(3y^2 - 1)(x^2 - 1)^2(y^2 + 3x^2) \\
 & \quad \left. + 128y^2(x^2 - 1)^2(y^2 - 1) + 16(x^2 - 1)^2(y^2 - 1)^2 \right] \\
 & + a_4 \left[ 24(y^2 - 1)^2(x^2 + y^2)^3 + 576x^2(y^2 - 1)^2(x^2 + y^2)^2 + 144(3x^2 - 1)(y^2 - 1)^2(5x^4 + 6x^2y^2 + y^4) \right. \\
 & \quad + 16 \times 24x^2(x^2 - 1)(y^2 - 1)^2(5x^2 + 3y^2) + 72(x^2 - 1)^2(y^2 - 1)^2(5x^2 + y^2) \\
 & \quad + 24(x^2 - 1)^2(x^2 + y^2)^3 + 576y^2(x^2 - 1)^2(x^2 + y^2)^2 + 144(3y^2 - 1)(x^2 - 1)^2(5y^4 + 6x^2y^2 + x^4) \\
 & \quad + 16 \times 24y^2(y^2 - 1)(x^2 - 1)^2(5y^2 + 3x^2) + 72(x^2 - 1)^2(y^2 - 1)^2(5y^2 + x^2) \\
 & \quad + 32(3x^2 - 1)(3y^2 - 1)(x^2 + y^2)^3 + 384y^2(3x^2 - 1)(y^2 - 1)(x^2 + y^2)^2 \\
 & \quad + 48(3x^2 - 1)(y^2 - 1)^2(x^4 + 6x^2y^2 + 5y^4) + 384x^2(x^2 - 1)(3y^2 - 1)(x^2 + y^2)^2 \\
 & \quad + 128 \times 24x^2y^2(x^2 - 1)(y^2 - 1)(x^2 + y^2) + 16 \times 24x^2(x^2 - 1)(y^2 - 1)^2(x^2 + 3y^2) \\
 & \quad + 48(x^2 - 1)^2(3y^2 - 1)(y^4 + 6x^2y^2 + 5x^4) + 16 \times 24y^2(y^2 - 1)(x^2 - 1)^2(y^2 + 3x^2) \\
 & \quad \left. + 144(x^2 - 1)^2(y^2 - 1)^2(x^2 + y^2) \right] - 1.
 \end{aligned}$$

Set

$$\int_0^{.25} \int_0^1 (\nabla^4 w_4 - 1) dx dy = 0,$$

$$\int_{.25}^{.50} \int_0^1 (\nabla^4 w_4 - 1) dx dy = 0,$$

$$\int_{.50}^{.75} \int_0^1 (\nabla^4 w_4 - 1) dx dy = 0,$$

and

$$\int_{.75}^{1.0} \int_0^1 (\nabla^4 w_4 - 1) dx dy = 0,$$

To simplify the preceding integrations, the following approximation was made:

$$0 = \int_{y_i}^{y_{i+1}} \int_0^1 (\nabla^4 w_4(x, y) - 1) dx dy \approx \int_0^1 (\nabla^4 w_4(x, \bar{y}_i) - 1) dx$$

where  $\bar{y} = y_i + \frac{.25}{2}$ ,  $y_i = .25, .5, .75, 1.0$

When the four algebraic equations obtained were solved simultaneously, the following a's were obtained:

$$a_1 = .020264$$

$$a_2 = .0068144$$

$$a_3 = -.0003590$$

$$a_4 = -.0000302$$

A digital computer program based upon a similar integration and averaging approximation on the y's was developed. It was used to generate higher number-of-parameter solutions and to see if the technique converged as more and more terms were included. Results from this program are presented in the comparisons presented later.

## 5. Collocations Applied to Example 1

The differential equation is satisfied point wise within the domain. This results in one algebraic equation for each internal point selected. The trial solution used, once again, satisfies the boundary conditions. Thus:

$$w_n = (x^2 - 1)^2 (y^2 - 1)^2 \sum_{k=1}^n a_k (x_k^2 + y_k^2), \quad k = 1, 2, \dots, n$$

# Contrails

$$\nabla^4 w_n(x_{\bar{k}}, y_{\bar{k}}) = 1$$

where  $(x_{\bar{k}}, y_{\bar{k}})$  are the points collocated.

For only a one-term solution, collocation of the central point  $(x, y) = (0, 0)$  yields:

$$\nabla^4 w_1(0, 0) = a_1 [24(0-1)^2 + 24(0-1)^2 + 32(0-1)(0-1)] = 1$$

Thus:

$$a_1 = \frac{1}{80} = .0125$$

A four-term solution, collocated at the four points  $(x, y) = (0, 0), (.5, 0), (.5, .25), (.5, .5)$ , yields:

$$a_1 = .019166$$

$$a_2 = .004878$$

$$a_3 = .001422$$

$$a_4 = -.001876$$

Note that symmetry of the trial solution also guarantees, for this solution, a zero equation residual at the thirteen additional points  $(x, y) = (-.5, 0), (0, \pm .5), (.5, -.25), (-.5, \pm .25), (\pm .25, \pm .5), (.5, -.5), (-.5, \pm .5)$ .

If  $\nabla^4 w_n(x_{\bar{k}}, y_{\bar{k}}) - 1$  is collocated at  $\bar{k} = 1, 2, \dots, m > n$  points, an over-determined system results. This can then be solved in a least-square sense, thus resulting in a least-squares collocations procedure. Several cases of this were tried using various points and retaining different numbers of terms in the trial family. One four-parameter example, for which detailed results are presented later, yielded:

$$a_1 = .020393$$

$$a_2 = .0051165$$

$$a_3 = .000518$$

$$a_4 = .000171$$

for which the following twelve points were used:

$$(x, y) = (0, 0), (.2, 0), (.4, 0), (.6, 0), (.8, 0)$$

$$(.2, .2), (.4, .2), (.6, .3), (.4, .4), (.8, .4)$$

$$(.6, .6), (.8, .8)$$

It should be noted these points are located within an octant of the plate and because of symmetrical reflections, are representative of the entire plate.

A digital computer program based on this approach and using up to 30 parameters was developed.

## 6. Mikhlin Applied to Example 1

This method begins with the trail family that satisfies the differential equation but not the boundary conditions. The boundary conditions are then satisfied in a least-squares sense. To demonstrate this with four-parameter solution, the first several terms of Equation (59) are first converted to cartesian coordinates

$$w_4 = \frac{x^4 + 2x^2y^2 + y^4}{64} + a_1 + a_2(x^2 + y^2) + a_3(x^4 - 6x^2y^2 + y^4) + a_4(x^6 - 5x^4y^2 - 5x^2y^4 + y^6) \quad (60)$$

Setting the following integrals to zero:

$$\int_0^1 \frac{\partial w_4}{\partial a_k} w_4 dx = 0, \quad k = 1, 2, 3, 4$$

$$\int_0^1 \frac{\partial(\frac{\partial w_4}{\partial x})}{\partial a_k} \frac{\partial w_4}{\partial x} dx = 0, \quad k = 1, 2, 3, 4$$

(the symmetry of  $w$  and  $w_4$  permits consideration of only 1/8th the total boundary) results in seven algebraic equations (because  $k=1$  for the second integral simply yields the equation  $0 = 0$ ). When solved in an algebraic least-squares sense, these equations yield:

$$a_1 = .021635$$

$$a_2 = -.03641$$

$$a_3 = -.002708$$

$$a_4 = .003128$$

## 7. Point-Matching Applied to Example 1

Once again, this method starts with an exact solution to the differential equation. However, as with collocations, the boundary conditions are matched at discrete points. When the number of undetermined parameters coincides with the number of conditions matched at various points, the parameters are solved for through a direct inversion of the coefficient matrix. If, however, the resulting system is over-determined, the algebraic equations are solved in a least-squares sense.

# Contraails

To develop a four-term solution, start with Equation (59) to obtain:

$$w_4 = \frac{r^4}{64} + a_1 + a_2 r^2 + a_3 r^4 \cos 4\theta + a_4 r^6 \cos 4\theta \quad (61)$$

$$\frac{\partial w_4}{\partial r^4} = \frac{r^3}{16} + 2a_2 r + 4a_3 r^3 \cos 4\theta + 6a_4 r^5 \cos 4\theta \quad (62)$$

Proceed to set  $w_4$  and  $\frac{\partial w_4}{\partial r} = 0$  at

$$(r, \theta) = (1, 0) \text{ and } (\sqrt{2}, \frac{\pi}{4}).$$

Once again, as a result of the problems and  $w_4$ 's symmetries, this is equivalent to matching the slope and deflection at the additional six points

$$r, \theta = (1, \pm \frac{\pi}{2}), (1, \pi), (\sqrt{2}, \pm \frac{3\pi}{4}), (\sqrt{2}, -\frac{\pi}{4})$$

as well.

It can be objected that the boundary condition imposed should be  $\partial w_4 / \partial x$  rather than  $\partial w_4 / \partial r$ . However, because  $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cos \theta + \frac{\partial w}{\partial y} \sin \theta$  and both  $\frac{\partial w}{\partial x}$  and  $\frac{\partial w}{\partial y}$  vanish separately on B, a necessary condition on  $w$  is that  $\partial w / \partial r = 0$  on B. Thus, use of the condition  $\frac{\partial w_4}{\partial r}$  can be justified here.

Results of the point matching indicated and the simultaneous solution for the parameters, yields:

$$a_1 = .018939$$

$$a_2 = -.034090$$

$$a_3 = -.004261$$

$$a_4 = .003788$$

A digital computer program based upon least-squares point-matching (i. e., the  $a$ 's are obtained from an over-determined system of equations) was developed for this problem, using the simpler condition  $\partial w / \partial r$  at discrete points on B. Numerical results, based on retaining various numbers of terms, were obtained and are presented later.

It has been suggested that selection of an equal spacing of points, with an accounting for symmetries on B, could result in more accurate solutions. Thus, a second solution using four parameters, was obtained, in which the deflection and radial slope were collocated at the points

$$(x, y) = (1, .25), (1, .75)$$

Considering symmetry, this resulted in satisfaction of the boundary conditions at the additional fourteen points

$$(x, y) = (1, -.25), (1, -.75), (-1, \pm.25), (-1, \pm.75), \\ (\pm.25, \pm 1), (\pm.75, \pm 1)$$

The solution for the case was:

$$a_1 = .020252$$

$$a_2 = -.035266$$

$$a_3 = -.005491$$

$$a_4 = .004539$$

8. Comparisons for Example 1 (Solutions are presented in Figure 38 and Tables XI through XIX)

For purposes of comparing the aforementioned results, it is desirable to have a uniform measure of accuracy for the various quantities of interest. Thus, dimensionless errors were defined as ratios with respect to the maximum values, and these are converted to percentages,  $\epsilon_\chi$ , as follows:

$$\epsilon_\chi = \frac{\chi_n(x, y) - \chi(x, y)}{\chi_M} \times 100\%$$

where  $\chi_n$  is the quantity whose accuracy is under evaluation,  $\chi$  is the exact value at the same point  $x, y$  and  $\chi_M$  is the maximum (or minimum depending on sign) value the exact solution attains in  $D$ . Thus,  $\chi$ , can represent deflection, bending moment, slope, or any other plate quantity of interest with perhaps one exception; that exception is the domain equation-residual. Physically nonvanishing domain residuals can be viewed as errors in the external loading. Thus, a more suitable dimensionless residual error is used

$$\epsilon_R = \frac{R \times 100\%}{(q/D)_{\max}} = R \times 100\%$$

Although the exact (closed form) solution for the problem is not known, it is known numerically to a high degree of confidence. The method used here to obtain it was to use the least-squares point-matching computer program mentioned earlier and to collocate the boundary conditions at 60 equally spaced points while using the first thirty parameters in the trial family. Upon resubstitution of the parameters into the boundary conditions. The average-absolute integrated boundary errors were:

$$\text{Deflection: } \frac{\int_B |\epsilon_{w_{30}}| dS}{\int_B dS} \approx .000005\%$$



$$\text{Slope: } \frac{\int_B \epsilon_{SL30} dS}{\int_B dS} \approx .00006\%$$

Because the  $w_{30}$  trial solution satisfied the differential equation exactly, and the boundary errors were extremely small, it seemed reasonable to accept this as the exact solution (see Figure 38).

The  $\chi$  quantities used to compare some of the hand solutions obtained are deflection ( $w$ ), radial moment ( $M$ ), radial slope ( $\partial w / \partial r$ ), and domain equation residual ( $R$ ). It has been found that the domain accuracy can be ascertained by simply plotting these curves along the  $x$ -axis because similar curves plotted along the lines  $x=y$  and  $x=2y$  lead to identical conclusions. However, because the over-all boundary accuracy leads to useful results, these are presented.

Figures 9 through 16 compare  $\epsilon_w$ ,  $\epsilon_M$ ,  $\epsilon_{SL}$ , and  $\epsilon_R$  along the plate's  $x$ -axis (shown by solid curves) and clamped boundary (dotted curves) for the following solution methods:

- 1) One-term Ritz-Galerkin
- 2) One-term least-squares
- 3) Four-term Biezeno-Koch
- 4) Four-term collocation
- 5) Four-term twelve point least-squares collocation
- 6) Four-term Mikhlin
- 7) Four-term point-matching
- 8) Four-term equally spaced point-matching

The trial families of the latter three methods satisfy the biharmonic equation whereas the previous ones satisfy the homogeneous deflection and slope boundary conditions.

The curves of these figures reveal that the errors  $\epsilon_w$ ,  $\epsilon_M$ , and  $\epsilon_{SL}$  for any one method are, quantitatively speaking, quite similar within the plate interior. Boundary errors are generally higher, except where the particular method in question ensures that certain boundary errors vanish (e.g., the Ritz-Galerkin method yields  $\epsilon_w = 0$  on the boundary). This is reflected in the biharmonic residuals on the boundaries, which are several times those in the interior (it should be noted that different scales are used in the graphs for equation residuals on the boundary from those occurring in the domain).

Note: Solid Lines Indicate Interior Errors ( $y = 0$ , Absissa =  $x/a$ )  
 Dashed Lines Indicate Boundary Errors ( $x=a$ , Absissa =  $y/a$ )

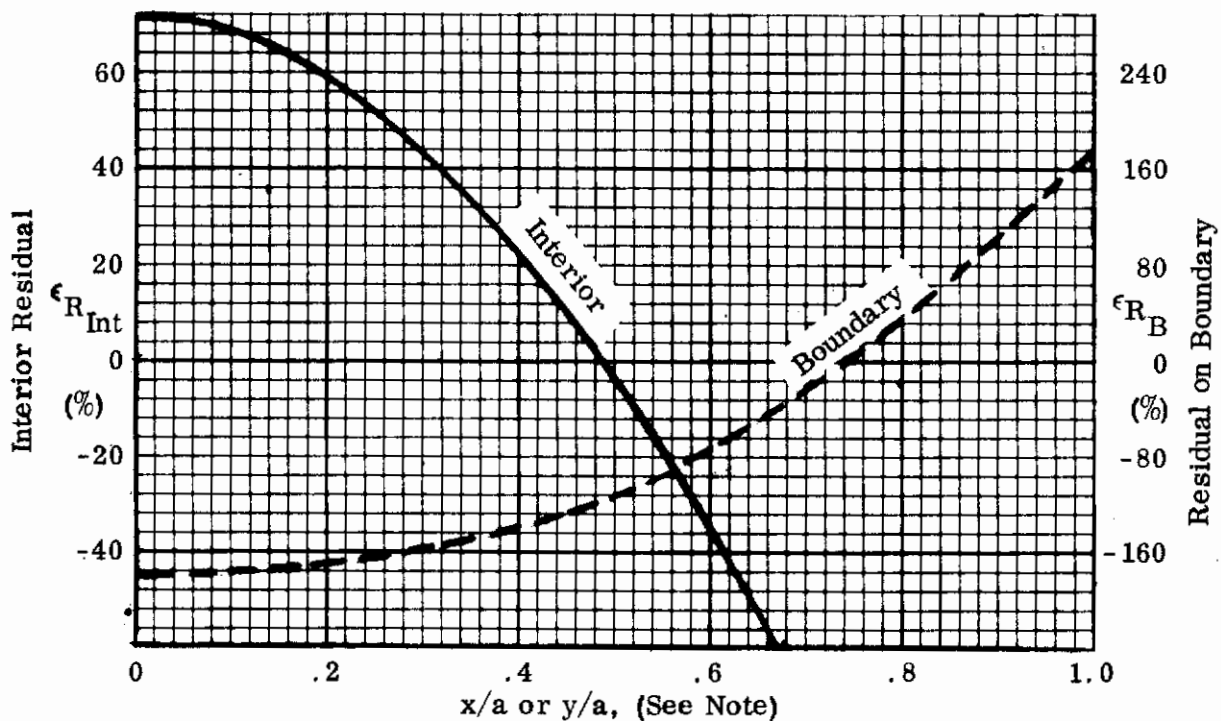
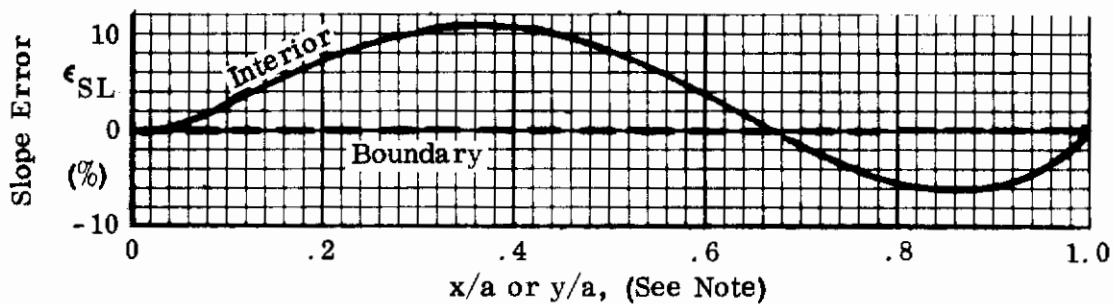
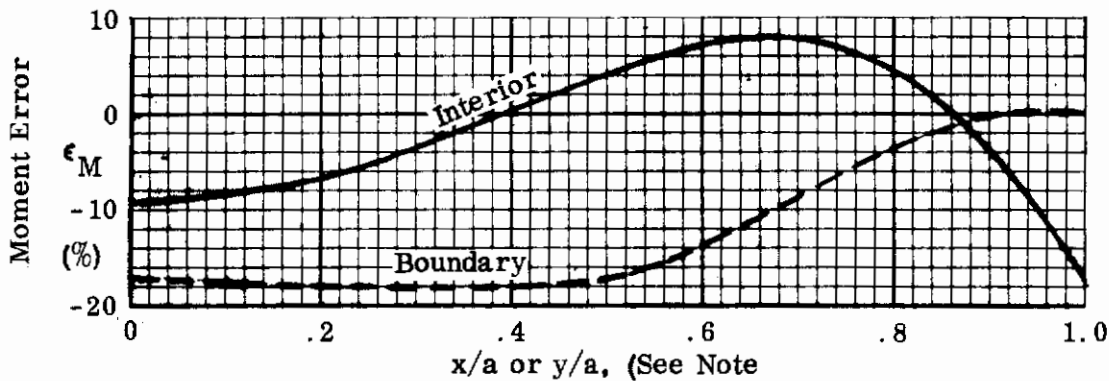
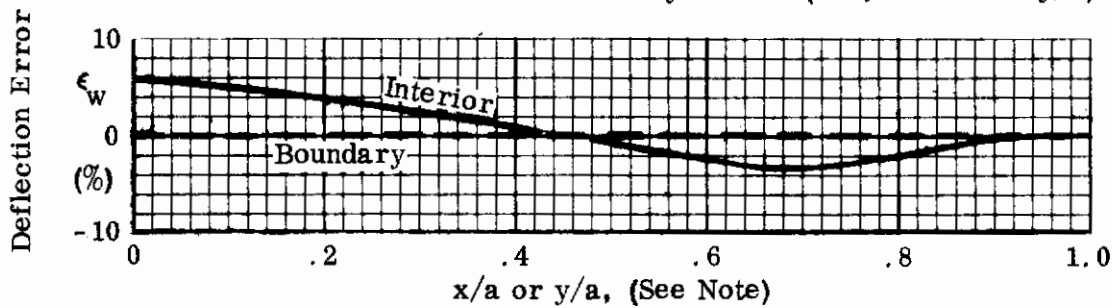


Figure 9. One Parameter Ritz-Galerkin Boundary and Interior Errors for Example 1

Note: Solid Lines Indicate Interior Errors ( $y = 0$ , Absissa =  $x/a$ )  
 Dashed Lines Indicate Boundary Errors ( $x = a$ , Absissa =  $y/a$ )

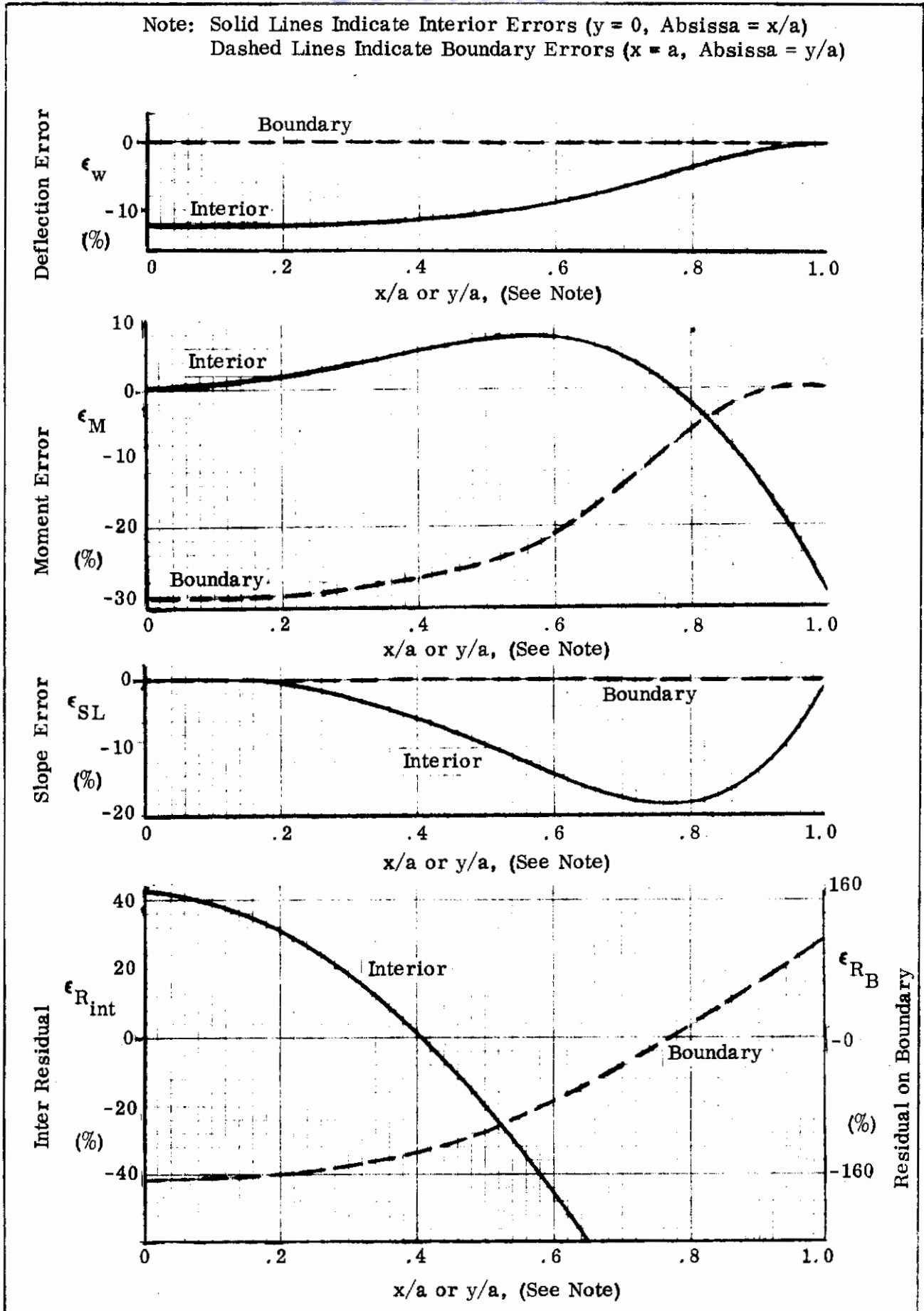


Figure 10. One Parameter Least Squares Boundary and Interior Errors for Example 1.

Note: Solid Lines Indicate Interior Errors ( $y = 0$ , Absissa =  $x/a$ )  
 Dashed Lines Indicate Boundary Errors ( $x = a$ , Absissa =  $y/a$ )

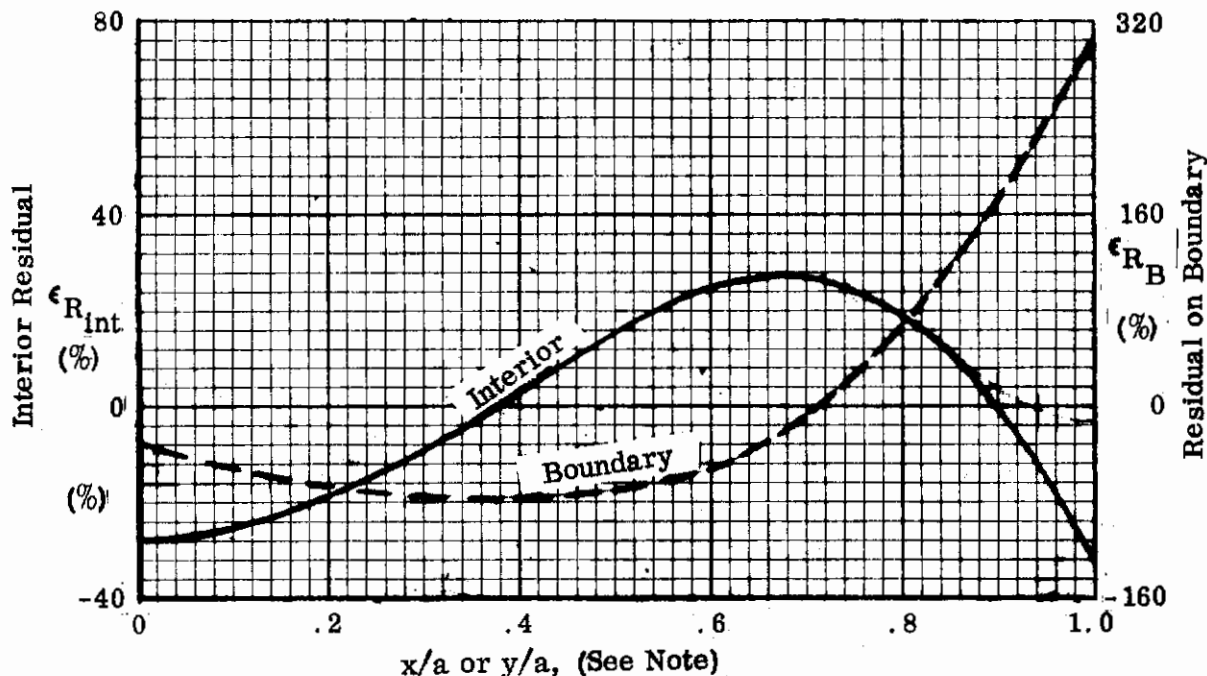
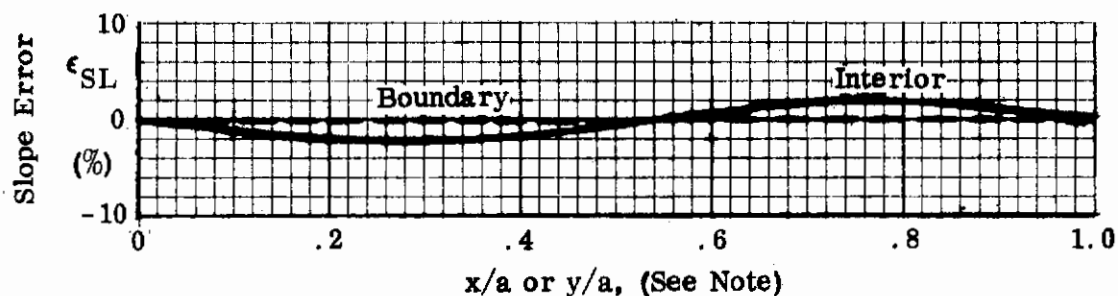
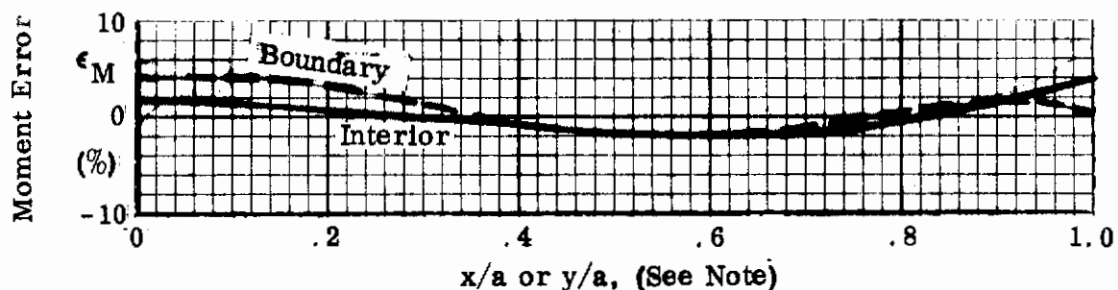
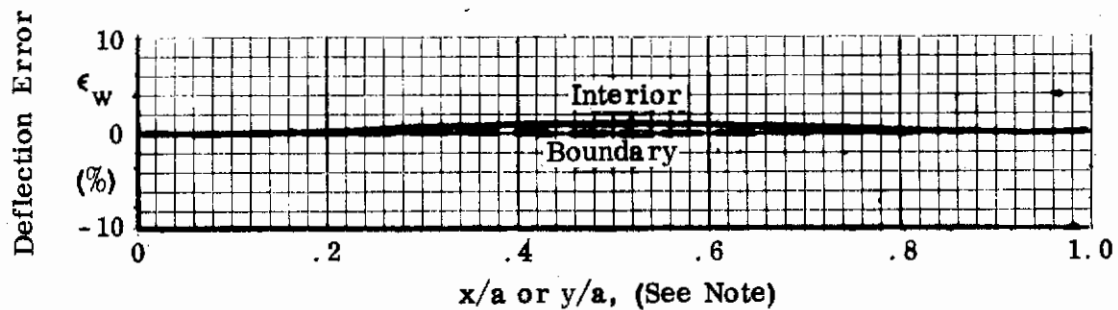


Figure 11. Biezeno-Koch Boundary and Interior Errors for Example 1

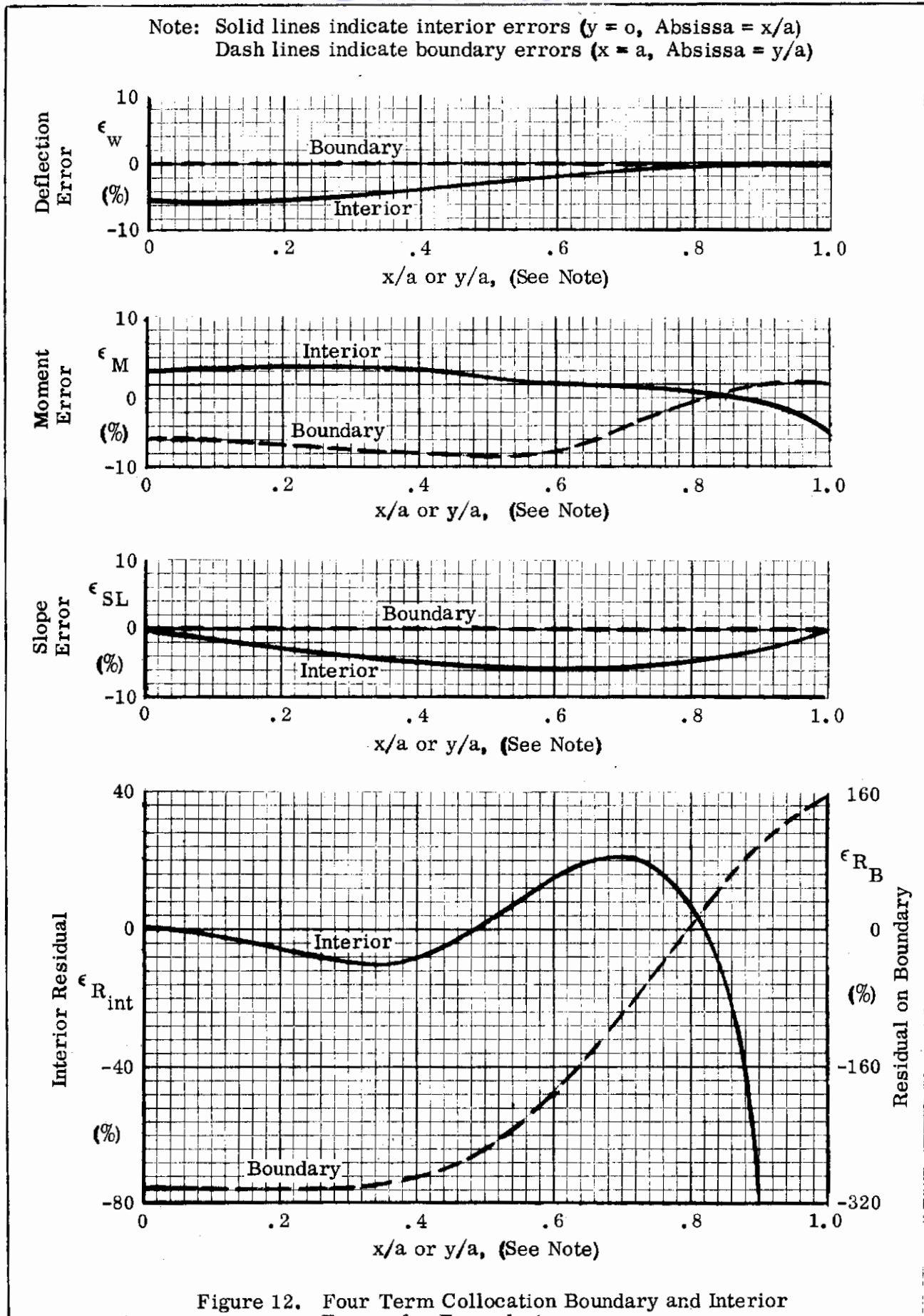


Figure 12. Four Term Collocation Boundary and Interior Errors for Example 1.

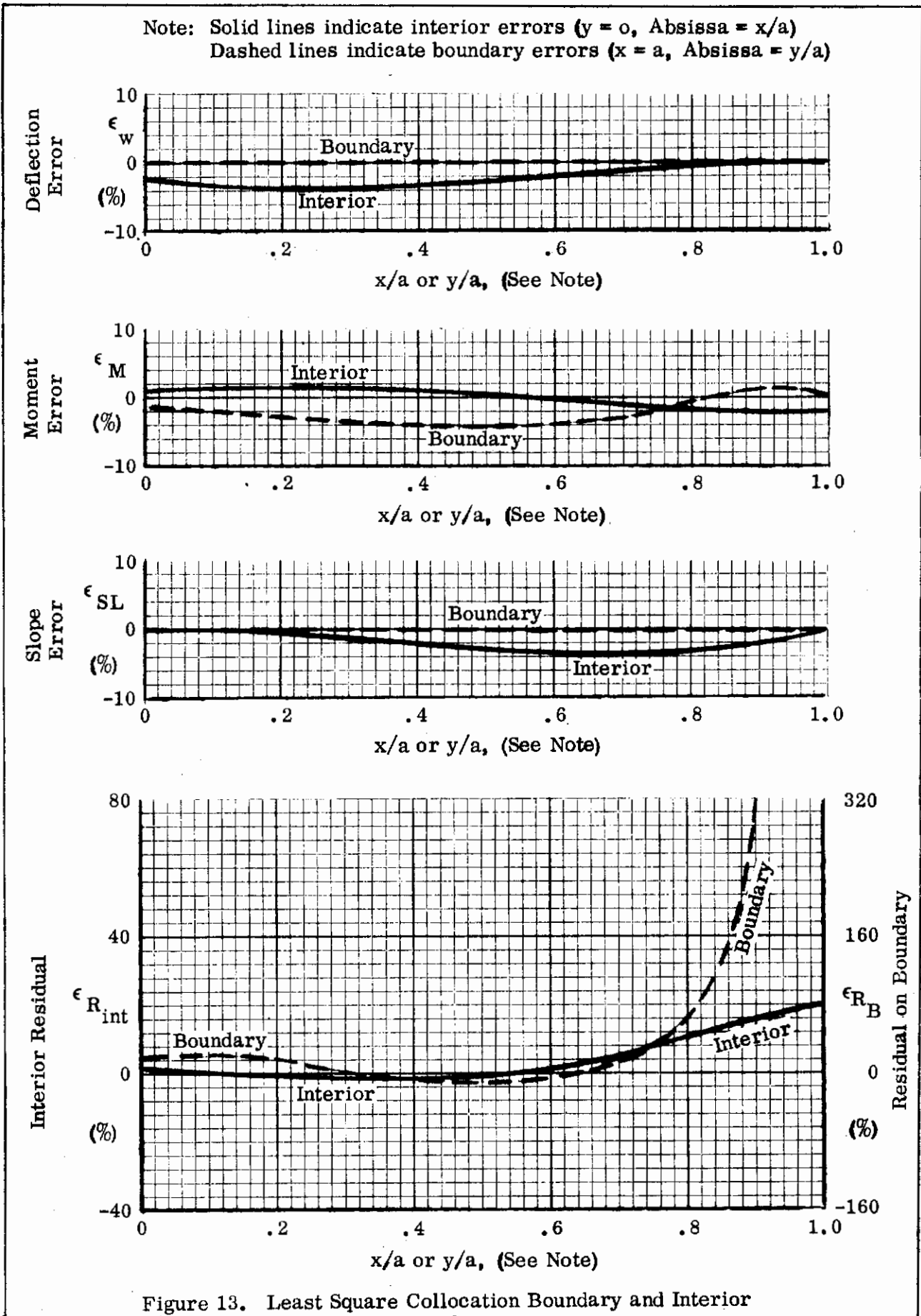


Figure 13. Least Square Collocation Boundary and Interior Errors for Example 1.

# Contrails

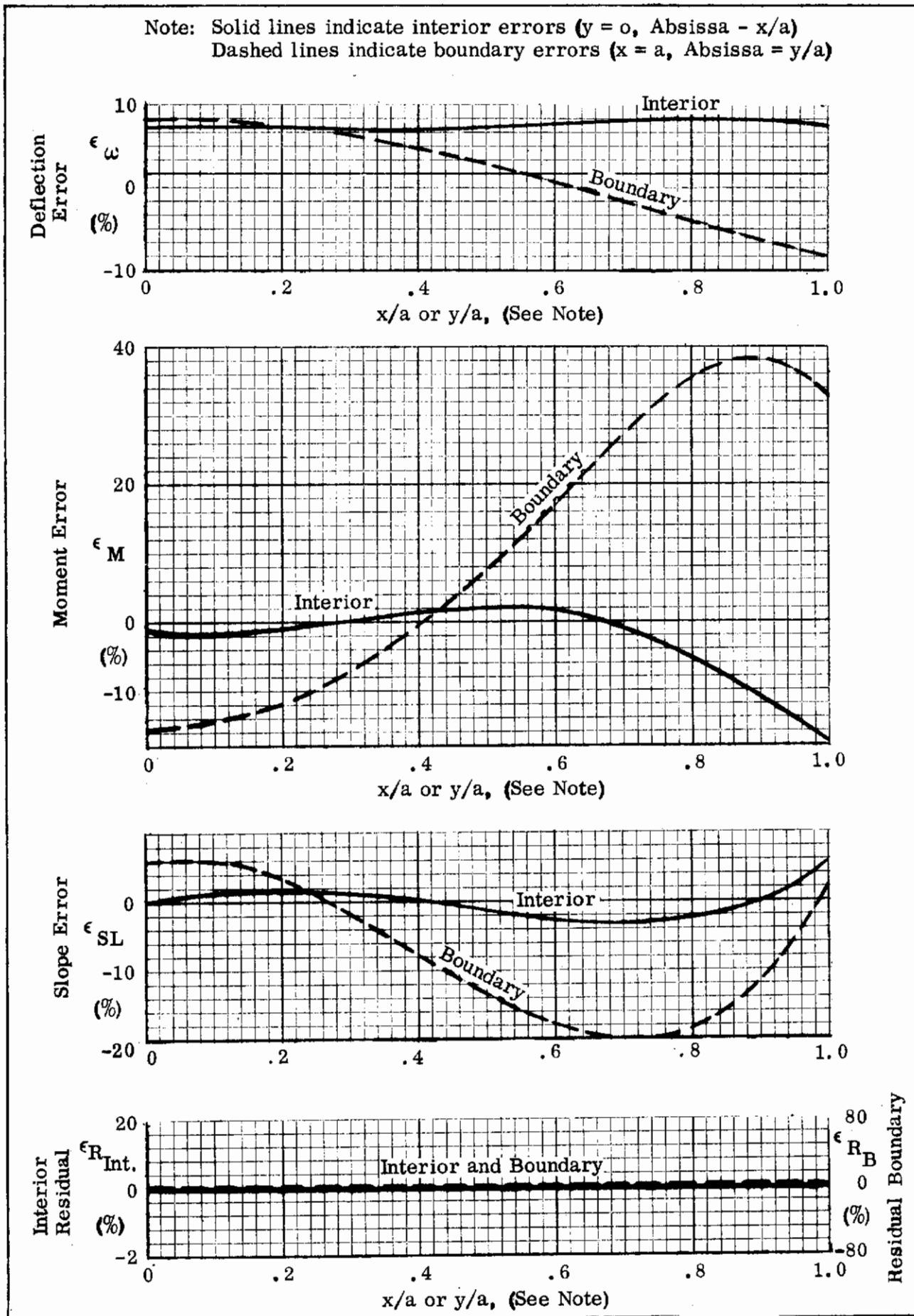


Figure 14. Mikhlin Boundary and Interior Errors for Example 1.

# Contrails

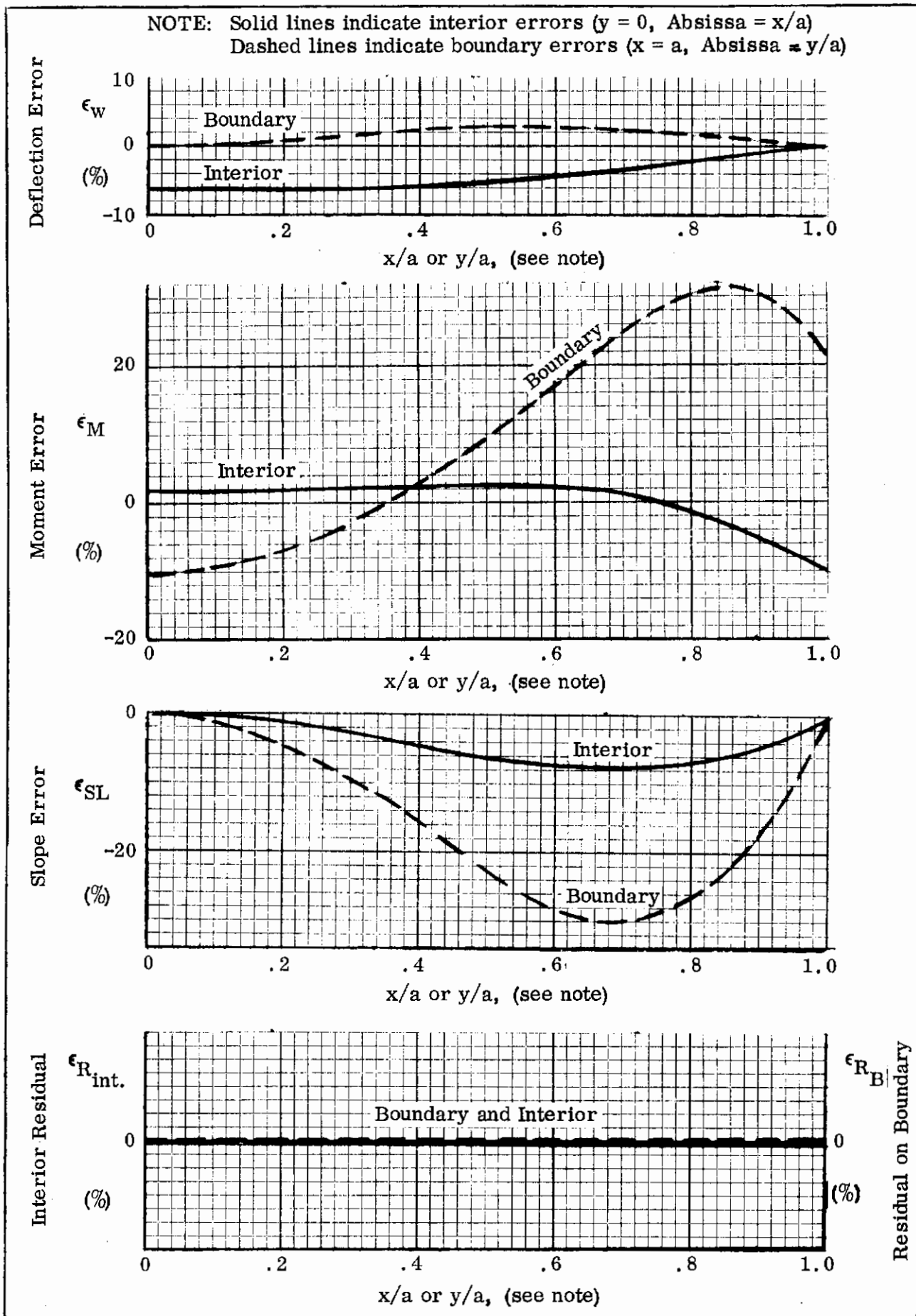


Figure 15. 4 x 4 Point Matching Interior and Boundary Errors for Example 1



NOTE: Solid Lines Indicate Interior Errors ( $y = 0$ , Absissa =  $x/a$ )  
 Dashed Lines Indicate Boundary Errors ( $x = a$ , Absissa =  $y/a$ )

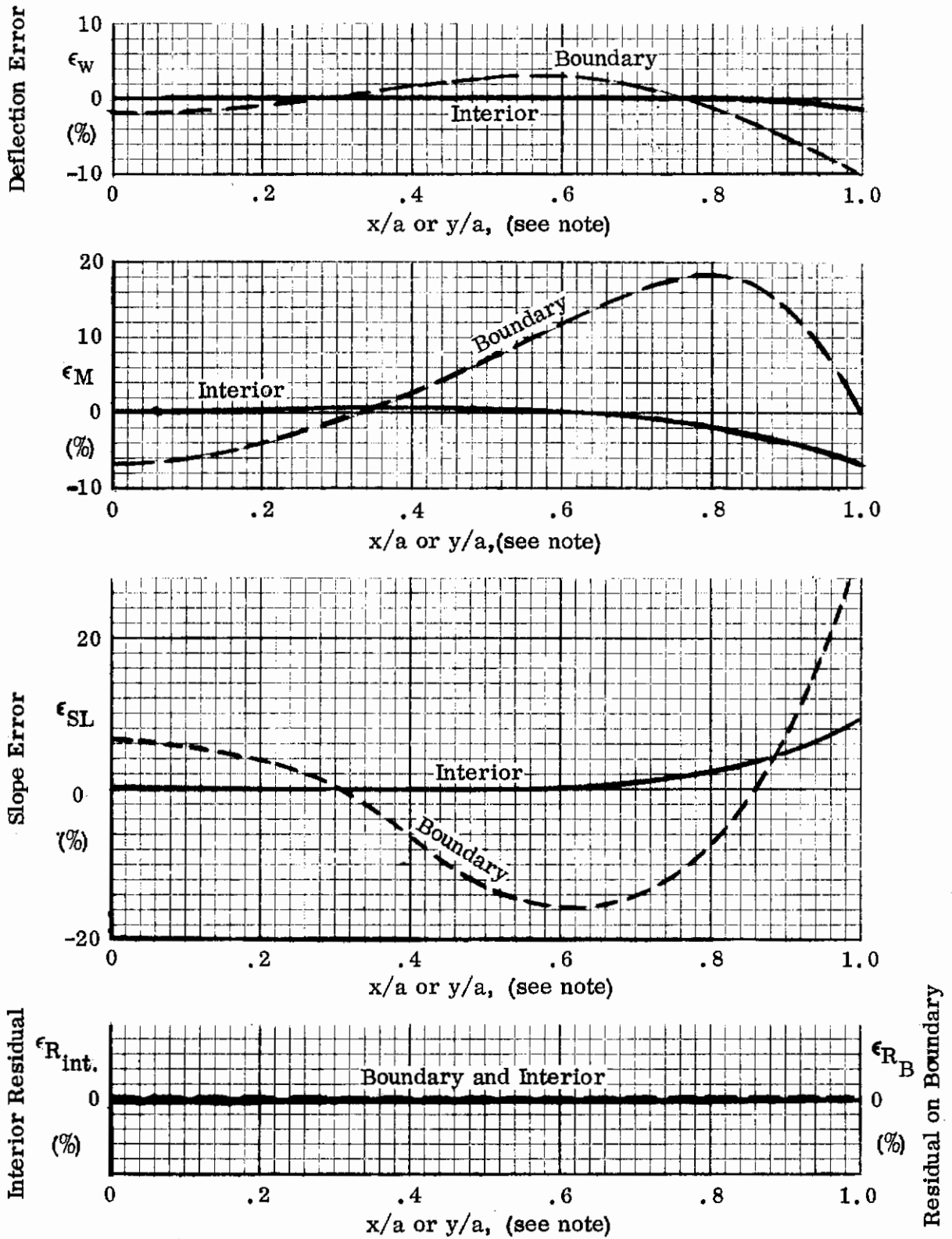


Figure 16. 4 x 4 Equal-Spaced Point Matching Interior and Boundary Errors for Example 1

# Contrails

The residuals can be thought of as the additional plate load ( $q/\bar{D}$ ) required to make these corresponding solutions exact. Thus, while the collocation residual was small throughout most of the plate interior, it gave rather low deflections because it was drastically off (almost 300%) in the loading (residuals) near the edges (see Figure 12).

Of interest also, is that solutions that satisfy the domain equation exactly (1) but not the boundary conditions (Equation (2)) yield their largest errors on or near the boundary (reference Figures 14, 15, and 16).

Using the area under the error curves (Figures 17 through 20) as a measure of overall accuracy, comparison of the methods leads to the following ranking (in order of most accurate to least accurate):

1. Least Square Collocation, Biezeno-Koch, and Equal Spaced Pointing-Matching (comparable)
2. Collocation
3. Point-Matching, Mikhlin, and Ritz-Galerkin (comparable)
4. Least Squares

Equally spaced point-matching gave extremely accurate results within the domain interior; however, the errors became amplified on and near the boundary. The curves clearly indicate the poorer accuracy obtained with the Ritz-Galerkin and least-squares solution methods. This inferior performance is attributed to the fewer terms considered in the trial families of these two as compared to the other methods investigated.

On the basis of the amount of hand calculations required to arrive at a numerical solution for the undetermined parameters, the methods ranked as follows (from least amount to most):

1. Ritz-Galerkin (one term)
2. Least-Squares (one term)
3. Point Matching (four terms)
4. Equal Spaced Point Matching (four terms)
5. Mikhlin (four terms)
6. Collocation (four terms)
7. Least Squares Collocations (four terms)
8. Biezeno-Koch (four terms)

While Ritz-Galerkin are Least Squares and ranked first and second with regard to ease of application, it must be noted that only one undetermined parameter was retained in the trial solutions. As mentioned earlier and shown on page 42, an extension to more than one term by either of these methods is considerably more time consuming.

The minimum number of terms required to indicate the salient features of any particular method; while still yielding preliminary engineering results, was the basis used for selecting the number of terms in the previous trial solutions. The ease of application of any single technique was strongly dependent upon the number of undetermined parameters retained, and this will always be the case because each method eventually results in a system of simultaneous equations requiring solutions.

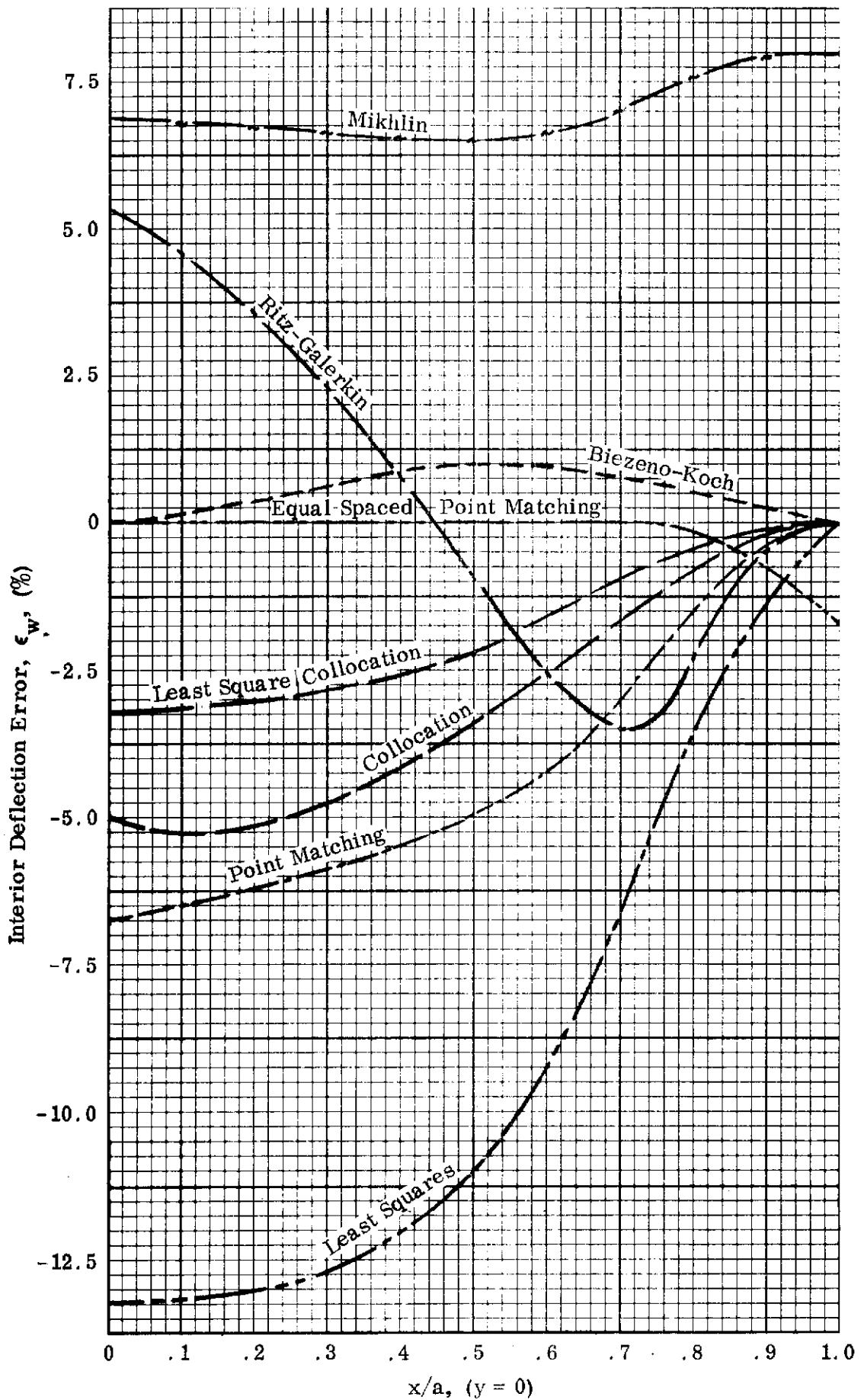


Figure 17a. Comparison of Interior Deflection Errors for Example 1

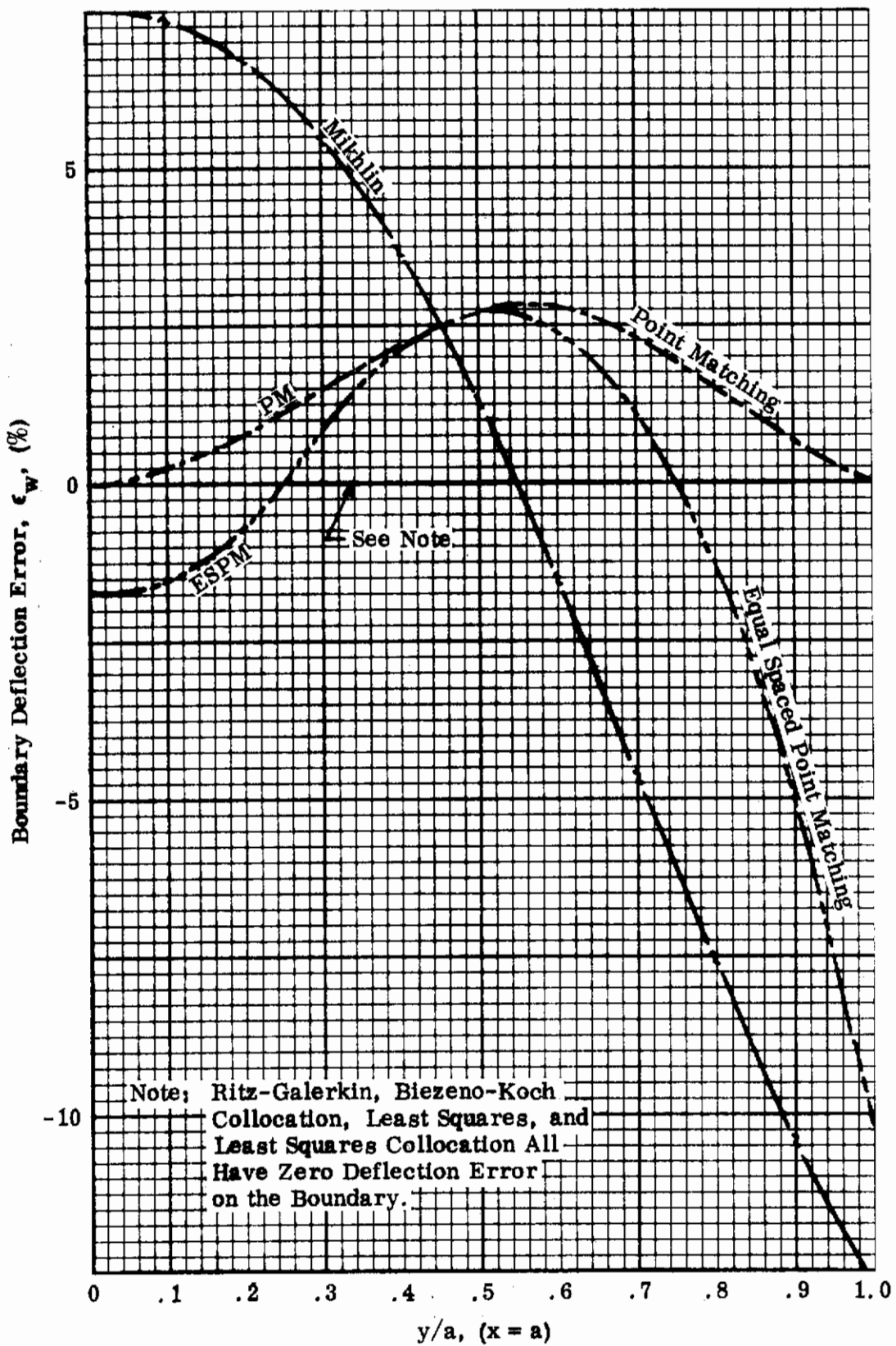


Figure 17b. Comparison of Boundary Deflection Errors for Example 1

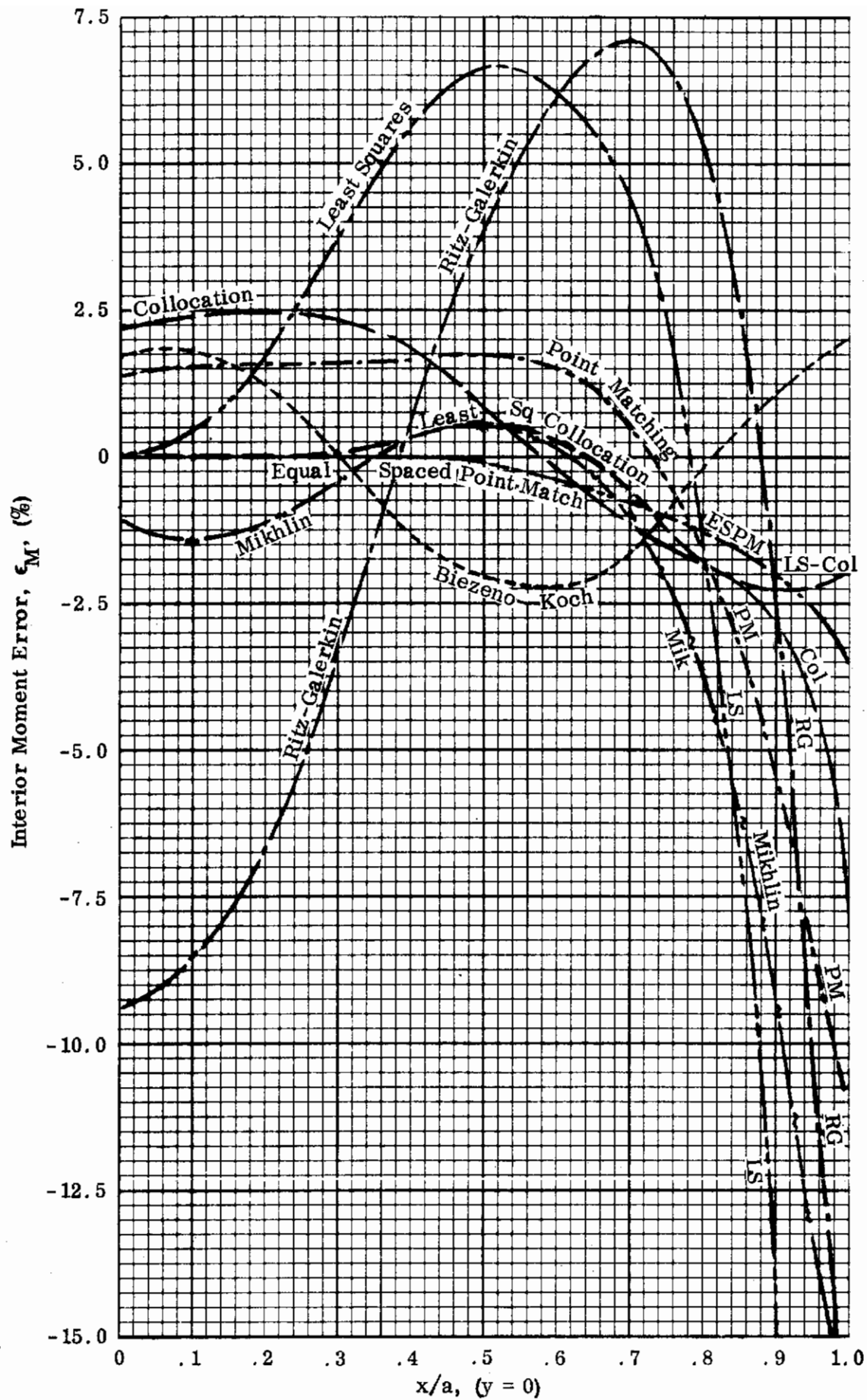


Figure 18a. Comparison of Internal Moment Errors for Example 1

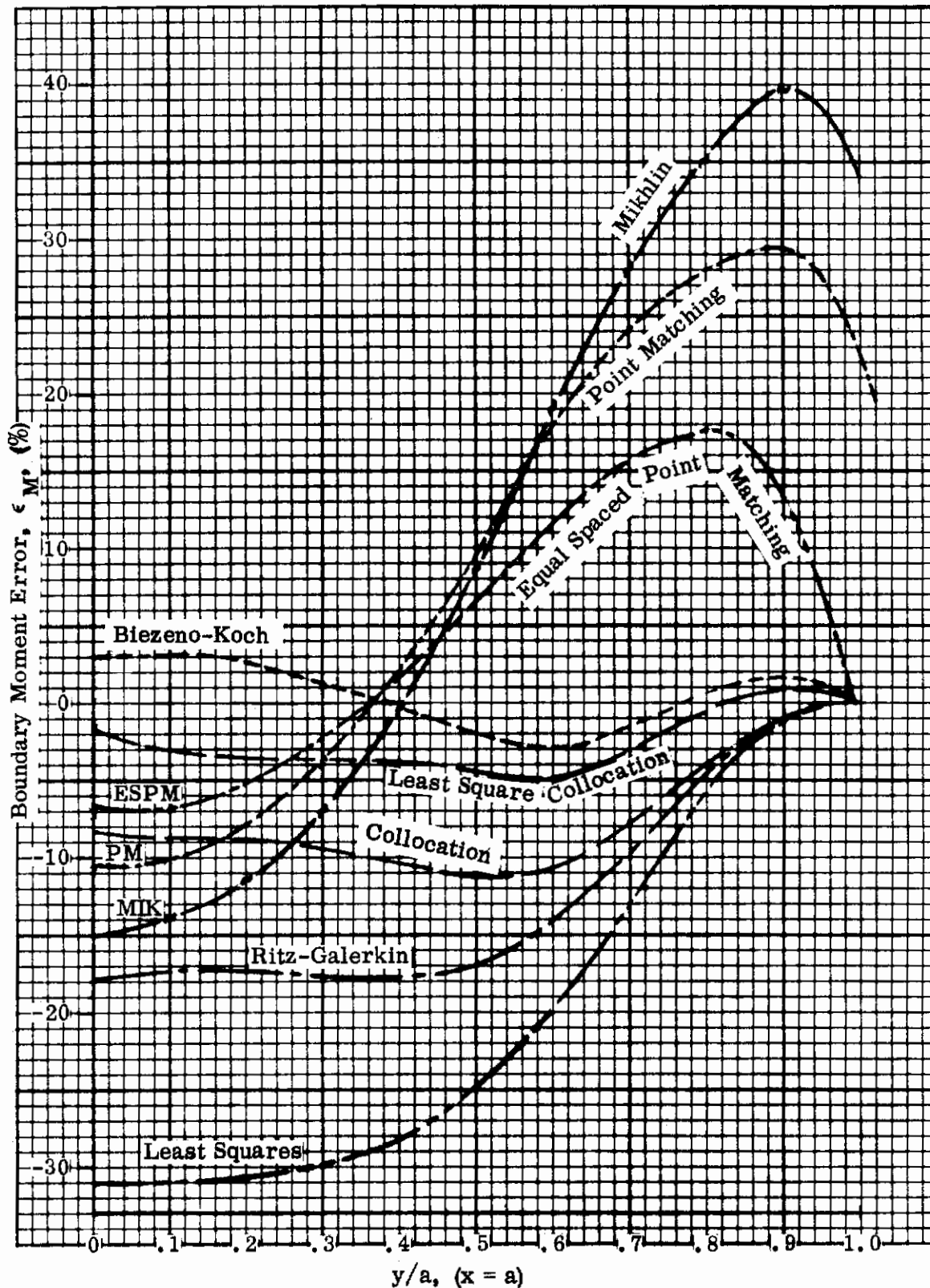


Figure 18b. Comparison of Boundary Moment Errors For Example 1.

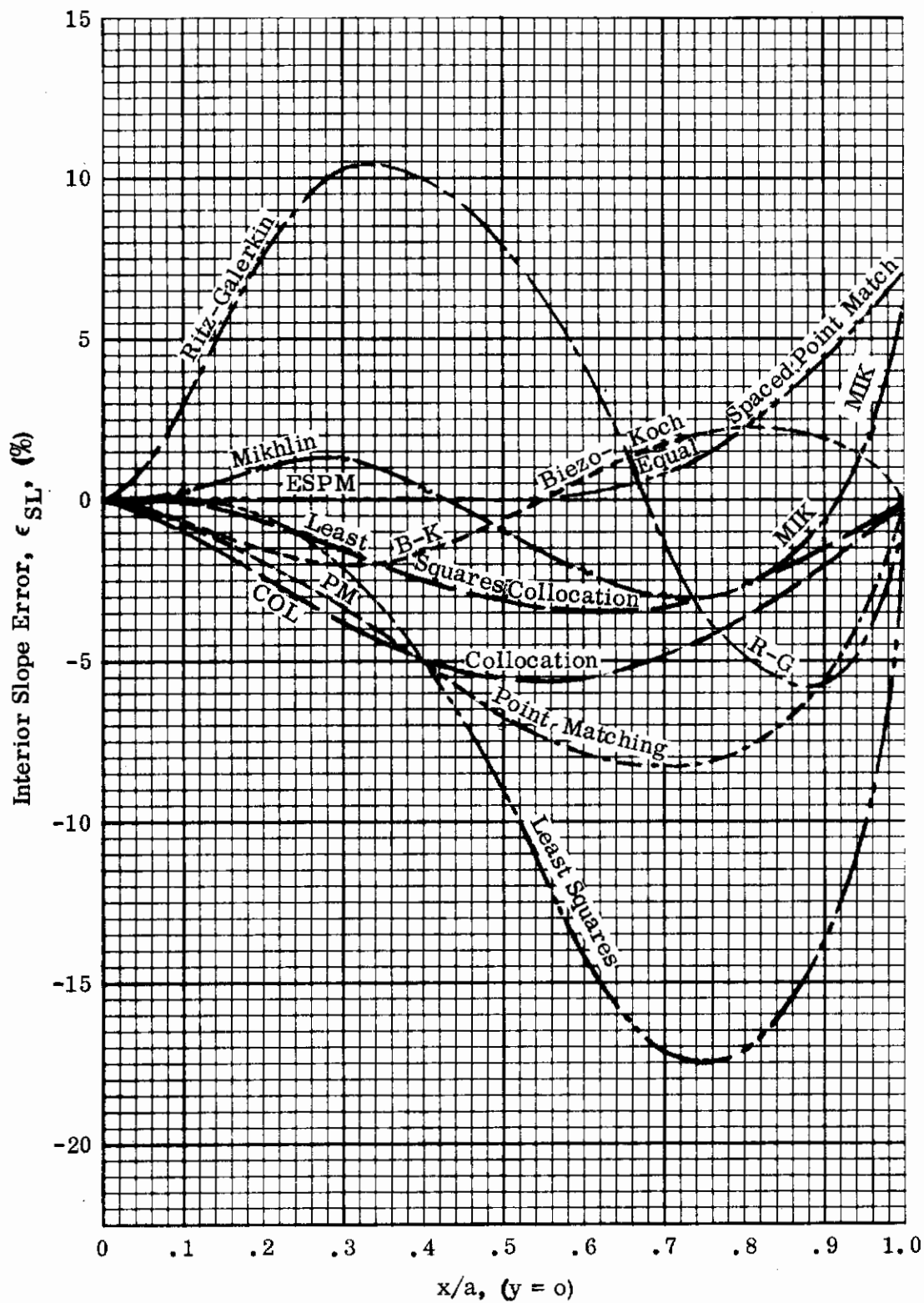


Figure 19a. Comparison of Interior Slope Errors for Example 1.

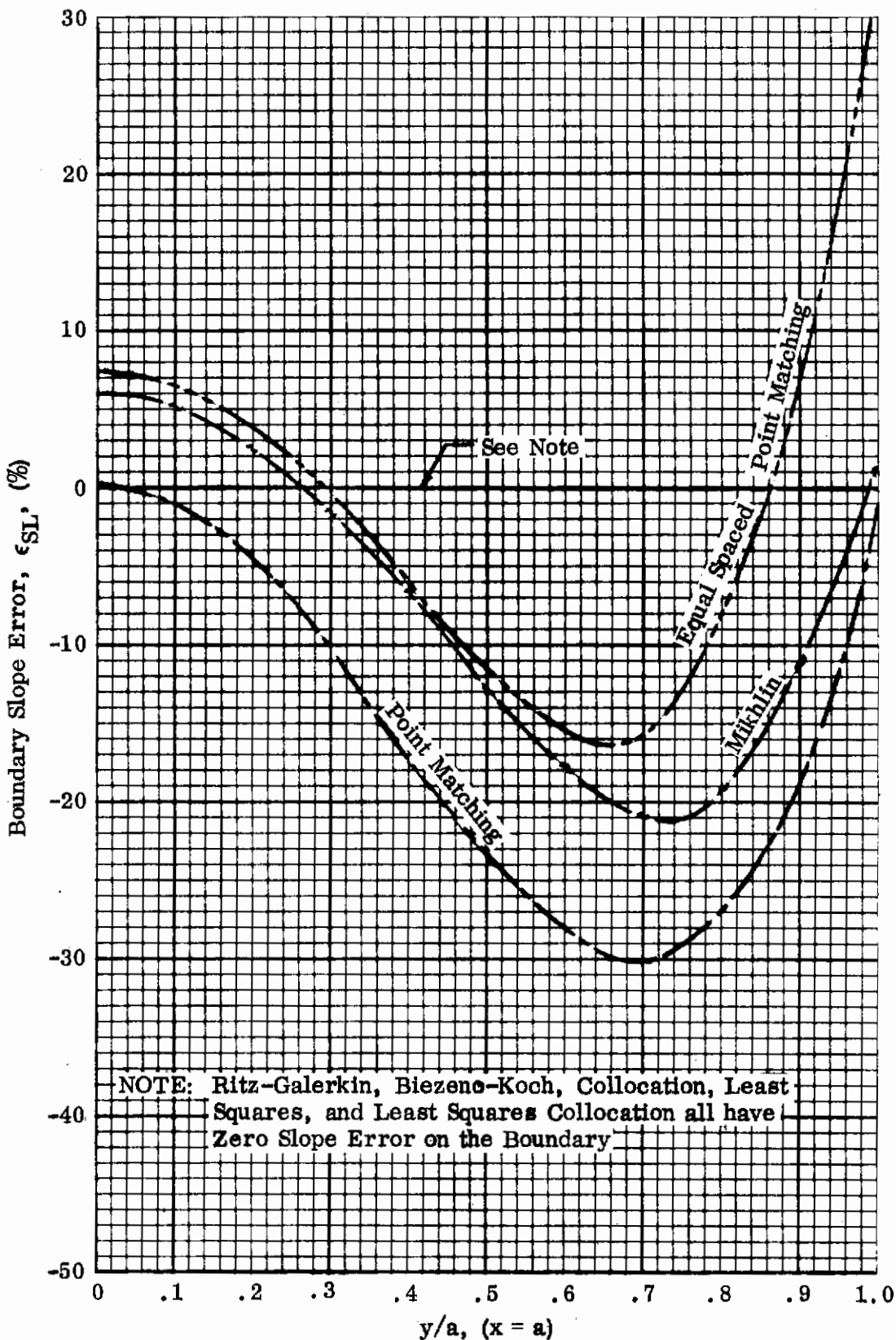


Figure 19b. Comparison of Boundary Slope Errors for Example 1



# Contrails

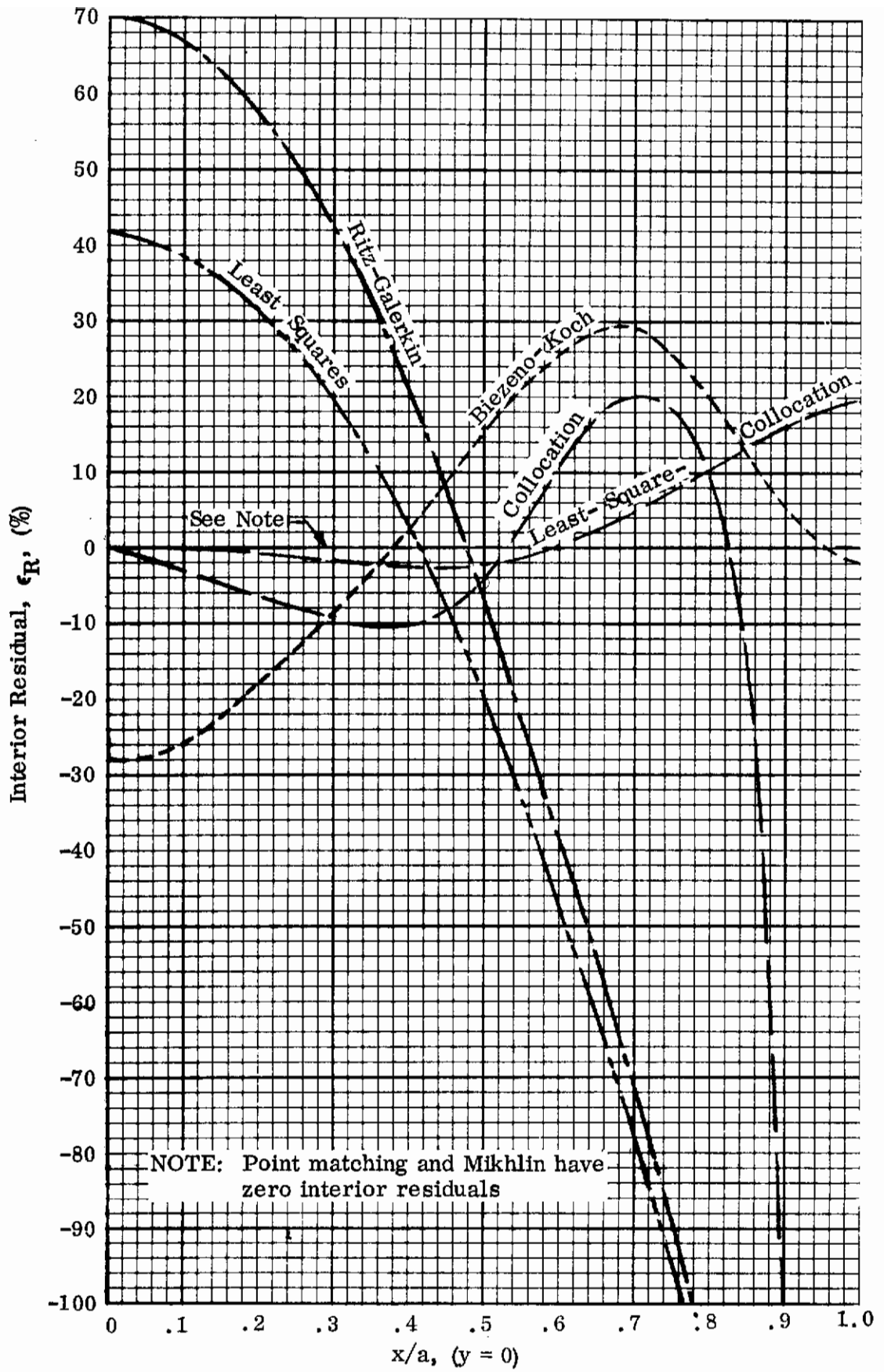


Figure 20a. Comparison of Interior Residuals for Example 1

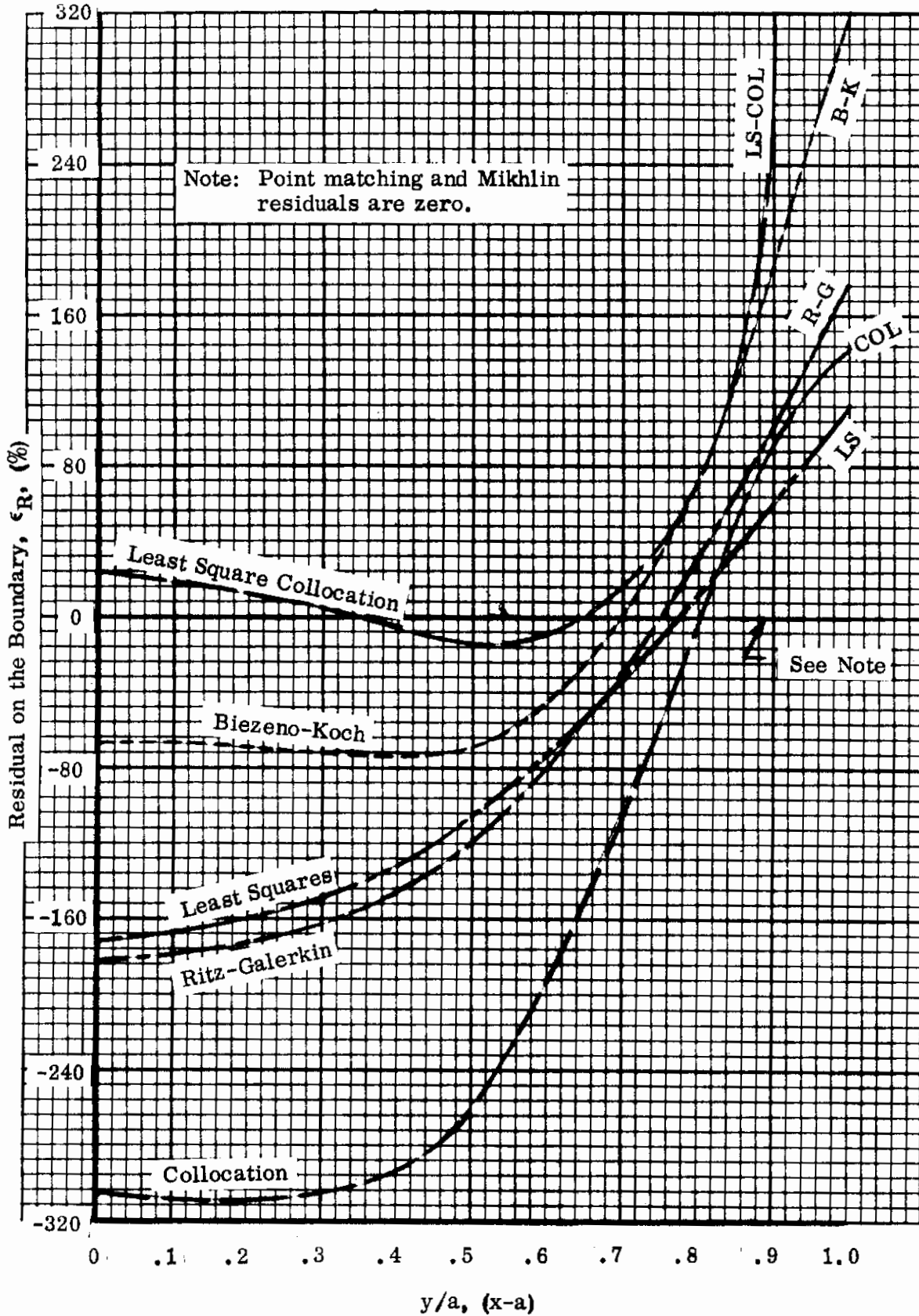


Figure 20b. Comparison of Residuals on the Boundary for Example 1

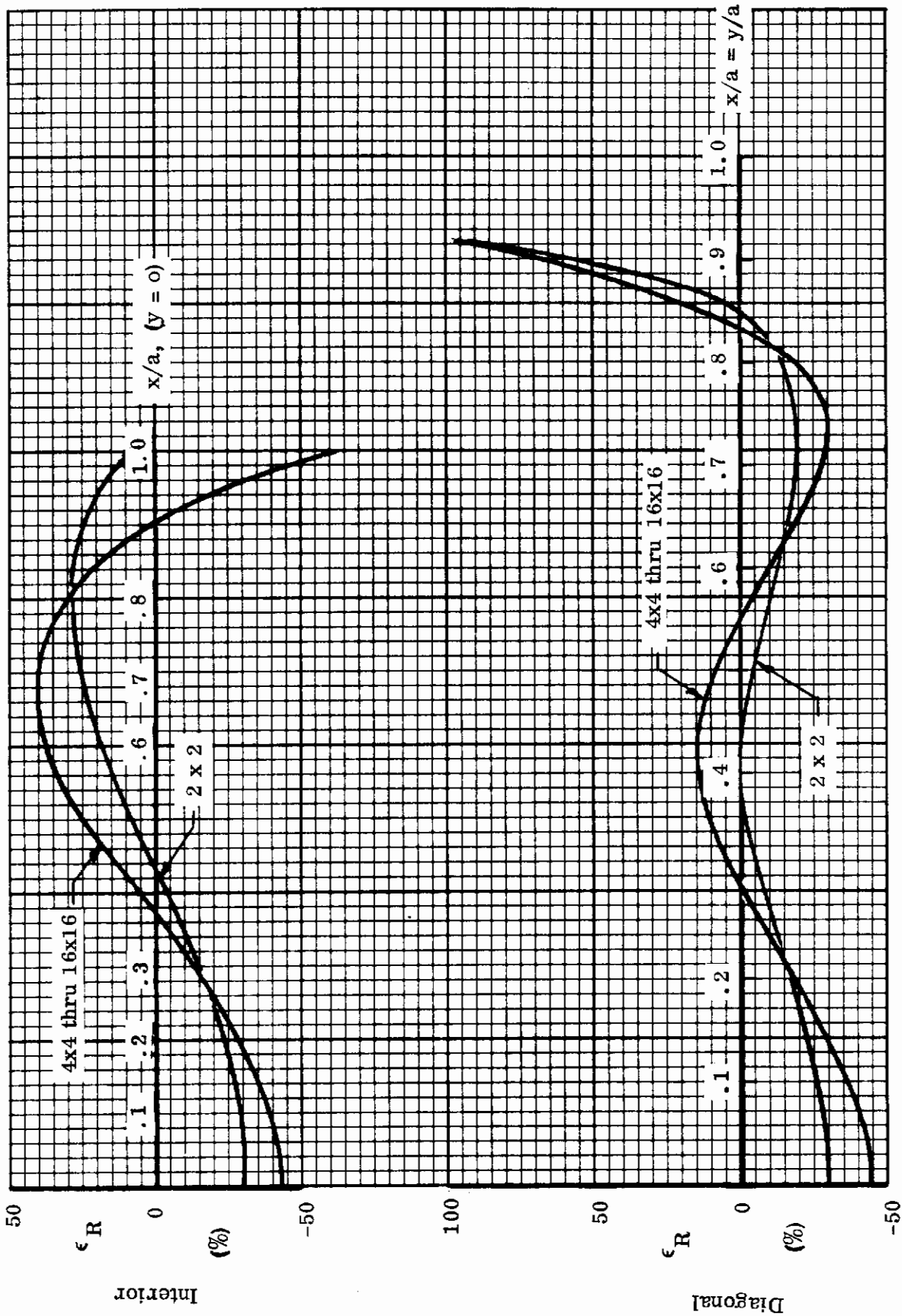
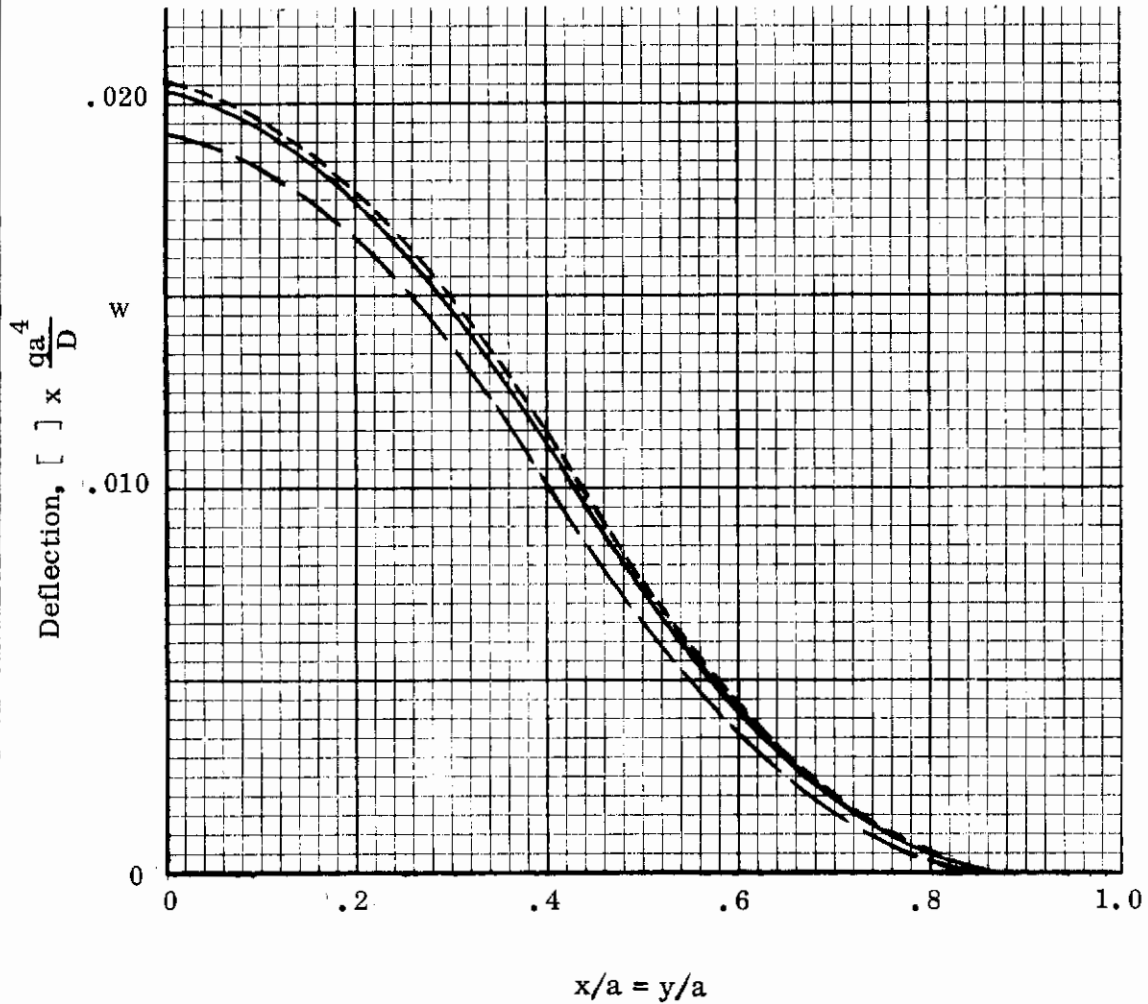


Figure 21. Biezeno-Koch Residuals for Example 1



Key: ----- Biezeno Koch (3 to 17 terms)  
————— "Exact", Point Matching (5 to 30 terms), and  
          least Sq. Collocations (12 x 3 thru 12 x 7)  
- · - · - Collocation (6 to 8 terms)

Figure 22. Deflection Along Diagonal for Example 1

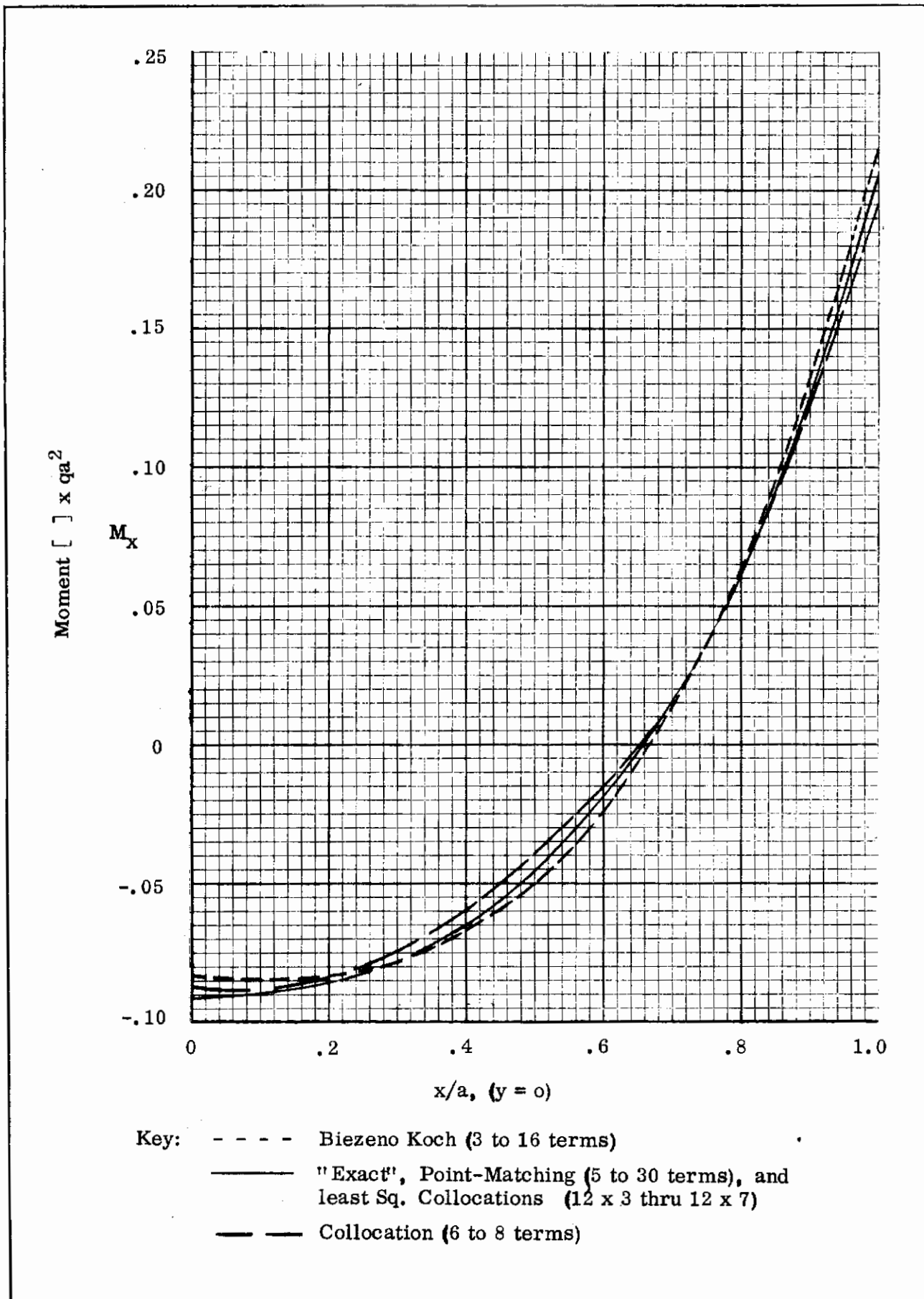


Figure 23. Approximate Moments Along X-Axis for Example 1

# Contrails

While none of these solutions could be deemed very accurate, this preliminary effort did yield an indication of the feasibility of automating the point-matching, collocation, and Biezeno-Koch methods conveniently. Consequently, digital programs with a capacity to include many more undetermined parameters were written for the clamped square plate. These, in turn, were used to investigate the numerical convergence and accuracy of solutions by the respective techniques.

Each computer program was capable of handling up to 30 terms with an option to accommodate up to 60 points, (in a least-square sense). Of the three, the point-matching and collocation programs were simplest to develop and served complimentary functions, in the following sense.

A point-matching program is most suitable when the program is designed to handle various shaped plates because the same trial family is used and the plate geometry simply becomes input. However, if the loading varies from one application to the next, such a program has the disadvantage that a different particular solution must be inserted for each type loading. Of course, it is possible to build up a complex particular solution by summing a series of elementary particular solutions. However, this can become a lengthy procedure for even modestly varying forcing functions (loadings). Nevertheless, this procedure is used in an existing program (Reference 26).

A collocation program has the ability of accommodating different loading conditions by simply varying the input. However, it does not provide any flexibility with trial functions because a new set of boundary conditions requires a different trial family.

The usefulness of an automated Biezeno-Koch program is limited because it requires more development, possesses disadvantages similar to collocations, and provides no real advantages over least-squares collocations.

Noteworthy is that it can be shown from a theoretical point of view that least-squares point-matching approaches Mikhlin's method and the least-squares collocations approaches least-squares, when points used are equally spaced. However, mechanical inaccuracies resulting from numerical truncation preclude this theoretical convergence.

In connection with these comments, it is noteworthy that a computer program based on the Ritz-Collocation technique with least-squares option (listed in Section III-C) could be developed to obtain results for which least-squares Ritz-Collocations approaches the Ritz-Galerkin method.

It has been found, for this example, that a good indication of a solution's accuracy can be obtained by simply listing the central deflection and maximum bending moments it yields. Thus, Tables II, III, and IV, present only these quantities as comparisons for computer based results.

Solutions achieved by Point-Matching, Table II, were obtained by selecting equally spaced boundary points on which to impose the conditions of zero deflection and radial slope. While there are certainly more propitious methods available for boundary point-selection (Reference 25), the results furnished by this obvious and simple scheme were sufficiently accurate to obviate the need for a more sophisticated approach.

In an attempt to minimize truncation errors, the coefficient matrix and right hand sides associated with each system of simultaneous equations, as well as their solutions, were generated in double precision.

The average approximate boundary error presented in Table II was obtained by averaging the absolute value of the errors along the boundary and multiplying slope errors by  $100\% / \frac{\partial w}{\partial w}$  and deflection errors by  $100\% / w_{\max}$ .

TABLE II  
POINT-MATCHING

SOLUTIONS TO EXAMPLE 1

Number of Undetermined Parameters	$[w(0, 0)] \times \frac{qa^4}{D}$	$[M_x(0, 0)] \times qa^2$ ( $\nu = .3$ )	$[M_x(1, 0)] \times qa^2$ (maximum moment) ( $\nu = .3$ )	Approximate Average Boundary Error Percentage	
				Deflection	Slope
2	.02441406	-.10156250	.1046875	20%	25%
3	.01785714	-.09285714	.1321428	12	28
4	.01893939	-.08863630	.1840909	1	16
5	.02025207	-.09163447	.2057089	.14	.50
6	.02024759	-.09165961	.2061334	.03	.57
7	.02024185	-.09160634	.2053435	.03	.19
8	.02024524	-.09162378	.2052817	.02	.10
12	.02024515	-.09162057	.2053346	.002	.020
16	.02024511	-.09162038	.2053351	.00018	.0011
20	.02024511	-.09162036	.2053350	.00002	.00048
30	.02024511	-.09162036	.2053350	.000011	.00036
30*	.02024511	-.09162036	.2053368	.0000047	.00006

\* Obtained for a 60 by 30 least-squares point matching matrix (points equally spaced), which approaches Mikhlín's method.

TABLE III  
COLLOCATIONS

SOLUTIONS TO EXAMPLE 1

Number of Undetermined Parameters	$[w(0,0)] \times \frac{qa^4}{D}$	$[M_x(0,0)] \times qa^2$ ( $\nu = .3$ )	$[M_x(1,0)] \times qa^2$ (Maximum Moment) ( $\nu = .3$ )	Approximate Average [Residual]	
				X Axis	$\times D/q \times 100\%$ Diagonal ( $x = y$ )
2	.021866	-.098485	.221763	9%	16%
3	.021225	-.092801	.244710	37	12
4	.019166	-.086984	.188726	10	14
5	.020350	-.089302	.228466	17	12
6	.019247	-.087292	.200260	8.5	10.5
7	.019208	-.088740	.196160	6.2	5.0
8	.019203	-.088719	.195990	8.0	9.5
10	.020148	-.090659	.209925	8.0	9.1
12	.023476	-.096809	.570983	70	520
16	.017939	-.081342	.213751	16	735
20	.013403	-.065471	.089824	-	-
30	.037560	-.127093	2.13750	-	-
Exact	.020245	-.091620	.205335	0	0



TABLE IV

LEAST-SQUARES COLLOCATION

SOLUTIONS TO EXAMPLE 1

Number of Collocation Points Times Number of Undetermined Parameters	$[w(0, 0)] x \frac{qa^4}{D}$	$[M_x(0, 0)] x qa^2$ ( $\nu = .3$ )	$[M_x(1, 0)] x qa^2$ (Maximum Moment) ( $\nu = .3$ )	Approximate Average [Residual] $x D/q x 100\%$	
				X Axis	Diagonal ( $x = y$ )
12 x 1	.017581	-.091421	.140650	43.0%	28.5%
12 x 2	.020685	-.093030	.210202	5.4	9.0
12 x 3	.020394	-.092817	.209248	5.7	9.5
12 x 4	.020393	-.092739	.209585	7.5	9.5
12 x 5	.020394	-.092747	.209585	6.6	10.0
12 x 6	.020402	-.092784	.209634	6.0	10.3
12 x 7	.020395	-.092719	.210052	6.0	11.1
16 x 8	.020846	-.094715	.212775	6.9	7.5
20 x 10	.020925	-.095271	.213030	7.5	7.4
30 x 12	.020653	-.093945	.210437	4.6	5.8
30 x 14	.020455	-.093204	.207405	3.4	5.9
30 x 16	.020403	-.092699	.207801	4.8	10.0
Exact	.020245	-.091620	.205335	0	0

# Contrails

As can be observed, the average percentage error produced in the boundary conditions decreased quite rapidly as more and more points were matched. Eventually, the solutions converged to  $w_{\max} = w(0, 0) = .0202451 \times \frac{qa^4}{D}$ ,  $M(0, 0) = -.0916204 q$ ,  $M_{\max} = M(1, 0) = .205335 q$ , for six significant figures and  $\nu$  (Poisson's ratio) = 0.3. This solution is exact for all intent and purposes because the trial functions selected satisfy the plate differential equation exactly, and boundary errors are extremely small for the last 4 cases of Table II, each of which give the above values.

The extreme accuracy and rapid convergence of the point-matched solution is attributable to the following reasons:

- 1) The boundary conditions are smooth and continuous (see Reference 25 for an example of a problem in which the boundary conditions are not continuous and note the numerical difficulties encountered)
- 2) The angle change between the boundary normal and reference axis is small (only  $\frac{\pi}{2}$  radians) at the plate corners
- 3) The boundary is compact (i. e., the ratio of maximum to minimum distance from coordinate-origin to boundary is sufficiently close to 1. See Reference 26 in connection with the importance of compactness.

It is interesting to note that the slope boundary errors are approximately an order of magnitude higher than their deflection counterparts for any given number of undetermined parameters.

Results for the collocation method are presented in Table III. As is readily observable, the solutions do not converge as more and more points are collocated. In fact, the results diverge from the exact values as the number of unknown parameters in the trial solutions is increased beyond 10. This situation is similar to what has been found to occur in point-matching applications (Reference 25) in that, although conditions are matched at discrete points, they, in some instances, experience large excursions in between matched points. The weakness of the  $n > 10$  collocation solutions can be discerned without recourse to the known exact solution by simply noting the average absolute residuals (or loading errors) with increasing  $n$ . As for solutions with  $n \leq 10$ , the average absolute residual hovers around 5 or 10%. Therefore, average deflection and moment errors of 5%, as obtained for  $6 \leq n \leq 10$ , seem reasonable.

An obvious extension of collocations is least-square collocations, which attempts to match the differential equation at more points than it can in a manner similar to its least-square point-matching counterpart. Numerical results, as presented in Table IV, reveals that answers do not quite stabilize and remain +4% away from the exact solution even for the cases with larger  $n$ . This behavior is attributed to the limiting nature of the trial solutions used.

The numerical comparisons given in Table V for the Biezeno-Koch method reveal a rather rapid stabilization of results and then a gradual divergence for the larger systems of equations ( $n > 20$ ). This latter behavior is probably caused by poorer matrix conditioning, beyond the  $n = 10$  term solution.

TABLE V

BIEZENO-KOCH

SOLUTIONS TO EXAMPLE 1

Number of Undetermined Parameters	$[w(0,0)] \times \frac{qa^4}{D}$	$[M_x(0,0)] \times qa^2$ ( $\nu = .3$ )	$[M_x(1,0)] \times qa^2$ (Maximum Moment) ( $\nu = .3$ )	Approximate Average [Residual] $\times D/q \times 100\%$	
				X Axis	Diagonal ( $x = y$ )
2	.019178	-.082502	.206424	16.7%	14.1%
3	.020287	-.085086	.217123	18.8	15.3
4	.020299	-.085126	.217321	18.8	15.3
5	.020305	-.085204	.217262	18.8	15.3
6	.020305	-.085204	.217262	18.8	15.3
7	.020305	-.085204	.217262	18.8	15.3
8	.020305	-.085204	.217262	18.8	15.3
12	.020290	-.085093	.217230	18.8	15.3
16	.020394	-.085654	.217795	18.8	15.3
20	.020577	-.087715	.216166	16.1	13.6
30	.021812	-.095850	.219371	13.9	32.9
Exact	.020245	-.091620	.205335	0	0

Another interesting feature of these solutions is that, even though the deflection and moment results are only 1 to 2% in error, the average domain equation residuals are approximately 18% off in most cases, (see Figure 21), thus, indicating that the residual should not be taken as an absolute criteria for testing the accuracy of a solution.

Deflection and moment results for some of the more accurate solutions obtained by each method, as determined by the average domain and boundary error criteria, are presented in Figures 22 and 23. These curves give the data of Tables I through V a realistic perspective, as to the accuracy possible when a sufficient number of trial terms are considered. The results also point to a shortcoming associated with the trial functions selected; they form an incomplete set of functions (as was the case for each of the methods applied to Example 1 which satisfied the boundary conditions exactly). Thus, results so obtained approached an inaccurate solution before they diverged; (as was noted earlier) divergence occurred because of numerical inaccuracies in the machine computations. This is contrasted by results for methods that satisfied the differential equation (with a complete set of functions), which converged to the exact solution rapidly. Furthermore, these did not diverge up to and including numerical solutions with 30 undetermined parameters.

A last look at this example with regard to trial solutions that satisfy the differential equation exactly reveals that solution accuracy is greatly improved when going from a four-parameter to a five-parameter point-matched solution (Table II), where the particular solution is  $r^4/64$ . Morley (Reference 30), via a variational procedure, obtained a very accurate five-parameter solution using the particular solution  $x^2y^2/8$  together with the homogeneous solution-form used in the current investigation. Briefly stated, his variational method minimizes the boundary energy-difference ( $V_{\text{error}}$ ) between the exact (unbarred quantities) and approximate (barred) solutions, i.e.,

$$\delta V_{\text{error}} = \int_B \left\{ (\bar{Q}_\eta - Q_\eta)(\bar{w} - w) + (\bar{M}_{\eta s} - M_{\eta s}) \left( \frac{\partial \bar{w}}{\partial \eta} - \frac{\partial w}{\partial \eta} \right) - (\bar{M}_\eta - M_\eta) \left( \frac{\partial \bar{w}}{\partial \eta} - \frac{\partial w}{\partial \eta} \right) \right\} ds = 0$$

where  $\nabla^4(w - \bar{w}) = 0$  in D,  $Q_\eta$ ,  $M_\eta$ , and  $M_{\eta s}$  are the plate shear, and normal and twisting moments on the boundary, respectively.

Therefore, a limited numerical investigation was undertaken to assess the effect upon solution accuracy of:

- a) Morley's variational method (as opposed to point matching and Mikhlin's method)
- b) Particular solution used  $\left( \frac{x^2 + y^2}{64} \right)^2$  as compared to  $\frac{x^2 y^2}{8}$
- c) Number of undetermined parameters used (up to 5)
- d) Relative importance of the different boundary conditions (deflection as opposed to slope)

TABLE VI  
EFFECT ON SOLUTION ACCURACY OF NUMBER OF PARAMETERS  
AND PARTICULAR SOLUTION FOR EXAMPLE 1.

Method	Number of Undetermined Parameters	Central Deflection $[ \ ]_x \frac{qa^4}{D}$	Maximum Bending Moment $[ \ ]_x qa^2$ ( $\nu = .3$ )	Particular Solution		x = 1 Boundary Points Matched ( $0 \leq y \leq 1$ )	
				$\frac{(x^2 + y^2)^2}{64}$	$\frac{x^2 y^2}{8}$	Deflection y	Slope y
Point-Matching	2	.02441	.1047	•		.5	.5
Point-Matching	2	.03125	-.0550		•	.5	.5
Point-Matching	3	.01786	.1321	•		0, 1	.5
Point-Matching	3	.02083	.1424	•		.5	0, 1
Point-Matching	3	.02531	.1062	•		.286, .857	.571
Point-Matching	3	.02531	.1062		•	.286, .857	.571
Point-Matching	3	.02749	.1467	•		.571	.286, .857
Point-Matching	3	.02749	.1467		•	.571	.286, .857
Point-Matching	4	.01894	.1841	•		0, 1	0, 1
Point-Matching	4	.01894	.1841		•	0, 1	0, 1
Point-Matching	4	.02025	.1910	•		.25, .75	.25, .75
Point-Matching	4	.02025	.1910		•	.25, .75	.25, .75
Mikhlin (with 8 to 1 deflection weighting)	4	.02066	.1768	•		-	-
Mikhlin (with 8 to 1 deflection weighting)	4	.02066	.1768		•	-	-
Point-Matching	5	.02025	.2057	•		0, .5, 1	0, .5
Point-Matching	5	.02025	.2057		•	0, .5, 1	0, .5
Point-Matching	5	.02023	.2053	•		0, .5	0, .5, 1
Point-Matching	5	.02019	.2086	•		.1, .5, .9	.3, .7
Point-Matching	5	.02019	.2086		•	.1, .5, .9	.3, .7
Point-Matching	5	.02026	.2050	•		.3, .7	.1, .5, .9
Point-Matching	5	.02026	.2050		•	.3, .7	.1, .5, .9
Mikhlin (with 8 to 1 deflection weighting)	5	.02030	.2051	•		-	-
Morley (Reference 2)	5	.02024	.208		•	-	-
Exact Solution (referring to Table XI)	-	.020245	.20534				

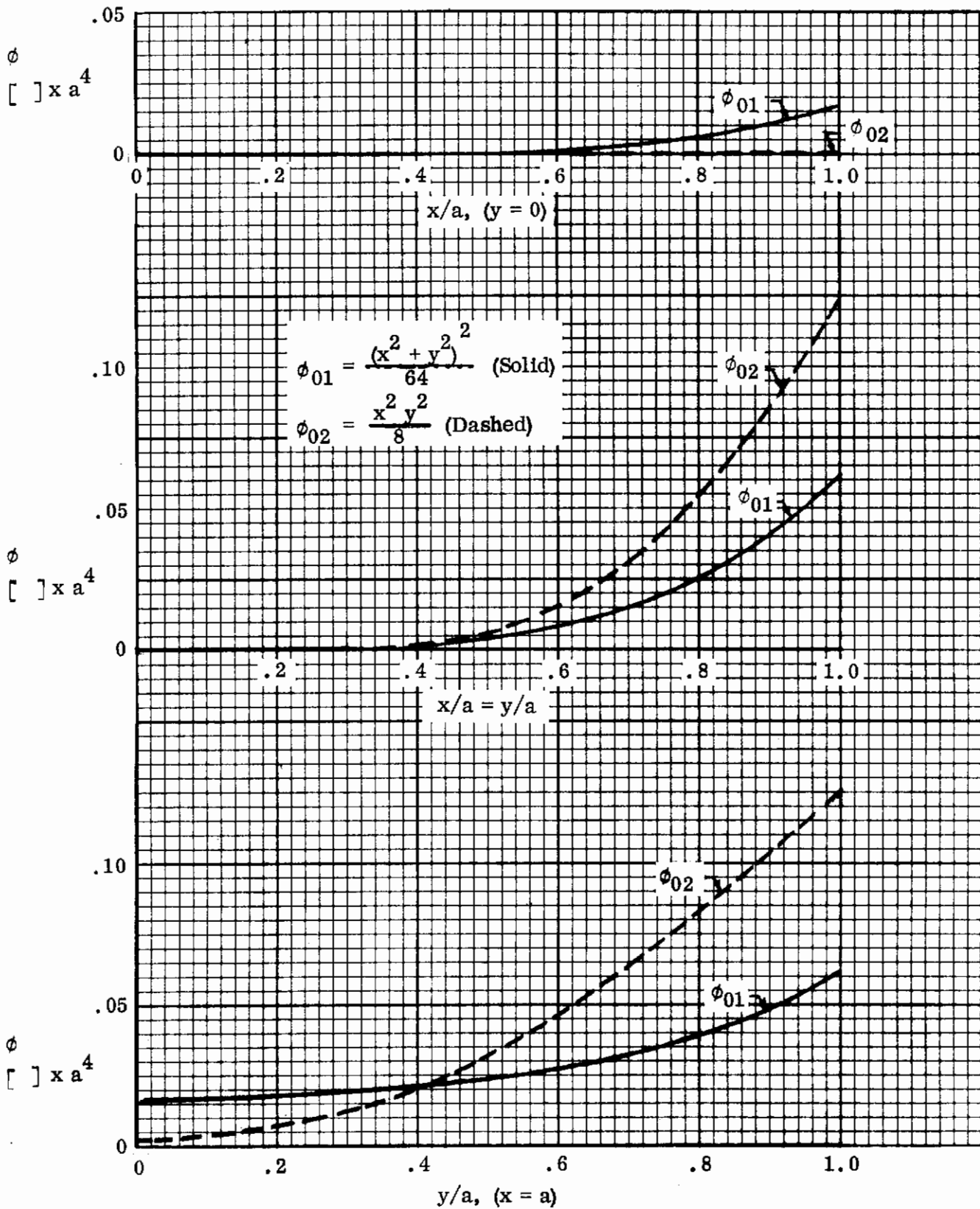


Figure 24. Comparison of Particular Solutions for Example 1

The results are presented in Table VI. They reveal that, although very different-type particular solutions were used, function-wise at least (see Figure 24) the results differ only slightly regardless of method as long as 5 undetermined parameters are used. The only significant difference in accuracy for this problem, using the same homogeneous solution, is caused by the number of terms retained in the approximate series solution. This seems to emphasize the importance of the trial functions used and the number of terms retained in the solution form, rather than the specific solution-method or particular solution.

## C. EXAMPLE 2 - HEAT CONDUCTION IN ANNULAR WEDGE

### 1. Problem Statement and Selection of Trial Functions

The equation governing the steady state heat conduction in a material with uniform and isotropic conductivity is:

$$\nabla^2 T = 0 \quad \text{in } D \quad (63)$$

where  $D$  is the two-dimensional domain contained within an annular sector with boundary,  $B$ , on which  $T$  is specified as:

$$T(a, \theta) = 0 \quad (64)$$

$$T(r, \pm\theta_0) = 0 \quad (65)$$

$$T(b, \theta) = \cos \frac{\pi\theta}{2\theta_0} \quad (66)$$

(see Figure 7b).

The exact solution to this example can be generated through a straight forward application of separation-of-variables and yields:

$$T = \frac{\left(\frac{r}{a}\right)^\alpha - \left(\frac{r}{a}\right)^{-\alpha}}{\left(\frac{b}{a}\right)^\alpha - \left(\frac{b}{a}\right)^{-\alpha}} \cos \alpha\theta \quad (67)$$

where  $\alpha = \frac{\pi}{2\theta_0}$ .

The exact solution for this problem in itself is not of interest here, but simply serves as a standard to which approximate solutions by the methods under investigation will be compared.

### 2. Construction of the Trial Solutions

For approximate methods that require exact satisfaction of boundary conditions (i.e., Ritz-Galerkin, least-squares, collocations, and subdomain), select:

$$T_n = \phi_0(r, \theta) + \sum_{k=1}^n a_k \phi_k(r, \theta) \quad (68)$$

where  $\phi_0$  satisfies the boundary conditions of Equations (64), (65), and (66),  $\phi_k$  satisfies homogeneous conditions on all boundaries of D, and the  $a_k$  are to be determined according to the particular approximate method used.

A method of constructing a suitable  $\phi_0$  is to start with the product form:

$$\phi_0(r, \theta) = \psi_0(r) \chi_0(\theta) \quad (69)$$

Substituting Equation (68) into (66) yields:

$$T_n(b, \theta) = \phi_0(b, \theta) = \psi_0(b) \chi_0(\theta) = \cos \frac{\pi\theta}{2\theta_0} \quad (70)$$

One way of satisfying Equation (70) is to have:

$$\chi_0 = \cos \frac{\pi\theta}{2\theta_0} \quad (71)$$

and

$$\psi_0(b) = 1 \quad (72)$$

This choice of  $\chi_0$  also satisfies Equation (65) automatically. To satisfy Equations (64) and (72), let:

$$\psi_0 = \frac{f(r) - f(a)}{f(b) - f(a)}, \quad (73)$$

where  $f$  is any function of  $r$ , but not a constant, defined in D. For simplicity,  $f(r)$  is taken as:  $f(r) = r$ . Therefore,  $\phi_0$  becomes:

$$\phi_0 = \frac{r - a}{b - a} \cos \frac{\pi\theta}{2\theta_0} \quad (74)$$

In a similar fashion, product-type trial functions,  $\phi_k$ , homogeneous on B, as follows:

$$\phi_k = \psi_k(r) \chi_k(\theta) \quad (75)$$

where

$$\psi_k(b) = \psi_k(a) = 0 \quad (76)$$

$$\chi_k(\theta_0) = 0 \quad (77)$$

and  $\chi_k(\theta)$  is an even function:

$$\chi_k(\theta) = \chi_k(-\theta) \quad (78)$$

Therefore, let:

$$\psi_k = (r - b)(r - a) \bar{\psi}_k(r) \quad (79)$$



$$\chi_k = (\theta^2 - \theta_0^2) \bar{\chi}_k(\theta) \tag{80}$$

where  $\chi_k$  is even in  $\theta$ , and the  $\bar{\chi}_k$  and  $\chi_k$  are arbitrary. However, for simplicity let these be polynomials in  $r$  and  $\theta$ . Therefore, Equation (54) becomes:

$$T_n = \frac{r-a}{b-a} \cos \frac{\pi\theta}{2\theta_0} + (r-b)^n (r-a)^n (\theta^2 - \theta_0^2)^n \sum_m \sum_p a_k r^{m_k} \theta^{2p_k} \tag{81}$$

$m, p = 0, 1, 2, \dots$

Because the exponent  $n$ , on  $(r-b)$ ,  $(r-a)$ , and  $(\theta^2 - \theta_0^2)$  is unnecessary to satisfy Equations (64), (65), and (66):

$$T_n = \frac{r-a}{b-a} \cos \frac{\pi\theta}{2\theta_0} + (r-b)(r-a)(\theta^2 - \theta_0^2) \sum_m \sum_p a_k r^{m_k} \theta^{2p_k} \tag{82}$$

$m, p = 0, 1, 2, \dots$

For methods that require exact satisfaction of the differential Equation (63), the well-known separation-of-variables solution (Reference 31) is:

$$T_n = (A_0 + B_0 \ln r)(C_0 + D_0 \theta) + \sum_k (A_k r^k + B_k r^{-k})(C_k \cos k\theta + D_k \sin k\theta) \tag{83}$$

This is reduced by taking:

$$D_0 = D_k = 0$$

because  $T$  must be symmetrical about  $\theta = 0$ ,

$$C_0 = C_k = 1$$

because the  $A$ 's and  $B$ 's are sufficiently general to absorb this normalization of the  $C$ 's without restricting  $T$ , and  $B_0 = 0$ , because the  $\ln r$  function is usually associated with axisymmetric solutions. Finally, Equation (83) reduces to:

$$T_2 = A_0 + \sum_k (A_k r^k + B_k r^{-k}) \cos k\theta \tag{84}$$

For convenience, the  $k$ 's are chosen as integers in Equation (84).

If  $\alpha$  (Equation (67)) is also an integer, comparison of Equation (67) with Equation (83) shows that it is possible for Equation (84) to result in an exact solution by taking:

$$A_0 = 0, \quad A_k = B_k = 0, \quad k \neq \alpha$$

$$\left. \begin{aligned} A_k &= \frac{\alpha^{-\alpha}}{\left(\frac{b}{a}\right)^\alpha - \left(\frac{b}{a}\right)^{-\alpha}} \\ B_k &= \frac{-a\alpha}{\left(\frac{b}{a}\right)^\alpha - \left(\frac{b}{a}\right)^{-\alpha}} \end{aligned} \right\} k = \alpha$$

Thus, Equation (83) would not be a suitable solution form for point-matching, Mikhlin, as well as a host of other reasonable methods because it would lead to approximate results that are exact.

Therefore, if  $\alpha$  is selected not as an integer, e.g.,

$$\alpha = \frac{\pi}{2\theta_0} = \frac{3}{2}$$

or

$$\theta_0 = \frac{\pi}{3}$$

the ratio  $\frac{b}{a}$  is quite arbitrary, and it is selected as

$$\frac{b}{a} = 1.6$$

Where required, the values of b and a are taken as 0.8 and 0.5, respectively.

### 3. Ritz-Galerkin Applied to Example 2

The equivalence of the Ritz and Galerkin methods for this example is demonstrated in Appendix II.

Inserting solution form Equation (82), which satisfies the boundary conditions, into differential Equation (63), yields:

$$\begin{aligned} \nabla^2 T_n = R = & \frac{\cos \alpha \theta}{r(b-a)} \left\{ 1 - \alpha^2 \left( 1 - \frac{a}{r} \right) \right\} \\ & + \sum_k a_k r^{m_k} \theta^{2p_k} \left\{ (\theta^2 - \theta_0^2) \left[ (m_k + 2)^2 - (m_k + 1)^2 \left( \frac{a+b}{r} \right) + \frac{m_k^2 ab}{r^2} \right] \right. \\ & \left. + \frac{1}{r^2} (r-a)(r-b) \left[ 2(2p_k^2 + 3p_k + 1) - 2p_k(2p_k - 1) \theta_0^2 \theta^{-2} \right] \right\} \quad (85) \\ & m_k, p_k = 1, 2, \dots \end{aligned}$$

For a one-term solution with:

$$\theta_0 = \frac{\pi}{3}$$

$$a = .5$$

$$b = .8$$

$$m_k = p_k = 0$$

R becomes simply

$$R = 2a_1 \left( 2\theta^2 - \frac{2\pi^2}{9} + 1 \right) + \frac{1}{r} \left[ -\frac{25}{6} \cos \frac{3\theta}{2} + 1.3 a_1 \left( -\theta^2 + \frac{\pi^2}{9} - 2 \right) \right] + \frac{1}{r^2} \left[ \frac{15}{4} \cos \frac{3\theta}{2} + .8 a_1 \right] \quad (86)$$

To apply the Galerkin method, the residual of Equation (86) is orthogonalized with respect to  $\varphi_1$  in D, i. e.,

$$\int_{.5}^{.8} 2 \int_0^{\pi/3} (r - .8)(r - .5) \left( \theta^2 - \frac{\pi^2}{9} \right) R r d\theta dr = 0 \quad (87)$$

This becomes a single algebraic equation in  $a_1$  that when solved for  $a_1$ , yields:

$$a_1 = 1.0776$$

Thus:

$$T_1 = \frac{r - .5}{.3} \cos \frac{3\theta}{2} + 1.0776 (r - .8)(r - .5) \left( \theta^2 - \frac{\pi^2}{9} \right)$$

A computer program based upon Equation (85) and the n conditions:

$$\int_{.5}^{.8} \int_0^{\pi/3} \varphi_k R r d\theta dr = 0, \quad k = 1, 2, \dots, n \quad (88)$$

where

$$\varphi_k = (r - b)(r - a) (\theta^2 - \theta_0^2) r^{m_k} \theta^{2p_k}$$

was written. Detailed integrations called for by Equation (88) were performed analytically and programmed. The integrations were rather lengthy but quite reasonable in comparison to those that would be required by an n-term ( $n > 1$ ) Ritz solution for Example 1. The program's input quantities include the radii a and b, semiopening angle  $\theta_0$ , and trial-function parameters  $m_k$  and  $p_k$ . The program generates the n algebraic equations indicated by Equation (88) and proceeds to solve them simultaneously. The solution is then numerically compared against its exact counterpart. The program also computes approximate solution residuals for comparison purposes.

#### 4. Least Squares Applied to Example 2

As was the case for Example 1, this method requires more labor than the Galerkin method for the same number of parameters retained in the trial family. This results from the fact that the same residual (R) is made orthogonal with respect to a weighting function that involves more terms (i. e.,  $L\varphi_k$  generally involves more terms than  $\varphi_k$ ). Thus, more algebraic manipulation and integration is required.

To demonstrate the least-squares method for a one-term solution, set:

$$\int_a^b \int_0^{\theta_0} (v^2 \phi_1) R \, r \, d\theta \, dr = 0 \quad (89)$$

where R is given by Equation (86) and:

$$v^2 \phi_1 = 2(2\theta^2 - \frac{2\pi^2}{9} + 1) + \frac{1}{r} \left[ 1.3 (-\theta^2 + \frac{\pi^2}{9} - 2) \right] + \frac{1}{r^2} (.8)$$

Performing the operations indicated by Equation (89) and solving for  $a_1$  yields:

$$a_1 = 1.095$$

## 5. Biezeno-Koch Applied to Example 2

Starting with the family given in Equation (82), which satisfies the boundary conditions, this method calls for a subdivision of the domain into  $n$  pieces and proceeds to satisfy equilibrium, on the average, for each piece.

Formally, it sets:

$$\int_{D_k} R \, dA_k = 0 \quad (90)$$

within subdomains,  $D_k$ , contained within  $D$ , and where R is given by Equation (86).

Application of Equation (90) to a one-term solution, where  $D_k$  is the entire domain, performance of the integration with respect to  $r$  first, and substitution into the limits  $r = 0.5$  and  $0.8$ , yields:

$$0 = 2 \int_0^{\pi/3} \left\{ \theta^2 [ .78 a_1 - .39 a_1 ] + [ .466 a_1 - .352 a_1 + .163 a_1 ] \right. \\ \left. + \cos \frac{3\theta}{2} [ -1.25 + .765 ] \right\} d\theta$$

Integrating this with respect to  $\theta$  gives:

$$.756(.39)a_1 + 2.095(.277)a_1 + 1.333(-.485) = 0$$

Thus:

$$a_1 = \frac{.646}{.879} = .736$$

and, therefore:

$$T_1 = \frac{r - .5}{.3} \cos \frac{3\theta}{2} + .736 (r - .8) (r - .5) \left( \theta^2 - \frac{\pi^2}{9} \right)$$

## 6. Collocations Applied to Example 2

If, once again, a trial family that satisfies the boundary conditions is used to develop a general expression for the domain residual with undetermined parameters, this can be collocated at  $n$  points to yield  $n$  algebraic equations in the  $n$  parameters. To demonstrate this for a simple one-parameter solution,  $R$  of Equation (86) is truncated at  $k = 1$  and then collocated at the central point  $(r, \theta) = (.65, 0)$  for Example 2. The constant  $a_1$  then becomes 1.072. Thus:

$$T_1 = \frac{r - .5}{.3} \cos \frac{3\theta}{2} + 1.072 (r - .8) (r - .5) \left( \theta^2 - \frac{\pi^2}{9} \right)$$

A computer program that permits the retention of up to thirty undetermined parameters and collocation of up to sixty points (i.e., up to 60 equations can be generated) was written. It solves the algebraic equations that result and in doing so minimizes the sum of the internal residual squared at the points collocated.

## 7. Milkhlin Applied to Example 2

Starting with a  $T_n$  that satisfies differential Equation (63), the boundary conditions (Equations (64) through (66)) are satisfied approximately by letting:

$$a \int_{\theta=0}^{\theta_0} \frac{\partial T_n}{\partial a_k} T_n d\theta + \int_a^b \frac{\partial T_n}{\partial a_k} T_n dr - b \int_{\theta_0}^0 (T_n - \cos \alpha \theta) d\theta = 0$$

$k = 1, 2, \dots, n$

Note, the boundary integrations must be carried out over positive arc-lengths. This is necessary because the quantity to be minimized must be a positive definite quantity. Hence, a negative sign appears before the last integral. In general then, Milkhlin's method, stated more precisely minimizes:

$$\int_B R_i^2 d|S|$$

where  $R_i$  is real on  $B$ .

Beginning with the first four terms of solution form Equation(84) , Milkhlin's method yields four simultaneous equations that, when solved for the parameters, yields the approximate solution:

$$T_4 = .417 + .584 r \cos \theta - .505 r^{-1} \cos \theta + 1.116 r^2 \cos 2\theta$$

## 8. Point-Matching Applied to Example 2

Starting with  $n$  terms in solution form Equation (84) and setting  $T_n = T$  at  $m$  points results in an  $m \times n$  system of algebraic equations for the undetermined parameters. If this system is of rank  $n$ , a unique solution for the  $n$   $a_k$ 's is possible. Limiting  $m = n = 4$ , where the four boundary points collocated are:

$$(r, \theta) = (.5, 0), (.5, \frac{\pi}{3}), (.8, \frac{\pi}{3}), (.8, 0),$$

and solving for the a's, results in:

$$T_4 = .641 + .286 r \cos \theta - \frac{.552}{r} \cos \theta + 1.282 \cos 2\theta$$

A computer program that solves heat conduction problems (for only temperature boundary conditions) by least squares point-matching ( $m > n$ ) was written and results obtained from it are presented later.

9. Comparisons for Example 2

Similar to the error definitions for Example 1,  $\epsilon_T$ , the percentage temperature error, is defined as:

$$\epsilon_T(r, \theta) = \frac{T_n(r, \theta) - T(r, \theta)}{T_{\max}} \times 100$$

where  $T$  is the exact temperature (its maximum value being unity). Thus:

$$\epsilon_T = (T_n - T) \times 100\%$$

The percentage residual error,  $\epsilon_R$ , is defined by:

$$\epsilon_R(r, \theta) = \frac{R(r, \theta)}{\left( \nabla^2 \phi_0 \right)_{\max}} \times 100$$

This definition was prompted by the fact that  $T_n$  consists of two parts,  $\phi_0$  and  $\bar{T}_n$ , where:

$$\bar{T}_n = \sum_k a_k \phi_k$$

the first of which ( $\phi_0$ ) does not contain any undetermined parameters. Thus, the second part  $\bar{T}_n$ , can be thought of as the approximate solution of the modified problem:

$$\nabla^2 \bar{T}_n = -\nabla^2 \phi_0 \quad \text{in } D$$

and

$$\bar{T}_n = 0 \quad \text{on } B$$

where  $-\nabla^2 \phi_0$  can be thought of as the ratio of distributed heat source, of strength  $\nabla^2 \phi_0$ , to conductivities. It can be shown from the expression for  $\nabla^2 T_n$  that:

$$\nabla^2 \phi_0 = [r(b-a)]^{-1} \left\{ 1 - \left(1 - \frac{a}{r}\right) \left(\frac{\pi}{2\theta_0}\right)^2 \right\} \cos \frac{\pi\theta}{2\theta_0} .$$

Therefore, the maximum  $\nabla^2 \phi_0$  occurs at  $r = a$ ,  $\theta = 0$  and is equal to:

$$\left( \nabla^2 \phi_0 \right)_{\max} = [a(b-a)]^{-1}$$

Temperature and residual error comparisons, as previously defined, are made in Figures 25 and 26, for the one-term Ritz-Galerkin, least-squares, collocations and subdomain (Biezeno-Koch) solutions. The curves reveal only slight differences among these methods. This is to be expected because all four approaches employ the same solution form; i. e.,

$$T_1(r, \theta) = \varphi_0(r, \theta) + a_1 \varphi_1(r, \theta)$$

where  $\varphi_0$  and  $\varphi_1$  are the same for all four cases and only  $a_1$  varies. Furthermore, the variation in the parameter  $a_1$  is very slight, as can be seen in the following table:

<u>Method</u>	<u><math>a_1</math></u>
Ritz-Galerkin	1.078
Least-squares	1.095
Collocation (of central point $r = .65, \theta = 0$ )	1.072
Biezeno-Koch	1.088

In addition to presenting these solutions, Figures 25 and 26 indicate the errors involved if there are no undetermined parameters, i. e.,  $T_0 = \varphi_0$  (dashed curves). The comparative accuracy of this particular solution indicates the reason for the high accuracy obtained with only a single undetermined parameter.

Once again, as with the square clamped plate problem, the residual errors are much higher than the temperature errors (being two orders of magnitude greater within the interior). However, unlike the previous problem-solutions by these methods, the domain equation-residuals evaluated on the boundary are not much higher but comparable to those found inside the domain.

The point-matching computer program was used to obtain a four-parameter least-squares solution. The errors for these solutions are compared in Figure 27 along with the straight point-matching and Mikhlin four-parameter solution errors. Results for a typical one-term solution by one of the previous methods (which used a solution form that satisfied the boundary conditions) is also presented for comparison purposes. As can be seen by the even-spaced dashed lines in Figure 27, these errors are almost imperceptible, except for the error curves for  $\theta = 0$  from  $r = 0.5$  to  $0.8$  (upper graphs of Figure 27a and Figure 27b), where the  $\epsilon_T$  and  $\epsilon_R$  were largest in Figures 25 and 26.

In addition to giving a quantitative measure of the relative accuracy of the methods as applied to Example 2, these curves demonstrate, systematically, how least-squares point-matching approaches Mikhlin's method. To accentuate this comparison, several additional least-squares point-matching cases were run. The results of these are presented in the following tabulation.

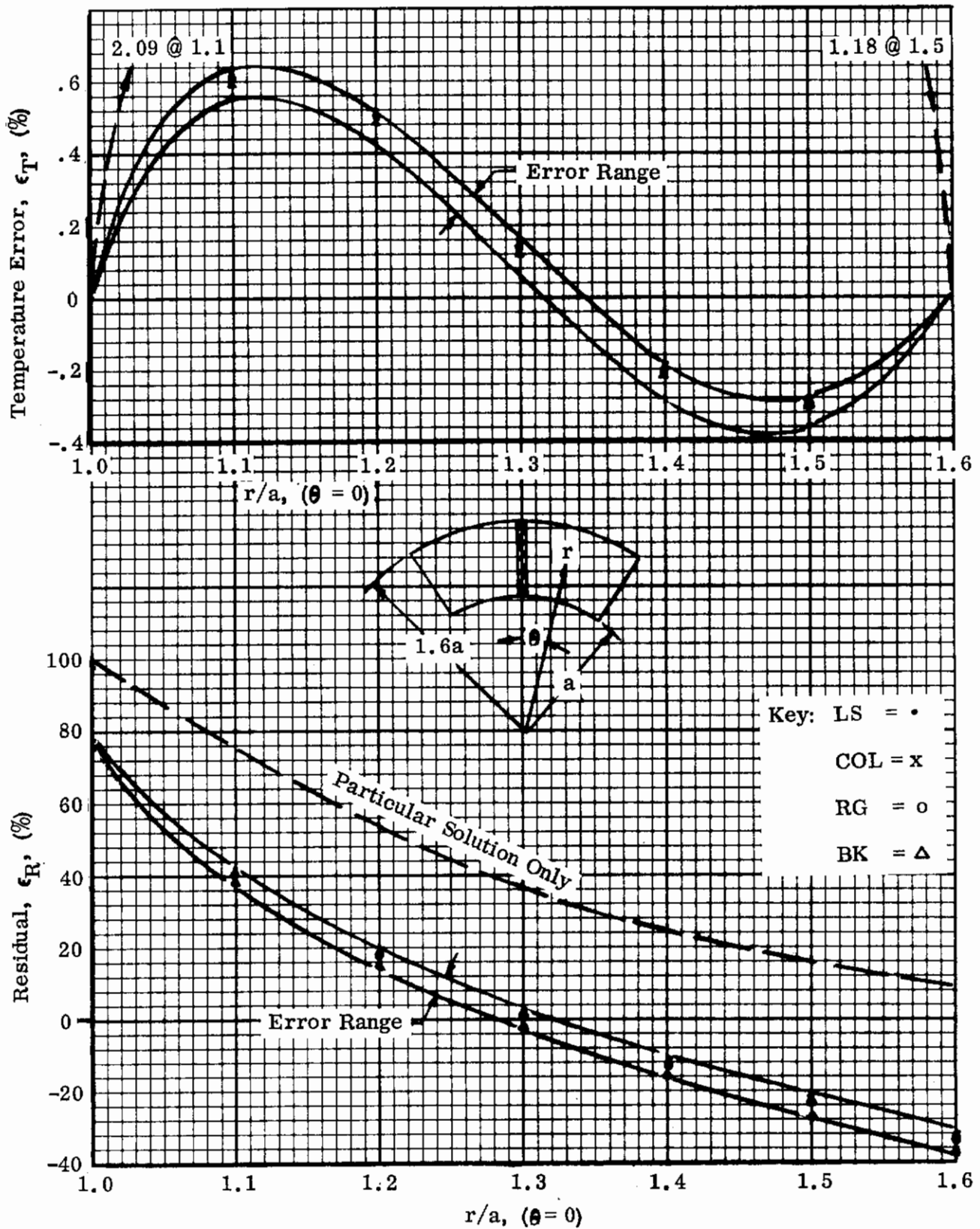


Figure 25a. Example 2, Range of Interior Temperature and Residual Errors ( $\epsilon_T$  and  $\epsilon_R$ ) for One-Term Ritz-Galerkin, Least Squares, Collocation and Biezeno-Koch Solutions



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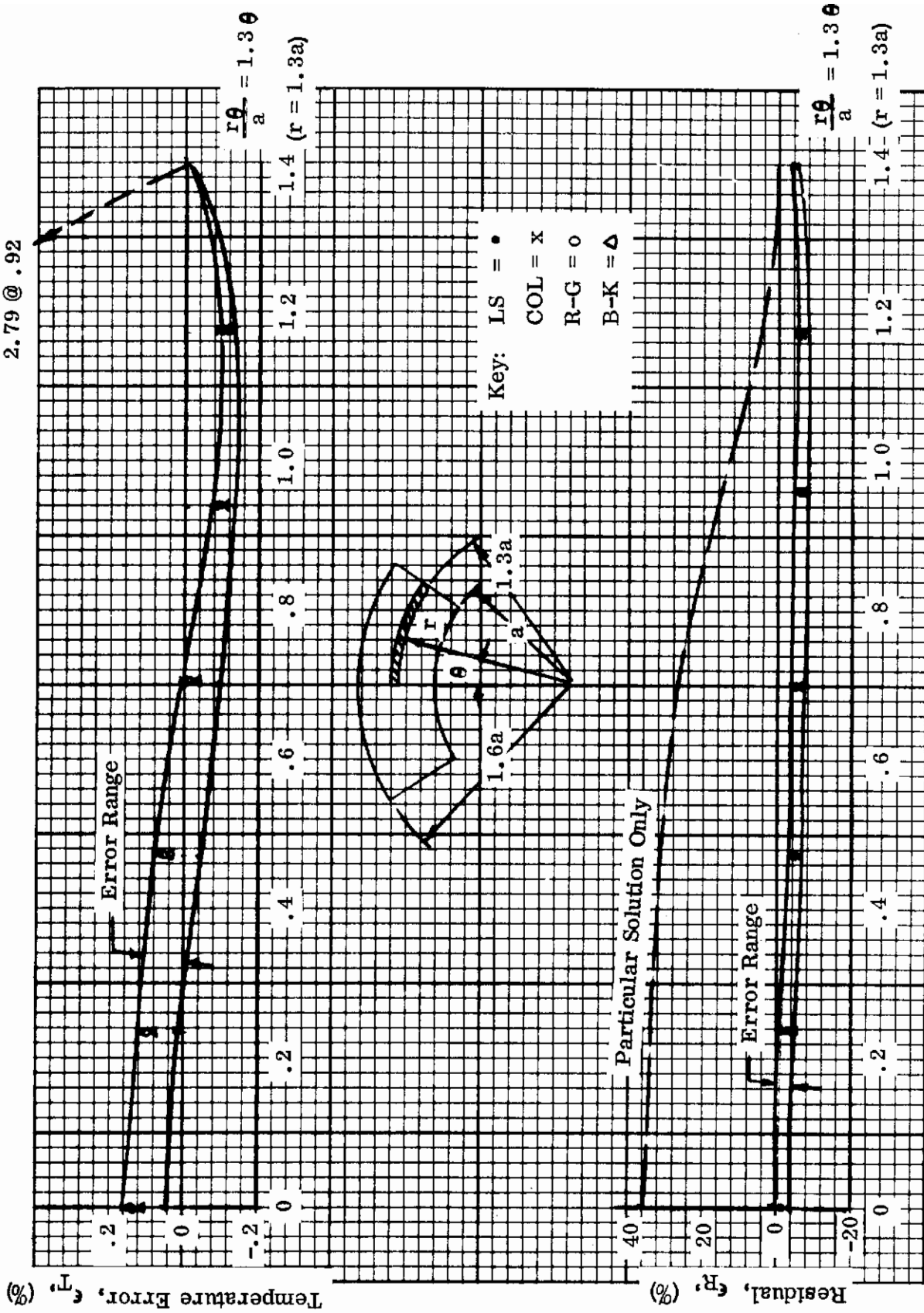


Figure 25b. Example 2, Range of Interior Temperature and Residual Errors for One-Term Ritz-Galerkin, Least-Squares, Collocation and Biezeno-Koch Solutions

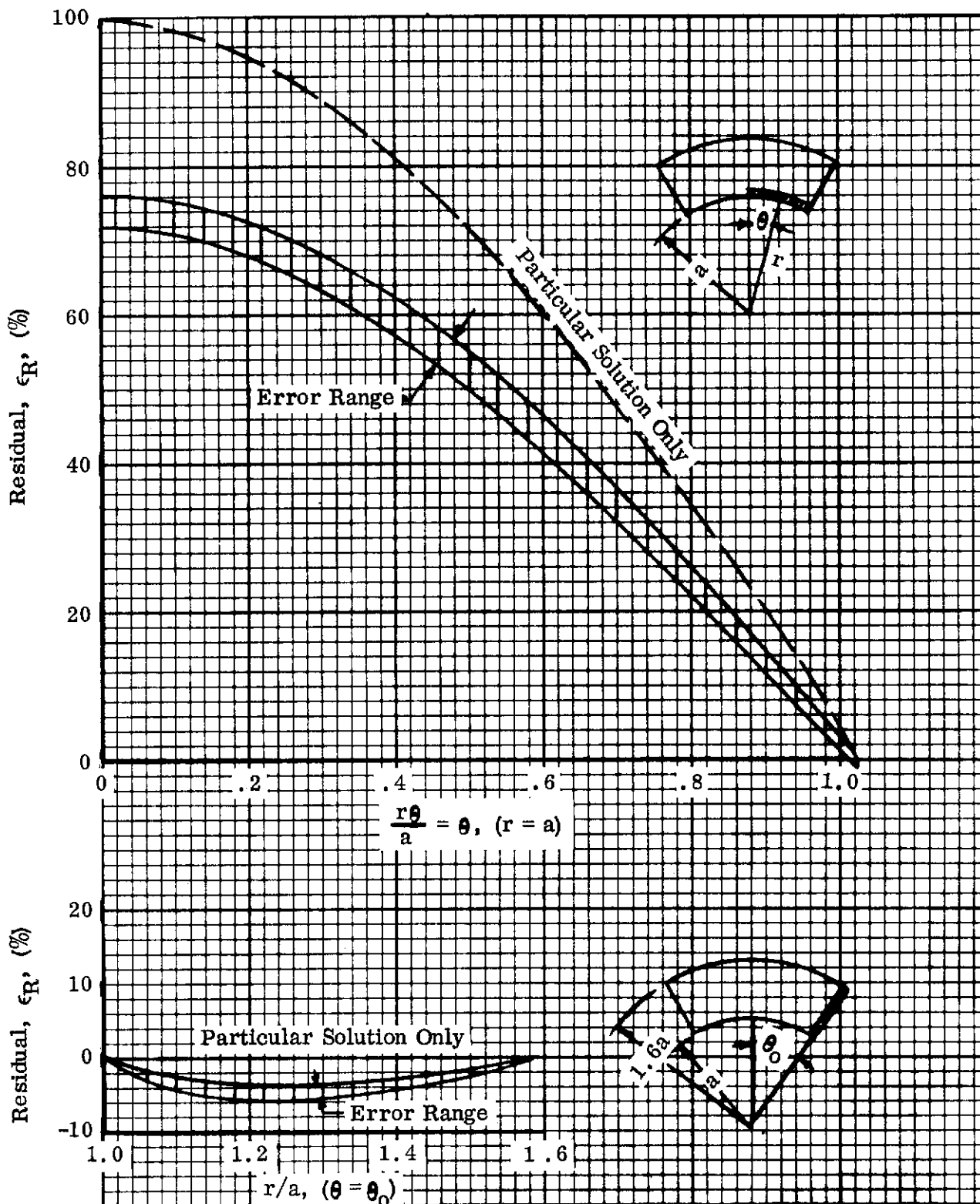


Figure 26a. Example 2, Range of Residual Errors ( $\epsilon_R$ ) on the Boundaries for One-Term Ritz-Galerkin, Least Squares, Collocation and Biezeno-Koch Solutions

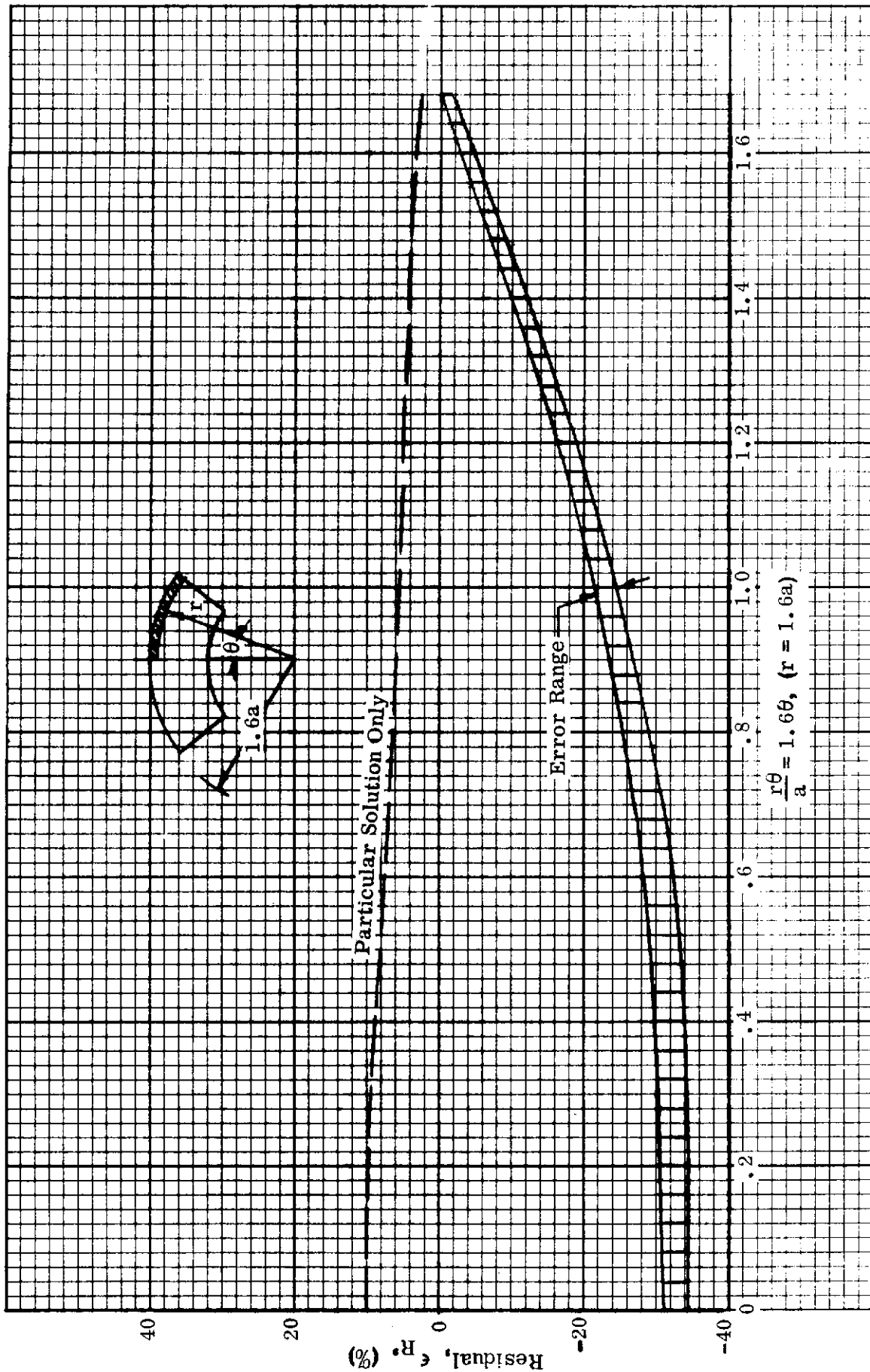


Figure 26b. Example 2, Range of Residual Errors ( $\epsilon_R$ ) on the Outer Boundary for One-Term Ritz-Galerkin, Least Squares, Collocation and Biezeno-Koch Solutions

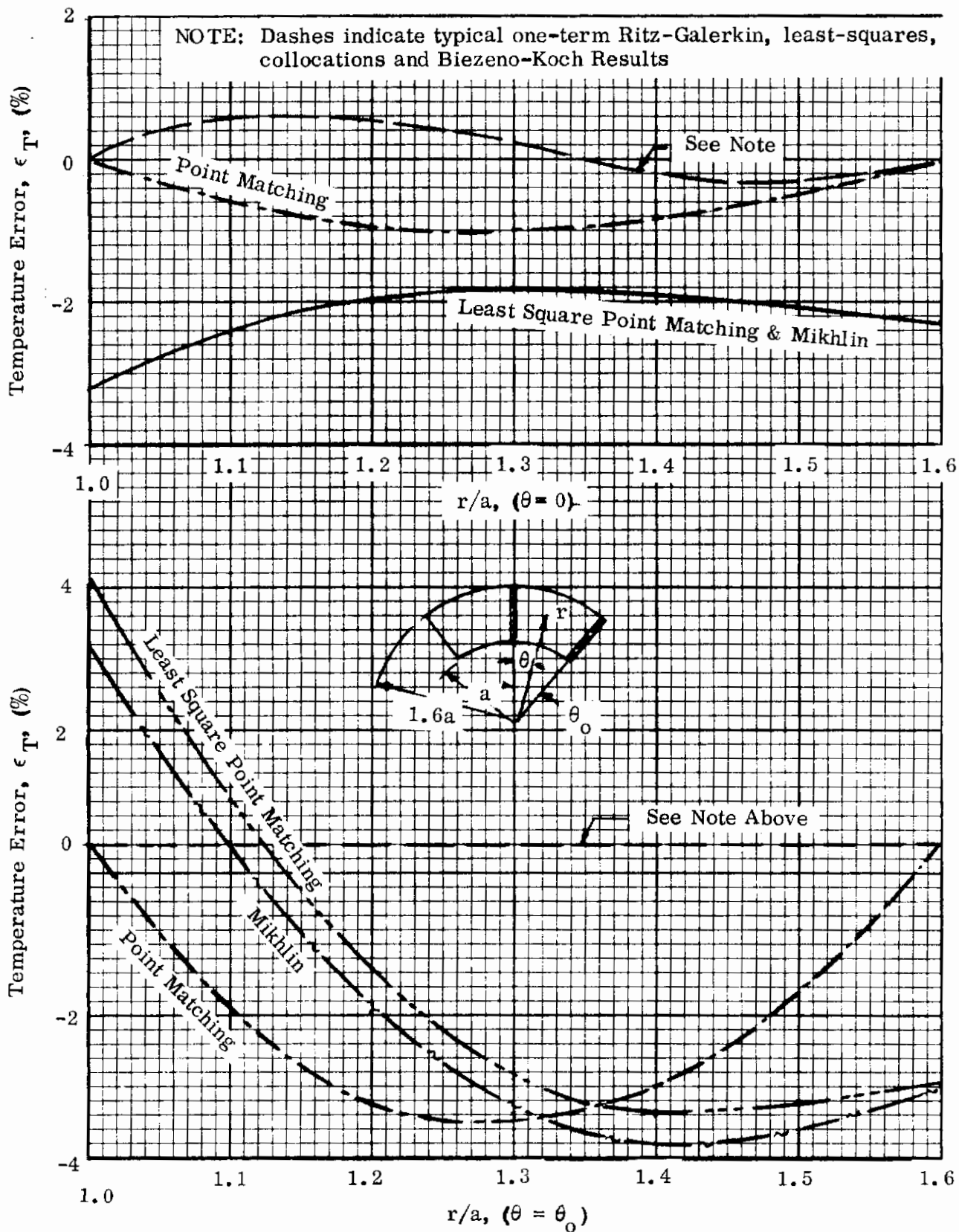


Figure 27a. Example 2, Comparison of Temperature Errors and Four-Term Point Matching, Mikhlin, and Least-Square Point Matching Solutions

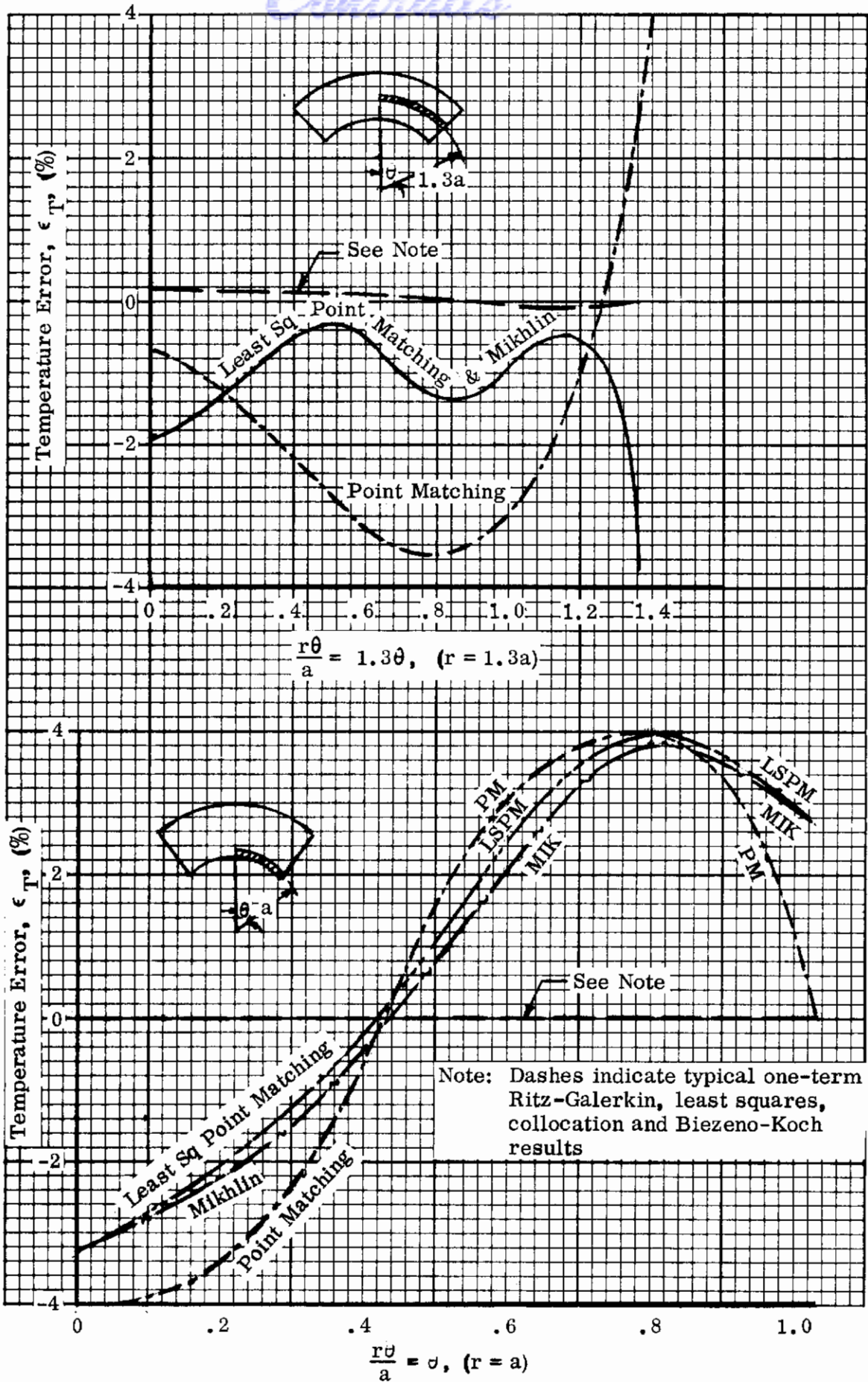


Figure 27b. Example 2, Comparison of Temperature Errors for Four-Term Point Matching, Mikhlín, and Least Square Point Matching Solutions

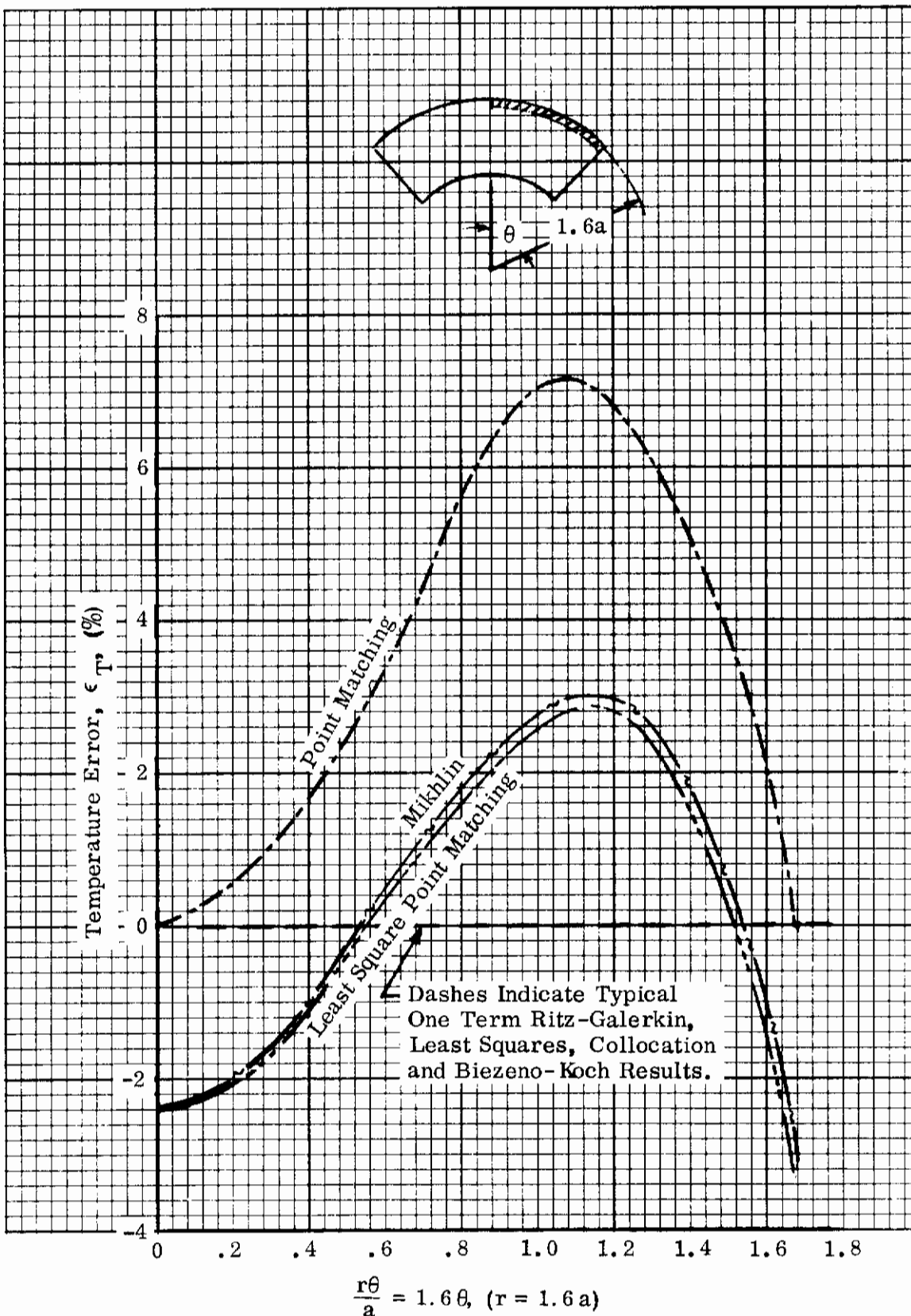


Figure 27c. Example 2, Comparison of Temperature Errors for Four-Term Point Matching, Mikhlín, and Least-Square Point Matching Solutions

# Contrails

Trial Family:  $T = A_1 + A_2 r \cos \theta + A_3 r^{-1} \cos \theta + A_4 r^2 \cos 2\theta$

Method	Coefficients			
	A <sub>1</sub>	A <sub>2</sub>	A <sub>3</sub>	A <sub>4</sub>
Point Matching (corner and $\theta = 0$ points matched)	.641	.286	-.552	1.282
Least-Squares Point Matching ( 8x4)	.521	.521	-.525	1.149
(16x4)	.473	.588	-.509	1.111
(32x4)	.459	.610	-.504	1.099
(points equally spaced)				
Mikhlin	.471	.584	-.505	1.116

Judging from the coefficients, it appears as if the 32x4 solution deviated further (from that obtained by Mikhlin's method) than the 16x4 least-squares solution. This was probably due to higher numerical inaccuracies associated with the larger matrices because theory predicts a greater correspondence between Mikhlin and a least-square attempt which uses more points.

Additional numerical results using the point-matching program are presented in Tables VII and VIII.

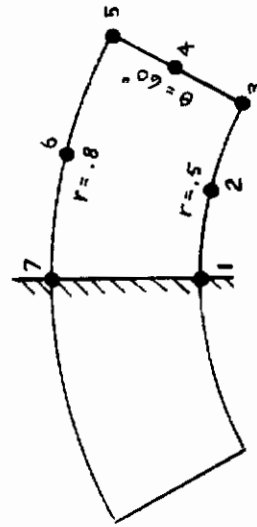
In contrast to results for Example 1, numerical studies performed with the collocation program, for the problem at hand, resulted in eventual convergence, practically speaking, to the exact solution (refer to Tables VII and VIII). This is attributed here, to the use of less limiting trial functions than was used for Example 1. In particular, note that the terms included within the summation form a complete set of functions, whereas those used in Example 1 do not permit flexibility in the  $\theta$  direction.

Because there are obviously many ways in which an n parameter solution can be selected by Ritz's method and collocations by varying the  $m_k$ 's and  $p_k$ 's, a wide variety of cases were run. Tables IX and X reveal that the best solutions (shown by asterisks), for any given number of parameters, improve as more trial terms are selected, until numerical precision problems set in and begin to destroy solution accuracy. As can be seen in Figure 28, this occurred rather early for the Ritz-Galerkin method. Note also, the correlation between the lowest maximum residuals and temperature errors, where  $|\epsilon_R|_{\max}$  was evaluated at least 10% within the domain boundaries.

More detailed comparisons of the effect of varying the number of parameters, trial functions, and points collocated and a comparison of Ritz-Galerkin with collocations is to be found in Figures 29, 30, and 31. In each case, Ritz is superior to collocations. However, least-square collocation solution accuracy is comparable to Ritz's. This is significant because a collocation program, with least-squares option, is generally simpler to develop. The selection of trial functions is important as evidenced by the negligible improvement of the two-term Ritz solution, over the one-term Ritz solution (Figure 29) when the (c, d) pairs chosen are (0, 0) and (1, 1) and the considerable im-

TABLE VII: POINT MATCHING RESULTS FOR EXAMPLE 2

Number of Parameters	$T(r, \theta)$		Average Absolute Boundary Error (Approximate)		Points Not Matched in Reference Figure
	$(0.65, 0)$	$(0.65, 40 \text{ degrees})$	$r = 0.5$	$\theta = 60 \text{ degrees}$	
4	.51895	.22663	.040	.020	.044
5	.53021	.27227	.030	.007	.0004
6	.52804	.26382	.00046	.0053	.0010
7	.52792	.26394	.00048	.00021	.00035
<b>Exact</b>	<b>.52791</b>	<b>.26395</b>	<b>0</b>	<b>0</b>	<b>0</b>



Reference Figure



TABLE VIII: LEAST SQUARES POINT MATCHING RESULTS FOR EXAMPLE 2

Matrix Size	T		Average Absolute Boundary Error (Approximate)	
	(0.65, 0)	(0.65, 40)	r = 0.5	$\theta = 60$ degrees r = 0.8
4 x 4	.51895	.22663	.040	.020
8 x 4	.54010	.24039	.034	.021
12 x 4	.54233	.24231		
16 x 4	.54266	.24623		
24 x 4	.54697	.24635		
32 x 4	.54604	.24819	.032	.023
5 x 5	.53021	.27227	.030	.007
12 x 5	.52851	.26294	.011	.003
8 x 8	.52789	.26408		
16 x 8	.52794	.26398	.00042	.00012
12 x 12	.52791	.26395	.00010	.00011
24 x 12	.52791	.26395	.00000	.00000
16 x 16	.52791	.26395	.00000	.00000
32 x 16	.52791	.26395	.00000	.00000
Exact	.52791	.26395	0	0

TABLE IX. COLLOCATION RESULTS FOR EXAMPLE 2

Number of Terms	$ \epsilon_T _{\max}$	$ \epsilon_R _{\max}$	c	d	Collocation Points $[r](\theta)$
1	.75	77.0	0	0	[.65](30)
* 1	.62	75.3	0	0	[.65](0)
2 (No. 1)	$.30 \times 10^2$	$30.1 \times 10^2$	0,1	0	[.65](15,45)
2 (No. 2)	.64	75.6	0	0,1	[.65](15,45)
2 (No. 3)	.64	75.6	0 1	0 1	[.65](15,45)
2 (No. 4)	$.49 \times 10^2$	$49.6 \times 10^2$	0,1	0	[.65](0,30)
2 (No. 5)	.63	75.6	0	0,1	[.65](0,30)
* 2 (No. 6)	.63	75.6	0 1	0 1	[.65](0,30)
3	$.37 \times 10^1$	$38.4 \times 10^1$	0 1	0,1 0	[.65](10,30,50)
3	$.54 \times 10^1$	$55.2 \times 10^1$	0 1	0,1 0	[.65](0,20,40)
3	.63	75.5	0 1	0,1 1	[.65](10,30,50)
* 3	.63	75.5	0 1	0,1 1	[.65](0,20,40)
* 4	.17	33.4	0 1	0,1 0,1	[.60, .70](15,45)
5	.79	20.8	0 1	0,1,2 0,1	[.55, .75](15,45), [.65](30)
5	.50	23.8	0 1 2	0,1 0,1 1	[.55, .75](15,45), [.65](30)
5	.79	29.9	0 1 1	0,1 0,1 2	[.55, .75](15,45), [.65](30)
* 5	.03	9.8	0 1 2	0,1 0,1 0	[.55, .75](15,45), [.65](30)
5	.78	24.9	0 1 2	0,1 0,1 2	[.55, .75](15,45), [.65](30)

TABLE IX (Cont'd)

Number of Terms	$ \epsilon_T _{\max}$	$ \epsilon_R _{\max}$	c	d	Collocation Points [r]( $\theta$ )
5	.91	23.8	0 1	0,1,2,3 0	[.55,.75](15,45), [.65](30)
5	.17	33.3	0 1 2	0,1 0,1 6	[.60,.70](10,30), [.65](50)
* 6	.03	9.4	0 1 2	0,1,2 0,1 0	[.55,.75](15,45), [.65](0,30)
6	$.36 \times 10^1$	$10.0 \times 10^1$	0 1 2	0,1,2 0,1 0	[.55,.75](10,30,50)
7	$.2 \times 10^1$	$11.3 \times 10^1$	0 1	0,1,2,3 0,1,2	[.55,.75](10,30,50), [.65](30)
* 7	.04	9.9	0 1 2	0,1,2 0,1,2 0	[.55,.75](10,30,50), [.65](30)
7	.06	10.5	0 1 2	0,1,2 0,1,2 0	[.55,.75](10,30,50), [.65](40)
* 8	.01	9.0	0 1 2	0,1,2 0,1,2 0,1	[.55,.75](10,30,50), [.65](20,40)
* 9	.01	9.0	0 1 2	0,1,2 0,1,2 0,1,2	[.55,.75](10,30,50), [.65](0,20,40)
9	.01	9.0	0 1 2	0,1,2 0,1,2 0,1,2	[.55,.65,.75](10,30,50)
* 12	.01	8.7	0 1 2 3	0,1,2,3 0,1,2,3 0,1,2 0	[.55,.65,.75](7.5,22.5,37.5,52.5)
16	.74	$4.4 \times 10^1$	0 1 2 3	0,1,2,3 0,1,2,3 0,1,2,3 0,1,2,3	[.55,.65,.75](5,15,25,35,45), .65 (.50)
* 16	.002	2.4	0 1 2 3	0,1,2,3 0,1,2,3 0,1,2,3 0,1,2,3	[.5375,.6125,.6875,.7625] (7.5,22.5,37.5,52.5)

TABLE IX (Cont'd)

Number of Terms	$ \epsilon_T _{\max}$	$ \epsilon_R _{\max}$	c	d	Collocation Points [r] ( $\theta$ )
* 24	.0008	1.6	0	0,1,2,3,4	[.60, .70, .75] (0, 15, 30, 45), [.55, .65, .75] (7.5, 22.5, 37.5, 52.5)
			1	0,1,2,3,4	
			2	0,1,2,3,4	
			3	0,1,2,3,4	
			4	0,1,2,3	

\* Indicates data plotted in Figure 28.

TABLE X. RITZ-GALERKIN RESULTS FOR EXAMPLE 2

Number of Terms	$ \epsilon_T _{\max}$	$ \epsilon_R _{\max}$	c	d
* 1	.61	75.2	0	0
2	.56	74.4	0 0	0 1
2	.52	73.6	0 1	0 1
* 2	.14	32.4	0 1	0 0
* 3	.07	31.5	0 0 1	0 1 0
* 4	.06	30.0	0 1	0,1 0,1
5	.08	30.0	0 1	0,1 0,1,2
5	.08	30.0	0 1 2	0,1 0,1 2
* 5	.01	9.6	0 1 2	0,1 0,1 0
5	.06	30.0	0 1 2	0,1 0,1 1
5	.89	23.7	0 1	0,1,2,3 0
5	.08	30.0	0 1	0,1,2 0,1
* 6	.02	9.4	0 1 2	0,1,2 0,1 0
* 7	.02	9.0	0 1 2	0,1,2 0,1 0,2

TABLE X (Cont'd)

Number of Terms	$ \epsilon_T _{\max}$	$ \epsilon_R _{\max}$	c	d
7	.02	9.1	0 1 2	0,1,2 0,1,2 0
* 8	.02	8.4	0 1 2	0,1,2 0,1,2 0,1
9	.02	8.4	0 1 2	0,1,2 0,1,2 0,1,2
12	.87	39.0	0 1 2 3	0,1,2,3 0,1,2 0,1,2 0,3
*12	.89	39.0	0 1 2 3	0,1,2,3 0,1,2,3 0,1,2 0
*16	.89	37.5	0 1 2 3	0,1,2,3 0,1,2,3 0,1,2,3 0,1,2,3
*24	$.92 \times 10^2$	$12.8 \times 10^3$	0 1 2 3 4	0,1,2,3,4 0,1,2,3,4 0,1,2,3,4 0,1,2,3,4 0,1,2,3

\* Indicates data plotted in Figure 28.

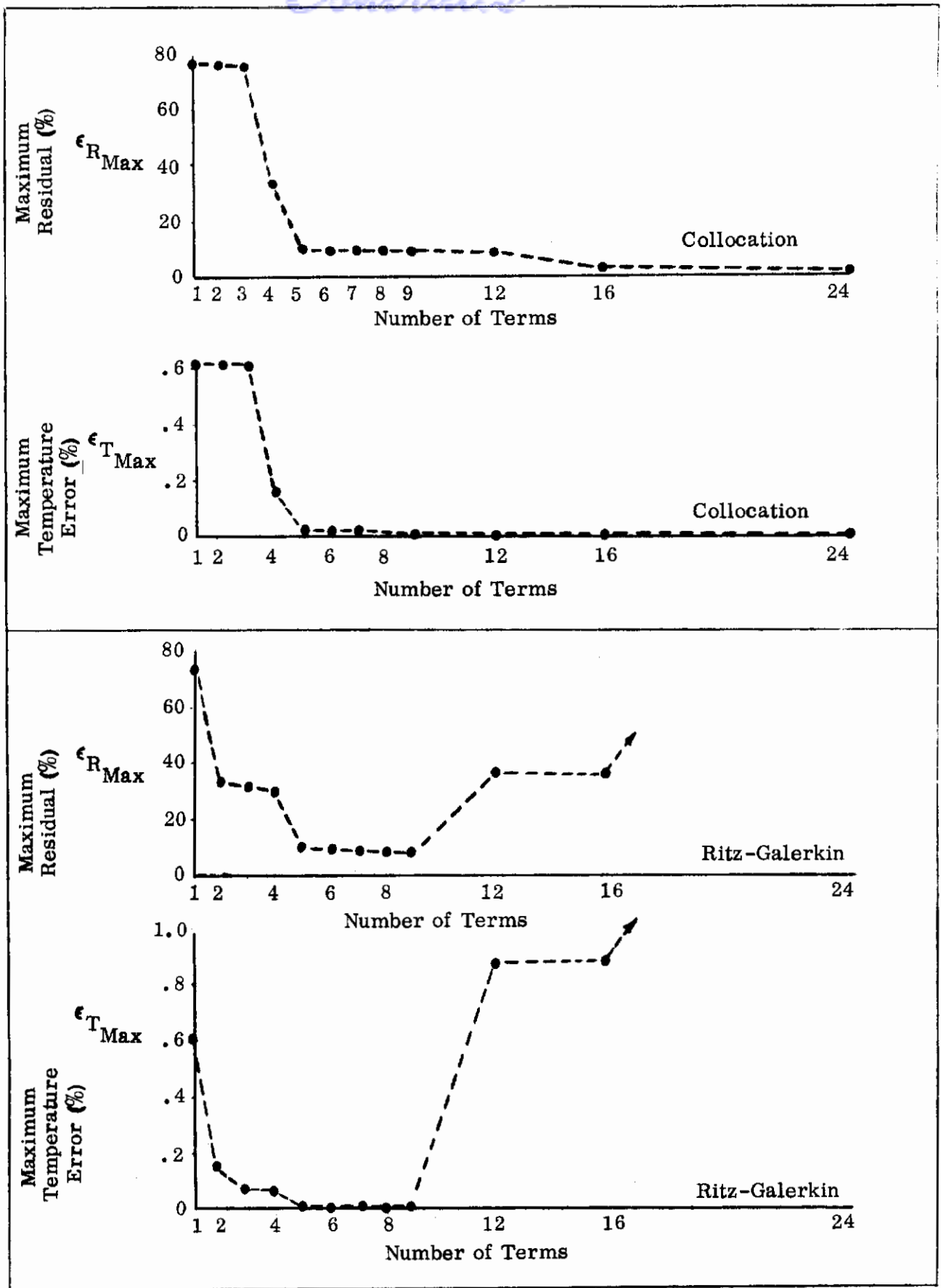


Figure 28. Example 2, Maximum Residuals and Temperature errors vs Number of Terms for Ritz-Galerkin and Collocation Solutions.

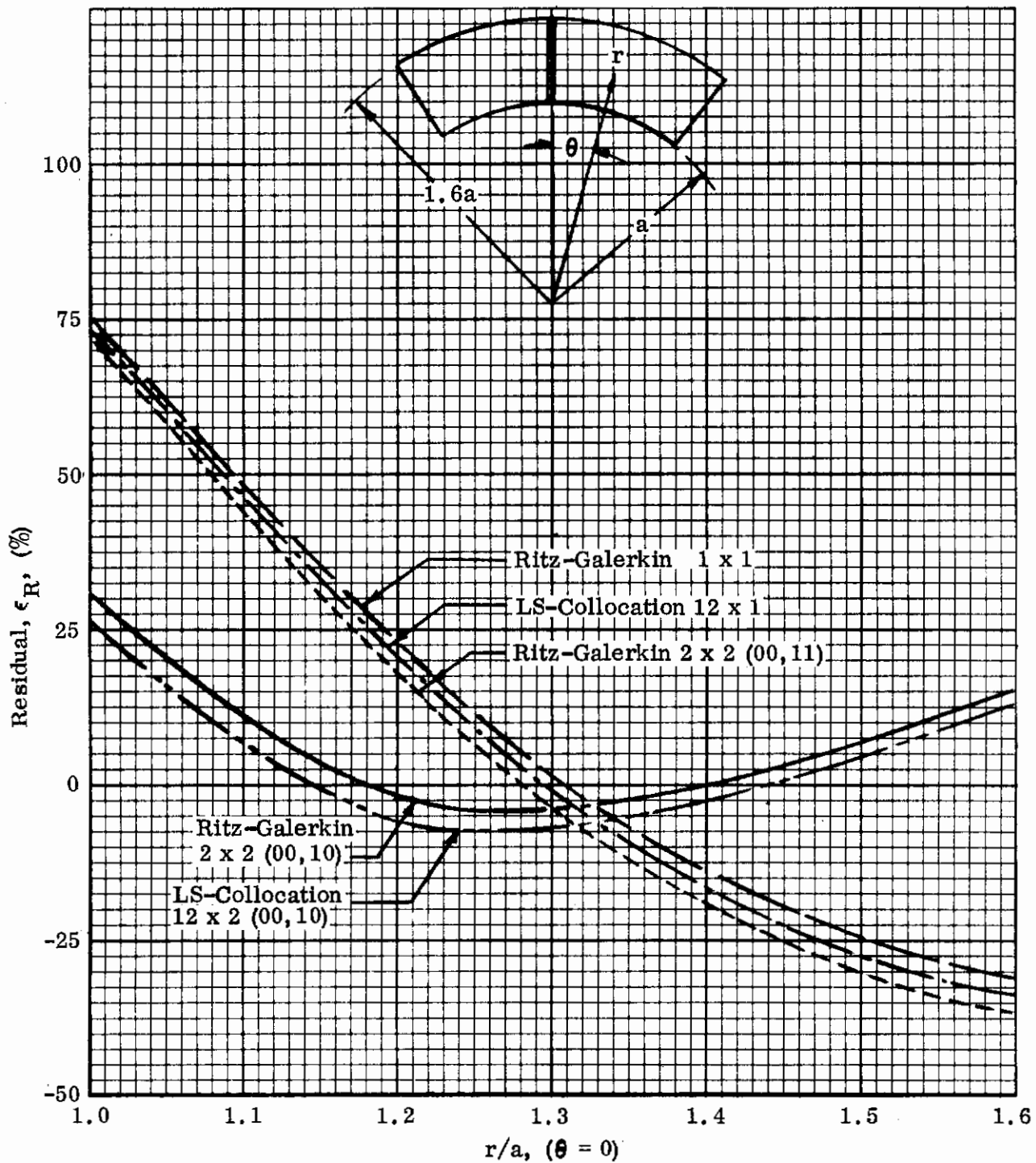


Figure 29a. Ritz-Galerkin and Least Squares-Collocation Residuals for Example 2



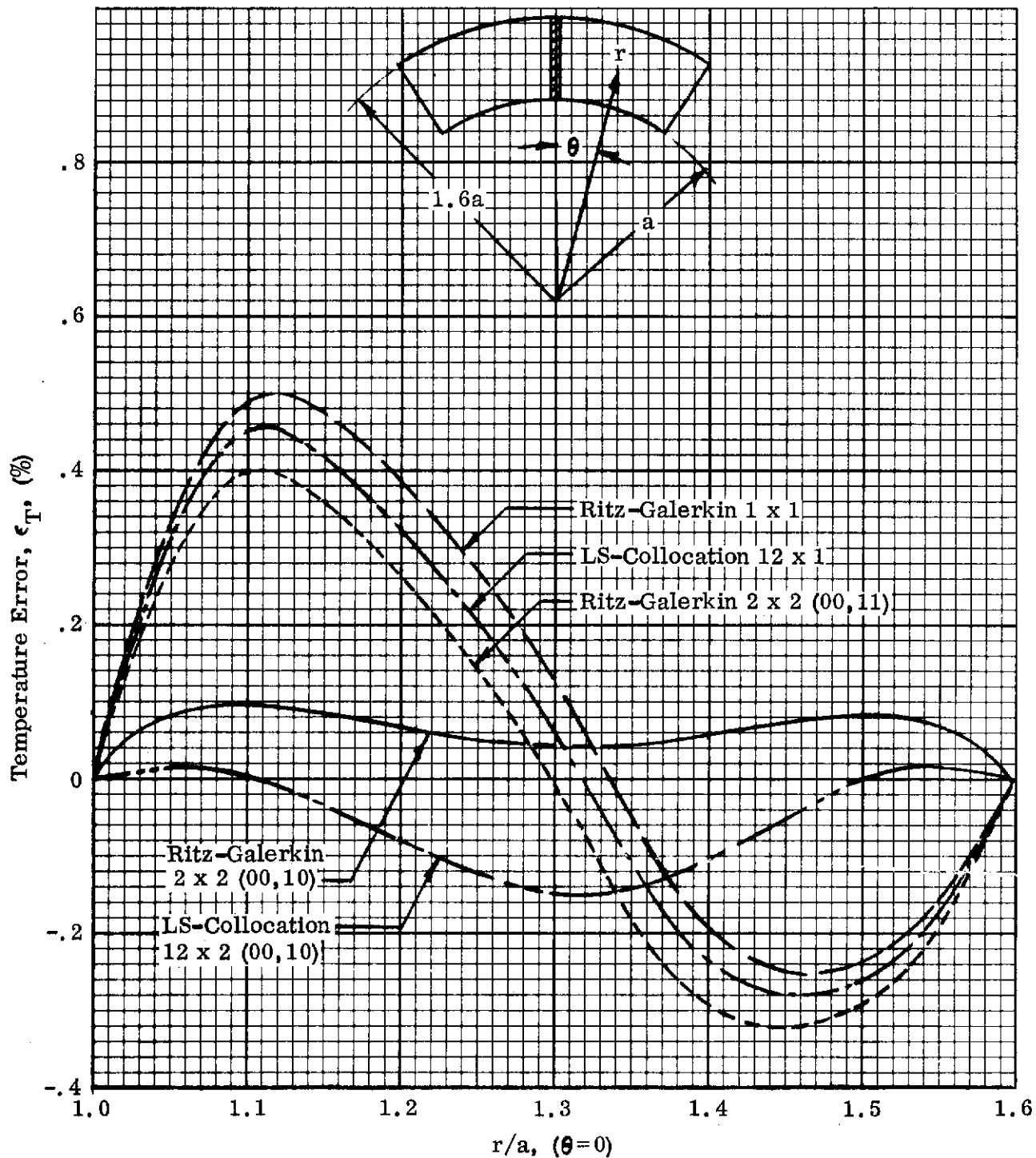


Figure 29b. Ritz-Galerkin and Least Squares-Collocation Temperature Errors for Example 2

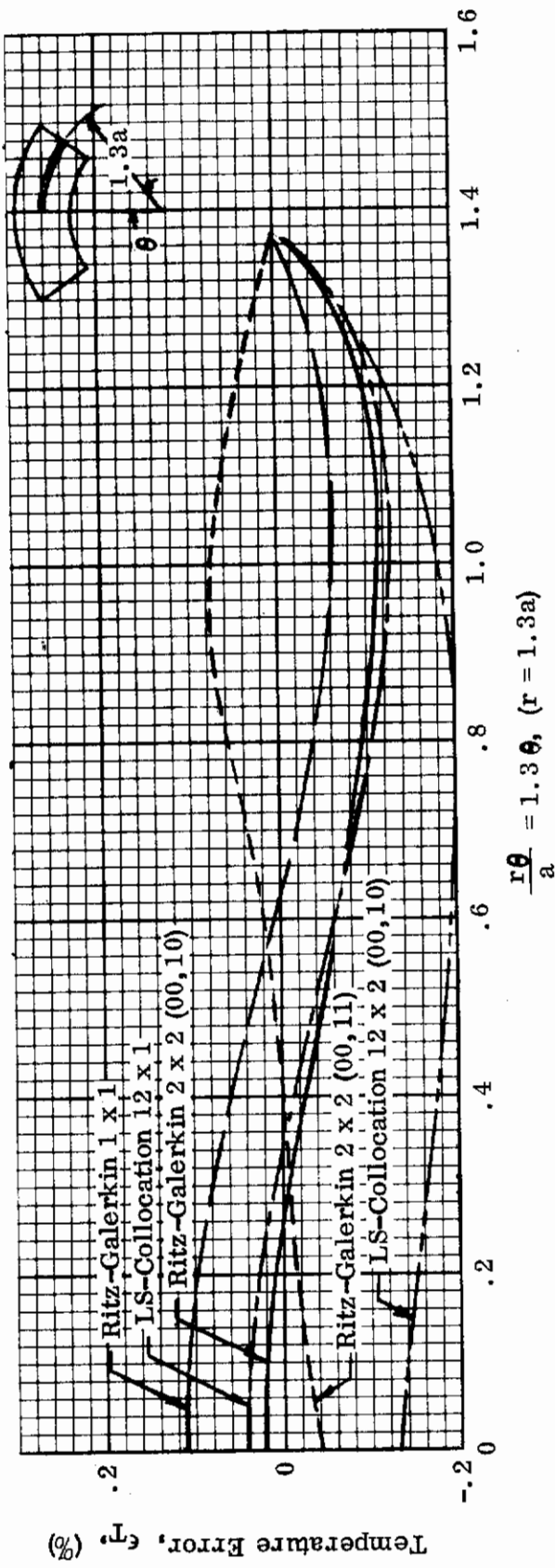
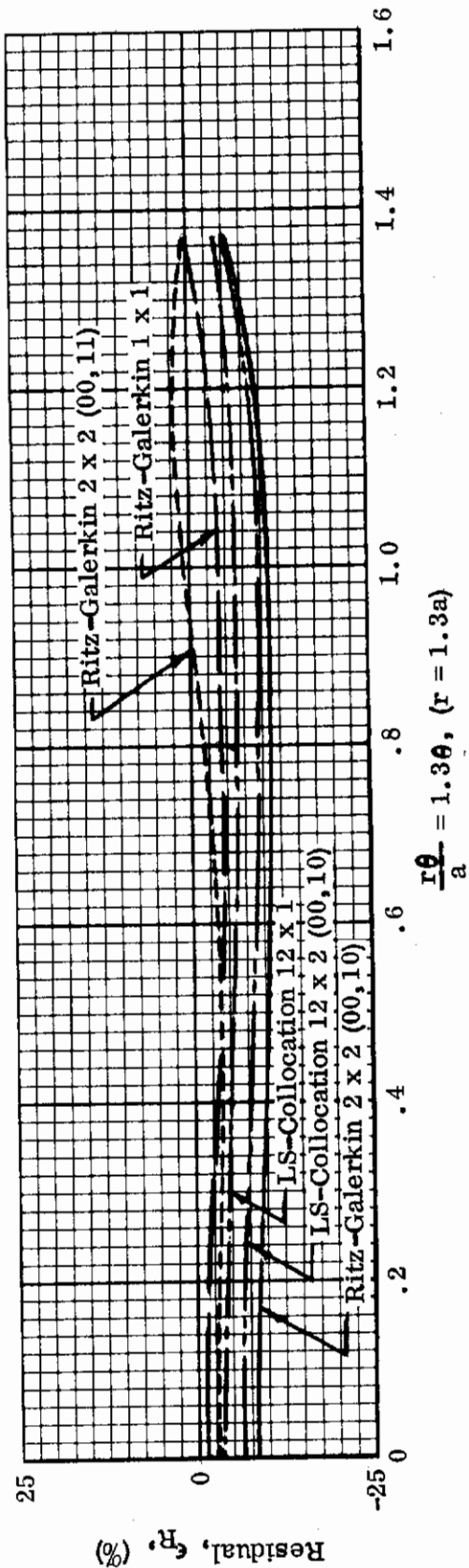


Figure 29c. Ritz-Galerkin and Least Squares-Collocation Residuals and Temperature Errors for Example 2

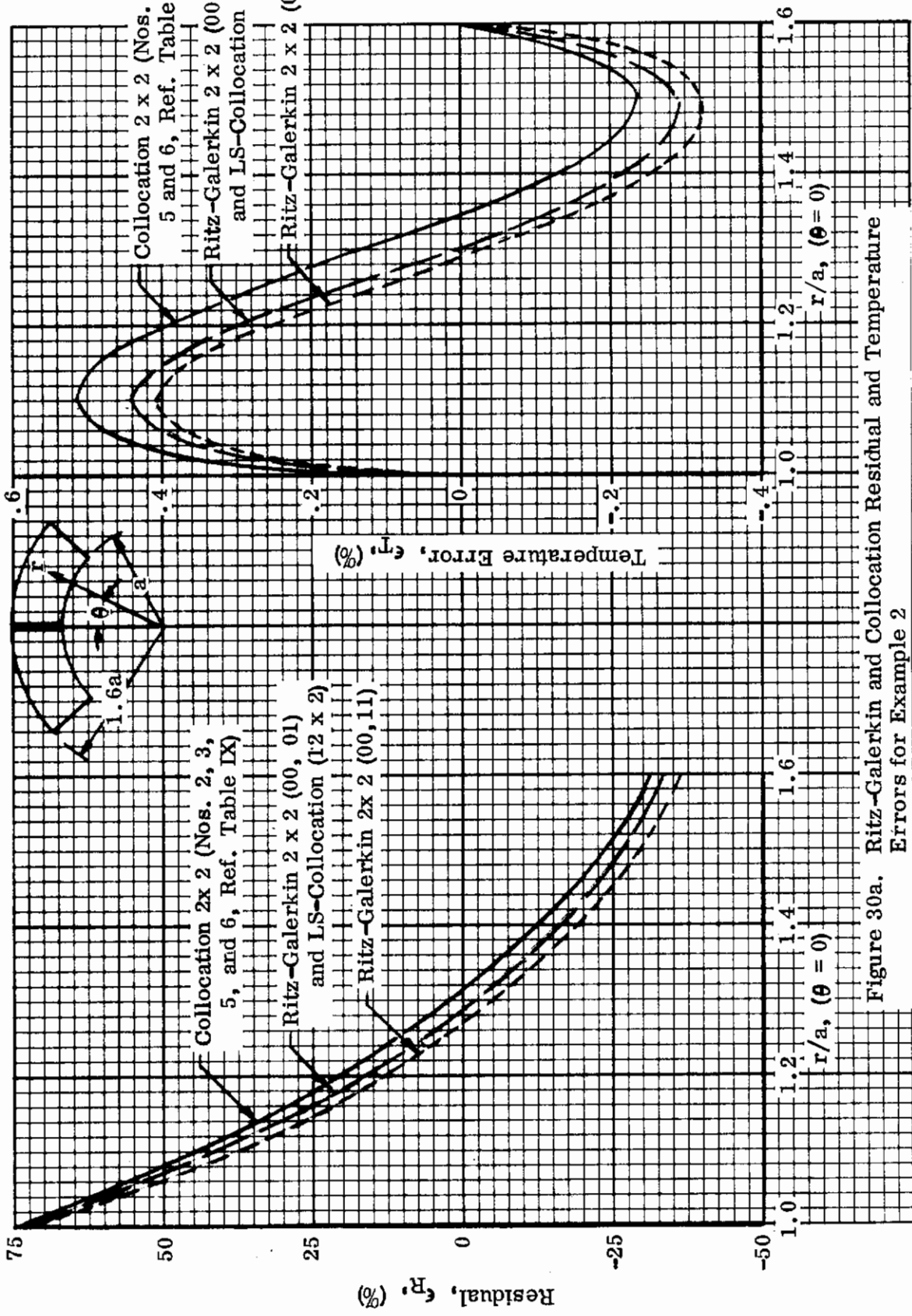


Figure 30a. Ritz-Galerkin and Collocation Residual and Temperature Errors for Example 2

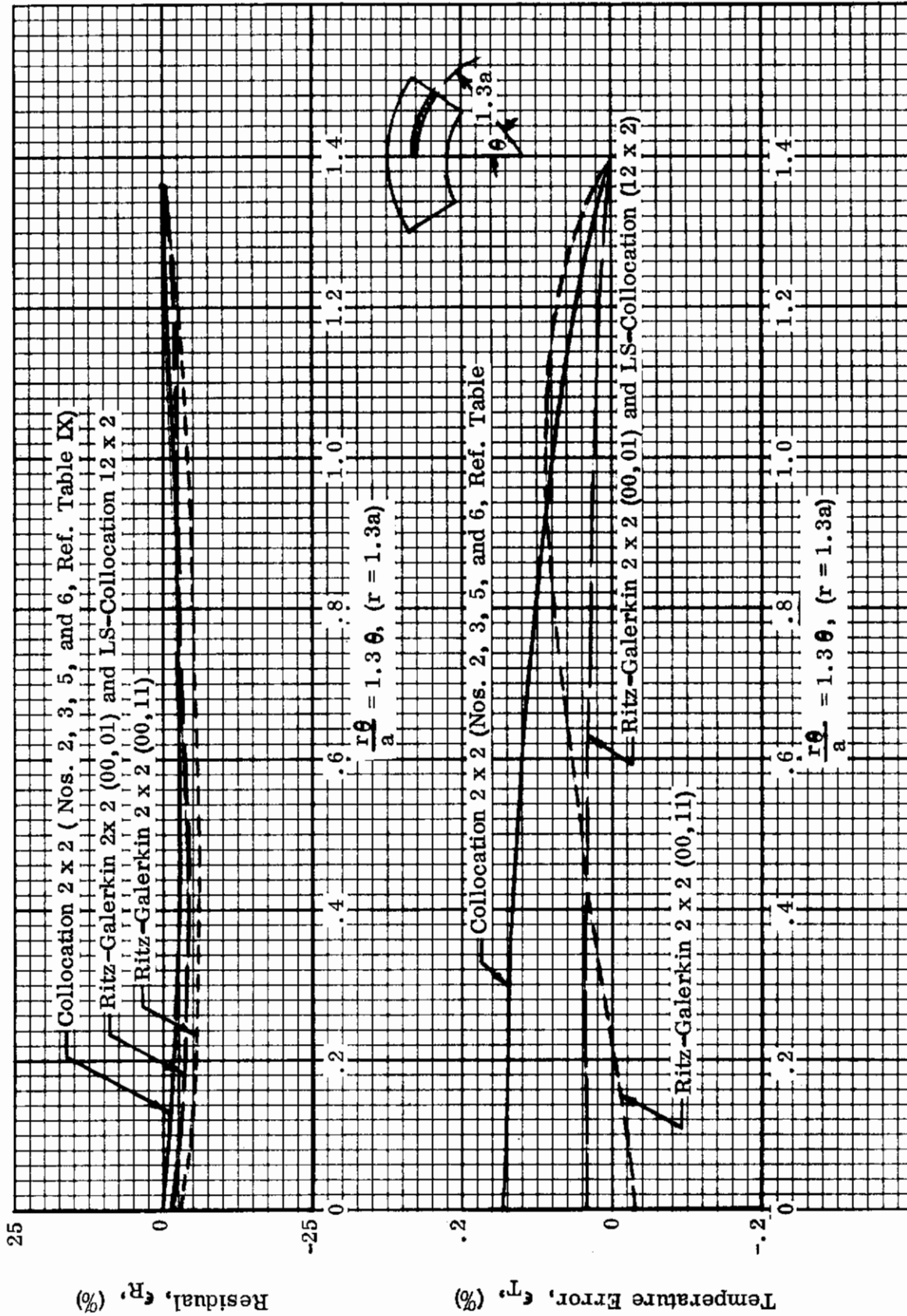


Figure 30b. Ritz-Galerkin and Collocation Residual and Temperature Errors for Example 2

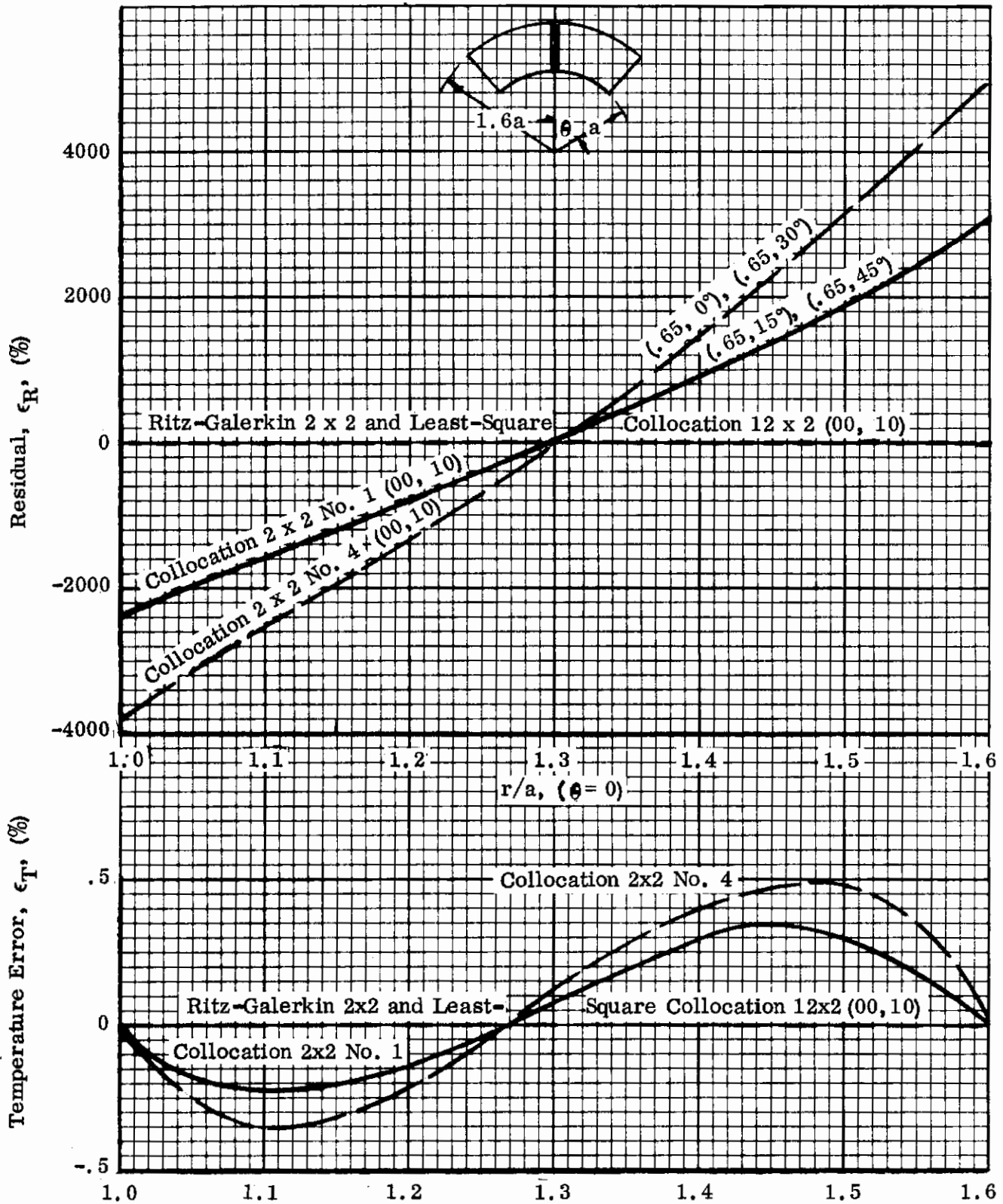


Figure 31. Ritz-Galerkin, Collocation and Least-Square Collocation Residual and Temperature Errors for Example 2

provement when the (c, d) pairs chosen are (0, 0) and (1, 0). However, because (c, d) = (0, 0) and (1, 0) gives better results for Ritz's but not collocations (Figure 31), the effect of varying trial functions is not as significant as the method used. The small effect, in some cases, of varying collocation points is again displayed in Figure 30 (solid lined curves).

#### D. EXAMPLE 3 - CLAMPED RECTANGULAR PLATE WITH A CENTRAL HOLE

##### 1. Problem Statement and Selection of Trial Functions

This example is concerned with the bending of a clamped rectangular plate, aspect ratio  $4 \times 1$ , with a clamped inner circular boundary (reference Figure 7c). The inner boundary is symmetrically located and has a diameter equal to one-half the shorter plate side. Mathematically stated, the domain equation of equilibrium is:

$$\nabla^4 w = f(x, y) \quad \text{in } D$$

where D is the region between the inner and outer boundaries, B,

$$x^2 + y^2 = R^2$$

and

$$x = \pm a, \quad y = \pm b$$

respectively, on which  $w = \frac{\partial w}{\partial \eta} = 0$  and where  $b = 2R = 1$  and  $a = 4$ .

The loading to stiffness ratio, f, is defined through the equation:

$$\nabla^4 w_{\text{exact}} = f$$

where

$$w_{\text{exact}} = (x^2 - a^2)^2 (y^2 - b^2)^2 (x^2 + y^2 - R^2)^2 \Omega(x)$$

and

$$\begin{aligned} \Omega(x) &= e^{-x}, & x > 0 \\ &= e^x, & x < 0 \end{aligned}$$

(see Figure 32 for the f which this  $\Omega$  produces).

A function that satisfied clamped boundary conditions on B, but was otherwise arbitrary in x, was used to generate the plate loading. The reasons for using this inverse technique for definition of the plate loading was simply that it was believed to be advantageous to have an exact solution to which approximate solutions could be compared.

In selecting the trial functions that satisfied the boundary conditions,  $w_{\text{exact}}$  was not included in this family. Its inclusion would obviously favor results obtained by methods that make use of that trial function and thus deem it unsuitable for its intended

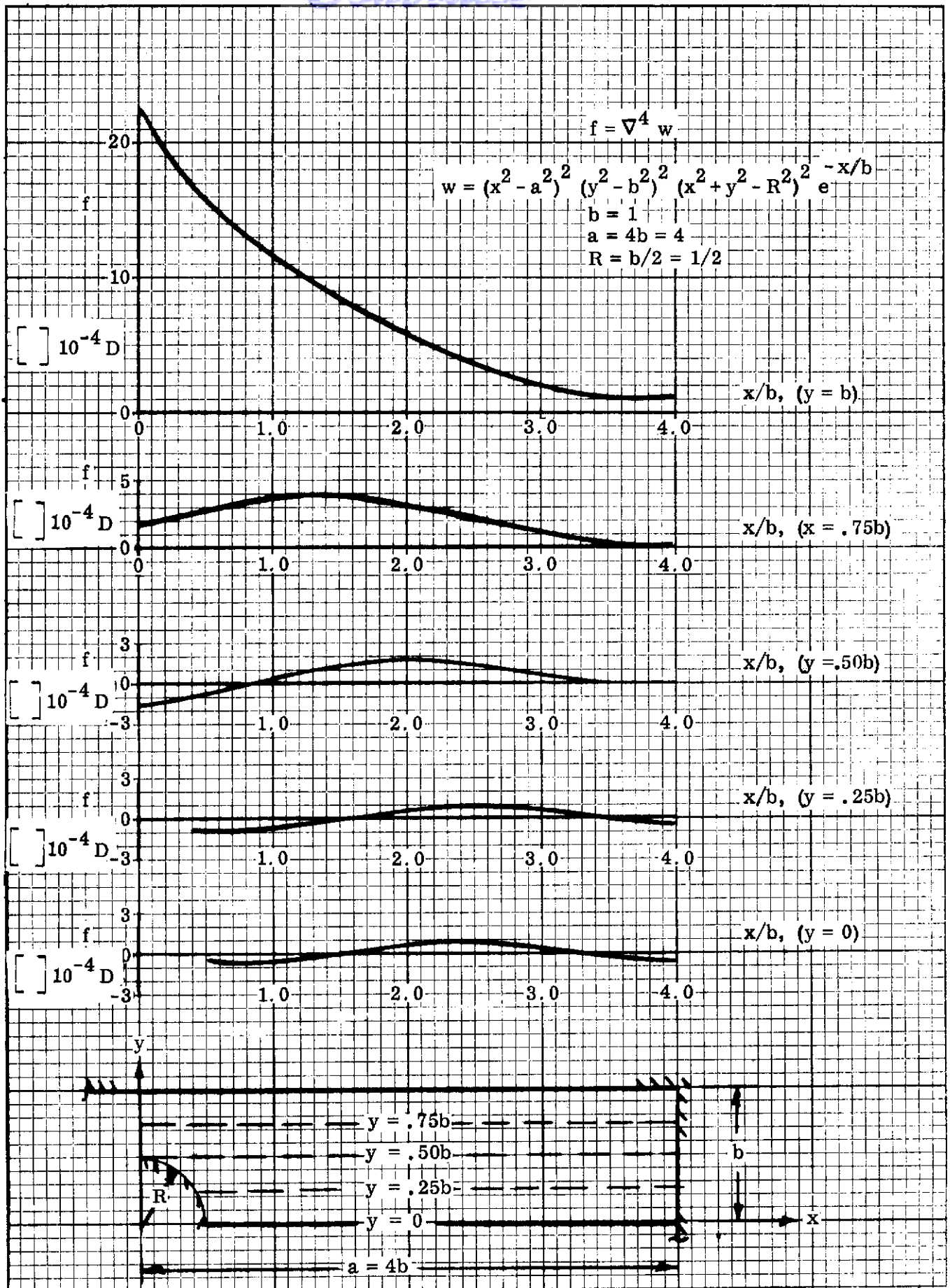


Figure 32: Loading Selected for Example 3.

purpose. Also, because the boundary conditions require functions that satisfy clamping on the inner and outer boundaries, arguments similar to those presented for Example 1 (Section III-B-1), lead to selection of  $w_n$  as:

$$w_n = (x^2 - a^2)^2 (y^2 - b^2)^2 (x^2 + y^2 - R^2)^2 \sum_{k=1}^n a_k \psi_k(x, y)$$

where the  $\psi_k$  and  $\Omega(x)$  were linearly independent. Specifically, the  $\psi_k$  are selected as:

$$\psi_k = x^{2m_k} y^{2p_k}, \text{ where } m_k \text{ and } p_k \text{ are integers.}$$

This choice maintained the problem's symmetry with respect to the coordinate axes.

## 2. Solution of Example 3 using Methods that Satisfy the Boundary Conditions

Example 3 is only treated by known methods that use trial families that satisfy the boundary conditions. These include:

- a) Ritz-Galerkin, for which the  $n$  conditions

$$\int_D (x^2 - a^2)^2 (y^2 - b^2)^2 (x^2 + y^2 - R^2)^2 \psi_k (\nabla^4 w_n - f) dA = 0$$

$$k = 1, 2, \dots, n$$

apply.

- b) Biezeno-Koch, for which

$$\int_{D_k} (\nabla^4 w_n - f) dA = 0$$

$$k = 1, 2, \dots, n$$

apply, where the  $D_k$  are in  $D$ .

- c) Least-squares collocations, for which

$$\nabla^4 w_n(x_k, y_k) - f(x_k, y_k) = 0$$

$$k = 1, 2, \dots, m > n$$

apply.

A digital computer program that generates  $n$ -term least-squares collocation solutions, where  $n$  could go as high as 20, was written. The program contains an option that calls a numerical integration subroutine and obtains one-term Ritz-Galerkin and Biezeno-Koch solutions.



The one-term Ritz-Galerkin, Biezeno-Koch, and least square collocations solutions are of the form:

$$w_1 = a_1 (x^2 - a^2)^2 (y^2 - b^2)^2 (x^2 + y^2 - R^2)$$

whereas the n-term least-squares collocation solutions are of the form:

$$w_n = (x^2 - a^2)^2 (y^2 - b^2)^2 (x^2 + y^2 - R^2)^2 \sum_k a_k x^{2m_k} y^{2p_k}$$

Deflection and moment results for one-parameter solutions are presented in Figures 33 and 34. In general, they reveal very similar values for Ritz-Galerkin and least-squares collocation solutions (which are low in comparison to the exact results) and Biezeno-Koch results, which are high compared to the exact reference solution. Defining

$$\epsilon_{R_1} = \frac{R}{f_{\max}} \times 100\%$$

and

$$\epsilon_{R_2} = \frac{R}{f_{\text{avg}}} \times 100\%$$

where  $f_{\max}$  occurs at  $(x, y) = (0, 1)$  and

$$f_{\max} = 226,400.0$$

and  $f_{\text{avg}} = \frac{1}{24} \sum_{j=1}^{24} f_j = 10,830$  with  $f_j$  computed at 24 equally spaced interior points in

a symmetrical quadrant of D, it can be seen that the Biezeno-Koch solution yields large positive residuals (Figure 35) whereas the Ritz-Galerkin and least-squares collocation residuals (which are very similar quantitatively) are negative in the region furthest from the clamped boundaries (where they have the greatest potential effect upon the overall solution). Recalling that domain residuals can be viewed as errors in the loading, it is reasonable to expect, qualitatively at least, high Biezeno-Koch solutions and low Ritz-Galerkin and least-square collocation results even without recourse to the exact solution.

Using the same loading condition, a  $15 \times 5$  least-square collocation solution and a  $5 \times 5$  ordinary collocation solution were obtained. The points collocated in each case are shown in Figure 36. Deflection, moment, and residual results (Figure 37) reveal that the least-square solution is quite accurate, while the ordinary collocation solution is largely in error. A similar behavior was observed for  $16 \times 4$  as compared to  $4 \times 4$  collocation solutions as well. The inaccuracy of the  $4 \times 4$  and  $5 \times 5$  cases is attributed to the fact that equilibrium collocation at only 4 or 5 interval points is inadequate for an accurate account of the highly varying load condition considered. Furthermore, use of more parameters with ordinary collocations would not remedy this situation because, as noted previously, this would only create solutions that oscillate greatly between matched points. Thus, the natural superiority of least-squares collocations has been demonstrated again.

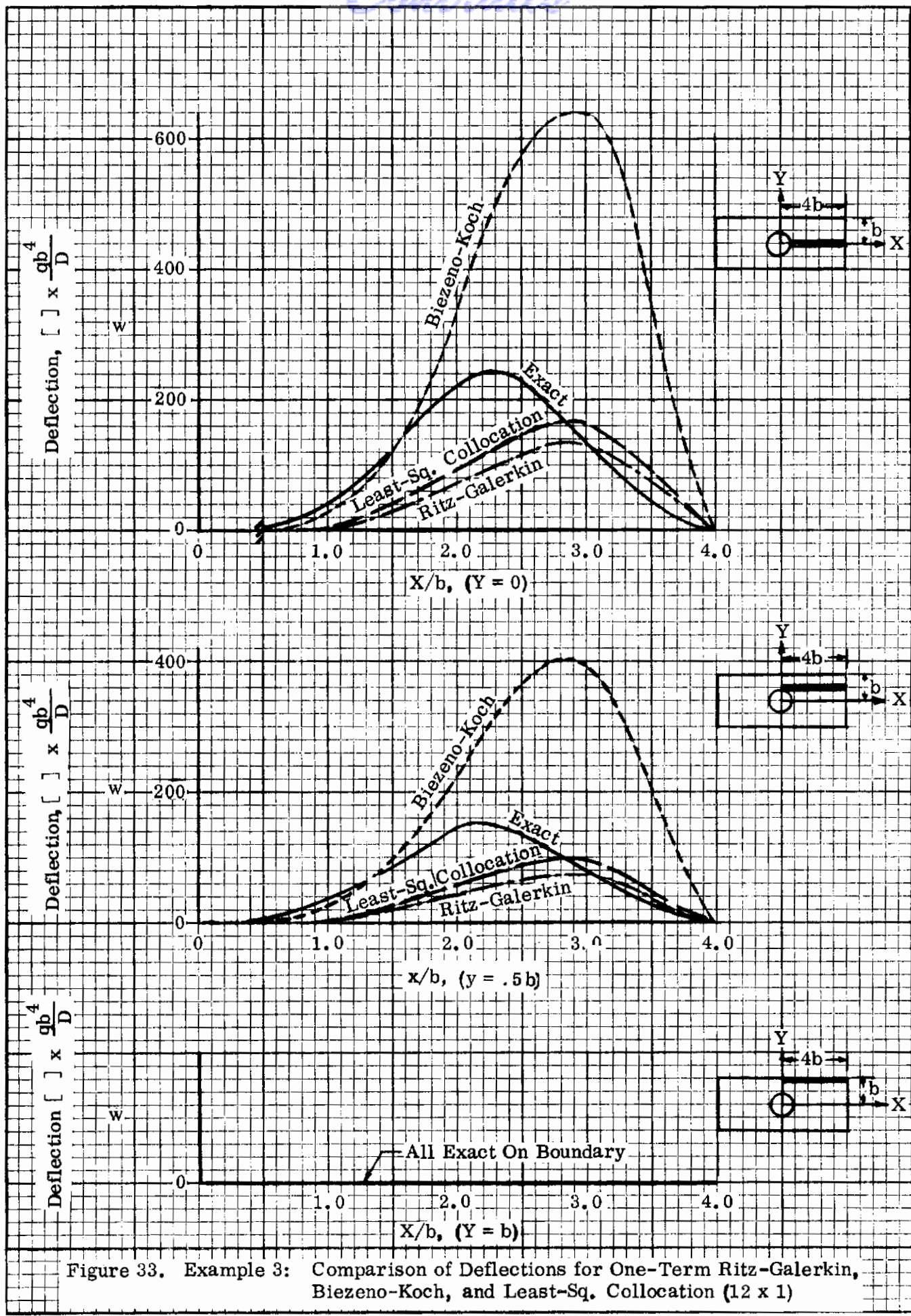


Figure 33. Example 3: Comparison of Deflections for One-Term Ritz-Galerkin, Biezeno-Koch, and Least-Sq. Collocation (12 x 1)

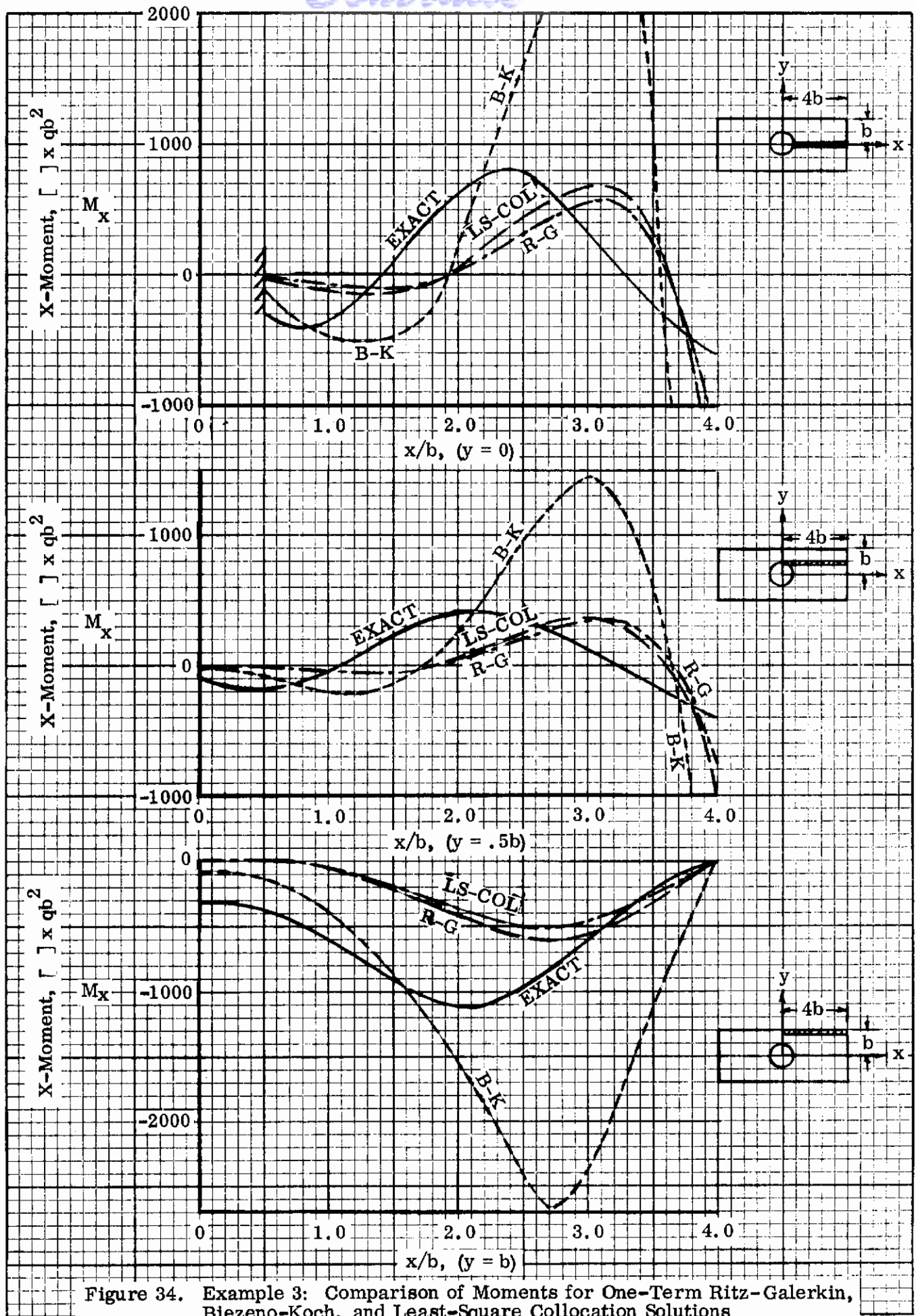


Figure 34. Example 3: Comparison of Moments for One-Term Ritz-Galerkin, Biezeno-Koch, and Least-Square Collocation Solutions

Centraide

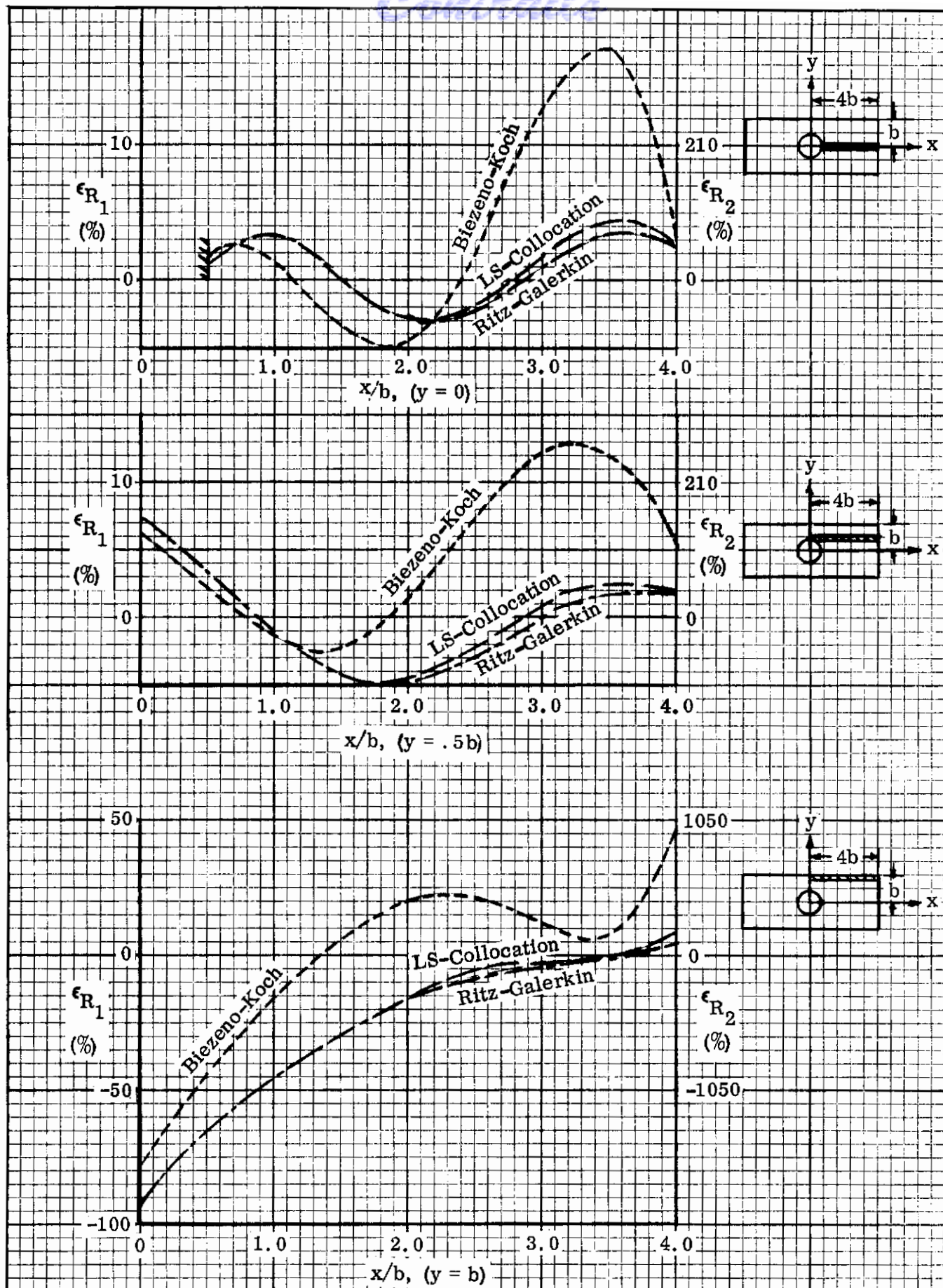


Figure 35. Example 3: Comparison of Residual Errors ( $\epsilon_R$ ) for One-Term Ritz-Galerkin, Biezeno-Koch, and Least-Square Collocation Solutions

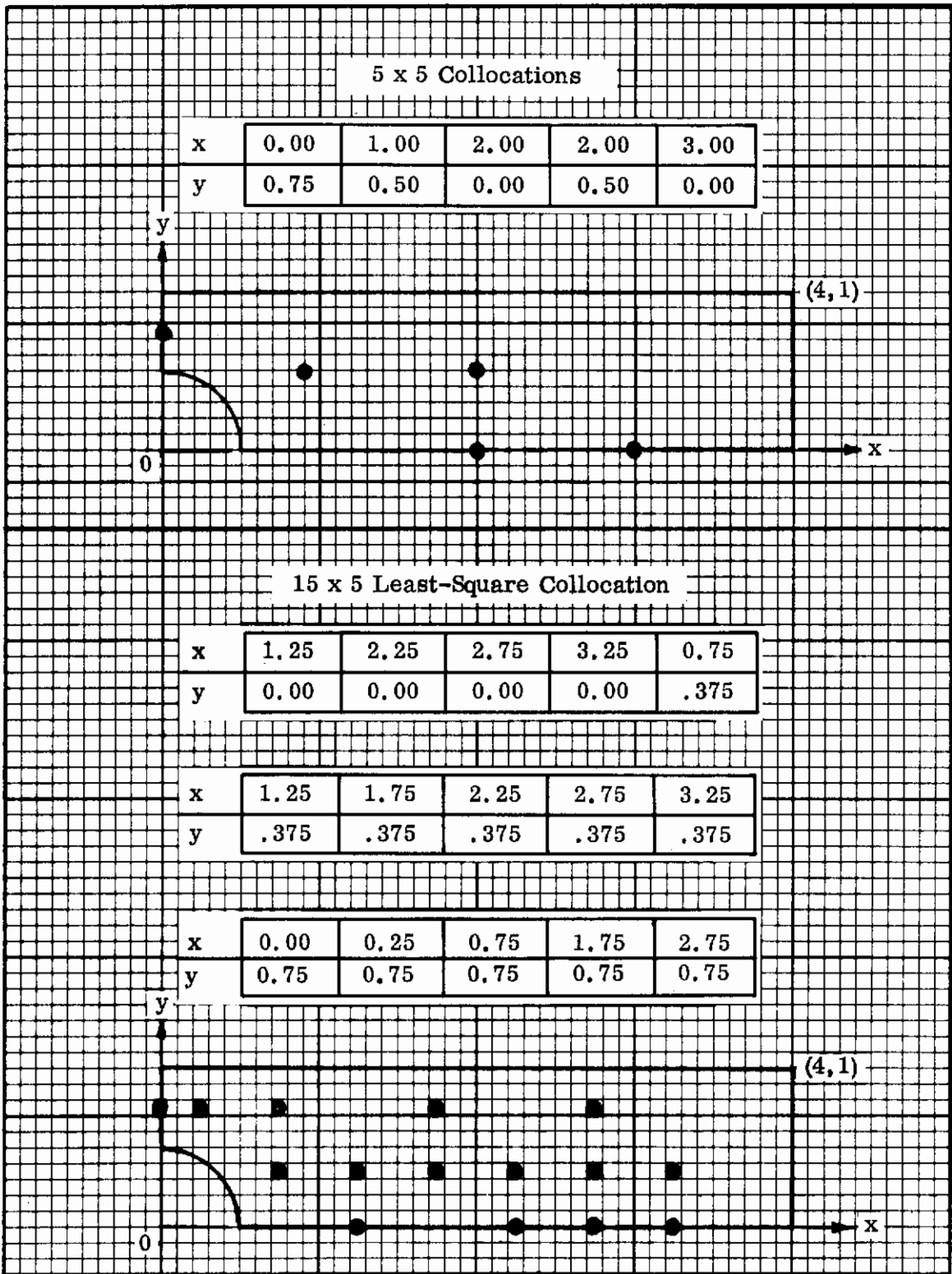


Figure 36. Collocation Points used for Example 3

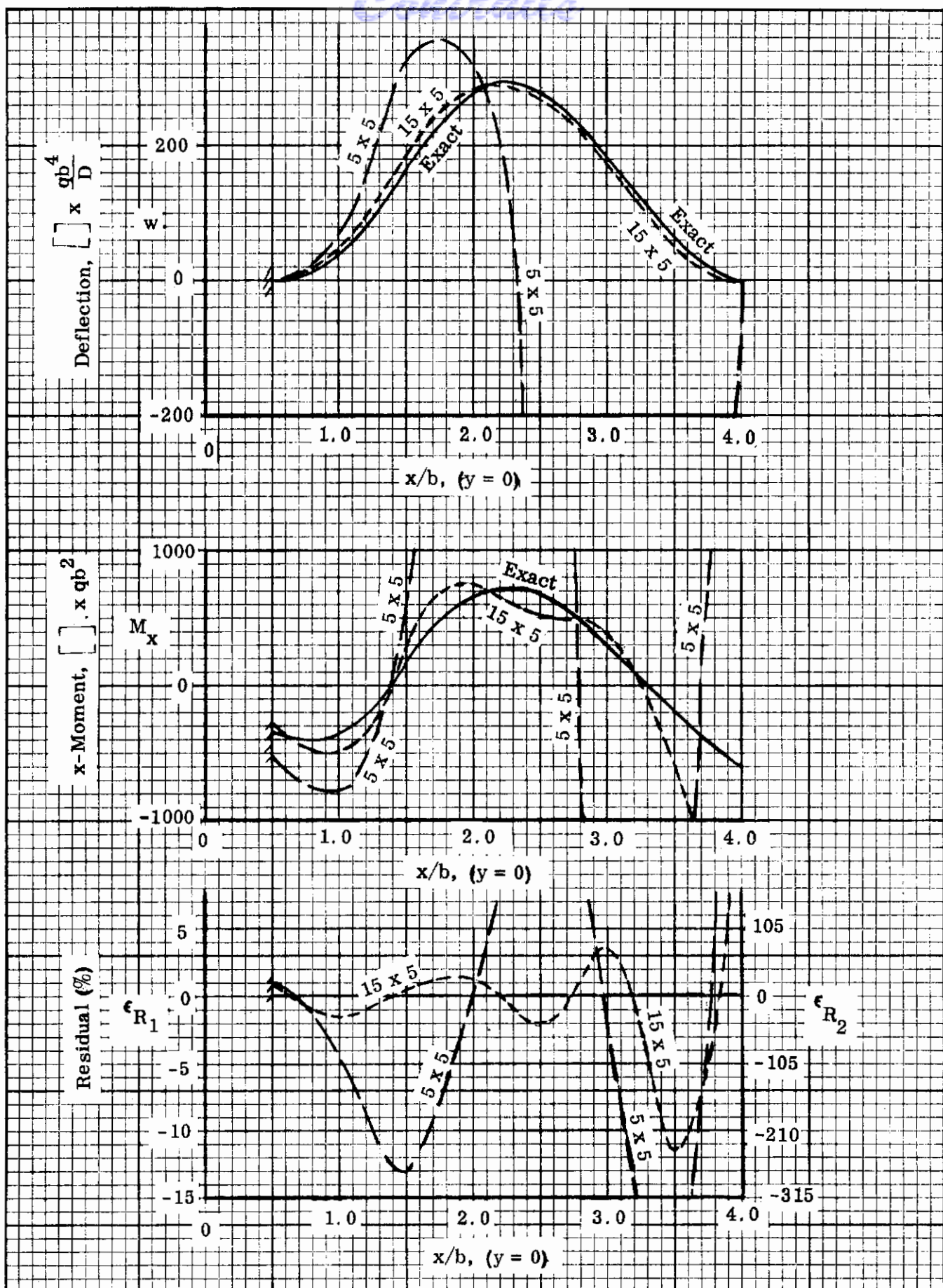


Figure 37. Example 3; Comparison of Exact Solution with 5 x 5 Collocation and 15 x 5 Least Square Collocation

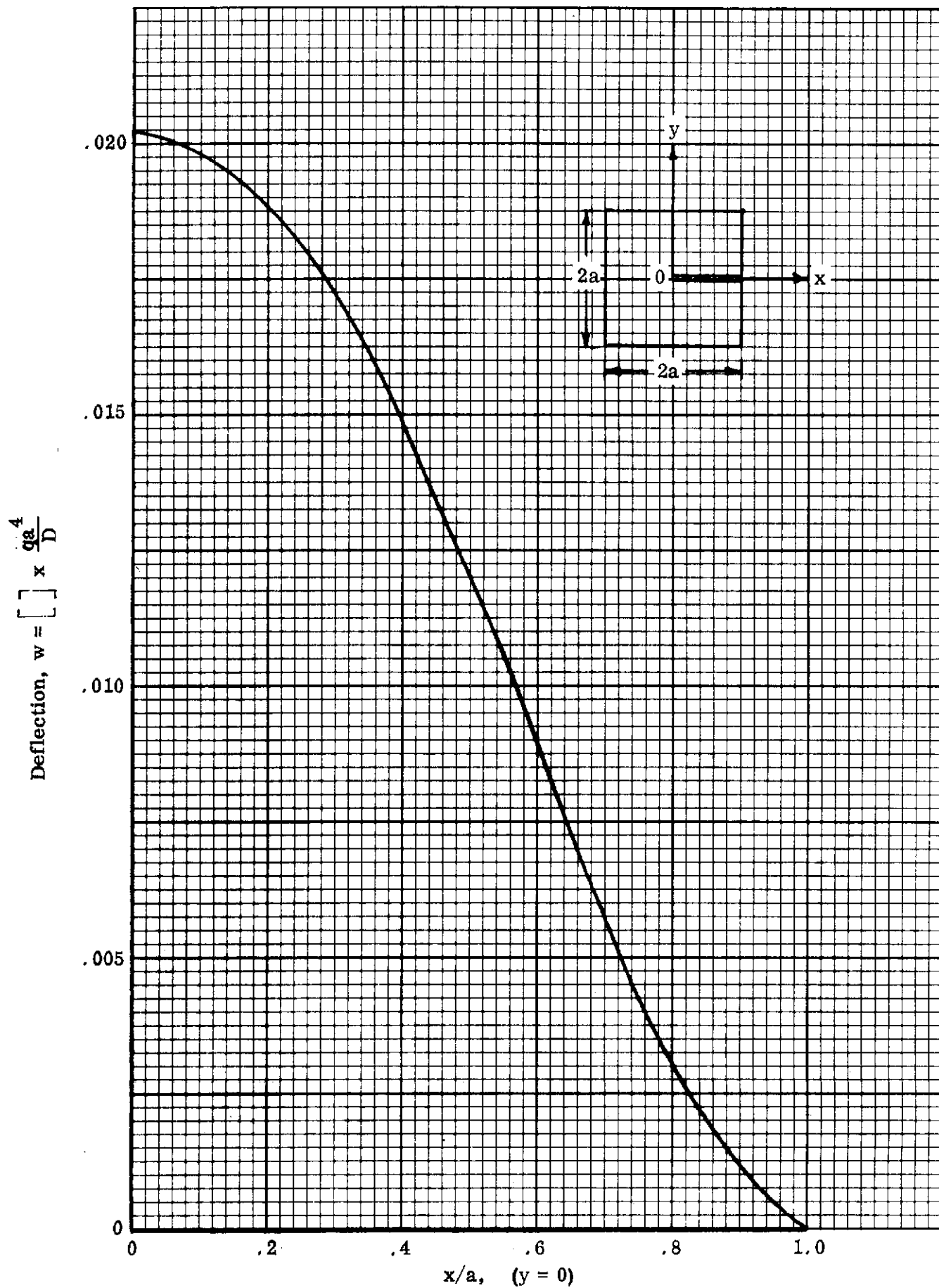


Figure 38a. Example 1, Exact Deflection Along  $x$ - Axis

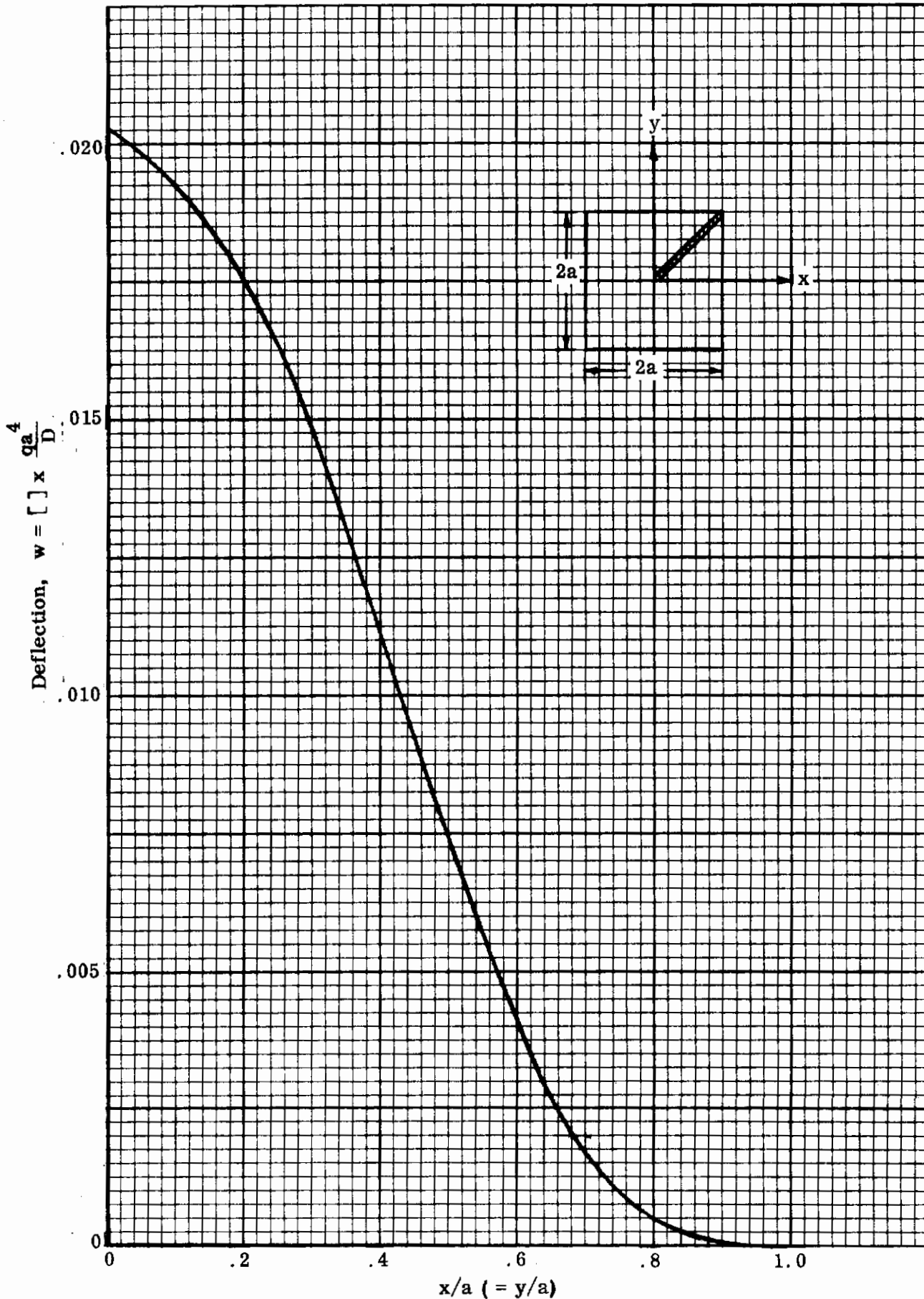


Figure 38b. Example 1, Exact Deflection Along Diagonal



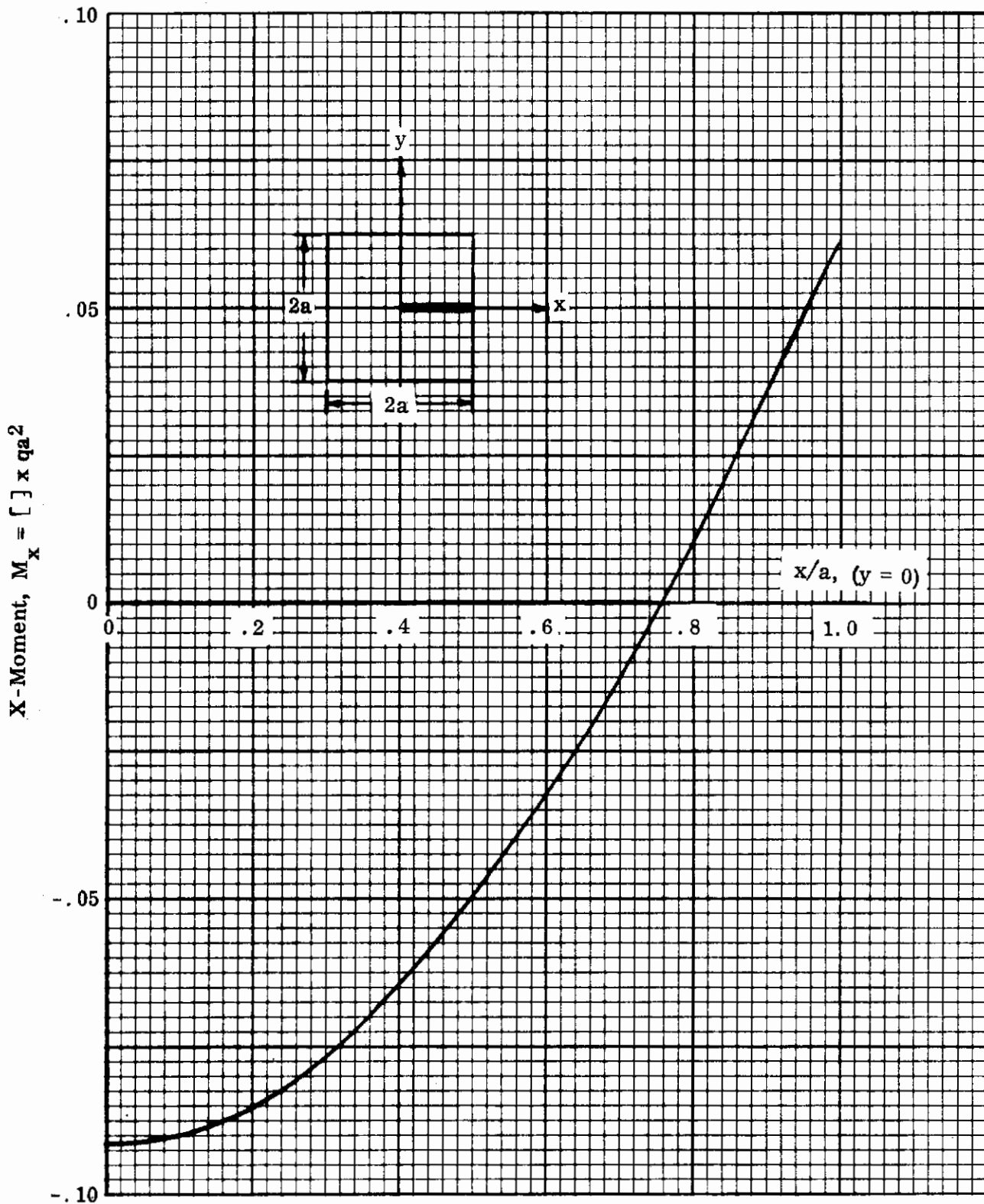


Figure 38c. Example 1, Exact X-Moment Along x-Axis

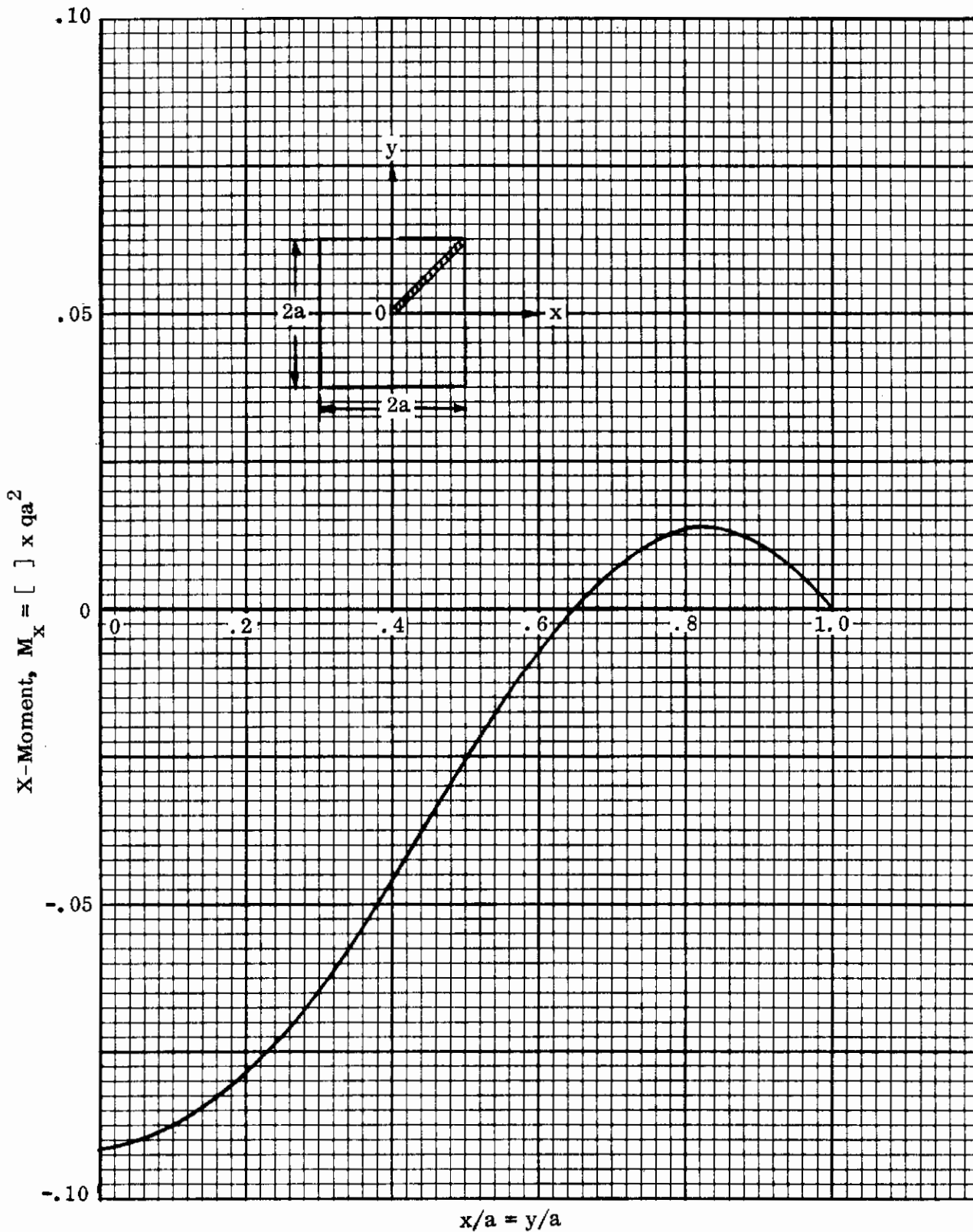


Figure 38d: Example 1, Exact X-Moment Along Diagonal

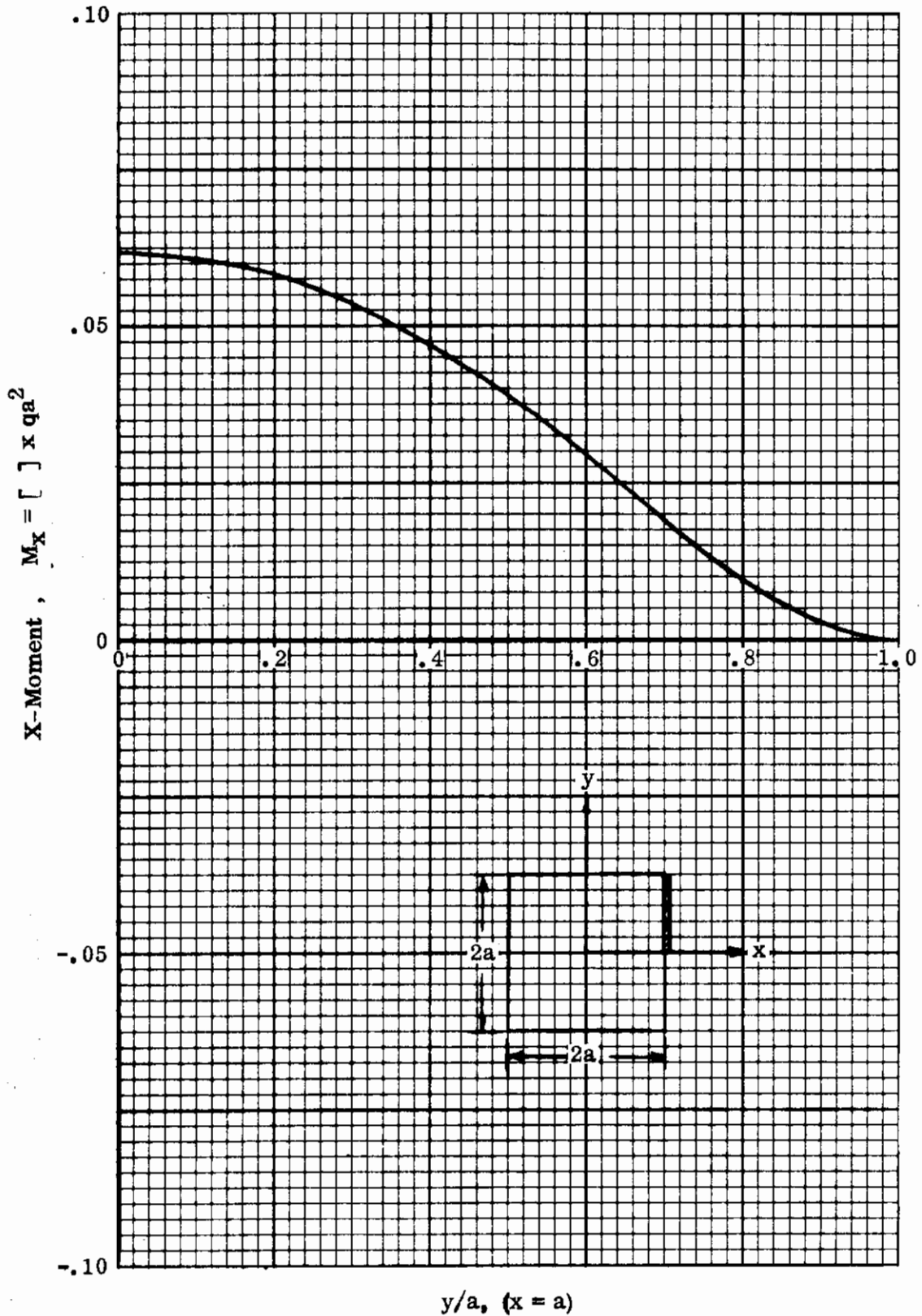


Figure 38c: Example 1, Exact X-Moment Along Boundary

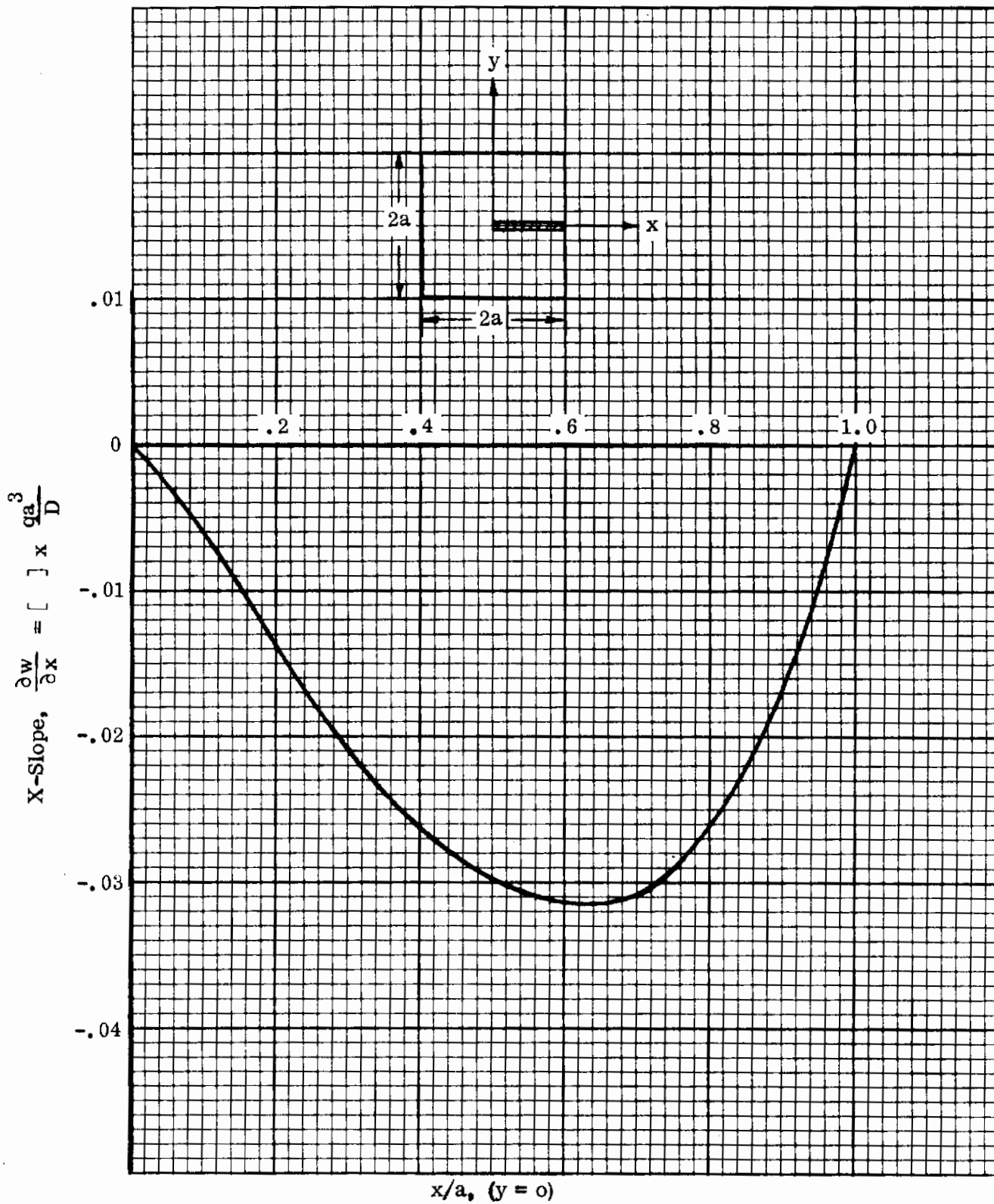


Figure 38f: Example 1, Exact X-Slope Along X-Axis

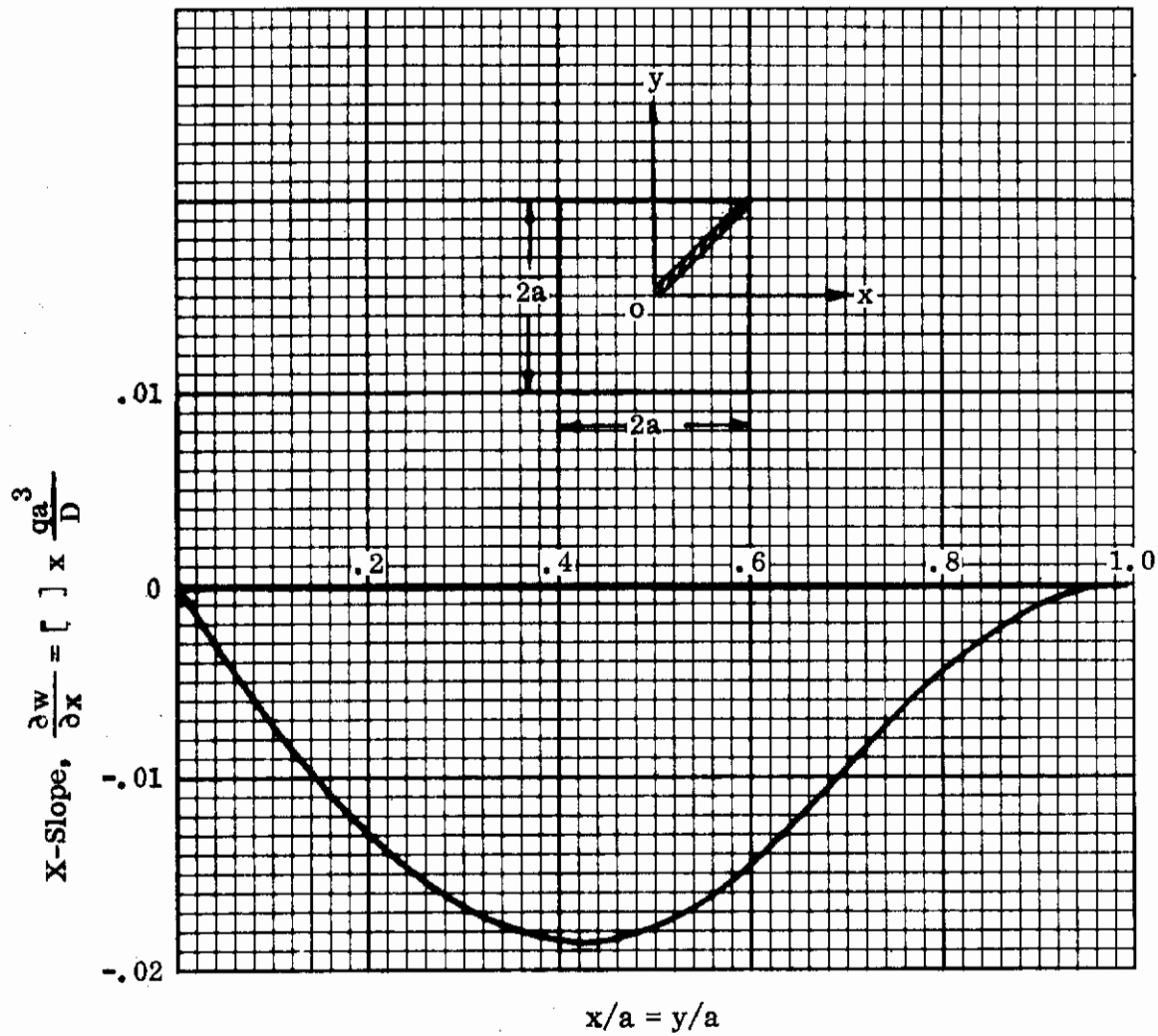


Figure 38g: Example 1, Exact X-Slope Along Diagonal

TABLE XI

"EXACT" SOLUTION FOR EXAMPLE 1  
(60 X 30 LEAST-SQ. POINT MATCHING)

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$
X-AXIS	0	0	.02024511	-.09162036	.00000000
	.1	↑	.01989372	-.09002241	-.00700767
	.2		.01885177	-.08507124	-.01376808
	.3		.01715768	-.07629167	-.1999880
	.4		.01488134	-.06288133	-.02534622
	.5		.01213313	-.04369554	-.02934771
	.6		.00907684	-.01722751	-.03139127
	.7		.00594671	.01841402	-.03067202
	.8		.00306903	.06552379	-.02614541
	.9	↓	.00088860	.12680430	-.01647820
1	0	.00000000	.20533680	.00000000	
DIAGONAL	0	0	.02024511	-.09162036	.00000000
	.1	.1	.01954882	-.08839552	-.00687839
	.2	.2	.01755994	-.07902302	-.01276851
	.3	.3	.01456465	-.06440904	-.01681907
	.4	.4	.01099477	-.04606509	-.01844561
	.5	.5	.00736250	-.02611120	-.01744671
	.6	.6	.00417383	-.00727747	-.01409995
	.7	.7	.00182549	.00712579	-.00922752
	.8	.8	.00049405	.01341957	-.00420121
	.9	.9	.00003848	.00874934	-.00076346
1	1	.00000001	-.00006157	.00000005	
BOUNDARY	1	0	.00000000	.20533680	.00000000
	↑	.1	↑	.20221700	.00000001
		.2		.19289860	.00000000
		.3		.17746800	.00000001
		.4		.15617190	-.00000001
		.5		.12949680	-.00000004
		.6		.09840279	-.00000004
		.7	↓	.06474635	.00000004
		.8	.00000000	.03193849	.00000008
	↓	.9	.00000001	.00661473	-.00000012
1	1	.00000001	-.00006157	.00000050	

TABLE XII  
ONE TERM RITZ-GALERKIN SOLUTION FOR EXAMPLE 1

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$	Residual [ ] D/q	
X-Axis	0	0	.0212660	-.11058319	.00000000	.70127998	
	.1	↑ ↓	.02084281	-.10752344	-.00842134	.67070796	
	.2		.01959874	-.09837481	-.01633229	.57960442	
	.3		.01761037	-.08322917	-.02322247	.42980671	
	.4		.01500529	-.06223963	-.02858150	.22437717	
	.5		.01196213	-.03562055	-.03189900	-.03239702	
	.6		.00871055	-.00364755	-.03266458	-.33500366	
	.7		.00553129	.03334253	-.03036785	-.67670574	
	.8		.00275607	.07495159	-.02449843	-1.0495412	
	.9		.00076770	.12072027	-.01454594	-1.4443232	
1.0	0	.00000000	.17012800	.00000000	-1.8506400		
Diagonal	0	0	.0212660	-.11058319	.00000000	.70127998	
	.1	.1	.02042803	-.10513111	-.00825375	.64074840	
	.2	.2	.01806220	-.08968386	-.01505184	.46772824	
	.3	.3	.01458315	-.06684898	-.01923053	.20794278	
	.4	.4	.01058773	-.04057430	-.02016711	-.09573568	
	.5	.5	.00672870	-.01555076	-.01794319	-.38328601	
	.6	.6	.00356784	.00362359	-.01337941	-.57753816	
	.7	.7	.00143869	.01351846	-.00789868	-.58417317	
	.8	.8	.00035719	.01318506	-.00317500	-.29172316	
	.9	.9	.00002771	.00570864	-.00052509	.42842862	
1.0	1.0	.00000000	.00000000	.00000000	1.7220479		
Boundary	1.0	0	↑ ↓	.00000000	.17012800	.00000000	-1.8506400
	↑ ↓	.1		.16674244	-1.8199658		
		.2		.15678996	-1.7273311		
		.3		.14088299	-1.5708985		
		.4		.12004231	-1.3476055		
		.5		.09569700	-1.0531650		
		.6		.06968443	-.68206484		
		.7		.04425029	-.22756790		
		.8		.02204859	.31828779		
	.9	.00614162		.96468912			
1.0	1.0	.00000000	.00000000	1.7220479			

TABLE XIII  
ONE TERM LEAST-SQUARES SOLUTION FOR EXAMPLE 1

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$	Residual [ ] D/q	
X-Axis	0	0	.01764100	-.09173320	.00000000	.41128000	
	.1	↑	.01728994	-.08919501	-.00698584	.38591927	
	.2		.01625794	-.08160585	-.01354829	.31034523	
	.3		.01460851	-.06904193	-.01926397	.18608204	
	.4		.01244749	-.05163027	-.02370950	.01566999	
	.5		.00992306	-.02954868	-.02646150	-.19733451	
	.6		.00722575	-.00302579	-.02709658	-.44835886	
	.7		.00458842	.02765897	-.02519135	-.73181444	
	.8		.00228627	.06217535	-.02032243	-1.0419064	
	.9		.00063684	.10014231	-.01206644	-1.3685840	
1.0	0		.00000000	.14112800	.00000000	-1.7056400	
Diagonal	0	0	.01764100	-.09173320	.00000000	.41128000	
	.1	.1	.01694587	-.08721048	-.00684682	.36106661	
	.2	.2	.01498332	-.07439636	-.01248610	.21753944	
	.3	.3	.01209731	-.05545391	-.01595249	.00203700	
	.4	.4	.00878295	-.03365801	-.01672943	-.24987646	
	.5	.5	.00558172	-.01289998	-.01488459	-.48841101	
	.6	.6	.00295967	.00300591	-.01109875	-.64955095	
	.7	.7	.00119345	.01121411	-.00655227	-.65505496	
	.8	.8	.00029630	.01093753	-.00263379	-.41245595	
	.9	.9	.00002299	.00473554	-.00043560	.18493883	
1.0	1.0	.00000000	.00000000	.00000000	1.2580480		
Boundary	1.0	0	↑	.00000000	.14112800	↑	-1.7056400
	↑	.1		.13831954	-1.6801946		
		.2		.13006356	-1.6033503		
		.3		.11686809	-1.4734832		
		.4		.09957992	-1.2883527		
		.5		.07938450	-1.0441025		
		.6		.05780603	-.73626003		
		.7		.03670739	-.35923658		
		.8		.01829019	.09357260		
	.9	.00509472		.62978844			
1.0	1.0	.00000000	.00000000	.00000000	1.2580480		



**TABLE XIV**  
**FOUR TERM BIEZENO-KOCH SOLUTION FOR EXAMPLE 1**

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$	Residual [ ] D/q
<b>X-Axis</b>	0	0	.02026381	-.08765432	.00000000	-.27411954
	.1	↑               ↓	.01992731	-.08656051	-.00671709	-.25053474
	.2		.01892580	-.08301374	-.01327008	-.18290450
	.3		.01728591	-.07622746	-.01944126	-.08054128
	.4		.01506089	-.06492289	-.02490697	.04123859
	.5		.01234378	-.04737892	-.02918781	.16143478
	.6		.00928525	-.02150537	-.03160309	.25383630
	.7		.00611576	.01505718	-.03123138	.28767332
	.8		.00317132	.06480672	-.02687946	.22847709
	.9		.00092170	.13021338	-.01706265	.03918272
1.0	0	.00000000	.21351151	.00000000	-.31848818	
<b>Diagonal</b>	0	0	.02026381	-.08765432	.00000000	-.27411954
	.1	.1	.01959611	-.08527476	-.00661121	-.22834326
	.2	.2	.01767206	-.07792932	-.01243131	-.11302362
	.3	.3	.01472894	-.06527065	-.01667338	.01308425
	.4	.4	.01115564	-.04753057	-.01862989	.07602033
	.5	.5	.00746006	-.02639355	-.01785815	.02133659
	.6	.6	.00418974	-.00555396	-.01445315	-.14068699
	.7	.7	.00179742	.00965064	-.00929770	-.28487400
	.8	.8	.00047591	.01419633	-.00407590	-.12624916
	.9	.9	.00003940	.00750281	-.00073381	.80239601
1.0	1.0	.00000000	.00000000	.00000000	3.1234545	
<b>Boundary</b>	1.0	0	.00000000	.21351151	.00000000	-.31848818
	↑               ↓	.1	.00000000	.20973317	↑               ↓	-.33604136
		.2	.00000000	.19853801		-.38181448
		.3	.00000000	.18036532		-.43530576
		.4	.00000000	.15601208		-.46288534
		.5	.00000000	.12672788		-.41866109
		.6	.00000000	.09434316		-.24580330
		.7	.00000000	.06142548		.12155803
		.8	.00000000	.03145688		.75571561
		.9	.00000000	.00902277		1.7318637
1.0	1.0	.00000000	.00000000	.00000000	3.1234545	

TABLE XV  
FOUR TERM COLLOCATION SOLUTION FOR EXAMPLE 1

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$	Residual [ ] D/q
X-Axis	0	0	.01916651	-.08698357	.00000000	.00000046
	.1	↓	.01883304	-.08532448	-.00664770	-.01856388
	.2	↓	.01784566	-.08034439	-.01303460	-.06161730
	.3	↓	.01624373	-.07193644	-.01889216	-.09613744
	.4	↓	.01409484	-.05963069	-.02391829	-.08305618
	.5	↓	.01150062	-.04228125	-.02772236	-.00000051
	.6	↓	.00860952	-.01787693	-.02974329	.12926875
	.7	↓	.00563830	.01627956	-.02916365	.20104900
	.8	↓	.00290034	.06271364	-.02486979	-.00358459
	.9	↓	.00083224	.12187018	-.01554204	-.85214097
	1.0	0	.00000000	.18872575	.00000000	-2.8883622
Diagonal	0	0	.01916651	-.08698357	.00000000	.00000046
	.1	.1	.01850552	-.08378262	-.00652936	-.03521176
	.2	.2	.01661736	-.07470316	-.01212554	-.09816689
	.3	.3	.01376962	-.06091840	-.01601510	-.10645296
	.4	.4	.01036150	-.04355484	-.01766031	-.03889232
	.5	.5	.00687441	-.02374141	-.01678350	-.00000045
	.6	.6	.00381106	-.00390118	-.01348446	-.14802466
	.7	.7	.00159301	.01089885	-.00853185	-.45831225
	.8	.8	.00039985	.01451897	-.00357312	-.47704640
	.9	.9	.00002975	.00663552	-.00057601	.45280376
	1.0	1.0	.00000000	.00000000	.00000000	1.5092491
Boundary	1.0	0	.00000000	.18872575	.00000000	-2.8883622
	↑	.1	↑	.18513102	↑	-2.8945061
	↑	.2	↑	.17449686	↑	-2.8997915
	↑	.3	↑	.15729789	↑	-2.8655514
	↑	.4	↑	.13440550	↑	-2.7307956
	↑	.5	↑	.10719985	↑	-2.4204675
	↑	.6	↑	.07770004	↑	-1.8631842
	↑	.7	↑	.04866273	↑	-1.0247079
	↑	.8	↑	.02355276	↑	.03471948
	↑	.9	↑	.00621813	↑	1.0690888
	1.0	1.0	.00000000	.00000000	.00000000	1.5092491

TABLE XVI  
LEAST SQUARE COLLOCATION (30 x 4) SOLUTION TO EXAMPLE 1

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$	Residual [ ] D/q
X-Axis	0	0	.0196149	-.0895639	.0000000	.0110277
	.1	↑ ↓	.0192715	-.0878827	-.0068458	.0071154
	.2		.0182546	-.0827428	-.0134246	-.0028596
	.3		.0166051	-.0738349	-.0194465	-.0139798
	.4		.0143951	-.0605812	-.0245716	-.0192556
	.5		.0117347	-.0420474	-.0283753	-.0113604
	.6		.0087824	-.0168342	-.0303030	-.0150843
	.7		.0057611	.0170394	-.0296119	.0608190
	.8		.0029803	.0622431	-.0253002	.1190255
	.9		.0008661	.1222126	-.0160229	.1723895
1.0	0	.0000000	.2011733	.0000000	.1903802	
Diagonal	0	0	.0196149	-.0895639	.0000000	.0110277
	.1	.1	.0189342	-.0863039	↑ ↓	.0032100
	.2	.2	.0169893	-.0768306		-.0171531
	.3	.3	.0140596	-.0620503		-.0448219
	.4	.4	.0150690	-.0434675		-.0809753
	.5	.5	.0070258	-.0232758		-.1365431
	.6	.6	.0039360	-.0045323		-.2166618
	.7	.7	.0016929	.0088518		-.2727105
	.8	.8	.0004522	.0130032		-.1140455
	.9	.9	.0000380	.0070763		.7156450
1.0	1.0	.0000000	.0000000	.0000000	3.070908	
Boundary	1.0	0	.0000000	.2011733	.0000000	.1903802
	↑ ↓	.1	↑ ↓	.1976556	-.0067250	.1680163
		.2		.1872328	-.0124879	.1060219
		.3		.1703108	-.0164513	.0200990
		.4		.1476158	-.0180233	-.0622528
		.5		.1202731	-.0169779	-.0998820
		.6		.0899249	-.0135820	-.0356707
		.7		.0588926	-.0087102	.2059550
		.8		.0303917	-.0038452	.7213186
		.9		.0088022	-.0007046	1.629130
1.0	1.0	.0000000	.0000000	.0000000	3.070908	

TABLE XVII  
FOUR TERM MIKHLIN SOLUTION FOR EXAMPLE 1

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$
<b>X-AXIS</b>	0	0	.0216351	-.0939683	.0000000
	.1	↑                 ↓	.0212750	-.0921248	-.0071765
	.2		.0202103	-.0864930	-.0140373
	.3		.0184893	-.0767689	-.0202444
	.4		.0161959	-.0624458	-.0254144
	.5		.0134559	-.0428142	-.0290966
	.6		.0104441	-.0169621	-.0307503
	.7		.0073951	.0162253	-.0297219
	.8		.0046152	.0580655	-.0252229
	.9		.0024975	.1100786	-.0163070
1.0	0		.0015384	.1739874	-.0018479
<b>DIAGONAL</b>	0	0	.0216351	-.0939683	.0000000
	.1	.1	.0209196	-.0907346	-.0070824
	.2	.2	.0188595	-.0812285	-.0133074
	.3	.3	.0157053	-.0660358	-.0179075
	.4	.4	.0118446	-.0461321	-.0202956
	.5	.5	.0077565	-.0228839	-.0201546
	.6	.6	.0039494	.0019524	-.0175280
	.7	.7	.0008793	.0262296	-.0129095
	.8	.8	-.0011491	.0474106	-.0073334
	.9	.9	-.0020996	.0625676	-.0024646
1.0	1.0	-.0023394	.0683825	-.0006885	
<b>BOUNDARY</b>	1.0	0	.0015384	.1739874	-.0018479
	↑                 ↓	.1	.0014953	.1729623	-.0015266
		.2	.0013629	.1698833	-.0006004
		.3	.0011331	.1647391	.0008182
		.4	.0007962	.1575110	.0025414
		.5	.0003463	.1481726	.0043067
		.6	-.0002105	.1366903	.0057761
		.7	-.0008461	.1230226	.0065367
		.8	-.0014980	.1071209	.0061007
		.9	-.0020549	.0889288	.0039050
1.0		1.0	-.0023394	.0683825	-.0006885

TABLE XVIII  
FOUR TERM POINT MATCHING SOLUTION FOR EXAMPLE 1

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$
X-Axis	0	0	.0189394	-.0886364	.0000000
	.1	↑ ↓	.0185996	-.0869216	-.0067725
	.2		.0175942	-.0816545	-.0132655
	.3		.0159660	-.0724670	-.0191720
	.4		.0137913	-.0587454	-.0241309
	.5		.0111861	-.0396307	-.0276987
	.6		.0083161	-.0140182	-.0293236
	.7		.0054089	.0194420	-.0283166
	.8		.0027687	.0623454	-.0238254
	.9		.0007945	.1165330	-.0148070
1.0	0	.0000000	.1840909	.0000000	
Diagonal	0	0	.0189394	-.0886364	.0000000
	.1	.1	.0182655	-.0854061	-.0066600
	.2	.2	.0163374	-.0759515	-.0123927
	.3	.3	.0134253	-.0609818	-.0163800
	.4	.4	.0099425	-.0416788	-.0180218
	.5	.5	.0063920	-.0196970	-.0170454
	.6	.6	.0032892	.0028364	-.0136146
	.7	.7	.0010640	.0233212	-.0084382
	.8	.8	-.0000589	.0386848	-.0028800
	.9	.9	-.0002024	.0453818	-.0009327
1.0	1.0	.0000000	.0393939	.0000000	
Boundary	1.0	0	.0000000	.1840909	.0000000
	↑ ↓	.1	.0000371	.1826814	.0003750
		.2	.0001396	.1784485	.0014545
		.3	.0002823	.1713784	.0031023
		.4	.0004276	.1614485	.0050909
		.5	.0005327	.1486269	.0071023
		.6	.0005585	.1328727	.0087273
		.7	.0004828	.1141360	.0094659
		.8	.0003142	.0923576	.0087273
		.9	.0001108	.0674693	.0058295
1.0	1.0	.0000000	.0393939	.0000000	

TABLE XIX  
FOUR TERM EQUAL SPACED POINT MATCHING  
SOLUTION FOR EXAMPLE 1

	x/a	y/a	Deflection [ ] $\frac{qa^4}{D}$	X-Moment [ ] $qa^2$	X-Slope [ ] $\frac{qa^3}{D}$
X-AXIS	0	0	.0202516	-.0916918	.0000000
	.1	↑                 ↓	.0199000	-.0900783	-.0070124
	.2		.0188575	-.0850908	-.0137734
	.3		.0171631	-.0762882	-.0199990
	.4		.0148871	-.0629350	-.0253398
	.5		.0121394	-.0440017	-.0293482
	.6		.0090809	-.0181650	-.0314462
	.7		.0059383	.0161930	-.0308920
	.8		.0030219	.0609841	-.0267481
	.9		.0007469	.1184143	-.0178480
1.0	0	-.0003419	.1909836	-.0027638	
DIAGONAL	0	0	.0202516	-.0916918	.0000000
	.1	.1	.0195547	-.0884654	-.0068854
	.2	.2	.0175632	-.0790694	-.0127898
	.3	.3	.0145614	-.0643536	-.0168632
	.4	.4	.0109801	-.0457342	-.0185168
	.5	.5	.0073303	-.0251936	-.0175539
	.6	.6	.0041126	-.0052812	-.0143010
	.7	.7	.0016990	.0108876	-.0097379
	.8	.8	.0001892	.0196306	-.0056291
	.9	.9	-.0007592	.0166994	-.0046538
1.0	1.0	-.0021269	-.0027210	-.0105371	
BOUNDARY	1.0	0	-.0003419	.1909836	-.0027638
	↑                 ↓	.1	-.0002808	.1890915	-.0023922
		.2	-.0001122	.1834097	-.0013318
		.3	.0001210	.1739220	.0002540
		.4	.0003531	.1606009	.0020929
		.5	.0005033	.1434085	.0038035
		.6	.0004874	.1222958	.0048957
		.7	.0002325	.0972027	.0047703
		.8	-.0003052	.0680585	.0027192
		.9	-.0011189	.0347815	-.0020746
1.0	1.0	-.0021269	-.0027210	-.0105371	

## SECTION V DISCUSSION

### A. SUMMARY OF ADVANTAGES AND DISADVANTAGES

Well known methods for which the trial functions must satisfy either the boundary conditions or differential equation exactly, are given in Table I (Section III). A list of advantages and disadvantages associated with methods included within each of these rather broad categories are as follows:

- 1) Approximate methods that satisfy boundary conditions exactly:
  - a) Advantages:
    - Errors of quantities satisfied exactly on the boundary are small
    - It is not required that the trial solution satisfy the domain equation
    - The problem is easily resolved for the same boundary conditions but different loadings
  - b) Disadvantages:
    - Difficulties in finding functions that satisfy the boundary equations exactly
    - Evaluation of the analytical expression for domain residual can be laborious
    - Difficulty in correlating domain residuals with solution accuracy
    - Little effort can be salvaged from a previous solution if a boundary condition is altered
- 2) Approximate methods that satisfy the differential equation exactly:
  - a) Advantages:
    - Domain equilibrium is satisfied and, therefore, results at interior points can be accurate even if boundary errors are sizable
    - Problem is easy to re-solve if boundary conditions are changed
  - b) Disadvantages
    - Difficulty in finding solutions that satisfy the domain differential equation

- A new particular solution is required each time the forcing function is changed
- Boundary accuracy is, generally, poorer than accuracy within the interior

## B. ACCURACY AND EASE OF APPLICATION

In general, it can be said that, based upon the examples, solutions with smaller residuals are more accurate on an overall basis for the entire domain and boundary. For methods that satisfied the differential equation, but not the boundary conditions, the largest errors occurred on the boundary. However, it is difficult to correlate a boundary error in one function of the dependent variable (e. g., deflection or slope) with an error in a different function (e. g., bending moment) on the boundary. Similarly, with solutions that satisfied the boundary conditions exactly, but not the domain equation, the nondimensional residuals were difficult to correlate quantitatively with the solutions accuracy. However, these residuals did serve an important function by qualitatively distinguishing between different solution accuracies for the same problem.

With regard to collocation and point-matching techniques, the use of algebraic least-squares was found to provide a definite improvement. This was particularly the case where the retention of a large number of parameters caused erratic behavior between matched points with ordinary collocations and point-matching. Also, the point-matching and collocation methods were easier to apply than methods requiring domain or boundary integrations.

The Ritz-Galerkin technique was found to yield somewhat more accurate results than least-squares. A similar result was reported in a limited investigation applied to heat-conduction problems by Costello (Reference 32). Furthermore, the least-square technique generally required more analytical and computational effort.

No firm criteria regarding the number of parameters required for accuracy can be given. However, as more and more terms were considered for any given method, it soon became obvious if results were converging. This was reflected in the solutions, as well as the residuals they produced. Also in evidence, was the gradual divergence as numerical precision errors set in for solutions with large numbers of terms, and the inadequacy of an incomplete set of functions to converge beyond a value that differed from the exact solution (Example 1 - Section IV).

## C. DIGITAL COMPUTER AUTOMATION

The method found easiest to automate were point-matching and collocations. These techniques can be made to yield results that approach Mikhlin and least-squares if an algebraic least-squares option is incorporated in the program. Furthermore, the computer logic associated with this addition is relatively simple. In certain cases, a Ritz-collocation procedure (Section IV) can be automated that, when coupled with an algebraic least-squares procedure, permits an approximation to the Ritz method.

Since each of these methods results, ultimately, in a system of linear algebraic equations requiring simultaneous solution, any improvements towards accurately handling large systems of such equations become important considerations for the automation of these methods. Niedenfuhr, Leissa, and Lo (Reference 26) have found empirically



that the least-squares point-matching of  $m \times n$  size matrices yields best results when the ratio of  $m/n$  is approximately two. On the average, this ratio has been found to yield the best least-squares collocation results also. In other words, this ratio yields the best compromise between the number of points approximated and the number of terms retained. Limitation of the original matrix size is thus necessitated by precision errors resulting from the machine matrix operations.

## D. CONCLUSIONS

The foregoing examples and evaluations by specific methods included in the general method of undetermined parameters, although by no means exhaustive, should provide the engineer with an indication of the applicability and accuracy that can be expected by the various techniques.

In general, it has been found desirable to use a trial solution that satisfies exactly as much as possible of the problem's mathematical requirements, with due regard to an individual problem's idiosyncrasies (e.g., use of singular solutions for concentrated load problems). Once a trial solution has been selected, it sometimes results that several specific methods discussed no longer remain suitable candidates for further consideration. Often, as with examples 1, 2, and 3 of Section IV, it will be found that alternative trial solutions can be used. One alternative leads to solutions by methods that satisfy the boundary conditions but not the differential-equation. Another alternate leads to exact satisfaction of the domain equation but not the boundary conditions. In such cases, criteria must be established as to which method leads to more efficient results.

If a precise solution is desired, least-squares point-matching is simplest to apply and automate. This method also permits wide flexibility in altering the boundary conditions as well. However, if the boundary conditions do not change for a variety of loadings, or high accuracy is desired near the boundaries, least-squares collocations is more suitable. If a quick solution with reasonable accuracy is desired, the Ritz-Galerkin technique works well with a single undetermined parameter and appears superior to least-squares.

APPENDIX I  
ALGEBRAIC LEAST-SQUARES

Given the  $m \times n$  coefficient matrix  $[c]$ ,  $m > n$ , of rank  $n$ , and vector  $\{b\}$  where:

$$\{\epsilon\} = [c] \{a\} - \{b\}$$

To find the vector  $\{a^0\}$  that minimizes the scalar  $S$  (composed of squared elements of  $\{\epsilon\}$ )

$$S = \{\epsilon\}^T \{\epsilon\}$$

make the first variation of  $S$ , with respect to the elements  $a_j$  of  $\{a\}$ , stationary:

$$\delta S = \frac{\partial S}{\partial a_j} \delta a_j = 0, \quad j = 1, 2, \dots, n$$

Since the  $\delta a_j$  are arbitrary, this requires that:

$$\frac{\partial S}{\partial a_j} = 0, \quad j = 1, 2, \dots, n$$

Therefore,

$$\frac{\partial S}{\partial a_j} = \frac{\partial}{\partial a_j} [\{\epsilon\}^T \{\epsilon\}] = \frac{\partial}{\partial a_j} [(\{a\}^T [c]^T - \{b\}^T) ([c] \{a\} - \{b\})] = 0$$

If  $\frac{\partial \{a\}}{\partial a_j} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$ , then  $\begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}^T [c]^T \{\epsilon\} + \{\epsilon\}^T [c] \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} = 0$

Since

$$\begin{Bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{Bmatrix}^T [c]^T = \begin{Bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{Bmatrix}^T,$$

$$[c] \begin{Bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{Bmatrix} = \begin{Bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{Bmatrix}.$$

The preceding becomes

$$\frac{\partial S}{\partial a_j} = 2 \begin{Bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{Bmatrix}^T \quad \{\epsilon\} = 0, \quad j = 1, 2, \dots, n$$

Expanding  $\{\epsilon\}$  in terms of  $[c]$ ,  $\{a^0\}$ , and  $\{b\}$  and combining all  $n$  equations, of the aforementioned type, in matrix form yields

$$[c]^T [c] \{a^0\} - [c]^T \{b\} = 0$$

If  $[c]$  is of rank  $n$ , then the system  $[c]^T [c]$  will also be of rank  $n$ . Thus, the solution of this last matrix equation is unique.

To prove that this  $\{a^0\}$  not only makes  $S$  stationary but also minimizes it, consider the Taylor expansion of  $S$  about  $\{a^0\}$ :

$$S(a) = S(a^0) + \sum_{j=1}^n \frac{\partial S(a^0)}{\partial a_j} (a_j - a_j^0) + \frac{1}{2} \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2 S}{\partial a_k \partial a_j} (a_j - a_j^0) (a_k - a_k^0)$$

This series terminates because  $S$  is only quadratic in the  $a$ 's and so the higher derivatives vanish. Also, since:

$$\begin{aligned} \frac{\partial^2 S}{\partial a_k \partial a_j} &= \frac{\partial}{\partial a_k} \left( \frac{\partial S}{\partial a_j} \right) = \frac{\partial}{\partial a_k} \left[ 2 \begin{Bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{Bmatrix}^T \{\epsilon\} \right] \\ &= 2 \begin{Bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{Bmatrix}^T [c] \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} = 2 \begin{Bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{Bmatrix}^T \begin{Bmatrix} c_{1k} \\ \vdots \\ c_{mk} \end{Bmatrix} \end{aligned}$$

$$\frac{1}{2} \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2 S}{\partial a_k \partial a_j} (a_j - a_j^0) (a_k - a_k^0) = \{\delta\}^T [c]^T [c] \{\delta\}$$

where:

$$\{\delta\} = \{a\} - \{a^0\}$$

Thus,

$$S(a) - S(a^0) = \{\delta\}^T [c]^T [c] \{\delta\} > 0$$

for  $\delta \neq 0$

Therefore,  $S(a^0)$  is an absolute minimum.

APPENDIX II

EQUIVALENCE OF THE RITZ AND GALERKIN METHODS FOR THE  
HARMONIC AND BIHARMONIC OPERATORS

For differential equations that are self-adjoint, i. e. ,

$$\int_D \psi_2 L \psi_1 dV = \int_D \psi_1 L \psi_2 dV$$

where  $\psi_1$  and  $\psi_2$  are admissible functions that satisfy the homogeneous boundary conditions, and

$$Lu = 0$$

is the differential equation in domain D, the Ritz and Galerkin methods are equivalent. However, this equivalence is only demonstrated for the harmonic and biharmonic problems considered in this report. Both methods require a solution form that satisfies the boundary conditions. However, the description of each is different. Ritz's method, for Example 2 requires the minimization of the integral:

$$I_n = \frac{1}{2} \int_D \bar{\nabla} \cdot T_n \bar{\nabla} T_n dA$$

i. e. ,

$$\delta I = 0 \tag{II-1}$$

whereas Galerkin's method sets

$$\int_D \phi_k \nabla^2 T_n dA = 0, \quad k = 1, 2, \dots, n \tag{II-2}$$

To show the equivalence of Equations (II-1) and (II-2), note that:

$$\begin{aligned} \delta I_n = 0 \quad \text{implies} \quad & \int_D \bar{\nabla} \cdot T \bar{\nabla} (\delta T) dA \\ & = \sum_{k=1}^n \delta a_k \int_D \bar{\nabla} \cdot \phi_k \bar{\nabla} T_n dA = 0 \end{aligned} \tag{II-3}$$

Since the  $\delta a_k$  are arbitrary, minimization of  $I_n$  requires

$$\int_D \bar{\nabla} \cdot \phi_k \bar{\nabla} T_n dA = 0 \quad k = 1, 2, \dots, n \tag{II-4}$$

Applying the divergence theorem to Equation (II-4),

$$\int_D \bar{\nabla} \cdot \phi_k \bar{\nabla} T_n dA = \int_B \phi_k \frac{\partial T_n}{\partial \eta} dS - \int_D \phi_k \nabla^2 T_n dA = 0 \tag{II-5}$$

as well as the boundary condition

$$\phi_k = 0 \quad \text{on } B,$$

results in the equivalence of Equations (II-1) and (II-2).

Applied to Examples 1 and 3, Galerkin's method sets

$$\int_D \phi_k (\nabla^4 w_n - \frac{q}{D}) dA = 0, \quad k = 1, \dots, n \quad (\text{II-6})$$

with

$$w_n = \sum_{k=1}^n a_k \phi_k \quad \text{and} \quad \phi_k = \frac{\partial \phi_k}{\partial \eta} = 0 \quad \text{on } B \quad (\text{II-7})$$

where B includes both inner and outer boundaries.

Ritz's method for the clamped plate required the stationariness of  $V_n$  with respect to the parameters  $a_k$ , i. e.,

$$\frac{\partial V_n}{\partial a_k} = 0, \quad k = 1, \dots, n \quad (\text{II-8})$$

where:

$$V_n = \frac{1}{2} \int_D \left[ (\nabla^2 w_n)^2 - \frac{2q}{D} w_n \right] dA \quad (\text{II-9})$$

Applying the "second form of Green's theorem" (Reference 39) to Equation (II-9) yields:

$$\begin{aligned} 2V_n = & \int_D w_n \left( \nabla^4 w_n - \frac{2q}{D} \right) dA + \int_B w_n \frac{\partial (\nabla^2 w_n)}{\partial \eta} dS \\ & - \int_B \frac{\partial w_n}{\partial \eta} \nabla^2 w_n dA \end{aligned} \quad (\text{II-10})$$

where  $\eta$  is the boundary normal directed away from D. Substituting the boundary conditions satisfied by  $w_n$  into Equation (II-10)

$$V_n = \frac{1}{2} \int_D w_n \left( \nabla^4 w_n - \frac{2q}{D} \right) dA \quad (\text{II-11})$$

Applying Equation (II-8) to Equation (II-11) yields:

$$\frac{\partial V_n}{\partial a_k} = \int_D \left[ \frac{\partial w_n}{\partial a_k} \left( \frac{\nabla^4 w_n}{2} - \frac{q}{D} \right) + \frac{w_n}{2} \nabla^4 \left( \frac{\partial w_n}{\partial a_k} \right) \right] dA = 0 \quad (\text{II-12})$$

Employing Equation (II-7), we note that:

$$\frac{\partial w_n}{\partial a_k} = \varphi_k.$$

It is also possible to show that:

$$\int_D w_n \nabla^4 \varphi_k dA = \int_D \varphi_k \nabla^4 w_n dA$$

through two successive applications of Greens Theorem and Equation (II-7). Thus, substituting these last two expressions into Equation (II-12) yields:

$$\frac{\partial V}{\partial a_k} = \int_D \varphi_k (\nabla^4 w_n - \frac{q}{D}) dA = 0, \quad (\text{II-13})$$

which establishes the equivalence of these two methods for both clamped plate problems.

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## SECTION VI

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