THE ABSOLUTE VALUE MODAL STRAIN ENERGY METHOD

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ABSTRACT

The Modal Strain Energy (MSE) method has become a significant design tool in the last decade. The MSE method allows the approximate solution to a complex eigenvalue problem using the solution to a real eigenvalue problem and knowledge of element loss factors and the percentage of MSE contained in each element.

While the MSE method has been shown to be relatively accurate in many instances, particularly for design purposes, it can produce high errors even in simple cases with relatively low damping levels. This paper discusses a new MSE method which allows the approximate solution to complex eigenvalue problems while providing more accurate approximations.

The Absolute Value MSE (AVMSE) method incorporates the absolute value of the individual element impedances to form a real "stiffness" matrix. From the solution of the eigenvalue problem associated with the system mass matrix and this real "stiffness" matrix; approximate mode shapes, natural frequencies, and damping ratios may be obtained.

The new method provides greater accuracy than the standard MSE method in most instances, and more importantly provides conservative damping estimates while the standard method tend to overpredict damping ratios. The AVMSE provides more accurate results while remaining innexpensive to implement.

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Introduction

- **Damping Devices Can Be Modeled by Impedances**
  - **Viscoelastics**
  - **D-Struts**
  - **Passive Piezoceramic Members**

- **Impedance Is a Frequency-Dependent Complex Quantity**

$$Z(s) = Z_R(s) + i\, Z_I(s)$$

- **Rearranging the Impedance Results In a Frequency-Dependent "Complex Stiffness"**

$$\equiv K_d(s) \cdot (1 + i\, \eta_d(s))$$

- **Modes of Systems That Contain Damping Devices Are Complex**

Passive damping applied to structures for vibration control has become an accepted practice in the past decade. Passive damping devices are efficiently modeled by their mechanical impedance for incorporation into structures. Such devices include viscoelastic members, viscous damping members, piezoceramic passive devices, and many others.

Mechanical impedance is a frequency dependent complex quantity which describes the relationship between an applied force to a member and the resulting displacement.

By writing the impedance in the proper form, a frequency dependent complex stiffness results. This form of the impedance is typically used to model viscoelastic material, however, it can be used for other damping members. The modes of a system which incorporate damping devices with complex stiffnesses are generally complex.

Introduction (concl)

- MSE Method Is Well-Known for Approximate Solution of Damped Systems

$$(-M\omega^2 + K_R)\cdot\Phi = 0$$

$$\zeta_i = \frac{1}{2}\Phi_i^T \cdot K_I \cdot \Phi_i / \omega_i^2$$

- MSE Method Provides Insight into Damping Design Improvements
- MSE Method Can Sometimes Provide Inaccurate Results–It Tends To
  - Overpredict Damping Ratios
  - Underpredict Natural Frequencies
  - Overpredict Modal Motion across Damped Elements
- New Method That Improves Approximations Has Been Developed

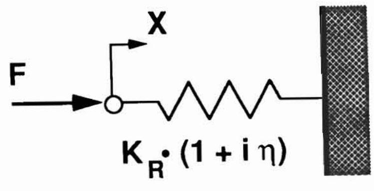
The Modal Strain Energy Method is a well known method of approximating the solution of the complex eigenvalue problem using real eigenvalue problems.

In addition to the computational advantage of the MSE method as compared to the complex eigenvalue problem, it has the advantage that it provides insight in effective improvements in damping design for a particular system. However, the MSE method can sometimes provide inaccurate results. It tends to overpredict the damping ratios, underpredict natural frequencies, and overpredict the motion across damped elements.

This presentation discusses a new approximation method with improved accuracy which also has the computation advantages of the MSE method.

Typical Complex Impedance Element

- The Ratio of the Motion Across a Complex Spring to Applied Force Is the Absolute Value of the Impedance



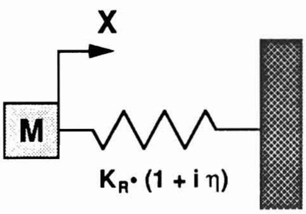
$$F e^{i\theta_F} = K e^{i \tan^{-1}(\eta)} \cdot X e^{i\theta_X}$$

$$\frac{|X|}{|F|} = K = K_R \cdot \sqrt{1+\eta^2}$$

For a typical complex impedance element, the complex spring constant describes the ratio of the displacement to applied force. The ratio of the magnitude of the displacement to the magnitude of the force is the absolute value of the impedance.

Single Degree-of-Freedom Complex System

- **The Natural Frequency and Damping Ratio Are Found from the Location of the Pole**



$$\lambda = \omega \cdot e^{i\left[\tan^{-1}\left(\frac{2\zeta}{\sqrt{1+\zeta^2}}\right) + \frac{\pi}{2}\right]}$$

$$\omega = \sqrt{\frac{\left|K_R \cdot (1 + i\eta)\right|}{M}}$$

$$\zeta = \sqrt{\frac{1 - \sqrt{\frac{1}{\eta^2 + 1}}}{2}}$$

For a single degree of freedom system with a complex spring, the natural frequency and damping ratio can be obtained from the location of the pole in the system transfer function. The eigenvalue can be written in terms of its magnitude and phase angle, which are a function of the natural frequency and viscous damping ratio.

Equating the pole location to the eigenvalue, we find that the natural frequency is the square root of the ratio of the absolute value of the impedance and the mass. The modal damping ratio is a function of the element loss factor.

The MSE Method

- **Assume the Real Modes Derived from the Real Part of the Stiffness Matrix Are Good Approximate Eigenvalues**

$$(-M\,\omega^2 + K_R)\cdot\Phi = 0$$

- **Compute the Rayleigh Quotient To Find the Eigenvalue**

$$\lambda_i^2 \cong \frac{\Phi_i^T\cdot(K_R + K_I)\cdot\Phi_i}{\Phi_i^T\cdot M\cdot\Phi}$$

- **Assume That the Real Part of the Rayleigh Quotient Is the Natural Frequency Squared**

$$|\lambda_i|^2 \cong \frac{\Phi_i^T\cdot K_R\cdot\Phi_i}{\Phi_i^T\cdot M\cdot\Phi_i} \equiv \omega_i^2$$

- **Calculate the Damping Ratio Using the Approximation Formula**

$$\zeta_i = \frac{1}{2}\,\Phi_i^T\cdot K_I\cdot\Phi_i\,/\,\omega_i^2$$

In the well known MSE method, the real system modes which are derived from the mass matrix and the real part of the complex stiffness matrix are assumed to be accurate approximations to the complex system eigenvectors.The Rayleigh Quotient is then used to approximate the eigenvalues of the system.

As opposed to the absolute value of the Rayleigh Quotient, using the MSE method the natural frequency squared is assumed to be the real part of the Rayleigh Quotient. The damping ratio is then calculated using an approximation formula which is one half the ratio of the real and imaginary parts of the Rayleigh Quotient.

Note that the MSE method does not yield the correct result for the single degree of freedom system. It is relatively accurate, however, for small damping ratios.

The Absolute Value MSE Method

- **Assemble the Absolute Value Matrix**

$$\overline{K} = \sum_{j=1}^{NE} K_{R_j} \sqrt{1+\eta_j^2}$$

- **Find the Eigenvectors and Eigenvalues Associated with This Matrix and the Mass Matrix**

$$(-M\,\omega^2 + \overline{K}) \cdot \Phi = 0$$

- **Assume That These Vectors Are Approximate Eigenvectors**

- **Compute the Rayleigh Quotient To Find the Approximate Eigenvalue**

$$\lambda^2 \cong \frac{\Phi^T \cdot (K_R + iK_I) \cdot \Phi}{\Phi^T \cdot M \cdot \Phi}$$

- **Calculate the Modal Strain Energy in Each Element**

$$MSE_{ij} = \frac{\Phi_i^T \cdot K_{R_j} \sqrt{1+\eta^2} \cdot \Phi_i}{\omega_i^2}$$

A new energy method of has been developed for the approximation of the complex eigenvalue problem, which is termed the Absolute Value Modal Strain Energy (AVMSE) method. In the AVMSE method, a stiffness matrix is assembled with the absolute value of each elements impedance, which is consistent with the single degree of freedom system.

The eigenvectors and eigenvalues associated with the system mass matrix and this stiffness matrix are then computed. These eigenvectors are assumed to be a good approximation to the complex system eigenvectors. The Rayleigh Quotient is then used to determine the location of the eigenvalue.

By defining the modal strain energy in the elements of the system, in a manner similar to the standard MSE method, we can determine the eigenvalue location in terms of element real and complex energies. This provides the same insight for design development as the standard MSE method.

The Absolute Value MSE Method (concl)

- **Calculate the Approximate Natural Frequency As the Absolute Value of the Rayleigh Quotient**

$$\left|\lambda_i^2\right| = \left[\left(\omega_A^2 \cdot \sum_{j=1}^{NE} \frac{MSE_{ij}}{\sqrt{1+\eta_j^2}}\right) + \left(\omega_A^2 \cdot \sum_{j=1}^{NE} \frac{MSE_{ij} \cdot \eta_j}{\sqrt{1+\eta_j^2}}\right)\right] \cong \omega_A^2$$

- **Use the Phase Angle of the Rayleigh Quotient To Calculate the Damping Ratio**

$$\eta_i = \frac{\displaystyle\sum_{j=1}^{NE} \frac{MSE_{ij} \cdot \eta_j}{\sqrt{1+\eta_j^2}}}{\displaystyle\sum_{j=1}^{NE} \frac{MSE_{ij}}{\sqrt{1+\eta_j^2}}} \qquad \zeta_i = \sqrt{\frac{1-\sqrt{\dfrac{1}{\eta^2+1}}}{2}}$$

From the complex Rayleigh Quotient, the natural frequency is the square root of its absolute value. This can be shown to be approximately the eigenvalue which is computed from the real eigenvalue problem.

By defining the modal loss factor, the viscous damping ratio can be determined from the phase angle of the eigenvalue similar to the single degree of freedom system. The modal loss factor can be written in terms of the MSE in various elements and their loss factors.

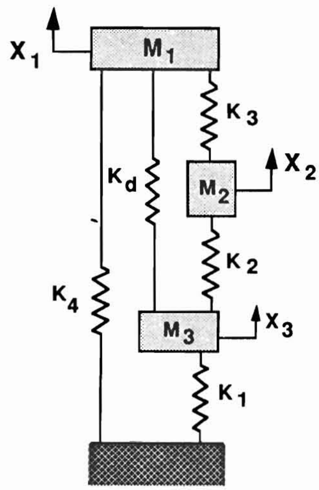
A Simple Example Problem

- Simple 3-DOF System with One Complex Spring

- Vary the Spring Loss Factor

- Compute Modal Properties Using Various Methods
  - Complex Eigenproblem
  - MSE Method
  - Absolute Value MSE Method

- Compare Results

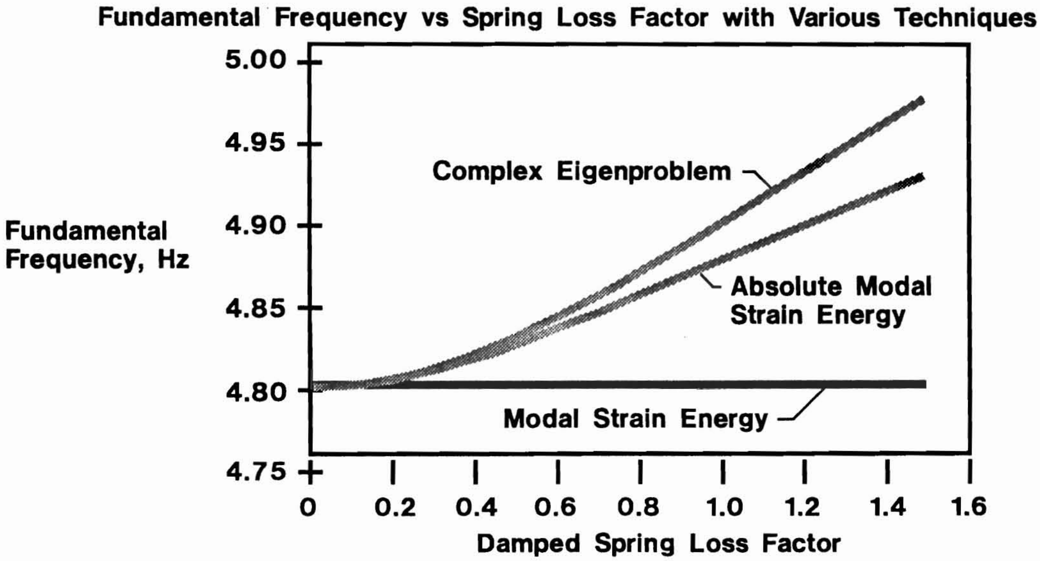


To demonstrate the behavior of the standard MSE method, the AVMSE method, and the complex eigenvalue problem, a simple system was used. This system consists of three masses interconnected by springs, one of which is complex.

For the example, the loss factor of the complex spring was varied and the modal properties were computed using the various methods, so that the results could be compared.

Fundamental Frequency vs Spring Loss Factor

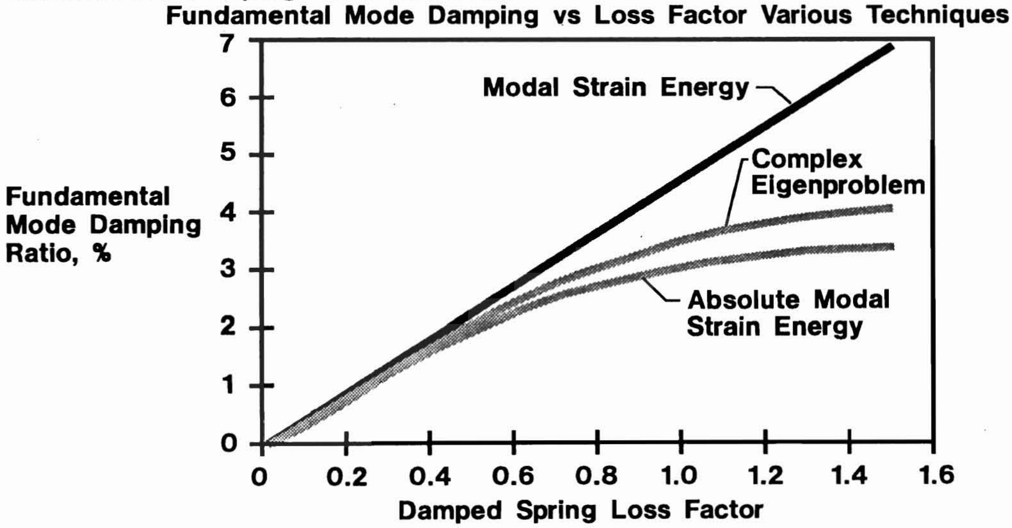
- **Absolute Value MSE Shows Frequency Increase with Spring Loss Factor**



Plotted here is the variation in the system fundamental frequency with the damped element loss factor. The MSE solution, of course shows no variation in natural frequency with element loss factor. The complex eigenproblem shows a frequency increase as the loss factor of the element increases, implying "stiffer" behavior of the damped element. Similarly, the AVMSE method also shows an increase in the frequency with element loss factor, and is more accurate than the standard MSE method.

Fundamental Mode Damping vs Spring Loss Factor

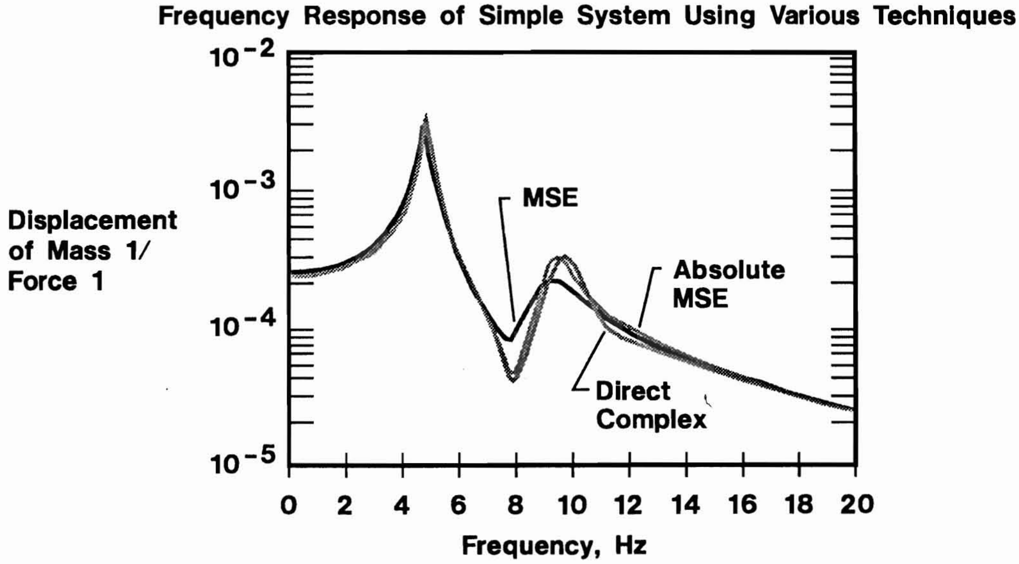
- **Absolute Value MSE More Accurate and Has Desirable Characteristics**
  - **Peak Damping at Specific Loss Factor Value**
  - **Conservative Damping Ratio Estimates**



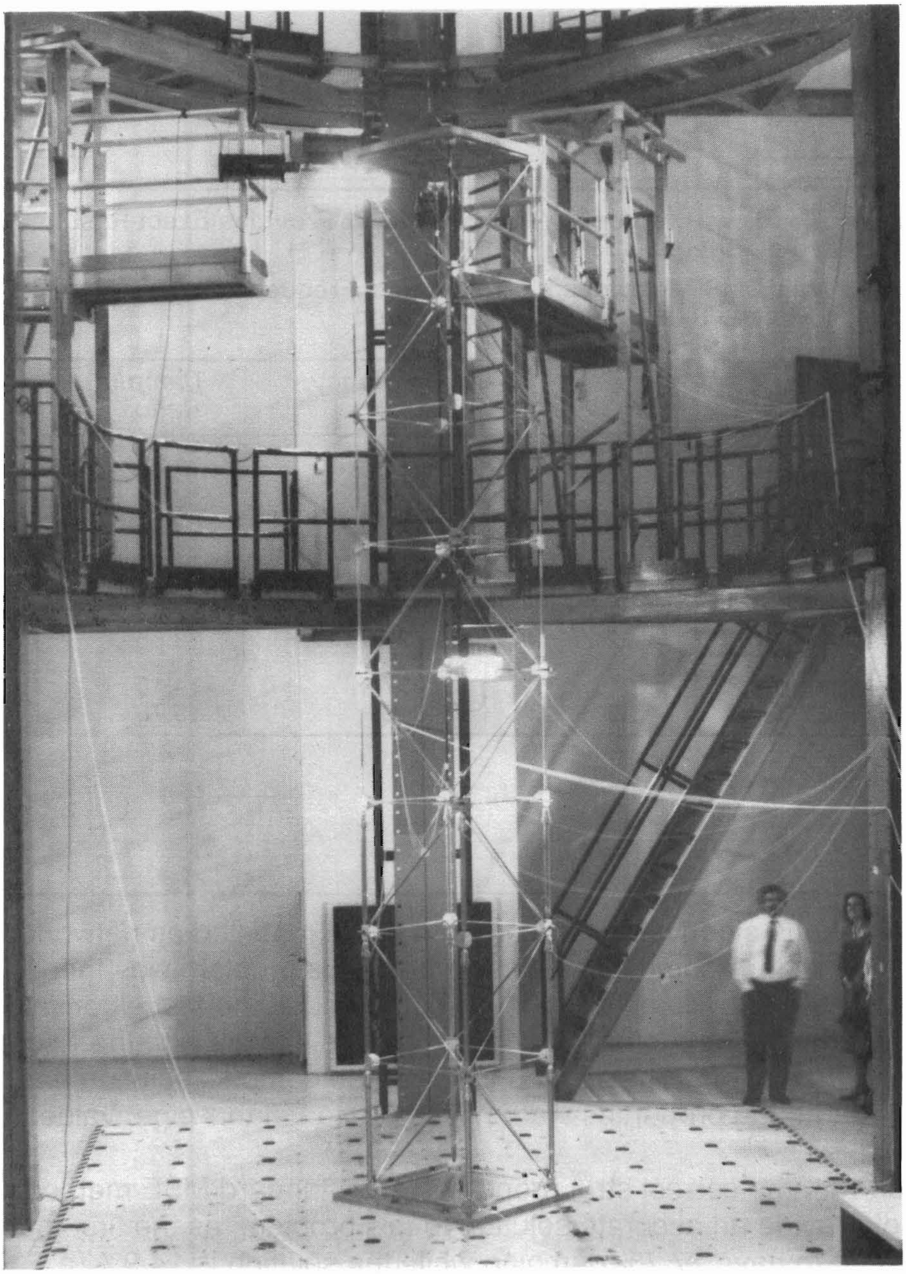
This plot shows the damping in the fundamental mode of the system as the loss factor is varied. The MSE solution predicts that the damping is linear in the element loss factor. The complex eigenvalue problem, however, shows that as the loss factor becomes large, the rate of damping increase becomes smaller. The complex eigenvalue problem also shows that if the loss factor is raised too high, the damping can actually decrease. The AVMSE solution duplicates this behavior, while providing a conservative estimate to the damping ratio. Notice that the problem was selected such that the damping ratio is low, on the order of 3%.

Frequency Response Using Various Methods

- **Absolute Value MSE Provides More Accurate Frequency Response**



This plot compares the frequency response for displacement of the first mass for forces applied to the same mass computed using the various methods. Notice that the absolute value MSE method is more accurate for the first mode, and very much better for the two coupled modes at 9 and 11 Hz. The standard MSE method has much greater error in this frequency region.



To show a comparison of the methods on a realistic problem, the PACOSS D-Strut truss was selected. This truss includes damping members which can be modeled with complex impedances. The exact solution can be generated by the solution of a viscous damping eigenvalue problem and compared with alternative solution methods.

Results of Application to D-Strut Truss Model

- Complex Modes with Complex Stiffness Matrix Provides Exact Results If Computed at Eigenvalue
- Absolute Value MSE Provides More Accurate Frequencies, Damping, and Mode Shapes than Standard MSE Method

Analysis Method	*Mode No.*	*Frequency, Hz*	*Damping Ratio*
Complex Network	1 2	5.493 6.142	14.93 15.63
Complex Stiffness	1 2	5.493 6.093	14.93 16.51
MSE	1 2	5.220 5.720	17.69 18.85
Absolute MSE	1 2	5.457 6.000	14.10 15.78

The viscous eigenvalue problem with struts modeled as networks was solved first. A complex stiffness eigenvalue problem was then solved with the strut complex stiffness computed at the first eigenvalue. Of course, the first mode of this solution agrees exactly with the viscous damping formulation. The second mode of the complex solution has some error, as the properties were not computed at this eigenvalue.

Even with the correct strut properties, the standard MSE method does not provide an accurate solution to this problem, as the first mode is predicted to have 17.7% damping while the solution is 14.9%, a relative error of 18.5%. Similarly the frequency is in error by almost 5%. The AVMSE method, however, is much more accurate. This method underpredicts the damping ratio by 5.5% relative error, and underpredicts the frequency by 0.6% relative error. This shows the much greater accuracy which can be achieved with the AVMSE method on difficult damping problems.

Conclusions

- New Method of Complex Stiffness Eigenproblem Approximation Developed
  - Uses Real Eigenproblem
  - Stiffness Matrix Is Assembled with Absolute Value of Impedance
- Absolute Value MSE Method Behaves As Complex Eigenproblem
  - Elements Appear "Stiffer" As Loss Factor Increases
  - Damping Decreases As Loss Factors Raised Too High
  - Can Model Systems with Dashpots As Only Connection between DOFs

A new method of determining approximate properties of damped structures has been developed and was presented. This method is based on a real eigenvalue problem which uses an assemblage of the absolute value of the element impedance matrices as the stiffness matrix.

The behavior of the new method is more consistent with the complex eigenvalue problem. Similar to the complex solution, elements appear stiffer as loss factors increase, and the natural frequency increases. The AVMSE method does not predict a linear relationship between element loss factors and system damping, similar to the complex eigenvalue problem.

Additionally, the new method can model systems with dashpots as the sole connection between degrees of freedom. As a dashpot has an infinite loss factor and no real part to its stiffness, the standard MSE method would not place a stiffness term in the MSE stiffness matrix. The new method places a term equal to the natural frequency multiplied by the dashpot coefficient in the stiffness matrix, and allows the approximate solution to this problem.

Conclusions (concl)

- Absolute Value MSE Method Provides More Accurate Results for Examples
  - Single-DOF System
  - Three-DOF Example Problem
  - D-Strut Truss
- Absolute Value MSE Method Is As Easy To Implement As Standard MSE Method
- New Method Should Be Used for Accurate Damping Design Approximations
  - More Accurate
  - Same Expense As MSE

The AVMSE method provides more accurate results for example problems, including a single degree of freedom system, the three degree of freedom example system, and the D-Strut truss.

The AVMSE method is as easy and inexpensive to implement as the standard MSE method, as element stiffnesses need only be raised by a factor dependent on their loss factors. Damping is readily determined from the MSE distribution.

Based on these results and conclusions, the new method AVMSE method should be used for damping design and system analysis, as it provides more accurate solutions at the same expense as the MSE method. Errors due to the AVMSE approximation tend to be conservative.