FINITE ELEMENT ANALYSIS OF PLATE BUCKLING
USING A MIXED VARIATIONAL PRINCIPLE*

by

D. J. Allman**

(Royal Aircraft Establishment, Farnborough, UK)

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A (mixed) variational principle for stresses and displacements is presented from which the governing differential equations of the bifurcational buckling of thin elastic plates are derived. Subsequent finite element analysis, using a linear approximation for the bending moments and a cubic approximation for the displacements within each element, leads to the familiar linear eigenvalue problem for determining the critical intensities of in-plane loading. Numerical solutions are given to a number of simple examples involving square plates using a recently published eigenvalue algorithm which takes advantage of the banded properties of the final matrix formulation. Comparisons are made, where possible, with results from the finite element analyses of other authors and with exact or classical Rayleigh-Ritz solutions.

**Senior Scientific Officer, Structures Department.

1. INTRODUCTION

Although the analysis of plate buckling problems using the finite element method is closely related to the analysis of plate bending it has not received the same enthusiastic attention. The displacement formulation is favoured by most authors[1-5], although their attention is often restricted to rectangular plates[1-3] and in many cases[1,2,4] the principle of minimum potential energy is applied in a manner which is not strictly correct. Convergence to the exact solution with consistent refinement of the finite element mesh size is thus unassured. Examples of the associated difficulties can be seen in the work of Anderson *et al.*[4], as reported by Clough and Felippa[5], in the context of the non-compatible triangular element of Bazeley *et al.*[6]

The development of a satisfactory triangular element is an important step in dealing with the buckling of plates of arbitrary shape. However, the possibility of using compatible triangular elements for this purpose is unattractive. The simplest displacement elements[6,7] use, as unknown parameters, the values of the transverse displacement w and its partial derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ at the element corners together with the normal derivatives $\frac{\partial w}{\partial n}$ at the mid-points of the element sides; a total of twelve degrees of freedom for each element. Now, in the interests of computational efficiency, it is desirable to eliminate the element mid-side nodes. This can be achieved quite simply by assuming a linear variation of normal slope along the element boundaries, but unfortunately the resulting triangular element with nine degrees of freedom produces an excessively over-stiff idealisation. In an effort to overcome this difficulty Clough and Felippa[5] derive a quadrilateral 'compound' element which is synthesized from four compatible triangular elements. Solutions to the plate buckling problem using this element show an improvement in accuracy over the former triangular element with nine degrees of freedom, but this is, to some extent, offset by the corresponding increase in computation required in the calculation of the element stiffness matrix.

In the present work, a finite element solution to the plate buckling problem is presented which is based on a mixed variational principle of the type given by Reissner[8]. This approach has also been previously considered by Cook[9] using a mixed formulation[10,11] of the constant bending moment equilibrium element[12,13], but the accuracy of the numerical results was found to be poor. The element 'stiffness' matrix used here is identical to that derived by Allman[12] for plate bending problems, where a linear bending moment field, defined inside a triangular element, is used in conjunction with a cubic displacement field on the element boundary. The so-called element 'geometric stiffness' matrix, which represents the contribution of the in-plane loading to the transverse stiffness of the plate, is calculated using the relatively simple non-compatible cubic displacement field given by Bazeley *et al.*[6] This is shown, in Appendix A, to provide a correct application of an associated mixed variational principle. The unknown parameters of the finite element model are the values of w, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ at the corners of the triangular element, so that the connection properties are identical to those of the simple displacement elements[6,7] mentioned previously. However, the elimination of the mid-side nodes does not here affect the numerical accuracy to such a marked degree.

In the numerical solution of the buckling problem, advantage is taken of the banded form of the linear eigenvalue problem, as described by Peters and Wilkinson[14]. This is believed to be a new feature in the solution of plate buckling problems and a brief description of the method is included in the present paper. A number of simple numerical solutions involving uniform plane stress states is calculated to illustrate the accuracy which can be attained using the mixed finite element formulation. These are compared, where possible, to the classical solutions given by Timoshenko and Gere[15] and to results from the finite element analyses of other authors.

Although the numerical examples given in the paper apply specifically to cases of uniform plane stresses, the finite element analysis can also be used to calculate the buckling loads of plates which are subjected to more complex in-plane loading systems. For these problems, the distribution of plane stress resultants in the plate may be calculated to an accuracy which is probably adequate for the buckling analysis using a mesh of compatible triangular elements with linearly varying displacements; the same mesh being subsequently used for the eigenvalue formulation. This procedure, which gives a uniform stress field in each element, has the two-fold advantage of making the calculation of the geometric stiffness matrix a relatively simple matter and ensuring that the homogeneous equations of plane stress equilibrium are satisfied identically in each element, as required by the associated mixed variational principle for plate buckling.

2. A MIXED VARIATIONAL PRINCIPLE FOR THE BUCKLING OF THIN PLATES

The finite element analysis, which is presented in this paper, is based on a mixed variational principle of the type given by Reissner[8]. Here, the governing differential equations of plate buckling, together with their homogeneous kinematic and traction boundary conditions, are derived by considering arbitrary variations of an appropriate mixed functional.

Consider a thin elastic plate of area A, lying in the plane of the rectangular Cartesian coordinates x_i $(i = 1,2)$, which is in equilibrium under a system of applied membrane loads. In the absence of body forces, the distribution of stress resultants λN_{ij} $(i,j = 1,2)$ in the plane of the plate satisfies the homogeneous equations of equilibrium, so that, employing the usual summation convention,

$$\lambda \frac{\partial N_{ij}}{\partial x_i} = 0 \quad , \tag{2-1}$$

where the parameter λ is the ratio of a critical intensity of applied loading to the actual intensity of loading. The critical values of λ are now sought at which the plate can buckle out of the plane $x_1 - x_2$ and still remain in equilibrium under the action of the applied membrane loads.

To avoid ambiguity, the plate is assumed to be singly connected and bounded by a piecewise smooth contour $C = C_T \cup C_k$, C_T and C_k being respectively those parts of the total boundary C on which traction and kinematic boundary conditions are prescribed. The distance s around C is measured anti-clockwise starting from some suitable reference point, while n_i and t_i are the Cartesian components of the unit exterior normal n and the anti-clockwise unit tangential vector t (see Fig.1). On the boundary C the

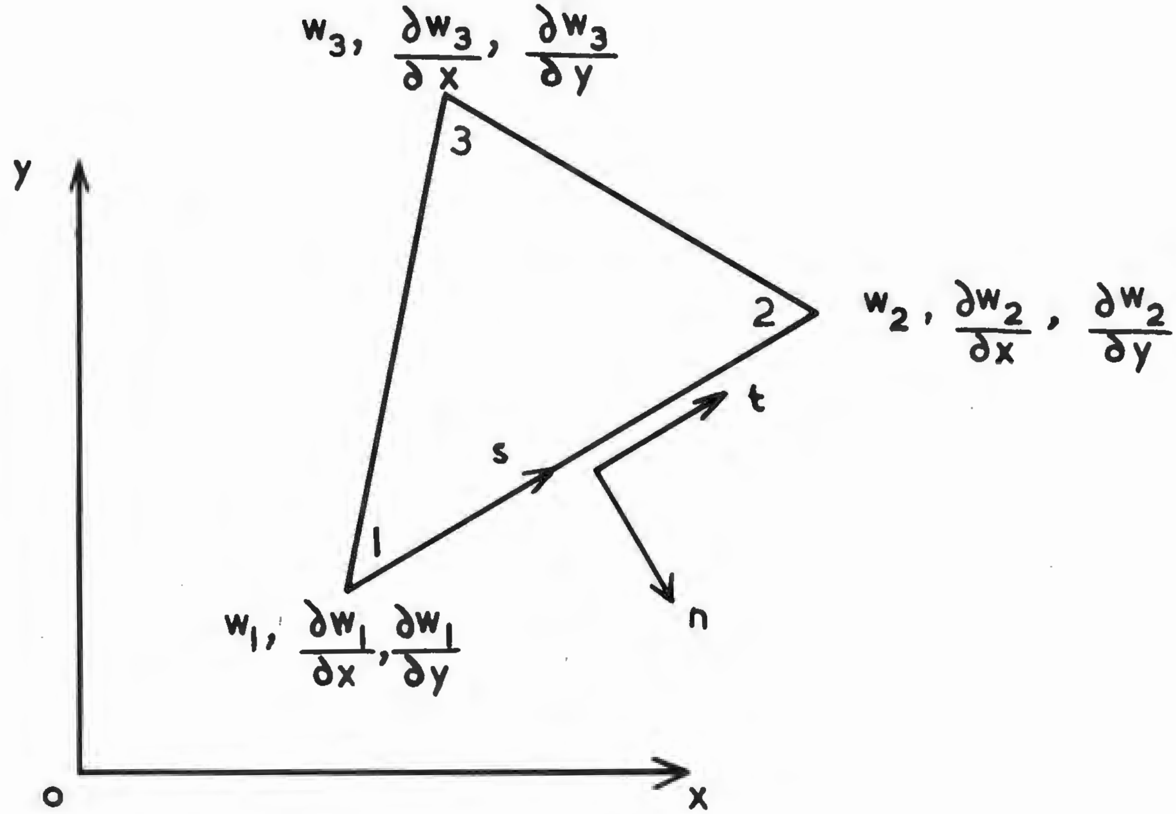


FIG.1 CONNECTION PROPERTIES OF THE MIXED TRIANGULAR ELEMENT

field of bending and twisting moments M_{ij} gives rise to a normal bending moment M_n, a twisting moment M_{ns} and a shearing force Q_n given by

$$\left.\begin{aligned} M_n &= M_{ij}n_in_j \ , \\ M_{ns} &= M_{ij}n_it_j \ , \\ Q_n &= \frac{\partial M_{ij}}{\partial x_i} n_j \ . \end{aligned}\right\} \tag{2-2}$$

The transverse Kirchhoff force is given by

$$V_n = Q_n + \frac{\partial M_{ns}}{\partial s} \ , \tag{2-3}$$

while at 'corners' N, which occur on the piecewise smooth contour C, the twisting moment M_{ns} produces concentrated forces

$$R_N = [M_{ns}]_{s^-}^{s^+} \ , \tag{2-4}$$

where the points s^- and s^+ immediately precede and succeed the corner point N respectively.

Furthermore, because of the rotation of the mid-surface of the plate which occurs in the change from the unbuckled to the buckled position of equilibrium, the distribution of plane stress resultants contributes to the total transverse force on the boundary C. This resultant force is defined as

$$\bar{V}_n = V_n + \lambda N_n \frac{\partial w}{\partial n} + \lambda N_{ns} \frac{\partial w}{\partial s} \ , \tag{2-5}$$

where λN_n and λN_{ns} are the normal and tangential components of the membrane stress resultants λN_{ij}, on the boundary C, defined by

$$\left.\begin{aligned} \lambda N_n &= \lambda N_{ij}n_in_j \ , \\ \lambda N_{ns} &= \lambda N_{ij}n_it_j \ . \end{aligned}\right\} \tag{2-6}$$

Using the above notation, the functional of the mixed variational principle is now defined as

$$\begin{aligned} J = \ & \tfrac{1}{2} \iint_A \left\{ C_{ijk\ell} M_{ij} M_{k\ell} + \lambda N_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \right\} dA \\ & + \iint_A \left\{ \frac{\partial^2 M_{ij}}{\partial x_i \partial x_j} + \lambda N_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} \right\} w dA - \sum_{C_T} [R_N w_N] \\ & - \int_{C_T} \bar{V}_n w ds + \int_{C_T} M_n \frac{\partial w}{\partial n} ds \ , \end{aligned} \tag{2-7}$$

where $C_{ijk\ell}$ is the elastic compliance tensor which, for an isotropic plate of thickness h and Young's modulus E, is written

$$C_{ijk\ell} = \frac{1}{D(1-\nu^2)}\left[(1+\nu)\delta_{ik}\delta_{j\ell} - \delta_{ij}\delta_{k\ell}\right] \quad ; \qquad (2\text{-}8)$$

ν being Poisson's ratio and $D = \dfrac{Eh^3}{12(1-\nu^2)}$ the flexural rigidity of the plate. The Kronecker delta $\delta_{ij} = 0$ for $i \neq j$, while $\delta_{ii} = 1$ (no sum on i). The functional J has no subsidiary conditions and is to be made stationary by considering arbitrary variations in both the bending moment and displacement fields.

Recalling the Green's formulae

$$\iint_A \frac{\partial^2 \delta M_{ij}}{\partial x_i \partial x_j} w\,dA - \iint_A \frac{\partial^2 w}{\partial x_i \partial x_j} \delta M_{ij}\,dA = \sum_C [w_N \delta R_N] + \oint_C w\delta V_n\,ds - \oint_C \frac{\partial w}{\partial n}\delta M_n\,ds \quad \ldots (2\text{-}9)$$

and

$$\iint_A \lambda N_{ij} \frac{\partial w}{\partial x_i}\frac{\partial \delta w}{\partial x_j}\,dA + \iint_A \lambda N_{ij} \frac{\partial^2 \delta w}{\partial x_i \partial x_j} w\,dA + \iint_A \lambda \frac{\partial N_{ij}}{\partial x_i}\frac{\partial \delta w}{\partial x_j} w\,dA$$

$$= \oint_C w\delta \bar{V}_n\,ds - \oint_C w\delta V_n\,ds \qquad (2\text{-}10)$$

and noting that the last term on the LHS of equation (2-10) vanishes because of the plane equilibrium equations (2-1), the variational equation

$$\delta J = 0 \qquad (2\text{-}11)$$

is the required stationary condition of equation (2-7). The Euler equations and natural boundary conditions of equation (2-11) are the governing equations of the plate buckling problem; these are: the equation of equilibrium of the buckled configuration

$$\frac{\partial^2 M_{ij}}{\partial x_i \partial x_j} + \lambda N_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} = 0 \quad ; \qquad (2\text{-}12)$$

the three constitutive relations

$$\frac{\partial^2 w}{\partial x_i \partial x_j} + C_{ijk\ell}M_{k\ell} = 0 \quad ; \qquad (2\text{-}13)$$

the homogeneous traction boundary conditions on C_T,

$$\bar{V}_n = M_n = R_N = 0 \quad , \qquad (2\text{-}14)$$

and the homogeneous kinematic boundary conditions on C_k,

$$w = \frac{\partial w}{\partial n} = 0 \quad . \qquad (2\text{-}15)$$

3. FINITE ELEMENT ANALYSIS

For the finite element analysis, it is convenient to transform the functional of equation (2-7) into a slightly different form. This is achieved by using the identity of equation (2-10) with δw replaced by w and requiring that the kinematic boundary conditions (equation (2-15)) are satisfied *a priori* on C_k. It is then found that, for a typical element e, the contribution J_e of the element to the total functional J is given (in an unabridged notation) by

$$J_e = \tfrac{1}{2}\iint_\Delta \frac{1}{D(1-\nu^2)}\left[(M_x + M_y)^2 + 2(1+\nu)(M_{xy}^2 - M_x M_y)\right]dA$$
$$+ \iint_\Delta \left(\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2}\right) w\,dA$$
$$- \sum_{C_\Delta} [R_N w_N] - \oint_{C_\Delta} V_n w\,ds + \oint_{C_\Delta} M_n \frac{\partial w}{\partial n}\,ds$$
$$- \frac{\lambda}{2}\iint_\Delta \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + N_y\left(\frac{\partial w}{\partial y}\right)^2\right] dA \quad , \qquad (3\text{-}1)$$

with subsidiary conditions $w = \frac{\partial w}{\partial n} = 0$ on C_k. Here, the surface integrals are calculated over the area Δ of the finite element and the summation and line integrals are evaluated at the element corners and anti-clockwise around the finite element boundary C_Δ respectively. This formulation of the functional is more convenient than that given in equation (2-7) because, as in the conventional stiffness method, the process of numerical calculation of J_e is identical for each element, while the kinematic boundary conditions are applied as a separate operation after the calculation of the global matrices is complete.

The details of a mixed finite element for plate bending have been given previously by Allman[12]. Here, that analysis is extended to deal with plate buckling and, for convenience, some of the previous results are repeated below.

A linear field of bending moments is defined by

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = N_1 \beta \quad , \qquad (3\text{-}2)$$

where N_1 is a (3×9) matrix which renders the individual linear fields orthonormal with respect to the strain energy metric of an element and β is a vector of nine undetermined moment parameters $\beta_1, \ldots, \beta_9$. The twelve generalised loads at the boundaries of a triangular element are given by the components of the vector

$$Q = G^T\beta \quad , \tag{3-3}$$

where G^T is a (12 × 9) matrix calculated from the linear bending moment field of equation (3-2). Twelve generalised displacements corresponding to the generalised loads are given by the components of the vector

$$q = TW \quad , \tag{3-4}$$

where T is a (12 × 9) transformation matrix calculated from a compatible displacement field defined on the element boundary in terms of a vector of nine generalised displacements, viz.

$$W^T = \left[w_1, \frac{\partial w_1}{\partial x}, \frac{\partial w_1}{\partial y}, w_2, \frac{\partial w_2}{\partial x}, \frac{\partial w_2}{\partial y}, w_3, \frac{\partial w_3}{\partial x}, \frac{\partial w_3}{\partial y}\right] \quad . \tag{3-5}$$

The subscripts 1, 2, 3 refer to the corners of the triangular element numbered in anti-clockwise order (see Fig.1).

The calculation of the element 'geometric stiffness' matrix involves the determination of the first derivatives of the displacement w inside an element. However, for a mixed finite element analysis, it is unnecessarily restrictive for this purpose to employ a displacement field with both w and $\frac{\partial w}{\partial n}$ continuous across element boundaries. In fact, it is demonstrated in Appendix A that it is adequate to use the simple expression proposed by Bazeley *et al.*[6], denoted here by $\bar{w}$, which violates slope compatibility on the element boundary. The potential energy of the in-plane loads is written, in terms of the non-compatible field $\bar{w}$, as

$$\frac{\lambda}{2}\iint_\Delta \left[N_x\left(\frac{\partial \bar{w}}{\partial x}\right)^2 + 2N_{xy}\frac{\partial \bar{w}}{\partial x}\frac{\partial \bar{w}}{\partial y} + N_y\left(\frac{\partial \bar{w}}{\partial y}\right)^2\right] dA = \frac{\lambda}{2} W^T K_1 W \quad , \tag{3-6}$$

where

$$K_1 = \iint_\Delta \left[\frac{\partial \bar{w}}{\partial x}, \frac{\partial \bar{w}}{\partial y}\right] \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{w}}{\partial x} \\ \frac{\partial \bar{w}}{\partial y} \end{bmatrix} dA \tag{3-7}$$

is a (9 × 9) element 'geometric stiffness' matrix (otherwise known as the 'initial stress' matrix[2] or the 'stability coefficient' matrix[1]) and where W is the vector of nine generalised displacements defined previously.

The functional of equation (3-1) can now be written, in matrix notation, as

$$J_e = \tfrac{1}{2}\beta^T\beta - Q^Tq - \tfrac{1}{2}\lambda W^T K_1 W \tag{3-8}$$

and, substituting from equation (3-3) and equation (3-4), we have

$$J_e = \tfrac{1}{2}\beta^T\beta - \beta^T(GT)W - \tfrac{1}{2}\lambda W^T K_1 W \quad . \tag{3-9}$$

Considering variations $\delta\beta$ and δW, the contribution of J_e to the total system of equations is

$$\delta J_e = \delta\beta^T[\beta - (GT)W] - \delta W^T[(GT)^T\beta + \lambda K_1 W] \quad , \tag{3-10}$$

and since $\delta\beta$ is quite independent of inter-element continuity,

$$\beta = (GT)W \quad . \tag{3-11}$$

Substituting for β in equation (3-10) and summing over all elements, we find

$$\sum_e \delta W^T[K_0 + \lambda K_1]W = 0 \quad , \tag{3-12}$$

where

$$K_0 = (GT)^T(GT) \tag{3-13}$$

is a (9 × 9) elastic 'stiffness' matrix for the element. Finally, noting the arbitrary nature of δW when the kinematic boundary conditions of equations (2-15) are applied, and defining global matrices by

$$\bar{K}_0 = \sum_e K_0 \quad ; \quad \bar{K}_1 = \sum_e K_1 \quad ; \quad \bar{W} = \sum_e W \quad , \tag{3-14}$$

equation (3-12) may also be written as

$$-\bar{K}_1\bar{W} = \mu\bar{K}_0\bar{W} \tag{3-15}$$

where $\mu = \frac{1}{\lambda}$. Here, $\bar{K}_0$ and $\bar{K}_1$ are real banded symmetric matrices; $\bar{K}_0$ being positive definite when the rigid body degrees of freedom are restrained (see Appendix B). The critical intensities of the applied in-plane load are now determined from the eigenvalues μ of the above system of equations where, in view of the inverse relation between μ and λ, the first critical value of λ corresponds to the maximum positive value of μ.

4. DETERMINATION OF EIGENVALUES OF BAND SYMMETRIC MATRICES

The determination of the buckling loads from the finite element formulation presented previously involves the calculation of the eigenvalues of the linear matrix equation (3-15). In general, both of the global matrices $\bar{K}_0$ and $\bar{K}_1$ are real banded symmetric matrices with $\bar{K}_0$ positive definite. However, if equation (3-15) is reduced to the standard symmetric eigenvalue problem by using the Cholesky decomposition of $\bar{K}_0$ given by

$$\bar{K}_0 = LL^T \quad , \tag{4-1}$$

where L is a lower triangular matrix, we obtain

$$-(L^{-1}\bar{K}_1L^{-T})(L^T\bar{W}) = \mu(L^T\bar{W}) \quad , \tag{4-2}$$

where the matrix $(L^{-1}\bar{K}_1L^{-T})$ is full and the advantage of the banded property of the matrices $\bar{K}_0$ and $\bar{K}_1$ is lost.

In a recent paper, Peters and Wilkinson[14] describe how the eigenvalues of equation (3-15) can be calculated in a way which takes account of the intrinsic banded nature of $\bar{K}_0$ and $\bar{K}_1$. Their method is based on the fact that the leading principal minors of the matrix $(\bar{K}_1 + \mu\bar{K}_0)$ form a Sturm sequence. This implies that the number of eigenvalues which is greater than μ is equal to the number of agreements in sign between consecutive members of the sequence of leading principal minors

$$\det\,(\bar{K}_1 + \mu\bar{K}_0)_r \quad , \quad (r = 0,1,\ \ldots,n) \quad , \tag{4-3}$$

where n is the order of $\bar{K}_0$ and $\bar{K}_1$ and, by definition,

$$\det\,(\bar{K}_1 + \mu\bar{K}_0)_0 = 1 \ . \tag{4-4}$$

In the computational algorithm, the leading principal minors are not determined explicitly but only the sign of $\det\,(\bar{K}_1+ \mu\bar{K}_0)_{r+1}$ in terms of the sign of $\det\,(\bar{K}_1 + \mu\bar{K}_0)_r$. This is done by comparing the signs of the pivotal elements (the products of which give the minors) of the triangularisation of the matrix $(\bar{K}_1 + \mu\bar{K}_0)$ given by Martin and Wilkinson[16] which is specially designed for the band symmetric case.

The algorithm is, perhaps, best described by considering its use for the calculation of all the eigenvalues μ_i $(i = 1,2,\ldots,M)$ in the interval $a < \mu_i \leqslant b$. The Sturm sequence counts S_a and S_b are first determined at a and b. This shows that $M = S_b - S_a$ eigenvalues lie in the given range and that a and b can be taken as lower and upper bounds for all of them. To determine each eigenvalue, repeated bisection of the interval combined with the Sturm sequence count is used until upper and lower bounds are found which contain just one eigenvalue. Every time a Sturm sequence count is made, the upper and lower bounds for each eigenvalue are updated. When a single eigenvalue is isolated in this way, an alternative technique with a higher convergence rate is used to locate that eigenvalue to the required accuracy. For this purpose, Peters and Wilkinson[14] recommend the use of a method of successive linear interpolation and an effective procedure is described in their paper. Notice that this means all multiple eigenvalues are determined entirely by bisection, since an interval is never attained which contains only one such eigenvalue.

In the present work the kinematic boundary conditions are applied in such a way (see Appendix B) that the corresponding eigenvalues are determined as $\mu = -1$. Hence only the positive eigenvalues of equation (3-15) represent a solution to our plate buckling problem, and the initial lower limit can be conveniently chosen as $a = 0$. The choice of a suitable initial upper limit is, however, not straightforward, but a convenient procedure is described by Peters and Wilkinson[14] as follows. The quantity ρ is calculated which is given by

$$\rho = \frac{\|\bar{K}_1\|_\infty}{\|\bar{K}_0\|_\infty} \quad , \tag{4-5}$$

where the ∞-norms of $\bar{K}_0$ and $\bar{K}_1$ are defined as

$$\left.\begin{aligned} \|\bar{K}_0\|_\infty &= \max_i \sum_j |(\bar{k}_0)_{ij}| \ , \\ \|\bar{K}_1\|_\infty &= \max_i \sum_j |(\bar{k}_1)_{ij}| \end{aligned}\right\} \tag{4-6}$$

The Sturm sequence is then obtained for

$$\mu = \rho, 4\rho, 4^2\rho, \ \ldots . \tag{4-7}$$

until an upper bound is obtained to the largest eigenvalue (i.e. when $S_b = 0$).

Usually, in practical problems, it is necessary to calculate only the lowest intensity of loading λ at which a plate buckles; in consequence the maximum value of μ is to be determined. In this case the lower limit a is continuously updated to the value of the previous upper limit b during the process of calculating the Sturm sequence of equation (4-7). It is then a matter of simple programming to arrange for the largest eigenvalue to be located by bisection before transferring to the more rapidly convergent interpolation scheme. This process can be extended in an obvious way so that, if desired, several of the lower order critical intensities can be calculated.

5. NUMERICAL RESULTS

The results of a number of simple examples, computed with the mixed element, are given in Tables 1 and 2 for the cases of isotropic square plates subjected to various simple uniform stress distributions along their boundaries. The kinematic boundary conditions are assumed to be either all simply supported or all clamped edges and the finite element solutions are compared with the classical solutions quoted by Timoshenko and Gere[15]. All the examples are calculated using both a (4×4) mesh and an (8×8) mesh over the whole plate, taking advantage of symmetry as shown in Figs.2a, 2b and 2c. Poisson's ratio is taken as $\nu = 0.3$.

Where possible, the results are also compared to those given by Clough and Felippa[5] for their compatible element and to the results quoted by Anderson *et al.*[4] using the non-compatible displacement element of Bazeley *et al.*[6] Although all of these solutions involve the same final number of unknown parameters, it must be remembered that those given by Clough and Felippa involve an additional partial solution to build up their compound element. It is apparent that the results from the mixed element are, for the cases given, mainly superior to the other two. Furthermore, the results obtained from the non-compatible element appear to converge very slowly.

A comparison is also made with the solutions obtained using the more restrictive rectangular elements given by Kapur and Hartz[1], Dawe[2] and Carson and Newton[3]; of these, only the last is a correct application of the principle of minimum potential energy. The results given by Carson and Newton converge more rapidly than those given by Kapur and Hartz and Dawe and are also more accurate than those obtained from the present mixed element. However, their element involves an extra degree of freedom $\left(\text{namely the twist } \frac{\partial^2 w}{\partial x \partial y}\right)$ at each of the element corners.

Table 1

CRITICAL LOADS OF SIMPLY SUPPORTED SQUARE PLATE UNDER UNIFORM PLANE STRESSES

Loading case	Mesh size	Triangular elements			Rectangular elements			Classical solution
		Present analysis	Clough and Felippa	Anderson *et al.*	Kapur and Hartz	Dawe	Carson and Newton	
Uniform uni-axial compression	4 × 4	4.031	4.126	3.72*	3.770	3.978	4.001	4.00
	8 × 8	4.006	4.031	3.94*	3.933	3.993	4.000	
Uniform bi-axial compression	4 × 4	2.016				1.989		2.00
	8 × 8	2.003				1.997		
Uniform shear	4 × 4	10.131				9.481	9.418	9.34
	8 × 8	9.468				–	–	
Multiplier = $\pi^2 D/\ell^2$ for all cases.								

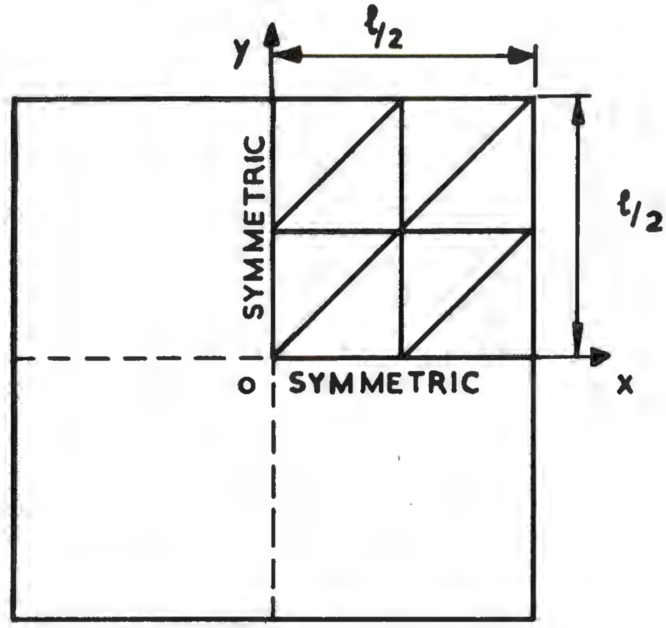
*Results communicated privately by Dr. R. G. Anderson.

Table 2

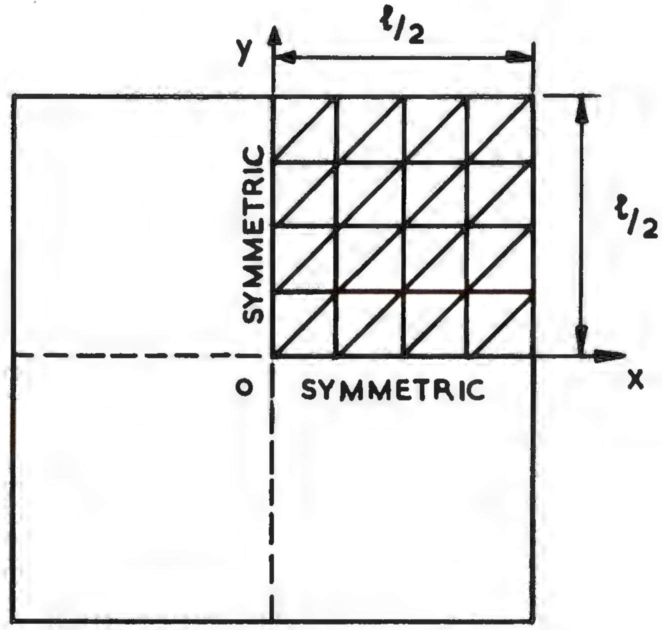
CRITICAL LOADS OF CLAMPED SQUARE PLATE UNDER UNIFORM PLANE STRESSES

Loading case	Mesh size	Triangular elements			Rectangular elements			Classical solution
		Present analysis	Clough and Felippa	Anderson *et al.*	Kapur and Hartz	Dawe	Carson and Newton	
Uniform uni-axial compression	4 × 4	10.990		9.30	9.284	10.147		10.07
	8 × 8	10.252		-	9.782	10.065		
Uniform bi-axial compression	4 × 4	5.602	5.625	5.043	4.975		5.327	5.31
	8 × 8	5.356	5.399	5.119	5.160		5.305	
Uniform shear	4 × 4	17.382					15.043	14.71
	8 × 8	15.172					-	

Multiplier = $\pi^2 D/\ell^2$ for all cases.

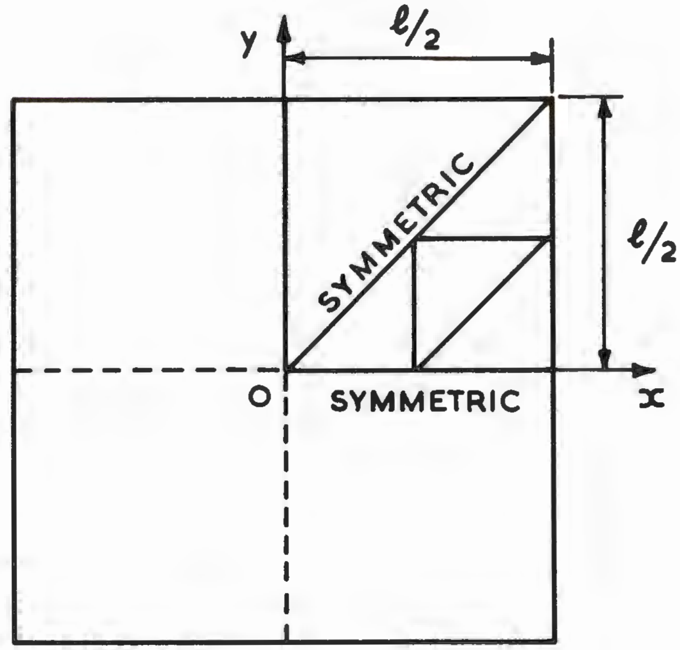


(4×4) MESH

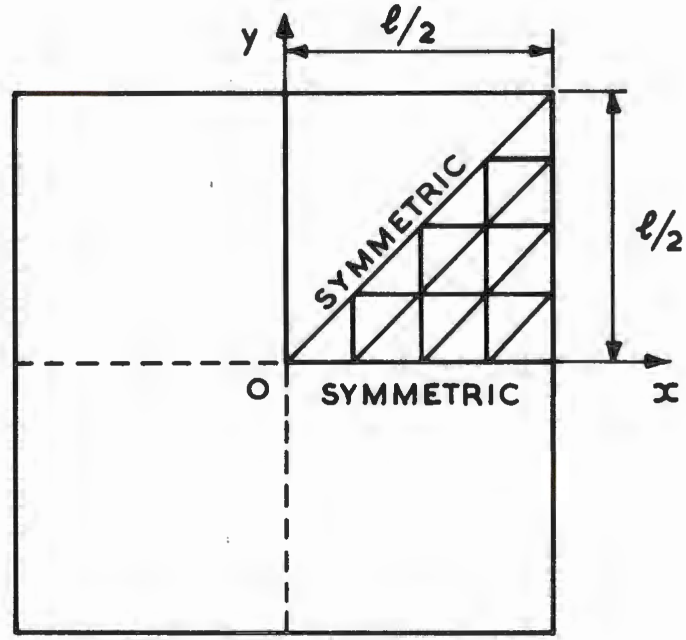


(8×8) MESH

FIG.2a MESH ARRANGEMENT FOR UNIFORM UNI-AXIAL COMPRESSION OF A SQUARE PLATE.

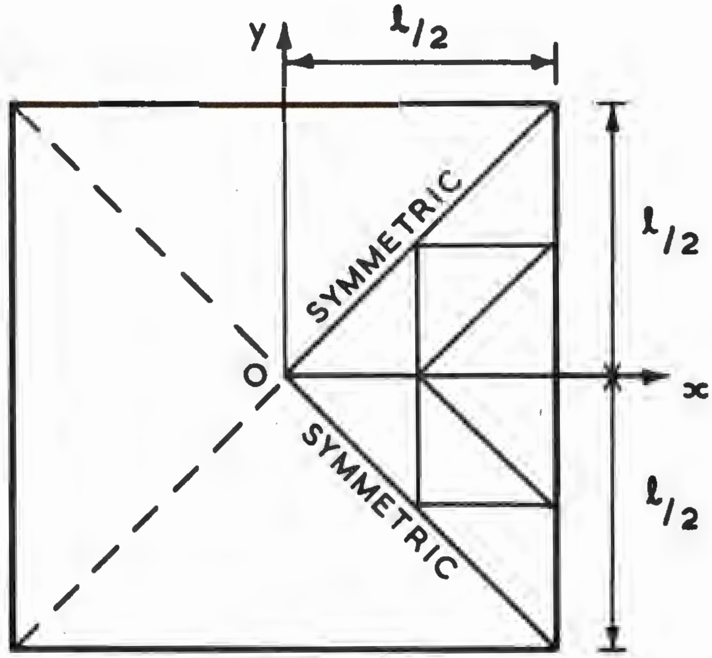


(4x4) MESH

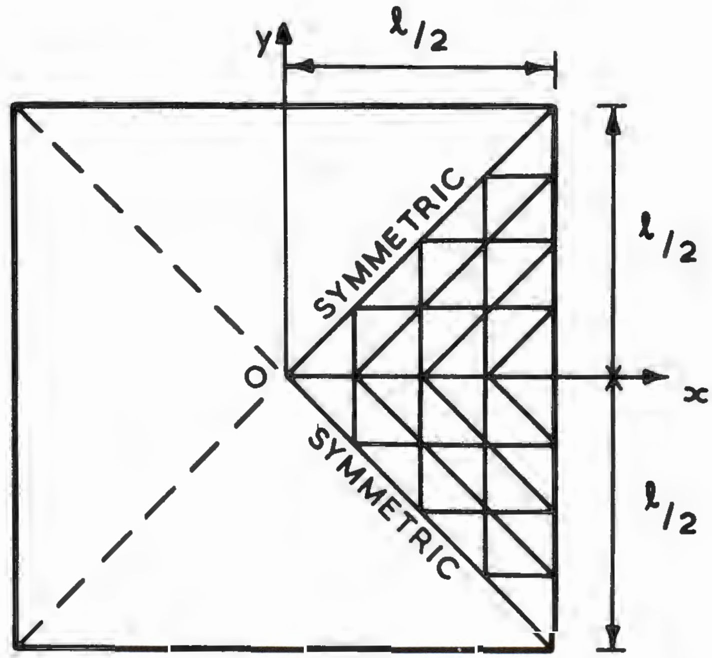


(8x8) MESH

FIG. 2b MESH ARRANGEMENT FOR UNIFORM BI-AXIAL COMPRESSION OF A SQUARE PLATE



(4 x 4) MESH



(8 x 8) MESH

FIG. 2c MESH ARRANGEMENT FOR UNIFORM SHEAR OF A SQUARE PLATE

6. CONCLUSIONS

It is shown that a variational principle for stresses and displacements can be used to derive the governing equations of the plate buckling problem. Solutions to simple numerical examples, using a finite element scheme based on this mixed variational principle, indicate that very satisfactory accuracy can be achieved for the critical states of in-plane loading. In the solution of the final eigenvalue problem use is made of a recently published algorithm which takes account of the banded nature of the final matrix formulation.

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Appendix A

CALCULATION OF THE GEOMETRIC STIFFNESS MATRIX USING A NON-COMPATIBLE DISPLACEMENT FIELD

In section 3 it is found convenient to calculate the element geometric stiffness matrix using the simple displacement field proposed by Bazeley *et al.*[6] which satisfies continuity of only the transverse displacement on the element boundaries. It is shown below that the use of this approximation still represents a valid application of a mixed variational principle.

Let the non-compatible displacement field inside an element be denoted by $\bar{w}$, while on the element boundary a second displacement field, denoted by w, is defined which satisfies the kinematic requirements of both displacement and normal slope continuity. The contribution J_e of an element e to the total functional J is now assumed to be given by (using the summation convention with $i,j = 1,2$)

$$J_e = \iint_\Delta \left\{ \tfrac{1}{2} C_{ijk\ell} M_{ij} M_{k\ell} + \frac{\partial^2 M_{ij}}{\partial x_i \partial x_j} \bar{w} \right\} dA - \sum_{C_\Delta} [R_N w_N]$$

$$- \oint_{C_\Delta} V_n w ds + \oint_{C_\Delta} M_n \frac{\partial w}{\partial n} ds - \frac{\lambda}{2} \iint_\Delta N_{ij} \frac{\partial \bar{w}}{\partial x_i} \frac{\partial \bar{w}}{\partial x_j} dA \quad , \qquad \text{(A-1)}$$

with subsidiary condition $w = \bar{w}$ on C_Δ. Considering variations in both the bending moment and displacement fields it is found that

$$\delta J_e = \iint_\Delta \left\{ \left(\frac{\partial^2 \bar{w}}{\partial x_i \partial x_j} + C_{ijk\ell} M_{k\ell} \right) \delta M_{ij} + \left(\frac{\partial^2 M_{ij}}{\partial x_i \partial x_j} + \lambda N_{ij} \frac{\partial^2 \bar{w}}{\partial x_i \partial x_j} \right) \delta \bar{w} \right\} dA$$

$$- \sum_{C_\Delta} [R_N \delta w_N] - \oint_{C_\Delta} \left(V_n + \lambda N_n \frac{\partial \bar{w}}{\partial n} + \lambda N_{ns} \frac{\partial \bar{w}}{\partial s} \right) \delta w ds$$

$$+ \oint_{C_\Delta} M_n \delta \left(\frac{\partial w}{\partial n} \right) ds + \oint_{C_\Delta} \left(\frac{\partial w}{\partial n} - \frac{\partial \bar{w}}{\partial n} \right) \delta M_n ds \quad . \qquad \text{(A-2)}$$

The stationary condition $\sum_e \delta J_e = 0$ thus provides the governing equations of the buckling problem and, in particular, the last term of equation (A-2) restores the condition of slope continuity $\frac{\partial w}{\partial n} = \frac{\partial \bar{w}}{\partial n}$ on the element boundary.

Appendix B

APPLICATION OF HOMOGENEOUS LINEAR CONSTRAINTS TO BANDED MATRIX EQUATIONS

A general and convenient procedure for applying non-homogeneous linear constraints to banded matrix equations is given by Morley[17]. This procedure can be simplified considerably for the plate buckling problem, which is formulated in terms of the banded global matrices $\bar{K}_0$ and $\bar{K}_1$, where the linear constraints correspond to the application of homogeneous kinematic boundary conditions given by equation (2-15).

It is required to solve the eigenvalue problem

$$\delta\bar{W}^T(\bar{K}_0 + \mu\bar{K}_1)\bar{W} = 0 \quad , \qquad \text{(B-1)}$$

where the variations $\delta\bar{W}$ are not arbitrary, but are subject to homogeneous linear constraints imposed by the kinematic boundary conditions

$$P\bar{W} = \Gamma\bar{W} \quad . \qquad \text{(B-2)}$$

It is always possible, by suitable row and column interchanges, to arrange that P is a non-singular tri-diagonal matrix (see Morley[17]), whereas Γ is a diagonal matrix with unit and zero elements on the leading diagonal; the zero elements corresponding to the boundary values. As an example, equation (B-2) is written out in full for typical homogeneous linear constraints, viz.

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & P_{i,i} & P_{i,i+1} & \\ & & & P_{i+1,i} & P_{i+1,i+1} & \\ & & & & & \ddots & \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ \bar{W}_i \\ \bar{W}_{i+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ \bar{W}_i \\ \bar{W}_{i+1} \\ \vdots \end{bmatrix} .$$

... (B-3)

Incorporation of equation (B-2) into equation (B-1) leads to the result

$$\delta\bar{W}^T(P^{-1}\Gamma)^T(\bar{K}_0 + \mu\bar{K}_1)(P^{-1}\Gamma)\bar{W} = 0 \quad , \qquad \text{(B-4)}$$

where $\delta\bar{W}$ is now an arbitrary quantity which may be deleted from equation (B-4). It is apparent, from the form of equation (B-3), that the calculation of the inverse matrix P^{-1} requires the inversion of submatrices of order 2 at most and that P^{-1} is, moreover, a tri-diagonal matrix. Consequently, the matrix operations of pre-multiplication by $(P^{-1}\Gamma)^T$ and post-multiplication by $(P^{-1}\Gamma)$ neither increase the bandwidth nor disturb the symmetry of the global matrices $\bar{K}_0$ and $\bar{K}_1$, and it is possible to take advantage of the banded symmetric form of equation (B-4) in computer programs. The effect of these matrix operations is to replace the elements of certain rows (and corresponding columns) in equation (B-1) by zero elements. This makes the matrix $\bar{K}_0$ singular. In the present work, in order that the eigenvalue algorithm of Peters and Wilkinson[14] may be used, the matrix $\bar{K}_0$ is made positive definite by substituting a unit element in the leading diagonal position of a zero row (or column); an identical substitution is also made in $\bar{K}_1$. All the eigenvalues of equation (B-4) which correspond to the homogeneous linear constraints of equation (B-2) now take the values $\mu = -1$. Thus, in our plate buckling problem, only the positive eigenvalues of equation (B-4) correspond to the critical intensities of applied in-plane loading. These are determined as described in section 4.