GENERAL SESSION IV

RESPONSE TO VIBRATION AND NOISE OF STRUCTURES WITH IDEALIZED PROPERTIES

Chairman : L. E. Goodman

DETERMINATE AND RANDOM EXCITATION OF LUMPED-LINEAR STRUCTURES

by

R. Plunkett

General Engineering Laboratory
General Electric Company
Schenectady, New York

I. INTRODUCTION

The basic mathematics for dealing with linear multi-degree of freedom systems has been known for more than a century[1]*; the problem was that the arithmetic has been too voluminous for practical computation. Until very recently, most of the emphasis in practical computation has been directed toward approximate or iteration methods for finding a few of the lowest frequencies and mode shapes. Among these are: Rayleigh, Rayleigh-Ritz, Holzer, Stodola, Myklstad-Prohl, Lanczos and related methods. When analog computers first came on the scene, an effort was made to use them as analogs of the primitive mechanical system[2]. While such an approach has been successful in some cases, the major problem has been in getting circuit elements sufficiently precise or computing accuracies sufficiently high to ensure stability. A similar approach was first used with digital computers; the major effort was devoted to coding the old familiar hand methods for computer operations. More recently we have come to realize that hand methods are not necessarily well suited to digital computers and vice-versa. By a stroke of poetic justice, a method that was devised by Jacobi more than a century ago and discarded as being too laborious for hand computation, is ideal for machine computation[3]. Using this method it is now practical to find all the mode shapes and frequencies of large multi-degree of freedom systems. Using classical transformations, these results in the frequency domain may be used to find responses in the time domain.

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* Numbers in parentheses refer to Bibliography at end of paper.

II. LINEAR MULTI-DEGREE OF FREEDOM SYSTEMS

In matrix form, the governing dynamic equations of a damped, passive, linear multi-degree of freedom system are(3):

$$M\ddot{x} + C\dot{x} + Kx = fe^{j\omega t} \qquad \text{II-1}^*$$

In these equations, x is the column vector whose elements are the displacement of the masses in the directions of the chosen geometrical coordinates; M is a square matrix whose elements are the rigid masses corresponding to those same coordinates. If there is no mass coupling, M is a diagonal matrix; in any case, M is symmetric, $M^T = M$. C is the matrix of viscous damping coefficients; we will assume that there are no gyroscopic terms so that $C^T = C$. K is the square, symmetric matrix of spring constants. f is the column matrix whose elements are complex numbers which are the amplitude and phase of the externally applied forces which are isochronous of circular frequency ω. From this, it is obvious that the particular solution for x is isochronous of frequency ω and each term has a common element $e^{j\omega t}$; the real part has the phase of $\cos\omega t$ and the imaginary that of $\sin\omega t$. The general solution is the solution of the homogenous equation:

$$M\ddot{x} + C\dot{x} + Kx = 0 \qquad \text{II-2}$$

As usual, we let:

$$x(t) = xe^{\alpha t} \qquad \text{II-3}$$

then

$$(\alpha^2 M + \alpha C + K)x = 0 \qquad \text{II-4}$$

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* A capital letter will indicate a square matrix (except for Q which is a column vector), a small letter a column vector or a scalar, and T superscript(^{T}) a transposed matrix, i.e. if the element of A is a_{ij}, the element of A^T is a_{ji}. I is the identity matrix whose element is δ_{ij}, (kroneker delta) and A^{-1} is the inverse of A; i.e. $A^{-1}A = AA^{-1} = I$.

If this is an n-degree of freedom system the governing determinant:

$$\| \alpha^2 M + \alpha C + K \| = 0 \qquad \text{II-5}$$

is reducible to a 2 n^{th} order polynomial which has 2n roots. Some of these roots are real and others occur as conjugate complex pairs since all the elements are real; it may be shown that the real portion of all roots must be negative(1). The eigenvectors corresponding to real roots must be real and are the shapes that decay exponentially and monotonically without oscillation. Those that correspond to complex roots are complex conjugates with negative real parts so that they vary timewise as damped exponentials. The eigenvectors corresponding to them are also complex conjugates with the imaginary part in quadrature to the real part.

In many cases the C matrix is small in comparison with the other two; i.e. αC is small compared to the larger of $\alpha^2 M$ or K. If we ignore C, we have the homogeneous equation for the natural frequencies and mode shapes of the undamped system:

$$| \alpha^2 M + K | \, x = 0 \qquad \text{II-6}$$

There are n eigenvalues of the form $\omega_r = \pm j\alpha_r$

$$| -\omega_r^2 M + K | \; x_r = 0 \qquad \text{II-7}$$

Let X be the square matrix whose columns are the x_r, then:

$$-\Omega^2 MX + KX = 0 \qquad \text{II-8}$$

where Ω^2 is the diagonal matrix of the ω_n^2. Premultiply by X^T

$$X^T KX = X^T \Omega^2 MX \qquad \text{II-9}$$

Transpose:

$$X^T K^T X = X^T M^T \Omega^{2T} X \qquad \text{II-10}$$

but $K^T = K$, $M^T = M$ and $\Omega^{2T} = \Omega^2$(diagonal) so:

$$X^T KX = X^T M \Omega^2 X \qquad \text{II-11}$$

Subtracting equation II-11 from equation II-9

$$X^T \Omega^2 MX - X^T M \Omega^2 X = 0 \qquad \text{II-12}$$

or $$X^T (\Omega^2 M - M \Omega^2) X = 0$$

The on-diagonal terms of the bracket are zero, therefore, the off-diagonal terms of $X^T MX$ must vanish (the singular case of two or more elements of Ω^2 being equal may also be handled the same way by a suitable constraint on the corresponding columns of X). The values of x_r may be scaled by any value desired so that the on-diagonal terms may be made equal to 1, therefore

$$X^T MX = I \qquad \text{II-13}$$

or from equation II-9

$$X^T KX = \Omega^2 \qquad \text{II-14}$$

The solution to equation II-1 takes the form:

$$-\omega^2 Mx + j\omega Cx + Kx = f \qquad \text{II-15}$$

If we now let

$$x = Xq$$

and substitute into equation II-15:

$$-\omega^2 MXq + j\omega CXq + KXq = f \qquad \text{II-16}$$

Premultiply by X^T

$$\left| -\omega^2 X^T MX + j\omega X^T CX + X^T KX \right| q = X^T f = Q \qquad \text{II-17}$$

or

$$\left| -\omega^2 I + j\omega X^T CX + \Omega^2 \right| q = Q \qquad \text{II-18}$$

If the off-diagonal terms of X^TCX are small in comparison to the on-diagonal terms, then X^TCX may be approximated by a diagonal matrix. In that case the square matrix which premultiplies q is a diagonal matrix and its inverse is the diagonal matrix of the reciprocals of its terms. Thus

$$q = \left|-\omega^2 I + j\omega X^T C X + \Omega^2\right|^{-1} Q \qquad \text{II-19}$$

is easy to form. But $x = Xq$ so that:

$$x = X\left|-\omega^2 I + j\omega X^T C X + \Omega^2\right|^{-1} Q \qquad \text{II-20}$$

which may be written in the form:

$$x_i = \sum_{r=1}^{n} \frac{x_{i,r} \sum_{j=1}^{n} x_{j,r} f_j}{(\omega_r^2 - \omega^2) + j\omega \tilde{C}_r} \qquad \text{II-21}$$

where:

$$\tilde{C}_r = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,r} C_{ij} x_{j,r}$$

and $x_{i,r}$ is the i^{th} element of the r^{th} vector of X subject to the normalizing condition of equation II-13.

Another possibility would be to solve equation II-15, equation II-18 or equation II-20 directly for each value of ω.

Any of the above procedures is valid for deterministic values of f, i.e. where f is either a pure sinusoid or is a known function of time that may be expanded into a series of sinusoids by a Fourier analysis. A somewhat different approach must be used if the time variation of f is known only in a statistical fashion. It is obvious, however, that the necessary information is contained in equation II-21 which can be substituted into a Fourier Integral.

III. FREQUENCIES AND MODE SHAPES

It is clear that the usefulness of the above exposition depends very heavily on being able to find the mode shapes and frequencies, i.e. X and Ω^2. For many years efforts have been based on getting good approximations to the lowest frequencies and associated mode shapes in the fond hope that

equation II-21 could be truncated after the first few terms and still give a good approximation for those forcing frequencies appreciably lower than the highest ω_n considered. In general, this expectation is valid, but the number of modes which must be considered can become excessive.

The earliest method was the Rayleigh-Ritz[4] which used the stationary property of frequencies as a function of amplitude parameters to determine the best set of frequencies for an assumed set of shapes. This gives better frequencies than it does mode shapes and so is not a very good procedure for determining stresses.

Another approach has been that of iteration[3]. If we rewrite equation II-8 to read:

$$K^{-1}Mx = \frac{1}{\omega^2}\,x \qquad \text{III-1}$$

and substitute an assumed vector x on the left side, it may be shown that resubstituting the right side into the left will converge eventually to the eigenvector corresponding to the lowest natural frequency which will also be found thereby. There are various methods for finding higher modes, all of them based on purging the lower modes by means of equation II-13. The major drawback of this approach is the loss of significant figures. Normally about one significant figure will be lost per mode shape determined.

A method due to Lanczos[5] avoids this problem by building up a polynomial whose roots approximates the lowest frequencies of the system. This approach is based on generating vectors which are mutually orthogonal in the sense of equation II-13 and using these to generate a characteristic polynomial. The roots of a p^{th} order polynomial of this type will give acceptable values for the lowest $p/2$ eigenvalues of the governing system. The eigenvectors may be found from the roots and the original equations in the usual fashion by inversion of the reduced matrix.

Those systems in which the succession of masses is primarily sequential, such as beams or shafts (a topological tree), may be handled by the Holzer-Myklstad-Prohl method[6]. This involves using a trial frequency, assuming a displacement, and if necessary a slope, at one end and calculating the elastic distortion due to the inertia forces while proceeding from mass to mass along the length. This calculation determines an unbalanced force at the far end which is that force required to maintain the assumed displacement at the chosen frequency. This result is used as a guide to choose new frequencies in succession until one is found for which the terminal force is acceptably small. The associated frequency is a natural frequency and the shape is the mode shape. The method can be used with branched systems but not with those

that form a complex two or more dimensional grid. The other major drawback is the ever-present danger of missing some frequencies completely. This approach is very widely used because one need only cover the frequency range of interest and because one may calculate a few frequencies with an accuracy associated with the complete analysis of a much larger system.

Within the last few years, it has become apparent that the above methods, and other related ones, have been developed within the limitations of slide-rules and desk-calculators. With the advent of high speed digital calculators, the tendency has been to program these same methods with no change in fundamental approach. There has been a renewed effort lately to take advantage of their ability to do long sequences of simple arithmetic operations with limited storage. Out of this has come a method of successive rotation of orthogonal coordinates originally due to Jacobi[3]. Although known for about 100 years, it has not been useful because of the large number of numerical manipulations necessary; the calculations consist, however, of exactly the type well suited for digital calculators. The basic approach is that of pre and post multiplying by a unitary rotation matrix so as to make selected off-diagonal terms zero; while each such rotation will make previously reduced off-diagonal terms non-zero, the sequence will ultimately converge to an almost diagonal matrix whose elements are the eigenvalues of the primitive system. The product of these several rotations generates the eigenvectors. As an example of the time involved, the 48 eigenvalues and eigenvectors of a 48 x 48 matrix, were found on a 704 computer by eight full iterations in six minutes including input and tape read-out; using the enclosure theorem of Kohn-Kato[3] all 48 eigenvalues were shown to be valid to at least five significant figures with eight carried by the machine. The ratio of maximum to minimum eigenvalues, in this case, was 10^6 with one group of four repeated roots. The eigenvectors were orthogonal, in the sense of equation II-13, to within three or four units in the eighth digit. Other solutions have been found with three zero roots so that the matrix was three-fold singular; matrices as large as 112 square have been successfully handled.

IV. RESPONSE CURVES

In this presentation, we are actually not interested in mode shapes and frequencies, but in the response of the system to a set of applied forces. For the moment, let us consider the response to a harmonic force; the response to a transient or non-deterministic force may be found by means of some sort of a Fourier Transform from the frequency domain to the time domain.

Equation II-21 is the response in terms of the mode shapes and natural frequencies with certain assumptions as to the damping. Another approach would be to solve equation II-15 directly. If we write

$$-\omega^2 Mx + j\omega Cx + Kx = f \qquad \text{II-15 bis}$$

we may write

$$Ax = f \qquad \text{IV-1}$$

where $A = U + jV$, $x = y + jz$

Then:

$$\begin{vmatrix} U & -V \\ V & U \end{vmatrix} \begin{vmatrix} y \\ z \end{vmatrix} = \begin{vmatrix} f \\ 0 \end{vmatrix} \qquad \text{IV-2}$$

and one need only invert:

$$\begin{vmatrix} U & -V \\ V & U \end{vmatrix}$$

to find **x**. One might think that this would be preferable to solving for the eigenvalues and frequencies and substituting into equation II-21; actually inverting to solve equation IV-2 is almost as much work as solving the eigenvalue problem and only gives the value at one frequency. The actual calculation involved in equation II-21 is very simple and may be done on a smaller machine such as the 650. The results should be programmed to plot directly from the output onto an x-y recorder, preferably with log scales, since otherwise there is a tremendous amount of numbers to handle. Equation II-21 need only be calculated over the frequency range of interest and one might think that only the mode shapes whose frequencies fall in that same range need be considered; experience has shown, however, that such a truncation leads to serious error at very low frequency ($\ll\omega_1$) and at the anti-resonances (x_i very small).

It has been pointed out before that the Holzer-Myklstad-Prohl method is not subject to such limitations for those problems where it is applicable. One can calculate the damped case with little more difficulty than the undamped case and where it applies it is the method of choice. One might still wish to use equation II-21 to investigate the effect of changing certain parameters, such as bearing damping, since this can be

done without recomputing the whole problem, but there is no great saving if the changed parameter is mass or stiffness.

At our own Laboratory, we are working on marrying the best features of digital and analog computers for this problem. One might think of setting up equation II-1 directly on an analog computer, and this has been done in some cases(2,7). The problem arises when one uses an active feedback electronic differential analyzer and the system damping is small in one or more modes. In general, the matrix K will be a fairly full one, particularly so since one must usually reduce the number of degrees of freedom to get the problem on the machine; for the standard machine, one needs at least three stabilized amplifiers per degree of freedom. In this case there will be n^2 connections among 3n amplifiers; unless the damping is very large in every possible loop, at least one of them will have negative damping due to inaccurate phase shift and the problem will be unstable as set up. The solution is to put equation II-18 on the computer in the form:

$$\ddot{q} + X^T C X \dot{q} + \Omega^2 q = Q \qquad \text{IV-3}$$

The only coupling among the q loops is through the X^TCX matrix. The individual undamped loops:

$$\ddot{q}_i + \Omega_i^2 q = Q_i \qquad \text{IV-4}$$

will be automatically damped by an amount which depends on how good the resistors and capacitors are; even so, this damping will usually be less than the natural damping of a mechanical system. The addition of the X^TCX coupling will always be stabilizing so long as the C are small. The exact comparison is between k_{ij} and ωC_{ij}; if $\omega C_{ij} >> k_{ij}$, this has the effect of clamping x_i and x_j together and the vagaries of the feedback loops may make the reduced system unstable. Since such large damping is not a desirable state of affairs, it does not seem to be an important limitation. While this use of an electronic differential analyzer for the forced vibration problem has not been fully explored, it appears to have merit for investigating the effect of transients, complex or random excitation, non-linear damping of otherwise linear systems, or parametric changes in C values to minimize the response.

V. TRANSIENT AND RANDOM RESPONSE

It has been pointed out in many previous publications that the dynamic description of the system is contained in equation II-21 as well as it is in equation II-1. We have already discussed how this problem may be set up on an analog computer; in this section we shall examine its numerical treatment.

Let $H_{ij}(\omega)$ be the response at point i due to a unit sinusoidal force at j; $j\omega H_{ij}(\omega)$ is the transfer mobility of i with respect to j subject to the natural constraints of the system. $H_{ij}(\omega)$ is found from equation II-21 by letting $f_j = 1$; in passing $H_{ij} = H_{ji}$.

If this system is subjected to a transient force at j, $f_j(t)$, the response may be found by means of Fourier transforms and convolution integrals[8]. If we find the Fourier transform of $f_j(t)$ either numerically or by reed shock gages, then

$$F_j(\omega) = \int_{-\infty}^{\infty} f_j(t) e^{-j\omega t} dt \qquad \text{V-1}$$

Then

$$x_i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_j(\omega) H_{ij}(\omega) e^{j\omega t} d\omega \qquad \text{V-2}$$

or alternatively:

$$x_i(t) = \int_{0}^{\infty} h_{ij}(\tau) f_j(t-\tau) d\tau \qquad \text{V-3}$$

where

$$h_{ij}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{ij}(\omega) e^{j\omega t} d\omega \qquad \text{V-4}$$

is the time response of the system to a unit pulse and equation V-3 is the convolution integral. The effects of transient forces applied at several points may be found by superposition.

The response to random excitation may be found in a similar fashion from the mobility[9]. For example, if $S_f(\omega)$ is the power spectral density of the force, then:

$$S_x(\omega) = |H(\omega)|^2 S_f(\omega)$$

is the power spectral density of the displacement. In this case $H(\omega)$ is the appropriate transfer mobility between the point of application of the force and the point where the response is wanted. Powell indicates the necessary conditions on the force for it to be considered in this unified fashion[10]. A particular result of some interest is the mean square response to a uniform spectrum force; this is

$$x^2 = \frac{S_o}{2\pi} \int_o^\infty |H(\omega)|^2 \, d\omega$$

where S_o is the amplitude of the spectral density, i.e. in pounds^2 per cps.

For our purposes here, it is sufficient to point out that there are relatively simple methods of handling random excitation and transient response problems once the mobility, equation II-21, is found. These methods may be numerical, graphical, or by analog computer but are all approached in the same way.

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