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THE DETERMINATION OF TEMPERATURE, STRESSES AND DEFLECTIONS IN TWO-DIMENSIONAL THERMO-ELASTIC PROBLEMS

BY

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FOREWORD

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ABSTRACT

An analytical successive-approximation method for the solution of linear partial differential equations is presented first in general terms, and then applied to the solution of two-dimensional heat and thermal stress problems. The method is applicable when solutions are desired for bars or plates, i.e. for bodies with one dimension small compared to the others. The final expressions given by this procedure for example for the stress σ consist of a number of terms ($\sigma = \sum \sigma_i$), where the term σ_i is proportional to the quantity $[\beta^{(2i-1)} \partial^{(2i-1)} T/(\partial x)^{2i-1}]$; β is the ratio of height to length of the bar, and x measures the distance along the span. The number of terms required is thus small for thermal loadings varying smoothly along the span, and for thin bars. Similar results are obtained for the temperature and the deflections. Explicit formulas for the calculation of stresses and deflections are given. The validity of the Bernoulli-Euler hypothesis of beam-theroy is examined. Illustrative examples are presented for all the above developments. The use of the method in problems in which the material properties are functions of the temperature is outlined.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

Carl Reichert for
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LIST OF SYMBOLS

$2c$	height of beam
c_i	constant
D_{XT}, D_{XYT}, D_{YT}	partial differential operators
E	Young's modulus
F, F_i	functions of x, y ant t
f	function defined by eqs. (9c) or (10c)
G	shear modulus
H	function defined in eq. (26)
H_i	functions defined in eqs. (A7a)
I	moment of inertia
i, j	indices
k_i, k'_j, k''_j	constants
$2L$	length of bar
M	bending moment
n	order of differential equation
q	function of ξ
Q, Q^*	heat flow
Q_o	constant with dimensions of heat flow
S'_i	$= \sum_{n=1,3,5}^{\infty} [(-1)^n e^{-n^2 \pi^2 t_1}/(n^i \pi^i)]$
S''_i	$= \sum_{n=2,4,6...}^{\infty} [e^{-n^2 \pi^2 t_1}/(n^i \pi^i)]$
S_i	$= S''_i + S'_i = \sum_{n=1}^{\infty} [(-1)^n e^{-n^2 \pi^2 t_1}/(n^i \pi^i)]$
t	time
t_1	$= \kappa t/(4c^2)$

T, T^*	temperature
T_i	functions such that $T = \sum T_i$
u, v	displacements in x and y directions, respectively
x, y	coordinates
X_i	function of x and t
Y_i	function of y and t
z	dependent variable
z_i	functions such that $z = \sum z_i$
α	coefficient of thermal expansion; exponent in eq. (13)
β	$= c/L$
η	$= y/(2c)$
η_1	$= \eta + (1/2)$
κ	thermal diffusivity
ν	Poisson's ratio
ξ	$= x/(2c)$
σ_x, σ_y	normal stresses in directions indicated
τ	shear stress
φ	stress function
φ_i	functions such that $\varphi = \sum \varphi_i$

INTRODUCTION AND OUTLINE OF REPORT

The advent of high-speed flight has increased the importance of thermal effects in aircraft structures and has consequently accentuated the need for practical method for the calculation of the distribution of temperature, stress and deflection in many types of bodies under a variety of conditions. The present paper considers the effect of heating on a thin rectangular bar according to the two-dimensional theory of elasticity, and deals particularly with thermal loadings varying smoothly along the length of the bar. The analysis presented here may also be used in the case of plates, if the temperature distribution is independent of one of the coordinates in the plane of the plate. Such plates are often encountered, for instance, with the standard sheet-and-stringer type of construction exemplified in Fig. 6, which may be taken to represent a reinforced wing covering. As heat is applied, non-uniformity in the sheet temperature distribution in the x-direction is introduced because of the heat drawn off by the stringers. Incidentally, questions concerning the buckling of the sheet are not taken up in this report.

The proposed method of solution takes as its starting point the formulas for temperature, stresses and deflections valid for the case of heating independent of the span, and proceeds to evaluate successive approximations; the required number of these depends on the "smoothness" of the heating along the span, i.e. on the degree of its variation from uniformity. The precise meaning of the term "smooth" will be clarified during the detailed development of the method, but, roughly speaking, it may be said to refer to spanwise boundary conditions which are expressible either as polynomials or as rapidly converging power series. Also, the thinner the bar, the better the convergence. As has been just indicated, only two-dimensional systems are discussed in this paper; it will be seen, however, that the method is general enough to hold for three-dimensional systems as well. Work is in fact at present in progress in which the analysis described here is being extended to a general type of thin-walled section.

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The body of this report is divided into seven sections and three appendices, as follows:

1) General Considerations. The method is developed here in general terms without reference to any specific physical problem, and the approximations required to meet the boundary conditions are discussed.

2) Temperature Distribution-General Equations. The above method is applied to the specific case of the heat equation, and is illustrated in the following section, namely:

3) Temperature Distribution - Example, with the aid of Appendix 1.

4) Stress Distribution - General Equations. The same method is again employed, and leads to explicit formulas for the various stresses in terms of the temperature. Some of the results of this section are derived in detail in Appendix 2. The effect of visco-elastic or otherwise inelastic behavior of the material is not considered in this work, though it is known(1) to be often of overriding importance. It has been shown, however, that in many problems inelastic solutions are most easily obtained by starting with the elastic solution of the same problem(2,3). Furthermore, the influence of viscosity is small if the temperature level is not too high, or if high temperature are of short duration. The section which follows, namely:

5) Stress Distribution - Example illustrates the use of the formulas developed in section 4 by continuing the problem of section 3. Some of the details of the calculations are given in Appendix 3.

6) The Calculation of Deflections is taken up next; explicit formulas for the longitudinal and transverse displacements are given. Considerations on the applicability of the Bernoulli-Euler hypothesis of beam theory to thermal stress problems are included. It may be noted in this connection that theories for the thermal bending of plates have been derived under this assumption by several authors(12,13), though some doubts as to its validity have arisen(14). The present approach makes it possible to resolve this question within the scope of the solutions considered.

7) Stress Distribution - Variable Properties. This section shows that the general method of section 1 can be used also if the mechanical properties of the material (assumed constant in all previous sections) vary with the temperature.

1. GENERAL CONSIDERATIONS

Let the solution be required of a differential equation of any order n of the form

$$(D_{YT} + D_{XYT} + D_{XT})z = F(x, y, t) \qquad (1)$$

where z is the dependent variable, and where the linear partial differential operator $(D_{YT} + D_{XYT} + D_{XT})$ has been divided in three parts chosen so that no derivatives with respect to x appear in D_{YT}, none with respect to y in D_{XT}, and only terms containing both such derivatives appear in D_{XYT}. Let the solution be desired for the range $y_1(x) \leq y \leq y_2(x)$; $x_1(y) \leq x \leq x_2(y)$; $t \geq 0$; and assume the boundary conditions to be of the form:

$$\sum_{j=0}^{n-1} k_j \frac{\partial^j z}{\partial y^j} = X_{1,2}(x,t) \text{ when } y = y_{1,2}(x) \qquad (1a)$$

$$\sum_{j=0}^{n-1} k'_j \frac{\partial^j z}{\partial x^j} = Y_{1,2}(y,t) \text{ when } x = x_{1,2}(y) \qquad (1b)$$

$$\sum_{j=0}^{n-1} k''_j \frac{\partial^j z}{\partial t^j} = f_o(x,y) \text{ when } t = 0 \qquad (1c)$$

where the k's are constants. A method of solution of (1) under these boundary conditions will now be outlined; it will be useful when solutions are desired for a thin strip, i.e. when $(y_2 - y_1) \ll (x_2 - x_1)$. Let the solution of eq. (1) be written in the form

$$z(x, y, t) = \sum_{i=0}^{\infty} z_i(x, y, t) \qquad (2)$$

where the z_i functions are to be determined as indicated below. Substitution into eq. (1) gives

$$(D_{YT} + D_{XYT} + D_{XT})(z_o) + D_{YT}(z_1) + \sum_{i=2}^{\infty} [D_{YT}(z_i) + (D_{XYT} + D_{XT})z_{i-1}] = F(x,y,t) \qquad (2a)$$

It is therefore clear that eq. (2a), and therefore also eq. (1), will be satisfied if the z_i functions are chosen so as to satisfy the following relations:

$$D_{XT}\, z_1 = F_1\,(x, y, t) \tag{3a}$$

$$D_{YT}\,(z_i) = -\,(D_{XYT} + D_{XT})z_{i-1} + F_i(x, y, t) \qquad i = 2, 3, 4 \ldots \tag{3b}$$

$$(D_{YT} + D_{XYT} + D_{XT})\, z_o = 0 \tag{3c}$$

where the F_i functions can be chosen in any convenient manner provided of course that

$$F = \sum_{i=1}^{\infty} F_i \tag{3d}$$

The manner in which the portions z_i of the variable z are defined is of course not unique; for example, if they are chosen so as to satisfy the following relations

$$D_{YT}\, z_1 = F_1\,(x, y, t) \tag{3e}$$

$$D_{YT}\, z_2 = D_{XYT}\, z_1 + F_2(x, y, t) \tag{3f}$$

$$D_{YT}\, z_i = -D_{XYT}\, z_{i-1} - D_{XT}\, z_{i-2} + F_i\,(x, y, t) \qquad i = 3, 4, 5, \ldots \tag{3g}$$

$$(D_{YT} + D_{XYT} + D_{XT})\, z_o = 0 \tag{3h}$$

it is easily seen that eq. (1) will again be satisfied. Other manners of defining the z_i functions can of course be devised; the individual functions z_i will be different in each case, but their sum will be the same. In the solutions presented in the following sections the functions z_i will be defined in such a manner that eac contains derivatives with respect to x of a single order only; thus in the solution of the heat equation (section 2) the first of the methods just described will be employed, while for the determination of the stresses (section 4) the second was use In any case, both the system (3a)-(3c) or the system (3e)-(3h) have certain characteristics in common, which will now be discussed.

Solution of (3a) and (3b), or of (3e), (3f) and (3g) is evidently simpler than that of eq. (1) since no derivatives with respect to x appear in the left-hand sides of these equations, and therefore the role of this independent variable has been reduced to that of a parameter. However, it is possible to satisfy with each function $z_i (i \neq 0)$ individually only conditions (1a) and (1c); the function z_o must then be chosen so as to satisfy homogeneous boundary conditions in place of (1a) and (1c), as well as

$$\sum_{j=0}^{n-1} k'_j \frac{\partial^j z_o}{\partial x^j} = Y_{1,2}(y, t) - \sum_{i=1}^{\infty} \sum_{j=0}^{n-1} k'_j \frac{\partial^j z_i}{\partial x^j} \text{ at } x = x_{1,2}(y) \qquad (3i)$$

Clearly the exact determination of z_o is, in general, as difficult as the solution of the original problem; fortunately it is possible in the case of a thin-strip to obtain an approximate and at the same time adequate expression for this quantity.

Consider first an infinite narrow strip, that is the case of $x_{1,2} = \mp \infty$. Conditions (1b) would then ordinarily be replaced by the requirement that z and some of its derivatives remain finite at $x = \pm\infty$, and will be automatically satisfied if this requirement is consistent with the behavior at infinity of the functions Γ, X_1 and X_2. In this case then $z_o = 0$.

In the case of a long but finite strip, one may write $(y_2 - y_1) \ll (x_2 - x_1)$, and an approximation to z_o is the solution of

$$D_{XT} z_o = 0 \qquad (4)$$

With

$$\sum_{j=0}^{n-1} k'_j \frac{\partial^j z_o}{\partial x^j} = \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} [Y_{1,2} - \sum_{i=1}^{\infty} \sum_{j=0}^{n-1} k'_j \frac{\partial^j z_i}{\partial x^j}] \, dy \quad \text{when } x = x_{1,2} \qquad (4a)$$

The resulting expression for z_o will then be independent of y. The error introduced by this approximation can be found by solving Eqs. (3c) or (3h) under boundary condition (3i). Comparison of eqs. (3i) and (4a) shows that the average correction to the quantity on the left-hand side of these equations is zero. The actual general determination of this error is again a task of the same difficulty as the solution of the original equation; but in the problems of temperature and stress distribution the error is negligible except in short distances [of the order of $(y_2 - y_1)$] at each end of the strip. In the stress problem this follows from Saint-Venant's principle[4]; in the heat problem it is intuitively clear that an analogous principle will hold. As an example, the temperature T in a semi-infinite strip of height $(y_2 - y_1) = (2c)$ due to a suddenly applied "self-equilibrating" heat input $Q = Q_o \cos(\pi y/c)$ at $x = 0$ can be verified to be

$$[T\kappa/(2\ c\ Q)] = -\int_0^{t_1} (1/\pi t_1)^{1/2} \exp\left[-4\pi^2 t_1 - x^2/(16c^2 t_1)\right] d\,t_1 \qquad (5)$$

or, in a form more suitable for calculation:

$$(2\pi\,\kappa\,T/c\,Q) = e^{-\pi\,x/c}\left\{1 + (2/\sqrt{\pi})\ \mathrm{erf}\left[2\pi\sqrt{t_1} - x/(4c\sqrt{t_1})\right]\right\} - e^{\pi\,x/c}\left\{1 - (2/\sqrt{\pi})\ \mathrm{erf}\left[2\pi\sqrt{t_1} + x/(4c\sqrt{t_1})\right]\right\} \qquad (5a)$$

where the non-dimensional time $t_1 = \kappa t/4c^2$, and where κ is the thermal diffusivity of the material. A plot of these expressions, shown in Fig. 1, clearly indicates that only a limited region is appreciably affected by the heat applied.

The solution of Eq. (1) has thus been reduced to that of (3a), (3b) [or (3e), (3f) and (3g)] and (4), together with the appropriate boundary conditions; application of these equations in the specific problems of interest here is considered and illustrated below with reference to a rectangular region defined by

$$-c < y < c\ ; \quad -L < x < L; \ \beta = (c/L) \ll 1 \qquad (6)$$

2. TEMPERATURE DISTRIBUTION-GENERAL EQUATIONS

The two-dimensional heat equation is

$$\kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{\partial T}{\partial t} ; \qquad (7)$$

thus, in the notation of Eq. 1,

$$D_{YT} = \kappa \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t} ; \quad D_{XYT} = 0; \quad D_{XT} = \kappa \frac{\partial^2}{\partial x^2} ; \quad z = T; \quad F = 0 \qquad (7a)$$

The solution is then obtained from the previous equations as

$$T = T_o + T_1 + T_2 + \ldots . \qquad (8)$$

where $T_i (i \geq 1)$ is obtained from eqs. (3a) and (3b), namely

$$\kappa \frac{\partial^2 T_1}{\partial y^2} - \frac{\partial T_1}{\partial t} = 0 \qquad (8a)$$

$$\kappa \frac{\partial^2 T_i}{\partial y^2} - \frac{\partial T_i}{\partial t} = -\kappa \frac{\partial^2 T_{i-1}}{\partial x^2} \qquad i \geq 2 \qquad (8b)$$

The boundary and initial conditions under which these equations must be solved vary a great deal from problem to problem, and it therefore appears impractical to conside any general type of solution. Solutions of eq. (8a) under a variety of conditions are however available in the literature(5).

After all the T_i quantities except T_o have been found, the latter is obtained fro

$$\kappa \frac{\partial^2 T_o}{\partial x^2} = \frac{\partial T_o}{\partial t} \qquad (9)$$

together with the appropriate boundary condition given by (4a). If, for example, the ends $x = \pm L$ are insulated, the pertinent solution is

$$T_o = (1/\beta) \int_0^{t_1} \int_{-(1/2)}^{(1/2)} \left\{ f(t_1 \beta^2, -\xi\beta)\left(\sum_{i=1}^{\infty} T_i'\right)_{\xi = -(1/2\beta)} - f(t_1 \beta^2, +\xi\beta)\left(\sum_{i=1}^{\infty} T_i'\right)_{\xi = (1/2\beta)} \right\} d\eta \, dt_1 \qquad (9a)$$

where

$$\beta = c/L \; ; \; t_1 = k\, t/(4c^2); \; \xi = x/2c \; ; \; \eta = y/(2c) \qquad (9b)$$

and where dots and primes indicate differentiation with respect to t_1 and ξ, respectively. The quantity f is the temperature in the rectangular region (6) caused by a suddenly applied constant unit heat irput at x = L, all other edges being insulated, and is given by the expression[5]:

$$f(t_1\beta^2; \xi\beta) = t_1\beta^2 + \frac{1}{2}(\xi\beta + \frac{1}{2})^2 - \frac{1}{6} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} e^{-(n^2\pi^2 t_1\beta^2)} \cos n\pi(\xi\beta + \frac{1}{2}) \qquad (9c)$$

Relations analogous to (9a-c) could be written for other boundary conditions. Use of the above equations in the solution of a specific problem is illustrated in the example which follows.

3. TEMPERATURE DETERMINATION - EXAMPLE

Let it be desired to find the temperature distribution in the rectangular bar defined by eq. (6), when heat is applied to the side y = +c at the rate

$$Q = Q_o\, q(\xi) \tag{10}$$

all other edges being insulated. For the time being this heat application is assumed to be suddenly started at $t_1 = 0$, and to be continued thereafter at a constant rate; other types of time variation are considered later. The boundary conditions for eq. (8a) are then

$$K \frac{\partial T_1}{\partial y} = \begin{cases} Q & \text{when } y = c \\ 0 & \text{when } y = -c \end{cases} \tag{10a}$$

and for eqs. (8b)

$$\frac{\partial T_i}{\partial y} = 0 \quad \text{when } y = \pm c \qquad i \geq 2 \tag{10b}$$

The function T_1 is then readily found to be[5]

$$\frac{T_1}{2c\,Q_o} = q(\xi) f(t_1, \eta) = q(\xi) \left\{ t_1 + \frac{1}{2}\left(\eta + \frac{1}{2}\right)^2 - \frac{1}{6} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n^2\pi^2 t_1}}{n^2} \cos n\pi\left(\eta + \frac{1}{2}\right) \right\} \tag{10c}$$

where the function f is that defined by eq. (9c). The other functions T_i can then be determined from eqs. (8b) and (10b) in a straightforward manner (see Appendix 1); the result is

$$\frac{T_2 K}{2\,c\,Q_o} = q''(\xi) \left\{ \frac{t_1^2}{2} - \frac{(\eta + \frac{1}{2})^4}{24} + \frac{(\eta + \frac{1}{2})^2}{12} - \frac{7}{360} - \right.$$

$$\left. - \frac{2}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n^2\pi^2 t_1}}{n^4} (1 + n^2 \pi^2 t_1) \cos n\pi \left(\eta + \frac{1}{2}\right) \right\} \tag{10d}$$

$$\frac{T_3 \; K}{2\,c\,Q_o} = q''''(\xi) \left\{ \frac{t_1^3}{6} + \frac{(\eta+\frac{1}{2})^6}{720} - \frac{(\eta+\frac{1}{2})^4}{144} + \frac{7(\eta+\frac{1}{2})^2}{720} - \frac{31}{15120} - \right. \tag{10e}$$
$$\left. - \frac{2}{\pi^6} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n^2\pi^2 t_1}}{n^6} [1 + n^2\pi^2 t_1 + n^4\pi^4 t_1^2/2] \cos n\pi \left(\eta + \frac{1}{2}\right) \right\}$$

and so forth. Note that primes indicate differentiation with respect to ξ. It may be noticed that, in general, T_i is proportional to the 2(i-1)th derivative of q with respect to ξ ; thus it is clear that, if q is a polynomial, only a finite number of T_i functions will exist and so no question of convergence of the series of eq. (8) arises. Thus T_1 represents the entire solution if q (ξ) is polynomial of the first degree at most; $(T_1 + T_2)$ if q (ξ) is a polynomial of the third degree at most; and so forth. For q functions other than polynomials no such simple results can be obtained; but certainly if the function is sufficiently "smooth" (i.e. if its higher derivatives decrease sufficiently fast) convergence will be obtained.

Good convergence will also result for thin bars, even if the derivatives with respect x or (x/2L) do not decrease very rapidly. Replace in fact the derivatives with respect to ξ appearing in eqs. (10d) and (10e) by the corresponding derivatives with respect to (x/2L); then T_i is found to be proportional to

$$[\partial^{2(i-1)} q/(\partial\xi)^{2(i-1)}] = \beta^{2(i-1)}[\partial^{2(i-1)} q/\partial (x/2L)^{2(i-1)}] \tag{10f}$$

which will be smaller the smaller the aspect ratio β.

In order to proceed with the determination T_o it is impractical to leave q (ξ) in general terms; therefore a specific example was chosen, namely

$$q(\xi) = 1 + (q_o)\beta(\xi + 2\beta\xi^2); \qquad |\xi| \leq (1/2\beta) \tag{11}$$

where q_o is a constant. Then eq. (9a) yields, after some manipulation,

$$\frac{T_o K}{2 C Q_o} = q''(\xi)\left\{ -\frac{t_1^2}{2} + \frac{t_1}{4\beta^2}\left(\frac{1}{6} - \xi\beta - 2\xi^2\beta^2\right) - \frac{1}{96\beta^4}\left(\frac{7}{60} - 3\xi\beta - 2\xi^2\beta^2 + 4\xi^3\beta^3 + 4\xi^4\beta^4\right) - \right.$$

$$\left. - \frac{1}{2\beta^4\pi^4} \sum_{n=1}^{\infty} \frac{[3(-1)^n + 1]}{n^4} e^{-n^2\pi^2\beta^2 t_1} \cos\left[n\pi\left(\xi\beta + \frac{1}{2}\right)\right]\right\} \qquad (12)$$

In this case q is a polynomial of the second degree and therefore $T_i = 0$ for i larger than 2.

The variation with time of the temperature as given by the above formulas at $X = 0$ and $y = c$ is plotted in non-dimensional form in Fig. 2. The values $q_o = -(1/2)$ and $\beta = 1/5$ were chosen for the numerical example. Also shown on the same figure is the temperature at the same point due to a heat input Q^* varying with time according to the law

$$Q^*(\xi, t_1) = Q_o\, q(\xi)\, t_1 e^{-\alpha t_1} \qquad (13)$$

where α ix a non-dimensional parameter chosen here as equal to 2.5. A plot of eq. (13) is shown in the inset of Fig. 2. The temperature T^* due to Q^* is, from Duhamel's formula(11),

$$T^* = \int_0^{t_1} T(t^*) \frac{\partial Q^*}{\partial t_1}(\xi, t^* - t_1)\, d t^* \qquad (13a)$$

where $T(t_1)$ is the temperature previously calculated for heating suddenly applied. The integration indicated in eq. (13a) was carried out numerically.

An example of the variation of the temperature along the span of the bar is presented in Fig. 3, in which a non-dimensional plot of temperature against ξ for $y = c$ and $t_{1=1} = .05$ may be found. Again $\beta = 1/5$ and $q_o = -1/2$. Some comments on these numerical results are given at the end of section 5.

4. STRESS DISTRIBUTION - GENERAL EQUATIONS

The governing equation of two-dimensional thermo-elasticity is[6]

$$\frac{\partial^4 \varphi}{\partial x^2} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\alpha ET) \qquad (14)$$

where α and E are the coefficient of thermal expansion and Young's modulus, respectively (both taken as constants in this section), and where the stresses are related to the stress function φ by the relations

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} \;;\; \sigma_y = \frac{\partial^2 \varphi}{\partial x^2} \;;\; \tau = -\frac{\partial^2 \varphi}{\partial x \partial y} \qquad (14a)$$

In the notation of Eq. (1)

$$D_{YT} = \frac{\partial^4}{\partial y^4} \;;\; D_{XYT} = 2\frac{\partial^4}{\partial x^2 \partial y^2} \;;\; D_{XT} = \frac{\partial^4}{\partial x^4} \;;\; z = \varphi ;\; F = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\alpha ET) \qquad (14b)$$

The problem is the determination of φ for a given temperature distribution, under boundary conditions stipulating that

$$\sigma_x = \tau = 0 \quad \text{when} \quad x = \pm L \qquad (14c)$$

$$\sigma_y = \tau = 0 \quad \text{when} \quad y = \pm c \qquad (14d)$$

The method previously described gives

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \dots \qquad (15)$$

where φ_i $(i \geq 1)$ is obtained from the following relations, written in accordance with eqs. (3e), (3f) and (3g):

$$\frac{\partial^4 \varphi_1}{\partial y^4} = -\frac{\partial^2}{\partial y^2}(\alpha ET) = F_1(x, y, t) \qquad (15a)$$

$$\frac{\partial^4 \varphi_2}{\partial y^4} = -\frac{\partial^2(\alpha ET)}{\partial x^2} - 2\frac{\partial^4 \varphi_1}{\partial x^2 \partial y^2} \qquad (15b)$$

$$\frac{\partial^4 \varphi_i}{\partial y^4} = -\frac{\partial^4 \varphi_{i-2}}{\partial x^4} - 2\frac{\partial^4 \varphi_{i-1}}{\partial x^2 \partial y^2} \quad i \geq 3 \qquad (15c)$$

These equations must be solved under the boundary conditions (14d), which may be reduced to the form

$$\varphi_i = \frac{\partial \varphi_i}{\partial y} = 0 \quad \text{when } y = \pm c \qquad (15d)$$

by writing the stresses in terms of φ as in eq. (14a), and integrating the resulting expression with respect to x. The details of the solution of eqs. (15) are indicated in Appendix 2; the final results are listed in table I, together with the corresponding expressions for the stresses, for i = 1, 2 and 3. It may be noticed that (similarly to the solution previously presented for the heat equation) φ_i is proportional to the 2(i-1)th derivative of the temperature with respect to x; thus the discussion given following eqs. (10) applies to the stress distribution problem as well. In particular, it may be noticed that the expression for the stress σ_x given by φ_1 is identical with that given by Timoshenko[4], and is exact for temperatures either constant or varying linearly with the distance along the span; in the latter case of course the stress σ_x is accompanied by a shear stress τ.

The formulas given in Table I are analogous to those given by Seewald[7] to indicate the required correction to the elementary $\sigma_x = My/I$ formula in cases in which the bending moment M is not a linear function of the distance along the span. It is shown in the section on the calculation of deflections that the expression for φ_1 can be obtained on the basis of the Bernoulli-Euler hypothesis, and is therefore entirely analogous to the formulas of ordinary beam theory.

The expressions given in Table I can be simplified considerably if the function F of eq. (14b) is independent of y, as is for example true when the temperature is that given by eq. (9a), or its special case eq. (12). The expressions for φ_i and its stresses, pertaining to this case, have been collected in Table II. Note that in this case $\varphi_1 = 0$.

Formulas have been given in the tables just referred to for the determination of all φ_i's with the exception of φ_o. Combination of eqs. (14c) and (15d), however, shows that

$$\int_{-c}^{c} \sigma_x \, dy = \int_{-c}^{c} \tau \, dy = \int_{-c}^{c} \sigma_x y \, dy = 0 \qquad (16)$$

and that therefore the stresses of tables I and II are self-equilibrating. The validity of eqs. (16) may of course be checked by direct substitution of the expressions contained in these tables. It follows that the right-hand side of eqs. (4a) is zero; hence the value

$$\varphi_o = 0 \qquad (16a)$$

will satisfy all the necessary relations within the approximation afforded by use of Saint-Venant's principle. Tables I and II therefore represent the desired solution to the problem of stress distribution.

5. STRESS DISTRIBUTION - EXAMPLE

The stresses arising in a rectangular bar under the heat input of eq. (10) can now be calculated by direct use of the formulas of tables I and II. The temperature is given by eqs. (10c-e); direct substitution gives (for example the stress σ_x) the expression

$$[\sigma_x K/(2c\alpha EQ_o)] = q(\xi)\left\{\frac{1}{24} - \frac{\eta^2}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} e^{-n^2\pi^2 t_1}\cos[n\pi(\eta+\tfrac{1}{2})] - \right.$$

$$\left. - 48\,\eta\, S_4' \right\} + q''(\xi)\left\{ t_1\left(\frac{1}{24} - \frac{\eta^2}{2}\right) + \frac{\eta^4}{12} - \frac{\eta^2}{24} + \frac{7}{2880} + \right.$$

$$+ S_4'\,[16\,\eta^3 - (\tfrac{12}{5} + 48 t_1)\,\eta] - 96\,\eta\, S_6' +$$

$$\left. + \frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-n^2\pi^2 t_1}}{n^4}(2 + n^2\pi^2 t_1)\cos n\pi(\eta+\tfrac{1}{2})]\right\} \qquad (17)$$

where

$$S_i' = \sum_{n=1,3,5\ldots}^{\infty}[e^{-n^2\pi^2 t_1}/(n^i\pi^i)]$$

Some details of the derivation of eq. (17) may be found in Appendix 3. To the above expression for the stress must now be added the stress due to the temperature T_o of eq. (12). Table II gives this additional stress as

$$[\sigma_x K/(2c\alpha EQ_o)] = T_o''\left\{\frac{1}{24} - \frac{\eta^2}{2}\right\} + T_o''''\left\{\frac{\eta^4}{12} - \frac{\eta^2}{24} + \frac{7}{2880}\right\} + \ldots \qquad (17a)$$

The expression of eq. (17) is exact if q is that given by eq. (11), while eq. (17a) contains even in this case an infinite number of terms. Numerical calculations showed however that the series contained in the latter equation is very rapidly convergent, and that in fact even the first term provides a reasonable approximation. Some numerical results are presented in Figs. 4 and 5 both for suddenly applied heat [eq. (10)], and for heat application following eq. (13).

Figs. 3 and 5 show the variation of temperature and stress, respectively, with the distance along the span; also shown in these figures is the variation along the span of the applied heat Q according to eq. (11). Inspection of the calculations from which these results were obtained shows that the first terms of the expressions for these quantities (i.e. T_1 and φ_1) are of paramount importance, while even the second term may be neglected. The second term is in fact proportional to Q", a quantity which is, in the present example, constant and small with respect to Q. If the applied heat variation with followed a polynomial of higher degree, the situation just described would not take place, but terms other than the first would play an important role. The quantity T_o is of course of importance even in the present problem.

6. CALCULATION OF DEFLECTIONS

The deflections of the bar considered in the previous developments are easily calculated from the stresses given in table I. The displacements in the x and y directions are respectively denoted by u and v and must satisfy the equations[4]

$$E \frac{\partial u}{\partial x} = \sigma_x - \nu \sigma_y + \alpha ET$$

$$E \frac{\partial v}{\partial y} = \sigma_y - \nu \sigma_x + \alpha ET \qquad (18)$$

$$E \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 2(1 + \nu)\tau$$

The desired result is easily obtained by substituting in (18) the expressions for the stresses in terms of the stress function of eq. (15), and integrating the first and second of eqs. (18) with respect to x and y respectively. The arbitrary functions (respectively of y and x) arising during this process are then adjusted so as to satisfy the last of (18). The final result is:

$$\frac{u}{\alpha} = \left\{ c_1 + c_2 y \right\} + \int_{-L}^{x} \left\{ \frac{1}{2c} \int_{-c}^{c} T\, dy + \frac{3y}{2c^3} \int_{-c}^{c} T\, y\, dy \right\} dx + \qquad (19a)$$

$$\frac{\partial}{\partial x} \left\{ (1 + \nu)\left[\int_{-c}^{y} T\, dy - \int_{-c}^{y} T\, y\, dy\right] - \frac{c}{2}\int_{-c}^{c} T\, dy \left[\frac{1}{6} + \frac{\nu}{2} + (1 + \nu)\frac{y}{c} + \right.\right.$$

$$\left. + \left(1 + \frac{\nu}{2}\right)\left(\frac{y}{c}\right)^2\right] + \frac{1}{2}\int_{-c}^{c} T\, y\, dy \left[1 + \nu + \left(\frac{21}{10} + \frac{3\nu}{2}\right)\left(\frac{y}{c}\right) - \left(1 + \frac{\nu}{2}\right)\left(\frac{y}{c}\right)^3\right] -$$

$$\left. - \frac{1}{4c}\int_{-c}^{c} T\, y^2\, dy - \frac{y}{4c^3}\int_{-c}^{c} T\, y^3\, dy \right\} + \ldots\ldots$$

and

$$\frac{v}{\alpha} = \left\{ c_3 - c_2 x \right\} + \int_{-L}^{x} \int_{-L}^{x} \left[\frac{3}{2c^3} \int_{-c}^{c} T\, y\, dy \right] d\,x\,d\,x +$$

$$+ \left\{ (1 + \nu) \int_{-c}^{y} T\, dy - \frac{1}{2} \int_{-c}^{c} T\, dy \; [1 + \nu + \nu \frac{y}{c}] + \right. \tag{19b}$$

$$\left. + \frac{1}{c} \int_{-c}^{c} T\, y\, dy \left[\frac{7}{20} + \frac{3\nu}{4} - \frac{3\nu}{4} (\frac{y}{c})^2 \right] + \frac{1}{4c^3} \int_{-c}^{c} T\, y^3\, dy \right\} + \ldots .$$

Both the right-hand sides of eqs. (19a) and (19b) contain successively higher derivatives of the temperature with respect to x, and therefore will converge under conditions similar to those mentioned following eq. (10e). The first bracket in each of the above expressions depends on the constants c_1, c_2 and c_3, which are to be determined from the boundary conditions of the problem. The next bracket represents a contribution due entirely to the function φ_1 of eq. (15). This portion, in the case of the displacement u, is linear in y, and therefore follows the Bernoulli-Euler assumption that sections plane before bending remain plane after bending, and may be seen to be rigorously valid only for temperature distributions which are independent of the distance along the span. Formula (19a) can also be used to estimate the error in any elementary theory developed on the basis of the Bernoulli-Euler hypothesis. The corresponding term for the displacement v depends upon the integral $\int_{-c}^{c} T\, y\, dy$, which plays in fact a role similar to that of the bending moment in elementary beam theory. It is interesting in this connection to calculate the average curvature; it is

$$\frac{d^2 v_{av}}{d\,x^2} = \left\{ \frac{3}{2c^3} \int_{-c}^{c} \alpha\, T\, y\, dy \right\} + \left\{ - \frac{3}{20c} \int_{-c}^{c} \alpha \frac{\partial^2 T}{\partial x^2}\, y\, dy + \frac{1}{4c^3} \int_{-c}^{c} \alpha \frac{\partial^2 T}{\partial x^2}\, y^3\, dy \right\} + \ldots . \tag{20}$$

This equation is analogous to one developed by von Karman[8] to show that correction required to the elementary beam formula

$$M = EI \frac{d^2 v}{dx^2} \tag{20}$$

if the bending moment is not a linear function of x. Note that in the present case, the moment of inertia I of a thin rectangle of unit width is $I = 2\,c^3/3$. In view of the discussion following eq. (10e), one may conclude that the first term in the expression for the deflection (or for the stresses) will become more and more accurate as the loading becomes smoother, or as the bar becomes thinner,

7. STRESS DISTRIBUTION - VARIABLE PROPERTIES

In all the above developments it has been assumed that the mechanical propertie of the material are independent of the temperature. This was not found to be the case however in an experimental investigation conducted by the N.A.C.A.[9, 10], whic in fact led to the following approximate formulas for the variation of the modulus of elasticity E, the shear modulus G, and the coefficient of thermal expansion α for 75S-T6 aluminum alloy.

$$
\begin{aligned}
E &= E_o - (1470T + 15.1T^2) \\
G &= G_o - (1440T + 4.8T^2) \\
\alpha &= \alpha_o + 3.52 \times 10^{-9} T
\end{aligned}
\qquad (21)
$$

In these expressions the subscript o indicates the properties at T = 0. The values $E_o = 10.5 \times 10^6$ psi, $G_o = 4.0 \times 10^6$ psi and $\alpha_o = 12.52 \times 10^{-6}$/ F are suggested for the above material[10].

Examination of eqs. (21) shows a considerable variation of material properties with temperature. The variation of α provides no new difficulties, since it merely change the right-hand side of the governing eq. (14) and therefore corresponds to a differen applied temperature function. The variation of E and G (and therefore ν) on the other hand requires replacement of eq. (14) by

$$
\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\varphi}{E\partial y^2} - \frac{\nu}{E}\frac{\partial^2\varphi}{\partial x^2}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\varphi}{E\partial x^2} - \frac{\nu}{E}\frac{\partial^2\varphi}{\partial y^2}\right) + 2\frac{\partial^2}{\partial x\partial y}\left(\frac{1+\nu}{E}\frac{\partial^2\varphi}{\partial x\partial y}\right) =
$$

$$
= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\alpha T) \qquad (22)
$$

where φ is again defined by eq. (14a). This equation, though still linear, has variable coefficients and is therefore more difficult to handle than (14). Considerable doubt has been recently thrown, however, on the correctness of the variation of E and G as obtained above, and the opinion has been expressed[1] that a more critical interpretation of the experimental evidence might reveal that the moduli of elasticity vary in fact very little, the apparent variation being due to a neglect of the visco-elastic effects. It is clear that some attention should be given to this question, before extensive investigations of possible methods of solution of (22) are undertaken. This problem can of course not be dealt with here; nevertheless it is shown below how the general method employed throughout this paper is applicable to eq. (22) as well. It will then remain for future research to determine the type of problem in which such a solution is of importance.

If the left-hand side of eq. (22) is expanded, it may be put in the form of eq. (1); after a number of transformations, the result is:

$$D_{YT} = \frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2}{\partial y^2}\right)$$

$$D_{XT} = \frac{\partial^2}{\partial x^2}\left(\frac{1}{E}\frac{\partial^2}{\partial x^2}\right)$$

$$D_{XYT} = 2\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{E}\frac{\partial^2}{\partial x \partial y}\right) - \left[\frac{\partial^2(\nu/E)}{\partial x^2}\frac{\partial^2}{\partial y^2} + \frac{\partial^2(\nu/E)}{\partial y^2}\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2(\nu/E)}{\partial x \partial y}\frac{\partial^2}{\partial x \partial y}\right]$$

$$z = \varphi \; ; \quad F = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\alpha T)$$

The boundary conditions are again those of eqs. (14c) and (14d). Eq. (15) of course still holds, but (15a) becomes

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2 \varphi_1}{\partial y^2}\right) = -\frac{\partial^2(\alpha T)}{\partial y^2} \tag{24}$$

Integration gives

$$\varphi_1 = \int_{-c}^{y}\int_{-c}^{y}(-\alpha E T + c_1 E y + c_2 E)\, dy\, dy \tag{24a}$$

where the values of c_1 and c_2 required to satisfy the boundary conditions are

$$c_1 = \frac{\left(\int_{-c}^{c} E y\, dy\right)\left(\int_{-c}^{c}(\alpha E T) dy\right) - \left(\int_{-c}^{c} E\, dy\right)\left(\int_{-c}^{c}\alpha E T y\, dy\right)}{\left(\int_{-c}^{c} E y\, dy\right)^2 - \left(\int_{-c}^{c} E\, dy\right)\left(\int_{-c}^{c} E y^2\, dy\right)} \tag{24b}$$

$$c_2 = \frac{\left(\int_{-c}^{c} E\,y\,dy\right)\left(\int_{-c}^{c} \alpha E T\,y\,dy\right) - \left(\int_{-c}^{c} E\,y^2\,dy\right)\left(\int_{-c}^{c} \alpha E T\,dy\right)}{\left(\int_{-c}^{c} E\,y\,dy\right)^2 - \left(\int_{-c}^{c} E\,dy\right)\left(\int_{-c}^{c} E\,y^2\,dy\right)} \tag{24c}$$

Note that c_1 and c_2 are functions of x if T(and therefore E) vary along the span. With these quantities the first terms of the expressions for the stresses may be derived. In particular, it is interesting to note that the first term of the stress σ_x is

$$\sigma_{x_1} = -\alpha E T + c_1 E y + c_2 E \tag{24d}$$

and has therefore the same form as that given in table I and could be obtained on the basis of the Bernoulli-Euler assumption. Of course the constants c_1 and c_2 are different here from those of table I, though they become identical (and this is true of all corresponding expressions in the developments for variable and constant properties) if E is taken to be a constant. Examination of table II shows that if

$$\alpha T = a + b\,y \tag{25}$$

where a and b are constants (α is also constant in table II, though not now) the result

$$\varphi_1 = \sigma_{x_1} = 0 \tag{25a}$$

holds. Direct substitution of (25) into (24b) and (24c) shows that in this case $c_1 = b$ and $c_2 = a$ and therefore (25a) again results.

In the case of constant properties, the expression for σ_x analogous to (24a) was found to be exact for temperature distributions with $\partial^2(\alpha T)/\partial x^2 = 0$. In the present case, however, it will be seen φ_2 contains terms of the type $\partial(\alpha T)/\partial x$; hence $\varphi_i (i \geq 2)$ will vanish, and eq. (24a) will be exact, only for temperature distributions independent of the span.

The next step in the solution is the determination of φ_2 from eq. (3b), which in this case is

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{E}\frac{\partial^2 \varphi_2}{\partial y^2}\right) = -(D_{XYT} + D_{XT}(\varphi_1)) - \frac{\partial^2(\alpha T)}{\partial x^2} = H, \text{ say} \qquad (26)$$

Integration gives the second term in the expression for the stress σ_x as

$$x_2 = \frac{\partial^2 \varphi_2}{\partial y^2} = E\int_{-c}^{y}\int_{-c}^{y} H\, dy\, dy + c_3 E y + c_4 E \qquad (26a)$$

where the quantities c_3 and c_4 are found from eqs. (15d) or (16). After a number of integrations by parts, the results are as follows:

$$-4c^3 c_3 = 2c^3 \int_{-c}^{c} H\, dy - 3c^2 \int_{-c}^{c} H\, y\, dy + \int_{-c}^{c} H\, y^3\, dy \qquad (26b)$$

$$-4c\, c_4 = c^2 \int_{-c}^{c} H\, dy - 2 \int_{-c}^{c} H\, y\, dy + \int_{-c}^{c} H\, y^2\, dy \qquad (26c)$$

It appears impractical to calculate these quantities in general terms; it is clear however that in any particular problem the right-hand side of eq. (26)(i.e. H)is a known function of x, y and t, and that the necessary integrations can be performed without difficulty.

The determination of φ_2 itself will require the double integration of (26a); a procedure similar to the one just employed will lead to all subsequent functions φ_i of eq. (15). It may be noticed that by means of this method the necessity for solving explicitly the original equation with variable coefficients is eliminated. Finally, it will be noticed that relations (16) are still valid, and hence again $\varphi_o = 0$ as in eq. (16a).

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APPENDIX 1

Derivation of Eqs. (10d) and (10e)

The function T_1 of Eq. (10c) is given in ref. 5, p. 104; substitution of T_1 into eq. (8b) gives

$$\frac{k}{2 c Q_a} \left(\frac{\partial^2 T_2}{\partial \eta^2} - \frac{\partial T_2}{\partial t_1}\right) = - q''(\xi) \Big\{ t_1 + \frac{1}{2}(\eta + \frac{1}{2})^2 - \frac{1}{6} - \tag{A1}$$

$$- \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n^2\pi^2 t_1}}{u^2} \cos n\pi (\eta + \frac{1}{2}) \Big\}$$

in the notation of eq. (9b). The solution of this equation is the function T_2 given in eq. (10d) and may be obtained as follows. Assume T_2 to be of the form

$$\frac{k T_2}{2 c Q_c q''(\xi)} = c_1 t_1^2 + c_2 (\eta + \frac{1}{2})^4 + c_3 (\eta + \frac{1}{2})^2 + c_4 +$$

$$+ \frac{2}{\pi^2} \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2 t_1} (c_5 + c_6 t_1) \cos n\pi (\eta + \frac{1}{2}) \tag{A2}$$

which is of the same general nature as the right-hand side of (A1) and satisfies the boundary conditions (10b) at $y = -c$ (i.e. at $\eta = -\frac{1}{2}$). The similar condition at $\eta = \frac{1}{2}$ will be satisfied if

$$4 c_2 + 2 c_3 = 0 \tag{A2a}$$

Substitution of expression (A2) into (A1) and comparison of like terms on both sides of the resulting relation shows that eq. (A1) is compatible with (A2a), and furthermore yields the values of all constants $c_1, \ldots, c_6$. The final result is the temperatu function of eq. (10d). There remains now to verify the fact that this solution satisfies also the initial condition, namely

$$T_2 = 0 \quad \text{when } t_1 = 0 \tag{A3}$$

Substitution of the expression for T_2 into this equation gives

$$-\frac{\eta_1^4}{24}+\frac{\eta_1^2}{12}-\frac{7}{360}-\frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos n\pi\eta_1 = 0 \qquad \text{(A3a)}$$

where $\eta_1 = \eta + (1/2)$. The summation contained in (A3a) can be evaluated with the aid of the following well-known Fourier expansions:

$$\eta_1^2 = \frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi\eta_1$$

$$\text{(A3b)}$$

$$\eta_1^4 = \frac{1}{5}+\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi\eta_1-\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos n\pi\eta_1$$

Substitution into (A3a) shows that the latter is satisfied. It may be remarked that the solution of eq. (A1) can be readily determined with the aid of the Laplace transform technique, which allows the all boundary and initial conditions are immediately incorporated in the solution and it is therefore a priori known that they will be satisfied.

The derivation of eq. (10e) is entirely analogous to that just outlined and is therefore not presented in detail. It may be remarked, however, that the final check of the condition

$$T_3 = 0 \quad \text{when } t_1 = 0 \qquad \text{(A4)}$$

requires use of the series

$$\eta_1^6 = \frac{1}{7}+12\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^2\pi^2}-\frac{20}{n^4\pi^4}+\frac{120}{n^6\pi^6}\right)\cos n\pi\eta_1 \qquad \text{(A5)}$$

in addition to those given by eqs. (A3b).

APPENDIX 2

Derivation of the Expressions of Table I

The function φ_1 is the solution of eq. (15a) and may therefore be written as

$$\varphi_1 = -\int_{-c}^{y}\int_{-c}^{y} \alpha\, E\, T\, dy\, dy + c_0 + c_1 y + \frac{c_2 y^2}{2} + \frac{c_3 y^3}{6} \qquad (A6)$$

where the constants $c_0, \ldots, c_3$ are to be determined from the boundary conditions (15d). Substitution in (15d) gives

$$c_0 + c_1 c + c_2 (c^2/2) + c_3 (c^3/6) = \int_{-c}^{c}\int_{-c}^{y} \alpha\, E\, T\, dy\, dy$$

$$c_0 - c_1 c + c_2 (c^2/2) - c_3 (c^3/6) = 0$$

$$c_1 + c_2 c + c_3 (c^2/2) = \int_{-c}^{y} E\, T\, dy \qquad (A6a)$$

$$c_1 - c_2 c + c_3 (c^2/2) = 0$$

The solution of these four simultaneous equations is:

$$(c_0/\alpha E) = -\frac{c}{4}\int_{-c}^{c} T\, dy + \frac{1}{2}\int_{-c}^{c}\int_{-c}^{y} T\, dy\, dy = \frac{c}{4}\int_{-c}^{c} T\, dy - \frac{1}{2}\int_{-c}^{c} T\, y\, dy \qquad (A6b)$$

$$(c_1/\alpha E) = -\frac{1}{4}\int_{-c}^{c} T\, dy + \frac{3}{4c}\int_{-c}^{c}\int_{-c}^{y} T\, dy\, dy = \frac{1}{2}\int_{-c}^{c} T\, dy - \frac{3}{4c}\int_{-c}^{c} T\, y\, dy$$

$$(c_2/\alpha E) = \frac{1}{2}\int_{-c}^{c} T\, dy$$

$$(c_3/\alpha E) = \frac{3}{2c^2}\int_{-c}^{c} T\, dy - \frac{3}{2c^3}\int_{-c}^{c}\int_{-c}^{y} T\, dy\, dy = \frac{3}{2c^3}\int_{-c}^{c} T\, y\, dy$$

It may be noticed that for all the above constants with the exception of c_2 two expressions are given; the first of these is obtained by direct solution of the set (A6a), and the second is derived from the first with the aid of the identity

$$\int_{-c}^{c}\int_{-c}^{y} T\,dy\,dy = c\int_{-c}^{c} T\,dy - \int_{-c}^{c} T\,y\,dy \qquad \text{(A6c)}$$

which is easily verified by means of integration by parts. If now the double integral appearing in eq. (A6) is expanded with the aid of the relation

$$\int_{-c}^{y}\int_{-c}^{y} T\,dy\,dy = y\int_{-c}^{y} T\,dy - \int_{-c}^{y} T\,y\,dy \qquad \text{(A6d)}$$

of which (A6c) is a special case, and if the values found for the constants are substituted into eq. (A6) the entry of Table I for the function φ_1 will result.

The derivation of the other φ_i functions appearing in Table I can be performed in a manner entirely analogous to that just outlined. Consider for example the functions φ_2 and φ_3; they are given by eqs. (15b) and (15c) respectively as

$$\varphi_2 = -\int_{-c}^{y}\int_{-c}^{y} H_2\,dy\,dy + c_4 + c_5\,y + \frac{c_6 y^2}{2} + \frac{c_7 y^3}{6}$$

$$\varphi_3 = -\int_{-c}^{y}\int_{-c}^{y} H_3\,dy\,dy + c_8 + c_9\,y + \frac{c_{10} y^2}{2} + \frac{c_{11} y^3}{6} \qquad \text{(A7)}$$

where

$$H_2 = \int_{-c}^{y}\int_{-c}^{y} \frac{\partial^2(\alpha E T)}{\partial x^2}\,dy\,dy + 2\,\frac{\partial^2 \varphi_1}{\partial x^2}$$

$$H_3 = \int_{-c}^{y}\int_{-c}^{y} \frac{\partial^4 \varphi_1}{\partial x^4}\,dy\,dy + 2\,\frac{\partial^2 \varphi_2}{\partial x^2} \qquad \text{(A7a)}$$

Similar equations can be written for all functions φ_i. Eqs. (A7) are of the same form as (A6); therefore, the constants $c_4, \ldots, c_{11}$ can be immediately found from the appropriate one of eqs. (A6a) by substituting either H_2 or H_3 for $\alpha E T$. The expression for φ_i can then be obtained in terms of H_i directly from the expression for φ_1 given in Table I and is then

$$\varphi_i = -y\int_{-c}^{y} H_i\,dy + \int_{-c}^{y} H_i\,y\,dy + \frac{c}{4}\left[1 + 2\left(\frac{y}{c}\right) + \left(\frac{y}{c}\right)^2\right]\int_{-c}^{c} H_i\,dy - \frac{1}{4}\left[2 + 3\left(\frac{y}{c}\right) - \left(\frac{y}{c}\right)^3\right]\int_{-c}^{c} H_i\,y\,dy \qquad (A8)$$

The substitution $H_1 = \alpha E T$ gives φ_1 as previously found; use of H_2, H_3 and so forth gives the corresponding functions φ_2, φ_3, etc. It may be noticed that all H_i with the exception of H_1 contain terms in which a double integration is indicated; when substituted into eq. (A8) these terms will require a triple integration. These may be eliminated by successive integration by parts in a manner similar to that employed in eq. (A6d) in the case of double integration; the desired formula is in this case

$$\int_{-c}^{y}\int_{-c}^{y}\int_{-c}^{y} H_i\,dy\,dy\,dy = \frac{y^2}{2}\int_{-c}^{y} H_i\,dy - y\int_{-c}^{y} H_i\,y\,dy + \frac{1}{2}\int_{-c}^{y} H_i\,y^2\,dy \qquad (A)$$

The final form for φ_i which is most convenient for purposes of derivation of the expressions in Table I is then obtained with the aid of (A6c), (A6d), (A8), (A9) and identity of the form

$$\int_{-c}^{y} H_i\,dy = \int_{-c}^{y}\int_{-c}^{y}\int_{-c}^{y} \frac{\partial^2 H_i}{\partial y^2}\,dy\,dy\,dy \qquad (A9a)$$

The final result is

$$\varphi_i = \frac{y^3}{6}\int_{-c}^{y} \frac{\partial^2 H_i}{\partial y^2}\,dy - \frac{y^2}{2}\int_{-c}^{y} y\,\frac{\partial^2 H_i}{\partial y^2}\,dy + \frac{y}{2}\int_{-c}^{y} y^2\,\frac{\partial^2 H_i}{\partial y^2}\,dy - \frac{1}{6}\int_{-c}^{y} y^3\,\frac{\partial^2 H_i}{\partial y^2}\,d$$

$$+ \frac{c^3}{24}\left[1 - 3\left(\frac{y}{c}\right)^2 - 2\left(\frac{y}{c}\right)^3\right]\int_{-c}^{c} \frac{\partial^2 H_i}{\partial y^2}\,dy + \frac{c^2}{8}\left[\left(\frac{y}{c}\right) + 2\left(\frac{y}{c}\right)^2 + \left(\frac{y}{c}\right)^3\right]\int_{-c}^{c} y\,\frac{\partial^2 H_i}{\partial y^2}\,dy \qquad (A10)$$

$$- \frac{c}{8}\left[1 + 2\left(\frac{y}{c}\right) + \left(\frac{y}{c}\right)^2\right]\int_{-c}^{c} y^2\,\frac{\partial H_i}{\partial y^2}\,dy + \frac{1}{24}\left[2 + 3\left(\frac{y}{c}\right) - \left(\frac{y}{c}\right)^3\right]\int_{-c}^{c} y^3\,\frac{\partial^2 H_i}{\partial y^2}\,dy$$

Substitution of H_2 and H_3 of eqs. (A7a) into the above equation gives, after some simplification, the functions φ_2 and φ_3 of Table I.

APPENDIX 3

Derivation of Eq. (17)

Eq. (17) is derived by substituting the temperature $T = T_1 + T_2$ into the expressions of Table I, where T_1 and T_2 are given in eqs. (10b) and (10c) respectively; the function $q(\xi)$ appearing in the temperature expansion is found in eq. (11). The result of the above mentioned operation can be put in the following form:

$$\frac{\sigma_x K}{2c\ E\ Q_0} = \left\{ -T_1 + (1/2)\int_{-1}^{1} T_1\, d(y/c) + (3/2)(y/c)\int_{-1}^{1} T_1\,(y/c)d(y/c)\right\} +$$

$$+ \left\{ -T_2 + (1/2)\int_{-1}^{1} T_2\, d(y/c) + (3/2)(y/c)\int_{-1}^{1} T_2\,(y/c)d(y/c) + \right. \tag{A10}$$

$$+ (1/4)\left[(y/c)\int_{-1}^{(y/c)} T_1\, d(y/c) - \int_{-1}^{(y/c)} T_1\,(y/c)\, d\,(y/c) - \right.$$

$$-(1/12)(1/c^2)(c^2 + 6\,y\,c + 6y^2)\int_{-1}^{1} T_1''\, d(y/c) +$$

$$+(1/20)(1/c^3)(10c^3 + 21\,y\,c^2 - 10\,y^3)\int_{-1}^{1} T_1''\,(y/c)\, d\,(y/c) -$$

$$\left.\left. -(1/4)\int_{-1}^{1} T_1''\,(y/c)^2\, d\,(y/c) - (1/4)(y/c)\int_{-1}^{1} T_1''\,(y/c)^3\, d\,(y/c)\right]\right\}$$

where primes indicate differentiation with respect to $\xi = x/(2c)$. Inspection of the above equation and of the expressions for T_1 and T_2 will reveal that the first bracket of the right-hand side of eq. (A10) depends only on the function q, while the second bracket depends only on the function q". The details of the substitution are outlined below.

The various integrals appearing in eq. (A10) may be shown to be as follows:

$$(1/q) \int_{-1}^{1} T_1 \, d\,(y/c) = (1/q'') \int_{-1}^{1} T_1'' \, d\,(y/c) = 2\, t_1$$

$$(1/q) \int_{-1}^{1} T_1 \,(y/c)\, d\,(y/c) = (1/q'') \int_{-1}^{1} T_1'' \,(y/c)\, d\,(y/c) = (1/6) - 16\, S_4'$$

$$(1/q'') \int_{-1}^{(y/c)} T_1'' \, d\,(y/c) = t_1\,(2\eta + 1) + [(\eta^3/3) + (\eta^2/2) - (\eta/12) - (1/8)] -$$

$$-(4/\pi^3) \sum_{n=1}^{\infty} [(-1)^n/n^3] \exp \sin \qquad \text{(A11)}$$

$$(1/q'') \int_{-1}^{(y/c)} T_1'' \,(y/c)\, d\,(y/c) = t_1\,[2\eta^2 - (1/2)] + [(\eta^4/2) + 2\,(\eta^3/3) - (\eta^2/12)$$

$$+ (7/96)] - (8/\pi^4) \sum_{n=1}^{\infty} [(-1)^n/n^4] (\cos -1 + n\pi\eta \sin) \exp$$

$$(1/q'') \int_{-1}^{1} T_1'' \,(y/c)^2\, d\,(y/c) = (2t_1/3) + (1/45) - 32\, S_4''$$

$$(1/q'') \int_{-1}^{1} T_1 \,(y/c)^3\, d\,(y/c) = (1/10) - 48\, S_4' + 384\, S_6'$$

$$(1/q'') \int_{-1}^{1} T_2 \, d(y/c) = t_1^2$$

$$(1/q'') \int_{-1}^{1} T_2 \,(y/c)\, d\,(y/c) = (1/60) - 16\, S_6' - 16\, t_1\, S_4'$$

The following notation has been employed in writing the above equations

$$\eta = y/(2c) \; ; \quad \exp = e^{-n^2\pi^2 t_1}$$

$$\cos = \cos n\pi\,(\eta + \tfrac{1}{2}) \; ; \quad \sin = \sin n\pi\,(\eta + \tfrac{1}{2}) \qquad \text{(A11a)}$$

$$S_i' = (1/\pi^i) \sum_{n=1,3,5\ldots}^{\infty} (\exp/n^i)$$

$$S_i'' = (1/\pi^i) \sum_{2,4,6\ldots}^{\infty} (\exp/n^i)$$

$$S_i = S_i'' - S_i' = (1/\pi^i) \sum_{n=1}^{\infty} (-1)^n (\exp/n^i)$$

Substitution of the above integrals into eq. (A10) and simplification of the results gives the desired eq. (17) without further difficulty.

ESSES; $i = 1, 2, 3.$ $\qquad \sigma_{y_i} = \dfrac{\partial^2 \varphi_i}{\partial x^2}$

3

$$\left\{\frac{y^5}{120}\int T\,dy - \frac{y^4}{24}\int Ty\,dy + \frac{y^3}{12}\int Ty^2\,dy - \frac{y^2}{12}\int Ty^3\,dy + \frac{y}{24}\int Ty^4\,dy - \frac{1}{120}\int Ty^5\,dy + \right.$$
$$+ c^5\left[\frac{1}{180} - \frac{1}{90}\left(\frac{y}{c}\right)^2 + \frac{1}{288}\left(\frac{y}{c}\right)^4 - \frac{1}{240}\left(\frac{y}{c}\right)^5 - \frac{1}{480}\left(\frac{y}{c}\right)^6\right]\oint T\,dy +$$
$$+ c^4\left[-\frac{1}{1050}\left(\frac{y}{c}\right) + \frac{2}{175}\left(\frac{y}{c}\right)^3 + \frac{1}{48}\left(\frac{y}{c}\right)^4 + \frac{9}{800}\left(\frac{y}{c}\right)^5 - \frac{1}{1120}\left(\frac{y}{c}\right)^7\right]\oint Ty\,dy$$
$$+ c^3\left[-\frac{1}{48}\left(\frac{y}{c}\right)^2 - \frac{1}{24}\left(\frac{y}{c}\right)^3 - \frac{1}{48}\left(\frac{y}{c}\right)^4\right]\oint Ty^2\,dy +$$
$$+ c^2\left[\frac{1}{60}\left(\frac{y}{c}\right) + \frac{1}{24}\left(\frac{y}{c}\right)^2 + \frac{7}{240}\left(\frac{y}{c}\right)^3 - \frac{1}{240}\left(\frac{y}{c}\right)^5\right]\oint Ty^3\,dy -$$
$$\left. - \frac{c}{96}\left[1 + 2\left(\frac{y}{c}\right) + \left(\frac{y}{c}\right)^2\right]\oint Ty^4\,dy + \frac{1}{480}\left[2 + 3\left(\frac{y}{c}\right) - \left(\frac{y}{c}\right)^3\right]\oint Ty^5\,dy\right\}$$

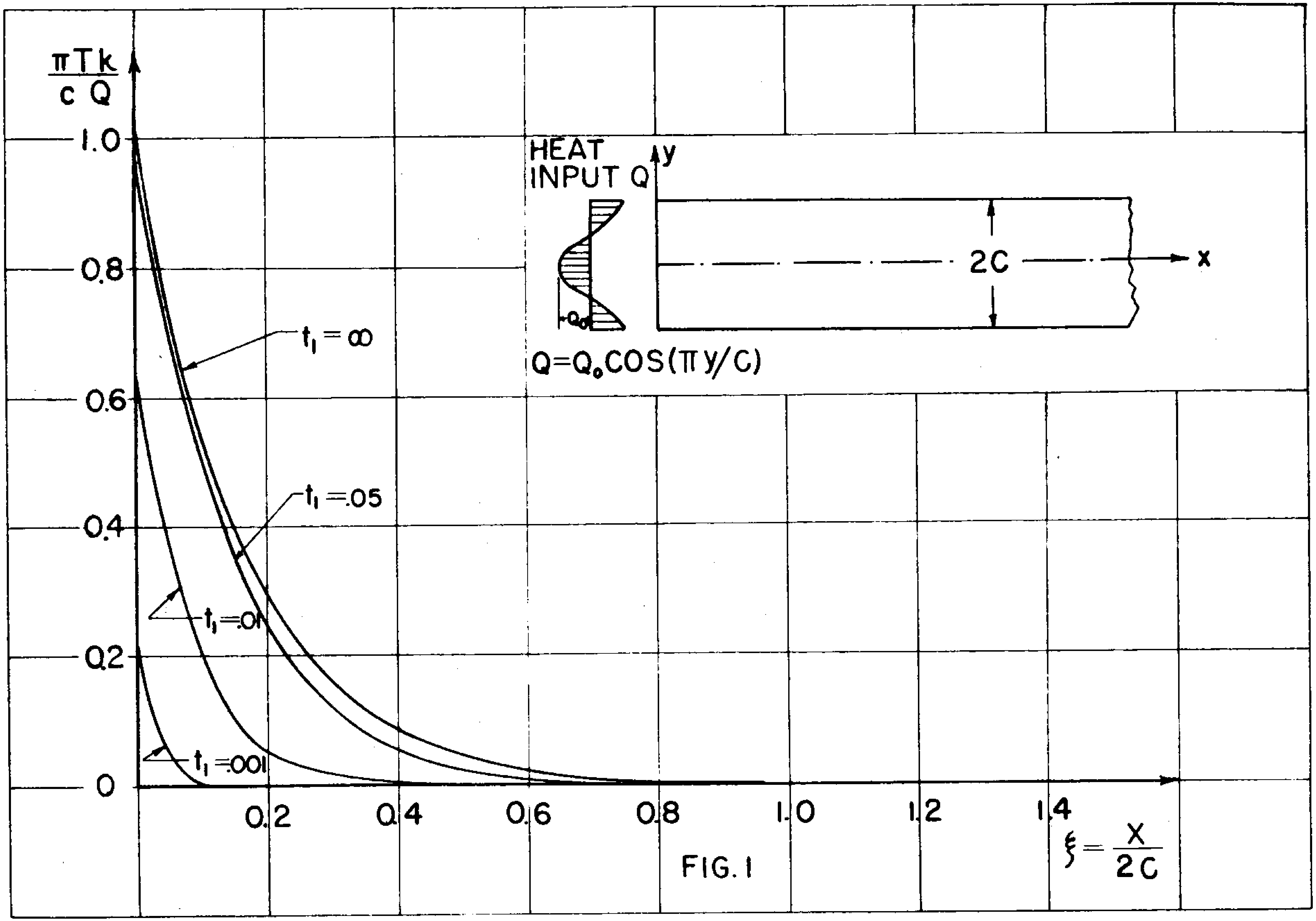
$$\left\{\frac{y^4}{24}\int T\,dy - \frac{y^3}{6}\int Ty\,dy + \frac{y^2}{4}\int Ty^2\,dy - \frac{y}{6}\int Ty^3\,dy + \frac{1}{24}\int Ty^4\,dy + \right.$$
$$+ c^4\left[-\frac{1}{45}\left(\frac{y}{c}\right) + \frac{1}{72}\left(\frac{y}{c}\right)^3 - \frac{1}{48}\left(\frac{y}{c}\right)^4 - \frac{1}{80}\left(\frac{y}{c}\right)^5\right]\oint T\,dy +$$
$$+ c^3\left[-\frac{1}{1050} + \frac{6}{175}\left(\frac{y}{c}\right)^2 + \frac{1}{12}\left(\frac{y}{c}\right)^3 + \frac{9}{160}\left(\frac{y}{c}\right)^4 - \frac{1}{160}\left(\frac{y}{c}\right)^6\right]\oint Ty\,dy +$$
$$+ c^2\left[-\frac{1}{24}\left(\frac{y}{c}\right) - \frac{1}{8}\left(\frac{y}{c}\right)^2 - \frac{1}{12}\left(\frac{y}{c}\right)^3\right]\oint Ty^2\,dy + c\left[\frac{1}{60} + \frac{1}{12}\left(\frac{y}{c}\right) + \frac{7}{80}\left(\frac{y}{c}\right)^2 - \right.$$
$$\left. - \frac{1}{48}\left(\frac{y}{c}\right)^4\right]\oint Ty^3\,dy - \frac{1}{48}\left[1 + \left(\frac{y}{c}\right)\right]\oint Ty^4\,dy + \frac{1}{160c}\left[1 - \left(\frac{y}{c}\right)^2\right]\oint Ty^5\,dy\left.\right\}$$

$$\left\{\frac{y^3}{6}\int T\,dy - \frac{y^2}{2}\int Ty\,dy + \frac{y}{2}\int Ty^2\,dy - \frac{1}{6}\int Ty^3\,dy - \right.$$
$$- c^3\left[\frac{1}{45} - \frac{1}{24}\left(\frac{y}{c}\right)^2 + \frac{1}{12}\left(\frac{y}{c}\right)^3 + \frac{1}{16}\left(\frac{y}{c}\right)^4\right]\oint T\,dy + c^2\left[\frac{12}{175}\left(\frac{y}{c}\right) + \frac{1}{4}\left(\frac{y}{c}\right)^2 + \right.$$
$$\left. + \frac{9}{40}\left(\frac{y}{c}\right)^3 - \frac{3}{80}\left(\frac{y}{c}\right)^5\right]\oint Ty\,dy - c\left[\frac{1}{24} + \frac{1}{4}\left(\frac{y}{c}\right) + \frac{1}{4}\left(\frac{y}{c}\right)^2\right]\oint Ty^2\,dy +$$
$$\left. + \left[\frac{1}{12} + \frac{7}{40}\left(\frac{y}{c}\right) - \frac{1}{12}\left(\frac{y}{c}\right)^3\right]\oint Ty^3\,dy - \frac{1}{48c}\oint Ty^4\,dy - \frac{1}{80c^3}\left(\frac{y}{c}\right)\oint Ty^5\,dy\right\}$$

TABLE II

STRESS FUNCTION AND STRESSES; TEMPERATURE INDEP. OF y

i	1	2	3
$\frac{\varphi_i}{\alpha E}$	0	$-\frac{1}{24}(y^2-c^2)^2\frac{d^2T}{dx^2}$	$\frac{1}{360}(y^2-c^2)^2(y^2-3c^2)\frac{d^4T}{dx^4}$
$\tau_i/(\alpha E) = -\frac{1}{\alpha E}\frac{\partial^2\varphi_i}{\partial x\partial y}$	0	$\frac{y}{6}(y^2-c^2)\frac{d^3T}{dx^3}$	$-\frac{y}{180}(y^2-c^2)(7c^2-3y^2)\frac{d^5T}{dx^5}$
$\sigma_{x_i}/(\alpha E) = \frac{1}{\alpha E}\frac{\partial^2\varphi_i}{\partial y^2}$	0	$-\frac{1}{6}(3y^2-c^2)\frac{d^2T}{dx^2}$	$\frac{1}{180}(15y^4-30y^2c^2+7c^4)\frac{d^4T}{dx^4}$
$\sigma_{y_i}/(\alpha E) = \frac{1}{\alpha E}\frac{\partial^2\varphi_i}{\partial x^2}$	0	$-\frac{1}{24}(y^2-c^2)^2\frac{d^4T}{dx^4}$	$\frac{1}{360}(y^2-c^2)^2(y^2-3c^2)\frac{d^6T}{dx^6}$



πTk / c Q
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
ξ = X / 2C
t1 = ∞
t1 = .05
t1 = .01
t1 = .001
HEAT INPUT Q
y
x
2C
Q0
Q=Q0COS(πy/C)

FIG. 1

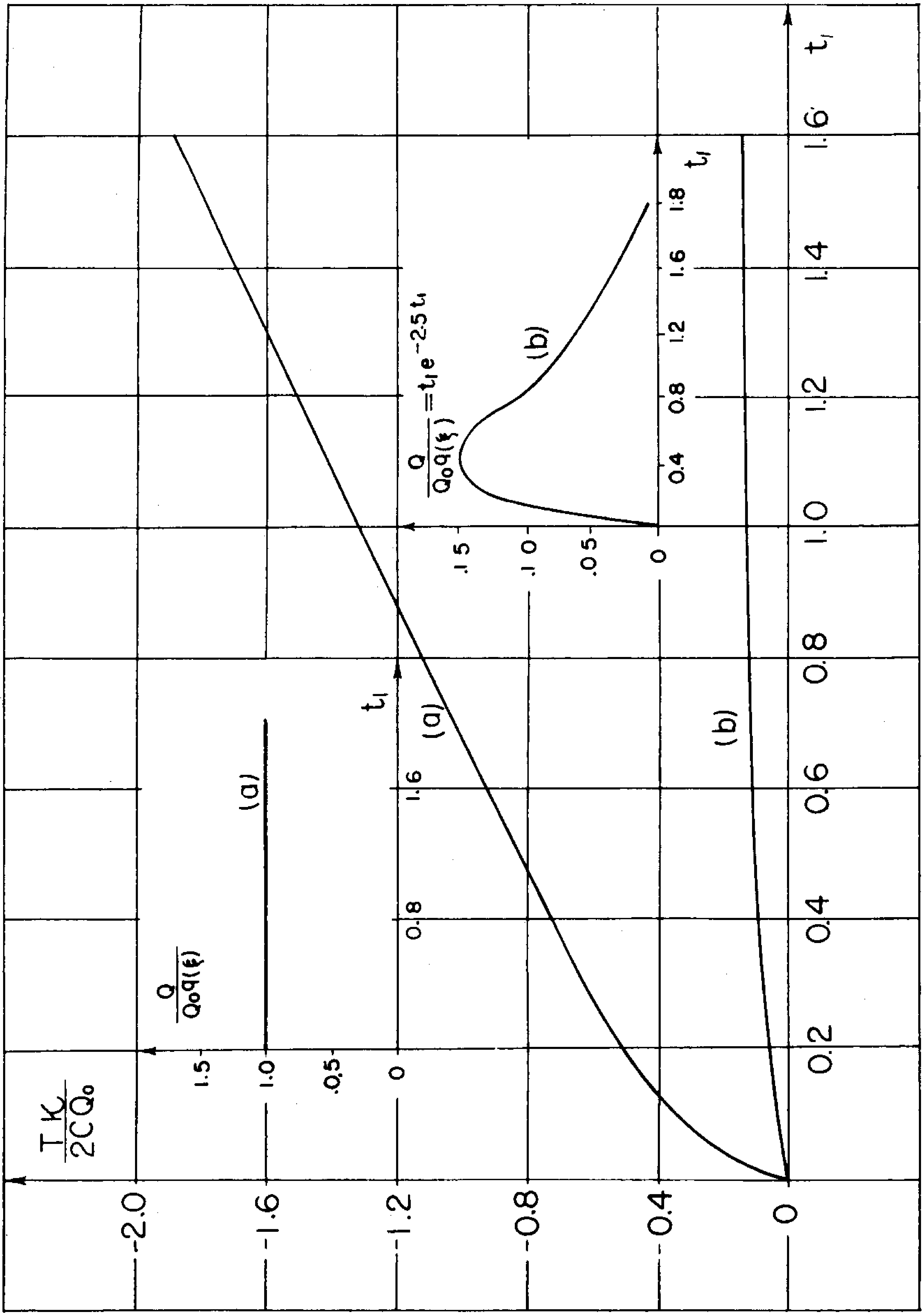
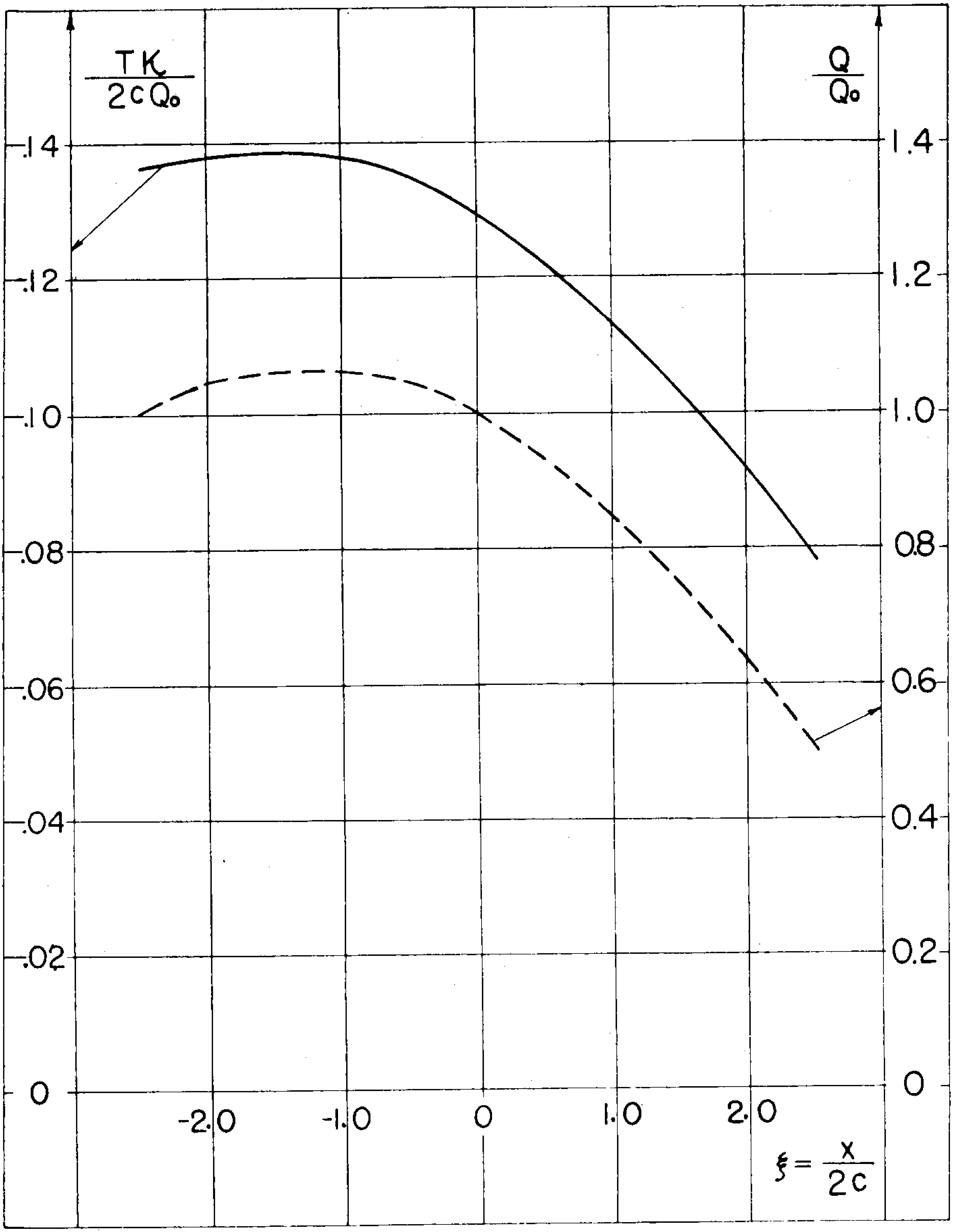

TK/2CQo
-2.0
-1.6
-1.2
-0.8
-0.4
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
t1
Q/Qoq(ξ)
1.5
1.0
.05
0
0.8
1.6
t1
(a)
(a)
(b)
Q/Qoq(ξ) = t1e^-2.5t1
.15
.10
.05
0
0.4
0.8
1.2
1.6
1.8
t1
(b)

FIG 2



$\frac{TK}{2cQ_0}$
$\frac{Q}{Q_0}$
.14
.12
.10
.08
.06
.04
.02
0
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
-2.0
-1.0
0
1.0
2.0
$\xi = \frac{x}{2c}$

FIG. 3

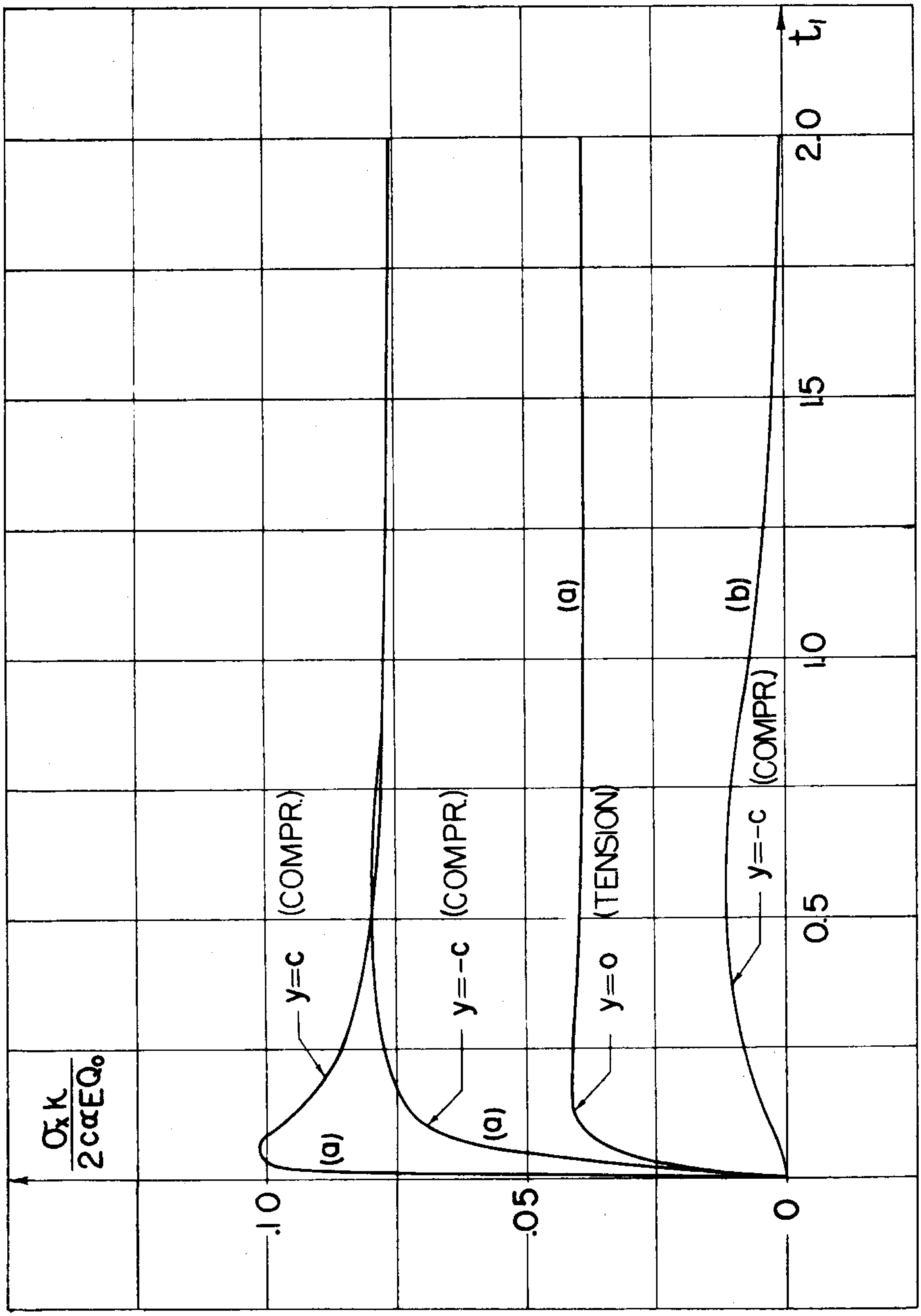
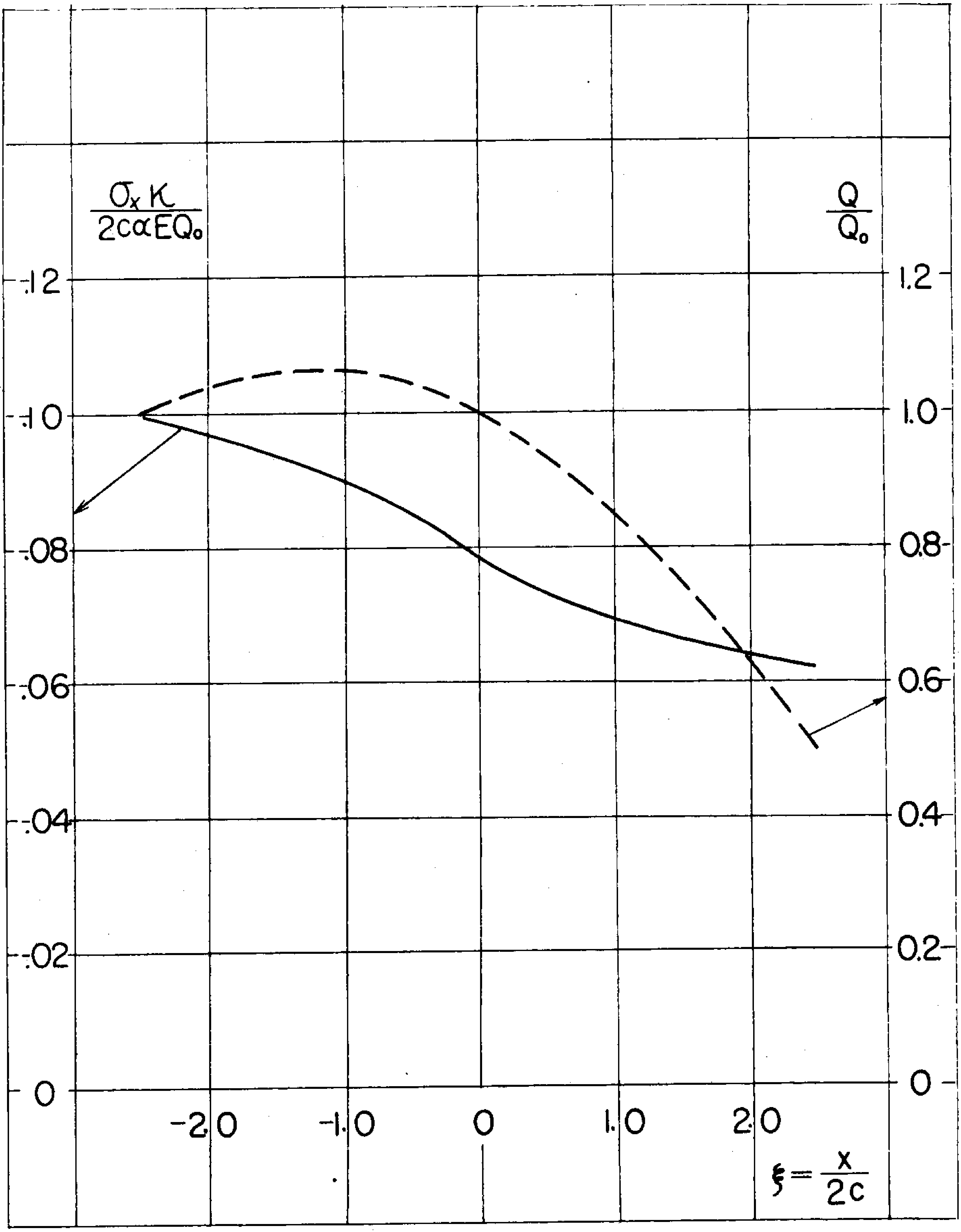

$\frac{\sigma_x K}{2c\alpha E Q_0}$
.10
.05
0
0.5
1.0
1.5
2.0
t_1
y=c
(COMPR.)
y=-c
(COMPR.)
y=o
(TENSION)
y=-c
(COMPR.)
(a)
(a)
(a)
(b)

FIG.4



$\frac{\sigma_x K}{2c\alpha E Q_0}$
$\frac{Q}{Q_0}$
.12
.10
.08
.06
.04
.02
0
1.2
1.0
0.8
0.6
0.4
0.2
0
-2.0
-1.0
0
1.0
2.0
$\xi = \frac{x}{2c}$

FIG. 5

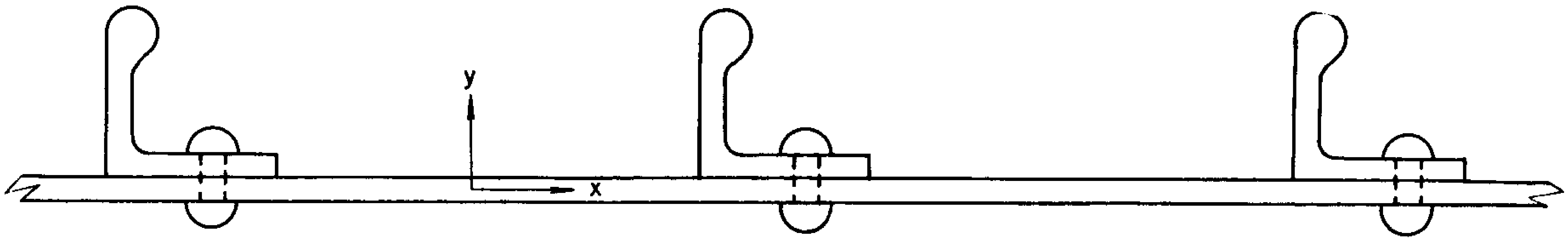


FIG. 6