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STATISTICAL CONTROL SYSTEM DESIGN

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FOREWORD

This report was prepared by the Research Department of Grumman Engineering Corporation, Bethpage, New York under USAF Contract AF33(615)-2431. This contract was initiated under Project Nr 8219 Task Nr 821904. The work was administered under the direction of the Flight Control Division, Air Force Flight Dynamics Laboratory, Research and Technology Division; Mr. D. Bowser was the Air Force Project Engineer.

The report covers work conducted from May 1965 through May 1966, under the direction of Mr. Robert Reiss, the principal investigator. The primary aim of the project was the development of techniques to measure the effect of parameter uncertainty on the performance of a system for which the control law was derived using the nominal parameter values. A secondary aim was to investigate the problem of deriving an optimal stochastic control law in the context of a specific problem suggested by the Flight Dynamics Laboratory. This portion of the work was directed toward demonstrating some of the difficulties involved in obtaining stochastic control laws.

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ABSTRACT

This report discusses the performance of optimal control systems with random parameters. The plants treated are described by time-invariant ordinary differential equations whose parameters are random variables with known distributions. The control law remains fixed as the nominal optimal control, and it is assumed known. Because the parameters of the differential equations are random variables, the performance is a random variable and can only be described in terms of its statistics. Several methods of evaluating the ensemble statistics are presented and compared. Alternatives better suited to stochastic systems are given to the classical notion of sensitivity. Numerical results are presented for specific examples, and the computer programs used in the study are discussed.

The report also includes an analysis of a particular optimal stochastic control problem, to indicate some of the difficulties involved in deriving optimal stochastic control laws.

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SECTION I

INTRODUCTION

In deterministic optimal control theory, the assumption is made that the plant is known exactly. In practice, however, uncertainty often exists concerning the values of the parameters that describe the system. For example, variations in manufacturing tolerances and environmental conditions may cause deviation of parameters from their nominal values. Therefore, a significant and practical problem is the determination of the statistical characteristics of optimally designed systems that use the assumption of nominal parameter values, but with parameters that are in fact random variables. (An analysis such as this could be used, for example, in the case of a high-performance aircraft, because environmental changes are slow relative to the time constants in the aircraft dynamics). However, once such a nominally optimal system is evaluated, the performance may be considered unsatisfactory. In that event we would seek the optimal stochastic law that takes into account the a priori knowledge of the distribution of the random parameter, rather than just the mean or nominal value.

This report is divided into two major sections, Section II and Section III. In Section II, we concern ourselves with the definition and computation of adequate measures of the performance of nominally optimal systems. Reliability analysts have treated problems of this nature (cf., Ref. 1). It is our intention to present some of these concepts in a setting more familiar to the control engineer, as well as to define new concepts. Several methods of computing distribution functions of the performance, given the distribution of the parameters, are discussed.

Next, several sensitivity coefficients are defined and their utility indicated. Some numerical results for particular systems are given, and the computer programs that were used are described.

In Section III, a particular optimal stochastic control problem is analyzed. Both the continuous and discrete time versions are treated. Various types of control laws are defined, depending on the data available to the controller. It is shown that the conventional method of handling stochastic systems, viz., dynamic programming, is not applicable to this problem, and the reasons are given.

SECTION II

PERFORMANCE OF CONTROL SYSTEMS WITH RANDOM PARAMETERS

In this section we begin with a discussion of those aspects of "performance" that arise due to the stochastic nature of the system. In the deterministic case, performance is usually measured as some function whose domain is the space of admissible controls. It is not our concern to justify the use of a particular performance index as being any more physically meaningful than others. Rather, it is to point out that if the system has stochastic elements, any measure of performance is a random variable (because it is a function of a random variable) and therefore, the description of the characteristics of the performance can only be given in terms of distribution functions or statistics. Our problem then is twofold; first, to derive, either analytically or numerically, the distribution function of the performance based on the known distributions of the parameters; and second, to select sensitivity measures and meaningful statistics (moments, percentiles, etc.) that will adequately represent the behavior of the system. While an assumption of normality may make the second question somewhat trivial (because the mean and variance completely represent the distribution function), it is generally true that the input-output (parameter-performance index) relationship is nonlinear; therefore, even assuming normal inputs, we cannot assume normal outputs. In such cases, which statistics will best represent system performance and how these shall be computed are the questions we attempt to answer.

1. METHODS OF COMPUTING THE DISTRIBUTION FUNCTION

Suppose we are given the system dynamics

$$\dot{x} = f(x,u^*,\alpha) \tag{2-1}$$

and the performance index

$$J = \int_0^T g(x,u^*,t)\, dt \quad , \tag{2-2}$$

where

- x is the vector state of the system;
- u^* is the optimal control;
- T is fixed, and
- α is a random variable.

The performance V is given by

$$V = \int_0^T g(x,u_a,t)\, dt \quad , \tag{2-3}$$

where u_a, the "nominal optimal control," is obtained by replacing α with α^0 in the expression for the optimal control u^*, and α^0 is the nominal value of α. Thus

$$V = V(\alpha,\alpha^0,x_0) \quad , \tag{2-4}$$

where x_0 = initial state, and

$$V = J \quad \text{for} \quad \alpha = \alpha^0 \quad .$$

The problem is to determine the distribution function of the performance, P(V). It would of course be desirable to derive P(V) analytically, but for most cases this is an extremely difficult, and sometimes impossible, task. In general, a closed form solution of the differential equation for the state is necessary. From this, the resulting performance V is evaluated. This function must then be inverted in order to solve for the distribution of V (Ref. 2). Because V is in general extremely complicated (if at all available in closed form), this procedure can present formidable problems. To illustrate the difficulties involved, the procedure was carried out for the sample problem (see Sec. II.3).

Because of the apparent difficulty in using analytical methods, we sought other techniques that would yield the desired results and still remain applicable in nontrivial cases. The methods that we selected can be categorized as: a) a numerical version of a known analytical technique, and b) methods based on sampling techniques.

a. Numerical Version of an Analytical Technique

These methods have some advantage over sampling methods because they yield (up to the limits of computer accuracy) exact values of the distribution function and its parameters. The disadvantage is that as the complexity of the problem grows, the difficulty of implementation increases more rapidly than in sampling methods.

If α is a scalar in Eq. (2-1), P(V) can be obtained numerically in the following way. Increment α throughout its range; at each point simulate the system dynamics (using the nominal optimal control) and compute the performance V. In this way an array V versus α is obtained. As an example, suppose V is computed as shown in Fig. 1. Then the distribution function of V is given by

$$Pr(V \leq V_1) = Pr(\alpha_a \leq \alpha \leq \alpha_b) + Pr(\alpha_c \leq \alpha \leq \alpha_d) \quad . \tag{2-5}$$

(See for example Ref. 3). Thus, to calculate $P(V)$, V_1 is incremented throughout the range of V. At each point the corresponding values of α are found by using interpolation if necessary, and the probabilities in Eq. (2-5) are computed from the given $P(\alpha)$. The accuracy of this procedure is limited only by the step sizes in α and V. Once $P(V)$ is determined, any of its moments can be computed.

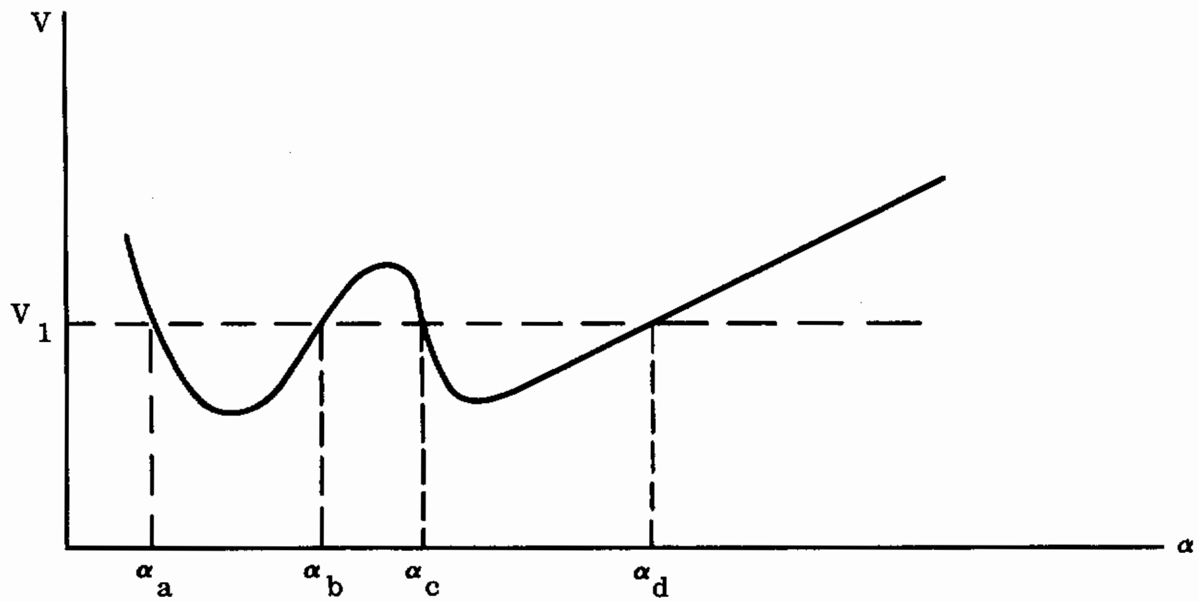


Fig. 1 Performance Function

b. Sampling Techniques

While the method we have discussed above seems quite adequate and accurate for the one-dimensional case, the difficulties associated with it become quite severe as the number of dimensions (random parameters) increase. Therefore, consider the Monte Carlo simulation procedure (cf. Ref. 4) for obtaining the distribution function of the performance and its statistics. For this procedure it is generally true that dimensionality is not a limitation. The only real limitation is the number of case histories one must obtain to get sufficiently accurate estimates of the statistics of interest; the computer time for the required number of runs may be excessive.

However, problems do arise that are peculiar to the simulation of control systems. For instance, if there is a region within the range of parameter variation that causes the performance to have a large value (it is assumed that unstable parameter values are excluded), and if this region has a relatively low probability, any sampling technique will be highly sensitive to the number of samples taken from this region. For example, in a small random sample there is a high probability that values of the parameters from this region would not enter the sample at all, thereby causing an underestimate of the mean and variance. Therefore, in such cases strict random sampling may not yield accurate estimates. In order to circumvent this it is necessary to use sampling methods that guarantee a representative sample for any given sample size. One such method is stratified random sampling. With this technique the region of parameter variation is divided into intervals such that the parameter has an equal probability of occurrence in each interval. An equal number of samples is then drawn from each interval, thus guaranteeing a representative sampling over the entire range of parameter variation. This technique will guarantee that those parameter values yielding large values of performance with low probability will enter the sample in their proper proportion, and thus have the correct effect on the computation of the statistics.

Having discussed some techniques for constructing distribution functions of the performance, the next section (II.2) will discuss the problem of deriving meaningful statistical measures of the performance.

2. PERFORMANCE MEASURES

While the distribution function does contain all the statistical information about the random variable V, it is desirable to have some numerical figures of merit that the control systems engineer can use.

a. Sensitivity

In deterministic optimal control theory, sensitivity is usually defined as $S_\alpha^V = \partial V/\partial\alpha|_{\alpha^0}$. Because this definition only takes into account the variations in a neighborhood of the nominal value, it is in a sense saying that only small deviations from the nominal are expected, and that these are uniformly distributed (since the deviations are weighted uniformly everywhere). When more is known concerning the variation in the input parameters, certainly a more adequate definition of sensitivity can be made. Some alternatives and their relative merits are given below.

First, because $\partial V/\partial \alpha$ is a random variable, its mean value

$$\bar{S}^V_{\alpha_i} = E\left\{\frac{\partial V}{\partial \alpha_i}\right\} = \int_{-\infty}^{+\infty} \frac{\partial V}{\partial \alpha_i}\, p(\alpha_i)\, d\alpha_i$$

is a worthwhile statistic to employ as a definition of sensitivity. Note that if $\partial V/\partial \alpha_i$ is a linear function of α_i, then

$$\frac{\partial V}{\partial \alpha_i} = k_1 \alpha_i + k_2$$

$$\bar{S}^V_{\alpha} = E\left(\frac{\partial V}{\partial \alpha_i}\right) = k_1 E(\alpha_i) + k_2 \quad .$$

If $\alpha^0 = E(\alpha_i)$, i.e., the nominal is equal to the mean value, which is the usual case, then

$$S^V_{\alpha} = \left.\frac{\partial V}{\partial \alpha}\right|_{\alpha^0} = k_1 \alpha^0 + k_2 = \bar{S}^V_{\alpha} \quad ,$$

and the definition seems quite natural in light of this agreement with the classical notion.

Also, it is interesting to note that the mean partial derivative has the same value as the coefficient of a linear least squares fit to the function $V(\alpha)$ when α is normally distributed (see Appendix A). Intuitively, the variance of the performance should also be a measure of sensitivity, because those distributions with small σ_V^2 (for a given $\sigma_{\alpha_i}^2$) are those which yield a small spread of output values over the entire range of inputs. Therefore, a sensitivity coefficient can be defined as

$$S_{\sigma_{\alpha_i}}^{\sigma_V} = \frac{\sigma_V}{\sigma_{\alpha_i}}\bigg|_{\sigma_{\alpha_j}=0,\ j\neq i}$$

This of course ignores interaction effects in multidimensional systems. The two notions, the standard deviation ratio and the partial or mean partial derivatives, can be related by noting that, to the first order, the output variance is given (Ref. 5) by

$$\sigma_V^2 \cong \sum_i \left(S_{\alpha_i}^V\right)^2 \sigma_{\alpha_i}^2 \quad ,$$

or, choosing the average rather than nominal value of $\partial V/\partial\alpha$,

$$\sigma_V^2 \cong \sum_i \left(\bar{S}_{\alpha_i}^V\right)^2 \sigma_{\alpha_i}^2 \quad .$$

However, if $V(\alpha)$ is a very nonlinear function, then neither of these approximations to the standard deviation ratio may be adequate. This does not imply that the ratio is a poor statistical substitute for classical sensitivity; it simply means that methods not as closely linked to the classical sensitivity must be used for evaluating it. Simple or stratified Monte Carlo sampling, from which variance estimates can be made, are possible alternates. The difficulty of determining the accuracy of these variance estimates can be further complicated by nonlinear and non-Gaussian assumptions. In addition, for a nonnormal $P(V)$, the variance is not necessarily useful as an indicator of any particular percentile. For that case a percentile estimate that is valid and independent of the nature of the distribution is desirable.

b. Percentile Estimation

To be more specific, suppose we desire the bounds on the performance such that with a given confidence those bounds will contain the actual value of the Xth percentile (i.e., that value of V which will not be exceeded X percent of the time). Such nonparametric estimates are available (Ref. 6) for percentile estimation. The notion of estimating percentiles in a nonparametric way is simply another way of obtaining the "spread" of the performance, and then using this as a measure of sensitivity. Again, the main point is that a stochastic system demands a statistical quantity as a measure of its sensitivity.

3. NUMERICAL RESULTS

a. One-Dimensional Case

In order to demonstrate some of the techniques previously discussed, the first example selected is a simplified pitch controller, described in Ref. 7.

The plant is given (in canonical form) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad , \qquad (2\text{-}6)$$

and the output is given by

$$y = \begin{bmatrix} c & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The performance index is

$$V = \lim_{T \to \infty} \int_0^T \left\{ q\left[cx_1 + x_2\right]^2 + ru^2 \right\} dt \quad , \qquad (2\text{-}7)$$

where q and r are weighting coefficients, and c is a deterministic system parameter. It can be shown (Ref. 7) that the nominal optimal control is

$$u_a = - k_1 x_1 - k_2 x_2 \quad , \qquad (2\text{-}8)$$

where

$$k_1 = - \alpha_2^0 + \sqrt{\left(\alpha_2^0\right)^2 + \frac{c^2 q}{r}}$$

$$k_2 = - \alpha_1^0 + \sqrt{\left(\alpha_1^0\right)^2 - 2\alpha_2^0 + \frac{q}{r} + 2\sqrt{\left(\alpha_2^0\right)^2 + \frac{c^2 q}{r}}} \quad .$$

Because of its simple nature, this example afforded some opportunity for analytical treatment. Some of the numerical procedures were verified.

If we have a single parameter variation,

$$p(\alpha_1)d\alpha_1 = p[\alpha_1(V)] \, |\alpha_1'(V)| \, dV \quad , \qquad (2\text{-}9)$$

where $p(\alpha_1)$ is the probability density of α_1. (See for example Ref. 2, p. 294.) This analytical procedure requires that we obtain the function $V(\alpha_1, \alpha_2^0)$, which can be found in closed form by using Eqs. (2-6), (2-7), and (2-8).

$$V = (qc^2 + rk_1^2)\,\frac{x_1^2(0) + \dfrac{[(\alpha_1 + k_2)x_1(0) + x_2(0)]^2}{\alpha_2^0 + k_1}}{2(\alpha_1 + k_2)}$$

(2-10)

$$+ (q + rk_2^2)\left[\frac{x_2^2(0) + (\alpha_2^0 + k_1)x_1^2(0)}{2(\alpha_1 + k_2)}\right] - (cq + rk_1k_2)x_1^2(0) \quad .$$

Solving for α_1 one obtains

$$k_2 + \alpha_1 = \frac{\alpha_2^0 + k_1}{x_1^2(0)\delta}\left[V + \beta x_1^2(0) - \frac{x_1(0)x_2(0)\delta}{\alpha_2^0 + k_1}\right] \qquad (2\text{-}11)$$

$$\pm \sqrt{\frac{(\alpha_2^0 + k_1)^2}{x_1^4(0)\delta^2}\left[\frac{x_1(0)x_2(0)\delta}{\alpha_2^0 + k_1} - \beta x_1^2(0) - V\right]^2 - \left[1 + \frac{(\alpha_2^0 + k_1)\gamma}{\delta}\right](\alpha_2^0 + k_1 + \lambda^2)}$$

where

$$\lambda = \frac{x_2(0)}{x_1(0)} \quad ,$$

$$\beta = cq + rk_1k_2 \quad ,$$

$$\gamma = q + rk_2^2 \quad , \quad \text{and}$$

$$\delta = qc^2 + rk_1^2 \quad .$$

The derivative is given by

$$\frac{d\alpha_1}{dV} = \frac{\alpha_2^0 + k_1}{x_1^2(0)\delta}\left[1 \pm \frac{1}{\sqrt{1 - \dfrac{x_1^4(0)\delta^2\left[1 + \dfrac{\gamma(\alpha_2^0 + k_1)}{\delta}\right]\left[\alpha_2^0 + k_1 + \lambda^2\right]}{\left[(\alpha_2^0 + k_1)V + (\alpha_2^0 + k_1)\beta x_1^2(0) - x_1(0)x_2(0)\delta\right]^2}}}\right] .$$

Because the inversion is nonunique, the range of parameter variation must be separated into regions before Eq. (2-9) is applied. This was done with $q = 0.2$, $r = 1.0$, $c = 0.5$ (these values were used throughout most of the study) and for $p(\alpha_1)$ uniformly distributed, $0.0 < \alpha_1 < 2.3$. The other data used were $x_1(0) = x_2(0) = 5.0$, $\alpha_2^0 = 10.0$. The density function $p(V)$ is shown in Fig. 2. A 100-sample Monte Carlo simulation is also shown in Fig. 2, where the ordinate scale is normalized to $f/(N \cdot \Delta V)$, f = frequency of occurrence, N = total number of runs, and ΔV = interval size. This gives units of probability density.

Figures 3 and 4 show $V(\alpha_1,\alpha_2^0)$ and $V(\alpha_1^0,\alpha_2)$, respectively, when

$$\alpha_1^0 = \alpha_2^0 = 15.0 \quad , \quad \alpha_1,\alpha_2 > 0$$

$$x_1(0) = x_2(0) = 5.0 \quad .$$

The curves indicate the high sensitivity to parameter variation in the neighborhood of the origin. In general, this can cause serious computational difficulties when attempts are made to obtain statistics of the performance.

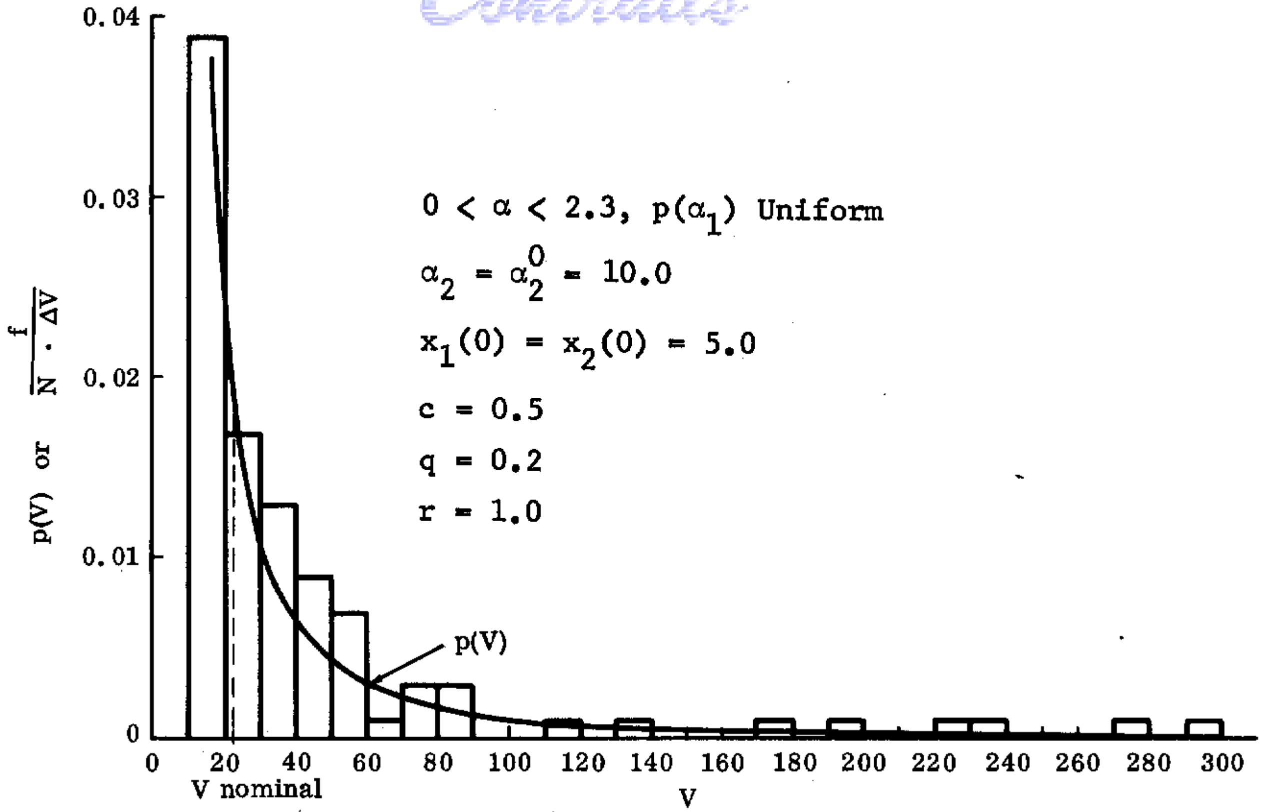


Fig. 2 Performance Density Function and Histogram with 100 Monte Carlo Samples

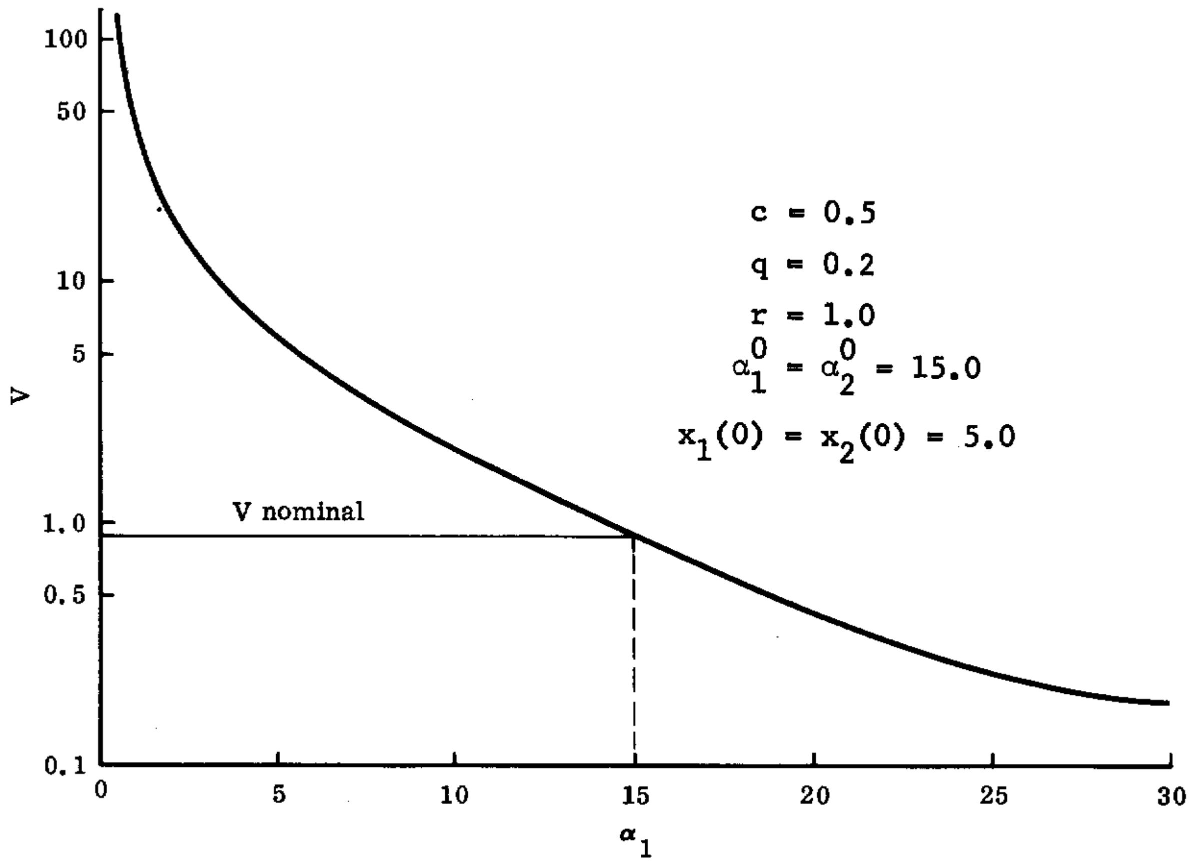


Fig. 3 Performance versus α_1 with α_2 Constant

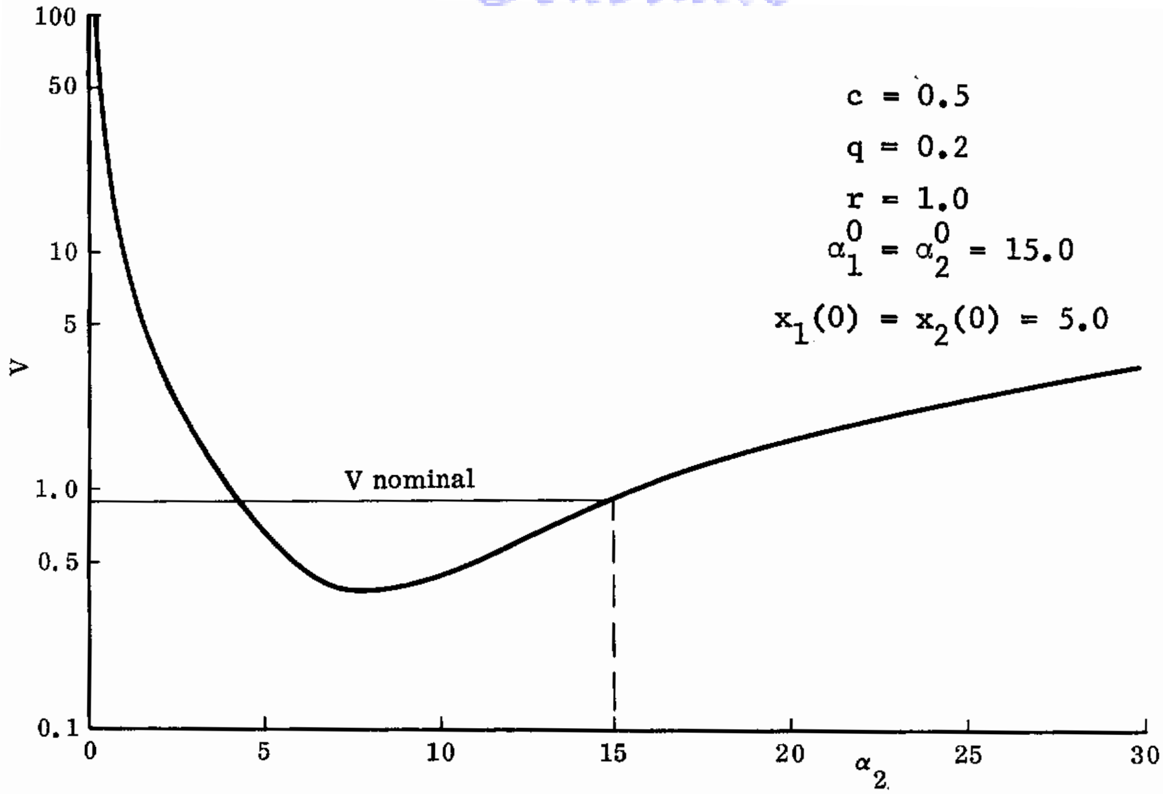


Fig. 4 Performance versus α_2 with α_1 Constant

Figure 5 is an example of a density function $p(V)$, which was obtained by differentiating the distribution function computed via the method of Eq. (2-5) with $P(\alpha_2)$ Gaussian, $\sigma_{\alpha_2} = 3.0$, and α_1 = constant = 15.0. The singularity occurring near $V = 0.39$ is explained by referring to Fig. 4 and noting that a minimum occurs at that value of V. Because Eq. (2-9) is valid (with α_2 inserted for α_1), a singularity is expected when $\partial V/\partial\alpha_2 = 0$. This computation, while instructive for the single dimensional case is not readily extendable to higher dimensions. The only reasonable technique in the higher dimensional case is Monte Carlo simulation or some variation of it.

For the same data as in Fig. 5, Figs. 6 and 7 illustrate 2 histograms of 100 Monte Carlo runs each, with simple random sampling, and Fig. 8 is a histogram of 100 runs with stratified

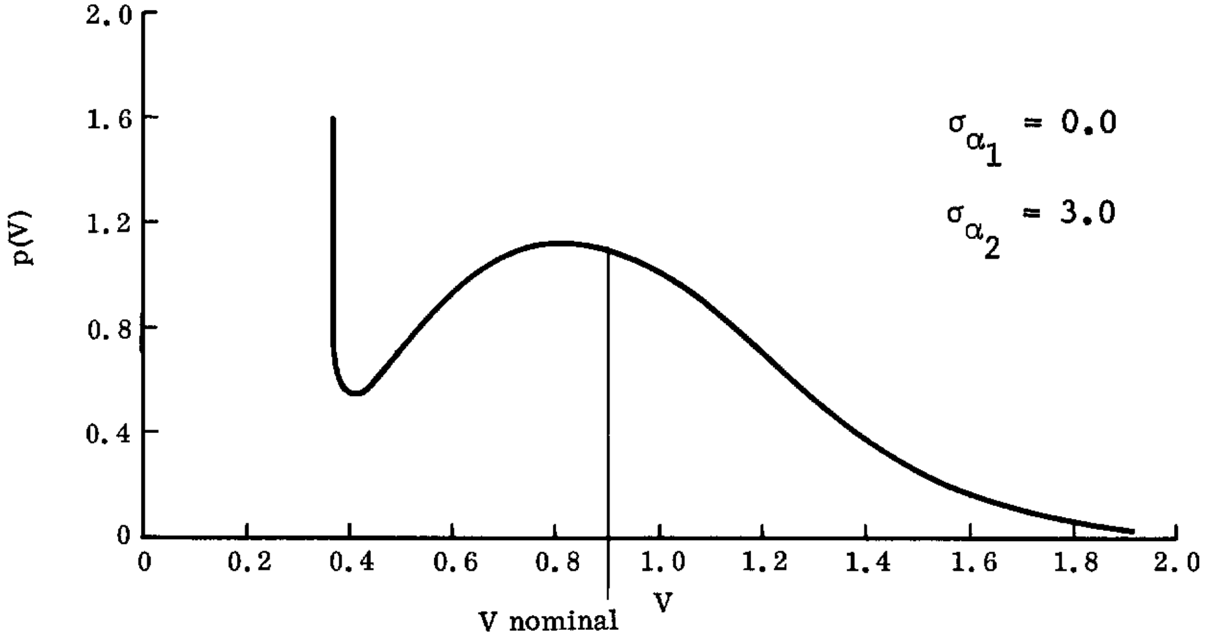


Fig. 5 Performance Density Function $\left(p(\alpha_2) \text{ Normal}\right)$

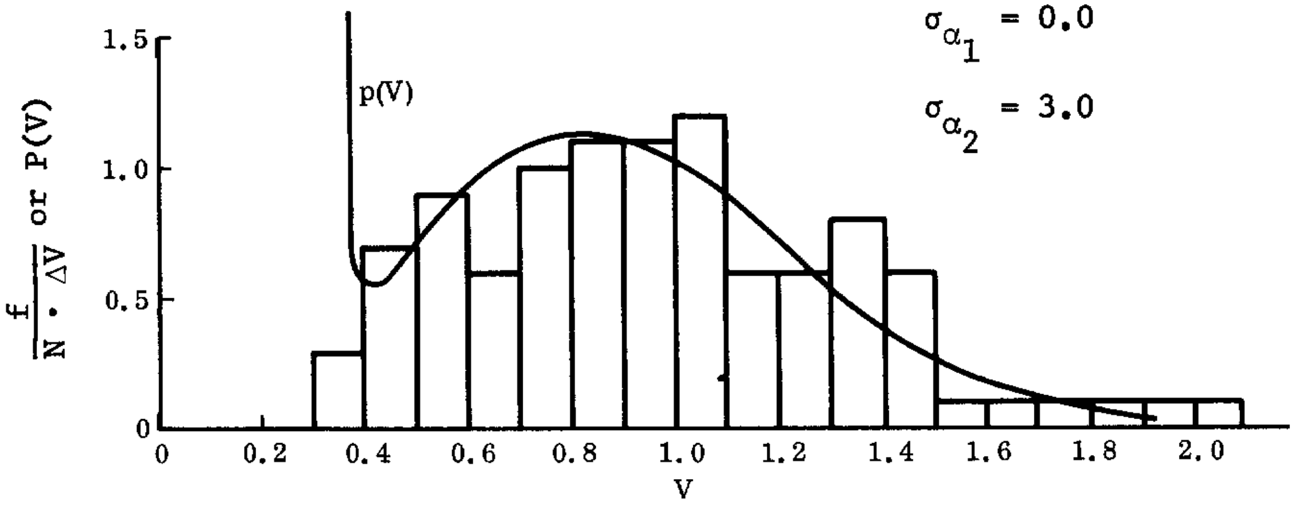


Fig. 6 Performance Histogram No. 1 (100 Random Samples)

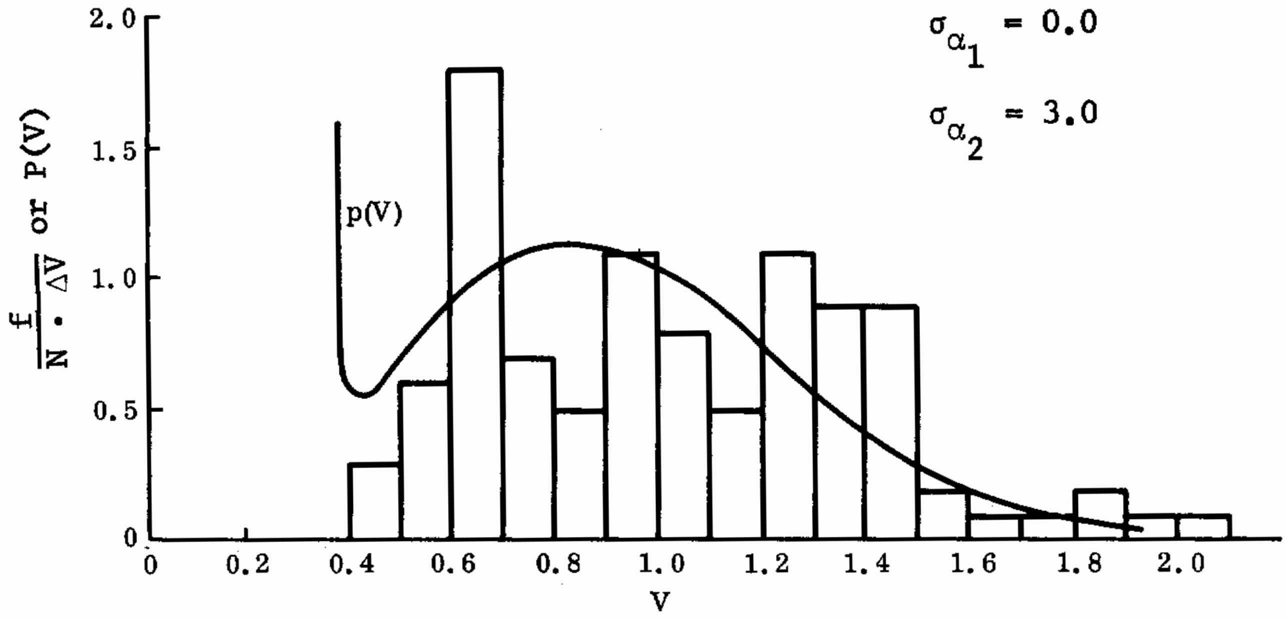


Fig. 7 Performance Histogram No. 2 (100 Random Samples)

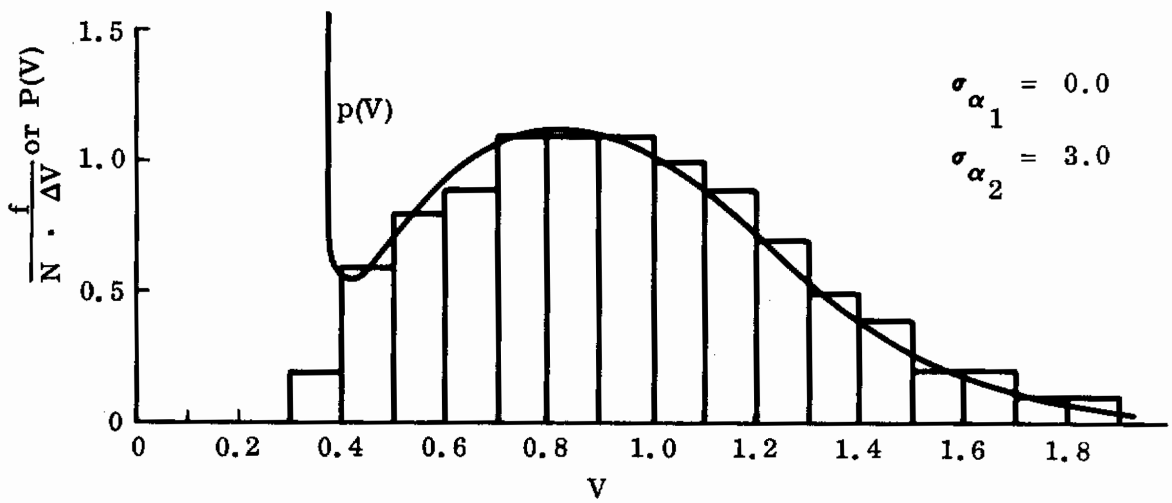


Fig. 8 Performance Histogram (100 Stratified Samples)

sampling (one sample from each of 100 equal probability intervals). If sampling is to be used, clearly stratified sampling is the preferred technique because it gives better agreement with the computed density function. The limitation on its use will depend on the dimensionality of the problem. For n random parameters and one sample from each of k equal probability intervals, the number of samples is k^n. For a particular problem, if this number is prohibitive in terms of computer time, one may have to resort to straight sampling and accept less confidence in the results.

The singular behavior would be observed in a sampling process only for a large number of samples and a very fine grid size in the histogram plot. This demonstrates that care must be taken in the choice of the particular sampling technique used, or some system behavior of interest may not be discovered.

(1) <u>Variance Estimation</u> In this section we report some numerical results in the estimation of certain statistics of the first sample problem.

Table I lists values of output variance computed on the basis of the various techniques we have discussed. The values in column 1 were calculated via integrating with the computed $p(V)$ and are taken as the actual values. Using an equal number of runs, it is seen that stratified sampling is the more reliable sampling technique. As previously noted, the classical notion of sensitivity has deficiencies with regard to stochastic systems. This is emphasized by comparing the last two columns: the output variances estimated with the mean partial derivative linearization give better agreement with the actual system variance. Therefore this linearization is preferred over the classical sensitivity coefficient.

TABLE I

COMPUTED AND ESTIMATED VARIANCES

DATA		COMPUTED VARIANCES			ESTIMATED VARIANCES	
σ_{α_1}	σ_{α_2}	①	②	③	④	⑤
0	1	.014	.015	.014	.014	.014
0	2	.054	.059	.055	.057	.054
0	3	.116	.102	.116	.127	.112
1	0	.020	.019	.020	.019	.020
2	0	.096	.122	.095	.077	.090
3	0	.319	.206	.290	.173	.252

$p(\alpha_1, \alpha_2) = p(\alpha_1)p(\alpha_2)$

$p(\alpha_1)$, $p(\alpha_2)$ Truncated (0.1, 30.0) Gaussian

$\bar{\alpha}_1 = \bar{\alpha}_2 = 15.0$, $x_1(0) = x_2(0) = 5.0$, $q = 0.2$, $r = 1.0$, $c = 0.5$

Col. 1: $\sigma_V^2 = \int (V - \bar{V})^2 \, p(V) \, dV$

2: σ_V^2 , 100 runs, uniform sampling

3: σ_V^2 , 100 runs, stratified sampling

4: $\sum \left(S_{\alpha_i}^V\right)^2 \sigma_{\alpha_i}^2$

5: $\sum \left(\bar{S}_{\alpha_i}^V\right)^2 \sigma_{\alpha_i}^2$

(2) Nonparametric Percentile Estimation As an example of nonparametric percentile estimation we estimate a range of V that contains the true P^{th} percentile. The formula given in Ref. 6 determines with 90 percent confidence the range R in number of observations about the sample P^{th} percentile:

$$R = \pm\, 0.0164 \sqrt{P(100 - P)N}$$

where

R = range about the sample P^{th} percentile

N = number of samples.

The above formula is arrived at from the following considerations: When observations are arranged in order of size, the position of the observation which defines the sample P^{th} percentile is given by $\frac{P(N+1)}{100}$. The value of the observation yielding the P^{th} percentile will vary from sample to sample. For large N, the probability that the true P^{th} percentile will lie in the interval defined by

$$\frac{PN}{100} \pm k \sqrt{P(100 - P)N}$$

is obtained by using the normal approximation to the binomial distribution (valid for $N > 50$). The value of k is obtained from a table of the normal distribution, and is simply the number of standard deviations on each side of the mean that contain the desired confidence level. This number is divided by 100 because our variable is percentile rather than probability. In the above example, 1.64 standard deviations contain 90 percent of the normal distribution. If we desired a 95 percent confidence level, we would use $k = \frac{1.96}{100} = .0196$. Thus we estimate a range of V that contains the true P^{th} percentile. If $P = 90$, $N = 100$, then $R \cong \pm 5$ and the true 90th percentile is located between the 85th and 95th observations (with 90 percent confidence). In the Monte Carlo history of Fig. 6, this is equivalent to the range $1.35 < V < 1.68$, while in the history of Fig. 7, this range is $1.4 < V < 1.6$.

(3) Correlated Coefficients The second example used in this study consists of a set of longitudinal equations of motion of an aircraft and elevator actuator. In the previous example a single parameter variation was treated. In this example there is still only one random variable in the problem, but several of the system coefficients are functionally dependent on this random variable. Thus, the coefficients are correlated in this case through their dependence on dynamic pressure ($q_V = \frac{1}{2}\rho V^2$, V = velocity) changes, which are considered to be the major cause of system parameter variation.

The incremental equations of motion can be written in stability axes with the assumption of no speed change as:

$$\begin{bmatrix} \dot{\alpha} \\ \ddot{\theta} \\ \dot{\delta}_e \end{bmatrix} = \begin{bmatrix} -L_\alpha & 1 & -L_{\delta_e} \\ M_\alpha - M_{\dot{\alpha}} L_\alpha & M_{\dot{\theta}} + M_{\dot{\alpha}} & M_{\delta_e} - M_{\dot{\alpha}} L_{\delta_e} \\ 0 & 0 & -1/\tau \end{bmatrix} \begin{bmatrix} \alpha \\ \dot{\theta} \\ \delta_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/\tau \end{bmatrix} \begin{bmatrix} \delta_c \end{bmatrix}, \tag{2-12}$$

where

α = incremental angle of attack,

θ = pitch rate,

δ_e = incremental elevator angle,

δ_c = command input to actuator,

τ = elevator actuator time constant, and

$M_{\dot\theta}$, M_α, $M_{\dot\alpha}$, M_{δ_e}, L_α, and L_{δ_e} are longitudinal stability derivatives expressed as acceleration coefficients: L/m, M/I_y. α and θ are system outputs, and the performance index is chosen as

$$2V = \lim_{T \to \infty} \int_t^T \left[q_{11}\alpha^2 + q_{22}\theta^2 + \delta_c^2 \right] dt \quad ,$$

and the nominal optimal control is

$$u_a = -k_1\alpha - k_2\dot\theta - k_3\delta_c \quad ,$$

where k_1, k_2, k_3 can be found by solving the matrix Riccati equation (see Ref. 7).

Typical parameter data for a high performance vehicle are given in Tables II and III. As a first approximation, the dependence of the parameters on q_V was assumed linear, i.e.,

$$L_\alpha = L_\alpha^0 + K_1(q_V - q_V^0) \quad ,$$

$$L_{\delta_e} = L_{\delta_e}^0 + K_2(q_V - q_V^0) \quad ,$$

$$M_\alpha = M_\alpha^0 + K_3(q_V - q_V^0) \quad ,$$

$$M_{\dot\alpha} = M_{\dot\alpha}^0 + K_4(q_V - q_V^0) \quad ,$$

$$M_{\dot\theta} = M_{\dot\theta}^0 + K_5(q_V - q_V^0) \quad ,$$

$$M_{\delta_e} = M_{\delta_e}^0 + K_6(q_V - q_V^0) \quad ,$$

where the superscripts denote nominal values. If better fits to available data are desired, higher-order terms may be added with slight modification to the existing computer programs.

TABLE II

FLIGHT PATH PARAMETERS FOR TYPICAL HIGH PERFORMANCE VEHICLE (Ref. 8)

t sec	h 1000 ft	M	q_V* lbs/ft^2	α deg	n_z† g	V ft/sec
0	226	5.6	22.1	0	0	5690
20	211	5.3	26.0	0	0	5690
40	147	5.5	92.2	15	.4	5940
60	102	5.9	676.0	15	2.6	5930
74	80	5.4	1043.3	12.6	5.0	5200
90	77	4.8	858.5	3	1.1	4700

* $q_V = \frac{1}{2} \rho V^2$

† n_z = normalized load factor

TABLE III

NUMERICAL DATA FOR TYPICAL HIGH PERFORMANCE VEHICLE (Ref. 8)

t sec	$M_{\dot{\theta}}$	$M_{\dot{\alpha}}$	M_α	M_δ	L_α	L_{δ_e}
0	-.0011	-.0004	- 0.1569	- 0.1131	.0016	.0002
20	-.0020	-.0007	- 0.3173	- 0.2317	.0034	.0004
40	-.0078	-.0027	- 1.5083	- 1.1664	.0114	.0022
60	-.0595	-.0208	-11.11	- 8.51	.1120	.0172
74	-.1541	-.0539	-26.41	-17.5	.2795	.0417
90	-.1322	-.0463	-17.1	-12.2	.2767	.0372

These data correspond to those of Table II

The condition of $h = 77{,}000$ feet in Table II was assumed nominal. For this case, with $q_{11} = q_{22} = 2.0$, the nominal optimal feedback coefficients are (see Ref. 7)

$$k_1 = 1.72$$

$$k_2 = -1.32$$

$$k_3 = 1.36 \quad .$$

It was assumed that q_V was normally distributed with a mean value of 900.0 lbs/ft^2. Figures 9 and 10 are histograms for the cases where $\sigma_{q_V} = 300$ lbs/ft^2 and $\sigma_{q_V} = 500$ lbs/ft^2, respectively. Each represents a 100 run sample. Although the random variable now enters into several of the system coefficients, the behavior of the performance density function is qualitatively similar to that of system (2-6) when only one coefficient is affected (as shown in Fig. 2).

b. <u>Two-Dimensional Case</u>

In the previous section (II.3.a) we treated two cases of single parameter variation. We now allow two parameters in the plant of Eq. (2-6) to be random variables with nonzero variances. For this case a new sampling method was derived in order to avoid the difficulties previously discussed, viz, high sensitivity, low probability regions. These problems now exist to a greater degree because both random variables take on values that cause rapid increase of the performance index. The sampling method we have used is a variable density stratified sampling which is implemented as follows. The parameter plane is sectioned into an equal probability grid by dividing each axis into equiprobability intervals (see for example Fig. 11). For normal stratified sampling, equal numbers of samples are taken from each square. However, in order

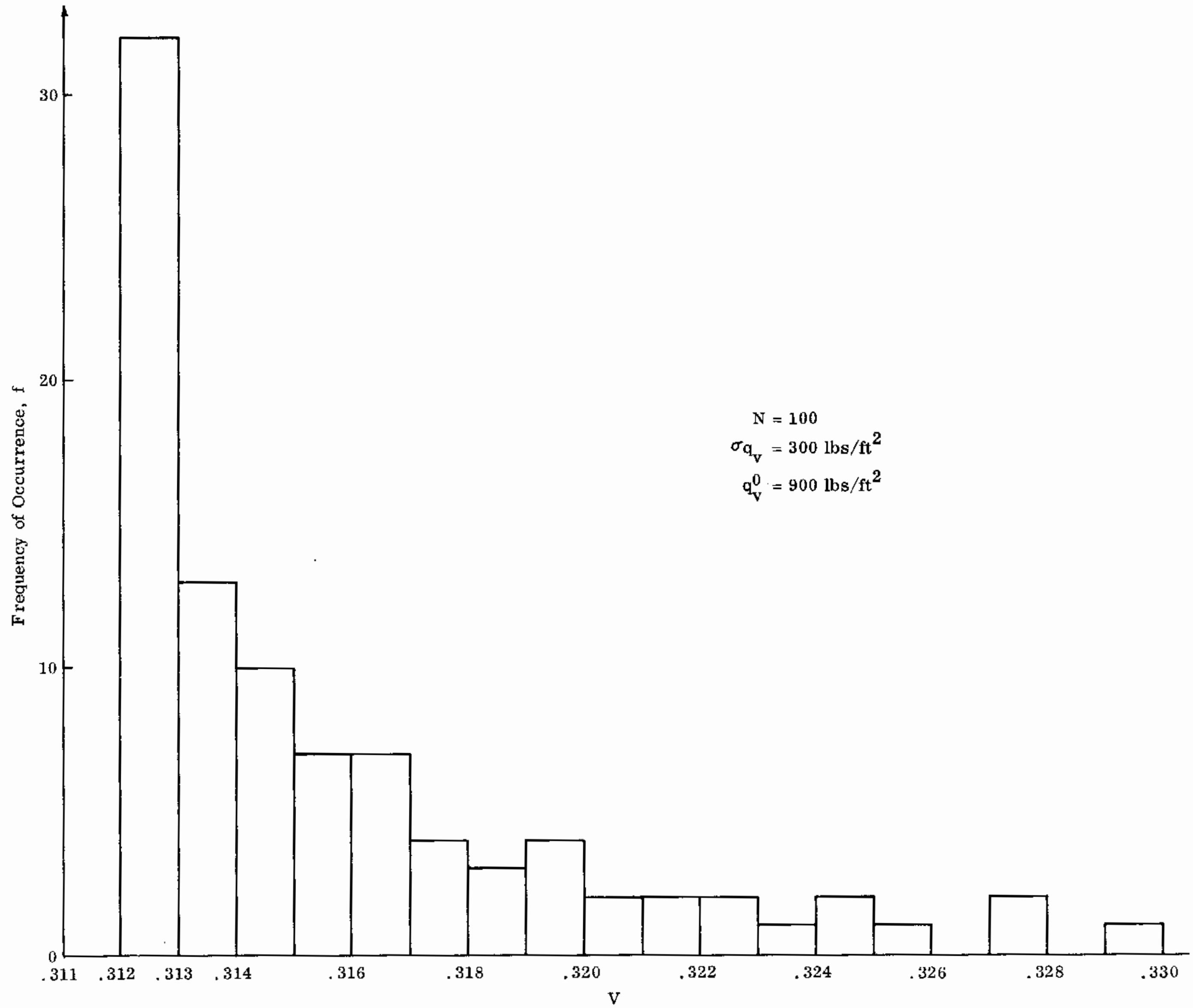


Fig. 9 Performance Histogram, Correlated Coefficients No. 1

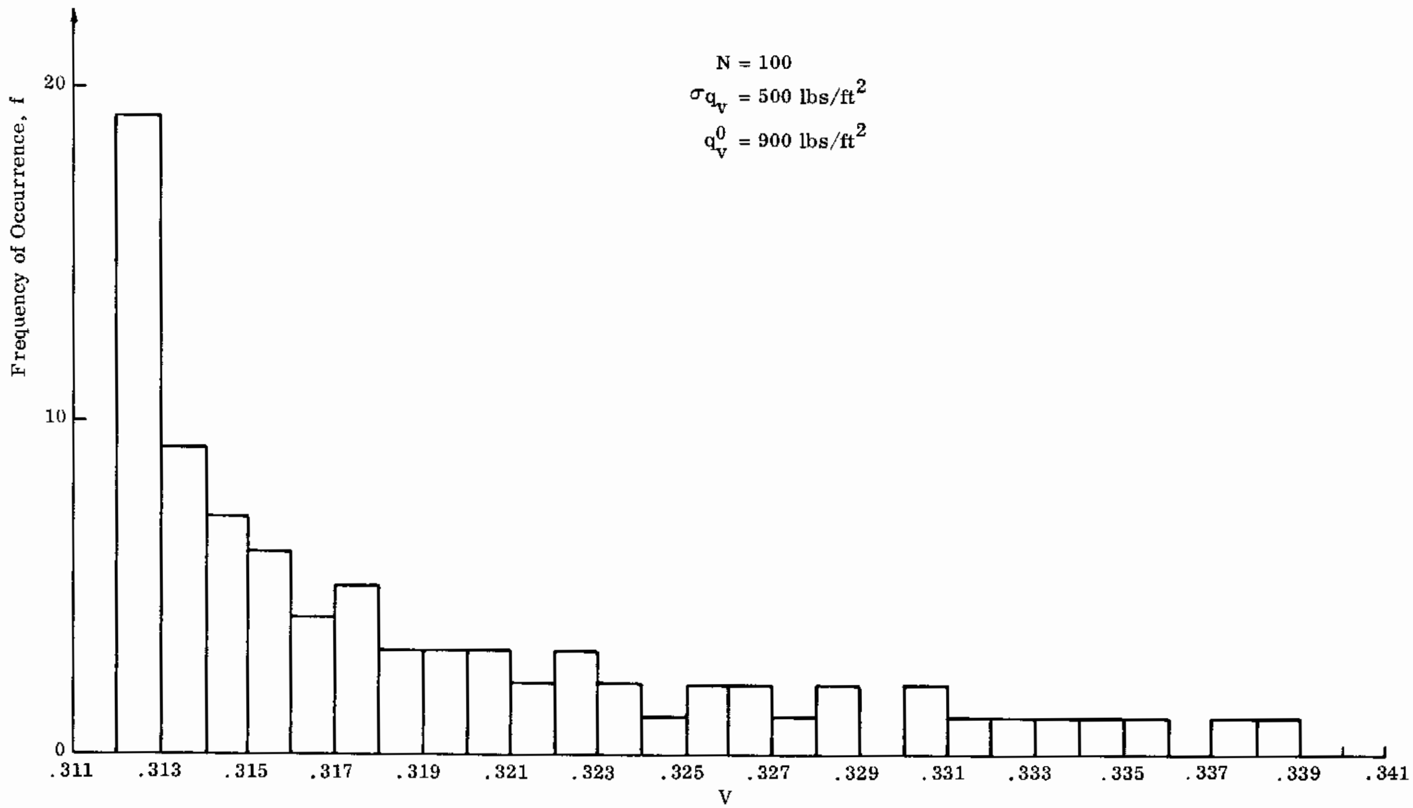


Fig. 10 Performance Histogram, Correlated Coefficients No. 2

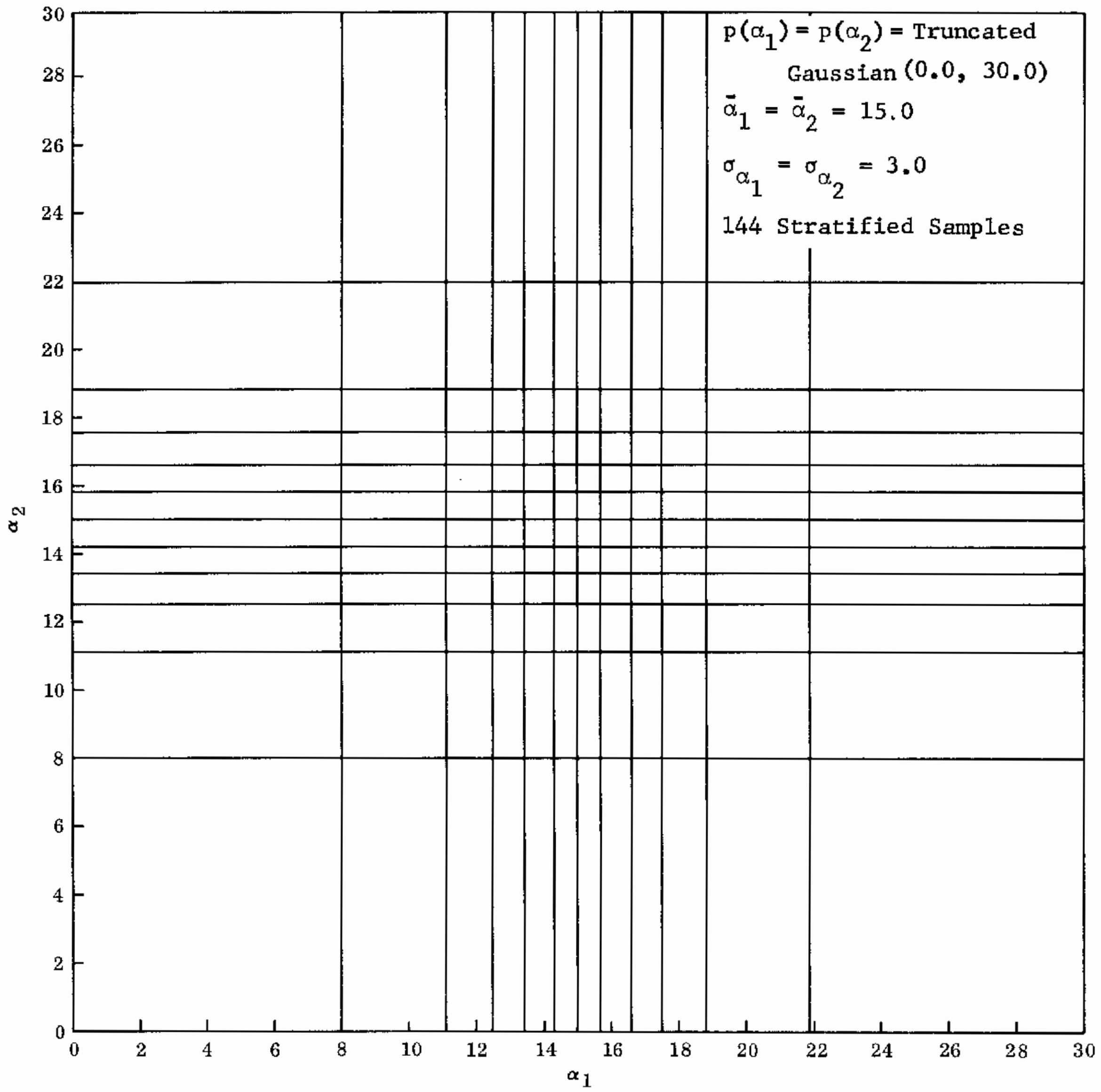


Fig. 11 Two-Dimensional Equiprobability Grid

to improve accuracy, one can sample more densely in some squares and weight the samples accordingly. In our case, the sensitive region near $\alpha_1 = \alpha_2 = 0$, was sampled ten times more densely than the remainder of the region of parameter variation, and improvement over the nonvariable-density stratified sampling in variance computation was noted.

In Table IV a comparison of the results for the variance estimated by various techniques is given. In the first column the results of the integration $\iint [V(\alpha_1,\alpha_2) - \bar{V}]^2 \, p(\alpha_1,\alpha_2) \, d\alpha_1, d\alpha_2$ are given in order to have a standard of comparison for the sampling scheme. The results clearly favor the variable density stratified sampling for estimation of the variance (the direct integration is used as the standard of comparison). The straightforward Monte Carlo sampling has a tendency to underestimate the variance consistently. This again points out the danger in using simple random sampling indiscriminately. Note that any fixed number of Monte Carlo runs will eventually give poor results as the variance increases, regardless how sophisticated the sampling scheme.

Table IV also shows the results of estimating the variance via the Taylor series expansions with the ordinary partials and mean partials. While the one-dimensional results were very good in this respect, the two-dimensional results leave much to be desired. It is clear that $V(\alpha_1,\alpha_2)$ is not sufficiently linear for a first order expansion to yield a valid approximation of the variance. This procedure for estimating variances is definitely not to be used unless it can be justified by some a priori knowledge of the function.

4. CONCLUSIONS

Some problems in describing the properties of a class of stochastic flight control systems have been discussed. Various

TABLE IV

VARIANCE ESTIMATES TWO-DIMENSIONAL CASE

σ_{α_1}	σ_{α_2}	1	2	3	4	5
3	3	.833	.618	.438†	.364	.300
3	2	.505	.539	.378	.306	.230
3	1	.451	.489	.486	.266	.187
2	3	.466	.270	.229	.202	.204
1	3	.378	.155	.131	.132	.146

1 Integration $\iint (V - \bar{V})^2 \, p(\alpha_1, \alpha_2) \, d\alpha_1 \, d\alpha_2$

2 Variable-density stratified sampling 109 x 109 grid

3 Monte Carlo 1000 runs

†11,881 runs

4 Variance Estimate using Mean Partial Derivatives

5 Variance Estimate using partial derivatives evaluated at nominal

Note: $p(\alpha_1, \alpha_2) = p(\alpha_1) \, p(\alpha_2)$

$p(\alpha_1)$, $p(\alpha_2)$ Gaussian

$\bar{\alpha}_1 = \bar{\alpha}_2 = 15.0$, $x_1(0) = x_2(0) = 5.0$

$q = 0.2$, $r = 1.0$, $c = 0.5$

$\sigma_{\alpha_1} = \sigma_{\alpha_2} = 3.0$

numerical methods of computing performance statistics and measuring sensitivity have been developed and compared.

From the results of this study it is evident that in the absence of closed form solutions each available numerical technique suffers from one deficiency or another. In the least complex case (one-dimensional random parameter), the most reliable data can be obtained by methods other than sampling, namely transformation of variables (analytical or numerical). However, because of small increases in complexity of the stochastic components of the problem, there is no alternative but to use sampling techniques. In two dimensions an inversion of the kind described in Eq. (2-11) is possible, although difficult. In three or more dimensions, such a method would be out of the question. On the other hand, sampling, while far less sensitive to dimensionality, presents another set of problems. The fundamental problem is that it is usually not possible to assess the accuracy of the estimates derived from the sample data unless we are willing to use nonparametric techniques such as the percentile estimation of Section II.2.b. However, nonparametric methods yield very wide confidence bands around the estimate, and this tends to reduce the significant information one can draw from the data. The solution then is to increase the sample size, but this increase is, of course, limited by the economics of computation.

While the absolute accuracy obtainable under a Monte Carlo simulation may be less than we would like, the solutions obtained with it certainly have value both as qualitative indicators of the system performance and as initial estimates for design information. With clever programming and careful initial planning, program running time can be made reasonable. Sophisticated sampling techniques may be designed to obtain the maximum information for a given sample size.

With regard to sensitivity of stochastic systems it was shown that the standard deviation ratio and the mean partial derivative are worthwhile alternatives to the classical notion of sensitivity. These measures provide the designer with useful indicators of system behavior. Further experience is needed for their thorough evaluation.

5. COMPUTER PROGRAMS

a. Program Manual - Program I

Program I submitted in fulfillment of Contract AF33(615)-2431 was written in FORTRAN IV language for the IBM 7090/7094 IBJOB Processor Component. The program performs a statistical analysis of the performance of dynamic systems whose parameters are functions of one or two random variables. Various outputs are available, including calculations of sample moments and numerical ranking of performance for histogram preparation.

The random variables are obtained from either a Gaussian or uniform distribution (via random number generators), although with slight modification other distributions are possible. For the Gaussian case, the samples can be stratified with an attendant increase in accuracy. Two sets of dynamics are programmed, a second and a third order case, both linear. The performance index is the integral of the weighted sum of squares of the state variables and the controls. Again with slight modification, different dynamics, and a different performance index (assuming one has available the nominal feedback coefficients) can be used. It is assumed here that specification of the dynamics includes the functional relation between system parameters and the random variables.

The program is a composite of several smaller programs developed during the course of the study. There are in fact nine options available to the operator (Table V). The choice among these options requires the specification of two control words that direct the logical flow through the sections used in common and through those sections unique to each option.

TABLE V

PROGRAM OPTIONS, PROGRAM I

OPTION NUMBER	SAMPLING OPTION	DYNAMICS OPTION	CONTROL WORD 1	CONTROL WORD 2
1	BOXNO	CHECK V	+1	+1
2	BOXNO	DYN 1	+1	0
3	BOXNO	DYN 2	+1	-1
4	RDM	CHECK V	0	+1
5	RDM	DYN 1	0	0
6	RDM	DYN 2	0	-1
7	STRAT	CHECK V	-1	+1
8	STRAT	DYN 1	-1	0
9	STRAT	DYN 2	-1	-1

The program has storage for seventy data words composed of problem data, flow control words, control constants for certain of the subroutines used, print control, and several extra words. All nine options use a portion of this data block, and individual options require various combinations of the remainder. Table VI defines all the data words and lists a set of sample data. Table VII breaks down the total data matrix into sections required by each of the options. It is intended that the operator take the assembled program and included subroutines, choose the option to be computed, then assemble the required data from Table VII, and submit the deck to the computer.

TABLE VI

DATA DEFINITION, PROGRAM I

DATA	DEFINITION	WHERE USED	SUBROUTINE	TYPICAL VALUE
1	q/r	ALL OPTIONS		.2
2	r			1.0
3	c			.5
4	σ_{α_1}			3.0
5	σ_{α_2}			3.0
6	$x_1(0)$			5.0
7	$x_2(0)$			5.0
8	α_1^0			15.0
9	α_2^0	ALL OPTIONS		15.0
10	$x_3(0)$	WITH DYNAMICS		0.0
12	STEP COUNTER			109.
13	DTAU		MAGIC	1.
14	DELTA			9.0×10^{-7}
15	EPSIL			3.0×10^{-5}
16	ERR(1)			1.0×10^{-4}
17	ERR(2)			1.0×10^{-4}
18	ERR(3)	WITH DYNAMICS	MAGIC	1.0×10^{-4}
24	CONTROL WORD 1 (SAMPLING)	ALL OPTIONS		-1, 0, +1
26	FINE GRID LIMITS	OPTIONS 7,8,9		0.
27				0.095
28				0.
29	FINE GRID LIMITS			0.095
30	DY RATIO			1.0
31	COUNTER			1.0
32	DY (ROUGH GRID INC)		(½ coarse grid)	.005
33	COUNTER	OPTIONS 7,8,9		109.0
36	CHECK OUT COUNTER	ALL OPTIONS		-1000.
37	CONTROL WORD 2 (DYNAMICS)	ALL OPTIONS		+1, 0, -1
38	$x_4(0)$	DYN 2		5.0
39	L_α NOMINAL			.0016
40	L_{δ_ϵ} NOMINAL			.0002
41	M_α NOMINAL			-.1569
42	$M_{\dot\alpha}$ NOMINAL			-.0004
43	$M_{\dot\theta}$ NOMINAL			-.0011
44	M_{δ_ϵ} NOMINAL			-.1131
45	τ			.15
46	q_{11}			2.0
47	q_{22}			2.0
48	q_{33}			1.0
49	k_1			-.5024
50	k_2			-3.5722
51	k_3			.0588
52	ERR(4)	DYN 2	MAGIC	1.0×10^{-4}
53	% CHANGE STOP TEST	DYN 1 & 2		.0001
54	TIME MAX	DYN 1 & 2		20.
55	MIN Δ_t TEST	DYN 1 & 2		1.0×10^{-8}
61	K_1	DYN 2		2.5×10^{-4}
62	K_2			3.36×10^{-5}
63	K_3			-1.55×10^{-2}
64	K_4			-4.18×10^{-5}
65	K_5			-1.19×10^{-4}
66	K_6	DYN 2		-1.10×10^{-2}

TABLE VII

PROGRAM OPTION DATA REQUIREMENTS, PROGRAM I

OPTION	DATA WORDS
1	1 - 9, 12, 13, 24, 33, 37, 54
2	1 - 10, 12 - 18, 24, 33, 37, 54
3	1 - 10, 12 - 18, 24, 33, 37 - 52, 54
4	same as 1
5	same as 2
6	same as 3
7	same as 1
8	same as 2 plus 26 - 29, 31, 32, 33
9	same as 3 plus 26 - 29, 31, 32, 33

(1) Dynamics Option "Check V" and "Dynamics I" refer to the simplified pitch controller example described by Eqs. (2-6), (2-7), and (2-8). Because the performance V (called check V in the program) can be readily determined as a closed-form function of the parameters in this example, it is inefficient to use computer time for the numerical integration of the system dynamics. Therefore, check V, which is the closed-form function of Eq. (2-10), will be used in most statistical analyses of this example. However, when check V is not a valid solution (e.g., finite optimization interval or nonzero weighting of final miss), then the Dynamics I option should be employed. The Dynamics II option gives the more complex pitch controller model of Eq. (2-12). Subroutine MAGIC used in integrating the dynamics is discussed in Ref. 9.

(2) Sampling Options The BOXNO option refers to the case where one or two normally distributed random variables are required, and simple random sampling is to be employed. The subroutine used for this option is discussed in Appendix B.

RDM refers to the case where one or two uniformly distributed random variables are required. The subroutine used for this option is discussed in Appendix B.

STRAT is an option that divides the α_1, α_2 plane into equiprobability regions for normally distributed random variables. A sample is then selected at the midpoint of the region. Subroutines INV and PHI, which are discussed in Appendix C, are called in STRAT.

The function of STRAT can be explained with the aid of Fig. 12. Let $\Phi(x)$ be the distribution function of a normally distributed random variable with zero mean and unit variance, i.e.,

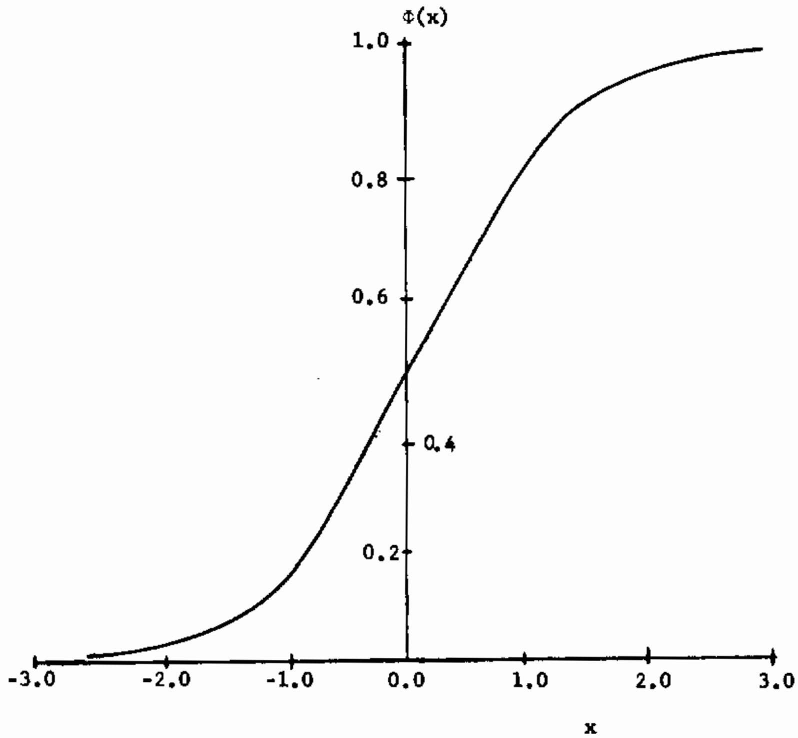


Fig. 12 Normal Distribution Function

$P(x < x_o) = \Phi(x_o)$. If the probability axis of $\Phi(x)$ is divided into equal intervals, and if from each of these intervals an equal number of values of the function $\Phi^{-1}(P) = x$ are obtained, these values of x will be normally distributed with zero mean and unit variance. This is a simple transformation of the uniformly distributed P to the normally distributed x via

$$\Phi(x) = P, \quad 0 \leq P \leq 1$$

$$\Phi^{-1}(P) = x, \quad [x \text{ normally distributed with } \bar{x} = 0, \ \sigma_x^2 = 1] \ .$$

Now to scale x to the appropriate normal distribution we simply multiply x by the standard deviation and add the mean to get

$$\alpha_1 = \sigma_{\alpha_1} x + \bar{\alpha}_1 \ .$$

Similarly, the α_2-axis can be subdivided to produce a grid as in Fig. 11. A further refinement is the option to sample a portion of each variable in greater detail (see Fig. 13). That is, the operator may wish to sample from (c,d) more densely than in the interval exterior to (c,d). This region of fine grid is to lie between constants 26 and 27 in one direction, and between constants 28 and 29 in the other. Constant 32 is one-half the magnitude of the rough grid step, and constant 30 is the ratio of fine grid to rough grid. Constants 12 and 33 are the total number of grid steps to be taken in the respective directions. The operator must calculate these constants based upon the choices of the above constants. For example, with a rough grid step of 0.1, a ratio of 100 between fine and rough grids, and limiting the fine grid between 0.0 and 0.1, there are 109 total steps in the chosen direction. The odd number

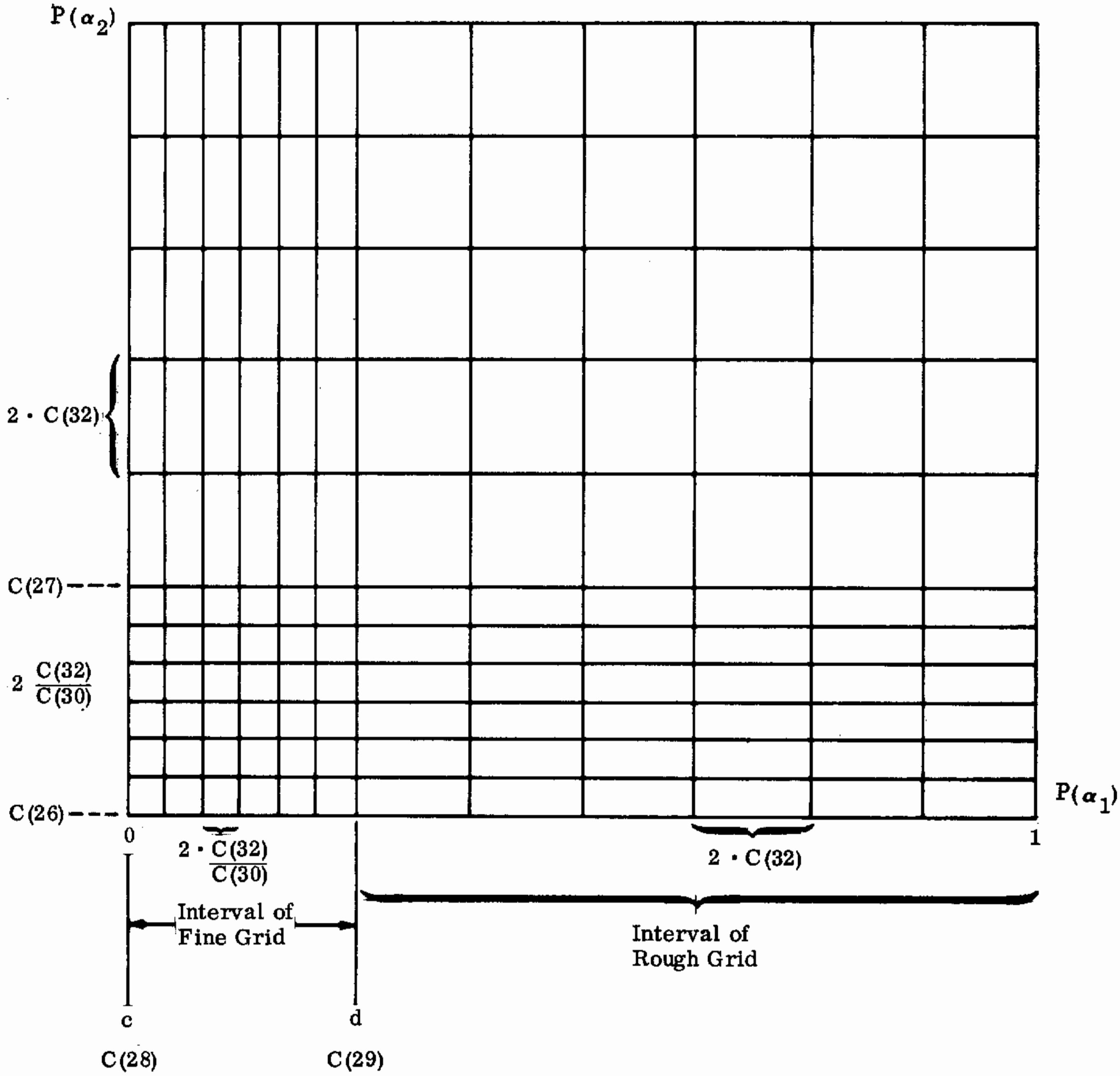


Fig. 13 Variable Density Grid

develops from the fact that we sample at the midpoint of each step.

(3) Data Input Data are to be submitted with the following format, I1, I4, 1E14.7. This format defines three words (I, II, D) per card. Word I is a key word which determines whether to store D in location II or if the entire data packet has been read. If I = 2, D is stored. If I = 1, D is ignored and execution is initiated. A further option is included. If I = 0, the program will branch to a section where the contents of the array V will be ranked and listed. This is normally done only at the conclusion of a series of runs. A data input packet then consists of data cards with a 2 in column 1, an address in columns 2-5, and a data word in columns 6-19. Following this packet is a card with a 1 in column 1 to initiate execution.

(4) Control Words There are in Program I a total of twelve control words. The functions of control words 1 and 2, located in positions 24 and 37, have been covered in Table I. In options 2, 3, 5, 6, 8, and 9 (control word 1, 0, -1), where the integrating routine MAGIC is used, a print control DTAU is available. Output is printed for the integrated sections only every DTAU time units.

Constant 36 is a control word to obtain additional printout for the verification of results. Under normal control, constant 36 is zero, and only the final values of each sample scan is printed. Changing constant 36 from zero will cause full detail scans to be listed up to constant 36.

b. Program Manual - Program II

Program II, submitted in fulfillment of Contract AF33(615)-2431, was written in FORTRAN IV language for the IBM 7090/7094 IBJOB Processor Component. This program performs a nonsampling analysis of dynamic systems whose parameters are functions of one random variable, α.

The program initially steps α through its range of variation; at each step the performance V is calculated either by integrating the system dynamics or by using a closed-form function if available. Thus a numerical function $V(\alpha)$ is obtained, stored, and printed if desired. With this function and a given distribution of α, (the program is equipped for either Gaussian or uniform distribution, although others can be handled with slight modification) the mean and variance are computed by direct integration. Also the probability density function $p(V)$ is computed at the operator's option. (Subroutine PHI is used in this part.)

Various sensitivity coefficients, such as the first partial derivative, mean partial derivative, and least squares linearization can be computed according to the operator's option. (Although the latter two are equivalent for Gaussian distributions, the least squares option is available for comparison in the case of other distributions.) Variance estimates that use these coefficients can be computed.

The program is a composite of several smaller specialized programs used during the course of the study. Table VIII outlines the options and the functions of three control words.

The program has storage for ninety data words composed of problem, data, flow control words, control constants for subroutines, print control, and extra and unused words. Table IX lists all the data words, defines them, and includes a set of sample data.

TABLE VIII

PROGRAM OPTIONS, PROGRAM II

OPTIONS	DYNAMICS OPTION	ANALYSIS OPTION	CONTROL WORD 1	CONTROL WORD 2
1	CHECK V	GAUSSIAN	+1	+1
2	DYN 1	GAUSSIAN	0	+1
3	DYN 2	GAUSSIAN	-1	+1
4	CHECK V	UNIFORM	+1	-1
5	DYN 1	UNIFORM	0	-1
6	DYN 2	UNIFORM	-1	-1

Control Word 3 determines whether variable α_1 or α_2 is being incremented.

When the CHECK V dynamics option is chosen, the program will calculate the sensitivity coefficients, mean partial derivative, least squares linearization (equivalent to mean partial derivative for Gaussian distribution), and the performance mean and variance. When either of the dynamics options are chosen, the program computes the mean, variance, and the least squares coefficient.

There are in Program II all of the control words of Program I and one additional word. The function of word 1, location 24, is now to choose between a Gaussian and Uniform distribution. Word 2, location 37, remains as the control for CHECK V, DYN 1, and DYN 2. The new word mentioned, word 3, location 38, has the function of controlling whether α_1 or α_2 is being incremented. If word 3 is positive, the program will increment α_1;

TABLE IX

DATA DEFINITION, PROGRAM II

DATA WORD	DEFINITION	WHERE USED	SUBROUTINE	TYPICAL VALUE
1	q/r	ALL OPTIONS		.2
2	r			1.0
3	c			.5
4	σ_{α_1}			3.0
5	σ_{α_2}			3.0
6	$x_1(0)$			5.0
7	$x_2(0)$			5.0
8	α_1^0			15.0
9	α_2^0	ALL OPTIONS		15.0
10	$x_3(0)$	WITH DYNAMICS		0.
12	STEP COUNTER			3000.
13	DTAU		MAGIC	1.
14	DELTA			9.0×10^{-7}
15	EPSL			3.0×10^{-5}
16	ERR(1)			1.0×10^{-4}
17	ERR(2)			1.0×10^{-4}
18	ERR(3)	WITH DYNAMICS	MAGIC	1.0×10^{-4}
19	UPPER LIMIT	UNIFORM DIST.		
20	LOWER LIMIT			
21	UPPER LIMIT			
22	LOWER LIMIT	UNIFORM DIST.		
24	CONTROL WORD 1	UNIFORM/GAUSSIAN		+1, -1
37	CONTROL WORD 2	DYNAMICS		+1, 0, -1
38	$x_4(0)$	DYN 2		5.0
39	L_α			.0016
40	L_{δ_ϵ}			.0002
41	M_α			-.1569
42	$M_{\dot{\alpha}}$			-.0004
43	$M_{\dot{\theta}}$			-.0011
44	M_{δ_ϵ}			-.1131
45	τ			.15
46	q_{11}			2.0
47	q_{22}			2.0
48	q_{33}			1.0
49	k_1			-.5024
50	k_2			-3.5722
51	k_3			.0588
52	ERR(4)	DYN 2	MAGIC	1.0×10^{-4}
55	% CHANGE STOP TEST			.0001
56	TIME MAX			20.
57	MIN Δt TEST			1.0×10^{-8}
61	K_1	DYN 2		2.5×10^{-8}
62	K_2			3.36×10^{-5}
63	K_3			-1.55×10^{-2}
64	K_4			-4.18×10^{-5}
65	K_5			-1.19×10^{-4}
66	K_6	DYN 2		-1.10×10^{-2}

if word 3 is negative, the program will increment α_2. The control words having to do with the integration routine MAGIC are explained in Ref. 9.

c. Program Manual - Program III

Program III submitted in fulfillment of Contract AF33(615)-2431, was written in FORTRAN IV language from the IBM 7090/7094 IBJOB Processor Component. This program computes P from the 3 x 3 Riccati matrix equation

$$-\dot{P}(t) = F^T P(t) + P(t)F - P(t)\, GR^{-1}G^T P(t) + H^T QH \quad . \qquad (2\text{-}13)$$

$$P(t_1) = S$$

where

G = input matrix,

F = the system matrix,

H = output matrix,

R = a positive definite matrix which weights each input in the performance index,

Q = a positive definite matrix which weights each output in the performance index, and

S = nonnegative definite matrix which weights each of the state variables at the final time, t_1.

Once P is known, the control which minimizes V is obtained via

$$u^o = -R^{-1}G^T Px \quad ,$$

where

$$\dot{x}(t) = F(t)\, x(t) + G(t)\, u(t)$$

$$y(t) = H(t)\, x(t)$$

$$x^T P(t_o)x = 2V = \|x\|^2_{P(t_o)} = \|x(t_1)\|^2_S + \int_{t_o}^{t_1} \left(\|y(\tau)\|^2_Q + \|u(\tau)\|^2_R\right) d\tau \quad .$$

For further details the reader is referred to Ref. 7. If $t_1 \to \infty$, $S \to 0$, $P(t)$ approaches a constant, and for a constant F matrix, then the result will be a linear closed-loop system with feedback parameters whose gains are constants.

The program solves for P by replacing Eq. (2-13) with a set of 6 simultaneous ordinary differential equations (P is symmetric and therefore has only 6 independent elements), and solving them backwards in time with $P(t_1) = 0$. This was accomplished by substituting $\dot{P}$ for $-\dot{P}$ in Eq. (2-13). The integration is terminated when a steady-state condition is reached.

The actual system under consideration is that of Eq. (2-12) so that matrices of the following form were programmed:

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} , \quad R = I \quad , \quad S = 3 \times 3 \ \text{null matrix}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 1/\tau \end{bmatrix}$$

For this case,

$$u^o = - \begin{bmatrix} 0 & 0 & \frac{1}{\tau} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$u^o = - \begin{bmatrix} \frac{P_{13}}{\tau} & \frac{P_{23}}{\tau} & \frac{P_{33}}{\tau} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$u^o = - \frac{1}{\tau} \left(P_{13}x_1 + P_{23}x_2 + P_{33}x_3 \right) \quad .$$

For the sample data in Table X

$$u^o = (.507x_1 + 3.59x_2 - 0.0592x_3) \quad .$$

The data input format is identical to that of Program I with the single exception that a blank card terminates each data packet and initiates execution. The program solves six differential equations by the use of the subroutine MAGIC (see Ref. 9). The data are defined in Table X.

The output of Program III is a listing of the data for verification and a complete time history of the integration.

d. Program Manual - Program IV

Program IV, submitted in fulfillment of Contract AF33(615)-2431, was written in FORTRAN IV language for the IBM 7090/7094 IBJOB Processor Component. This program evaluates the definite integral

$$\bar{V} = \int_{b_1}^{b_2} \int_{a_1}^{a_2} V(\alpha_1,\alpha_2)\ p(\alpha_1)\ p(\alpha_2)\ d\alpha_1\ d\alpha_2$$

and

$$\sigma^2 = \int_{b_1}^{b_2} \int_{a_1}^{a_2} \left(V(\alpha_1,\alpha_2) - \bar{V}\right)^2 p(\alpha_1)\ p(\alpha_2)\ d\alpha_1\ d\alpha_2 \quad ,$$

where $V(\alpha_1,\alpha_2)$ is defined in Eq. (2-10) with α_2 substituted for α_2^0. The above equations assume α_1 and α_2 to be uncorrelated. $p(\alpha_1)$ and $p(\alpha_2)$ are selected to be either uniform or Gaussian (in any combination) by the operator. This program can be used as a check on the means and variances computed in the sampling Program I.

TABLE X

DATA DEFINITION, PROGRAM III

DATA	DEFINITION	WHERE USED	TYPICAL VALUE
1	f_{11}†	MAIN PROGRAM	-.0016
2	f_{21}	↓	-.1569
3	f_{31}	↓	0.
4	f_{12}	↓	1.
5	f_{22}	↓	-.0015
6	f_{32}	↓	0.
7	f_{13}	↓	-.0002
8	f_{23}	↓	-.1131
9	f_{33}	↓	-6.67
10	τ	↓	.15
11	q_{11}	↓	2.0
12	q_{22}	↓	2.0
13	$p_{11}(0)$	↓	0.0
14	$p_{12}(0)$	↓	0.0
15	$p_{13}(0)$	↓	0.0
16	$p_{22}(0)$	↓	0.0
17	$p_{32}(0)$	↓	0.0
18	$p_{33}(0)$	MAIN PROGRAM	0.0
19	DELTA	MAGIC	
20	ERR(1)	↓	
21	ERR(2)	↓	
22	ERR(3)	↓	
23	ERR(4)	↓	
24	ERR(5)	↓	
25	ERR(6)	↓	
26	EPSIL	MAGIC	
27	DTAU	MAGIC PRINT CONTROL	
28	T STOP	MAIN PROGRAM	
29	Δt*	MAGIC	

*Δt must be defined if it is desired to proceed with a fixed grid. In this case DELTA is set to 0 and the six ERR are not used. (See Ref. 9 for more information).

†f_{ij} = entries in system matrix

The program evaluates the above integrals by converting to a set of two ordinary differential equations, and integrating over a two-dimensional grid via subroutine MAGIC. The program integrates over a strip of constant α_2, and then adds the contributions of each strip, working outwards from the mean α_2^0. The data input format is identical to that of Program I and is defined in Table XI. The output is a listing of the data for verification and values for $\bar{V}$ and σ^2.

TABLE XI

DATA DEFINITION, PROGRAM IV

DATA	DEFINITION		TYPICAL VALUE
1	q/r		.2
2	r		1.0
3	c		0.5
4	σ_{α_1}		3.0
5	σ_{α_2}		3.0
6	$x_1(0)$		5.0
7	$x_2(0)$		5.0
8	α_1^0		15.0
9	α_2^0		15.0
12	NUMBER OF GRID POINTS		
14	DELTA	MAGIC	9.0×10^{-7}
15	EPSIL	MAGIC	3.0×10^{-5}
16	ERR(1)	MAGIC	1.0×10^{-4}
17	ERR(2)	MAGIC	1.0×10^{-4}
25	GRID SIZE		.01

SECTION III

OPTIMAL STOCHASTIC CONTROL

1. INTRODUCTION

In the previous section, techniques of evaluating dynamic systems whose parameters are random variables were considered. The control laws, however, were derived on the basis of the nominal values of these parameters. That is, in order to derive the optimal feedback coefficients, the system was considered deterministic. We now shift our emphasis to the problem of the derivation of optimal control laws for stochastic systems. Such laws take into account the a priori knowledge of the statistical distribution of the parameters.

The particular problem chosen is one which was suggested by the Flight Dynamics Laboratory. While apparently simple, it demonstrates some of the pathology of stochastic systems.

The general procedure for treating stochastic systems has been to derive a recursion relation for the optimal loss function via dynamic programming. In this example it is shown that such a recursion equation cannot be derived.

The calculus of variations is used to analyze the continuous version of the problem for both open-loop and feedback control laws. Explicit optimal control laws are obtained for both finite and infinite optimization intervals in the open-loop case. In the feedback case, only control laws that are linear functions of the state are considered, for a reason that is explained in the treatment of this case. For the finite optimization interval, a differential equation for the optimal feedback gain is derived. A sufficient condition for the existence of a steady state law for the infinite interval is also obtained.

A hybrid case, called augmented feedback control, is also examined. It is shown that the controller has sufficient information to infer the value of the random variable arbitrarily soon after the starting point and proceed as if it were a deterministic problem. However, the supposedly optimal control fails to satisfy a Lipschitz condition, and is thereby invalidated.

It is shown that, in general, the optimal control law is a function of the initial time. That is, an optimal control from some intermediate time to the final time is different for different starting times. This behavior is linked to the way the random variable enters the problem and not because of any specification on the control law. It is this curious behavior that causes the failure of dynamic programming. This is demonstrated clearly in the treatment of the discrete version of the same problem.

In this section, t denotes the beginning of the optimization interval, T denotes the end of the optimization interval, and τ denotes the running time variable between these limits. For the discrete case t, T, and τ are replaced by n, N, and k, respectively.

2. PROBLEM STATEMENT

Consider the stochastic dynamic system,

$$\dot{x}(\tau) = bu(\tau) \quad , \quad x(t) = c \; , \tag{3-1}$$

where x is the state, u is the control, and b is a time invariant random variable with density function $p(b)$. The trajectory generated by Eq. (3-1) is $x(\tau|c,b)$. We seek the control that minimizes the performance index, defined by

$$S(c,t,T) = E \int_t^T \left[x^2(\tau|c,b) + \lambda u^2(\tau)\right] d\tau \ . \qquad (3\text{-}2)$$

Here, and in what follows, E denotes the expectation operator. Since b is the only random variable, E denotes integration with respect to b, with p(b) as a weighting function. The corresponding discrete problem is

$$x_{k+1} = x_k + bu_k\Delta \quad , \quad x_n = c \ , \qquad (3\text{-}3)$$

$$S(c,n,N) = E \sum_n^N \left(x_k^2 + \lambda u_k^2\right)\Delta \ , \qquad (3\text{-}4)$$

where

$$\Delta = (T - t)/(N - n) \ . \qquad (3\text{-}5)$$

To complete the problem statement it is necessary to specify what data are available to the controller at each time τ, or each step k. Three cases will be considered:

Case I. The controller is of the open-loop type, being a function of the initial state, initial time, final time, and present time.

Case II. The controller may depend on the same variables as in Case I, and in addition, on the present state. This will be called an augmented feedback controller.

Case III. The controller, of the feedback type, may depend on the present state, the initial time, final time, and present time.

In the analysis of the continuous cases, it will sometimes be necessary to interchange expectation and integration with respect to time. For the functions encountered in this problem, the interchange is valid for any sensible $p(b)$. This question is not relevant to anything in the problem and will not be commented on further.

Throughout the discussion we denote the moments of $p(b)$ by

$$m_k = Eb^k = \int_{-\infty}^{\infty} b^k p(b)\, db \ . \tag{3-6a}$$

Also, σ denotes the standard deviation:

$$\sigma^2 = m_2 - m_1^2 \ . \tag{3-6b}$$

3. CONTINUOUS TIME VERSION

a. Case I

The open-loop control law that minimizes Eq. (3-2) is sought. To derive the optimal control, t, T, and c are held fixed. It is convenient to define

$$w(\tau) = \int_{t}^{\tau} u(\eta)\, d\eta \qquad t \leq \tau \leq T \ . \tag{3-7}$$

Then

$$x(\tau|c,b) = c + bw(\tau) \tag{3-8}$$

and

$$S(c,t,T) = E\int_t^T \left[(c + bw(\tau))^2 + \lambda u^2(\tau)\right] d\tau \ . \tag{3-9}$$

The expected value of the integrand is denoted by $H(w(\tau), u(\tau),\tau)$:

$$H(w,u,\tau) = c^2 + 2cm_1 w + m_2 w^2 + \lambda u^2 \ . \tag{3-10}$$

The problem is in standard form for the calculus of variations:

$$S(c,t,T) = \int_t^T H(w(\tau),u(\tau),\tau)\ d\tau \tag{3-11a}$$

$$u(\tau) = \dot{w}(\tau) \tag{3-11b}$$

$$w(t) = 0 \tag{3-11c}$$

$$w(T) \qquad \text{is free.} \tag{3-11d}$$

The optimal solution satisfies the Euler-Lagrange equation

$$\frac{d}{d\tau}\left[\frac{\partial H}{\partial u}\right] - \frac{\partial H}{\partial w} = 0 \ . \tag{3-12}$$

Using Eq. (3-11b) and simplifying yields

$$\ddot{w}(\tau) - \frac{m_2}{\lambda}\, w(\tau) = \frac{cm_1}{\lambda} \ . \tag{3-13}$$

One boundary condition is provided by Eq. (3-11c). Equation (3-11d) gives the transversality condition

$$\dot{w}(T) = 0 \tag{3-14}$$

for the second boundary condition. The solution is

$$w(\tau) = \frac{cm_1 \cosh[\alpha(T - \tau)]}{m_2 \cosh[\alpha(T - t)]} - \frac{cm_1}{m_2}, \tag{3-15a}$$

$$\alpha = \sqrt{m_2/\lambda}. \tag{3-15b}$$

The derivative gives the optimal open-loop control, valid for all c, t, and T (provided $t \leq \tau \leq T$):

$$u^o(c,t,T,\tau) = -\frac{c\alpha m_1 \sinh[\alpha(T - \tau)]}{m_2 \cosh[\alpha(T - t)]}. \tag{3-16}$$

As $T \to \infty$, the optimal open-loop control approaches a limit,

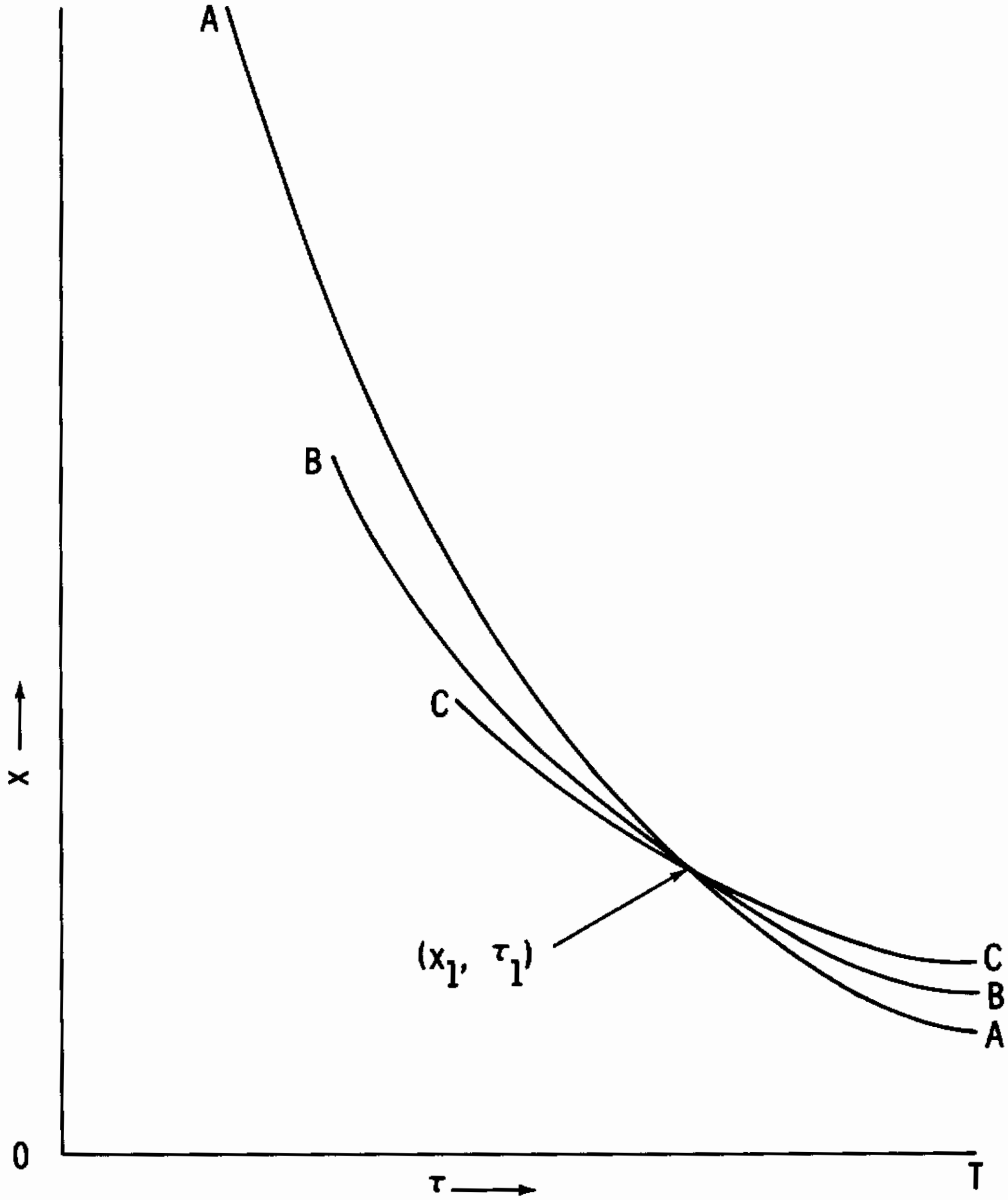
$$u^o(c,t,\infty,\tau) = -\frac{c\alpha m_1 e^{-\alpha(\tau-t)}}{m_2} \tag{3-16a}$$

The optimal open-loop performance index can be evaluated from Eqs. (3-10), (3-11a), (3-15), and (3-16):

$$S^o(c,t,T) = c^2 \left(1 - \frac{m_1^2}{m_2}\right)(T - t) + \frac{c^2 m_1^2}{m_2 \alpha} \tanh[\alpha(T - t)]. \tag{3-17}$$

From Eq. (3-6b) it is evident that S^o goes to infinity linearly with T, unless σ^2 of b is zero.

The optimal open-loop control law has a very curious property. Consider an arbitrary point in the x-τ plane, (x_1, τ_1), a fixed final time T, and consider all the optimal trajectories that pass through (x_1, τ_1) with b taking on the same value, b_1, for all trajectories (see Fig. 14). All such trajectories are characterized by having a starting point, $t < \tau_1$, and an initial state,



NOTE: ALTHOUGH TRAJECTORIES A, B, AND C COINCIDE AT (x_1, τ_1), THEIR OPTIMAL CONTROLS, REFLECTED BY THE TRAJECTORIES' SLOPES, ARE DIFFERENT IN THE INTERVAL $[\tau_1, T]$. SEE SECTION III.3.a, FOR DISCUSSION.

Fig. 14 Open-Loop-Optimum Trajectories

$$c = x_1 \Bigg/ \left[1 + \frac{b_1 m_1}{m_2}\left(\frac{\cosh[\alpha(T - \tau_1)]}{\cosh[\alpha(T - t)]} - 1\right)\right] . \qquad (3\text{-}18)$$

Unless $b_1 = m_2/m_1$, the optimal control at (x_1, τ_1) depends not only on x_1, τ_1, and T, but also on the starting time t:

$$u^o = - \frac{x_1 \alpha \sinh[\alpha(T - \tau_1)]}{b_1 \cosh[\alpha(T - \tau_1)] + \left(\frac{m_2}{m_1} - b_1\right)\cosh[\alpha(T - t)]} . \qquad (3\text{-}19)$$

That is, the optimal control depends not only on where you are (x_1) and how much time remains $(T - \tau_1)$, but also on how you got there!

Notice that the open-loop law, Eq. (3-16), in general cannot be converted to an equivalent closed-loop law by considering x_1 and τ_1 to be variables in Eq. (3-19), because the quantity b_1 is not available to the controller. The deterministic case, $\sigma^2 = 0$ in Eq. (3-6b), is an exception. Another exception occurs when both c and x are known; then it may be possible to determine b_1. This possibility is discussed in Case II. In this section it is assumed that only c is known; in Case III, it is assumed that only x is known.

b. Case II

In this case the controller may be a function of c, x, t, T, and τ. At time τ, the knowledge of these variables plus knowledge of the control law from t to τ is generally sufficient to determine b exactly. Knowing b exactly, one would naturally implement the deterministic optimal control law. The deterministic optimal control law, in feedback form, is found by adaptation of Eq. (3-19). Note that $m_2 = b^2$, $m_1 = b$, and $b_1 = b$ in the deterministic case, thus yielding

$$u(x,\tau) = -\frac{x}{\sqrt{\lambda}} \tanh[\alpha(T - \tau)] , \tag{3-20}$$

where

$$\alpha = b/\sqrt{\lambda} . \tag{3-21}$$

The resulting trajectories are

$$x(\tau) = c \frac{\cosh[\alpha(T - \tau)]}{\cosh[\alpha(T - t)]} , \tag{3-22}$$

and the performance index becomes

$$S = \frac{\sqrt{\lambda}\, c^2}{b} \tanh[\alpha(T - t)] . \tag{3-23}$$

The control, Eq. (3-20), can be implemented in the stochastic case if α can be computed from the data available to the controller. On the other hand, α can be computed from Eq. (3-22) if the past control has been Eq. (3-20). In other words, let $\alpha^*(c,x,\tau)$ be the solution of Eq. (3-22); then the optimal control law would be

$$u^*(c,x,\tau) = -\frac{x}{\sqrt{\lambda}} \tanh[\alpha^*(c,x,\tau)(T - \tau)] . \tag{3-24}$$

Should this work, it would be like pulling yourself up by your own bootstraps. It does not seem reasonable that the control can depend on α^* for all $\tau > t$ and that the expression for α^* at τ can depend on what the control was during the entire interval (t,τ). Either the chicken or the egg has to come first!

This suspicion is borne out by a closer mathematical inspection of Eq. (3-1), with Eq. (3-24) substituted for u. It can be shown that $\partial u^*/\partial x$ becomes unbounded as τ approaches t. First, note that, by Eq. (3-22), $\partial x/\partial\alpha$ approaches zero as τ

approaches t. It follows that $\partial\alpha^*/\partial x$ becomes unbounded there, and after a few more computations, that $\partial u^*/\partial x$ also becomes unbounded. Therefore u^* does not satisfy a Lipschitz condition, or any of the weaker conditions that guarantee uniqueness, in any region bordering on the line $\tau = t$ in the $(x-\tau)$ plane, and Eq. (3-1), with u^* inserted, may not have a unique solution. In other words, a value of S cannot be calculated for u^* with any certainty.

To try to overcome this difficulty, we define a new family of control laws, u_ϵ, that are independent of α in the interval $(t, t + \epsilon)$. For simplicity assume $b > 0$ with probability one. If ϵ is sufficiently small, the choice of u in $(t, t + \epsilon)$ has negligible effect on S; the important thing is that it be independent of α. Since the hyperbolic tangent in Eq. (3-24) approaches one for large positive arguments, a choice as good as any is to define u_ϵ in the interval $(t, t + \epsilon)$ by

$$u_\epsilon(c,x,\tau) = -\frac{x}{\sqrt{\lambda}}, \qquad t \leq \tau \leq t + \epsilon . \tag{3-25}$$

Now, knowing c at the beginning of the interval and x at the end, we can find α from the relationship:

$$x(t + \epsilon) = c\, e^{-\alpha\epsilon} . \tag{3-26}$$

Using the deterministic optimal control for $\tau > t + \epsilon$ gives the trajectory

$$x(\tau) = c\, e^{-\alpha\epsilon} \frac{\cosh[\alpha(T - \tau)]}{\cosh[\alpha(T - t - \epsilon)]} . \tag{3-27}$$

Under the hypotheses of this section, the controller cannot "remember" a previously computed value of α, but must continuously evaluate it by solving Eq. (3-27). Let the solution be the function $\alpha_\epsilon(c,x,\tau)$. Then u_ϵ is defined for $\tau > t + \epsilon$ by

$$u_\epsilon(c,x,\tau) = -\frac{x}{\sqrt{\lambda}} \tanh[\alpha_\epsilon(c,x,\tau)(T - \tau)]. \tag{3-28}$$

Let $S_\epsilon(c,t,T)$ be the performance index of u_ϵ, defined by Eq. (3-2), and define

$$\underline{S}(c,t,T) = \underset{\epsilon > 0}{\text{g.l.b.}}[S_\epsilon(c,t,T)] . \tag{3-29}$$

(g.l.b. stands for greatest lower bound). As ϵ approaches zero, S_ϵ approaches the expectation of the right side of Eq. (3-23). Therefore,

$$\underline{S}(c,t,T) = \sqrt{\lambda}\, c^2 E\left[\frac{1}{b} \tanh \frac{b}{\sqrt{\lambda}}(T - t)\right] . \tag{3-30}$$

This lower bound can be approached, but cannot be achieved by any valid control law, i.e., one that guarantees unique solutions of Eq. (3-1). Note that the expectation is always less than infinity for finite T, regardless of the distribution of b, because the integrand is bounded by $(T - t)/\sqrt{\lambda}$.

Note that $\underline{S}$ represents the best that can be achieved, even with perfect information about b; that is, assuming that b varies from sample to sample according to p(b), but is somehow known for each sample, $\underline{S}$ is the minimum <u>average</u> loss. Moreover, with perfect information about b available, the optimal control, Eq. (3-20), is unique. As $\epsilon \to 0$, u_ϵ approaches this optimum. Therefore, any sequence of control laws whose performance indices approach $\underline{S}$ must themselves approach the same limit as u_ϵ. This limit, Eq. (3-24), is <u>nonlinear in x</u> and <u>a function of c</u>.

The following conclusions summarize the previous discussion.

1) In the augmented feedback formulation, the problem stated in III.2 has no optimal control. However, a generalized optimal performance index can be defined via Eq. (3-29).

2) To approach $\underline{S}$ with a valid control law, it is necessary to use a law that is nonlinear in x and a function of c.

3) For finite T, $\underline{S}$ is finite, regardless of the distribution of b. Moreover, $\underline{S}$ increases no faster than logarithmically as T goes to infinity.

c. Case III

In Case III, the controller knows the present state, x, but not the initial state, c. As in all cases, t, T, and τ are also available.

In the previous section (III.3.b), we saw that, when nonlinear feedback laws were allowed, the near-optimal laws were functions of c. Therefore, it does not make sense to seek a feedback law that is functionally independent of c, and simultaneously optimal for all c, unless further restrictions are put on the class of allowable feedback laws. It turns out that if only feedback laws that are linear functions of the present state are allowed, then the same law is optimal (within the linear class) for all starting values. This result is made evident by the fact that the K to be minimized (see Eqs. (3-34) and (3-35) below) is independent of c. (Compare this with Eq. (3-11a) in Case I, where the integral to be minimized depended on c, and therefore the optimal control did also.) Therefore, a sensible formulation of this case, where the control must be independent of c, is to seek the optimal control law of the form

$$u^o(t,T,x(\tau),\tau) = x(\tau)f^o(t,T,\tau) \ . \tag{3-31}$$

As usual, we suppress dependence on t and T in the notation and consider the class of linear feedback controls

$$u(x(\tau),\tau) = x(\tau)f(\tau) \ . \tag{3-32}$$

It is convenient to define

$$y(\tau) = \int_t^\tau f(\eta)\, d\eta \ . \tag{3-33}$$

It follows that

$$S(c,t,T) = K\, c^2 \ , \tag{3-34}$$

where K is defined by

$$K = E \int_t^T (1 + \lambda f^2(\tau))\exp[2by(\tau)]\, d\tau \ . \tag{3-35}$$

To minimize K, the calculus of variations is employed. It is convenient to define

$$F(y) = E\exp[2by] \ , \tag{3-36}$$

$$H(y,f,\tau) = (1 + \lambda f^2)F(y) \ . \tag{3-37}$$

Now

$$K = \int_t^T H(y(\tau), f(\tau), \tau)\ d\tau \ , \tag{3-38a}$$

$$f(\tau) = \dot{y}(\tau) \ , \tag{3-38b}$$

$$y(t) = 0 \ , \tag{3-38c}$$

$$y(T) \qquad \text{is free.} \tag{3-38d}$$

The optimal solution satisfies

$$\frac{d}{d\tau}\left[\frac{\partial H}{\partial f}\right] - \frac{\partial H}{\partial y} = 0 \ , \tag{3-39}$$

which becomes

$$\left[\frac{2F(y)}{F'(y)}\right]\ddot{y} + \dot{y}^2 = \frac{1}{\lambda} \ . \tag{3-40}$$

One boundary condition comes from Eq. (3-38c), the other from the application of the transversality condition to Eq. (3-38d):

$$\dot{y}(T) = 0 \ , \tag{3-41a}$$

$$y(t) = 0 \ . \tag{3-41b}$$

An analytical solution for y does not exist in general, or for any commonly used distribution. Nevertheless, several pertinent observations about the solution can be made.

Keeping in mind that $\dot{y}(\tau)$ is the optimal feedback gain, let $y_1(\tau)$ be the solution of Eqs. (3-40), (3-41a), and (3-41b) for some interval (t,T), and let $t < t_1 < T$. Now set

$$y_2(\tau) = y_1(\tau) - y_1(t_1) \qquad t_1 \leq \tau \leq T . \tag{3-42}$$

Then $y_2(t_1) = 0$, $\dot{y}_2(T) = 0$, and $\dot{y}_2(\tau) = \dot{y}_1(\tau)$, for $t_1 \leq \tau \leq T$. However, $y_2(\tau)$ does not satisfy Eq. (3-40). It follows that $\dot{y}_1(\tau)$ is the optimal feedback gain when t is the starting point, but not when $t_1 > t$ is, despite the fact that the final times are the same. This is similar to the situation with the open-loop control and shows that the dependence of the optimal control on the initial time, as well as the time to go, arises because of the way the randomness enters the system, and not because of any specifications on the control.

It is possible to synthesize a closed-loop control from the optimal open-loop control, Eq. (3-16):

$$\begin{aligned} u(x,\tau,T) &= u^o(x,\tau,T,\tau) \\ &= - x \frac{\alpha m_1}{m_2} \tanh \alpha(T - \tau) . \end{aligned} \tag{3-43}$$

In some types of problems, this synthesis yields an optimal closed-loop control. However, because Eq. (3-43) is a linear control, and does not satisfy Eq. (3-40), it is not optimal.

The factor $2F(y)/F'(y)$ frequently takes a simple form. Here the forms are worked out for two common types of distribution. For the normal distribution,

$$p(b) = \exp\left[- \tfrac{1}{2}(b - m_1)^2/\sigma^2\right] \Big/ \sqrt{2\pi}\, \sigma . \tag{3-44}$$

$F(y)$ is found from the characteristic function of the normal distribution (see Ref. 10, p. 159) by the substitution $\omega = -2iy$. $F(y)$ for the gamma distribution, below, is found in the same way:

$$F(y) = \exp\left[2\sigma^2 y^2 + 2ym_1\right] , \qquad (3\text{-}45)$$

$$2F/F' = (m_1 + 2\sigma^2 y)^{-1} . \qquad (3\text{-}46)$$

For the gamma distribution,

$$p(b) = \mu^{n+1} b^n \exp[-\mu b] \;/\; \Gamma(n+1) , \qquad b \geq 0 ,$$
$$= 0 \qquad , \qquad b < 0 , \qquad (3\text{-}47)$$

where

$$\mu = m_1/\sigma^2 \quad \text{and} \quad n + 1 = m_1^2/\sigma^2 \qquad (3\text{-}48)$$

For m_1 positive:

$$F(y) = \left(\frac{\mu}{\mu - 2y}\right)^{n+1} \qquad y < \mu/2 , \qquad (3\text{-}49)$$

$$2F/F' \begin{cases} = \dfrac{1}{m_1} - \dfrac{2\sigma^2}{m_1^2} y & y < \dfrac{m_1}{2\sigma^2} \\ = 0 & y \geq \dfrac{m_1}{2\sigma^2} \end{cases} . \qquad (3\text{-}50)$$

Although Eq. (3-40) cannot be solved analytically, nevertheless, much qualitative information can be gleaned about the optimal control and its associated performance index for various distributions, $p(b)$. [This information is summarized at the end of this section (III.3.c).]

The first step is to rewrite Eq. (3-40) in the form

$$\dot{y} = \left(\frac{1}{\lambda} - \dot{y}^2\right) \frac{F'(y)}{2F(y)} \ . \tag{3-51}$$

From the definition of $F(y)$ it follows that

$$F'(y)/2F(y) = Ebe^{2by}/E\ e^{2by} \tag{3-52}$$

and

$$\frac{d}{dy}[F'/2F] = 2\left[E\ e^{2by}\ \ Eb^2e^{2by} - (Ebe^{2by})^2\right] \Big/ (E\ e^{2by})^2 \ . \tag{3-53}$$

By the Schwartz inequality,

$$\frac{d}{dy}[F'/2F] \geq 0 \tag{3-54}$$

Therefore, $[F'(y)/2F(y)]$ is an increasing function on the interval where it is finite. Furthermore, let B be the set of points for which $p(b) > 0$. Then, from Eq. (3-52), it can be shown that

$$\lim_{y \to -\infty} \left(\frac{F'(y)}{2F(y)}\right) = \inf B \equiv \underline{B} \ , \tag{3-55a}$$

$$\lim_{y \to +\infty} \left(\frac{F'(y)}{2F(y)}\right) = \sup B \equiv \overline{B} \ . \tag{3-55b}$$

If $\underline{B} < 0 < \overline{B}$, there is a unique point, y^*, where $F'(y) = 0$. At this point $F(y)$ takes on its minimum value. It is sufficient to consider only this case with $y^* \leq 0$, and the case with $0 \leq \underline{B} < \overline{B}$. This covers all cases where $m_1 \geq 0$. The other cases can be put into these categories by redefining: $b = -b$, $y = -y$. We shall investigate solutions of Eq. (3-51) with $y(t) = 0$ and various values of $\dot{y}(t)$, and see under what circumstances it is possible to satisfy the final condition, Eq. (3-41a).

Consider first the case, $0 \leq \underline{B} < \bar{B}$. (See Eqs. (3-47) through (3-50) for an example of this type.) $F'/2F$ is positive throughout its range and has a finite value (m_1) at zero. Examination of Eq. (3-51) reveals the following consequences. If $\dot{y}(t) \geq 0$, then $\dot{y}(\tau) > 0$ for all $\tau > t$, and $\dot{y}(\tau) \rightarrow \lambda^{-\frac{1}{2}}$. If $\dot{y}(t) \leq -\lambda^{-\frac{1}{2}}$, then $\dot{y}(\tau)$ never increases. In both cases, the final condition, $\dot{y}(T) = 0$, cannot be satisfied for any finite T. However, if $-\lambda^{-\frac{1}{2}} < \dot{y}(t) < 0$, then $\ddot{y}(\tau)$ is positive and remains positive as $\dot{y}(\tau)$ increases through negative values. The question remains whether $\dot{y}(\tau)$ reaches zero at some t_1 (in which case the optimal solution for $T = t_1$ has been generated) or whether $\dot{y}(\tau)$ only approaches some negative limiting value.

It can be shown that, if $p(b)$ is bounded at zero, then, whenever $-\lambda^{-\frac{1}{2}} < \dot{y}(t) < 0$, $\ddot{y}(\tau)$ is bounded away from zero for all $\tau > t$. Thus every starting value of $\dot{y}$ in this range generates the optimal solution for some finite value of T. As $\dot{y}(t)$ approaches $-\lambda^{-\frac{1}{2}}$ from the right, the T, for which the corresponding $\dot{y}(T) = 0$, approaches ∞. Also, $\dot{y}(\tau) \rightarrow -\lambda^{-\frac{1}{2}}$ and $y(\tau) \rightarrow -\lambda^{-\frac{1}{2}}(\tau - t)$ for all finite $\tau > t$. The minimum value of K [see Eqs. (3-34) and (3-35)] approaches $\lambda^{\frac{1}{2}}E(1/b)$. This leads to conclusions 1) and 2) at the end of this section (III.3.c).

Now consider the case where $\underline{B} < 0 < \bar{B}$ and $y^* < 0$. [See Eqs. (3-44) through (3-46), with $m_1 > 0$, for an example of this type.] Reference to Fig. 15 will be helpful in the following discussion. As before, if $\dot{y}(t) \geq 0$, then $\dot{y}(\tau) > 0$ for all $\tau > 0$,

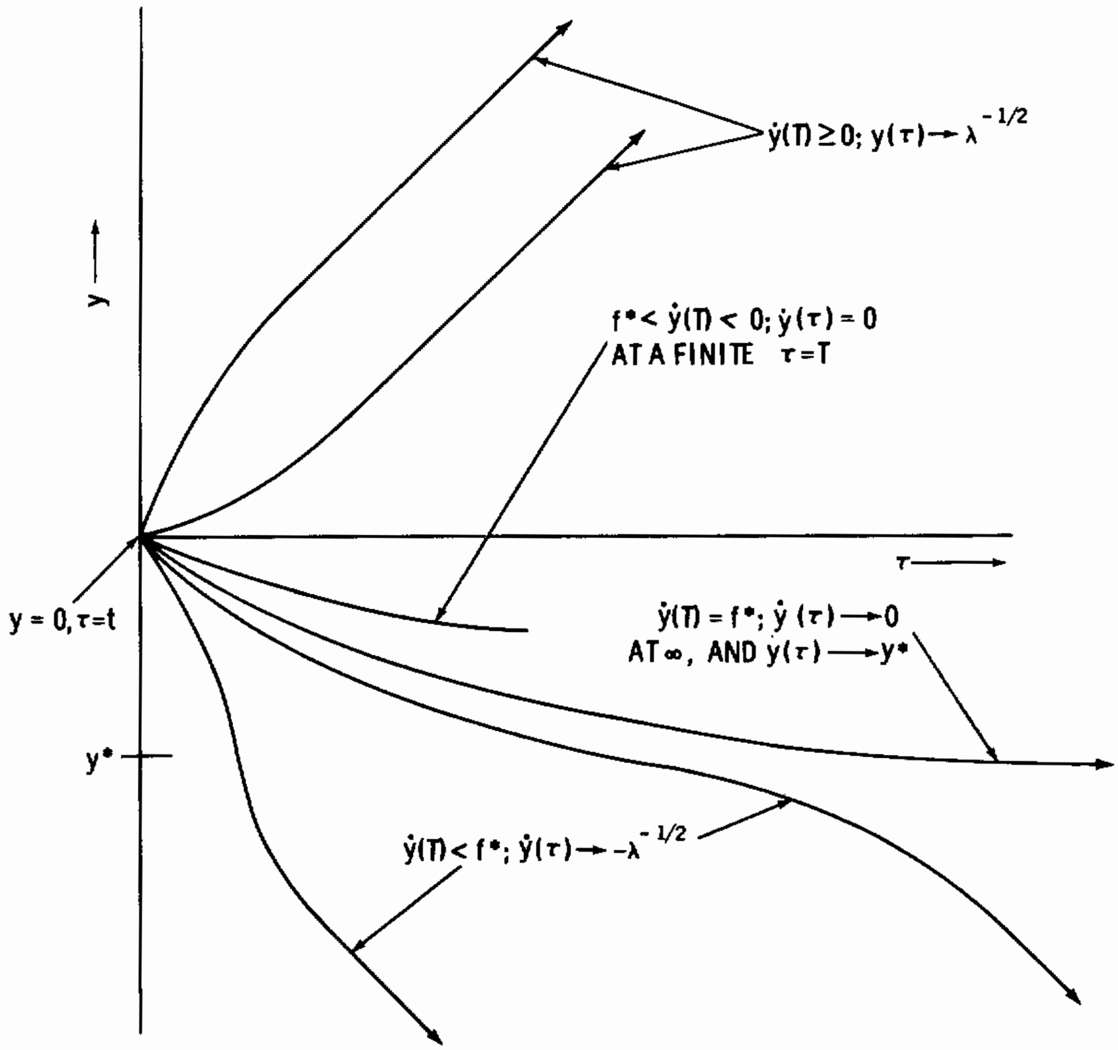


Fig. 15 Behavior of the Solution of Eq. (3-51) for Various Initial Values of $\dot{y}(\tau)$

and $\dot{y}(t) \to \lambda^{-\frac{1}{2}}$. Also, if $\dot{y}(t) \leq -\lambda^{-\frac{1}{2}}$, then $\dot{y}(\tau)$ decreases until $y(\tau) < y^*$. Thereafter, it increases, but approaches $-\lambda^{-\frac{1}{2}}$ in the limit. Finally, if $\dot{y}(t) > -\lambda^{-\frac{1}{2}}$, but sufficiently close to it, then $\dot{y}(\tau)$ increases at first, but not rapidly enough to prevent $y(\tau)$ from falling below y^* at some finite time. Thereafter, $\ddot{y}$ changes sign, and $\dot{y}(\tau)$ decreases to a limiting value of $-\lambda^{-\frac{1}{2}}$. If $\dot{y}(t)$ is negative, but sufficiently close to zero, then $y(\tau) > y^*$ for all $\tau > t$, and $\dot{y}(\tau)$ reaches zero at some finite time. Therefore, there is a critical negative value, f^*, strictly greater than $-\lambda^{-\frac{1}{2}}$, such that: if $f^* < \dot{y}(t) < 0$, the optimal control corresponding to some finite final time T is generated; if $\dot{y}(t) = f^*$, the optimal control for $T = \infty$ is generated.

Note that the solution with $\dot{y}(t) = f^*$ is <u>not</u> a steady state solution. The optimal feedback gain for the interval (t,∞) is a time varying function $f^o(\tau)$, which actually approaches zero as $\tau \to \infty$. Note also that $y(\tau) \to y^*$, the value that minimizes $F(y)$. It follows from Eqs. (3-34) and (3-35) that the optimal loss for interval (t,∞) is infinite. This leads to conclusions 3) and 4), below.

The conclusions of the qualitative analysis are:

1) If $p(b) = 0$ for $b \leq 0$ and $E(1/b)$ is finite, then the optimal feedback control for the interval (t,∞) is

$$u^o(x,\tau) = -\lambda^{-\frac{1}{2}}x \,, \tag{3-56}$$

and the optimal loss is finite,

$$S^o(c,t,\infty) = \lambda^{\frac{1}{2}}E(1/b)c^2 \,. \tag{3-57}$$

2) If $p(b) = 0$ for $b \leq 0$ and $p(b)$ is bounded in some neighborhood of 0, but $E(1/b)$ is infinite, then the optimal feedback control for the interval (t,∞) is still given by Eq. (3-56), and the optimal loss is infinite. However, the optimal mean loss rate, defined by

$$R^o(c,t,\infty) = \lim_{T \to \infty} \left[\frac{S^o(c,t,T)}{T - t}\right] \tag{3-58}$$

is zero.

3) If $\underline{B} < 0 < \overline{B}$ and $y^* < 0$ $(m_1 > 0)$, then the optimal feedback control for the interval (t,∞) is $x(\tau)f^o(\tau)$, where

$$f^o(\tau) = \dot{y}^o(\tau) \tag{3-59}$$

and $y^o(\tau)$ is the solution of Eq. (3-51) such that $\dot{y}^o(\tau) \to 0$ as $\tau \to \infty$, $y^o(t) = 0$, and $-\lambda^{-\frac{1}{2}} < f^* = \dot{y}^o(t) < 0$. The optimal loss is infinite, and the optimal mean loss rate, defined by Eq. (3-58), is

$$R^o(c,t,\infty) = c^2F(y^*) \ . \tag{3-60}$$

4) If $\underline{B} < 0 < \overline{B}$ and $y^* = 0$ $(m_1 = 0)$, then the optimal control is 0; the optimal loss is infinite, and the optimal mean loss rate is given by Eq. (3-60).

4. DISCRETE TIME VERSION

We shall restrict our attention to Case III because it is the natural one on which to try to apply dynamic programming.

Consider the optimal feedback control and resulting performance for a system described by Eqs. (3-3) and (3-4). In this discussion n, N, and k denote the initial, final, and present steps, respectively; c denotes the initial state and x_k denotes the k^{th} state. Here the arguments of u^o are in the following order: 1st - initial time, 2nd - final time, 3rd - present state, and 4th - present time. The arguments of S^o are: 1st - initial state, 2nd - initial time, and 3rd - final time. It is easy to show that

$$u^o(N,N,x,N) = 0 \tag{3-61a}$$

$$S^o(c,N,N) = c^2\Delta , \tag{3-61b}$$

$$u^o(N - 1, N, x, N - 1) = - x \left(\frac{\Delta m_1}{\lambda + m_2\Delta^2}\right) , \tag{3-62a}$$

$$u^o(N - 1, N, x, N) = 0 , \tag{3-62b}$$

$$S^o(c, N - 1, N) = c^2\Delta \left(2 - \frac{\Delta^2 m_1^2}{\lambda + m_2\Delta^2}\right) . \tag{3-62c}$$

When $n = N - 2$, by definition,

$$S^o(c, N - 2, N) = \min_{u_k} E \sum_{k=N-2}^{N} \left[x_k^2 + \lambda u_k(x_k)^2\right]\Delta , \tag{3-63a}$$

where

$$x_{N-2} = c \ , \tag{3-63b}$$

$$x_{N-1} = c + bu_{N-2}(c)\Delta \ , \tag{3-63c}$$

$$x_N = x_{N-1} + bu_{N-1}(x_{N-1})\Delta \ . \tag{3-63d}$$

Let us define

$$Q(c,u_{N-2},u_{N-1},u_N,b) = \left[x_{N-1}^2 + \lambda u_{N-1}(x_{N-1})^2 + x_N^2 + \lambda u_N(x_N)^2\right]\Delta \ , \tag{3-64}$$

where x_{N-1} and x_N are defined by Eqs. (3-63c) and (3-63d). Then

$$S^o(c,N-2,N) = \min_{u_k}\left[(c^2 + \lambda u_{n-2}(c)^2)\Delta + E\ Q\right] \ . \tag{3-65}$$

The method of dynamic programming is to state that u_{N-1} and u_N may be assumed optimal, i.e., given by Eqs. (3-62a) and (3-62b), so that Q in Eq. (3-65) may be replaced by $S^o(x_{N-1}, N-1, N)$ [see Eq. (3-62c)], thereby giving the recursion equation,

$$S^o(c,N-2,N) = \min_{u_{N-2}}\left[(c^2 + \lambda u_{N-2}^2)\Delta + ES^o(c + bu_{N-2}\Delta, N-1, N)\right] \ . \tag{3-66?}$$

The reason for the question mark is that Eq. (3-66?) is incorrect as a result of a subtle error, induced by the notation. Let us assume that the replacement of u_{N-1} by u_{N-1}^o is correct — which it isn't — and rewrite Q and $S^o(x_{N-1}, N-1, N)$ in such a way that the dependence on b is clearly shown. In fact it is instructive to write the expressions for the generalized problem where b may take different values, b_k, at each step. Let (b_{N-2}, b_{N-1}) have the joint density function $p_2(b_{N-2}, b_{N-1})$. We know $u_N^o = 0$, so omit that term. Therefore,

$$Q(c,u_{N-2},u_{N-1}^o,b_{N-2},b_{N-1}) =$$

$$= \Big\{(c + b_{N-2}u_{N-2}\Delta)^2 + \lambda\Big[u_{N-1}^o(c + b_{N-2}u_{N-2}\Delta)\Big]^2 \tag{3-67}$$

$$+ \Big[c + b_{N-2}u_{N-2}\Delta + b_{N-1}u_{N-1}^o(c + b_{N-2}u_{N-2}\Delta)\Delta\Big]^2\Big\}\Delta ,$$

where u_{N-1}^o is given by Eq. (3-62a). For all x, by definition,

$$S^o(x,N - 1,N) =$$

$$= \Big\{x^2 + \lambda\Big[u_{N-1}^o(x)\Big]^2 + \int \Big[x + b_{N-1}u_{N-1}^o(x)\Delta\Big]^2 p(b_{N-1})\,db_{N-1}\Big\}\Delta \ . \tag{3-68}$$

Setting $x = c + b_{N-2}u_{N-2}\Delta$, we find that

$$S^o(c + b_{N-2}u_{N-2}\Delta,N - 1,N) =$$

$$= \int Q(c,u_{N-2},u_{N-1}^o,b_{N-2},b_{N-1})\,p(b_{N-1})\,db_{N-1} \ . \tag{3-69}$$

So that

$$E\ S^o(c + b_{N-2}u_{N-2}\Delta,N - 1,N) =$$

$$= \iint Q(c,u_{N-2},u_{N-1}^o,b_{N-2},b_{N-1})\,p(b_{N-1})\,p(b_{N-2})\,db_{N-1}\,db_{N-2} \ . \tag{3-70}$$

But

$$E\, Q(c,u_{N-2},u^{o}_{N-1},b_{N-1},b_{N-2}) =$$

$$= \iint Q(c,u_{N-2},u^{o}_{N-1},b_{N-2},b_{N-1})\,p_2(b_{N-2},b_{N-1})\,db_{N-1}\,db_{N-2} \;. \qquad (3\text{-}71)$$

In order that the right sides of Eqs. (3-70) and (3-71) be the same function of u_{N-2}, the obvious requirement is that

$$p_2(b_{N-2},b_{N-1}) = p(b_{N-2})\,p(b_{N-1}) \;. \qquad (3\text{-}72)$$

This is precisely the condition for b_{N-2} and b_{N-1} to be statistically independent. In the problem we are concerned with, this is not the case. In fact b_{N-2} and b_{N-1} are totally dependent in that the probability that they are equal is one. For this situation, the joint p.d.f. contains a delta function:

$$p_2(b_{N-2},b_{N-1}) = p(b_{N-2})\,\delta(b_{N-1} - b_{N-2}) \;. \qquad (3\text{-}73)$$

The question arises whether, under any circumstances other than statistical independence, the right sides of Eqs. (3-70) and (3-71) can be equal for all values of u_{N-2}. For terms of Q that do not involve <u>both</u> b_{N-1} and b_{N-2}, the two integrals are automatically equal. However, examination of Eq. (3-67) reveals that there are four terms that contain both b_{N-1} and b_{N-2} in various combinations of powers. These terms contain u_{N-2} as well. Complicated conditions, involving moments up to the fourth order, for which equality holds can be written. But these conditions would be tailored to this particular form of Q, which only applies to the two-stage problem. For a three-stage problem additional conditions must be met, and these conditions would involve still higher moments. In short, although conditions weaker than independence can be found, they are too cumbersome to be a practical tool for analysis.

With no way to develop a recursion equation, the dynamic programming method must be abandoned; therefore, we fall back on simultaneous minimization with respect to u_{N-2} and u_{N-1} (u_N^o obviously is zero). Two simultaneous equations in u_{N-2}^o and u_{N-1}^o result from setting the partial derivatives equal to zero. The equations are nonlinear, and cannot be solved analytically. However, direct substitution of Eq. (3-62) proves that it is not the optimal u_{N-1} for the problem starting at $N - 2$.

5. CONCLUSIONS

The dynamic programming method, although applicable to the deterministic version of this problem, could not be used for the stochastic version. The reason for this is closely related to the fact that the optimal control, even in the feedback case, depends on the starting time as well as the time to go. It is not easy to see physically why this should be, but the equations cannot be disputed. If we consider a limiting case, where the distribution of b approaches an impulse, we find that the stochastic results approach the deterministic ones. The conclusion is that the way in which randomness enters the problem, particularly the presence of time correlation in the random parameter b, causes the peculiar behavior.

The extension of the methods employed in this section to the vector case should not pose formidable problems as far as the dynamics are concerned. The chief trouble spot would be handling several correlated random variables. The main emphasis of such a study should be on limiting cases as the final time becomes infinite. As was seen, the behavior in the limit depends on the distribution of b in some ways that are not immediately obvious.

Another possibility to be further investigated is the existence of nonlinear solutions to the augmented feedback control problem (Case II). Note that, in the discrete problem, b can be determined exactly at step $(n+1)$, if x_n (the initial state), x_{n+1}, and $u_n(x_n)$ are known. This is the case for the augmented feedback problem. It is conjectured that the optimal solution for the discrete case is nonlinear, but that it does not approach a limit that can be carried over to the continuous case, and that there is <u>no</u> optimal solution for the continuous problem.

SECTION IV

SUMMARY

This report has discussed aspects of analysis and synthesis of a class of stochastic control systems. Various analysis techniques including numerical inversion, and several sampling schemes were explored in Section II. For the single random parameter case, numerical inversion is preferred because of its accuracy and relative ease of application. For the multiple parameter cases, sampling is usually necessary; however, care should be exercised in the particular sampling scheme used in order to obtain maximum accuracy for a given sample size. It was also shown that the standard deviation ratio and the mean partial derivatives are worthwhile alternatives to the classical notion of sensitivity in the class of systems under study.

In Section III it was shown that the usual synthesis approach to problems of this type, i.e., dynamic programming, could not be used. This was because of the particular manner in which randomness entered the system, that is, the presence of time correlation in the random parameter. The calculus of variations was used in synthesizing open-loop, augmented feedback and feedback controllers.

APPENDIX A

EQUIVALENCE OF MEAN PARTIAL DERIVATIVE AND LEAST-SQUARES FIT

Given the function

$$y = f(x)$$

and a linear approximation

$$\hat{y} = k_1 x \ .$$

The value of k_1 that minimizes

$$E(y - \hat{y})^2$$

is known to be

$$k_1 = \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} x f(x) p(x) \, dx \ ,$$

where $p(x)$ is the probability density of x, with mean $\bar{x} = 0$, and variance σ^2.

The mean partial derivative is

$$k_2 = \int_{-\infty}^{+\infty} f'(x) p(x) \, dx \ .$$

Integrating by parts, one obtains

$$k_2 = f(x)p(x) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(x) p'(x) \, dx \ .$$

Because $p(x)$ vanishes at the upper and lower limits, the first term vanishes, and further, if

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} ,$$

then $k_2 = 1/\sigma^2 \int xf(x)p(x)\,dx$, which is identical to k_1. The same holds true for a nonzero mean, if the least-squares fit is constrained to pass through $f(\bar{x})$.

APPENDIX B

SUBROUTINES: RDM, BOXNO, RDMOUT, RDMIN

Source Language: MAP

Purpose: RDM - To generate pseudo-random numbers satisfying the rectangular distribution on (0,1). The numbers are in normalized floating point form.

BOXNO - To generate pseudo-random numbers satisfying the normal distribution with mean zero and standard deviation one. The numbers are in normalized floating point form.

RDMOUT - To enter RDM and return with the i^{th} element of the fixed point sequence which RDM has generated, and to return this number in the form of a 12-digit octal word to the calling program.

RDMIN - To enter RDM and re-store the i^{th} element of the fixed point sequence which RDM has previously generated. This element will be in the form of a 12-digit octal word.

Method: RDM - The sequence of computations is:

$$r_{i+1} = (2^7 + 1)r_i + 311715164025_{(8)} .$$

The resulting fixed point number is converted to normalized floating point form. The method is that of A. Rotenberg (Ref. 11). The constant 311715164025 was chosen since it is odd and approximates

$$(.5 + 3/6)2^{35} \quad (10)$$

which is shown by R. R. Coveyou (Ref. 12) to be that constant which causes the least serial correlation.

BOXNO - Generates two independent normal deviates, r_1 and r_2 with each execution:

$$r_1 = (-2 \log_e U_1)^{\frac{1}{2}} \cos(2\pi U_2)$$

$$r_2 = (-2 \log_e U_1)^{\frac{1}{2}} \sin(2\pi U_2)$$

with U_1 and U_2 random numbers from the rectangular distribution on (0,1). The method is that of G. E. Box and M. E. Muller (Ref. 13).

<u>Use</u>: RDM - The following FORTRAN statement is required:

```
R = RDM (DUMMY);
```

R will be the desired pseudo-random number. DUMMY need not be defined in the calling program, and will not be changed by RDM.

BOXNO - BOXNO requires no input from the calling program. The following FORTRAN statement is to be used:

```
CALL BOXNO (R1, R2).
```

The normal deviates will be stored in R1 and R2.

RDMOUT - RDMOUT requires no input from the calling program. The following FORTRAN statement is to be used:

```
CALL RDMOUT (OCT).
```

The required octal number is stored in OCT. The main program can then write out OCT in octal format so that it can be used in reinitializing RDM.

RDMIN - The following FORTRAN statement is to be used:

CALL RDMIN (OCT)

where OCT is the required word. OCT can be read into the calling program by means of an octal format.

Timing: RDM approximately 25 cycles; RDMIN and RDMOUT approximately 5 cycles each, BOXNO approximately 58 cycles plus the time required for one execution of each of its subroutines.

Checkout: The results are identical to those of the original routines, where satisfactory results were obtained from chi-square tests, sample means, and sample variances.

APPENDIX C

SUBROUTINE INV(Y,X)

Source Language: FORTRAN IV.

Purpose: To find the value of X such that

$$Y = \int_{-\infty}^{X} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt .$$

Use: CALL INV(Y,X), where $0.0 < Y < 1.0$ and X is an initial value for iteration. X = 0.0 is always a suitable starting value. The subroutine returns Y unchanged, and X equal to the solution of the above equation. Both X and Y are interpreted as REAL quantities. If Y is outside the allowable range, or if the iteration does not converge, execution is terminated.

Subroutines Needed: PHI, EXP, SQRT.

FUNCTION PHI(X)

Source Language: MAP.

Purpose: To compute the value of

$$\int_{-\infty}^{X} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt .$$

Use: The argument X is interpreted as a REAL (single precision floating point) quantity.

Subroutines Needed: EXP.

APPENDIX D

SOURCE LISTING PROGRAMS

PROGRAM I

```
C     C1=Q/R,C2=R,C3=C,C4=K3,C5=K4,C6=X1(0),C7=X2(0),C8=A0,C9=B0,C10=X3(0)
C
      DIMENSICN C(90)
      DIMENSION V(6000)
      DIMENSION YICS(4),YP(4),YC(4),YD(4),YNEW(4),YDERV(4),YDEV(4),
     1 FDEL(4),BDEL(4),TDEL(4),PARTY(4),YNEB(4),ERR(4)
      DOUBLE PRECISION YNEW,YP,YDERV
    2 FORMAT(I1,I4,1E14.7)
      WRITE (6,50)
    1 READ(5,2)I,II,D
      IF(I-1) 4,4,3
    3 C(II) = D
      GO TO 1
    4 EV = 0.
      EV2 = 0.
      ME=1
      A=0.
      BT = 0.
      BO = 0.
      YY=0.
      B=0.
    5 WRITE(6,7)C
  100 Y = 0.
    7 FORMAT(1H 1P12E9.2///)
      TOP1 = 0.
      TOP2 = 0.
      BOT  = 0.
      IF(C(37)) 20,  6,901
   20 NEQNS = 4
      GO TO 21
    6 NEQNS = 3
   21 CONTINUE
      DTAU =C(13)
      TIC=0.
      DELTA = C(14)
      EPSL  = C(15)
      ERR(1)= C(16)
      ERR(2)= C(17)
      ERR(3)= C(18)
      ERR(4) = C(52)
      YICS(1) = C(6)
      YICS(2) = C(7)
      YICS(3) = C(10)
      YICS(4)= C(38)
      YP(1)=C(6)
      YD(1)=C(7)
      CON1 = C(41) - C(42)*C(39)
      CON2 = C(43) + C(42)
      CON3 = C(44) - C(42)*C(40)
  901 Q  = C(1)*C(2)
      SR=SQRT(C(9)**2 +C(1)*C(3)**2)
      CA1 = -C(9) + SR
      CA2 = -C(8) + SQRT(C(8)**2 -2.*C(9) + C(1) + 2.*SR)
    8 IF(C(24))902,903,904
```

```
  904 CALL BOXNO(R1,R2)
      B = C(5)*R2 + C(9)
      A = C(4)*R1 + C(8)
      GO TO 905
  903 CALL RDM(R1)
      A  = C(4) *R1 + C(8)
      B  = C(5)*R1 + C(9)
      GO TO 905
  902 CALL STRAT(A,B,ME,C,Y,YY,DY,DYY)
  905 IF(C(37))33,33,14
   33 WRITE(6,53)
      A1=   A -C(8)
      B1=   B -C(9)
      X1 = C(39) + C(61)*A1
      X2 = C(40) + C(62)*A1
      X3 = C(41) + C(63)*A1
      X4 = C(42) + C(64)*A1
      X5 = C(43) + C(65)*A1
      X6 = C(44) + C(66)*A1
    9 CALL SETUP(MAGIN,MAGOUT,TIC,STEP,NEQNS,DTAU,EPSL,DELTA,ERR,TIME
     1,DTIME,YICS,YP,YC,YD,YNEW,YDERV,YDEV,FDEL,BDEL,TDFL,PARTY,YNEB)
   10 IF(C(37))200,12,1
  200 DC=-(C(49)*YP(1) + C(50)*YP(2) + C(51)*YP(4))
      YD(1) = -X1*YP(1) +YP(2) - X2*YP(4)
      YD(2) =  (X3-X4*X1)*YP(1) + (X5+ X4)*YP(2) +(X6-X4*X2)*YP(4)
      YD(3) = C(46)*YP(1)**2 + C(47)*YP(2)*YP(2) + C(48)*DC*DC
      YD(4) = -YP(4)/C(45) + DC/C(45)
      GO TO 13
   12 U = -CA1*YP(1) - CA2*YP(2)
      YD(1) = YP(2)
      YD(2) = -B*YP(1) - A*YP(2) + U
      YD(3) = Q*(C(3)*YP(1) + YP(2))**2 + C(2)*U**2
      IF(TIME)13,14,13
C
   14 CHEV = (Q*C(3)**2 + C(2)*CA1**2)*((YP(1)**2 + ((A+CA2)*YP(1)+YD(1)
     1 )**2/(B+CA1))/(2.*(A+CA2)))
     2 +(Q+C(2)*CA2**2)*((YD(1)**2 + (B+CA1)*YP(1)**2)/(2.*(A+CA2)))
     3 - (C(3)*Q + C(2)*CA1*CA2)*YP(1)**2
C
      SRO=SQRT(B**2 + C(1)*C(3)**2)
      CA10= -B + SRO
      CA20= -A + SQRT(A**2 - 2.*B + C(1)+ 2.*SRO)
      VOPT= (Q*C(3)**2 +C(2)*CA10**2)*((YP(1)**2 +((A+CA20)*YP(1)+
     1YD(1))**2/(B+CA10))/(2.*(A+CA20)))
     2+ (Q+C(2)*CA20**2)*((YD(1)**2 +(B+CA10)*YP(1)**2)/(2.*(A+CA20)))
     3- (C(3)*Q + C(2)*CA10*CA20)*YP(1)**2
C
      IF(C(37))13,13,35
   35 CAPV = CHEV
      GO TO 36
   13 MAGIN = -1
      IF(DTIME-C(55))  1,  1,702
  702 CONTINUE
   15 CALL MAGIC
      IF(MAGOUT)10,10,16
   16 MAGIN =0
```

```
      WRITE(6,18)TIME,YC
      IF(TIME)15,15,600
  600 STOP = ABS((YC(3)-YNEW(3))/YC(3))
      IF(STOP - C(53))19,19,701
  701 IF(TIME-C(54))15,1,1
   19 CAPV = YC(3)
   36 BOT  = BOT + C(31)
      DY = 2.*DY
      DYY= 2.*DYY
      TOP1= TOP1+ CAPV
      TOP2 = TOP2 + CAPV**2
      IF(C(24))906,907,907
  906 EV = EV + CAPV*DY*DYY
      EV2 = EV2 + CAPV**2*DY*DYY
      GO TO 908
  907 EV = TOP1/BOT
      FV2= TOP2/BOT
  908 SIG2 = EV2 - EV**2
      SIG  = SQRT(SIG2)
      MEG = BOT
   66 FORMAT(I4,1P7E14.7)
      V(MEG)= CAPV
      IF(BOT - C(33))502,500,500
  502 IF(BO - C(36))503,503,8
  503 WRITE(6,66)MEG,A,B,EV,SIG2,Y,YY,BO
      GO TO 8
  500 ME=0.
      IF(C(24))910,909,909
  910
C     STRAT SAMP OUTPUT
```

```
      DO 24M=1,MEG
      VL = 0.
      DO 22N=1,MEG
      IF(VL-V(N))23,22,22
   23 VL = V(N)
      NN = N
```

```
   22 CONTINUE
      MM= 0
      WRITE(6,2)MM,NN,VL
      V(NN) = 0.
   24 CONTINUE
C     WRITE(6,66)MEG,A,B,EV,SIG2,Y,YY,BT
      BOT = 0.
      BO = BO + 1.
      IF(BO-C(12))100,501,501
  501 WRITE(6,66)MEG,A,B,EV,SIG2,Y,YY,BT
      GO TO 1
   32 FORMAT(1P5E16.7//)
   51 FORMAT( 81H          R1                    R2                    A
     1     B          V(CLOSED FORM))
   50 FORMAT(107H     Q/R          R           C          K3          K4          X1(0)
     1   X2(0)        A0         BO         X3(0)        M           N    )
   11 FORMAT(///3H N=I4//)
   17 FORMAT(33H RUN NUMBER 1 IS THE NOMINAL CASE)
   18 FORMAT(1H 1P1E9.3,1P4E15.7)
   52 FURMAT(79H          E(V)        E(VSQUARED)           VARIANCE          SIG
     1MA          V=PERF  INDEX)
   53 FORMAT(51H    TIME          X1                 X2                  X3  )
   65 FORMAT(7H VOPT= 1P1E14.7///)
      END
```

```
C       FOR USE IN EQUAL PROBABILITY INTERVALS
C       EQUAL PROBABILITY ROUTINE
        SUBROUTINE STRAT(A,B,ME,C,Y,YY,DY,DYY)
        DIMENSION C(25)
        IF(Y.GE.C(26).AND.Y.LE.C(27))GO TO 20
        DY = C(32)
        GO TO 21
     20 DY = C(32)/C(30)
     21 IF(YY.GE.C(28).AND.YY.LE.C(29))GO TO 22
        DYY= C(32)
        GO TO 23
     22 DYY= C(32)/C(30)
     23 CONTINUE
        IF(ME)3,5,4
      4 YY = -DYY
        AAP = 0.
      5 YY =  YY + 2.*DYY
        ME=-1
        IF(YY-DYY/2.)5,5,9
      9 IF(YY-1. + DYY/2.)10,10,1
     10 X=0.
      7 CALL INV(YY,X)
        AA = C(4)*X + C(8)
        A = (AA +AAP)/2.
        AAP = AA
        Y = -DY
        BBP = 0.
        IF(A)5,5,3
      3 B=C(9)
      6 Y =Y + 2.*DY
        IF(Y-DY/2.)6,6,11
     11 IF(Y-1. + DY/2.)12,12,1
     12 X=C.
      8 CALL INV(Y,X)
        BB  = C(5)*X + C(9)
        B = (BB+BBP)/2.
        BBP= BB
        IF(B)6,6,1
      1 CONTINUE
        IF(C(4))30,31,30
     31 A = C(8)
        GO TO 32
     30 IF(C(5))32,33,32
     33 B = C(9)
     32 RETURN
        END
```

PROGRAM II

```
      SUBROUTINE STEPP(C,A,B,ME)
      DIMENSION C(36)
      IF(ME)3,4,2
    4 A=-C(25)
      ME=-1
    3 A=A+C(25)
      B = C(9)
      GO TO 1
    2 A= C(8)
      B=B+C(25)
    1 CONTINUE
      RETURN
      END
```

```
C     C1=Q/R,C2=R,C3=C,C4=K3,C5=K4,C6=X1(0),C7=X2(0),C8=A0,C9=B0,C10=X3(0)
C
      DIMENSION PR(6000)
      DIMENSION C(70)
      DIMENSION V(6000),TAY(6000)
     2
      DIMENSION YICS(4),YP(4),YC(4),YD(4),YNEW(4),YDERV(4),YDEV(4),
     1FWDEL(4),CKDEL(4),TBDEL(4),PARTY(4),YNEWB(4),ERR(4)
      DOUBLE PRECISION YNEW,YP,YDERV
    2 FORMAT(I1,I4,1E14.7)
    1 READ(5,2)I,II,D
      IF(I-1)4,4,3
    3 C(II) = D
      GO TO 1
    4 Z=Z
      WRITE (6,50)
      ME = 1
    5 WRITE(6,7)C
      JAY = C(36)
  100 CONTINUE
      B=-C(25)
    7 FORMAT(1H 1P12E9.2///)
      TOP1 = 0.
      TOP2 = 0.
      BOT  = 0.
      TIC=0.
      DELTA = C(14)
      EPSI  = C(15)
      ERR(1)= C(16)
      ERR(2)= C(17)
      ERR(3)= C(18)
      YICS(1) = C(6)
      YICS(2) = C(7)
      YICS(3) = C(10)
      YP(1)=C(6)
      YD(1)=C(7)
```

```
      NEQNS = 3
      Q= C(1)*C(2)
      SR =SQRT(C(9)**2 + C(1)*C(3)**2)
      CA1 = -C(9) + SR
      CA2 = -C(8) + SQRT(C(8)**2 -2.*C(9) + C(1) + 2.*SR)
      VAAB=(Q*C(3)**2+C(2)*CA1**2)*((C(6)**2/(2.*(C(9)+CA1))
     1 - (C(6)**2*(C(9)+CA1) + C(7)**2)/(2.*(C(9)+CA1)*(C(8)+CA2)**2)))
     2 - (Q + C(2)*CA2**2)*((C(7)**2 + (C(9)+CA1)*C(6)**2)/(2.*(C(8)+CA2
     3 )**2))
      VBAB = -(Q*C(3)**2 + C(2)*CA1**2)*((C(8)+CA2)*C(6)+C(7))**2/(2.*
     1  (C(8)+CA2)*(C(9)+CA1)**2) + (Q+C(2)*CA2**2)*C(6)**2/(2.*(C(8)
     2  + CA2))
      IF(JAY-2)305,306,306
  305 SIGV2 = C(5)**2*VBAB**2
      GO TO 307
  306 SIGV2 = C(4)**2*VAAB**2
  307 CONTINUE
      WRITE(6,300)VAAB,VBAB,SIGV2

  300 FORMAT(16H VAAB,VBAB,SIGV2/1P3E20.7)
    8 CALL STEPP(C,A,B,ME)
      IF(C(37))33,33,14
   33 CONTINUE
      GO TO 20
   20 Z=Z
      WRITE (6,53)
      TF = DTAU
      DTAU =C(13)
      A1=    A -C(8)
      B1=    B -C(9)
      X1 =   C(39) + C(61)*A1
      X2 =   C(40) + C(62)*A1
      X3 =   C(41) + C(63)*A1
      X4 =   C(42) + C(64)*A1
      X5 =   C(43) + C(65)*A1
      X6 =   C(44) + C(66)*A1
    9 CALL SETUP(MAGIN,MAGOUT,TIC,STEP,NEQNS,DTAU,EPSL,DELTA,ERR,TIME
     1,DTIME,YICS,YP,YC,YD,YNEW,YDERV,YDEV,FDEL,BDEL,TDEL,PARTY,YNEB)
   10 IF(C(37))200,12 ,1
  200 DC=-(C(49)*YP(1) + C(50)*YP(2) + C(51)*YP(4))
      YD(1) = -X1*YP(1) +YP(2) - X2*YP(4)
      YD(2) =  (X3-X4*X1)*YP(1) + (X5+ X4)*YP(2) +(X6-X4*X2)*YP(4)
      YD(3) = C(46)*YP(1)**2 + C(47)*YP(2)*YP(2) + C(48)*DC*DC
      YD(4) = -YP(4)/C(45) + DC/C(45)
      GO TO 13
   12 U = -CA1*YP(1) - CA2*YP(2)
      YD(1) = YP(2)
      YD(2) = -B*YP(1) - A*YP(2) + U
      YD(3) = Q*(C(3)*YP(1) + YP(2))**2 + C(2)*U**2
      IF(TIME)13,14,13
C
   14 CHEV = (Q*C(3)**2 + C(2)*CA1**2)*((YP(1)**2 + ((A+CA2)*YP(1)+YD(1)
     1 )**2/(B+CA1))/(2.*(A+CA2)))
     2 +(Q+C(2)*CA2**2)*((YD(1)**2 + (B+CA1)*YP(1)**2)/(2.*(A+CA2)))
     3 - (C(3)*Q + C(2)*CA1*CA2)*YP(1)**2
```

```
      IF(C(24))13,13,35
   35 CAPV = CHEV
      GO TO 36
   13 MAGIN = -1
      IF(DTIME-C(55))  1,  1,702
  702 CONTINUE
   15 CALL MAGIC
      IF(MAGOUT)10,10,16
   16 MAGIN =0
      WRITE(6,18) TIME,YC(1),YC(2),YC(3)
      IF(TIME)15,15,600
  600 STOP = ABS((YC(3)-YNEW(3))/YC(3))
      IF(STOP - C(53))19,19,701
  701 IF(TIME-C(54))15,1,1
   19 CONTINUE
      CAPV = YC(3)
   36 CONTINUE
      BOT = BOT + 1.
      MEG = BOT
      V(MEG)= CAPV
      IF(BOT - C(12))8,70,70
   70 STP = SQRT(6.2831853)
      DELTA = 0.
      ME = 0
      NEQNS = 3
      YICS(1) = 0.
      YICS(2) = 0.
      YICS(3) = 0.
      K = 10
      STEP=.01
      TIC= 0.1
      DTAU = 0.
      KKK=C(12)/2.
   71 CALL SETUP(MAGIN,MAGOUT,TIC,STEP,NEQNS,DTAU,EPSL,DELTA,ERR,TIME
     1,DTIME,YICS,YP,YC,YD,YNEW,YDERV,YDEV,FDEL,BDEL,TDEL,PARTY,YNEB)
      IF(C(37))13,13,35
   72 CONTINUE
      IF(C(24))400,400,401
  401 GO TO(402,403),JAY
  402 C45 = C(4)
      C89 = C(8)
      GO TO 404
  403 C45 = C(5)
      C89 = C(9)
  404 P = -(TIME-C89)*(TIME-C89)/((2.*C45*C45))
      P = 1./(STP*C45)*EXP(P)
      GO TO 405
  400 GO TO(406,407),JAY
  406 C45 = C(19)
      C89 = C(20)
      GO TO 408
  407 C45 = C(21)
      C89 = C(22)
  408 P = 1./(C45 + C89)
```

```
      TAY(K)  =  P
      YD(1)     = V(K)*P
      IF(C(37))409,409,410
  410 IF(JAY-2)303,302,302
  303 YD(2)=-P*
     1           (Q*C(3)**2  +C(2)*CA1**2)*((C(8)+CA2)*C(6)+C(7))**2/(2.*
     2(C(8)+CA2)*(TIME+CA1)**2)+(Q+C(2)*CA2**2)*C(6)**2/(2.*(C(8)
     3 +CA2)) *P
      GO  TO  304
  302 YD(2)=P*(Q*C(3)**2+C(2)*CA1**2)*(C(6)**2/(2.*(C(9)+CA1))
     1 - (C(6)**2*(C(9)+CA1)+ C(7)**2)/(2.*(C(9)+CA1)*(TIME+CA2)**2))
     2 - (Q+C(2)*CA2**2)*((C(7)**2  +(C(9)+CA1)*C(6)**2)/(2.*(TIME+CA2
     3 )**2))*P
      GO  TO  304
  409 YD(2)= 0.
      YD(3)  =  (TIME-C89)*(V(K)-V(KKK))*P
  304 CONTINUE
      MAGIN  =  -1
   73 CALL  MAGIC
      IF(MAGOUT)72,72,74

   74 MAGIN  =  0
C     WRITE(6,83)YC(1),P,TIME,K
      K=  K  +  1
      BOT  =  K
   75 IF(BOT  -  C(12))73,73,76
   76 VBAR  =  YC(1)
      CA5  =  YC(3)/(C45*C45)
      WRITE(6,301)(YC(K),K=2,4),CA5

  301 FORMAT(24H  YC(2),YC(3),YC(4),CA5   /1P4E20.7)
      WRITE(6,77)VBAR
   77 FORMAT(  8H  VBAR  =  1P1E20.7)
      K=  10
      CALL  SETUP(MAGIN,MAGOUT,TIC,STEP,NEQNS,DTAU,EPSL,DELTA,ERR,TIME
     1,DTIME,YICS,YP,YC,YD,YNEW,YDERV,YDEV,FDEL,BDEL,TDEL,PARTY,YNEB)
   79 YD(1)  =  (V(K)-  VBAR)**2*TAY(K)
      MAGIN  =  -1
   78 CALL  MAGIC
      IF  (MAGOUT)79,79,80
   80 MAGIN  =  0
C     WRITE(6,83)YC(1),V(K),TAY(K),K
   83 FORMAT(1H  1P3E15.7,I5)
      K=K+1
      BOT  =  K
   81 IF(BOT-C(12))78,78,82
   82 SV2  =  YC(1)
      WRITE(6,84)SV2
   84 FORMAT(5H  SV2=1P1E20.7)
      NN=C(12)
       I=NN-1
      ANN=NN
      ABAR  =  C89
      A2AV  =  C45
      DO  105    J=10,I
      K=J+1
      AJ=J
```

```
103 IF (V(K)-V(J))104,104,106
106 IF(K-J-1)205,205,108
107 PR(J) =PHI((ANN*STEP-ABAR)/A2AV)- PHI((AJ*STEP-ABAR)/A2AV)
    GO TO 105
108 AK=K
    PR(J) =PHI((AK*STEP-ABAR)/A2AV) - PHI((AJ*STEP-ABAR)/A2AV)
    GO TO 105
104 K=K+1
    IF (K-NN)103,107,107
105 CONTINUE
205 WRITE (6,202)
    VBPD=0.
    V2AVPD=0.
    DO 201  JJ= 11,NN
    RAR=SQRT((PR(JJ)-PR(JJ-1))**2)
    EVA=SQRT((V(JJ)-V(JJ-1))**2)
    DE= RAR/EVA
    VTRP=V(JJ)
    VBPD=VTRP*RAR+VBPD
    V2AVPD=VTRP**2*RAR+V2AVPD

C   WRITE(6,203) VTRP,DE,PR(JJ)
201 CONTINUE
    VARPD=V2AVPD -VBPD**2
    WRITE(6,204)VBPD,VARPD,SIGPD
202 FORMAT( 43H          V              VDENSITY          VDISTRIB.)
203 FORMAT(1H 1P3E15.7)
204 FORMAT( 36H VBAR CALCULATED FROM DENSITY FCN  = 1P1E15.7///,  35H
   1VARIANCE CALC. FROM DENSITY FCN  = 1P1E15.7///,  9H SIGMA  =
   21P1E15.7)
    GO TO 5
 32 FORMAT(1P5E16.7//)
 51 FORMAT( 81H          R1                  R2                  A
   1   R          V(CLOSED FORM))
 50 FORMAT(107H     Q/R        R          C          K3         K4         X1(0)
   1   X2(0)      A0         B0        X3(0)       M          N    )
 11 FORMAT(///3H N=I4//)
 17 FORMAT(33H RUN NUMBER 1 IS THE NOMINAL CASE)
 18 FORMAT(1H 1P1E9.3,1P3E15.7)
 52 FORMAT(79H          E(V)         E(VSQUARED)              VARIANCE             SIG
   1MA          V=PERF INDEX)
 53 FORMAT(51H    TIME          X1                  X2                  X3  )
 65 FORMAT(7H VOPT= 1P1E14.7///)
601 FORMAT(2H  1P2E15.7)
     END
```

PROGRAM III

```
      DIMENSION C(30),YICS(6),YPRED(6),YCURR(6),YDOT(6),YNEW(6),
     1   YDERV(6),YDEV(6),FWDEL(6),BKDEL(6),TBDEL(6),PARTY(6),YNEWB(6)
     2   ,ERR(6)
      DOUBLE PRECISION YNEW,YPRED,YDERV
    5 WRITE(6,6)
    6 FORMAT(1H1)
    1 READ(5,2)I,J,D
      IF(I)4,4,3
    3 C(J) = D
      GO TO 1
    2 FORMAT(I1,I4,1E14.7)
    4 WRITE(6,7)C
    7 FORMAT(5H DATA/(1P5E20.7))
      NEQNS = 6
      DELTA = C(19)
      EPSIL = C(26)
      DTAV  = C(27)
      DO 8K=1,NEQNS
      YICS(K) = C(K+12)
    8 ERR(K)  = C(K+19)
      STEP = C(29)
      TSTOP = C(28)
      TIC = 0.
      CALL SETUP(MAGIN,MAGOUT,TIC,STEP,NEQNS,DTAV,EPSIL,DELTA,ERR,
     1  TIME,DTIME,YICS,YPRED,YCORR,YDOT,YNEW,YDERV,YDEV,FWDEL,BKDEL,
     2  TBDEL,PARTY,YNEWB)
    9 IF(TIME-TSTOP)10,5,5
   10 YDOT(1) = 2.*(YPRED(1)*C(1) + YPRED(2)*C(2) + YPRED(3)*C(3))
     1    - YPRED(3)**2/C(10)**2 + C(11)
      YDOT(2) = YPRED(1)*C(4) + YPRED(2)*C(5) + YPRED(3)*C(6) + YPRED(2)
     1    *C(1) + YPRED(4)*C(2) + YPRED(5)*C(3) - YPRED(3)*YPRED(5)/
     2    C(10)**2
      YDOT(3) = YPRED(1)*C(7) + YPRED(2)*C(8) + YPRED(3)*C(9) + YPRED(3)
     1   *C(1) + YPRED(5)*C(2) + YPRED(6)*C(3) + YPRED(6)*YPRED(3)/C(10)
     2   **2
      YDOT(4) = 2.*(YPRED(2)*C(4) + YPRED(4)*C(5) + YPRED(5)*C(6))
     1    - YPRED(5)**2/C(10)**2 + C(12)
      YDOT(5) = YPRED(2)*C(7) + YPRED(4)*C(8) + YPRED(5)*C(9) + YPRED
     1    (3)*C(4) + YPRED(5)*C(5) + YPRED(6)*C(6) - YPRED(6)*YPRED(5)
     2    /C(10)**2
      YDOT(6) = 2.*(YPRED(3)*C(7) + YPRED(5)*C(8) + YPRED(6)*C(9))
     1    - YPRED(6)**2/C(10)**2
      MAGIN = -1
   12 CALL MAGIC
      IF(MAGOUT)9,9,13
   13 MAGIN = 0
      WRITE(6,14)TIME,YCORR
   14 FORMAT(1P7E15.7)
      GO TO 12
      END
```

PROGRAM IV

```
C      TWO-DIMENSIONAL DETERMINISTIC
C      C1=Q/R,C2=R,C3=C,C4=K3,C5=K4,C6=X1(0),C7=X2(0),C8=A0,C9=B0,C10=X3(0)
C
       DIMENSION C(36)
       COMMON / MGIC/MAGIN,MAGOUT,TIC,STEP,NEQNS,DTAU, EPSL,DELTA,TIME,
      1 DTIME,YICS(4)/YPRED/YP(4)/YCORR/YC(4)/YDOT/YD(4)/YNEW/YNEW(4)/
      2 YDERV/YDERV(4)/YDEV/YDEV(4)/FWDEL/FDEL(4)/BKDEL/BDEL(4)/TBDEL/
      3 TDEL(4)/ERR/ERR(4)/PARTY/PARTY(4)/YNEWB/YNEB(4)
       DOUBLE PRECISION YNEW,YP,YDERV
     2 FORMAT(I1,I4,1E14.7)
     1 READ(5,2)I,II,D
       IF(I-1) 1,4,3
     3 C(II) = D
       GO TO 1
     4 Z=Z
       WRITE (6,50)
     5 WRITE(6,7)C
     7 FORMAT(1H 1P12E9.2///)
       TEMPO =0.0
       BOT   = 0.
       TIC=0.
       DELTA = C(14)
       EPSL  = C(15)
       ERR(1)= C(16)
       ERR(2)= C(17)
       YICS(1)=0.0
       YICS(2) = 0.
       E= C(6)
       F= C(7)
       H =C(31)
       HH=C(32)
       NEQNS =2
       Q= C(1)*C(2)
       SR =SQRT(C(9)**2 + C(1)*C(3)**2)
       CA1 = -C(9) + SR
       CA2 = -C(8) + SQRT(C(8)**2 -2.*C(9) + C(1) + 2.*SR)
       TEMP=0.0
       TF= C(30)
       DTAU =C(13)
       STP = SQRT(6.2831853)
       MY=-1
       MYN=0
       GO TO 407
   410 B=C(9)
       MY=0
       GO TO 405
   407 B=C(9)
   409 IF(MY)410,402,403
   402 B=B+C(25)
       GO TO 405
   403 B=B-C(25)
   405 CONTINUE
     9 CALL SETUP(MAGIN,MAGOUT,TIC,STEP,NEQNS,DTAU,EPSL,DELTA,ERR,TIME
      1,DTIME,YICS,YP,YC,YD,YNEW,YDERV,YDEV,FDEL,BDEL,TDEL,PARTY,YNEB)
    10 Z=Z
```

```
  401 A=TIME
      PA=-(A-C(8))**2/(2.*C(4)**2)
      PA=1./(STP*C(4))*EXP(PA)
      PB=-(B-C(9))**2/(2.*C(5)**2)
      PB=1./(STP*C(5))*EXP(PB)
   14 CHEV=(Q*C(3)**2+C(2)*CA1**2)*((E**2+((A+CA2)*E+F
     1)**2/(B+CA1))/(2.*(A+CA2)))
     2+(Q+C(2)*CA2**2)*((F**2+(B+CA1)*E**2)/(2.*(A+CA2)))
     3 -(C(3)*Q + C(2)*CA1*CA2)*E**2
C
      YD(1)= CHEV * PA*PB
      YD(2)= CHEV**2*PA*PB
   13 MAGIN = -1
   15 CALL MAGIC
      IF(MAGOUT)10,10,16
   16 MAGIN =0
      IF(TIME-TF)15,19,19
   19 CONTINUE
      TEMP=TEMP+YC(1)
      TEMPO =TEMPO +YC(2)
      G=SQRT(YC(1)**2)
      GG=SQRT(YC(2)**2)
      BOT=BOT+1.
      IF(BOT-C(12)/2.)69,408,408
   69 IF(G.LE.H.AND.GG.LE.HH)GO TO 408
      GO TO 409
  408 MYN=MYN+1
      MY=1
      BOT=0.
      IF(MYN-1)407,407,70
   70 CONTINUE
      TEMP=C(25)*TEMP
      TEMPO =C(25)*TEMPO
      SIGAB =TEMPO -TEMP**2
      WRITE(6,404) TEMP,SIGAB
  404 FORMAT( 15H TWO-DIM MEAN =1P1E16.7,16H   TWO-DIM VAR = 1P1E16.7)
      GO TO 1
   50 FORMAT(107H    Q/R          R          C          K3          K4          X1(0)
     1   X2(0)        AO          BO          X3(0)         M          N     )
      END
```

APPENDIX E

LITERATURE SEARCH

The following literature search concentrates on articles in the field of stochastic control with specific emphasis on papers discussing multiplicative disturbances. They are divided into two major categories: 1) Optimization and 2) Study of Stochastic Differential Equations Including Stability Theory. Also included are several papers reviewing the entire area of stochastic control (with lengthy lists of references). The reader may consult these if interested in other topics in stochastic control theory. While not complete, the list is representative of the important current efforts in these areas.

REVIEW PAPERS

1. H. J. Kushner, "Some Problems and Some Recent Results in Stochastic Control," 1965 IEEE International Convention Record, Part 6, p. 108. General review of stochastic control theory with 58 references.

2. W. M. Wonham, "Stochastic Problems in Optimal Control," IEEE Convention Record (1963).

3. Symposium on Monte Carlo Methods, H. A. Myers, Ed., John Wiley & Sons, Inc., 1956. Discusses all aspects of Monte Carlo simulation techniques with emphasis on random number generation and sampling methods.

4. Random Number Generation and Testing, IBM Reference Manual C20-8011, N.Y., 1959. Discusses various techniques of random number generation including a section on programming aspects.

OPTIMIZATION

1. R. E. Kalman, "Control of Randomly Varying Linear Dynamical Systems," Proceedings of Symposia in Applied Mathematics, Vol. 13. American Mathematical Society, 1962. Existence of optimal solutions for random parameter linear plants.

2. P. Dorato, R. F. Drenick, and L. Shaw, Optimal Stochastic Control Theory, A Short Course, Polytechnic Institute of Brooklyn, January 1964. General discussion of stochastic optimal control; parametric random fluctuations are considered via dynamic programming formalism.

3. R. F. Drenick and L. Shaw, "Optimal Control of Linear Plants with Random Parameters," IEEE Transactions on Automatic Control, Vol. AC-9, No. 3. Treats, in discrete and continuous time, optimal (unconstrained) control of linear plants with random coefficients.

4. R. F. Drenick and R. A. Reiss, "Realization of an Optimal Control System," IFAC International Symposium on Sensitivity, Yugoslav Committee for Electronics, ETAN. Implementation of scheme to control variable parameter plants.

5. J. B. Farison, Identification and Control of Random Parameter Discrete Systems, Stanford Electronic Labs, TR 6302, January 1964. Learning system to identify random parameters correlated in time.

6. T. L. Gunckel, Optimum Design of Sampled-Data Systems with Random Parameters, Stanford Electronic Labs., Report 2102-2. Obtains feedback coefficients for systems with random variable parameters.

7. W. Gersch, and F. Kozin, "Optimal Control of Multiplicatively Perturbed Stochastic Systems," Proc. 1963 Allerton Conference on Circuit and System Theory, University of Illinois, Urbana, 1963.

8. R. A. Rohrer and M. Sobral, Jr., "Sensitivity Considerations in Optimal System Design," IEEE Transactions on Automatic Control, Vol. AC-10, No. 1, January 1965.

9. M. Aoki, "On Performance Losses in Some Adaptive Systems," Journal of Basic Engineering, Vol. 87, pp. 90-94, March 1965.

STOCHASTIC DIFFERENTIAL EQUATIONS AND STABILITY THEORY

1. S. Ariaratnam and P. Graefe, "Linear Systems with Stochastic Coefficients, Parts I, II, III," International Journal of Control, Vol. 1, Issue 1, p. 239; Vol. 2, Issue 2, p. 161; Vol. 3, Issue 3, p. 205. Correlation functions and spectral densities of linear systems with coefficients subjected to Gaussian white noise or incremental Brownian type process.

2. T. K. Caughey and J. K. Dienes, Journal of Mathematics and Physics, Vol. 40-41, p. 300, 1962. Finds autocorrelation and spectral density for linear n^{th} order differential equation whose zeroth derivative term contains random coefficient.

3. C. Adomian, "Linear Stochastic Operators," Review of Modern Physics, Vol. 35, No. 1, January 1963. Develops stochastic Green's functions for linear systems with stochastic parameters.

4. A. I. Ressin, "The Probability Characteristics of Error in Automatic Control Systems with Random Parameters," Technical Cybernetics, No. 3, U.S.S.R., 1964; U.S. Department of Commerce, Office of Technical Services, Joint Publication Research Service, TT 64-41595, 10, September 1964.

5. E. Wong and M. Zakai, On the Relation Between Ordinary and Stochastic Differential Equations, Electronics Research Laboratory, University of California, Berkeley, Report 64-26, 1964. Properties of solutions to stochastic differential equations.

6. H. J. Kushner, "On the Stability of Stochastic Dynamical Systems," Proceedings National Academy of Sciences, Vol. 53, 1965, p. 8.

7. F. Kozin, "On Relations Between Moment Properties and Almost Sure Lyapunov Stability for Linear Stochastic Systems," Journal of Mathematical Analysis and Applications, Vol. 10, 1965. Theorem on almost sure asymptotic Liapunov stability.

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3. Parzen, E., Modern Probability Theory and Its Applications, John Wiley, 1960.

4. Symposium on Monte Carlo Methods, H. A. Myers, ed., John Wiley & Sons, Inc., 1956.

5. Bass, J. J., "Statistical Flight Control System Design," Masters Thesis, Air Force Institute of Technology, GGC/EE/63-1, June 1963.

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9. Ball, D. J., and Stoodley, G. R., An IBM 704/7090 Program for the Numerical Solution of Ordinary Differential Equations, Grumman Research Department Computing Report CR61-2, October 1961.

10. Papoulis, A., Probability, Random Variables and Stochastic Processes, McGraw-Hill Book Company, 1965, p. 159.

11. Rotenberg, A., "A New Pseudo-Random Number Generator," Journal of the Association for Computing Machinery, January 1960.

12. Coveyou, R. R., "Serial Correlation in the Generation of Pseudo-Random Numbers," Journal of the Association for Computing Machinery, January 1960.

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13. ABSTRACT

This report discusses the performance of optimal control systems with random parameters. The plants treated are described by time-invariant ordinary differential equations whose parameters are random variables with known distributions. The control remains fixed as the nominal optimal control, and it is assumed known. Because the parameters of the differential equations are random variables, the performance is a random variable and can only be described in terms of its statistics. Several methods of evaluating the ensemble statistics are presented and compared. Alternatives to the classical notion of sensitivity, which are better suited to stochastic systems, are given. Numerical results for specific examples are presented, the the computer programs used in the study are discussed.

The report also includes an analysis of a particular optimal stochastic control problem for the purpose of indicating some of the difficulties involved in deriving optimal stochastic control laws.

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