APPLICATION OF FINITE ELEMENT METHOD FOR CONTINUUM MECHANICS PROBLEMS

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General nonlinear equations for a continuum are formulated such that they are convenient for discretization by the finite element method. The resulting set of nonlinear algebraic or ordinary differential equations are linearized and solved by the parametric differentiation method. The scope of application of the procedure is very wide. Numerical examples are presented to indicate the scope. The results, in general, show excellent agreement with known solutions for one step or short range processes. For long range solutions, the incremental step becomes a critical factor.

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INTRODUCTION

The major tasks in the development of a procedure for solving problems of a wide class may be listed as:

a) The formulation of the basic equations for this wide class of problems in a variational, differential or integro-differential form. The form of the equations should be convenient for the envisaged method of solution.

b) The evaluation of physical constants appearing in the field equations. Even though general theories being rapidly put forward, their use is inhibited by the lack of available experimental values. Also, experimental procedures have to keep pace with sophisticated theories so that the experimental errors do not nullify the accuracy of the formulation.

c) The development of efficient mathematical and numerical tools for solving the equations.

In this paper, an attempt is made to solve a wide class of problems in the field of continuum mechanics. Attention is focused on tasks (a) and (c). It is assumed that values of physical constants would be available in some form.

The general field equations of continuum mechanics has been derived from thermodynamic principles and the hypothesis of material frame indifference[1]. The procedure for formulation follows the concepts forwarded by Green and Rivlin[2]. Monopolar and dipolar stress fields and temperature field are considered.

Constitutive equations which should follow certain basic axioms[3] form the most difficult part of the general formulation. These equations bring out the particular behavior of a given material under prescribed environmental conditions. In this study, these equations are kept in a general form and, later, in the illustrative examples, particular forms are specified.

Finite element method is employed to annihilate the space dependency, and express the field equations and constitutive relations in terms of chosen field variables at discrete points. The field equations are put in a form convenient for finite element idealization. It should be noted that it is not necessary to have the equations in variational form for use of finite element method[4]. The discretized algebraic or differential equations, depending on whether or not time derivatives are involved, are, in general, simultaneous and nonlinear. This obviates the use of any direct method of solution. Parametric differentiation method[5,6,7] is employed to obtain a stepwise solution.

Two numerical examples are presented to indicate the generality of the approach.

GOVERNING EQUATIONS

The complete mathematical statement describing the state of an ideal deformable medium can be expressed through

a) law of conservation of mass

b) thermodynamic laws

c) constitutive relations

d) environmental stipulations expressed as boundary and initial conditions on loads, displacements and heat flow.

The variables commonly employed to describe these changes of state are: stress and displacement components, temperature, entropy and heat flow. A complete description covering a large class of problems and materials would be rather cumbersome to handle besides being intractable even for some of the simpler cases. Hence, in this paper, simplifying assumptions are made which would yet be general for a fairly wide class of problems.

Assumptions

1. The hypothesis of material frame indifference is applicable.
2. Concept of force defined through work rate principle is valid.
3. Displacement derivatives up to second order will be active for the internal energy expression (Grade II material)
4. Supply of non mechanical energy is only due to thermal effects.
5. Multipolar kinetic energy and heat flux has been neglected.
6. Antisymmetric parts of dipolar stresses i.e. $T^{|kj|i}$ are non-existent.

7. Only monopolar and dipolar stress fields are considered.

Under these restrictions, the foregoing laws would be invoked to arrive at the final set of equations.

Since the derivation of these equations is rather lengthy, only a few salient steps would be indicated. The detailed derivations could be found in Reference 8.

All quantities are referred in terms of the undeformed curvilinear coordinate system X^i or the initial position vector $\bar{X}$ (at time $t = 0$). Any point X with position vector $\bar{X}$ will be denoted after deformation as x and its position vector as $\bar{x}$ where

$$\bar{x} = \bar{x}(\bar{X},t) \tag{1}$$

The deformed position vector $\bar{x}$ can also be expressed in terms of the deformation vector $\bar{u}$, such that

$$\bar{x} = \bar{X} + \bar{u} \tag{2}$$

where $\bar{u} = \bar{u}(\bar{X},t)$ and $\bar{u}(\bar{X},o) = 0$

Material Frame Indifference[1,2]:

Any event has to be specified with respect to a reference frame for space as well as time. This specification is not unique and depends upon different observers. But the fundamental measurable quantities such as distances, angles and time intervals are independent of the observer. Consequently, any change in frame must preserve them along with the temporal orders of all events. The most general relation for such a change of frame may be expressed as,

$$\bar{x}^*(\bar{X},t^*) = \bar{R}(t)\ \bar{x}(\bar{X},t) + \bar{D}(t)\ ;\ t^* = t - a \tag{3}$$

where $(\bar{x}^*,t^*)$ and $(\bar{x},t)$ represent the same event in different frames, $\bar{D}(t)$ is a constant vector, $\bar{R}(t)$ is a proper orthogonal tensor and a is a real number. Two motions of a given medium related by Eq.(3) are said to be equivalent.

For a change of frame,

(a) a scalar remains unchanged

(b) a vector $\bar{V}$ transforms to $\bar{V}^*(\bar{X},t^*) = \bar{R}(t)\ \bar{V}(\bar{X},t)$ (4)

(c) a tensor $\bar{S}$ transforms to $\bar{S}^*(\bar{X},t^*) = \bar{R}(t)\ \bar{S}(\bar{X},t)\ R^{-t}(t)$

(5)

Functions whose values are scalars or tensors will be called frame indifferent, if both dependent and independent variables transform according to the above laws.

Definition of Forces:

The quantitative definition of forces is not unique nor straight forward, though it can be conveniently derived through the concept of work rate. The basic idea is that if P^i is a component of vector, and V_i is an arbitrary velocity component and if the scalar P^iV_i is the rate of work corresponding to the velocity vector V, then P^i may be defined as force in the direction of X^i. In continuum mechanics, this idea may be readily extended to define forces of more general nature[2,8]. Considering only the monopolar and dipolar cases, the work rate for body forces F^i and F^{ij} per unit mass, surface forces P^i and P^{ij} per unit area and stresses T^{ij} and T^{ijk} per unit area normal to axis X^i are given respectively as,

$$\rho_o(F^iV_i + F^{ij}V_{j|i}),\ P^iV_i + P^{ij}V_{j|i} \text{ and } T^{ij}V_j + T^{ijk}V_{k|j} \qquad (6)$$

where ρ_o is the initial mass density and bar indicates covariant differentiation.

Basic Laws:

An element of a continuum will obey the following three laws through any stage of evolution[1,9].

a) Conservation of mass: The mathematical expression for conservation of mass may be expressed as

$$P(\bar{x},t) = \rho_o(\bar{X},0) \ / \ \left|\frac{\partial x^k}{\partial X^L}\right| \qquad (7)$$

This relation may be taken as a definition of density, $\rho(\bar{x},t)$, at time t and for the point $\bar{x}$.

b) Second law of thermodynamics: This law asserts the existence of a quantity called entropy, η, such that

$$\eta = \eta(e,\ \nu_\sigma,\ \bar{X}) \qquad (8)$$

where e is the internal energy density and ν_σ are a certain number of mechanical parameters influencing the motion.

The absolute temperature T is defined through entropy as

$$\frac{1}{T} = \frac{d\eta}{de} \tag{9}$$

If dQ^* is the increase of heat energy at temperature T, then the increase of entropy can be written as,

$$d\eta = d\eta_i + \frac{dQ^*}{T} \tag{10}$$

where $d\eta_i$ is the production of entropy inside the system.

The second law asserts that for any process in nature, the production of entropy inside the system cannot be negative, i.e.

$$d\eta_i \geqslant 0 \tag{11}$$

where the equality stands for reversible process and the inequality for irreversible process.

The second law does not provide any additional equation but restricts the form of the constitutive relations.

c) Balance of energy: This law is expressed as

$$K^o + E^o = P + F \tag{12}$$

where K and E are the kinetic and internal energies respectively, P and F are the power supplied by external mechanical forces and the total non-mechanical power, in the present study, due to heat. Eq. (12) may also be termed as the first law of thermodynamics.

Considering an arbitrary material volume, **v**, of a continuum bounded by a surface, **s**, having a unit normal, $\bar{n}$, in the reference configuration, Eq. (12) can be written as

$$\rho_o(V^i V_i + e)dv = \rho_o(q + F^i V_i + F^{ij} V_{j|i})dv + (h + P^i V_i + P^{ij} V_{j|i})ds \tag{13}$$

Letting the arbitrary internal volume shrink to a point, and then applying the frame indifference principle, it can be shown that Eq. (13) reduces to

$$P^i = n_j T^{ji} \; ; \; P^{ji} = n_k T^{kji} \text{ and } h = n_i Q^i \tag{14}$$

Substituting relations (14) into Eq. (13) and transforming surface integrals to volume integrals and again applying the same principle, it is found that,

$$T^{ki}{}_{|k} + P_o F^i - \rho_o V^i = 0 \tag{15}$$

$$t^{*(ji)} = 0 \tag{16}$$

$$\rho_o \ell = t^{*(ji)} V_{i|j} + T^{i(kj)} V_{i|jk} + \rho_o q + Q^i{}_{|i} \tag{17}$$

where $t^{*ji} = T^{kji}{}_{|k} + \rho_o F^{ji} + T^{ji}$

In Eqs. (15), (16) and (17), the volume integral has been dropped, since these relations are true for any arbitrary volume. Eqs. (15) and (16) are the Cauchy's laws of motion whereas Eq. (17) has to be satisfied by constitutive relations for a medium.

If sufficient smoothness requirements are assumed, Eq. (8) may be inverted to give

$$e = e(\eta, \nu_\sigma, \bar{X}) \tag{18}$$

Since Eq. (17) involves terms upto second derivative of the mechanical parameters, expression (18) can be prescribed in the form

$$e = e(\eta, E_{ij}, E_{ijk}, \bar{X}) \tag{19}$$

with the restriction on E_{ij} and E_{ijk} that the scalar function e should be invariant under rigid body rotation.

One of the several ways is to prescribe the classical strain tensor for E_{ij} and Toupin's strain gradient[10] for E_{ijk}, i.e.

$$E_{ij} = u_{i|j} + u_{j|i} + u_{k|i}\, u^k{}_{|j} \tag{20}$$

and $$E_{ijk} = E_{ij|k} + E_{ik|j} - E_{jk|i}$$

The dependence of the density function e on $\bar{X}$ accounts for the non-homogeneity of the material.

Comparison of the time derivatives of e in Eqs. (17) and (18) yields,

$$(t^{*(ij)} - \hat{T}^{(ij)})\, A_{ij}\, dv = 0 \tag{21}$$

$$(T^{(kj)i} - \hat{T}^{(kj)i})\, V_{i|jk}\, dv = 0 \tag{22}$$

$$\rho T \overset{o}{\eta} = {}_d\hat{T}^{(ij)}\, A_{ij} + {}_d\hat{T}^{(kj)i}\, V_{i|jk} + \rho_o q + Q^i{}_{|i} \tag{23}$$

where, $A_{ij} = (V_{i|j} + V_{j|i})/2$,

the dissipative part of the stresses,

$${}_d\hat{T}^{(ij)} = t^{*(ij)} - {}_e\hat{T}^{(ij)} \; ; \; {}_d\hat{T}^{(kj)i} = T^{(kj)i} - {}_e\hat{T}^{(kj)i}$$

and the thermodynamic tensions,

$${}_e\hat{T}^{ij} = \rho_o \frac{\partial e}{\partial u_{j|i}} \; ; \; {}_e\hat{T}^{(kj)i} = \rho_o \frac{\partial e}{\partial u_{i|jk}}$$

Here, prefixes e and d are used to denote recoverable and irrecoverable parts of the stresses. Eqs. (15), (16), (21), (22) and (23) constitute a set of 40 equations in 31 unknowns and hence do not form a determinate set. However, the part of the stress $T^{|kj|i}$ does not contribute to the equations of motion nor produces any work[8], although it may have an important role for the boundary conditions[2]. For the sake of simplicity, this part will be assumed to be non existent within the body or on the bounding surfaces. This makes the set determinate.

Eq. (23) may be put in a slightly different form as:

$$(\rho_o \overset{o}{T} T \frac{d}{dt}(\frac{\partial \psi}{\partial T}) + \rho_o \overset{o}{T} q - \overset{o}{T}_{|i} Q^i + W_d \overset{o}{T}) dv + \eta_i \overset{o}{T} Q^i\, ds = 0 \tag{24}$$

where $W_d = {}_d\hat{T}^{(ji)} A_{ij} + {}_d\hat{T}^{(kj)i}\, V_{i|jk}$ and $\psi = \eta T$ representing Helmholtz free energy. Eq. (24) furnishes the thermal equation of state.

Constitutive Equations:

The general physical laws in themselves do not suffice to determine the deformation of a body subjected to given loading. A vast amount of literature is available on the mathematical derivations of constitutive equations based on the concepts of irreversible thermodynamics[11,12] as well as those obtained from considering the differential type[13] and the axiom of fading memory[14]. Any discussion on their general nature is beyond the scope of the present paper.

To solve any problem, it is necessary to prescribe the constitutive relations for:

$$ {}_{e}\hat{T}^{(ij)},\ {}_{e}\hat{T}^{(kj)i},\ {}_{d}\hat{T}^{(ij)},\ {}_{d}\hat{T}^{(kj)i},\ \psi \text{ and } Q^{i}. $$

One of the forms may be of the following type:

$$ \hat{T}^{(ij)} = \bar{T}^{(ij)} \cdot \left(\frac{\partial E_{ij}}{\partial u_{i|j}} + \frac{\partial E_{ij}}{\partial u_{j|i}} \right) /2 \qquad (25) $$

$$ \hat{T}^{(kj)i} = \bar{T}^{(kj)i} \cdot \left(\frac{\partial E_{ijk}}{\partial u_{i|jk}} \right) \qquad (26) $$

$$ \psi = \bar{\psi}(E_{ij},\ E_{ijk},\ A_{ij},\ A_{ijk},\ T,\ \overset{o}{T},\ t^{P}) \qquad (27) $$

$$ Q^{i} = \bar{Q}^{i}(E_{kl},\ E_{klm},\ T_{|k},\ t^{P}) \qquad (28) $$

where $\bar{T}^{(ij)}$ and $\bar{T}^{(kj)i}$ are Piola type stresses,

A^{i}s are the Rivlin-Eriksen tensors[3]

and t^{P} denotes the dependence of the function on their history of their arguments for $t \geq t^{P} \geq -\infty$ where t is the present value of time.

The two illustrative examples presented at the end use particular forms of Eqs. (25) to (28).

Summary:

Eq. (14) may be combined with Eq. (15) and (16) to yield:

$$(T^{ji}{}_{|j} + \rho_o F^i - \rho_o \overset{o}{V}{}^i)\, V_i dv + (P^i - \eta_j T^{ji})\, V_i ds = 0 \tag{29}$$

$$(T^{ji} - t^{*(ji)} + T^{(kj)i}{}_{|k} + \rho_o F^{ji})\, V_{i|j} dv + (P^{ji} - \eta_k T^{(kj)i})\, V_{i|j} ds = 0 \tag{30}$$

Eqs. (24) to (30) form the final set of equations. The final form is quite convenient for discretization by the finite element method.

The surface integrals in Eqs. (24), (29) and (30) would include the work done by the unbalanced surface stresses or heat flux vector. Thus, the discontinuity in stresses or heat flux vector would be taken into account in 'the mean'.

SOLUTION TECHNIQUE

Once the general equations are obtained, it is a matter of technique to get a solution out of them for a definite problem. Theoretically, it is possible, but the amount of complexities encountered for the exact solution are virtually prohibitive. Hence, finite element method will be employed here, since it may be considered as one of the most powerful discretization techniques available among the approximate techniques. Application of this method annihilates the space dependency and expresses the field equations in terms of the chosen field variables at discrete points. The resulting equations, in general, will be nonlinear and simultaneous which obviate the use of any direct method of solution. Hence, a parametric differentiation scheme[5,6,7,8] will be applied to obtain a stepwise solution.

Parametric differentiation technique is a systematic and straightforward 'marching' approach. The method consists of differentiating the governing equations with respect to a chosen parameter, λ, (say load) reducing the original set of nonlinear equations to linear equations in the differentials. The reduced set can be solved straightaway using any integration technique for a small step increment through a quadrature formula. This way the solution will march out from λ to the required value of the parameter, say λ_1. In essence, the nonlinearity of the problem is restricted only in the quadrature manipulation which virtually poses no difficulty. The method can be best demonstrated by taking a simple example. Say it is required to solve the nonlinear differential equation,

$$\dot{C}U + L = 0 \qquad (A)$$

with boundary condition

$$U = U_{t_o} \quad \text{at} \quad t = t_o \qquad (B)$$

where, $C = C(U,\dot{U})$, $L = L(t)$, $U = U(t)$ and $\dot{U} = dU/dt$

Choose the forcing parameter λ such that

$$\bar{L} = \bar{L}(t,\lambda) \text{ where } L(t) = \bar{L}(t, \lambda_1)$$

and $$\bar{U} = \bar{U}(t,\lambda) \text{ where } U(t) = \bar{U}(t,\lambda_1)$$

Consequently, it is possible to construct

$$\bar{C} = \bar{C}(\bar{U}(t,\lambda), \dot{\bar{U}}(t,\lambda))$$

where, $C(t) = \bar{C}(\bar{U}(t,\lambda_1), \dot{\bar{U}}(t,\lambda_1))$

Hence, the new differential equation can be written as,

$$\bar{C}\,\dot{\bar{U}} + \bar{L} = 0 \qquad (C)$$

with boundary condition,

$$\bar{U}(t_o,\lambda) = U_{t_o} \qquad (D)$$

The function $\bar{U}$ has to be chosen in such a way that it satisfies (C) and (D) at $\lambda = \lambda_o$. Now differentiation of (C) and (D) with respect to λ yields,

$$\frac{\partial \bar{C}}{\partial \bar{U}}\,\bar{U}^* + \left(\frac{\partial \bar{C}}{\partial \dot{\bar{U}}} + \bar{C} \right) \dot{\bar{U}}^* + \bar{L}^* = 0 \qquad (E)$$

and $$\bar{U}^*(t_o,\lambda) = 0 \qquad (F)$$

where, $\bar{U}^* = \frac{\partial \bar{U}}{\partial \lambda}$ and $\bar{L}^* = \frac{\partial \bar{L}}{\partial \lambda}$

The modified equation (E) and (F) is linear with respect to $\bar{U}^*$ and with variable coefficients. It can be solved, at least numerically, and the solution will be:

$$\bar{U}^*(t,\lambda_o).$$

From this, $\bar{U}(t,\lambda_o + \Delta\lambda)$ can be easily obtained by the integration,

$$\bar{U}(t,\lambda_o + \Delta\lambda) = \bar{U}(t,\lambda_o) + \int_{\lambda_o}^{\lambda_o+\Delta\lambda} \bar{U}^*(t,\lambda_o)\,d\lambda \qquad (G)$$

In similar way, the solution can be marched upto $\lambda = \lambda_1$ when $\bar{U}$ will be the same as U which is the solution of (A) and (B). The advantages of this method are that it allows the solution to be obtained throughout the range of λ and no iterative technique is needed.

For efficient execution of the quadrature in (G) the following two forward-integration methods are recommended:

1. Improved Euler-Cauchy method
2. Runge-Kutta-Gill method.

Both schemes are easily available in any text of numerical analysis.

APPLICATIONS

To show the application of the above general procedure, two simple problems have been illustrated with the assumptions that acceleration terms are neglected and dipolar stress field is not existent. Modified Euler method has been used for integration in the following problems.

Problem 1: The first problem has been selected as the heat conduction across a half space (second Danilovskaya) for a coupled thermoelastic illustration. This is solved considering rectangular finite element model having linear distribution of variables in the coordinate directions. The variables are two displacements, three stresses and temperature. For simplicity, Fourier law of heat conduction and only large rotation but small strains are considered here.

The finite element solution and the values as given by Nickell and Sackman[15] have been compared in Fig. 1. For the comparison, the following values are assumed:

$$P = 0.008 \ , \ \sigma = 15.3 \times 10^{-6} \ , \ PC_v = 0.96 \ , \ k = 0.04 \ ,$$

$$\lambda = 42.0 \times 10^{4} \ , \ u = 4.0 \times 10^{4}$$

where P, σ, C_v, k, λ and u are respectively mass density, linear thermal expansion coefficient, specific heat per unit mass, thermal conductivity and isothermal Lame's coefficients. The convective heat transfer across the surface is assumed to as, $Q^i = b(T-T_o)$ where, direction i is normal to the boundary surface, b is the film constant which is taken as 0.78, T is the temperature of the surface and T_o is the outside ambient temperature, which is 300.0. The boundary surface temperature is prescribed to vary linearly from 300 at t = 0 to 600 at $t = 1.45 \times 10^{-10}$ and then it is kept constant.

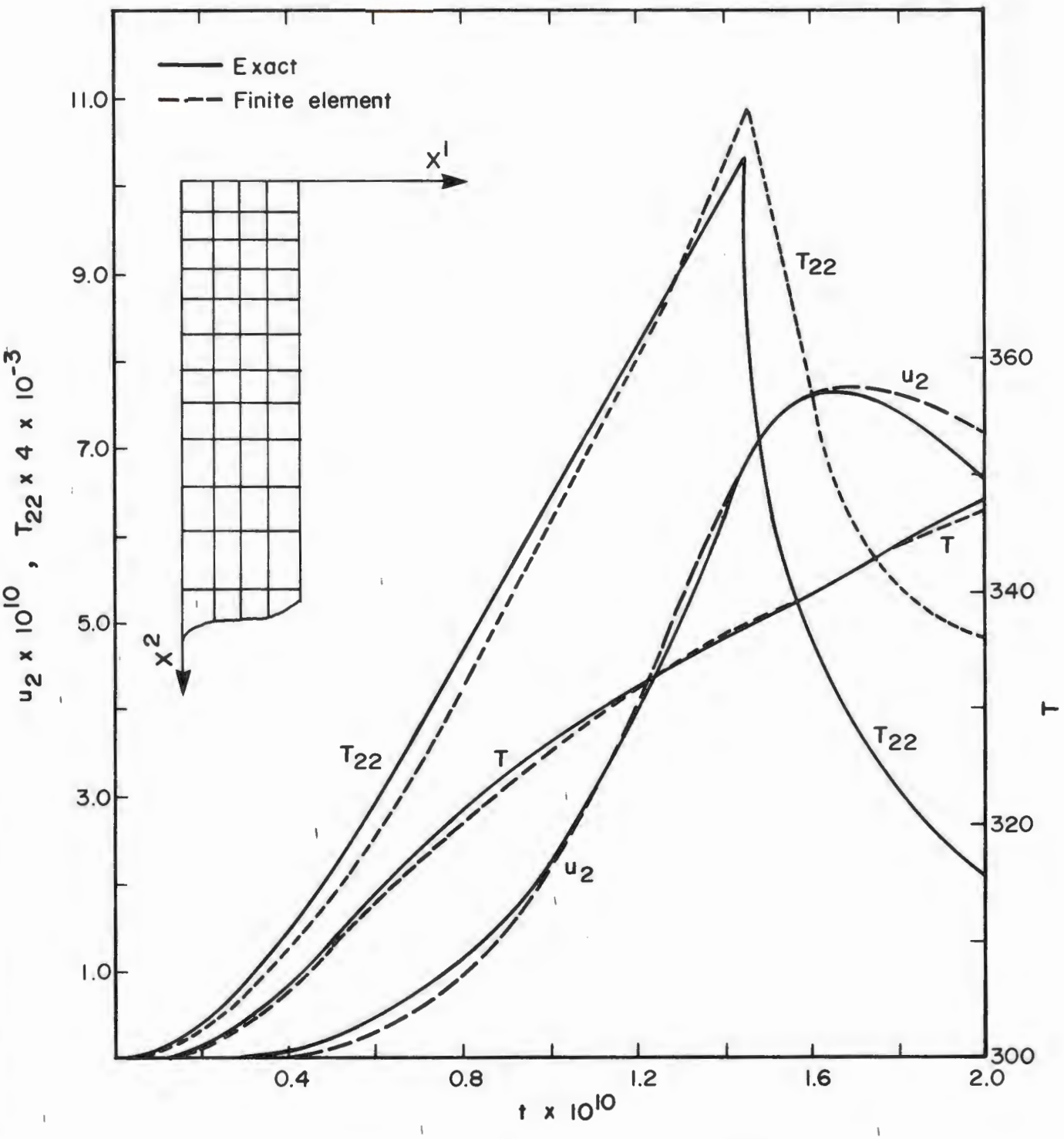


Fig. 1 - THERMAL LOADING ON AN ELASTIC HALF SPACE.

It is seen that u_2 and T agree satisfactorily up to $t = 1.60 \times 10^{-10}$. After that, they start deviating rather sharply. But, even before $t = 1.45 \times 10^{-10}$, the accuracy of stress is rather poor, the maximum error being about 8%.

Problem 2: A viscoelastic cylinder stiffened by an external elastic encasing with ablating inner surface has been chosen as the second example. This problem is analyzed using ring shaped element having triangular cross section with linear distribution of displacements in cylindrical polar coordinate system. The Young's modulus and the poisson's ratio of the elastic encasing are respectively 10^6 and 0.25. The ablating rate for the inner surface has been taken as [16] $a(t) = 25/(1-3t/4)$, where a(t) is the inner radius at time t. The pressure variation with respect to time is assumed to be $p(t) = 1 - e^{-5t}$. The constitutive relations for the viscoelastic material are prescribed as,

$$(T_{11}, T_{22}) = \int_0^t J(t-t') \quad \dot{E}_{11} + \dot{E}_{22} \quad dt'$$

$$+ \int_0^t K(t-t') \quad \dot{E}_{11}, \dot{E}_{22} \quad dt'$$

and $$T_{33} = \int_0^t J(t-t') \quad \dot{E}_{11} + \dot{E}_{22} \quad dt'$$

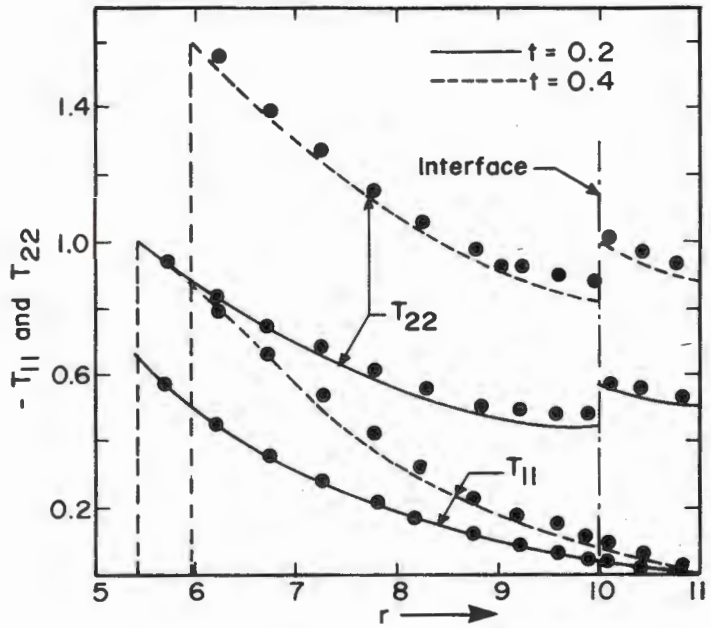
where K(t) and J(t) for this problem are taken as,

$$K(t) = 82.0 + 9282.0\, e^{-1.126t} \quad \text{and}$$

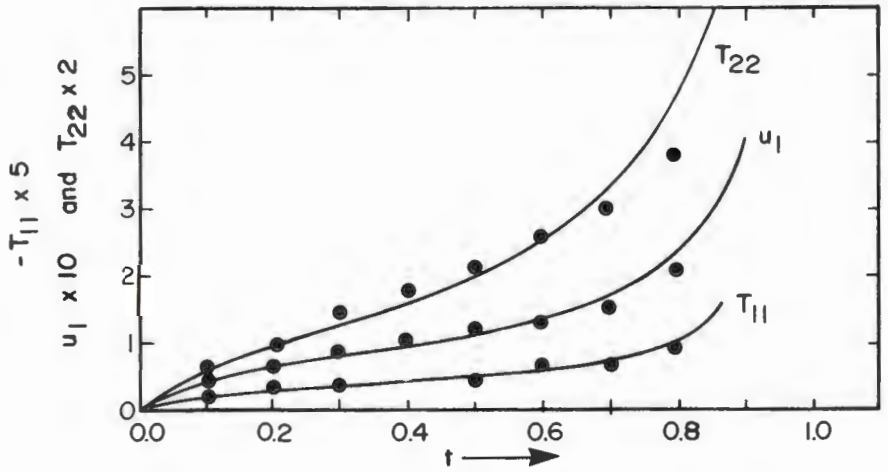
$$J(t) = 12472.7 - 3094.0\, e^{-1.126t}$$

Due to ablation of inner surface, at every new integration step, the arrangement of the element has to be modified. This is done by reducing the inner most column of elements and interpolating the previous values of strains at the centre of gravity of new reduced elements. The integrations necessary for obtaining the material coefficients have been achieved by trapezoidal rule.

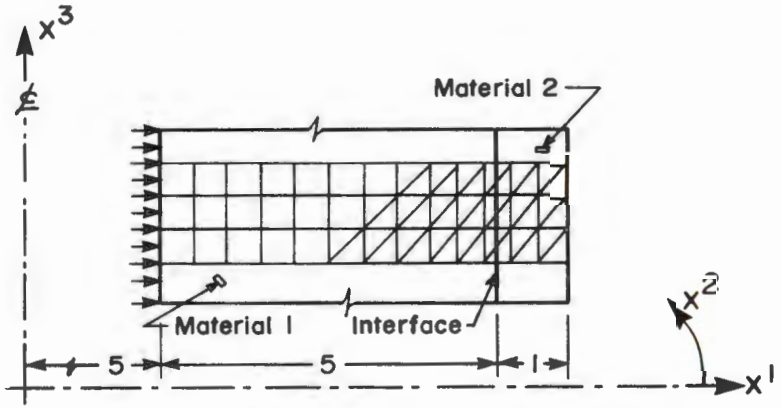
The results obtained using finite element method are compared in Fig. 2 with analytical solution presented by Ting[16]. It may be observed that stresses and displacement at the interface are agreeable up to t = 0.6. Also, the stress distributions are quite satisfactory except near the interface.



B. STRESS DISTRIBUTION IN CYLINDER



C. HISTORY AT INTERFACE



A. INITIAL ARRANGEMENT OF ELEMENTS

FOR V.E. MATERIAL I AT t = 0

Y.M. = 1.25×10^4, P.R.=0.25

FOR ELASTIC ENCASING

Y.M. = 10.0×10^4, P.R.=0.25

● FINITE ELEMENT SOLUTION

═══ EXACT

Fig. 2 - ANALYSIS OF VISCOELASTIC ENCASED CYLINDER

CONCLUSION

In this paper, following the concept forwarded by Green and Rivlin[2], the general nonlinear equations for a continuum are derived. These equations are directly applied to finite element discretization of space variable and parametric differentiation for linearization.

The scope of application of these equations is very wide. With minor modifications, they can be applied to any particular case of continuum problems, for example multipolar cases, viscoelasticity, viscoplasticity, coupled thermoelasticity with dissipative properties etc. The application to problems of nonlinear stability or complicated fluid flow with thermal effects may also be possible.

In this study, utilizing this general technique, two particular cases of simpler types have been solved, though many more have been given in Reference 8. The results, in general, show excellent agreement for short range processes. For long range solutions, the incremental step length becomes a critical factor. Unfortunately, due to the limitation in available computer time, no further refinement in this direction has been possible. From the limited experimentation it appears that the approach offers no more complexities than the numerical solution of nonlinear initial value problems. Moreover, Davidenko[6] has shown that the convergency of the process can be improved considerably by employing more efficient integration technique, such as Runge-Kutta procedure. It may be concluded that the finite element discretization coupled with parametric differentiation procedure for nonlinear equations forms a powerful tool for solving complex continuum mechanics problems.

NOMENCLATURES

All quantities in this study are defined with respect to the coordinate system at undeformed state.

A_{ij}, A_{ijk}	Rivlin-Ericksen tensors for mono and dipolar fields
E_{ij}, E_{ijk}	Strains
e	Internal energy density
F^{i}, F^{ij}	Body forces
G^{ij}, g^{ij}	Metric tensors for undeformed and deformed state
h	Heat flux
K^{ij}	Conductivity tensor
n_{k}	Components of unit outward normal
P^{i}, P^{ij}	Surface loads
Q^{i}	Heat flux vector
q	Distributed energy sources inside the body per unit mass
T^{ij}, T^{ijk}	Stresses
${}_{d}T^{ij}$, ${}_{d}T^{ijk}$	Dissipative parts of the stresses
T	Temperature
$T^{(ij)}$ etc.	Functionals for stresses
$\hat{T}^{(ij)}$, $T^{\|ij\|}$	Symmetric and antisymmetric parts of T^{ij}
t	Time
u_{i}, V_{i}	Displacements and velocities
$V_{i\|j}$	Velocity gradient where bar denotes covariant derivative

X^i, $\bar{X}$	Curvilinear coordinates and position vector
x^i, $\bar{x}$	Coordinates and position vector for deformed state
α	Indices
η	Entropy
ν_α	Thermodynamic affinities
ρ_o, ρ	Mass densities at undeformed and deformed state
τ^α	Thermodynamic stresses
ψ	Helmholtz free energy

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