THE ANALYSIS OF THIN SHELLS WITH TRANSVERSE SHEAR STRAINS BY THE FINITE ELEMENT METHOD

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An analysis of thin shells with transverse shear strain by the finite element method is developed. A shell theory with transverse shear strains is used in order to reduce the continuity requirements on the finite elements displacement assumption. The shell theory used requires only continuous displacement assumptions as contrasted to a Kirchhoff shell theory which requires continuous derivatives for the displacement normal to the shell. Since the majority of aerospace structures are based on shells of revolution, the equations are specialized to shells whose reference surfaces are a portion of an axisymmetric surface. A doubly-curved arbitrary quadrilateral element is developed with bilinear displacement assumptions for the inplane displacements and the transverse deformation variables and a bicubic variation in the out-of-plane displacement. Examples of calculations for flat plates, cylinders with cutouts, and spheres with cutouts are included.

* This work was supported by the United States Atomic Energy Commission

SECTION I

INTRODUCTION

The finite element method is a numerical solution technique which has been applied to the analysis of thin shells with considerable success. Linear elastic axisymmetric shells subjected to axisymmetric loads were the first ones treated (References 1 and 2). Subsequent work led to the analysis of linear elastic axisymmetric shells subjected to asymmetric loads. This was accomplished by using a sine and cosine eigenfunction expansion in the circumferential variable and applying the finite element method to the remaining meridional differential equations in each eigenvalue (References 3 through 8). Further work led to the treating of axisymmetric shells with curved meridians directly rather than analyzing conical frustum idealizations, (References 9 through 12 and 15). Other work has led to the elastic-plastic analysis of axisymmetric shells subjected to axisymmetric loads, (References 13 through 15), and rudimentary geometric nonlinearities have been considered for axisymmetric shells (References 16 through 21).

Concurrently, efforts have been directed towards the solution of thin shells of arbitrary geometry. Many authors have treated thin shells of arbitrary geometry as a sequence of flat plates or facets and used a flat plate finite element for each facet, (References 22 through 27). There are several treatments of arbitrary shells where the shell is considered locally to be a shallow shell and single finite elements developed based on this assumption, (References 28 through 31).

With the exception of four papers covering special cases, there has not been a treatment by the finite element method of the general equations governing the behavior of thin shells. Gallagher (Reference 32) considered a doubly curved rectangular element defined by the lines of principal curvature. The element was based on constant but distinct principal curvatures. Bogner, Fox and Schmit (Reference 33) treated a right circular cylinder of finite length with a singly curved rectangular element defined by the lines of principal curvature. Cantin and Clough (Reference 34) considered a rectangular cylindrical shell element defined by the lines of principal curvature with explicit rigid body modes present, and recently, Oden (Reference 35) discussed a class of shell elements again rectangular and defined by lines of an orthogonal coordinate system.

In order to develop a two-dimensional curved shell element, four major difficulties must be overcome. An element geometry must be selected that provides the needed flexibility in fitting the problem boundaries. A representation of the shell reference surface must be obtained that provides satisfactory curvatures. A displacement function must be found that provides the needed continuity, and for practical reasons the presence of rigid body modes must be assured. The character of these four difficulties is, to a large degree, dependent on the differential equations, or equivalently the variational principle used to describe the shell.

The equilibrium equations of thin shells with the Kirchoff hypothesis imposed written in terms of displacements are a set of fourth order differential equations. The related minimum potential energy functional requires second derivatives in the unknown displacements in order to be evaluated. In the finite element method this means that the element displacement assumptions must lead to continuous displacement fields which have continuous first derivatives. This is a very difficult proposition when dealing with irregular grids on the curved reference surfaces of thin shells.

The equibilrium equations of thin shells with transverse shear strains written in terms of displacements are a set of second order differential equations. The related minimum potential energy functional requires only first derivatives in the unknown displacements in order to be evaluated. In the finite element method this means that the element displacement assumptions need only provide continuous displacement fields. This is a much easier proposition with which to deal. This system of equations is also much more accurate than those above when examining the dynamic response of thin shells.

It is this second set of differential equations that is treated here. The equations are considered in general for clarity. Since the majority of aerospace structures are based on shells of revolution, the equations are specialized to shells whose reference surfaces are a portion of an axisymmetric surface. A curved quadrilateral element is developed which meets three out of the four requirements. The element is an arbitrary quadrilateral and satisfies the continuity requirements, while having rigid body freedoms only for vanishing curvature or size.

The continuity requirements of both the above variational principles can be relaxed if additional line integrals are added to the functional. These added integrals are essentially the continuity requirements times Lagrange multipliers integrated along lines of discontinuity. Several treatments of this in the finite element context are available, (References 36 through 40).

The use of a shell theory with transverse shear strains to reduce the continuity requirements imposed on the finite element method is not entirely new, (References 22, 25, 28, 55, and 56). However, the approach taken in these references is to modify the strain energy due to the transverse shear strains or to eliminate it altogether and impose a discrete version of the Kirchhoff hypothesis. Neither of these approaches is taken here.

SECTION II

THIN SHELL EQUATIONS

The treatment of the thin shell equations is that presented by K. Washizu in a series of lectures at the University of Washington in 1962. A later treatment of this problem appears in his monograph (Reference 41). Figure 1 shows the geometry of the reference surface.

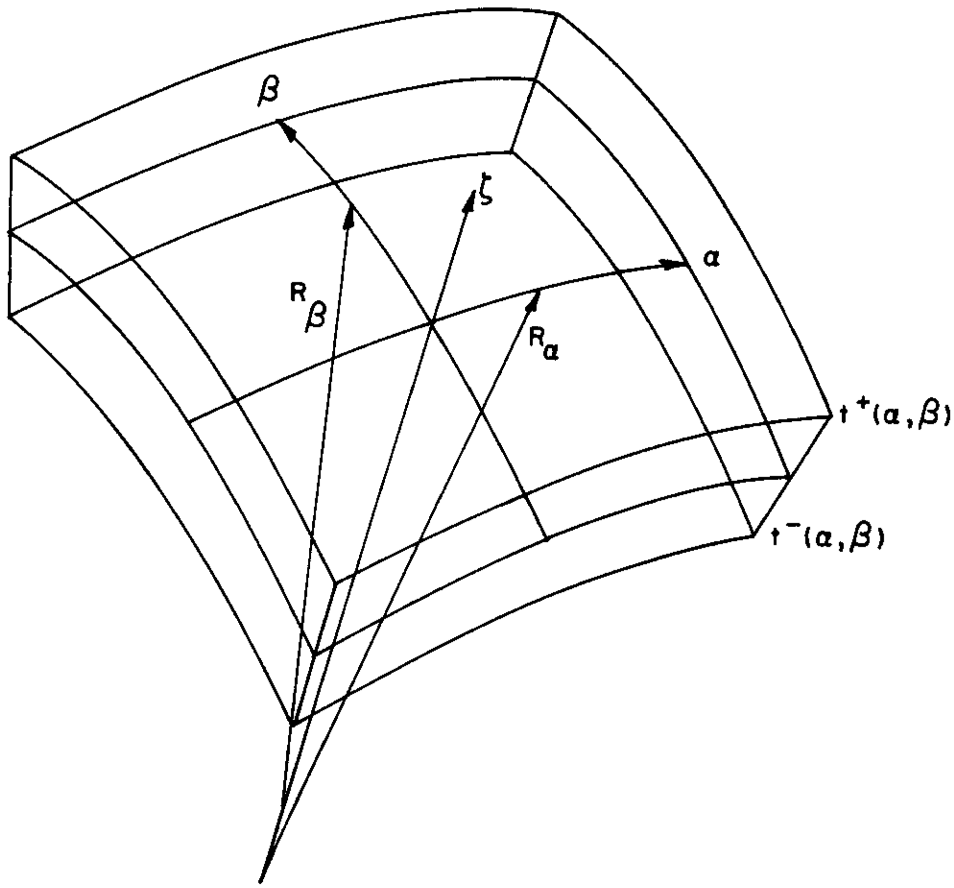

Figure 1

The principle coordinates in the reference surface are α and β. Normal to the reference surface is the coordinate ζ. The principle curvatures of the reference surface, $1/R_\alpha$ and $1/R_\beta$, are positive as shown. The upper surface of the shell is given by $t^+(\alpha, \beta)$ and the lower surface by $t^-(\alpha, \beta)$. An element of length in the shell coordinate system is given by

$$ds^2 = A^2 \left(1 + \frac{\zeta}{R_\alpha}\right)^2 d\alpha^2 + B^2 \left(1 + \frac{\zeta}{R_\beta}\right)^2 d\beta^2 + d\zeta^2 \tag{1}$$

where A and B are the Lamé coefficients in the α,β-plane.

In terms of Cartesian components of the displacement vector, the deformations allowed in the shell are of the form

$$\begin{aligned} u_\alpha(\alpha,\beta,\zeta) &= u(\alpha,\beta) + \zeta\, f(\alpha,\beta) \\ u_\beta(\alpha,\beta,\zeta) &= v(\alpha,\beta) + \zeta\, g(\alpha,\beta) \\ u_\zeta(\alpha,\beta,\zeta) &= w(\alpha,\beta) \end{aligned} \tag{2}$$

The resulting Cartesian components of the strain tensor are given by

$$\begin{aligned} e_{\alpha\alpha} &= \frac{\epsilon_{\alpha o} + \zeta K_\alpha}{\left(1+\frac{\zeta}{R_\alpha}\right)} \\ e_{\beta\beta} &= \frac{\epsilon_{\beta o} + \zeta K_\beta}{\left(1+\frac{\zeta}{R_\beta}\right)}, \qquad e_{\zeta\zeta} = 0 \\ 2e_{\alpha\beta} &= \frac{\gamma_{\alpha\beta o} + 2\zeta K_{\alpha\beta}}{\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)} \\ 2e_{\alpha\zeta} &= \frac{\gamma_{\alpha\zeta o}}{\left(1+\frac{\zeta}{R_\alpha}\right)} \\ 2e_{\beta\zeta} &= \frac{\gamma_{\beta\zeta o}}{\left(1+\frac{\zeta}{R_\beta}\right)} \end{aligned} \tag{3}$$

where

$$\begin{aligned} \epsilon_{\alpha o} &= \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{v}{AB}\frac{\partial A}{\partial \beta} + \frac{w}{R_\alpha} \\ \epsilon_{\beta o} &= \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{u}{AB}\frac{\partial B}{\partial \alpha} + \frac{w}{R_\beta} \\ \gamma_{\alpha\beta o} &= \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{v}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{u}{AB}\frac{\partial A}{\partial \beta} \\ \gamma_{\alpha\zeta o} &= f + \frac{1}{A}\frac{\partial w}{\partial \alpha} - \frac{u}{R_\alpha} \\ \gamma_{\beta\zeta o} &= g + \frac{1}{B}\frac{\partial w}{\partial \beta} - \frac{v}{R_\beta} \end{aligned} \tag{4}$$

$$K_\alpha = \frac{1}{A}\frac{\partial f}{\partial \alpha} + \frac{g}{AB}\frac{\partial A}{\partial \beta}$$

$$K_\beta = \frac{1}{B}\frac{\partial g}{\partial \beta} + \frac{f}{AB}\frac{\partial B}{\partial \alpha}$$

$$2K_{\alpha\beta} = \frac{1}{B}\frac{\partial f}{\partial \beta} - \frac{f}{AB}\frac{\partial A}{\partial \beta} + \frac{1}{A}\frac{\partial g}{\partial \alpha} - \frac{g}{AB}\frac{\partial B}{\partial \alpha} + \frac{1}{R_\alpha}\left(\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{v}{AB}\frac{\partial B}{\partial \alpha}\right) + \frac{1}{R_\beta}\left(\frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{u}{AB}\frac{\partial A}{\partial \beta}\right) \tag{4}$$

The equations of equilibrium in terms of the physical components of the stress resultants are

$$\frac{\partial}{\partial \alpha}(BN_{\alpha\alpha}) + \frac{\partial}{\partial \beta}(AN_{\beta\alpha}) + N_{\alpha\beta}\frac{\partial A}{\partial \beta} - N_{\beta\beta}\frac{\partial B}{\partial \alpha} + Q_{\alpha\zeta}\frac{AB}{R_\alpha} + AB\,\overline{Y}_\alpha = 0$$

$$\frac{\partial}{\partial \alpha}(BN_{\alpha\beta}) + \frac{\partial}{\partial \beta}(AN_{\beta\beta}) + N_{\beta\alpha}\frac{\partial B}{\partial \alpha} - N_{\alpha\alpha}\frac{\partial A}{\partial \beta} + Q_{\beta\zeta}\frac{AB}{R_\beta} + AB\,\overline{Y}_\beta = 0$$

$$\frac{\partial}{\partial \alpha}(BQ_{\alpha\zeta}) + \frac{\partial}{\partial \beta}(AQ_{\beta\zeta}) - AB\left(\frac{N_{\alpha\alpha}}{R_\alpha} + \frac{N_{\beta\beta}}{R_\beta}\right) + AB\,\overline{Y}_\zeta = 0 \tag{5}$$

$$\frac{\partial}{\partial \alpha}(BM_{\alpha\alpha}) + \frac{\partial}{\partial \beta}(AM_{\alpha\beta}) + M_{\alpha\beta}\frac{\partial A}{\partial \beta} - M_{\beta\beta}\frac{\partial B}{\partial \alpha} - Q_{\alpha\zeta}AB = 0$$

$$\frac{\partial}{\partial \alpha}(BM_{\alpha\beta}) + \frac{\partial}{\partial \beta}(AM_{\beta\beta}) + M_{\alpha\beta}\frac{\partial B}{\partial \alpha} - M_{\alpha\alpha}\frac{\partial A}{\partial \beta} - Q_{\beta\zeta}AB = 0$$

where

$$N_{\beta\alpha} = S_{\alpha\beta} + \frac{M_{\alpha\beta}}{R_\alpha}$$

$$N_{\alpha\beta} = S_{\alpha\beta} + \frac{M_{\alpha\beta}}{R_\beta} \tag{6}$$

The reference surface tractions are denoted by $\overline{Y}_\alpha$, $\overline{Y}_\beta$, and $\overline{Y}_\zeta$, and the stress resultants $N_{\alpha\alpha}$, $N_{\beta\beta}$, $S_{\alpha\beta}$, $M_{\alpha\alpha}$, $M_{\beta\beta}$, $M_{\alpha\beta}$, $Q_{\alpha\zeta}$ and $Q_{\beta\zeta}$ are defined by

$$N_{\alpha\alpha} = \int_{t-}^{t+} \sigma_{\alpha\alpha}\left(1 + \frac{\zeta}{R_\beta}\right) d\zeta$$

$$N_{\beta\beta} = \int_{t-}^{t+} \sigma_{\beta\beta}\left(1 + \frac{\zeta}{R_\alpha}\right) d\zeta \tag{7}$$

$$S_{\alpha\beta} = \int_{t-}^{t+} \sigma_{\alpha\beta} \, d\zeta$$

$$Q_{\alpha\zeta} = \int_{t-}^{t+} \sigma_{\alpha\zeta} \left(1 + \frac{\zeta}{R_\beta}\right) d\zeta$$

$$Q_{\beta\zeta} = \int_{t-}^{t+} \sigma_{\beta\zeta} \left(1 + \frac{\zeta}{R_\alpha}\right) d\zeta \qquad (7)$$

$$M_{\alpha\alpha} = \int_{t-}^{t+} \sigma_{\alpha\alpha} \, \zeta \left(1 + \frac{\zeta}{R_\beta}\right) d\zeta$$

$$M_{\beta\beta} = \int_{t-}^{t+} \sigma_{\beta\beta} \, \zeta \left(1 + \frac{\zeta}{R_\alpha}\right) d\zeta$$

$$M_{\alpha\beta} = \int_{t-}^{t+} \sigma_{\alpha\beta} \, \zeta \, d\zeta$$

On the curve C bounding the reference surface S, the boundary conditions are

$$\begin{aligned} N_\alpha &= \overline{N}_\alpha \quad \text{or} \quad u = \overline{u} \\ N_\beta &= \overline{N}_\beta \quad \text{or} \quad v = \overline{v} \\ Q &= \overline{Q} \quad \text{or} \quad w = \overline{w} \\ M_\alpha &= \overline{M}_\alpha \quad \text{or} \quad f = \overline{f} \\ M_\beta &= \overline{M}_\beta \quad \text{or} \quad g = \overline{g} \end{aligned} \qquad (8)$$

where the bar denotes a specified quantity and referring to Figure 2, the force resultants are defined by

$$\begin{aligned} N_\alpha &= N_{\alpha\alpha} \cos\theta + N_{\beta\alpha} \sin\theta \\ N_\beta &= N_{\alpha\beta} \cos\theta + N_{\beta\beta} \sin\theta \\ Q &= Q_{\alpha\zeta} \cos\theta + Q_{\beta\zeta} \sin\theta \\ M_\alpha &= M_{\alpha\alpha} \cos\theta + M_{\alpha\beta} \sin\theta \\ M_\beta &= M_{\alpha\beta} \cos\theta + M_{\beta\beta} \sin\theta \end{aligned} \qquad (9)$$

In Figure 2, n and s are coordinates tangent to the reference surface which are perpendicular and tangent to the edge of the shell, respectively. Assuming the shell material to be anisotropic, the stress-strain-temperature relation is given by

$$e_{ij} = B_{ijrs} \quad \sigma_{rs} + \alpha_{ij} \quad \Delta T \tag{10}$$

Here, B_{ijrs} are the Cartesian components of the elastic flexibility tensor, α_{ij} are the Cartesian components of the thermal expansion tensor and ΔT is the temperature change. The elastic flexibility tensor has the symmetries $B_{ijrs} = B_{jirs} = B_{ijsr} = B_{rsij}$.

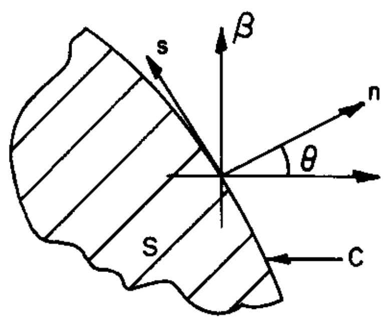

Figure 2

Assuming the shell to be in a state of plane stress, $\sigma_{\zeta\zeta} = 0$, results in

$$\begin{bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\beta\zeta} \\ \sigma_{\alpha\zeta} \\ \sigma_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} B_{1111} & B_{1122} & 2B_{1123} & 2B_{1113} & 2B_{1112} \\ B_{1122} & B_{2222} & 2B_{2223} & 2B_{2213} & 2B_{2212} \\ 2B_{1123} & 2B_{2223} & 4B_{2323} & 4B_{2313} & 4B_{2312} \\ 2B_{1113} & 2B_{2213} & 4B_{2313} & 4B_{1313} & 4B_{1312} \\ 2B_{1112} & 2B_{2212} & 4B_{2312} & 4B_{1312} & 4B_{1212} \end{bmatrix}^{-1} \left[\begin{bmatrix} e_{\alpha\alpha} \\ e_{\beta\beta} \\ 2e_{\beta\zeta} \\ 2e_{\alpha\zeta} \\ 2e_{\alpha\beta} \end{bmatrix} - \begin{bmatrix} \alpha_{\alpha\alpha} \\ \alpha_{\beta\beta} \\ 2\alpha_{\beta\zeta} \\ 2\alpha_{\alpha\zeta} \\ 2\alpha_{\alpha\beta} \end{bmatrix} \Delta T \right] \tag{11}$$

and carrying out the inversion gives

$$\begin{bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\beta\zeta} \\ \sigma_{\alpha\zeta} \\ \sigma_{\alpha\beta} \end{bmatrix} = \underset{\text{sym.}}{\left[C_{ij} \right]} \begin{bmatrix} e_{\alpha\alpha} \\ e_{\beta\beta} \\ 2e_{\beta\zeta} \\ 2e_{\alpha\zeta} \\ 2e_{\alpha\beta} \end{bmatrix} - \left[B_i \right] \Delta T \tag{12}$$

The stress resultant-strain-temperature relation for the shell is obtained by using the Definition 7. It is

$$\begin{bmatrix} N_{\alpha\alpha} \\ N_{\beta\beta} \\ Q_{\beta\zeta} \\ Q_{\alpha\zeta} \\ S_{\alpha\beta} \\ M_{\alpha\alpha} \\ M_{\beta\beta} \\ M_{\alpha\beta} \end{bmatrix} = \underset{\text{sym}}{\left[D_{ij} \right]} \begin{bmatrix} \epsilon_{\alpha o} \\ \epsilon_{\beta o} \\ \gamma_{\beta\zeta o} \\ \gamma_{\alpha\zeta o} \\ \gamma_{\alpha\beta o} \\ K_{\alpha} \\ K_{\beta} \\ 2K_{\alpha\beta} \end{bmatrix} - \left[M_i \right] \tag{13}$$

where

$$\begin{aligned}
D_{11} &= \int_{t-}^{t+} C_{11} \left(1 + \frac{\zeta}{R_\alpha}\right)^{-1} \left(1 + \frac{\zeta}{R_\beta}\right) d\zeta \\
D_{12} &= \int_{t-}^{t+} C_{12} \, d\zeta \\
D_{13} &= \int_{t-}^{t+} C_{13} \, d\zeta \\
D_{14} &= \int_{t-}^{t+} C_{14} \left(1 + \frac{\zeta}{R_\alpha}\right)^{-1} \left(1 + \frac{\zeta}{R_\beta}\right) d\zeta \\
D_{15} &= \int_{t-}^{t+} C_{15} \left(1 + \frac{\zeta}{R_\alpha}\right)^{-1} d\zeta \\
D_{16} &= \int_{t-}^{t+} C_{11} \, \zeta \left(1 + \frac{\zeta}{R_\alpha}\right)^{-1} \left(1 + \frac{\zeta}{R_\beta}\right) d\zeta \\
D_{17} &= \int_{t-}^{t+} C_{12} \, \zeta \, d\zeta \\
D_{18} &= \int_{t-}^{t+} C_{15} \, \zeta \left(1 + \frac{\zeta}{R_\alpha}\right)^{-1} d\zeta \\
D_{22} &= \int_{t^-}^{t^+} C_{22} \left(1 + \frac{\zeta}{R_\alpha}\right) \left(1 + \frac{\zeta}{R_\beta}\right)^{-1} d\zeta
\end{aligned} \tag{14}$$

$$D_{23} = \int_{t-}^{t+} C_{23} \left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{24} = \int_{t-}^{t+} C_{24}\, d\zeta$$

$$D_{25} = \int_{t-}^{t+} C_{25} \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{26} = D_{17}$$

$$D_{27} = \int_{t-}^{t+} C_{22}\,\zeta \left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{28} = \int_{t-}^{t+} C_{25}\,\zeta \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{33} = \int_{t-}^{t+} C_{33} \left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{34} = \int_{t-}^{t+} C_{34}\, d\zeta \qquad (14)$$

$$D_{35} = \int_{t-}^{t+} C_{35} \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{36} = \int_{t-}^{t+} C_{13}\,\zeta\, d\zeta$$

$$D_{37} = \int_{t-}^{t+} C_{23}\,\zeta \left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{38} = \int_{t-}^{t+} C_{35}\,\zeta \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{44} = \int_{t-}^{t+} C_{44} \left(1+\frac{\zeta}{R_\alpha}\right)^{-1}\left(1+\frac{\zeta}{R_\beta}\right) d\zeta$$

$$D_{45} = \int_{t-}^{t+} C_{45} \left(1+\frac{\zeta}{R_\alpha}\right)^{-1} d\zeta$$

$$D_{46} = \int_{t-}^{t+} C_{14}\,\zeta \left(1+\frac{\zeta}{R_\alpha}\right)^{-1}\left(1+\frac{\zeta}{R_\beta}\right) d\zeta$$

$$D_{47} = \int_{t-}^{t+} C_{24}\,\zeta\, d\zeta$$

$$D_{48} = \int_{t-}^{t+} C_{45}\, \zeta \left(1+\frac{\zeta}{R_\alpha}\right)^{-1} d\zeta$$

$$D_{55} = \int_{t-}^{t+} C_{55} \left(1+\frac{\zeta}{R_\alpha}\right)^{-1} \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{56} = D_{18}$$

$$D_{57} = D_{28}$$

$$D_{58} = \int_{t-}^{t+} C_{55}\, \zeta \left(1+\frac{\zeta}{R_\alpha}\right)^{-1} \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{66} = \int_{t-}^{t+} C_{11}\, \zeta^2 \left(1+\frac{\zeta}{R_\alpha}\right)^{-1} \left(1+\frac{\zeta}{R_\beta}\right) d\zeta$$

$$D_{67} = \int_{t-}^{t+} C_{12}\, \zeta^2 d\zeta$$

$$D_{68} = \int_{t-}^{t+} C_{15}\, \zeta^2 \left(1+\frac{\zeta}{R_\alpha}\right)^{-1} d\zeta \qquad (14)$$

$$D_{77} = \int_{t-}^{t+} C_{22}\, \zeta^2 \left(1+\frac{\zeta}{R_\alpha}\right) \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{78} = \int_{t-}^{t+} C_{25}\, \zeta^2 \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$D_{88} = \int_{t-}^{t+} C_{55}\, \zeta^2 \left(1+\frac{\zeta}{R_\alpha}\right)^{-1} \left(1+\frac{\zeta}{R_\beta}\right)^{-1} d\zeta$$

$$M_1 = \int_{t-}^{t+} B_1 \left(1+\frac{\zeta}{R_\beta}\right) \Delta T(\zeta)\, d\zeta$$

$$M_2 = \int_{t-}^{t+} B_2 \left(1+\frac{\zeta}{R_\alpha}\right) \Delta T(\zeta)\, d\zeta$$

$$M_3 = \int_{t-}^{t+} B_3 \left(1+\frac{\zeta}{R_\alpha}\right) \Delta T(\zeta)\, d\zeta$$

$$M_4 = \int_{t-}^{t+} B_4 \left(1+\frac{\zeta}{R_\beta}\right) \Delta T(\zeta)\, d\zeta$$

$$M_5 = \int_{t-}^{t+} B_5\, \Delta T(\zeta)\, d\zeta$$

$$
\begin{aligned}
M_6 &= \int_{t-}^{t+} B_1\, \zeta \left(1 + \frac{\zeta}{R_\beta}\right) \Delta T(\zeta)\, d\zeta \\
M_7 &= \int_{t-}^{t+} B_2\, \zeta \left(1 + \frac{\zeta}{R_\alpha}\right) \Delta T(\zeta)\, d\zeta \\
M_8 &= \int_{t-}^{t+} B_5\, \zeta\, \Delta T(\zeta)\, d\zeta
\end{aligned} \tag{14}
$$

In using these equations for dynamic response calculations, it is customary to scale the constants D_{3i} and D_{4i} with a shear coefficient, (Reference 42). For the static analysis considered here a shear coefficient equal to one is used.

The equations of Equilibrium 5, the stress resultant-strain-temperature Equations 13 and the strain-displacement Equations 4 combined, result in a set of five second order partial differential equations in the five unknowns u, v, w, f and g to be satisfied in the region S subject to the Boundary Conditions 8. These equations form a self adjoint system and are equivalent to the following minimum potential energy principle. The functional π (u, v, w, f, g) given by

$$
\pi = \iint_S \left(\frac{1}{2} \begin{bmatrix} \epsilon_{\alpha o} \\ \epsilon_{\beta o} \\ \gamma_{\beta\zeta o} \\ \gamma_{\alpha\zeta o} \\ \gamma_{\alpha\beta o} \\ \kappa_\alpha \\ \kappa_\beta \\ 2\kappa_{\alpha\beta} \end{bmatrix}^T \left[D_{ij}\right] \begin{bmatrix} \epsilon_{\alpha o} \\ \epsilon_{\beta o} \\ \gamma_{\beta\zeta o} \\ \gamma_{\alpha\zeta o} \\ \gamma_{\alpha\beta o} \\ \kappa_\alpha \\ \kappa_\beta \\ 2\kappa_{\alpha\beta} \end{bmatrix} - \begin{bmatrix} \epsilon_{\alpha o} \\ \epsilon_{\beta o} \\ \gamma_{\beta\zeta o} \\ \gamma_{\alpha\zeta o} \\ \gamma_{\alpha\beta o} \\ \kappa_\alpha \\ \kappa_\beta \\ 2\kappa_{\alpha\beta} \end{bmatrix}^T \left[M_i\right] - \begin{bmatrix} u \\ v \\ w \end{bmatrix}^T \begin{bmatrix} \bar{Y}_\alpha \\ \bar{Y}_\beta \\ \bar{Y}_\zeta \end{bmatrix} \right) AB\, d\alpha\, d\beta
$$
$$
- \int_{C_2} \begin{bmatrix} u \\ v \\ w \\ f \\ g \end{bmatrix}^T \begin{bmatrix} \bar{N}_\alpha \\ \bar{N}_\beta \\ Q \\ \bar{M}_\alpha \\ \bar{M}_\beta \end{bmatrix} ds \tag{15}
$$

is to be minimized on those functions which satisfy the displacement boundary conditions on $C_1 = C - C_2$ and which are continuous and have piecewise continuous derivatives. Here the strains are interpreted as functions of u, v, w, f and g.

In what follows, the specialization is made for shells whose reference surfaces are portions of an axisymmetric surface. In Figure 3 the reference surface coordinates are $\alpha = \theta$ circumferentially and $\beta = s$, a length coordinate, meridionally.

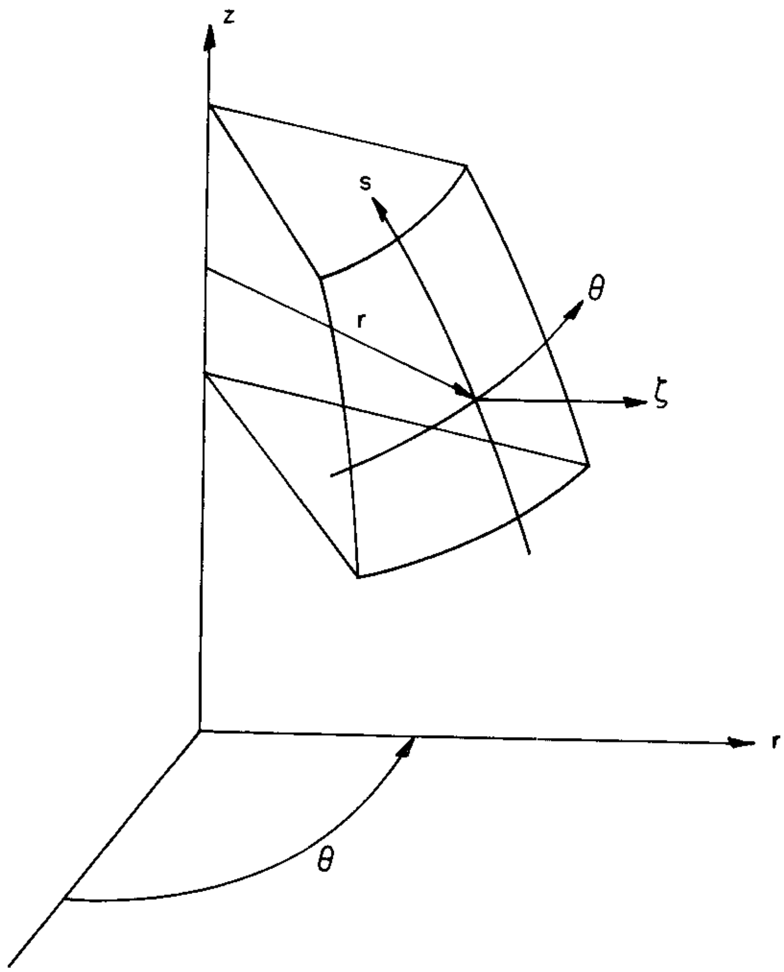

Figure 3

The Lamé coefficients in this case become

$$A = r(s), \quad B = 1 \tag{16}$$

With the r and z coordinates of a meridional line in terms of s, the curvatures are given by the expressions

$$\frac{1}{R_\theta} = \frac{1}{r}\frac{dz}{ds}$$
$$\frac{1}{R_s} = \frac{d^2z}{ds^2}\frac{dr}{ds} - \frac{d^2r}{ds^2}\frac{dz}{ds} \qquad (17)$$

SECTION III

FINITE ELEMENT METHOD

The finite element method is similar to the Ritz method and is applied to the variational statement of the problem. The object in the finite element method is to select a simple geometric shape, called an element, and to choose how the unknown functions are to vary in these regions. For example, triangles could be chosen as elements and the unknowns varied linearly in each element. The elements are "Fitted together" to form what is known as polyhedral approximations to the unknown functions of the variational principle. The polyhedral functions are characterized by unknown mesh point values and parameters yet to be determined. For a given problem the polyhedral approximations are substituted into the variational principle and the unknown mesh point values and parameters are determined by extremizing the functional. A polyhedral approximation must satisfy certain conditions before convergence to the actual solution can be obtained by reducing the element size. They must be admissible functions for the variational principle being considered, and they must be complete in the function space on which the variational principle is defined. For a complete discussion of the convergence properties of the finite element method, either Melosh, McLay, Key, Johnson and McLay, McLay, or Tong and Pian may be consulted respectively in References 43, 44, 45, 46, 47 and 48.

For the minimum potential energy principle considered here, admissible polyhedral approximations must be continuous and have piecewise continuous first derivatives, and they must be complete with respect to first derivatives. In addition, practical considerations require the presence of rigid body degrees of freedom, element by element, for coarse meshes. In what follows a doubly curved quadrilateral element is developed. It satisfies the above continuity requirements but has rigid body modes only for vanishing curvature or vanishing element size.

Referring to Figure 4, the element is a quadrilateral in the reference surface of the shell.

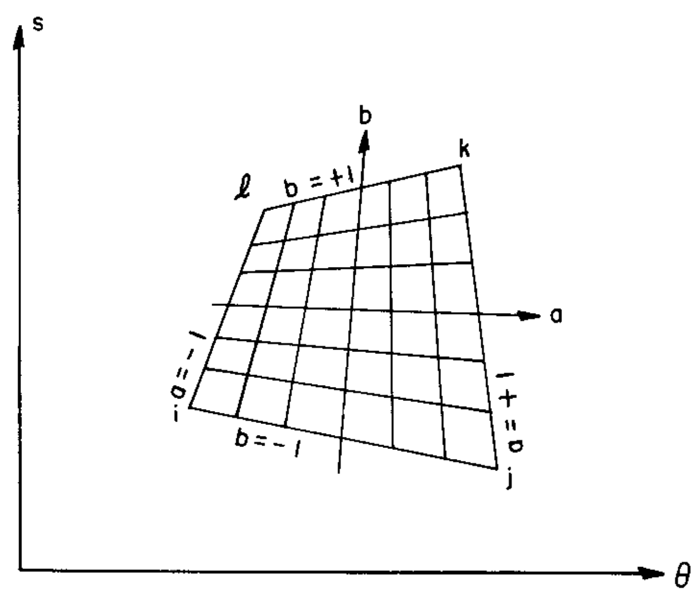

Figure 4

Following Irons (Reference 49) and Ergatoudis, Irons and Zienkiewicz (Reference 50) an oblique coordinate system a, b is introduced in the quadrilateral elements defined by the element geometry. The resulting coordinate transformation is

$$\theta(a,b) = \theta_i (1-a)(1-b)/4 + \theta_j (1+a)(1-b)/4 + \theta_k (1+a)(1+b)/4 + \theta_\ell (1-a)(1+b)/4$$
$$s(a,b) = s_i (1-a)(1-b)/4 + s_j (1+a)(1-b)/4 + s_k (1+a)(1+b)/4 + s_\ell (1-a)(1+b)/4 \qquad (18)$$

The axisymmetric reference surface for the element is given by

$$r = \hat{r}(a,b)$$
$$\theta = \hat{\theta}(a,b) \qquad (19)$$
$$z = \hat{z}(a,b)$$

The function $\hat{\theta}$ is treated exactly and is given by Equation 18. The functions $\hat{r}$ and $\hat{z}$ are treated approximately and replaced with bicubic polynomials in a and b. The result is

$$\begin{aligned} q(a,b) = {} & h_1(a)h_1(b)q_i + h_2(a)h_1(b)q_j \\ & + h_2(a)h_2(b)q_k + h_1(a)h_2(b)q_\ell \\ & + h_3(a)h_1(b)\left(\frac{\partial q}{\partial a}\right)_i + h_4(a)h_1(b)\left(\frac{\partial q}{\partial a}\right)_j \\ & + h_4(a)h_2(b)\left(\frac{\partial q}{\partial a}\right)_k + h_3(a)h_2(b)\left(\frac{\partial q}{\partial a}\right)_\ell \\ & + h_1(a)h_3(b)\left(\frac{\partial q}{\partial b}\right)_i + h_2(a)h_3(b)\left(\frac{\partial q}{\partial b}\right)_j \\ & + h_2(a)h_4(b)\left(\frac{\partial q}{\partial b}\right)_k + h_1(a)h_4(b)\left(\frac{\partial q}{\partial b}\right)_\ell \\ & + h_3(a)h_3(b)\left(\frac{\partial^2 q}{\partial a\,\partial b}\right)_i + h_4(a)h_3(b)\left(\frac{\partial^2 q}{\partial a\,\partial b}\right)_j \\ & + h_4(a)h_4(b)\left(\frac{\partial^2 q}{\partial a\,\partial b}\right)_k + h_3(a)h_4(b)\left(\frac{\partial^2 q}{\partial a\,\partial b}\right)_\ell \end{aligned} \qquad (20)$$

where h_1, h_2, h_3, and h_4 are second order Hermite interpolation functions given for both a and b by

$$h_1(\eta) = \frac{1}{4}(\eta^3 - 3\eta + 2)$$
$$h_2(\eta) = -\frac{1}{4}(\eta^3 - 3\eta - 2)$$
$$h_3(\eta) = \frac{1}{4}(\eta^3 - \eta^2 - \eta + 1) \qquad (21)$$
$$h_4(\eta) = \frac{1}{4}(\eta^3 + \eta^2 - \eta - 1)$$

and

$$\left(\frac{\partial q}{\partial a}\right)_m = \left(\frac{\partial s}{\partial a}\right)_m \left(\frac{\partial q}{\partial s}\right)_m , \quad m = i, j, k, \ell$$

$$\left(\frac{\partial q}{\partial b}\right)_m = \left(\frac{\partial s}{\partial b}\right)_m \left(\frac{\partial q}{\partial s}\right)_m , \quad m = i, j, k, \ell \tag{22}$$

$$\left(\frac{\partial^2 q}{\partial a \partial b}\right)_m = \left(\frac{\partial^2 s}{\partial a \partial b}\right)_m \left(\frac{\partial q}{\partial s}\right)_m + \left(\frac{\partial s}{\partial a}\right)_m \left(\frac{\partial s}{\partial b}\right)_m \left(\frac{\partial^2 q}{\partial s^2}\right)_m , \quad m = i, j, k, \ell$$

The function q represents either $\hat{r}$ or $\hat{z}$. For the r coordinate the derivatives in s are obtained from

$$\left(\frac{\partial r}{\partial s}\right)_m = -\operatorname{Sin} \phi_m \; ; \quad m = i, j, k, \ell \tag{23}$$

and for the z coordinate the derivatives in s are obtained from

$$\left(\frac{\partial z}{\partial s}\right)_m = \operatorname{Cos} \phi_m , \quad m = i, j, k, \ell \tag{24}$$

For the second derivatives in s, a value is used based on the change in ϕ from mesh point to mesh point. The derivatives needed to calculate the Curvatures 17 are obtained from the bicubic Expression 20. This treatment of the reference surface is suitable for arbitrary surfaces, however, the Expressions 22 will then contain additional terms.

These steps provide the needed flexibility in element geometry and result in an approximate reference surface from which satisfactory curvatures are calculated.

ELEMENT KB1

In the a, b coordinate system the unknown functions u, v, f and g are taken to vary bilinearly, defined by their values at the mesh points,

$$\begin{aligned}
u &= u_i(1-a)(1-b)/4 + u_j(1+a)(1-b)/4 + u_k(1+a)(1+b)/4 + u_\ell(1-a)(1+b)/4 \\
v &= v_i(1-a)(1-b)/4 + v_j(1+a)(1-b)/4 + v_k(1+a)(1+b)/4 + v_\ell(1-a)(1+b)/4 \\
f &= f_i(1-a)(1-b)/4 + f_j(1+a)(1-b)/4 + f_k(1+a)(1+b)/4 + f_\ell(1-a)(1+b)/4 \\
g &= g_i(1-a)(1-b)/4 + g_j(1+a)(1-b)/4 + g_k(1+a)(1+b)/4 + g_\ell(1-a)(1+b)/4
\end{aligned} \tag{25}$$

and w is taken as a bicubic

$$
\begin{aligned}
w = \; & h_1(a)h_1(b)w_i + h_2(a)h_1(b)w_j \\
& + h_2(a)h_2(b)w_k + h_1(a)h_2(b)w_\ell \\
& + h_3(a)h_1(b)\left(\frac{\partial w}{\partial a}\right)_i + h_4(a)h_1(b)\left(\frac{\partial w}{\partial a}\right)_j \\
& + h_4(a)h_2(b)\left(\frac{\partial w}{\partial a}\right)_k + h_3(a)h_2(b)\left(\frac{\partial w}{\partial a}\right)_\ell \\
& + h_1(a)h_3(b)\left(\frac{\partial w}{\partial b}\right)_i + h_2(a)h_3(b)\left(\frac{\partial w}{\partial b}\right)_j \\
& + h_2(a)h_4(b)\left(\frac{\partial w}{\partial b}\right)_k + h_1(a)h_4(b)\left(\frac{\partial w}{\partial b}\right)_\ell \\
& + h_3(a)h_3(b)\left(\frac{\partial^2 w}{\partial a \partial b}\right)_i + h_4(a)h_3(b)\left(\frac{\partial^2 w}{\partial a \partial b}\right)_j \\
& + h_4(a)h_4(b)\left(\frac{\partial^2 w}{\partial a \partial b}\right)_k + h_3(a)h_4(b)\left(\frac{\partial^2 w}{\partial a \partial b}\right)_\ell
\end{aligned}
\tag{26}
$$

where h_1, h_2, h_3, and h_4 are given by Equations 21. The polyhedral functions defined by the mesh point values of u, v, w, f, g, $\frac{\partial w}{\partial a}$, $\frac{\partial w}{\partial b}$, and $\frac{\partial^2 w}{\partial a \partial b}$ are continuous and have piecewise continuous derivatives and are complete with respect to first derivatives, Key (Reference 45). The derivation of the element stiffness matrix and load vector proceeds along the usual lines, (Reference 43).

The second derivatives in w are condensed out of the stiffness matrix before merger. They are retained as independent element deformation parameters since they do not contribute to interelement continuity and are difficult to interpret globally. The first derivatives in w are transformed to global derivatives before the elements are merged.

The stresses and stress resultants are calculated by evaluating the strain-displacement Relations 4 at the center of the element and using Equations 3 and 12 for the stresses and Equations 13 for the stress resultants.

In order to make the required integrations in Equation 15 tractable, additional approximations must be made. The integrations themselves are carried out numerically with a 25-point Gaussian quadrature in the a, b element coordinate system (Reference 49). Based on mesh point values various quantities in the interior of an element are obtained by interpolation. The result is

$$t^{+}(a,b) = t_i^{+}(1-a)(1-b)/4 + \cdots + t_\ell^{+}(1-a)(1+b)/4$$
$$t^{-}(a,b) = t_i^{-}(1-a)(1-b)/4 + \cdots + t_\ell^{-}(1-a)(1+b)/4 \tag{27}$$
$$\Delta T(a,b,\zeta) = \Delta T_i(\zeta)(1-a)(1-b)/4 + \cdots + \Delta T_\ell(\zeta)(1-a)(1+b)/4$$

For each quadrilateral element the meridional coordinate s is calculated from the nodal point r, z coordinates and the angle between the outward normal and the positive r-axis at each node. Referring to Figure 5, the arc length between two nodal points of the same element is approximated by a circular arc between them. Thus, the meridional distance between m and n is given by

$$s_n - s_m = \ell/(\sin\bar{\phi}/\bar{\phi}) \tag{28}$$

where

$$\ell = \left[(r_n - r_m)^2 + (z_n - z_m)^2\right]^{1/2}$$
$$\sin\bar{\phi}/\bar{\phi} \approx 1 - \frac{\bar{\phi}^2}{3!} + \frac{\bar{\phi}^4}{5!} - \frac{\bar{\phi}^6}{7!} + \frac{\bar{\phi}^8}{9!} \tag{29}$$
$$\bar{\phi} = \frac{1}{2}(\phi_n - \phi_m)$$

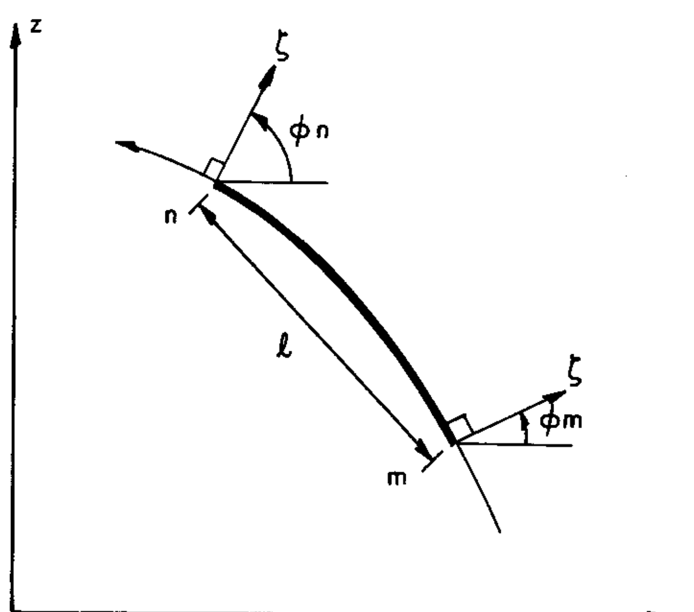

Figure 5

The integrals in Equations 14 are not easily evaluated. The following approximations to the integrands are made:

$$\left(1+\frac{\zeta}{R_\alpha}\right)^{-1} \approx 1-\frac{\zeta}{R_\alpha}+\left(\frac{\zeta}{R_\alpha}\right)^2$$

$$\zeta\left(1+\frac{\zeta}{R_\alpha}\right)^{-1} \approx \zeta-\frac{\zeta^2}{R_\alpha}$$

$$\zeta^2\left(1+\frac{\zeta}{R_\alpha}\right)^{-1} \approx \zeta^2$$

$$\left(1+\frac{\zeta}{R_\alpha}\right)^{-1}\left(1+\frac{\zeta}{R_\beta}\right) \approx 1-\frac{\zeta}{R_\alpha}+\frac{\zeta}{R_\beta}-\frac{\zeta^2}{R_\alpha R_\beta}+\left(\frac{\zeta}{R_\alpha}\right)^2 \tag{30}$$

$$\zeta\left(1+\frac{\zeta}{R_\alpha}\right)^{-1}\left(1+\frac{\zeta}{R_\beta}\right) \approx \zeta-\frac{\zeta^2}{R_\alpha}+\frac{\zeta^2}{R_\beta}$$

$$\zeta^2\left(1+\frac{\zeta}{R_\alpha}\right)^{-1}\left(1+\frac{\zeta}{R_\beta}\right) \approx \zeta^2$$

$$\left(1+\frac{\zeta}{R_\alpha}\right)^{-1}\left(1+\frac{\zeta}{R_\beta}\right)^{-1} \approx 1-\frac{\zeta}{R_\alpha}-\frac{\zeta}{R_\beta}+\left(\frac{\zeta}{R_\alpha}\right)^2+\frac{\zeta^2}{R_\alpha R_\beta}+\left(\frac{\zeta}{R_\beta}\right)^2$$

$$\zeta\left(1+\frac{\zeta}{R_\alpha}\right)^{-1}\left(1+\frac{\zeta}{R_\beta}\right)^{-1} \approx \zeta-\frac{\zeta^2}{R_\alpha}-\frac{\zeta^2}{R_\beta}$$

$$\zeta^2\left(1+\frac{\zeta}{R_\alpha}\right)^{-1}\left(1+\frac{\zeta}{R_\beta}\right)^{-1} \approx \zeta^2$$

The remaining approximations are obtained by interchanging R_α and R_β.

For axisymmetric shells, there have been numerous approaches taken in approximating the meridional arc length, the curvatures, and the reference surface, (References 10, 11, and 15). They all bear similarities to one another and to the treatment given here, since it is the same quantities which are being sought, based on a minimum of given information about the shell.

SECTION IV

EXAMPLE SOLUTIONS

To carry out the analysis, a computer program, SLADE I, has been written, Key and Beisinger (Reference 51). SLADE is designed specifically for shells whose reference surfaces are portions of axisymmetric surfaces. The program allows up to five separate layers and up to five separate elastic anisotropic materials with temperature dependent properties. The program handles variable thickness shells and allows thickness discontinuities along element boundaries. The program provides for both normal and tangential surface loads and for temperature changes through the thickness as well as varying over the reference surface. Several problems have been solved to evaluate the finite element introduced.

CURVATURE STUDY

The curvatures that result from the approximated reference surface are one of the important parts of the analysis that must be examined. This can be done by considering a spherical shell. In Figure 6 is the maximum meridional curvature error occurring as a function of the number of finite elements used along the meridian of the sphere. It can be seen that with eight elements spread over a 90-degree segment of a meridian, the curvature calculations begin to depend more on the accuracy of the input data than on the approximations to the reference surface. The input information for this study is accurate to five digits. The circumferential curvature errors are an order of magnitude smaller. A more difficult situation is encountered with an arbitrary element on a parabolic shell. Figure 7 shows such an element. For this element the outward normal turns through an angle of 9 degrees in the meridional direction and through an angle of ten degrees in the circumferential direction. The maximum error in the meridional curvature is 5.5% and the maximum error in the circumferential curvature is 0.075%.

PRESSURIZED SPHERE

A relatively simple problem requiring a curved element and involving a very nearly degenerate geometry is a sphere under internal pressure. Using the same geometry and mesh as in the curvature study, the errors in the radial displacement and the in-plane stress resultant are shown in Figures 8 and 9. The element at the pole is very nearly degenerate. The two nodal points at the pole have radial positions of $r = 10^{-4}$ inches. This situation does not present any difficulties for this problem.

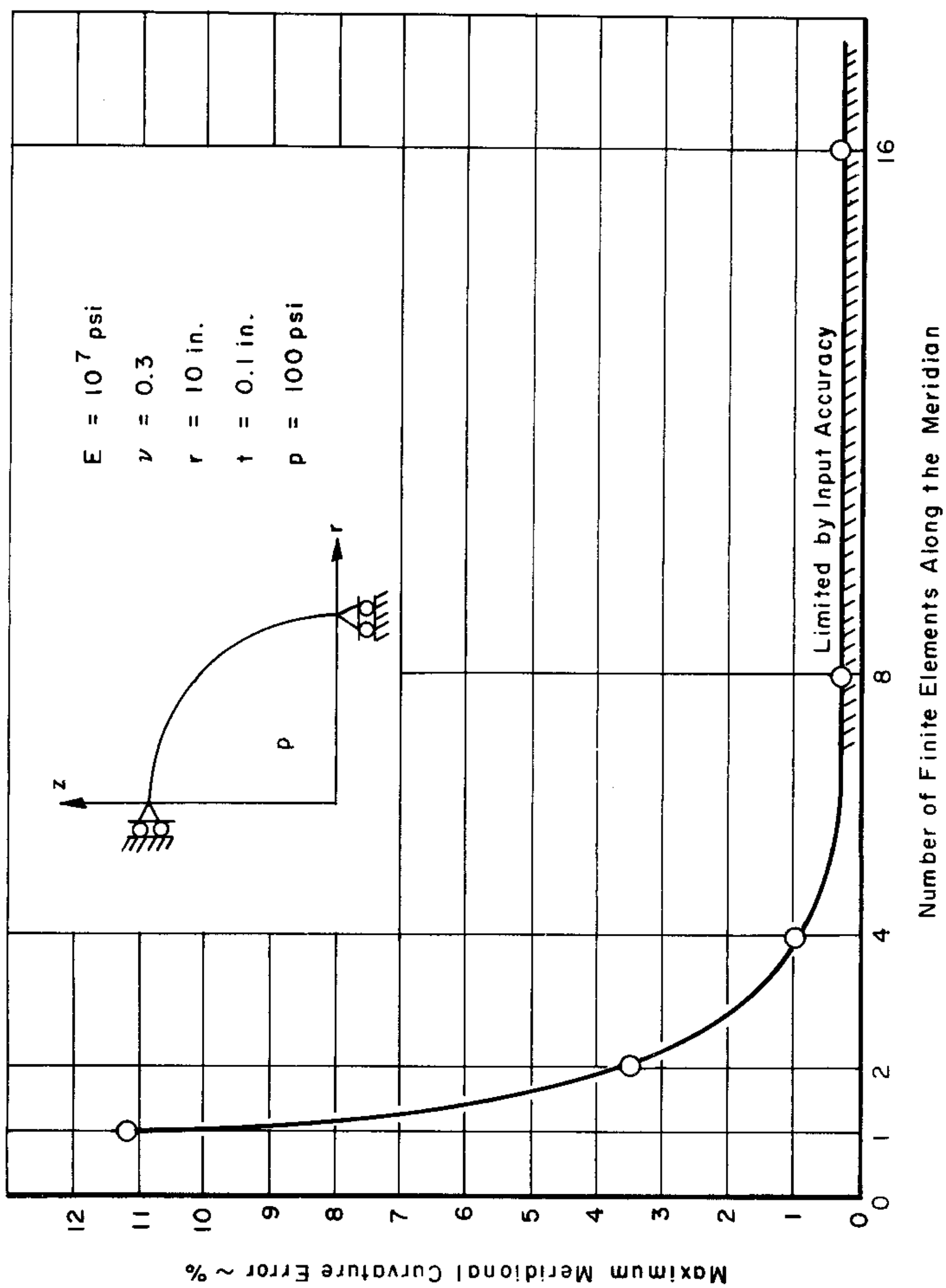

Figure 6. Curvature Behavior With Decreasing Element Size

Nodal Point Coordinates

	r in.	θ degrees	z in.	φ degrees
i	4.0	5.0	3.6	17.3
j	3.5	10.0	5.1	19.6
k	2.5	5.0	7.5	26.6
l	3.0	0.0	6.4	22.6

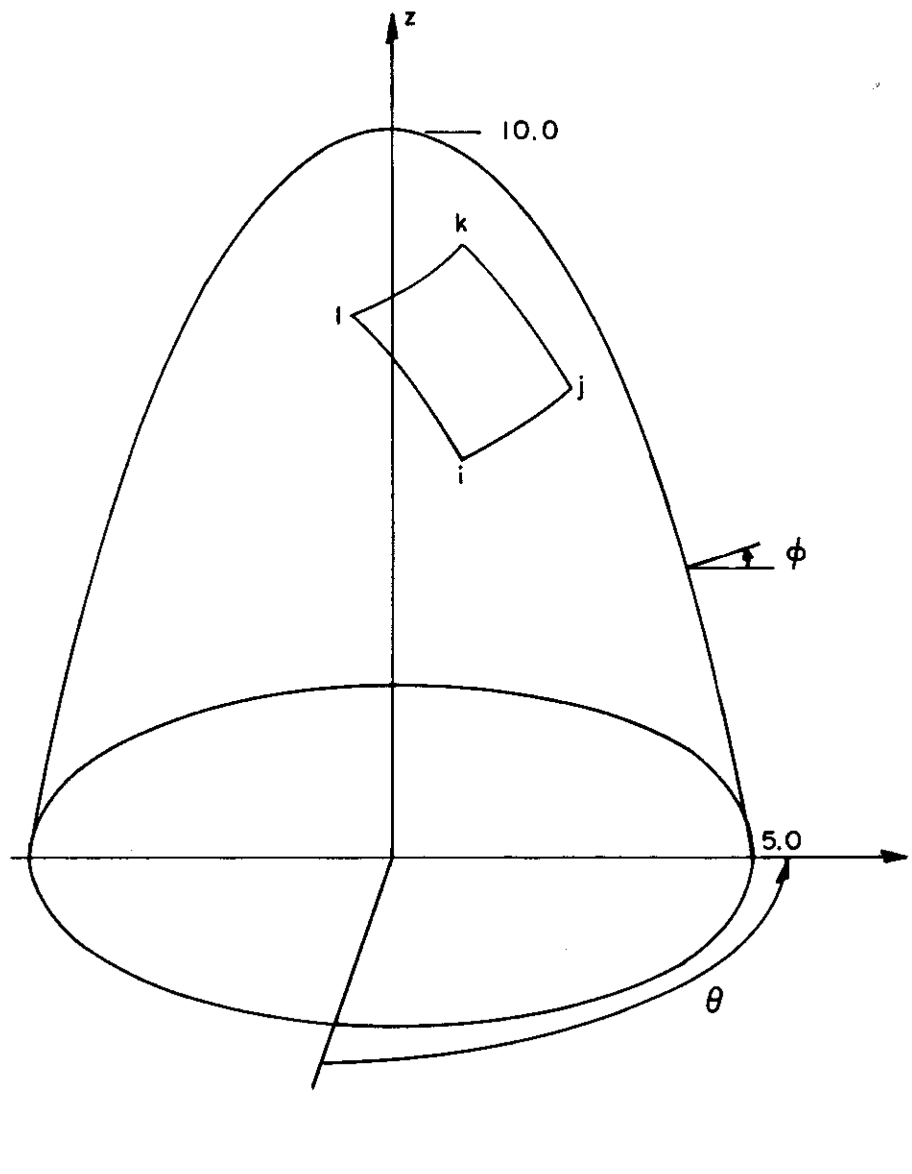

Figure 7. Parabolic Element

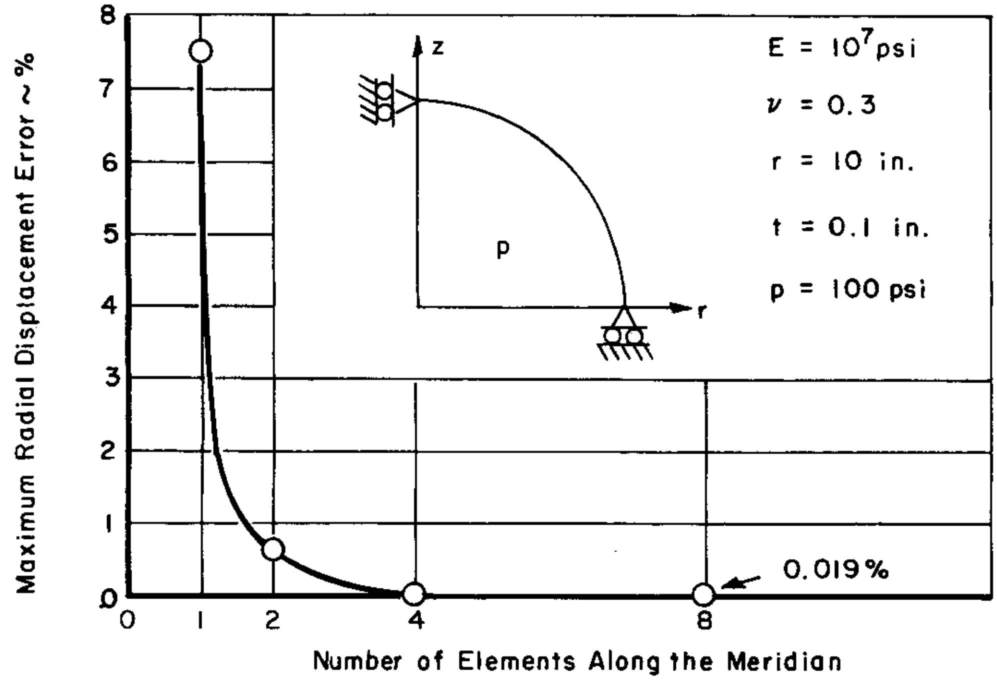

Figure 8. Displacement Errors For Decreasing Element Size

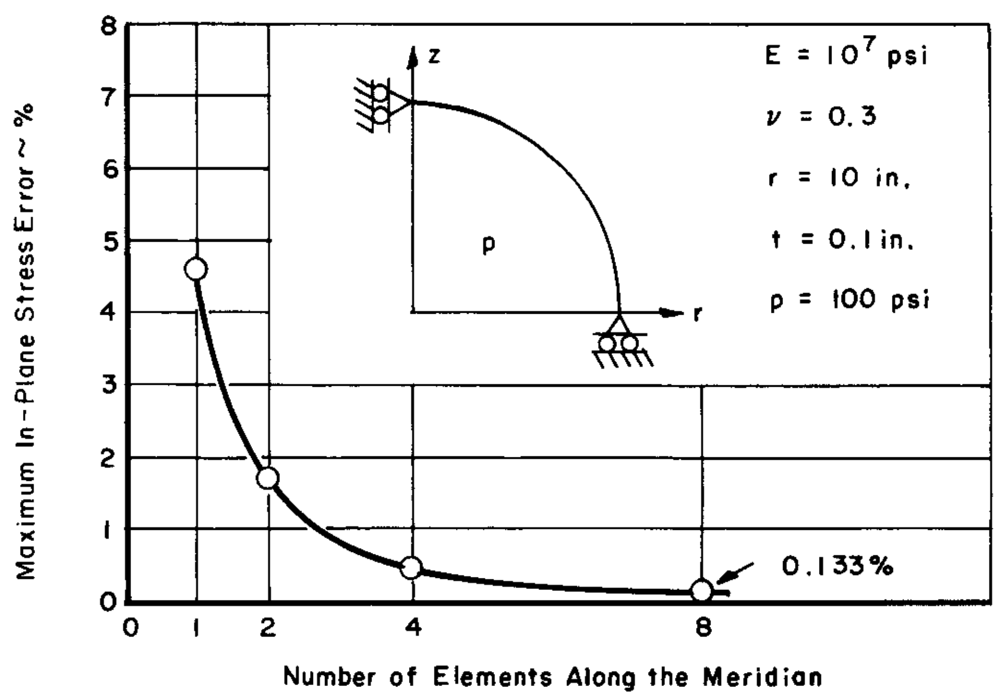

Figure 9. Stress Errors For Decreasing Element Size

FLAT PLATE ELEMENT

Contained in this analysis as a special case is an arbitrary quadrilateral flat plate element. As a comparison with existing flat plate elements, a portion of the very thorough study of Clough and Tocher (Reference 52) was repeated. Using the symmetry of the problem, a quadrant of a simply supported uniformly loaded square plate was analyzed. Shown in Figure 10 is the convergence behavior of the KB1 element compared with the elements examined by Clough and Tocher. The ACM element, Adini-Clough-Melosh, is a rectangular element based on a 12-term polynomial. The M element, Melosh, is a rectangular element based on physical reasoning. The P element, Papenfuss, is a rectangular element based on an incomplete bicubic polynomial. The HCT element, Hsieh-Clough-Tocher, is a triangular element based on three subtriangles with preferred polynomial expansions leading to a continuous displacement w with continuous derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. The complete study and references to these elements are contained in Reference 52. For this flat geometry all of the rigid body modes are present in the KB1 element.

CYLINDER WITH A CUTOUT

Lekkerkerker (Reference 53) has analyzed an infinite cylinder under axial load with a circular cutout. The problem has two planes of symmetry which are utilized in the analysis, see Figure 11. The meshes used are shown in Figures 12, 13, 14, and 15. These are irregular meshes on the surface of the cylinder with several nearly degenerate elements. These are located in relatively quiet regions of the problem and are not recommended for use where the answers are of interest. The results of these analyses are shown in Table I along with the results of Lekkerkerker (Reference 53). It should be noted that the plot programs connect the nodal points with straight lines and do not reflect the fact that the mesh is on the reference surface of the cylinder.

ANISOTROPIC CYLINDER

Gulati and Essenburg (Reference 54) examined the problem of an anisotropic cylindrical shell clamped at one end and displaced axially at the other end. They present a closed form solution to this problem for transverse shear strains and anisotropic elastic constants. Using the elastic constants presented by them, the problem was analyzed using the KB1 element. Figure 16 shows the results for the circumferential displacement. A uniform grid of 10 elements along the cylinder and 15 elements around the cylinder is used.

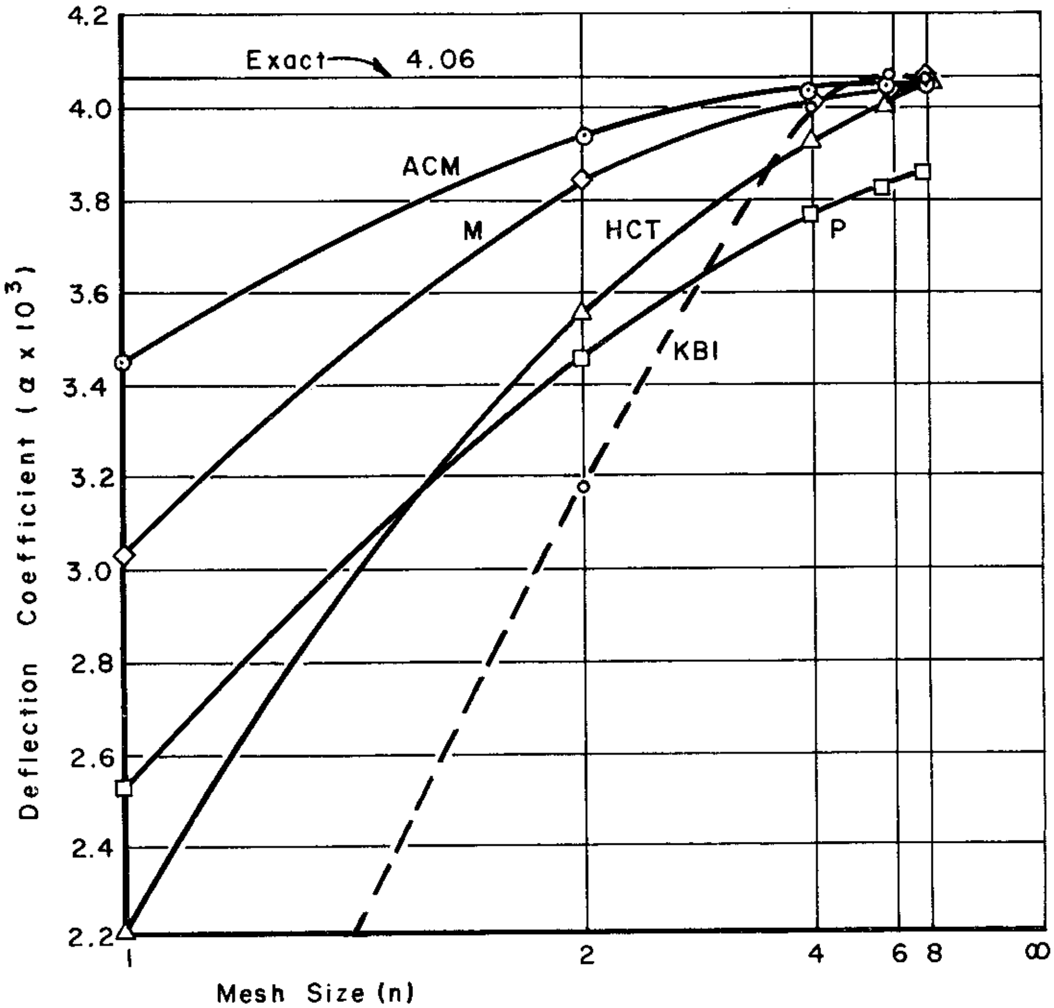

Figure 10. Deflection Coefficient Behavior With Decreasing Element Size,

$$\alpha = \frac{w_0 D}{qa^4}$$

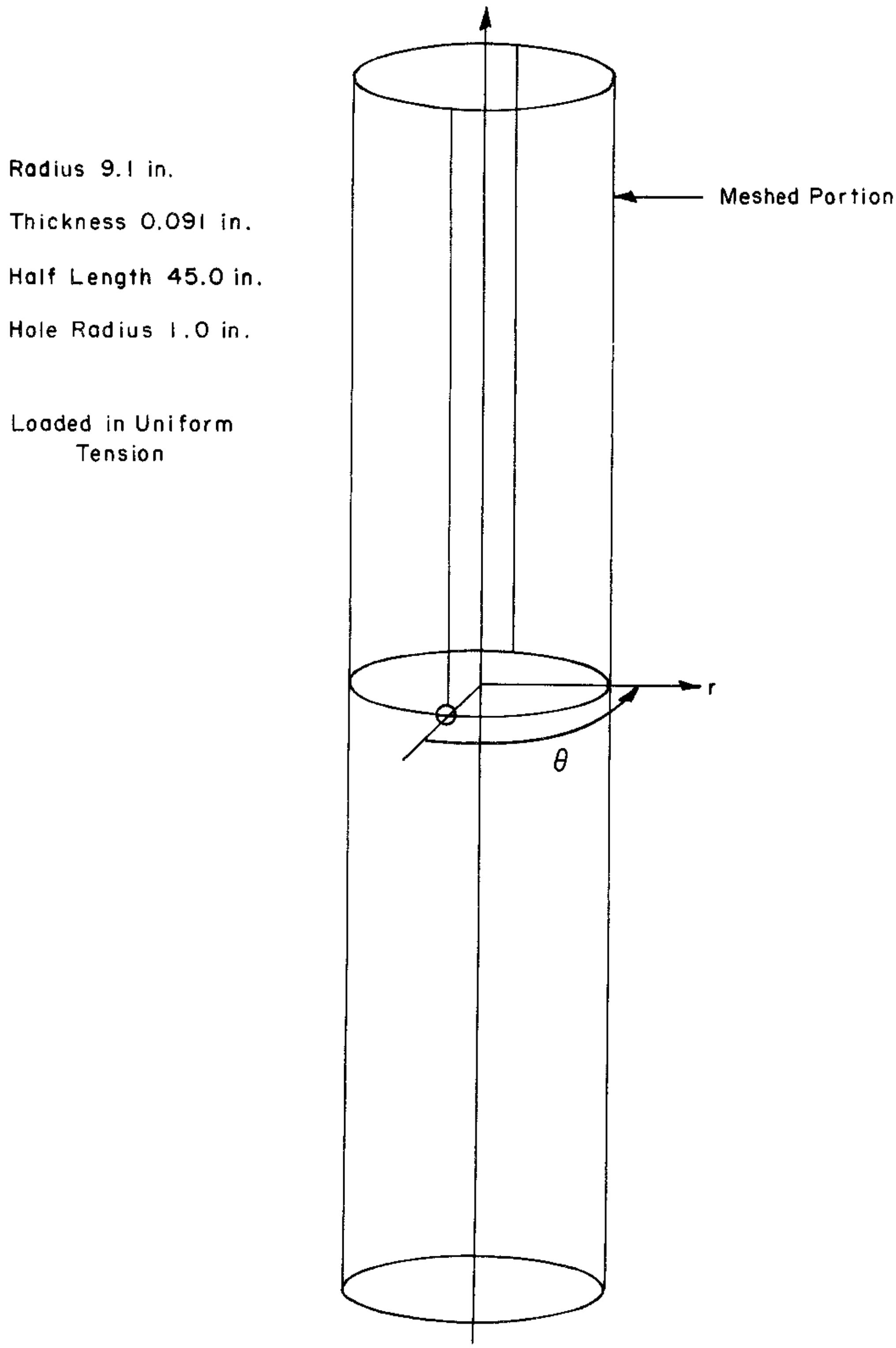

Figure 11. Cylinder With a Circular Cutout Loaded in Tension

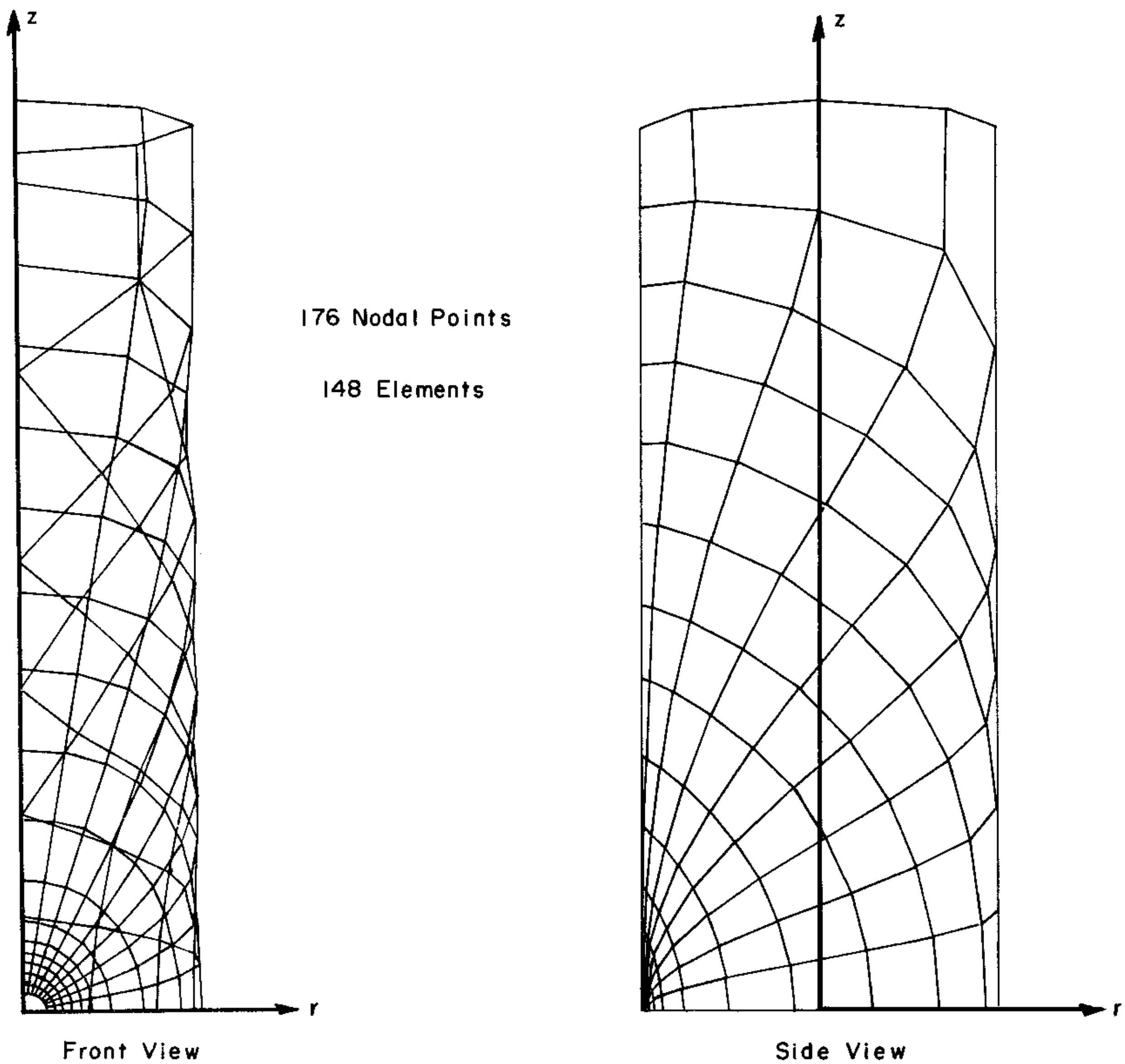

Figure 12. Coarse Mesh Layout

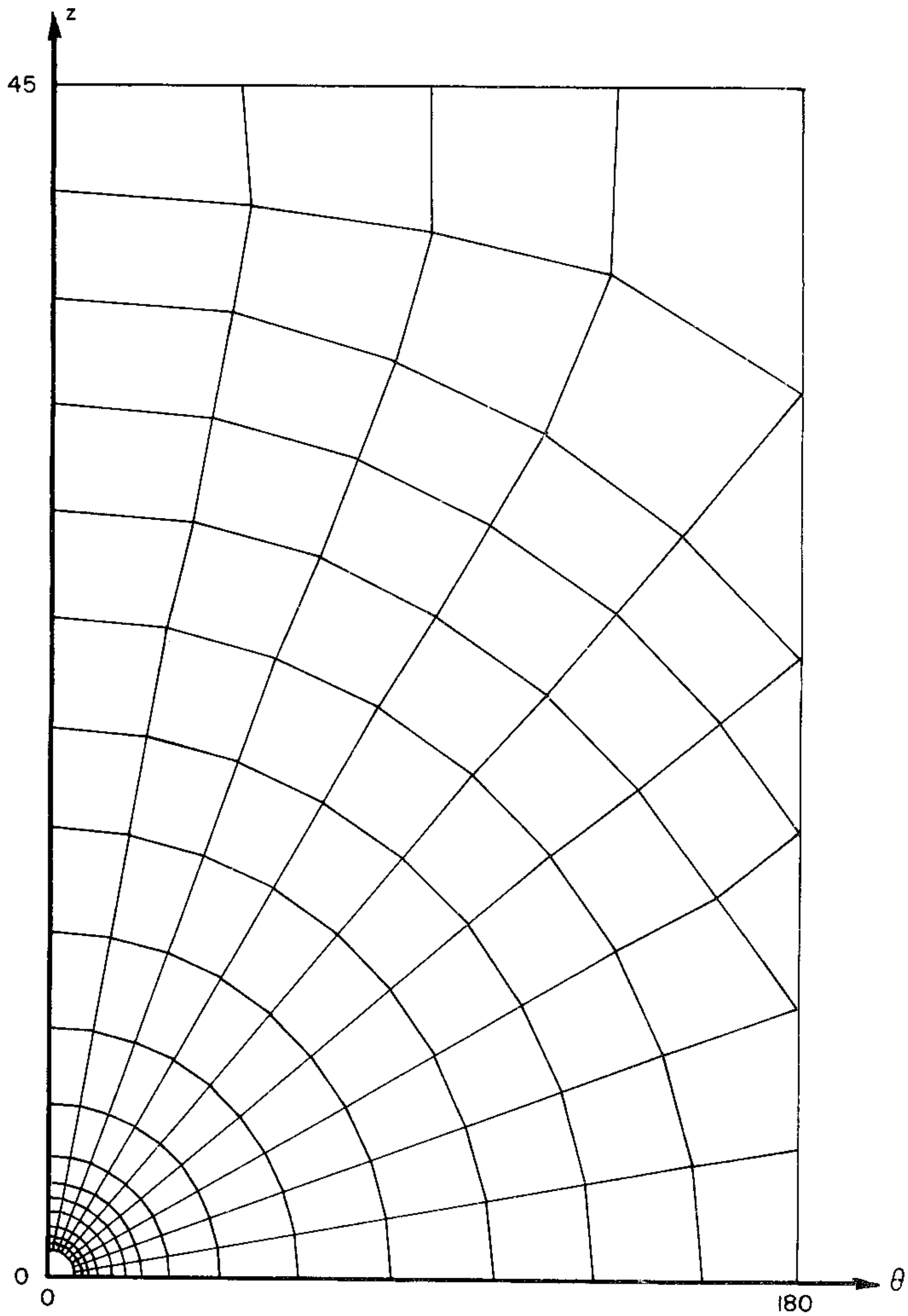

Figure 13. Developed Coarse Mesh

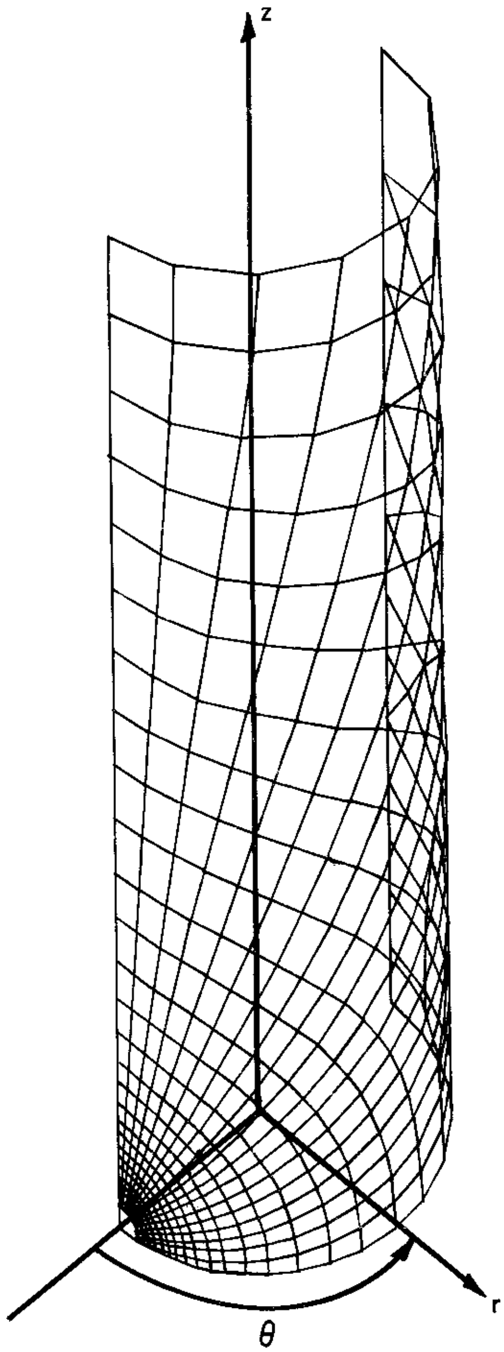

Figure 14. Fine Mesh Layout

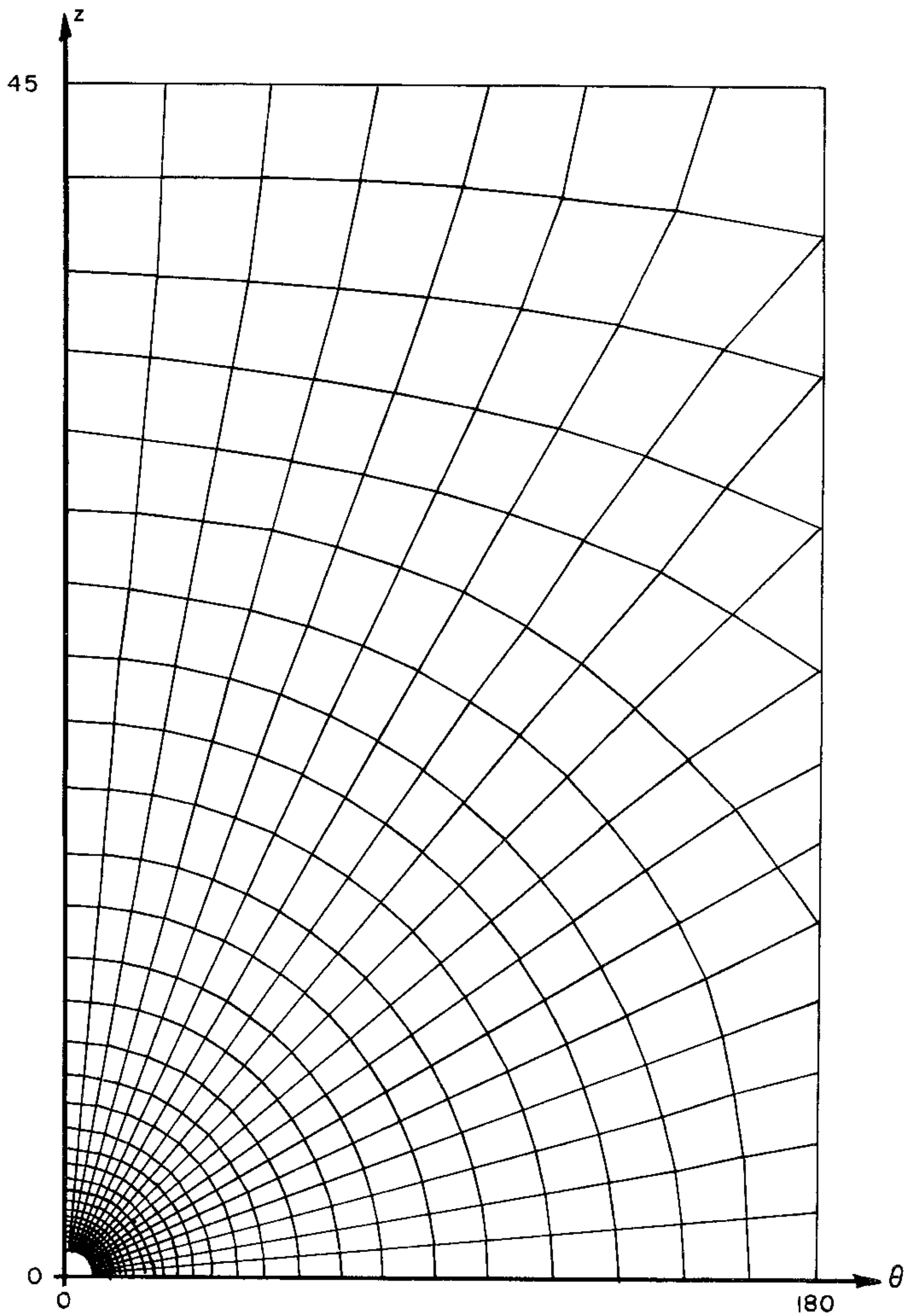

Figure 15. Developed Fine Mesh

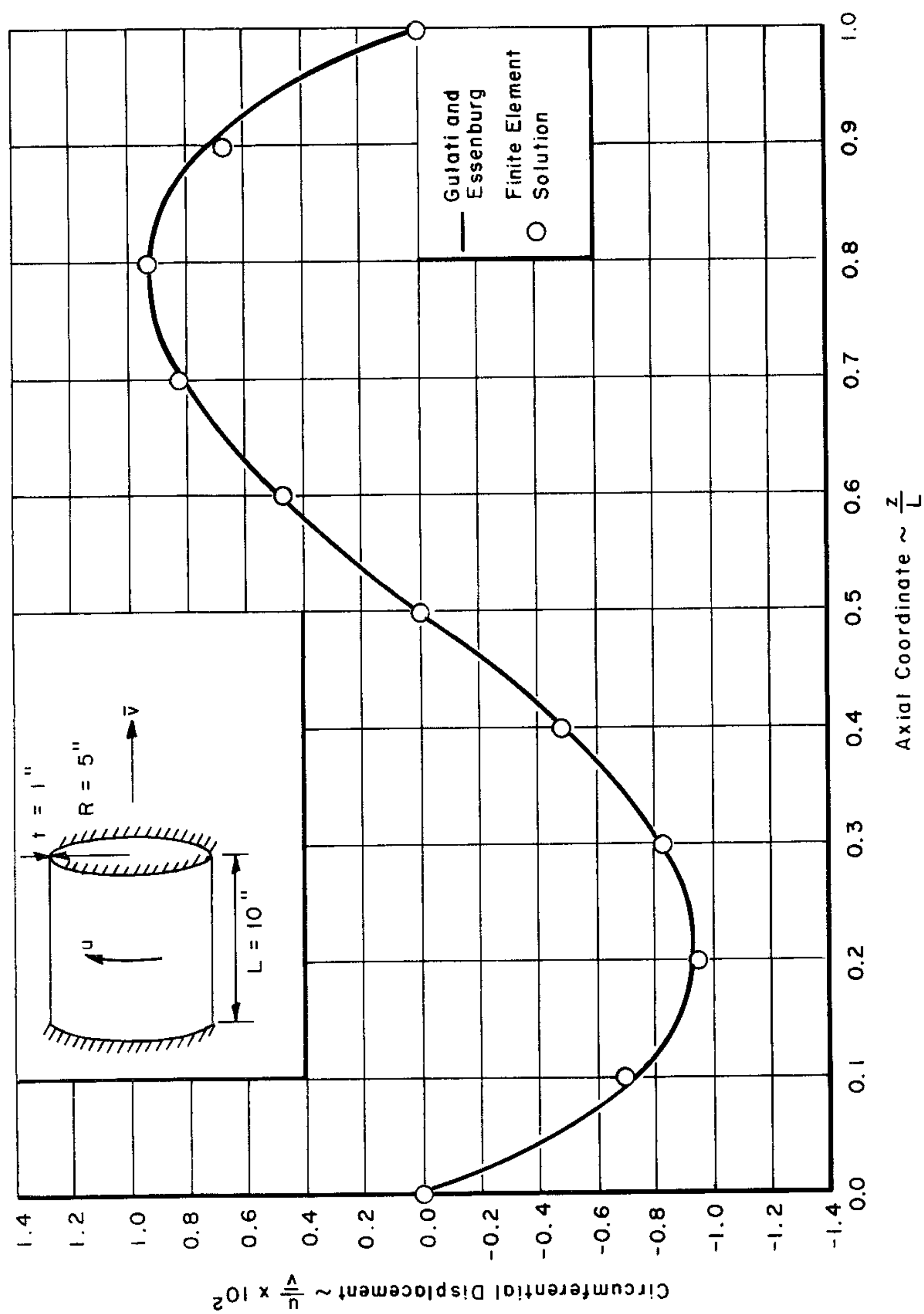

Figure 16. Circumferential Displacement for an Anisotropic Cylinder Uniformly Stretched

TABLE 1

STRESSES FOR A CYLINDER UNDER
AXIAL TENSION WITH A CIRCULAR CUTOUT

	Lekkerkerker 53 Graphical Results	Finite Element Solution[1] Coarse	Finite Element Solution[2] Fine
σ^{+}_{ss} @ A	3350 psi	3072 psi (-8%)	3338 psi (-.4%)
σ^{-}_{ss} @ A	4560 psi	4202 psi (-8%)	4506 psi (-1%)
$\sigma^{+}_{\theta\theta}$ @ B	-550 psi	-397 psi (-28%)	-469 psi (-15%)
$\sigma^{-}_{\theta\theta}$ @ B	-2200 psi	-1900 psi (-14%)	-2101 psi (-5%)

A is at the side of the hole at $r\theta = 1.0$, $z = 0.0$.

B is at the top of the hole at $r\theta = 0.0$, $z = 1.0$.

[1] Execution time for this mesh is 0.3 hours on a CDC 3600.

[2] Execution time for this mesh is 3.5 hours on a CDC 3600.

HEMISPHERE WITH AN OFF-CENTER HOLE

As an example of an irregular grid on a doubly curved shell, a 20-inch hemisphere with a 2-inch off-center hole was examined. Figure 17 shows the mesh used and the geometry of the problem. The hemisphere is clamped around the base and a symmetry plane is used to restrict the grid to the front half. The edge of the hole is free. The sphere is loaded with internal pressure. Figure 18 shows the inside and outside circumferential stresses along the line of symmetry. Figure 19 shows the inside and outside meridional stresses along the line symmetry. As a comparison, Figure 20 shows the inside and outside, circumferential and meridional stresses in a hemisphere with a centrally located hole.

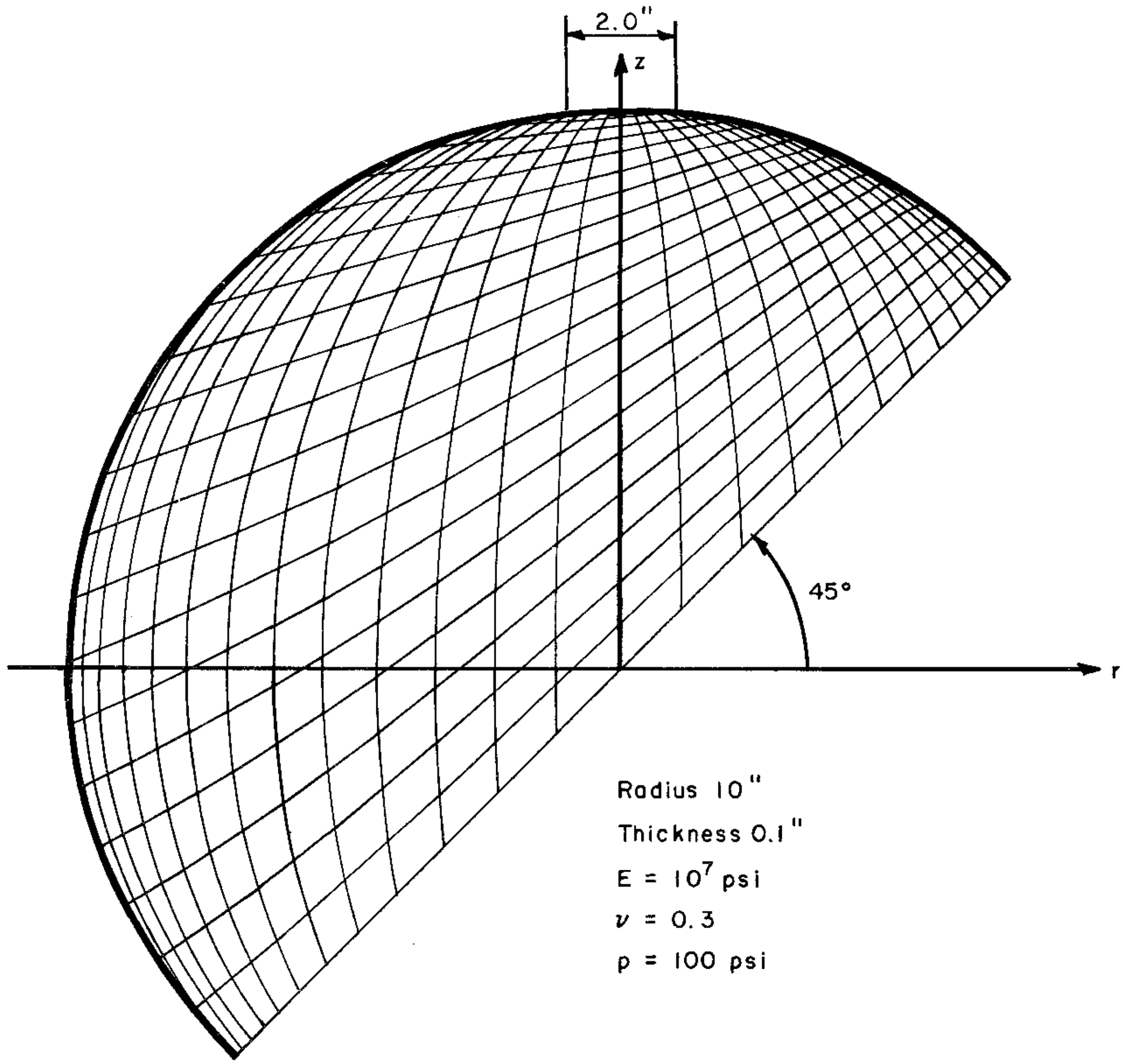

Figure 17. Hemisphere Mesh Layout

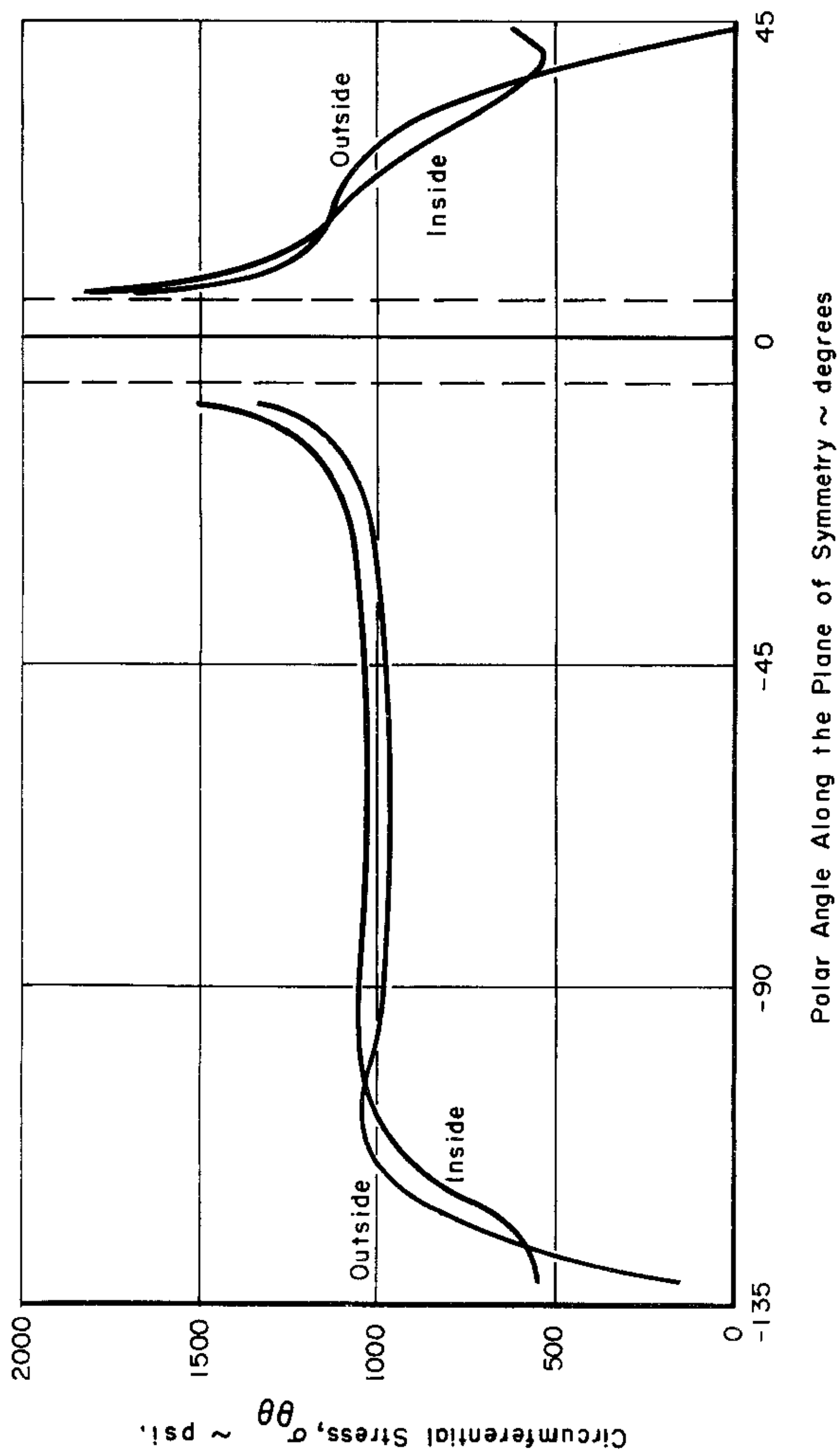

Figure 18. Circumferential Stresses in the Plane of Symmetry

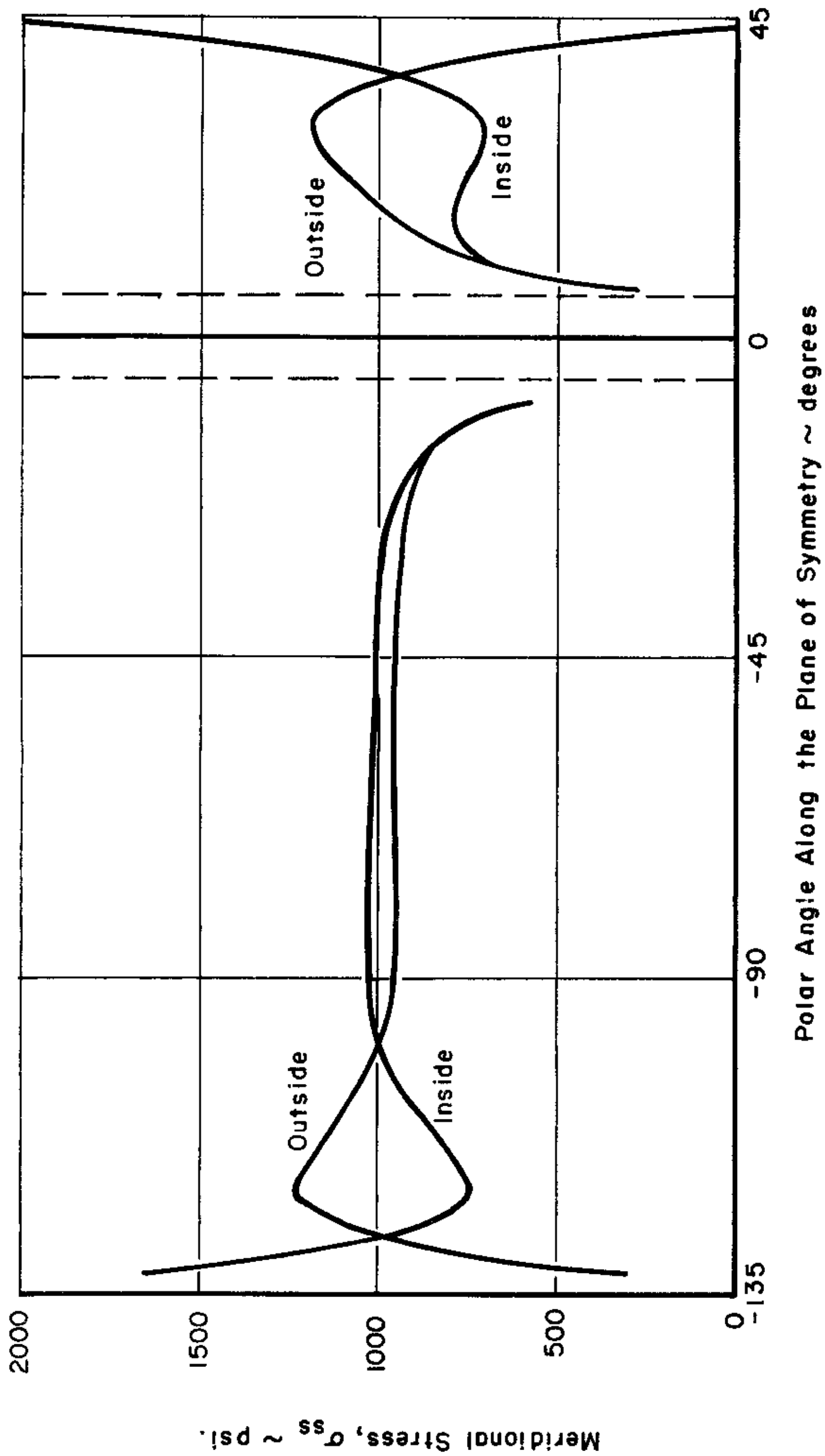

Figure 19. Meridional Stresses in the Plane of Symmetry

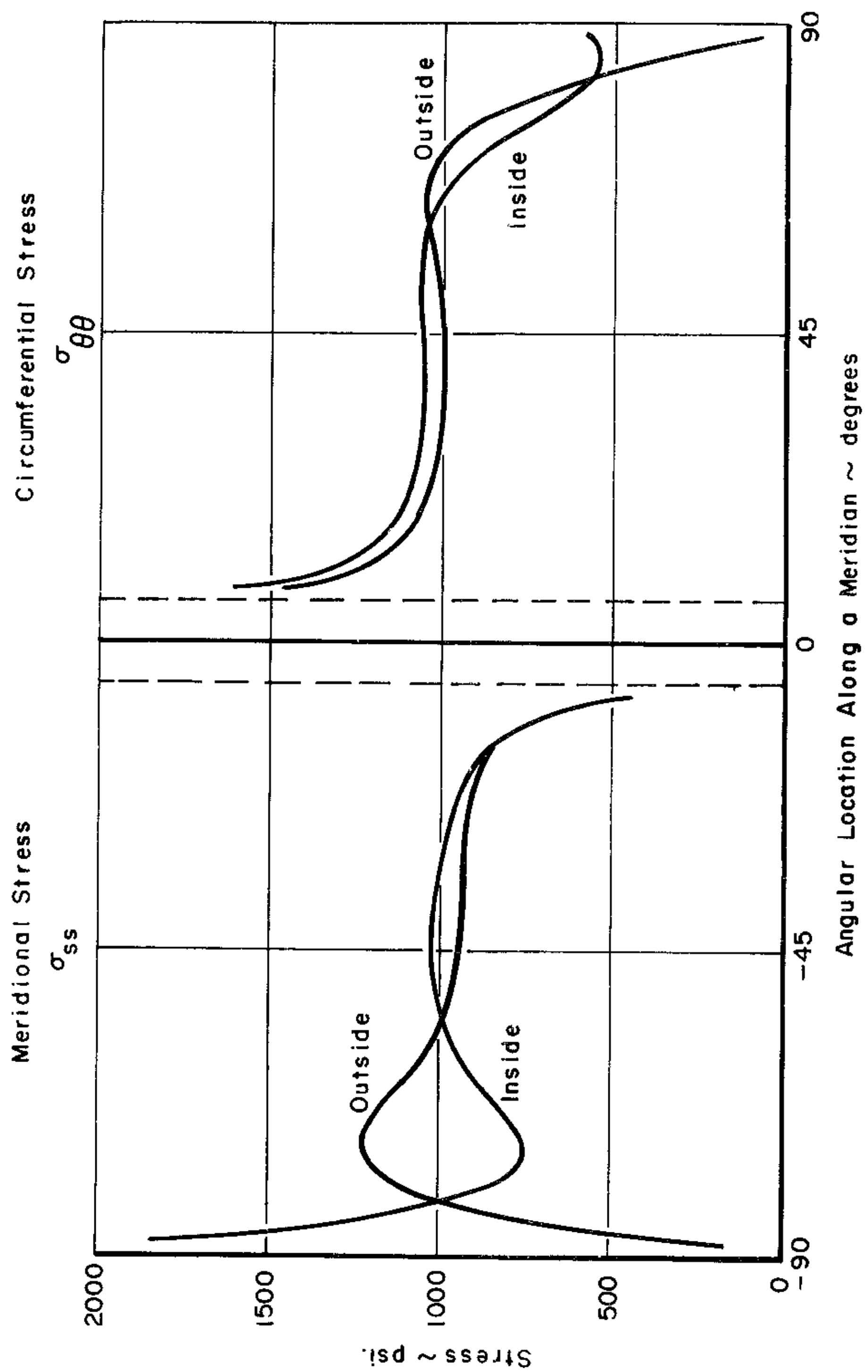

Figure 20. Stresses Due to Pressure in a Hemisphere With a Central Hole

RIGID BODY MODES

The deformation assumptions used to derive the element stiffness matrix do not contain accurate rigid body deformations. Rigid body deformations are trigonometric in character and are only approximated by the polynomials used. As a result, the stiffness matrix predicts restoring forces for rigid body displacements. For vanishing element size or curvature the restoring forces vanish and energy free rigid body motion is recovered. The strain-displacement relations used give zero strains for rigid body translations and infinitesimal rotations for any geometry. For the problems that have been examined, the number of elements needed to recover the needed rigid body freedom exceeds the number of elements needed to resolve the solution.

SECTION V

CONCLUSIONS

Along with the problems presented here, several extremely complicated practical problems have been analyzed with this element. The stresses have proven to be quite accurate and thermal stress analyses have been made with very good results. Shells with bulkheads, layered shells, shells with negative Gaussian curvature, shells with discontinuous meridional slope have all been analyzed with very good success. The only shortcoming discovered to date is the lack of rigid body degrees of freedom. (The rigid body degrees of freedom are recovered for either vanishing size or curvature). This has necessitated the use of large numbers of elements with the attendant execution time. Further work is under way in developing an element with all of the present features plus unrestrained rigid body motion. The computer program SLADE and the user's manual (Reference 51) will be available once this limitation has been removed.

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