ON A LINEAR PROPERTY OF LIGHTLY DAMPED SYSTEMS

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ABSTRACT

As one of the direct applications of complex damping theory, a useful property of structural damping is presented in this paper. If a structure is linear and lightly damped, (i.e. the maximum damping ratio < 0.3), then increased damping of the structure will result in proportional change in each modal damping ratio of the system. This property is particularly useful in damping re-design and damping measurement. A number of experimental and numerical examples are also presented.

INTRODUCTION

Quantities such as $\sqrt{1-\xi^2}$ and $\exp(-\xi\omega)$, where ξ is the damping ratio and ω is the undamped natural frequency, are often seen in the studies of dynamic systems. Direct treatments of these quantities are difficult. Furthermore, they are too complicated to be used in practice. Most engineering applications typically use approximated values for these quantities. In Table 1 some possible approximations of ξ and $\sqrt{1-\xi^2}$ together with the associated errors are given.

Table 1 Damping ratios and the approximations

approximation of ξ		approximation of $\sqrt{1-\xi^2}$		
ξ	sh(ξ) & error	$\sqrt{1-\xi^2}$	1 & error	$1-\xi^2/2$ & error
.001	1.0000005 0.%	0.9999995	1 5e- 5%	.9999995 0.0%
.01	.0100001 1.6e-5%	0.99995	1 5e- 3%	.999995 0.0%
.05	.0500208 .042%	0.9987492	1 0.125%	.99875 -8e-5%
.1	.1001667 .167%	0.9949874	1 0.501%	.995 .00126%
.2	.201336 .668%	0.9797959	1 2.020%	.98 .0204%
.25	.2526123 1.04%	0.9682458	1 3.175%	.96875 .0504%
.3	.3045202 1.51%	0.9539392	1 6.060%	.955 .106%

In this Table, the largest error appears when $\sqrt{1-\xi^2}$ is approximated by unity. If the value of ξ is less than 10% , then the error is no more than 0.5% . If the value of ξ is less than 30%, then this error is less than 5%. If we approximate $1 - \xi^2/2$ by unity, then the error is no more than 0.106%. These errors are tolerable in most engineering applications. We define a structural system to be *lightly damped* if the absolute value of the damping ratio for the system is less than 30% . The damping of most civil engineering structures such as buildings, bridges, dams and towers is usually less than 10%. Metal structures have even less damping. Theoretically speaking, for lightly damped systems, we have the following equations

$$\sqrt{1-\xi^2} \approx 1$$

$$\sqrt{1-\xi^2} \approx 1 - \xi^2/2$$
$$\exp(\xi) \approx 1 + \xi + \xi^2/2 \qquad (1)$$
$$ch(\xi) \approx 1$$
$$sh(\xi) \approx \xi$$

This paper is limited to the discussion to such systems.

COMPLEX DAMPING OF LIGHTLY DAMPED STRUCTURES

We first describe the complex damping ratios of lightly damped systems. Consider an MDOF system. For each virtual mode of the system, we can have an equation

$$\ddot{u} + (a + jb)\,\dot{u} + \omega_n^2 u = 0 \qquad (2)$$

The characteristic equation of (2) is given by

$$\lambda^2 + (a + jb)\,\lambda + \omega_n^2 = 0$$

with

$$\lambda = \frac{1}{2}\left[-(a+jb) \pm ((a+jb)^2 - 4\omega_n^2))^{1/2}\right]$$
$$= \omega_n \left[-\left(\frac{a+jb}{2\omega_n}\right) \pm \left(\left(\frac{a+jb}{2\omega_n}\right)^2 - 1\right)^{1/2}\right]$$
$$= j\,\omega_n \left[\left(\frac{a+jb}{2\omega_n} j\right) \pm \left(1 + \left(\frac{a+jb}{2\omega_n} j\right)^2\right)^{1/2}\right] \qquad (3)$$

Using (1) for lightly damped systems, we have

$$\left| \left(\frac{a+jb}{2\omega_n} j\right) \right| \ll 1 \qquad (4)$$

and

$$\left(1 + \left(\frac{a+jb}{2\omega_n} j\right)^2\right)^{1/2} \approx 1 + \frac{1}{2}\left(\frac{a+jb}{2\omega_n} j\right)^2 \qquad (5)$$

Without loss of generality, let us first take the positive sign of

$$\left(1 + \left(\frac{a+jb}{2\omega_n} j\right)^2\right)^{1/2}$$

in Equation (3). Then we have

$$\lambda = j\,\omega_n \left[\left(\frac{a+jb}{2\omega_n} j\right) + 1 + \frac{1}{2}\left(\frac{a+jb}{2\omega_n} j\right)^2\right] \qquad (6)$$

By using (1), we have

$$(\frac{a+jb}{2\omega_n} j) + 1 + \frac{1}{2} (\frac{a+jb}{2\omega_n} j)^2 \approx \exp(\frac{ja-b}{2\omega_n})$$

$$= \exp(j \frac{a}{2\omega_n}) \exp(\frac{-b}{2\omega_n})$$

$$\approx [1 + j \frac{a}{2\omega_n} - \frac{1}{2} (\frac{a}{2\omega_n})^2] \exp(\frac{-b}{2\omega_n})$$

$$\approx [j \frac{a}{2\omega_n} + (1 - (\frac{a}{2\omega_n} j)^2)^{1/2}] \exp(\frac{-b}{2\omega_n})$$

It follows that

$$\lambda = j\omega_n \left[j \frac{a}{2\omega_n} + (1 - (\frac{a}{2\omega_n} j)^2)^{1/2} \right] \exp(\frac{-b}{2\omega_n})$$

$$= - \frac{a}{2\omega_n} \exp(\frac{-b}{2\omega_n}) \omega_n + j(1 - (\frac{a}{2\omega_n} j)^2)^{1/2} \exp(\frac{-b}{2\omega_n}) \omega_n \qquad (7)$$

Now take the negative sign

$$- (1 + (\frac{a+jb}{2\omega_n} j)^2)^{1/2}$$

in Equation (3), we have

$$\lambda = - \frac{a}{2\omega_n} \exp(\frac{b}{2\omega_n}) \omega_n - j(1 - (\frac{a}{2\omega_n} j)^2)^{1/2} \exp(\frac{b}{2\omega_n}) \omega_n \qquad (8)$$

Combining Equations (7) and (8), we have

$$\lambda = - \frac{a}{2\omega_n} \exp(\frac{\mp b}{2\omega_n}) \omega_n \pm j(1 - (\frac{a}{2\omega_n} j)^2)^{1/2} \exp(\frac{\mp b}{2\omega_n}) \omega_n \qquad (9)$$

By comparing the Equation (9) with the standard form of λ,

$$\lambda = - \xi\omega \pm j\sqrt{1-\xi^2}\, \omega$$

we have

$$\frac{a}{2\omega_n} = \xi \qquad (a = 2 \xi \omega_n) \qquad (10)$$

and

$$\exp(\frac{\mp b}{2\omega_n}) \omega_n = \omega$$

By using (1)

$$\exp(\frac{\mp b}{2\omega_n}) \approx 1 \pm (1 \mp (\frac{b}{2\omega_n})^2)^{1/2}$$

Then

$$\frac{b}{2\omega_n} = \zeta \qquad (b = 2 \zeta \omega_n) \qquad (11)$$

In Equations (10) and (11), a and b are associated with the i^{th} virtual mode of the system. By assigning to a and b some proper subscripts, we have, for the i^{th} virtual mode of the system,

$$a_i = 2\,\xi_i \omega_{ni}$$
$$b_i = 2\,\zeta_i \omega_{ni} \tag{12}$$

and

$$\lambda_i = -\,\xi_i \exp(\mp\zeta_i)\,\omega_{ni} \pm j\sqrt{1-\xi^2}\,\exp(\mp\zeta_i)\,\omega_{ni} \tag{13a}$$

Sometimes, it is convenient to approximate λ_i by

$$\lambda_i = j\,\omega_n \exp(\,\xi \pm j\,\zeta\,) \tag{13b}$$

If we define the i^{th} *complex damping ratio* of a lightly damped system by

$$\vartheta_i = \frac{d_i}{2\omega_{ni}} = \xi_i \pm j\,\zeta_i \tag{14}$$

then we can make the following statements.

Theorem 1. For lightly damped MDOF system, The complex damping coefficient of the i^{th} virtual mode is

$$d_i = 2\,(\,\xi_i + j\,\zeta_i\,)\,\omega_{ni}$$

where the real part of the complex damping ratio, ξ_i, is the traditionally defined "damping ratio", ξ_i, i.e.

$$\xi_i = \frac{\text{real}\,(d_i\,)}{2\,\omega_{ni}} = \xi_i\,,$$

the imaginary part of the complex damping ratio, ζ_i, is associated with the change of undamped natural frequency ω_i from the zero-damping frequency ω_{ni}, i.e.

$$\omega_i = \exp(\zeta_i)\,\omega_{ni}.$$

Theorem 1 indicates that, for a lightly damped system, we can treat the real and imaginary parts of the complex damping ratio (or complex damping coefficient) separately. The Theorem is useful in energy analysis of real damping and imaginary damping.

THE LINEAR PROPERTY OF LIGHTLY DAMPED SYSTEMS

For lightly damped systems, the damped natural frequencies are approximately equal to the undamped natural frequencies. That is, if the value ζ_i is sufficiently small, then

$$\exp(\,\zeta_i) \approx 1$$

and

$$\omega_i = \exp(\zeta_i)\,\omega_{ni} \approx \omega_{ni}\,. \tag{15}$$

Equation (15) says that, if two lightly damped systems, H_1 and H_2, have the same mass and stiffness distribution, then

$$\Lambda^{(1)}_{Im} \approx \Lambda^{(2)}_{Im} \qquad (16)$$

and

$$\lambda^{(1)}_{iIm} \approx \lambda^{(2)}_{iIm}, \qquad i = 1,...2n \qquad (17)$$

To simplify the notations, we arrange the system eigenvalues in the following order

$$\lambda_1 , \lambda_2 , \ldots, \lambda_n$$

so that their corresponding natural frequencies satisfy

$$\omega_1 \le \omega_2 \le \ldots\ldots \le \omega_n .$$

For proportional systems, we now have the following lemma.

Lemma 1. If a lightly damped system H has proportional damping C_C which can be represented as the sum of two proportional damping C_{1C} and C_{2C}, i.e.

$$C_C = C_{1C} + C_{2C}$$

then, for the subsystem H_{1C} and H_{2C}, we have

$$\Lambda_{Re} = \Lambda^{(1C)}_{Re} + \Lambda^{(2C)}_{Re}$$

i.e.

$$\lambda_{iRe} = \lambda^{(1C)}_{iRe} + \lambda^{(2C)}_{iRe} \qquad i = 1,...2n \qquad (18)$$

and

$$\Lambda_{Im} \approx \Lambda^{(1C)}_{Im} \approx \Lambda^{(2C)}_{Im}$$

i.e.

$$\lambda_{iIm} \approx \lambda^{(1)}_{iIm} \approx \lambda^{(2)}_{iIm}, \qquad i = 1,...2n \qquad (19)$$

Lemma 1 says that, for a system with proportional damping, if it can be split into two subsystems both with proportional damping, then the imaginary part of the eigenvalues of the original system is the sum of the corresponding imaginary parts of the two subsystems. In other word, the damping ratios possess the following relationship

$$\xi^{(C)}_i = \xi^{(1C)}_i + \xi^{(2C)}_i, \qquad i = 1,.....2n \qquad (20)$$

where the superscript (.) stands for the corresponding system (.).

Lemma 1 can be used in damping identification. When dampers are added to a structure, the damping ratio of the structure is changed. By using equations (18) and (19), we can determine the damping ratio of the modified structure. In a later section, some examples will be given.

Lemma 1 can be extended to systems with general non-proportional damping. This is described in the following Lemma.

Lemma 2: For any lightly damped system with damping C, let H be the state matrix. If we separate the system into two subsystems H_P and H_N, then we have

$$\Lambda_{Re} \approx \Lambda_{Re}^{(P)}$$

i.e. $$\lambda_{iRe} \approx \lambda_{iRe}^{(P)}, \qquad i = 1, 2, \ldots 2n \qquad (21)$$

and $$\Lambda_{Im} \approx \Lambda_{Im}^{(P)} \approx \Lambda_{Im}^{(N)}$$

i.e. $$\lambda_{iIm} \approx \lambda_{iIm}^{(P)} \approx \lambda_{iIm}^{(N)}, \quad i = 1, 2, \ldots 2n \qquad (22)$$

This lemma is easily understood by noting that systems H and H_P have the identical damping ratios for their modes, and almost the same natural frequencies per each mode.

Theorem 2. If the damping matrix C of a lightly damped system H can be represented by the sum of two matrices C_1 and C_2, i.e. $C = C_1 + C_2$ then, for subsystems H_1 and H_2, we have

$$\Lambda_{Re} \approx \Lambda_{Re}^{(1)} + \Lambda_{Re}^{(2)}$$

i.e. $$\lambda_{iRe} \approx \lambda_{iRe}^{(1)} + \lambda_{iRe}^{(2)}, \qquad i = 1, 2, \ldots, 2n. \qquad (23)$$

and $$\Lambda_{Im} \approx \Lambda_{Im}^{(1)} \approx \Lambda_{Im}^{(2)}$$

i.e. $$\lambda_{iIm} \approx \lambda_{iIm}^{(1)} \approx \lambda_{iIm}^{(2)}, \qquad i = 1, 2, \ldots 2n \qquad (24)$$

PROOF.

Let $C_1 = C_{1P} + C_{1N}$ and $C_2 = C_{2P} + C_{2N}$. Then we have

$$C = C_1 + C_2 = (C_{1P} + C_{2P}) + (C_{1N} + C_{2N})$$

According to Lemma 2,

$$\Lambda_{Re} \approx \Lambda_{Re}^{(1P+2P)} = \Lambda_{Re}^{(1P)} + \Lambda_{Re}^{(2P)}$$

$$\approx \Lambda_{Re}^{(1)} + \Lambda_{Re}^{(2)}$$

The second half of the theorem is obvious.

Corollary 1. If lightly damped systems H_1 and H_2 have same mass and stiffness distribution and damping matrix C_2 of H_2 is β times of C_1 of H_1, (i.e. $C_2 = \beta C_1$), then, for H_1 and H_2, we have

$$\Lambda^{(1)} \approx \Lambda^{(2)}$$

i.e.

$$\lambda_{iIm}^{(1)} \approx \lambda_{iIm}^{(2)}, \qquad i = 1, 2, \ldots, 2n. \qquad (25)$$

and

$$\Lambda_{Re}^{(2)} \approx \beta\, \Lambda_{Re}^{(1)}$$

i.e.

$$\lambda_{iRe}^{(2)} \approx \beta\, \lambda_{iRe}^{(1)}, \qquad i = 1, 2, \ldots, 2n. \qquad (26)$$

APPLICATIONS AND EXAMPLES

Example I

Figure 1 shows a structure with 3 DOF. Before dampers are added, the system has the following damping ratios

Table 2 Damping Ratios of the Base Structure

Mode	I	II	III
damping ratio	.0102	.0087	.0079

By adding dampers to the base structure, the damping ratios are changed. Since the damping ratio of a damper is directly related to the physical parameters (such as the loss modulus and the volume of damping material), the ratio can be calculated when these parameters are given. Suppose we have already obtained the corresponding damping ratios contributed by the dampers (first row of Table 3). Now we would like to have the damping ratios of the structure after the dampers are incorporated. It is easy to see that the system is still lightly damped. So from Theorem 2 we can calculate the damping ratios using the linear property. The results are shown in the third row of Table 3. The last row in Table 3 gives of the experimental data to be directly compared with the calculated results.

Table 3 Calculation of damping ratio

mode	I	II	III
ξ_{add}	.275	.1010	.0744
ξ_{base}	.0102	.0087	.0079
calculated ξ	.2852	.1097	.07519
tested ξ	.2970	.0877	.06200

Example II

The second example is concerned with the **damping matrix decompositions**. Thus far, there are three popular damping matrix decompositions. (1). The Clough-Penzien decomposition

$$C = C_P + C_N$$

This decomposition gives a proportional damping matrix C_p. Consequently all the damping ratios of the system can be calculated.

(2). The pure proportional and non-proportional decomposition

$$C = C_d + C_o$$

This decomposition gives the pure non-proportional damping matrix C_o.

(3). The real-imaginary decomposition:

$$C = C_r + C_i$$

This decomposition gives the matrix C_r and the matrix C_i which provide the real part and the imaginary part of the complex damping ratio respectively.

Although decomposition (3) is in great use when dealing with energy analysis, its computations are intensive. With the help of Theorem 2 we can use the formula

$$C = C_d + C_o ,$$

to approximate $C = C_r + C_i$. This is a simple approach to obtain C matrix.

Suppose we have the following M-C-K system,

$$M = I, \quad C = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 4 & -2 \\ 0 & -1 & -2 & 5 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 180 & -48 & 0 & 0 \\ -48 & 136 & -88 & 0 \\ 0 & -88 & 180 & -92 \\ 0 & 0 & -92 & 92 \end{bmatrix}.$$

Since

$$C\,K = \begin{bmatrix} 408 & -144 & -92 & 92 \\ -324 & 544 & -352 & 0 \\ -132 & -440 & 992 & -552 \\ 48 & 40 & -732 & 644 \end{bmatrix}$$

we know that the system is non-proportionally damped. Using the pure proportional decomposition, we have

$$C_d = \begin{bmatrix} 3.8507 & -.2994 & .0301 & -.3307 \\ -.2944 & 3.6314 & -1.1827 & -.5486 \\ .0301 & -1.1827 & 3.6328 & -1.3303 \\ -.3307 & -.5486 & -1.3303 & 2.8851 \end{bmatrix}$$

and

$$C_o = \begin{bmatrix} -1.8507 & -.7006 & -1.0301 & .3307 \\ -.7006 & -.6314 & .1827 & -.4514 \\ -1.0301 & .1827 & .3672 & -.6697 \\ .3307 & -.4514 & -.6697 & 2.1149 \end{bmatrix}$$

The eigenvalues of the system are given by

$$-2.5182 \pm 16.5207j$$
$$-1.8893 \pm 13.6937j$$
$$-1.9617 \pm 9.8809j$$
$$-0.6307 \pm 3.0123j$$

their corresponding complex damping ratios are

$$.1507 \pm .0098j$$
$$.1367 \pm .0066j$$
$$.1947 \pm .0063j$$
$$.2050 \pm .0100j$$

The maximum damping ratio is about 21%. According to Corollary 1, if the damping matrix is reduced to one tenth of the original value, then the damping ratios will be approximately decreased to ten times smaller. Therefore the maximum damping ratio is about 2%.

In Table 4 we listed the results of $\lambda(H_{co})$ and $\vartheta(H_{co})$ from $C_d - C_o$ decomposition as well as the results from the system of C/10 damping.

Table 4 $\lambda(H_{co})$ and $\vartheta(H_{co})$

	Original system	System with C/10
eigenvalue λ	.0179 ± 16.7148j	.0000 ± 16.8771j
	-.0323 ± 13.8242j	.0000 ± 13.9147j
	.0148 ± 10.0737j	.0000 ± 10.0098j
	-.0004 ± 3.0769j	.0000 ± 3.0468j
complex damping ratio ϑ	1 ×	1e-4 ×
	.1507 ± .0098j	-.0100 ± .9811j
	.1367 ± .0066j	.0216 ± .6170j
	.1947 ± .0063j	-.0136 ± .6156j
	.2050 ± .0100j	.0012 ± .9824j

The numerical results in Table 4 show that the approximation is satisfactory. This is particularly obvious for the small damping ratios.

CONCLUDING REMARKS

Most engineering structures can be classified as lightly damped systems. Dynamic analyses of these structures could be different from and simpler than those of heavily damped systems. The nice linear property of the lightly damped systems presented in this paper is such an example. A further application of this property can be found in damper utilization design (see Liang et al 1991).

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